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On homotopy groups of quandle spaces and the quandle homotopy invariant of links

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Abstract

For a quandle X, the quandle space BX is defined, modifying the rack space of Fenn, Rourke and Sanderson [10], and the quandle homotopy invariant of links is defined in $\mathbb{Z}[\pi_2(BX)]$, modifying the rack homotopy invariant of [10]. It is known that the cocycle invariants introduced in [2, 4, 5] can be derived from the quandle homotopy invariant.

In this paper, we show that, for a finite quandle X, $\pi_2(BX)$ is finitely generated, and that, for a connected finite quandle X, $\pi_2(BX)$ is finite. It follows that the space spanned by cocycle invariants for a finite quandle is finitely generated. Further, we calculate $\pi_2(BX)$ for some concrete quandles. From the calculation, all cocycle invariants for those quandles are concretely presented. Moreover, we show formulas of the quandle homotopy invariant for the connected sum of knots and for the mirror image of links.

Keywords

link, rack, quandle, rack space, homotopy group, Postnikov tower, 2-cocycle invariant, Hurewicz homomorphism, group homology.

1 Introduction

A quandle is a set with a certain binary operation; a typical example is the set of elements of a conjugacy class in a group with the binary operation given by the conjugation of the group. For an oriented link L, the link quandle Q_L ([16]) is defined to be a quandle which is, roughly speaking, the conjugacy class of $\pi_1(S^3\backslash L)$ including meridians. For a finite quandle X, we call a quandle homomorphism $Q_L \to X$ an X-coloring of L, and call a link L with an X-coloring C an X-colored link. As an analog of the classifying space of a group, the quandle space BX of a quandle X is defined, modifying the rack space of Fenn, Rourke and Sanderson [10]. Further, the quandle homotopy invariant $\Xi_X(L;C) \in \pi_2(BX)$ of an X-colored link (L,C) is defined, modifying the rack homotopy invariant of [10], and the quandle homotopy invariant $\Xi_X(L) \in \mathbb{Z}[\pi_2(BX)]$ of a link L is defined to be the sum of $\Xi_X(L;C)$ for all X-colorings C of L. Independently, the quandle cocycle invariant $\Phi_\phi(L) \in \mathbb{Z}[A]$ of a link L was introduced in [4], using a 2-cocycle ϕ of X with a coefficient group A. Further, when X acts on A, the generalized cocycle invariant was introduced in [2]. Furthermore, for a 3-cocycle ϕ of X, the shadow cocycle invariant[1] was introduced in [5]. It is known that any of such cocycle invariants can be obtained from the quandle homotopy invariant by a homomorphism $\pi_2(BX) \to A$ induced by the 2-cocycle of the cocycle invariant, regarding the 2-cohomology of X as $H^2(BX;A)$.

[1]The shadow cocycle invariant can be regarded as a generalized cocycle invariant by regarding the 3-cocycle as a 2-cocycle with the coefficient group $\mathbb{Z}X$ on which X acts.

Therefore, it is important to know the concrete form of $\pi_2(BX)$ in order to study such cocycle invariants and the quandle homotopy invariant.

In this paper, we show that, for a finite quandle X, $\pi_2(BX)$ is finitely generated (Theorem 3.1), and that, for a connected finite quandle X, $\pi_2(BX)$ is finite (Theorem 3.6). It follows from Theorem 3.1 that, for a finite quandle X, the space spanned by cocycle invariants defined from X is finitely generated (Corollary 3.5). It follows from Theorem 3.6 that, for a connected finite quandle, any $\mathbb{Z}$-valued cocycle invariant is trivial (Corollary 3.9).

Theorems 3.1 and 3.6 are shown in the following way. It is known (see [20]) that, for a path connected CW complex W such that the action of $\pi_1(W)$ on $\pi_2(W)$ is trivial, there exists the following exact sequence,

$$H_3(\pi_1(W);\mathbb{Z}) \longrightarrow \pi_2(W) \longrightarrow H_2(W;\mathbb{Z}).$$

Hence, since $\pi_1(BX)$ acts on $\pi_2(BX)$ trivially ([10]), we have

$$H_3(\pi_1(BX);\mathbb{Z}) \longrightarrow \pi_2(BX) \longrightarrow H_2(BX;\mathbb{Z}).$$

Theorem 3.1 is shown, by showing that $H_3(\pi_1(BX);\mathbb{Z})$ and $H_2(BX;\mathbb{Z})$ are finitely generated for a finite quandle X. Theorem 3.6 is shown, by showing that $H_3(\pi_1(BX);\mathbb{Z})$ and $H_2(BX;\mathbb{Z})$ are finite for a connected finite quandle X.

Further, we calculate $\pi_2(BX)$ of some concrete quandles. For a dihedral quandle X of odd prime order p, we show that $\pi_2(BX) \cong \mathbb{Z}/p\mathbb{Z}$ (Proposition 4.1). It follows from Proposition 4.1 that any cocycle invariant defined from this quandle is equal to a scalar multiple of the shadow cocycle invariant defined from the Mochizuki 3-cocycle [22] (Corollary 4.2). For the tetrahedral quandle X, we show that $\pi_2(BX) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ (Proposition 4.4). It follows from Proposition 4.4 that any cocycle invariant defined from this quandle is equal to a linear sum of two particular cocycle invariants (Corollary 4.5). We also calculate $\pi_2(BX)$ for conjugation quandles of the symmetric group on 4 elements $\mathfrak{S}_4$ (Propositions 4.8 and 4.10). Moreover, we show that $\pi_2(BQ_K) \cong \mathbb{Z}$ for the knot quandle Q_K of a nontrivial knot K (Proposition 4.12). It follows from Proposition 4.12 that any natural transformation from the functor $\mathrm{Hom}(Q_K, \bullet)$ to the functor $\pi_2(B\bullet)$ is equal to a scalar multiple of the quandle homotopy invariant $\Xi_\bullet$ (Corollary 4.15).

Proposition 4.1 is shown in the following way. For a dihedral quandle X of odd prime order, it is known [22] that $H_2(BX;\mathbb{Z}) \cong 0$, and it is shown by direct calculation that $H_3(\pi_1(BX);\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$. Hence, the above mentioned exact sequence is rewritten,

$$\mathbb{Z}/p\mathbb{Z} \longrightarrow \pi_2(BX) \longrightarrow 0.$$

Hence, we can show that $\pi_2(BX) \cong \mathbb{Z}/p\mathbb{Z}$ by the existence of a non-trivial homomorphism $\pi_2(BX) \to \mathbb{Z}/p\mathbb{Z}$ induced by the Mochizuki 3-cocycle. We can calculate $\pi_2(BX)$ for other quandles X in a similar way, *i.e.*, we obtain an upper bound of the possibility of $\pi_2(BX)$ by the above mentioned exact sequence, and, further, we can determine the concrete form of $\pi_2(BX)$ by the existence of homomorphisms $\pi_2(BX) \to A$ induced by concrete cocycles.

Moreover, we show a formula of the quandle homotopy invariant for the connected sum of knots (Proposition 5.1) and a formula for the mirror image of links (Proposition 5.7). The connected sum formula is shown in the following way. For colored knots (K_1, C_1) and (K_2, C_2) whose colors at the connected arcs are equal, $\Xi_X(K_1 \sharp K_2; C_1 \sharp C_2) = \Xi_X(K_1; C_1)\Xi_X(K_2; C_2) \in \pi_2(BX)$ by definition. Hence, we can present $\Xi_X(K_1 \sharp K_2)$ in terms of $\Xi_X(K_1)$ and $\Xi_X(K_2)$ by counting colorings of $K_1 \sharp K_2$ in terms of colorings of K_1 and K_2.

The paper is organized as follows. In Section 2, we review the definitions of the quandle homotopy invariant and cocycle invariants. In Section 3, we show some properties of $\pi_2(BX)$. In Section 4, we calculate $\pi_2(BX)$ for some concrete quandles. In Section 5, we show formulas of the quandle homotopy invariant for the connected sum of knots and for the mirror image of links. In Section 6, as an Appendix we give a more geometric description of $\pi_2(BX)$ using the concordance relations.

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2 The quandle homotopy invariant

In this section, we review rack, quandle, coloring, the rack space, and define the quandle space modifying the rack space of [10, 11, 12]. Further, we define the quandle homotopy invariant of a link, modifying the rack homotopy invariant ([10]) of a framed link. We review that the cocycle invariant ([4, 2]) is derived from the quandle homotopy invariant.

A *rack* is a set X with a binary operation $* : X \times X \to X$ satisfying the following two conditions.

- For any $x, y \in X$, there is a unique element $z \in X$ such that $x = z * y$.

- For any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

A *quandle* is a rack X satisfying the following additional condition.

- For any $x \in X$, $x * x = x$.

A map $f : X \to Y$ between quandles or racks is called a *homomorphism* if $f(a*b) = f(a)*f(b)$ for any $a, b \in X$. We define the *associated group* of a rack R to be $\langle a \in R | ab = b(a*b),$ for $a, b \in R\rangle$ and denote it by $\mathrm{Ass}(R)$.

We review the link quandle Q_L of an oriented link L. Putting $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 | \ x^2 + y^2 \leq 1\} \cup [1, 2] \times \{0\} \subset \mathbb{R}^2$, we consider a continuous map $\mu : (\mathcal{D}, \{(0, 0)\}, \{(2, 0)\}) \to (S^3, L, \infty)$ such that $\mu^{-1}(L) = \{0, 0\}$. Then, the *link quandle* Q_L is the set of homotopy classes of such maps with an appropriate binary operation; see [16] for details.

For a quandle X, an X-*coloring* of an oriented link diagram D is a map $C : \{\text{arcs of } D\} \to X$ satisfying the condition of Figure 1 at each crossing of D. We denote the set of X-colorings of D by $\mathrm{Col}_X(D)$. It is known [16] that there is a canonical bijection between $\mathrm{Hom}(Q_L, X)$ and $\mathrm{Col}_X(D)$.

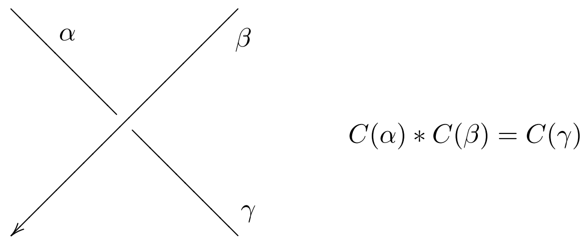

Figure 1: The condition of a coloring at each crossing

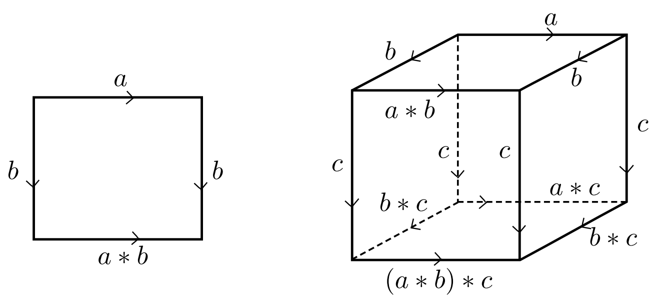

Figure 2: The 2-cell labeled by (a, b) and the 3-cell labeled by (a, b, c)

For a rack R, we review the *rack space* $\widehat{BR}$ introduced in [10]. The 0-skeleton of $\widehat{BR}$ consists of a single point. The 1-skeleton is obtained from the 0-skeleton by attaching 1-cells of the same number of elements of R, where we label the 1-cells by elements of R respectively. The 2-skeleton is obtained from the 1-skeleton by attaching 2-cells of squares labeled by (a, b) for any $a, b \in R$, where the 4 edges of a 2-cell are labeled by $a, b, a * b, b$

as shown in Figure 2, and the 4 edges are attached to the corresponding 1-cells. The 3-skeleton is obtained from the 2-skeleton by attaching 3-cells of cubes labeled by (a, b, c) for any $a, b, c \in R$, where the 6 faces of a 3-cell are labeled as shown in Figure 2, and the 6 faces are attached to the corresponding 2-cells. We omit the definition of the higher dimensional skeletons; see [10, 11, 12] for details, noting that the rack space is denoted BR there.

For a quandle X, we define the *quandle space* BX to be the space obtained from the rack space $\widehat{BX}$ by attaching 3-cells which bound 2-cells labeled by (a, a) for $a \in X$. By definition, BX and $\widehat{BR}$ are path connected and $\pi_1(BX) \cong \pi_1(\widehat{BX}) \cong \mathrm{Ass}(X)$.

We define the quandle homotopy invariant of an oriented link L, as follows, modifying the rack homotopy invariant [10] of a framed oriented link. For a diagram D of L and an X-coloring C of D, we construct a map $\xi_{D,C} : S^2 \to BX$, as follows. Regarding D as a decomposition of S^2 by immersed curves, we consider its dual decomposition. By $\xi_{D,C}$, we take the 0-cells of the dual decomposition to the single 0-cell of BX. We take a 1-cell at an arc colored by a to the 1-cell labeled by a. We take a 2-cell shown in Figure 3 to the 2-cell labeled by (a, b). By collecting them, we define a map $\xi_{D,C} : S^2 \to BX$. We denote the homotopy class of $\xi_{D,C}$ by $\Xi_X(D; C) \in \pi_2(BX)$. As shown in [10], $\Xi_X(D; C)$ is invariant under RII and RIII Reidemeister moves. Further, $\Xi_X(D; C)$ is invariant under the RI move, by a homotopy on the 3-cell bounding the 2-cell labeled by (a, a) (see Figure 4 for the corresponding 2-cell on S^2). Hence, $\Xi_X(D; C)$ is an invariant of (L, C), and we denote it by $\Xi_X(L; C)$. Further, we put

$$\Xi_X(L) \;=\; \sum_{C \in \mathrm{Hom}(Q_L, X)} \Xi_X(L; C) \;\in \mathbb{Z}[\pi_2(BX)],$$

and call it the *quandle homotopy invariant* of an oriented link L, where $\mathbb{Z}[\pi_2(BX)]$ is the integral group ring of $\pi_2(BX)$.

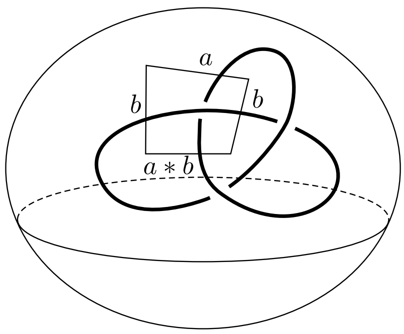

Figure 3: A 2-cell colored by (a, b) in the dual decomposition of D

We briefly review the quandle (co)homology groups ([4]) of a quandle X. We put $\mathcal{R} = \mathbb{Q}, \mathbb{Z}$ or $\mathbb{Z}/N\mathbb{Z}$. Let $C_n^R(X; \mathcal{R})$ be the free $\mathcal{R}$-module generated by X^n. Let $C_n^Q(X; \mathcal{R})$

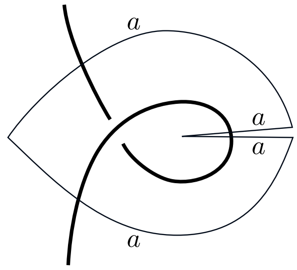

Figure 4: A 2-cell colored by (a, a) appearing in the R-I move

be the quotient of $C_n^R(X;\mathcal{R})$ by its submodule generated by n-tuples $(x_1, x_2, \dots x_n)$ with $x_i = x_{i+1}$ for some $i \in \{1, \cdots, n-1\}$. With appropriate boundary operators, $C_*^R(X;\mathcal{R})$ forms a chain complex, and $C_*^Q(X;\mathcal{R})$ forms a quotient chain complex of it. We denote their homology groups by $H_*^R(X;\mathcal{R})$ and $H_*^Q(X;\mathcal{R})$. From the dual complex $C_R^*(X;\mathcal{R}) = \mathrm{Hom}(C_*^R(X;\mathbb{Z}),\mathcal{R})$, we can define cohomology groups $H_R^*(X;\mathcal{R})$ and $H_Q^*(X;\mathcal{R})$. We call them the *rack (co)homology group* and the *quandle (co)homology group* of X. See [4] for details.

We review some properties of them. It is shown by definition that they are naturally isomorphic to the homology groups of $\widehat{BX}$ and BX,

$$H_2(\widehat{BX};\mathcal{R}) \cong H_2^R(X;\mathcal{R}), \qquad H_2(BX;\mathcal{R}) \cong H_2^Q(X;\mathcal{R}). \tag{1}$$

The universal coefficient theorem holds for the homology groups,

$$H_W^n(X;\mathcal{R}) \cong \mathrm{Hom}(H_n^W(X;\mathbb{Z}),\mathcal{R}) \oplus \mathrm{Ext}^1(H_{n-1}^W(X;\mathbb{Z}),\mathcal{R}), \quad W = R \text{ or } Q. \tag{2}$$

It is shown in [17] that

$$H_1^R(X;\mathbb{Z}) \cong \mathbb{Z}^m, \qquad H_2^R(X;\mathbb{Z}) \cong H_2^Q(X;\mathbb{Z}) \oplus \mathbb{Z}^m, \tag{3}$$

where m is the number of connected components of X (see Section 3.2 for the definition of a connected component).

We review the concept of a 2-cocycle and of the cocycle invariant, when a quandle X acts on the coefficient group. Let A be an abelian group on which X acts, *i.e.*, there is a map $\rho : X \to \mathrm{End}(A)$ satisfying that $\rho(a * b) = \rho(b)^{-1}\rho(a)\rho(b)$ for any $a, b \in X$. A map $\phi : X \times X \to A$ is called a *2-cocycle* if it satisfies the following formulas,

$$\begin{aligned}
&\phi(x,x) = 0,\\
&\rho(z)\cdot\phi(x,y) + \phi(x*y,z) + \rho((x*y)*z)\cdot\phi(y,z)\\
&\ = \phi(y,z) + \rho(y*z)\cdot\phi(x,z) + \phi(x*z,y*z).
\end{aligned}$$

Note that, when the action is trivial, ϕ is a 2-cocycle of the quandle cohomology group mentioned above. For a 2-cocycle ϕ, the cocycle invariant[2] $\Phi_\phi(L) \in \mathbb{Z}[A]$ of an oriented link L is defined [4, 2], and it is known [13, 24, 11] that the cocycle invariant is derived from the quandle homotopy invariant $\Xi_X(L)$ by

$$\Phi_\phi(L) = H_\phi\big(\Xi_X(L)\big) \ \in \mathbb{Z}[A], \tag{4}$$

where the homomorphism

$$H_\phi : \pi_2(BX) \longrightarrow A \tag{5}$$

is defined from ϕ, as follows. We present for an element $\gamma \in \pi_2(BX)$ as the union of 2-cells of BX labeled by (a,b) for some $(a,b) \in X^2$. We define $H_\phi(\gamma)$ to be the sum of $\big(\rho(x_n)\cdots\rho(x_1)\big)\phi(a,b)$ for such (a,b), where $x_1, \cdots, x_n$ are the labels of 1-cells of a path from the base point of γ to the 2-cell labeled by (a,b) along γ. Note that, when the action is trivial, the homomorphism (5) can alternatively be given by

$$H_\phi : \pi_2(BX) \xrightarrow{\mathfrak{H}} H_2(BX;\mathbb{Z}) \xrightarrow{\langle\phi,\cdot\rangle} A, \tag{6}$$

where $\mathfrak{H}$ is the Hurewicz homomorphism, and $\langle\phi,\cdot\rangle$ denotes the contraction by ϕ.

3 Properties of $\pi_2(BX)$

In this section, we show that $\pi_2(BX)$ is finitely generated for any finite quandle X in Theorem 3.1. Moreover, we show that $\pi_2(BX)$ is finite for any finite connected quandle X in Theorem 3.6. We show a relation between $\pi_2(BX)$ and $\pi_2(\widehat{BX})$ in Proposition 3.11.

3.1 $\pi_2(BX)$ is finitely generated

Theorem 3.1. *For a finite quandle X, $\pi_2(BX)$ and $\pi_2(\widehat{BX})$ are finitely generated.*

Proof. Since BX is obtained from $\widehat{BX}$ by attaching 3-cells, the inclusion $\widehat{BX} \hookrightarrow BX$ induces an epimorphism $\pi_2(\widehat{BX}) \to \pi_2(BX)$. It therefore suffices to show that $\pi_2(\widehat{BX})$ is finitely generated.

By Propositions 3.2 and 3.3 below, we obtain the following exact sequence,

$$H_3(\pi_1(\widehat{BX});\mathbb{Z}) \longrightarrow \pi_2(\widehat{BX}) \longrightarrow H_2(\widehat{BX};\mathbb{Z}).$$

Hence it is sufficient to show that $H_2(\widehat{BX};\mathbb{Z})$ and $H_3(\pi_1(\widehat{BX});\mathbb{Z})$ are finitely generated. Since $\widehat{BX}$ has finitely many 2-cells by definition, $H_2(\widehat{BX};\mathbb{Z})$ is finitely generated. Further

[2]This invariant was first defined in [4] when the action of X on A is trivial, where it is called the *quandle cocycle invariant*. Further, this invariant was defined for a general action in [2], where it is called the *generalized cocycle invariant*. The *shadow cocycle invariant* [5] (defined from a 3-cocycle) can also be regarded as a particular case of this invariant by regarding the 3-cocycle as a 2-cocycle with the coefficient group $\mathbb{Z}X$ on which X acts by (7).

it is known (see [20, Exercise 8.5$^{\rm bis}$]) that the homology group of a finitely generated group is finitely generated. Since the 1-skeleton of $\widehat{BX}$ consists of finitely many 1-cells, $\pi_1(BX)$ is finitely generated. Hence, $H_3(\pi_1(\widehat{BX});\mathbb{Z})$ is finitely generated. Therefore, $\pi_2(\widehat{BX})$ is finitely generated. $\square$

The following proposition is well known in the homotopy theory.

Proposition 3.2 (see [20, Lemma 8^{bis}.27], [1, Exercises II.5.1])**.** *Let W be a connected CW-complex such that the canonical action of $\pi_1(W)$ on $\pi_2(W)$ is trivial. Then, there is an exact sequence,*

$$\longrightarrow H_3(\pi_1(W);\mathbb{Z}) \xrightarrow{\tau} \pi_2(W) \xrightarrow{\mathfrak{H}} H_2(W;\mathbb{Z}) \longrightarrow H_2(\pi_1(W);\mathbb{Z}) \longrightarrow 0,$$

where $H_3(\pi_1(W);\mathbb{Z})$ denotes the group homology, τ denotes the transgressive map, and $\mathfrak{H}$ denotes the Hurewicz homomorphism.

Fenn, Rourke and Sanderson showed that the rack space satisfies the assumption of Proposition 3.2.

Proposition 3.3 ([11, 12])**.** *For any quandle X, the canonical action of $\pi_1(\widehat{BX})$ on $\pi_2(\widehat{BX})$ is trivial.*

Remark 3.4. The canonical action of $\pi_1(BX)$ on $\pi_2(BX)$ is also trivial. Because, since the inclusion $\widehat{BX} \hookrightarrow BX$ induces the isomorphism $\pi_1(\widehat{BX}) \cong \pi_1(BX)$ and an epimorphism $\pi_2(\widehat{BX}) \to \pi_2(BX)$, BX inherits the triviality of the action of $\pi_1(BX)$ on $\pi_2(BX)$ from $\widehat{BX}$.

We show a corollary of Theorem 3.1.

Corollary 3.5. *For a finite quandle X and a finitely generated abelian group A, the space spanned by A-valued cocycle invariants of X is finitely generated.*

Proof. By (4), an A-valued quandle cocycle invariant is derived from the quandle homotopy invariant by a homomorphism $\pi_2(BX) \to A$ of (5). As $\pi_2(BX)$ is finitely generated by Theorem 3.1, so is $\mathrm{Hom}(\pi_2(BX), A)$. $\square$

3.2 The rational homotopy group $\pi_2(BX) \otimes \mathbb{Q}$

In this section, we estimate the dimension of $\pi_2(BX) \otimes \mathbb{Q}$ for a finite quandle X, and show that $\pi_2(BX)$ is finite for a finite connected quandle X in Theorem 3.6.

For a quandle X, we consider a natural map,

$$X \longrightarrow \mathrm{Aut}(X), \qquad x \longmapsto (\bullet * x). \tag{7}$$

The *inner automorphism group* $\mathrm{Inn}(X)$ is the subgroup of $\mathrm{Aut}(X)$ generated by the image of the above map. A *connected component* of X is an orbit of the action of $\mathrm{Inn}(X)$ on X. A quandle X is called *connected*, when the action of $\mathrm{Inn}(X)$ is transitive.

Theorem 3.6.
(i) *Let X be a finite quandle with m connected components, let q be a prime $> |X|$ and let $\mathbb{F}_q$ be the finite field of order q. Then,*

$$\begin{aligned}\frac{m(m-1)}{2} \;&\leq\; \dim_{\mathbb{F}_q}(\pi_2(BX)\otimes\mathbb{F}_q) \;\leq\; \frac{m(m-1)(m+4)}{6},\\ \frac{m(m-1)}{2} \;&\leq\; \dim_{\mathbb{Q}}(\pi_2(BX)\otimes\mathbb{Q}) \;\leq\; \frac{m(m-1)(m+4)}{6}.\end{aligned}$$

(ii) *For a finite connected quandle X, $\pi_2(BX)$ is a finite abelian group whose elements are of order* $|\mathrm{Inn}(X)|\cdot|X|^2$.

Proof. We show (i), as follows. By changing the coefficients of the exact sequence of Proposition 3.2 to $\mathbb{F}_q$, we have the following exact sequence,

$$H_3(\mathrm{Ass}(X);\mathbb{F}_q) \longrightarrow \pi_2(BX)\otimes\mathbb{F}_q \longrightarrow H_2(BX;\mathbb{F}_q) \longrightarrow H_2(\mathrm{Ass}(X);\mathbb{F}_q) \longrightarrow 0,$$

noting that $\pi_1(BX)\cong\mathrm{Ass}(X)$. From the right part of the above sequence, we have the following inequality,

$$\begin{aligned}\dim(\pi_2(BX)\otimes\mathbb{F}_q) \;&\geq\; \dim\big(\mathrm{Ker}(H_2(BX;\mathbb{F}_q)\longrightarrow H_2(\mathrm{Ass}(X);\mathbb{F}_q))\big)\\ &\geq\; m(m-1)-\binom{m}{2} = \frac{m(m-1)}{2},\end{aligned}$$

where the second inequality is obtained from Lemmas 3.7 and 3.8 below. Further, we obtain an upper bound of the dimension of $\pi_2(BX)\otimes\mathbb{F}_q$ from the dimension of its adjacent terms in the above exact sequence,

$$\begin{aligned}\dim(\pi_2(BX)\otimes\mathbb{F}_q) \;&\leq\; \dim H_3(\mathrm{Ass}(X);\mathbb{F}_q)+\dim H_2(BX;\mathbb{F}_q)\\ &\leq\; \binom{m}{3}+m(m-1) \;=\; \frac{m(m-1)(m+4)}{6},\end{aligned}$$

where the second inequality is obtained from Lemmas 3.7 and 3.8 below. Therefore, we obtain the first formula of (i). We can obtain the second formula of (i) similarly.

We show (ii), as follows. It follows from the $m=1$ case of (i) that $\pi_2(BX)$ is a finite abelian group. It is known [17, Theorem 2] that any element of $H_2(BX;\mathbb{Z})$ is of order $|X|^2$. Further, by Lemma 3.8 below, any element of $H_3(\mathrm{Ass}(X);\mathbb{Z})$ is of order $|\mathrm{Inn}(X)|$. Hence, by the exact sequence of Proposition 3.2, any element of $\pi_2(BX)$ is of order $|\mathrm{Inn}(X)|\cdot|X|^2$. $\square$

Lemma 3.7. *Let X be a finite quandle with m connected components, and let $\mathbb{F}_q$ be a finite prime field of characteristic $> |X|$. Then,*

$$\dim_{\mathbb{F}_q} H_2(BX;\mathbb{F}_q) = \dim_{\mathbb{Q}} H_2(BX;\mathbb{Q}) = m(m-1).$$

Proof. It follows from [8, Corollary 4.3] that

$$\dim_{\mathbb{F}_q} H_i^R(X;\mathbb{F}_q) = \dim_{\mathbb{Q}} H_i^R(X;\mathbb{Q}) = m^i.$$

Hence, by (3) and (1),

$$\dim_{\mathbb{F}_q} H_2(BX;\mathbb{F}_q) = \dim_{\mathbb{F}_q} H_2^Q(X;\mathbb{F}_q) = \dim_{\mathbb{F}_q} H_2^R(X;\mathbb{F}_q) - m \ = \ m(m-1).$$

We can obtain $\dim_{\mathbb{Q}} H_2(BX;\mathbb{Q})$, similarly. $\square$

Lemma 3.8. *Let X be a finite quandle with m connected components, and let q be a prime number $> |X|$. Then,*

$$\dim_{\mathbb{F}_q} H_i(\mathrm{Ass}(X);\mathbb{F}_q) \le \binom{m}{i}, \qquad \dim_{\mathbb{Q}} H_i(\mathrm{Ass}(X);\mathbb{Q}) \le \binom{m}{i}.$$

Moreover, if X is connected and $i \ge 2$, then any element of $H_i(\mathrm{Ass}(X);\mathbb{Z})$ is of order $|\mathrm{Inn}(X)|$.

Proof. It is known [25, Theorem.2.10] that there is an exact sequence,

$$0 \longrightarrow \mathbb{Z}^m \longrightarrow \mathrm{Ass}(X) \xrightarrow{\varphi} \mathrm{Inn}(X) \longrightarrow 0,$$

where φ is the map obtained by extending the map (7) naturally. Since $|\mathrm{Inn}(X)| \le |\mathfrak{S}_{|X|}| = |X|!$ and $q > |X|$, by [1, Proposition III.10.4], $H_*(\mathrm{Ass}(X);\mathbb{F}_q)$ is isomorphic to the invariant subspace of $H_*(\mathbb{Z}^m;\mathbb{F}_q)$ with respect to the action of $\mathrm{Inn}(X)$. Hence,

$$\dim H_i(\mathrm{Ass}(X);\mathbb{F}_q) \le \dim H_i(\mathbb{Z}^m;\mathbb{F}_q) = \dim H_i\big((S^1)^m;\mathbb{F}_q\big) = \binom{m}{i},$$

and we obtain the first formula of the lemma. We can obtain the second formula similarly.

When X is connected, the above mentioned exact sequence is rewritten,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Ass}(X) \longrightarrow \mathrm{Inn}(X) \longrightarrow 0. \tag{8}$$

Hence, by [1, Proposition III.10.1], we can show that any element of $H_i(\mathrm{Ass}(X);\mathbb{Z})$ is of order $|\mathrm{Inn}(X)|$. $\square$

We show a corollary of Theorem 3.6.

Corollary 3.9. *Let X be a finite connected quandle, let q be a prime number $> |X|$, and let a be any natural number. Then, any $\mathbb{Z}$-valued cocycle invariant and any $\mathbb{Z}/q^a\mathbb{Z}$-valued cocycle invariant are trivial.*

Proof. An A-valued cocycle invariant is derived from the quandle homotopy invariant by a homomorphism $\pi_2(BX) \to A$ of (5). However, it follows from Theorem 3.6 that $\mathrm{Hom}(\pi_2(BX), \mathbb{Z}/q^a\mathbb{Z}) \cong \mathrm{Hom}(\pi_2(BX), \mathbb{Z}) \cong 0$. Therefore, such cocycle invariants are trivial. $\square$

Remark 3.10. It is known [16] that the knot quandle Q_K is connected. Further for a connected quandle Y the image of a quandle homomorphism $Y \to X$ is also connected by definition. Hence for the quandle homotopy invariant of knots the assumption X is connected is natural.

3.3 The relation between $\pi_2(BX)$ and $\pi_2(\widehat{BX})$

In this section, we show the relation between $\pi_2(BX)$ and $\pi_2(\widehat{BX})$.

Proposition 3.11. *Let X be a quandle with m connected components. Then, $\pi_2(\widehat{BX}) \cong \pi_2(BX) \oplus \mathbb{Z}^m$. The $\mathbb{Z}^m$-part is generated by the 2-cells labeled by (a, a) where $a \in X$ runs over representative elements of connected components of X.*

Proof. Since BX is obtained from $\widehat{BX}$ by attaching 3-cells, the canonical inclusion $E : \widehat{BX} \hookrightarrow BX$ induces an epimorphism $E_* : \pi_2(\widehat{BX}) \to \pi_2(BX)$, and we have the top exact sequence of the diagram below. Further, by (3), we have the bottom split exact sequence. Hence, we have the following commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\, E_* \arrow[r] \arrow[d, "\bar{H}"] & \pi_2(\widehat{BX}) \arrow[r, "E_*"] \arrow[d, "\widehat{H}"] & \pi_2(BX) \arrow[r] \arrow[d, "H"] & 0 & \text{(exact)} \\
0 \arrow[r] & \mathbb{Z}^m \arrow[r] & H_2^R(X;\mathbb{Z}) \arrow[r] & H_2^Q(X;\mathbb{Z}) \arrow[r] & 0 & \text{(split exact)}
\end{tikzcd}$$

where $\widehat{H}$ and H are the Hurewicz homomorphisms, and $\bar{H}$ is obtained by restricting $\widehat{H}$; it is shown by diagram chasing that $\bar{H}$ is well defined.

We show that $\bar{H}$ is an isomorphism, as follows. It follows from (3) that $\mathbb{Z}^m$ is generated by (a, a) where $a \in X$ runs over representative elements of connected components of X. Since $\mathrm{Ker}\, E_*$ is generated by the 2-cells labeled by (a, a) for $a \in X$ and the 2-cells labeled by (a, a) and $(a * b, a * b)$ are homotopic along the 3-cell labeled by (a, a, b), $\mathrm{Ker}\, E_*$ is generated by the 2-cells labeled by (a, a) where $a \in X$ runs over representative elements of connected components of X. Hence, $\bar{H}$ is an isomorphism.

Therefore, for a section $s : H_2^Q(X;\mathbb{Z}) \to \mathbb{Z}^m$, it is shown by diagram chasing that $\bar{H}^{-1} \circ s \circ \hat{H}$ gives a section $\pi_2(\widehat{BX}) \to \mathrm{Ker}\, E_*$. Hence, $\pi_2(\widehat{BX}) \cong \pi_2(BX) \oplus \mathbb{Z}^m$. $\square$

Proposition 3.11 is useful to obtain homotopy groups of quandle spaces from homotopy groups of rack spaces given in [11, 12, 27].

3.4 (Non)triviality of cocycle invariants

We give some criteria for (non)triviality of cocycle invariants.

Proposition 3.12. *Let X be a finite quandle, and let A be a finitely generated abelian group, on which X acts trivially. Let ϕ be an A-valued 2-cocycle of X, where its coefficient is trivial.*
(i) *If $H_2(\mathrm{Ass}(X);\mathbb{Z}) \cong 0$ and $[\phi] \neq 0 \in H^2_Q(X;A)$, then the cocycle invariant Φ_ϕ is non-trivial.*
(ii) *If the epimorphism $H_2(BX;\mathbb{Z}) \to H_2(\mathrm{Ass}(X);\mathbb{Z})$ in Proposition 3.2 is an isomorphism, then the cocycle invariant Φ_ϕ is trivial.*

Proof. We show (i), as follows. Since $H^Q_1(X;\mathbb{Z}) \cong \mathbb{Z}^m$ by (3), we have that $H^2_Q(X;A) \cong \mathrm{Hom}\big(H^Q_2(X;\mathbb{Z}), A\big)$ by the universal coefficient theorem. Since $[\phi] \neq 0 \in H^2_Q(X;A)$, the map $\langle \phi, \cdot \rangle$ of (6) is non-zero. Further, since $H_2(\mathrm{Ass}(X);\mathbb{Z}) \cong 0$, the Hurewicz homomorphism $\mathfrak{H}$ is surjective by Proposition 3.2. Therefore there exists $x \in \pi_2(BX)$ such that $\langle \phi, \mathfrak{H}(x) \rangle \neq 0 \in A$. Any element of $\pi_2(BX)$ can be presented by $\Xi_X(L;C)$ for some X-colored link (L, C) by Proposition 6.1 in Appendix. Hence, by (6), $H_\phi(\Xi_X(L;C)) = \Phi_\phi(C)$ is non-trivial.

We show (ii), as follows. Since $H_2(BX;\mathbb{Z}) \to H_2(\mathrm{Ass}(X);\mathbb{Z})$ is an isomorphism, the Hurewicz homomorphism is the zero map by Proposition 3.2. Therefore, by (6), H_ϕ is the zero map, and Φ_ϕ is trivial by (4). □

4 Examples of $\pi_2(BX)$ for some quandles

In this section, we calculate $\pi_2(BX)$ for some quandles: the dihedral quandle of odd prime order, the tetrahedral quandle, conjugation quandles of $\mathfrak{S}_4$ and the knot quandle.

4.1 The dihedral quandle of odd prime order

In this section, we calculate $\pi_2(BR_p)$ of the dihedral quandle R_p of odd prime order. We let p be an odd prime in this section.

The *dihedral quandle* R_p of order p is defined to be $\mathbb{Z}/p\mathbb{Z}$ with the binary operation $x * y = 2y - x$. This quandle is connected.

Proposition 4.1. *For a dihedral quandle R_p of odd prime order, $\pi_2(BR_p) \cong \mathbb{Z}/p\mathbb{Z}$. It is generated by $\Xi_{R_p}(T(2,p);C)$ where $T(2,p)$ is the $(2,p)$-torus knot and C is a non-trivial R_p-coloring of $T(2,p)$.*

Proof. By Proposition 3.2, we have the exact sequence,

$$H_3(\mathrm{Ass}(R_p);\mathbb{Z}) \longrightarrow \pi_2(BR_p) \longrightarrow H_2(BR_p;\mathbb{Z}),$$

noting that $\pi_1(BR_p) \cong \mathrm{Ass}(R_p)$.

We show that $H_3(\mathrm{Ass}(R_p);\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$, as follows. By direct calculation, we have the group isomorphism $\mathrm{Ass}(R_p) \cong \langle x, y | x^p = 1, xy = yx^{-1}\rangle (\cong \mathbb{Z} \ltimes \mathbb{Z}/p\mathbb{Z})$ given by $q \mapsto xy^q$ ($q \in R_p$) (see [7, Example 1.14]). Its integral cohomology groups are known by (8.14) of [14],

$$H^3(\mathrm{Ass}(R_p);\mathbb{Z}) \cong 0, \qquad H^4(\mathrm{Ass}(R_p);\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}.$$

Hence, it follows from the universal coefficient theorem that $H_3(\mathrm{Ass}(R_p);\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$. Hence, the above exact sequence is rewritten,

$$\mathbb{Z}/p\mathbb{Z} \longrightarrow \pi_2(BR_p) \longrightarrow H_2(BR_p;\mathbb{Z}).$$

We show that $H_2(BR_p;\mathbb{Z}) = 0$, as follows. It is known, see [22, Theorem 1.1 and Corollary 2.2], that $H_2^Q(R_p;\mathbb{F}_q) \cong 0$ for any prime number q. Hence, since $H_1(R_p;\mathbb{Z}) = \mathbb{Z}$ by (3), $H_Q^2(R_p;\mathbb{Z}) = 0$ by the universal coefficient theorem. Therefore, by (1), $H_2(BR_p;\mathbb{Z}) = 0$. Hence, the above exact sequence is rewritten,

$$\mathbb{Z}/p\mathbb{Z} \longrightarrow \pi_2(BR_p) \longrightarrow 0.$$

We give an epimorphism $\pi_2(BR_p) \to \mathbb{Z}/p\mathbb{Z}$, as follows. The Mochizuki 3-cocycle $\psi \in H_Q^3(R_p;\mathbb{Z}/p\mathbb{Z})$ [22] induces a homomorphism $H_\psi : \pi_2(BR_p) \to \mathbb{Z}/p\mathbb{Z}$ by (5), regarding ψ as a 2-cocycle with $\mathbb{Z}R_p$ coefficient (see footnote 2). It follows from a concrete calculation that $\Phi_\psi(T(2,p);C)$ is non-trivial in $\mathbb{Z}/p\mathbb{Z}$. Hence, we have an epimorphism $H_\psi : \pi_2(BR_p) \to \mathbb{Z}/p\mathbb{Z}$. Therefore, $\pi_2(BR_p) \cong \mathbb{Z}/p\mathbb{Z}$. □

Corollary 4.2. *Let R_p be a dihedral quandle of odd prime order. For any A-valued 2-cocycle ϕ of R_p, the cocycle invariant Φ_ϕ can be obtained from the shadow cocycle invariant of the Mochizuki 3-cocycle by some homomorphism $\mathbb{Z}/p\mathbb{Z} \to A$.*

Proof. By Proposition 4.1, we have an isomorphism $H_\psi : \pi_2(BR_p) \to \mathbb{Z}/p\mathbb{Z}$ induced by the Mochizuki 3-cocycle ψ, regarding ψ as a 2-cocycle. By (5), we have a homomorphism $H_\phi : \pi_2(BR_p) \to A$. Hence, by (4), Φ_ϕ is obtained from Φ_ψ by $H_\phi \circ H_\psi^{-1} : \mathbb{Z}/p\mathbb{Z} \to A$. □

Corollary 4.3. *Let R_p be a dihedral quandle of the order p. Then, any $\mathbb{Z}$-valued cocycle invariant of R_p is trivial. Further, if $N \in \mathbb{Z}_{\geq 0}$ is coprime to p, then any $\mathbb{Z}/N\mathbb{Z}$-valued cocycle invariant of R_p is also trivial.*

Proof. When p and N are coprime, any homomorphisms $\pi_2(BR_p) \to \mathbb{Z}$ and $\pi_2(BR_p) \to \mathbb{Z}/N\mathbb{Z}$ are zero. Hence, by Corollary 4.2, $\mathbb{Z}/N\mathbb{Z}$-valued and $\mathbb{Z}$-valued cocycle invariants are trivial. □

It is conjectured in [2, Page 527] that all $\mathbb{Z}/N\mathbb{Z}$-valued and $\mathbb{Z}$-valued cocycle invariants of R_3 are trivial. This conjecture holds as a particular case of Corollary 4.3.

4.2 The tetrahedral quandle

In this section, we calculate $\pi_2(BS_4)$ of the tetrahedral quandle S_4.

The *tetrahedral quandle* S_4 is defined to be the Alexander quandle $\mathbb{Z}[T]/(2, T^2+T+1)$ with the binary operation $x * y = Tx + (1-T)y$. This quandle is connected.

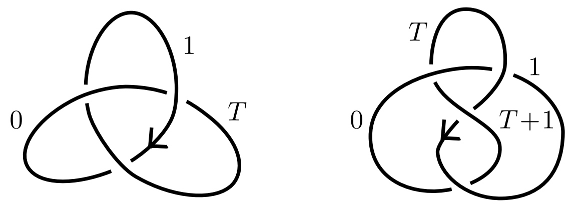

Figure 5: S_4-colorings C_1 and C_2 of the knots 3_1 and 4_1

Proposition 4.4. *For the tetrahedral quandle S_4, $\pi_2(BS_4) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. It is generated by $\Xi_{S_4}(3_1; C_1)$ and $\Xi_{S_4}(4_1; C_2)$, where C_1 and C_2 are the coloring of the knots 3_1 and 4_1 shown in Figure 5.*

Proof. We construct an epimorphism $\pi_2(BS_4) \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, as follows. For the $\mathbb{Z}/2\mathbb{Z}$-valued 2-cocycle ϕ given in [3], it follows from a concrete calculation that $\Phi_\phi(3_1; C_1) = 1$ and $\Phi_\phi(4_1; C_2) = 1$, where C_1 and C_2 are the S_4-colorings shown in Figure 5. For the 3-cocycle ψ given in [3, Page 64], it follows from a concrete calculation that $\Phi_\psi(3_1; C_1) = 1$ and $\Phi_\psi(4_1; C_2) = 0$, regarding ψ as a $\mathbb{Z}/4\mathbb{Z}$-valued 2-cocycle with $\mathbb{Z}S_4$ coefficient. Hence, we obtain an epimorphism,

$$H_\phi \oplus H_\psi : \pi_2(BS_4) \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z},$$

since it sends $\Xi_{S_4}(3_1; C_1)$ to $(1, 1)$ and $\Xi_{S_4}(4_1; C_2)$ to $(1, 0)$.

We show that the above epimorphism $H_\phi \oplus H_\psi$ is an isomorphism. It is sufficient to show that $|\pi_2(BS_4)| \leq 8$. By Proposition 3.2, we have the exact sequence

$$H_3(\mathrm{Ass}(S_4); \mathbb{Z}) \longrightarrow \pi_2(BS_4) \longrightarrow H_2^Q(S_4; \mathbb{Z}).$$

Since $H_2^Q(S_4; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ (see [23]) and $|H_3(\mathrm{Ass}(S_4); \mathbb{Z})| \leq 4$ by Lemma 4.7 below, we obtain $|\pi_2(BS_4)| \leq 8$. Therefore, $\pi_2(BS_4) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and it is generated by $\Xi_{S_4}(3_1, C_1)$ and $\Xi_{S_4}(4_1, C_2)$. $\square$

Corollary 4.5. *Any cocycle invariant of a tetrahedral quandle S_4 can be presented by a linear sum of the cocycle invariants Φ_ϕ and Φ_ψ given in the proof of Proposition 4.4.*

Proof. By Proposition 4.4, we have an isomorphism $H_\phi \oplus H_\psi : \pi_2(BS_4) \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. For any A-valued 2-cocycle η, we have a homomorphism $\pi_2(BS_4) \to A$ by (5). Hence, by (4), Φ_η is obtained from $\Phi_\phi \oplus \Phi_\psi$ by $H_\eta \circ (H_\phi \oplus H_\psi)^{-1}$. $\square$

Before passing to show Lemma 4.7, we prepare with Lemma 4.6 which we use later. Let X be a finite connected quandle. In general it is difficult to compute the integral group homology of infinite group $\mathrm{Ass}(X)$, We introduce the methods to estimate group homology $H_*(\mathrm{Ass}(X);\mathbb{Z})$.

We consider the following epimorphism given by the length of words;

$$\epsilon : \mathrm{Ass}(X) \longrightarrow \mathbb{Z}. \tag{9}$$

Then we have the group extension $0 \longrightarrow \mathrm{Ker}(\epsilon) \longrightarrow \mathrm{Ass}(X) \longrightarrow \mathbb{Z} \longrightarrow 0$. Since $\mathrm{Inn}(X)$ is finite, it follows from [1, Exercise IV.3.4] or [7, Remark 1.1] that $\mathrm{Ker}(\epsilon)$ is also finite.

Lemma 4.6. *For a natural number k, there exists the following short exact sequence.*

$$0 \longrightarrow \mathcal{H}_1(\mathbb{Z}; H_{k-1}(\mathrm{Ker}(\epsilon);\mathbb{Z})) \longrightarrow H_k(\mathrm{Ass}(X);\mathbb{Z}) \longrightarrow \mathcal{H}_0(\mathbb{Z}; H_k(\mathrm{Ker}(\epsilon);\mathbb{Z})) \longrightarrow 0$$

where $\mathcal{H}_p(G;A)$ is a group homology of G with local coefficients.

Proof. Using the Lyndon-Serre-Hochschild homology spectral sequence on the above group extension, we have a spectral sequence $E_r^{p,q}$ which satisfies $E_2^{p,q} \cong \mathcal{H}_p(\mathbb{Z}; H_q(\mathrm{Ker}(\epsilon);\mathbb{Z}))$ and converges strongly to $H_*(\mathrm{Ass}(X);\mathbb{Z})$.

Since for $p \geq 2$ we have $E_2^{p,q} \cong \mathcal{H}_p(\mathbb{Z}; H_q(\mathrm{Ker}(\epsilon);\mathbb{Z})) \cong 0$, the spectral sequence collapses at the E_2-term. Therefore, we obtain $H_k(\mathrm{Ass}(S_4);\mathbb{Z}) \cong E_k^{\infty} \cong E_k^2$. By the definition of strong convergence, we obtain the short exact sequence as desired. (For details see [20, Chapter 2,3].) □

Let us return the problem of computing $H_3(\mathrm{Ass}(S_4);\mathbb{Z})$.

Lemma 4.7. *$H_3(\mathrm{Ass}(S_4);\mathbb{Z})$ is of order ≤ 4.*

Proof. We decompose the associated group $\mathrm{Ass}(S_4)$. It is not difficult to obtain a group isomorphism $\mathrm{Ass}(S_4) \cong \langle A, B, C | AC = BCA, AC = CB, B = ABA, A = BAB \rangle$ given by $0 \mapsto AC, 1 \mapsto BC, T \mapsto C, 1+T \mapsto ABC$ and to check that $\mathrm{Ker}(\epsilon)$ of (9) is the quaternion group $Q_8 = \langle A, B | A = BAB, B = ABA \rangle$ using the Reidemeister Schreier method (see, for example, [19]). In summary we obtain a group extension $0 \longrightarrow Q_8 \longrightarrow \mathrm{Ass}(S_4) \longrightarrow \mathbb{Z} \longrightarrow 0$.

We claim $H_3(\mathrm{Ass}(S_4);\mathbb{Z})$ is a quotient group of $\mathbb{Z}/8\mathbb{Z}$. It is known (see 6.1(1) in [26].) that $H_2(Q_8;\mathbb{Z}) \cong 0$ *and* $H_3(Q_8;\mathbb{Z}) \cong \mathbb{Z}/8\mathbb{Z}$. It follows from Lemma 4.6 that $H_3(\mathrm{Ass}(S_4);\mathbb{Z}) \cong \mathcal{H}_0(\mathbb{Z}; H_3(Q_8;\mathbb{Z})) \cong \mathcal{H}_0(\mathbb{Z};\mathbb{Z}/8\mathbb{Z})$. Since in general the 0-th homology group of a group G with coefficients in M is $\mathcal{H}_0(G;M_\rho) \cong M/(1-\rho)M$, the homology $H_3(\mathrm{Ass}(S_4);\mathbb{Z})$ is a quotient of $\mathbb{Z}/8\mathbb{Z}$.

Further, we show that any element of $H_3(\mathrm{Ass}(S_4);\mathbb{Z})$ is of order 12. By direct calculation the inner group $\mathrm{Inn}(S_4)$ is the alternating group A_4. Since by (8) $\mathrm{Ass}(X)$ has an extension of $|A_4| = 12$ index, by Lemma 3.8 we directly estimate the order of elements of $H_3(\mathrm{Ass}(S_4);\mathbb{Z})$.

In summary we has shown that $H_3(\mathrm{Ass}(S_4);\mathbb{Z})$ is a quotient of $\mathbb{Z}/8\mathbb{Z}$ and that any element of $H_3(\mathrm{Ass}(S_4);\mathbb{Z})$ is of order 12. Hence, $H_3(\mathrm{Ass}(S_4);\mathbb{Z})$ is of order ≤ 4. $\square$

4.3 The conjugation quandle of a conjugacy class of $\mathfrak{S}_4$

In this section, we calculate $\pi_2(BS_6)$ of the quandle S_6, which is defined to be the set of elements of a conjugacy class in the symmetry group $\mathfrak{S}_4$ including $(12) \in \mathfrak{S}_4$ with the binary operation $x * y = y^{-1}xy$.

Proposition 4.8. *$\pi_2(BS_6)$ is either $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. The $\mathbb{Z}/3\mathbb{Z}$ part is generated by a S_6-colored trefoil knot.*

Proof. We consider the exact sequence of Proposition 3.2

$$H_3(\pi_1(BS_6);\mathbb{Z}) \longrightarrow \pi_2(BS_6) \longrightarrow H_2(BS_6;\mathbb{Z}) \longrightarrow H_2(\pi_1(BS_6);\mathbb{Z}) \longrightarrow 0.$$

We come now to calculate the terms. By direct calculation it is not difficult to show that $H_2(BS_6;\mathbb{Z}) \cong H_2^Q(S_6;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. On the other hand, it follows from Lemma 4.9 that $H_2(\pi_1(BS_6);\mathbb{Z})$ has a direct summand $\mathbb{Z}/2\mathbb{Z}$. Then the above exact sequence is rewritten,

$$H_3(\pi_1(BS_6);\mathbb{Z}) \longrightarrow \pi_2(BS_6) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

According to Lemma 4.9, as the order of $H_3(\pi_1(BS_6);\mathbb{Z})$ is a divisor of 12, so is the order of $\pi_2(BS_6)$.

It is sufficient to show that $\pi_2(BS_6)$ contains a direct summand $\mathbb{Z}/3\mathbb{Z}$. Let R_3 be a dihedral quandle of order 3. We define a homomorphism $i : R_3 \to S_6$ by $1 \mapsto (12), 2 \mapsto (13), 3 \mapsto (23)$ and a homomorphism $j : S_6 \to R_3$ by $(12),(34) \mapsto 1, (13),(24) \mapsto 2, (23),(14) \mapsto 3$. Then the composition $j\circ i = \mathrm{id}_{R_3}$ induces $j_*\circ i_* = \mathrm{id}_{\pi_2(BR_3)} : \pi_2(BR_3) \mapsto \pi_2(BR_3)$. Proposition 4.1 tell us that $\pi_2(BR_3) \cong \mathbb{Z}/3\mathbb{Z}$. Therefore $\pi_2(BS_6)$ has a direct summand $\mathbb{Z}/3\mathbb{Z}$ and its generator is represented by a colored trefoil. $\square$

Lemma 4.9. *The order of $H_3(\pi_1(BS_6);\mathbb{Z})$ is a divisor of 12 and $H_2(\pi_1(BS_6);\mathbb{Z})$ has a direct summand $\mathbb{Z}/2\mathbb{Z}$.*

Proof. Note that $\mathrm{Ass}(X) \cong \pi_1(BX)$. It is known [7, Example 1.15] that $\mathrm{Ass}(BS_6) \cong \mathbb{Z} \ltimes A_4$ where A_4 is an alternating group. It is known that $H_2(A_4;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and $H_3(A_4;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$ (e.g. see Lemma 4.2(4) in [26]). Since in general 0-th homology group of a group G with coefficients $\mathcal{H}_0(\mathbb{Z};M_\rho)$ is a quotient of M, we obtain the former applying Lemma 4.6 to $k = 3$.

We now show the latter. Applying Lemma 4.6 to $k = 2$, $H_2(\mathrm{Ass}(S_4);\mathbb{Z})$ contains $\mathcal{H}_0(\mathbb{Z};H_2(A_4;\mathbb{Z})) \cong \mathcal{H}_0(\mathbb{Z};\mathbb{Z}/2\mathbb{Z})$. However, in general the action of any group on $\mathbb{Z}/2\mathbb{Z}$

is trivial; $\mathrm{End}(\mathbb{Z}/2\mathbb{Z}) = \{1\}$. Namely the coefficient group of the group homology is trivial. Hence we have $\mathcal{H}_0(\mathbb{Z};\mathbb{Z}/2\mathbb{Z}) \cong H_0(\mathbb{Z};\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and $H_2(\mathrm{Ass}(S_4);\mathbb{Z})$ contains a direct summand $\mathbb{Z}/2\mathbb{Z}$. □

4.4 The conjugation quandle of another conjugacy class of $\mathfrak{S}_4$

In this section, we calculate $\pi_2(BS_6')$ of the quandle S_6', which is defined to be the set of elements of a conjugacy class in $\mathfrak{S}_4$ including $(1234) \in \mathfrak{S}_4$ with the binary operation $x * y = y^{-1}xy$.

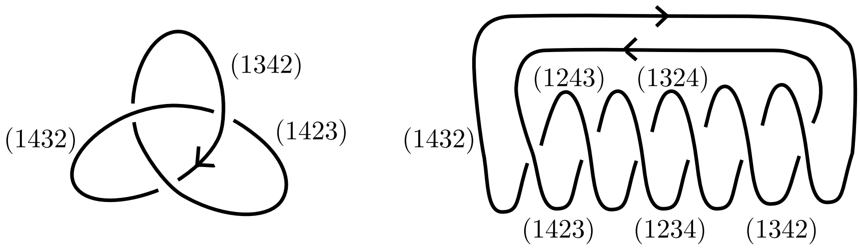

Figure 6: S_6'-colorings C_3 and C_4 of the knot 3_1 and of a link

Proposition 4.10. *For the conjugation quandle S_6', $\pi_2(BS_6') \cong \mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. It is generated by a $\Xi_{S_6'}(C_3)$ and $\Xi_{S_6'}(C_4)$, where C_3 is a coloring of the trefoil knot 3_1 and C_4 is a coloring of a link shown as Figure 6.*

Proof. We construct an epimorphism $\pi_2(BS_6') \to \mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, as follows. It is known [5, Example 7.5] that $H_Q^3(S_6';\mathbb{Z}/24\mathbb{Z}) \cong \mathbb{Z}/24\mathbb{Z}$. We denote its generator by ϕ. It follows from example 7.5 in [5] that the shadow 3-cocycle invariant of C is $\widehat{\Phi}_\phi(C) = 1 \in \mathbb{Z}/24\mathbb{Z}$. Hence the homomorphism $H_\phi : \pi_2(BS_6') \to \mathbb{Z}/24\mathbb{Z}$ derived from (5) is surjective. On the other hand, by direct calculation it is not difficult to show that $H_Q^2(S_6';\mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$ is generated by the following 2-cocycle;

$$\psi = 3\chi_{(1432),(1342)} + 3\chi_{(1234),(1243)} + 3\chi_{(1423),(1234)} + 3\chi_{(1243),(1432)}$$
$$+\chi_{(1423),(1432)} + \chi_{(1423),(1234)} + \chi_{(1423),(1342)} + \chi_{(1423),(1243)},$$

where we regard ψ as a map $S_6' \times S_6' \to \mathbb{Z}/4\mathbb{Z}$ and $\chi_{a,b}$ is a delta function. From a concrete calculation its 2-cocycle invariants are $\Phi_\psi(C_3) = 0$ and $\Phi_\psi(C_4) = 1$. Hence we obtain an epimorphism

$$H_\phi \oplus H_\psi : \pi_2(BS_6') \longrightarrow \mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$$

We show that the above epimorphism $H_\phi \oplus H_\psi$ is an isomorphism. It is sufficient to show that $|\pi_2(BS_6')| \leq 96$. We set up the usual exact sequence and calculate the terms,

$$H_3(\pi_1(BS_6');\mathbb{Z}) \longrightarrow \pi_2(BS_6') \longrightarrow H_2(BS_6';\mathbb{Z}).$$

It follows from $H_2^Q(S_6';\mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$ and universal coefficient theorem that $H_2(BS_6';\mathbb{Z}) \cong H_2^Q(S_6';\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$. Therefore the above sequence is rewritten,

$$H_3(\pi_1(BS_6');\mathbb{Z}) \longrightarrow \pi_2(BS_6') \longrightarrow \mathbb{Z}/4\mathbb{Z}.$$

Lemma 4.11 below says that $H_3(\pi_1(BS_6');\mathbb{Z})$ is a quotient of $\mathbb{Z}/24\mathbb{Z}$. Therefore $\pi_2(BS_6') \cong \mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. It is generated by a $\Xi_{S_6'}(C_3)$ and $\Xi_{S_6'}(C_4)$. □

Lemma 4.11. *$H_3(\pi_1(BS_6');\mathbb{Z})$ is a quotient of $\mathbb{Z}/24\mathbb{Z}$.*

Proof. We decompose the associated group $\mathrm{Ass}(S_6')$. It is shown by direct calculation that there is a group isomorphism $\mathrm{Ass}(S_6') \cong \langle A, y, Y | yYyA = AyYy = YAy = yAY,\ yYy = YyY, yY^2y = Y\ \rangle$ given by $(1234) \mapsto yYyA, (1432) \mapsto A, (1243) \mapsto yA, (1342) \mapsto YA, (1324) \mapsto Ay, (1423) \mapsto AY$. Noting that the homomorphism ϵ of (9) sends A, y, Y to $1, 0, 0$ respectively, by the Reidemeister Schreier method we can obtain $\mathrm{Ker}(\epsilon) \cong \langle y, Y | yYy = yYy,\ yY^2y = Y\rangle \cong D_{24}$, where D_{24} is the binary tetrahedral group. Then we have a group extention $0 \to D_{24} \to \mathrm{Ass}(S_6') \to \mathbb{Z} \to 0$.

It is known (see, for example, 6.1(3) in [26]) that the homology groups are $H_3(D_{24};\mathbb{Z}) \cong \mathbb{Z}/24\mathbb{Z}$ and $H_2(D_{24};\mathbb{Z}) \cong 0$. Hence, the result follows from Lemma 4.6 in the case $k = 2$. □

4.5 The knot quandle

In this section, we calculate $\pi_2(BQ_K)$ of the knot quandle Q_K of a knot K in Proposition 4.12. As a corollary of the proposition, we give a characterization of the quandle homotopy invariant as a natural transformation between some functors.

Regarding the identity map $\mathrm{id}_{Q_K} \in \mathrm{Hom}(Q_K, Q_K)$ as a Q_K-coloring of Q_K, as in [24], we call $\Xi_{Q_K}(K; \mathrm{id}_{Q_K}) \in \pi_2(BQ_K)$ the *fundamental class.*

Proposition 4.12. *For a non-trivial knot K, $\pi_2(BQ_K) \cong \mathbb{Z}$ and it is generated by the fundamental class $\Xi_{Q_K}(K; \mathrm{id}_{Q_K})$.*

Proof. Since $\mathrm{Ass}(Q_K) \cong \pi_1(S^3 \backslash K)$ ([16]), the exact sequence of Proposition 3.2 is as follows,

$$H_3(\pi_1(S^3\backslash K);\mathbb{Z}) \longrightarrow \pi_2(BQ_K) \longrightarrow H_2(BQ_K;\mathbb{Z}) \longrightarrow H_2(\pi_1(S^3\backslash K);\mathbb{Z}).$$

Let us calculate $H_*(\pi_1(S^3\backslash K);\mathbb{Z})$. Since $S^3\backslash K$ is Eilenberg-MacLane space, $H_*(\pi_1(S^3\backslash K);\mathbb{Z}) \cong H_*(S^3\backslash K;\mathbb{Z})$. It is shown by elementary calculation that $H_*(S^3\backslash K;\mathbb{Z}) = 0$ for $* = 2, 3$. Therefore, the above sequence is rewritten

$$0 \longrightarrow \pi_2(BQ_K) \xrightarrow{\mathfrak{H}} H_2^Q(Q_K;\mathbb{Z}) \longrightarrow 0.$$

Hence, the Hurewicz homomorphism $\mathfrak{H}$ induces $\pi_2(BQ_K) \cong H_2^Q(Q_K;\mathbb{Z})$. Further, it is known [6, Lemma 44] that $H_2^Q(Q_K;\mathbb{Z}) \cong \mathbb{Z}$ and it is generated by $\mathfrak{H}(\Xi_{Q_K}(K;\mathrm{id}_{Q_K}))$. Hence $\pi_2(BQ_K) \cong \mathbb{Z}$ and it is generated by $\Xi_{Q_K}(K;\mathrm{id}_{Q_K})$. $\square$

Remark 4.13. If K is the trivial knot, the knot quandle Q_K consists of a single element. Then, by definition, we have $H_2^Q(Q_K;\mathbb{Z}) \cong 0$, $\mathrm{Ass}(Q_K) \cong \mathbb{Z}$ and $H_3(\mathbb{Z};\mathbb{Z}) \cong 0$. Hence, by Proposition 3.2, we obtain $\pi_2(BQ_K) \cong 0$.

Remark 4.14. Wiest [27] showed that $\pi_2(\widehat{BQ_K}) \cong \pi_3(S^2 \vee S^3) \cong \mathbb{Z} \oplus \mathbb{Z}$ by geometric methods. We can verify this by $\pi_2(\widehat{BQ_K}) \cong \pi_2(BQ_K) \oplus \mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}$ by Propositions 3.11 and 4.12, since the knot quandle Q_K is connected (see [16]).

We regard $\mathrm{Hom}(Q_K, \bullet)$ and $\pi_2(B\bullet)$ as functors from the category of quandles. Then the quandle homotopy invariant $\Xi_\bullet$ can be regarded as a natural transformation from the former to the latter. It follows from Yoneda's lemma (see, for example, [21, Chapter 3]) that we have a bijection

$$\mathrm{Nat}\big(\mathrm{Hom}(Q_K, \bullet), \pi_2(B\bullet)\big) \cong \pi_2(BQ_K) \qquad \text{given by } \alpha \mapsto \alpha(\mathrm{id}_{Q_K}). \tag{10}$$

Through this bijection, the quandle homotopy invariant corresponds to the fundamental class $\Xi_{Q_K}(K;\mathrm{id}_{Q_K})$.

Corollary 4.15. *Let K be a non-trivial knot. Any natural transformation from the functor $\mathrm{Hom}(Q_K, \bullet)$ to the functor $\pi_2(B\bullet)$ is equal to a scalar multiple of the quandle homotopy invariant $\Xi_\bullet$.*

Proof. A natural transformation $\tau \in \mathrm{Nat}(\mathrm{Hom}(Q_K, \bullet), \pi_2(B\bullet))$ corresponds to $\tau(\mathrm{id}_{Q_K}) \in \pi_2(BQ_K)$ through (10). Since $\pi_2(BQ_K) \cong \mathbb{Z}$ is generated by the fundamental class $\Xi_{Q_K}(K;\mathrm{id}_{Q_K})$ by Proposition 4.12, $\tau(\mathrm{id}_{Q_K})$ is a scalar multiple of the fundamental class. Hence, the natural transformation τ is a scalar multiple of the quandle homotopy invariant $\Xi_\bullet$. $\square$

5 Formula for the connected sum and the mirror image

In this section, we show the formulas of the quandle homotopy invariant for the connected sum and the mirror image.

Proposition 5.1. *Let X be a connected finite quandle such that (7) is injective. Then, for knots K_1 and K_2,*

$$\Xi_X(K_1 \sharp K_2) = \frac{1}{|X|}\Xi_X(K_1) \cdot \Xi_X(K_2) \ \in \mathbb{Z}[\pi_2(BX)],$$

where $K_1 \sharp K_2$ denotes the connected sum of K_1 and K_2.

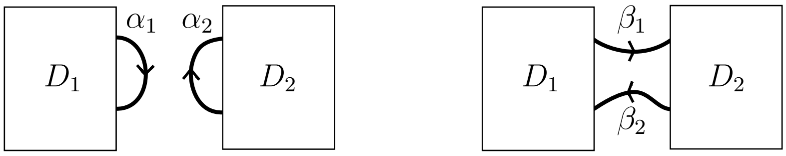

Figure 7: The arcs $\alpha_1, \alpha_2, \beta_1, \beta_2$

Proof. We choose diagrams D_1, D_2 of knots K_1, K_2 respectively. $D_1 \sharp D_2$ is a diagram of $K_1 \sharp K_2$. Let α_1, α_2, β_i be arcs of $D_1, D_2, D_1\sharp D_2$ shown in Figure 7. We define $\mathrm{Col}_X^a(D_i)$ to be the subset of $\mathrm{Col}_X(D_i)$ such that α_i is colored by $a \in X$ and define $\mathrm{Col}_X^{a,b}(D_1\sharp D_2)$ to be the subset of $\mathrm{Col}_X(D_1\sharp D_2)$ such that β_1, β_2 are colored by $a, b \in X$ respectively.

By Lemma 5.5 below, the right-hand side of the required formula is

$$\frac{1}{|X|}\Xi_X(K_1)\cdot\Xi_X(K_2) \;=\; |X| \sum_{(C_1,C_2)\in \mathrm{Col}_X^a(D_1)\times \mathrm{Col}_X^a(D_2)} \Xi(K_1;C_1)\cdot \Xi(K_2;C_2).$$

Further, by Lemma 5.5, the left-hand side is

$$\Xi_X(K_1\sharp K_2) \;=\; \sum_{x\in X}\sum_{C\in \mathrm{Col}_X^{a,x}(D_1\sharp D_2)} \Xi_X(K_1\sharp K_2;C) \;=\; |X| \sum_{C\in \mathrm{Col}_X^{a,a}(D_1\sharp D_2)} \Xi_X(K_1\sharp K_2;C),$$

where the second equality is obtained from Lemma 5.6. Hence, the required formula is obtained, since

$$\Xi_X(D_1\sharp D_2, C) = \Xi_X(D_1, C_1)\cdot \Xi_X(D_2, C_2) \in \pi_2(BX)$$

identifying C and (C_1, C_2) through the natural bijection between $\mathrm{Col}_X^{a,a}(D_1\sharp D_2)$ and $\mathrm{Col}_X^a(D_1)\times \mathrm{Col}_X^a(D_2)$. □

Corollary 5.2. *Let X be a finite connected quandle such that (7) is injective. For any A-valued 2-cocycle ϕ of X,*

$$\Phi_\phi(K_1\sharp K_2) = \frac{1}{|X|}\Phi_\phi(K_1)\cdot\Phi_\phi(K_2) \;\in \mathbb{Z}[A].$$

Proof. We obtain the corollary from Proposition 5.1 by (4). □

Remark 5.3. By the same argument, we can show the connected sum formula of cocycle invariants of 2-dimensional knots for any finite connected quandle, without the assumption of the injectivity of (7). However, for a 1-dimensional knot, we need such an assumption, because $|\mathrm{Col}_X(D_1\sharp D_2)|$ is not necessarily equal to $\frac{1}{|X|}|\mathrm{Col}_X(D_1)|\cdot|\mathrm{Col}_X(D_2)|$ for a general connected finite quandle X.

Remark 5.4. By Proposition 5.1, we can construct infinitely many knots with the same quandle homotopy invariants, as follows. For an appropriate Alexander quandle, it is known [15, Corollary 1] that, if the Alexander polynomial of a knot K is trivial, all X-colorings of K are trivial. Hence, for such K and X, $\Xi_X(K) = |X|$, and, by Proposition 5.1, $\Xi_X(K') = \Xi_X(K\sharp K')$ for any knot K'. Since there are infinitely many knots with trivial Alexander polynomial,[3] we have infinitely many $K\sharp K'$ whose quandle homotopy invariant are equal.

We show the following two lemmas, which are used in the proof of Proposition 5.1.

Lemma 5.5. *Let X be a finite connected quandle, let a be an element of X and let D_1 be a knot diagram of the type drawn in Figure 7. Then,*

$$\Xi(K_1) = |X| \sum_{C \in \mathrm{Col}_X^a(D_1)} \Xi(K_1; C).$$

Proof. We show that, for any $x \in X$, there exists a bijection $T_x : \mathrm{Col}_X^a(D_1) \cong \mathrm{Col}_X^x(D_1)$ such that $\Xi_X(K_1; C) = \Xi_X(K_1; T_x(C))$. Since X is connected, we can assume, without loss of generality, that $x = a * y$ for some $y \in X$. We define a bijection T_x by $T_x(C) = C * y$. Since the action of X on $\pi_2(BX)$ is trivial, we have $\Xi_X(K_1; C) = \Xi_X(K_1; C * y) = \Xi_X(K_1; T_x(C))$. Hence,

$$\sum_{C \in \mathrm{Col}_X^a(D_1)} \Xi_X(K_1; C) = \sum_{C \in \mathrm{Col}_X^x(D_1)} \Xi_X(K_1; C).$$

Therefore,

$$\Xi_X(K_1) = \sum_{x \in X} \sum_{C \in \mathrm{Col}_X^x(D_1)} \Xi_X(K_1; C) = |X| \sum_{C \in \mathrm{Col}_X^a(D_1)} \Xi_X(K_1; C).$$

$\square$

Lemma 5.6. *Let X be a quandle such that (7) is injective. Let C be an X-coloring of a 1-tangle, and let x, y be the colors of C at the two end points of the 1-tangle. Then, $x = y$.*

Proof. We consider a shadow coloring (see [5]) of a diagram of the 1-tangle; see Figure 8. When the right region is colored by a, the left region is colored by $a * x = a * y$. Hence, $a * x = a * y$ for any $a \in X$. Therefore this implies that the images of x and y by (7) are equal. Since (7) is injective, $x = y$. $\square$

We will deal with the quandle homotopy invariant of the mirror image.

[3] It is known [18] that a knot has trivial Alexander polynomial if and only if it can be undone by doubled-delta moves.

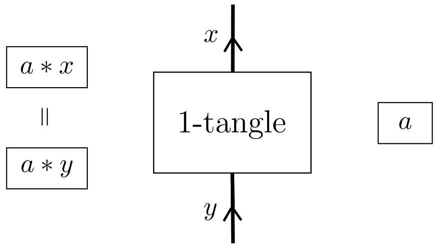

Figure 8: A shadow coloring around a 1-tangle

Proposition 5.7. *Let L be an oriented link, and let $-L^*$ denote the mirror image of L with opposite orientation. For any finite quandle X, $\Xi_X(-L^*) = \iota\big(\Xi_X(L)\big)$, where ι is the map of $\mathbb{Z}[\pi_2(BX)]$ induced by $\iota(x) = x^{-1}$ for any $x \in \pi_2(BX)$.*

Proof. We choose a diagram D of L. Let $-D^*$ denote the mirror image of D with opposite orienation; it is a diagram of $-L^*$. It follows from the definition of X-colorings that there exists a natural bijection $\varphi : \mathrm{Col}_X(D) \to \mathrm{Col}_X(-D^*)$. Since the mirror image of a diagram changes the orientation of S^2, we have that $\Xi_X(-D^*, \varphi(C)) = \iota\big(\Xi_X(D; C)\big)$. Hence, we obtain the required formula. □

6 Appendix: Concordance relations and the homotopy group $\pi_2(BX)$

In this appendix, we will study a more geometric description of the homotopy group $\pi_2(BX)$. The description is analogous to and follows from the results shown by Fenn, Rourke and Sanderson in [11] and [12].

For a quandle X, $\Pi_2(X)$ is defined to be the set of X-colorings of diagrams in S^2 modulo R-I,II,III moves and concordance relations, where the *concordance relations* are shown in Figure 9. The set $\Pi_2(X)$ has a binary operation given by disjoint union which we call addition. To be precise, given X-colorings C_1, C_2 choose copies in disjoint half spaces, then define $C_1 + C_2$ to be $C_1 \sqcup C_2$. The inverse element of C is the mirror image of C with opposite orientation and the zero element of $\Pi_2(X)$ is the empty set. This operation is well-defined and makes $\Pi_2(X)$ into an abelian group.

In section 2 we study the map $\Xi_X : \mathrm{Col}_X(D) \to \pi_2(BX)$. If X-colored links (L_1, C_1) and (L_2, C_2) are different from concordance relations, then we have $\Xi_X(L_1, C_1) = \Xi_X(L_2, C_2) \in \pi_2(BX)$ from the construction of Ξ_X. Hence it induces a homomorphism $\Phi_X : \Pi_2(X) \to \pi_2(BX)$ which is functorial from the category of quandles.

Proposition 6.1. *The homomorphism Φ_X is an isomorphism; $\Pi_2(X) \cong \pi_2(BX)$.*

Before proving the proposition, we shall review Theorem 6.2. In [11] and [12] Fenn,

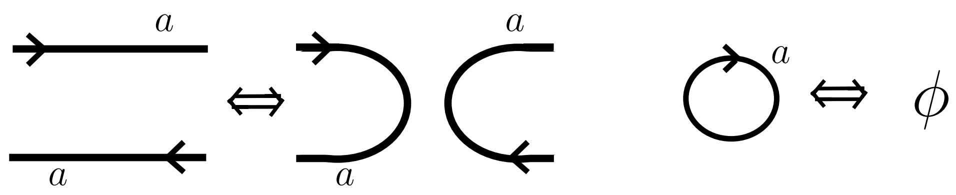

Figure 9: The concordance relations

Rourke and Sanderson examine the relation between homotopy group $\pi_2(\widehat{BX})$ and the concordance relations of framed link version. To begin with, we define $\widehat{\Pi}_2(X)$ to be a set of X-coloring of diagrams in S^2 modulo concordance relations and R-II,III moves without R-I move. The set $\Pi_2(X)$ has an addition given by disjoint union which makes $\Pi_2(X)$ into an abelian group. Similar to the above argument, we obtain the homomorphism, $\widehat{\Phi}_X : \widehat{\Pi}_2(X) \to \pi_2(\widehat{BX})$. Fenn, Rourke and Sanderson prove that

Theorem 6.2. *(Theorem 4.11 in [11] and Theorem 3.9 in [12]) The homomorphism* $\widehat{\Phi}_X$*, functorial from quandles, is an isomorphism;* $\widehat{\Pi}_2(X) \cong \pi_2(\widehat{BX})$.

We will prove Proposition 6.1 using their theorem.

Proof of Proposition 6.1. By Proposition 3.11 the canonical inclusion $E : \widehat{BX} \hookrightarrow BX$ induces an epimorphism $E_* : \pi_2(\widehat{BX}) \to \pi_2(BX)$. Further from the definition of Π, we have the canonical projection proj : $\widehat{\Pi}_2(X) \longrightarrow \Pi_2(X)$, since $\Pi_2(X)$ is defined by the stronger relation than the definition of $\widehat{\Pi}_2(X)$. From the construction of the map Φ_X we obtain the following commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(E_*) \arrow[r] \arrow[d, "\bar{\Phi}_X"] & \pi_2(\widehat{BX}) \arrow[r, "E_*"] \arrow[d, "\widehat{\Phi}_X"] & \pi_2(BX) \arrow[r] \arrow[d, "\Phi_X"] & 0 & \text{(exact)} \\
0 \arrow[r] & \mathrm{Ker}(\mathrm{proj}) \arrow[r] & \widehat{\Pi}_2(X) \arrow[r, "\mathrm{proj}"] & \Pi_2(X) \arrow[r] & 0 & \text{(exact)}
\end{tikzcd}$$

where $\bar{\Phi}_X$ is obtained by restricting $\widehat{\Phi}_X$.

We show that $\bar{\Phi}_X$ is an epimorphism, as follows. Let $\mathrm{Orb}(X)$ be a set of representatives of connected components of X. By definition Ker(Proj) is generated by the ± 1-framing unknot colored by $a \in X$ where $a \in X$ also runs over $\mathrm{Orb}(X)$. It follows from Proposition 3.11 that $\mathrm{Ker}(E_*)$ is generated by 2-cells labeled (a, a) where $a \in X$ runs over $\mathrm{Orb}(X)$. Note that $\Phi_X((a, a))$ is the 1-framing unknot colored by a (see Figure 3). Hence $\bar{\Phi}$ is an epimorphism.

Since $\widehat{\Phi}_X$ is isomorphism by Theorem 6.2, it follows from the five lemma that $\bar{\Phi}_X$ is an isomorphism. By applying the five lemma again in the diagram, Φ_X is an isomorphism.

□

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