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Abstract

A *bottom tangle* is a tangle in a cube consisting of arc components whose boundary points are on a line in the bottom square of the cube. A *ribbon bottom tangle* is a bottom tangle whose *closure* is a ribbon link. For every n-component ribbon bottom tangle T, we prove that the universal invariant J_T of T associated to the quantized enveloping algebra $U_h(sl_2)$ of the Lie algebra sl_2 is contained in a certain $\mathbb{Z}[q, q^{-1}]$-subalgebra of the n-fold completed tensor power $U_h^{\hat{\otimes} n}(sl_2)$ of $U_h(sl_2)$. This result is applied to the colored Jones polynomial of ribbon links.

1 Introduction.

For each ribbon Hopf algebra H, Reshetikhin and Turaev [14] defined invariants of framed links colored by finite dimensional representations. A *universal invariant* [9, 8, 13] associated to H is an invariant of framed tangles and links defined without using representations. The universal invariant has a universality property such that the colored link invariants constructed by Reshetikhin and Turaev are obtained from the universal invariants by taking trace in the representations attached to the components of links.

A quantized enveloping algebra $U_h := U_h(sl_2)$ of the Lie algebra sl_2 is a complete ribbon Hopf $\mathbb{Q}[[h]]$-algebra. By the *universal sl_2 invariant*, we mean the universal invariant associated to U_h. In [4], Habiro studied the universal invariant of *bottom tangles* (see Section 2) associated to an arbitrary ribbon Hopf algebra, and in [5], he studied the universal sl_2 invariant of bottom tangles (see Section 4). The universal sl_2 invariant of an n-component bottom tangle takes values in the n-fold completed tensor power $U_h^{\hat{\otimes} n}$ of U_h. For every oriented, ordered, framed link L, there is a bottom tangle whose *closure* is isotopic to L. The universal invariant of bottom tangles has a universality property such that the colored link invariants of a link L is obtained from the universal invariant of a bottom tangle T whose closure is isotopic to L, by taking the quantum trace in the representations attached to the components of links. In particular,

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one can obtain the colored Jones polynomials of links from the universal sl_2 invariant of bottom tangles.

An n-component link L is called a *ribbon link* if it bounds a system of n ribbon disks in S^3. Mizuma [11] derived an explicit formula for the first derivative at -1 for the Jones polynomial of 1-fusion ribbon knots, and in [12], she estimated the ribbon number of those knots by using the formula. Eisermann [2] proved that the Jones polynomial $V(L) \in \mathbb{Z}[v, v^{-1}]$ of an n-component ribbon link L is divisible by the Jones polynomial $V(O^n) = (v + v^{-1})^n$ of the n-component unlink O^n.

A *ribbon bottom tangle* is defined as a bottom tangle whose closure is a ribbon link. In this paper, we study the universal sl_2 invariant of ribbon bottom tangles.

1.1 Main result.

Set $v = \exp \frac{h}{2}$, $q = v^2$. We have $\mathbb{Z}[q, q^{-1}] \subset \mathbb{Z}[v, v^{-1}] \subset \mathbb{Q}[[h]]$. Let J_T denote the universal sl_2 invariant of a bottom tangle T.

Habiro [5] proved that the universal sl_2 invariant J_T of an n-component, algebraically-split, 0-framed bottom tangle T is contained in a certain $\mathbb{Z}[q, q^{-1}]$-subalgebra $(\tilde{\mathcal{U}}_q^{ev})^{\tilde{\otimes} n}$ of $U_h^{\hat{\otimes} n}$. He also defined another $\mathbb{Z}[q, q^{-1}]$-subalgebra $(\bar{U}_q^{ev})^{\tilde{}\tilde{\otimes} n} \subset (\tilde{\mathcal{U}}_q^{ev})^{\tilde{\otimes} n}$ and stated the following conjecture for boundary bottom tangle. (A bottom tangle is said to be *boundary* if it bounds mutually disjoint Seifert surfaces in $[0, 1]^3$, see [4] for the detail.)

Conjecture 1.1 (Habiro [5]). *Let T be an n-component boundary bottom tangle with 0-framing. Then we have $J_T \in (\bar{U}_q^{ev})^{\tilde{}\tilde{\otimes} n}$.*

We shall define another subalgebra $(\bar{U}_q^{ev})^{\hat{}\hat{\otimes} n} \subset (\bar{U}_q^{ev})^{\tilde{}\tilde{\otimes} n}$. (Here, we do not know whether the inclusion is proper or not, but the definition of $(\bar{U}_q^{ev})^{\hat{}\hat{\otimes} n}$ is more natural than that of $(\bar{U}_q^{ev})^{\tilde{}\tilde{\otimes} n}$ in our setting.) The main result of the present paper is the following, which we prove in Section 5.

Theorem 1.2. *Let T be an n-component ribbon bottom tangle with 0-framing. Then we have $J_T \in (\bar{U}_q^{ev})^{\hat{}\hat{\otimes} n}$.*

An n-component bottom tangle T is called a *slice bottom tangle* if T is *concordant* to the n-component trivial bottom tangle, where the trivial bottom tangle is the bottom tangle taking the shape as $\cap \ldots \cap$ (see Section 2 for the definition of the concordance of bottom tangles). The following is a generalization of Conjecture 1.1 and Theorem 1.2.

Conjecture 1.3. *If an n-component bottom tangle T is concordant to a boundary bottom tangle (in particular, if T is a slice bottom tangle), then we have $J_T \in (\bar{U}_q^{ev})^{\hat{}\hat{\otimes} n}$.*

1.2 An application to the colored Jones polynomial.

Here, we give an application of Theorem 1.2. We use the following q-integer notations.

$$\{i\}_q = q^i - 1, \quad \{i\}_{q,n} = \{i\}_q \{i-1\}_q \cdots \{i-n+1\}_q, \quad \{n\}_q! = \{n\}_{q,n},$$
$$[i]_q = \{i\}_q / \{1\}_q, \quad [n]_q! = [n]_q [n-1]_q \cdots [1]_q, \quad \begin{bmatrix} i \\ n \end{bmatrix}_q = \{i\}_{q,n} / \{n\}_q!,$$

for $i \in \mathbb{Z}, n \geq 0$.

For $l \geq 1$, let V_l denote the l-dimensional irreducible representation of U_h. Let $\mathcal{R}$ denote the representation ring of U_h over $\mathbb{Q}(v)$, i.e., $\mathcal{R}$ is the $\mathbb{Q}(v)$-algebra

$$\mathcal{R} = \mathrm{Span}_{\mathbb{Q}(v)}\{V_l \mid l \geq 1\}$$

with the multiplication induced by the tensor product.

Habiro [5] studied the following polynomials in V_2

$$\tilde{P}'_l = \frac{v^l}{\{l\}_q!} \prod_{i=0}^{l-1} (V_2 - v^{2i+1} - v^{-2i-1}) \in \mathcal{R},$$

for $l \geq 0$, and proved the following theorem.

Theorem 1.4 (Habiro [5])**.** *Let L be an n-component, algebraically-split, 0-framed link. We have*

$$J_{L;\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}} \in \frac{\{2l_j+1\}_{q,l_j+1}}{\{1\}_q} \mathbb{Z}[q,q^{-1}],$$

for $l_1,\dots,l_n \geq 0$, where j is a number such that $l_j = \max\{l_i\}_{1\leq i\leq n}$.

Here $J_{L;\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}}$ is the colored Jones polynomial of L associated to $\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}$ (see Section 4). The above theorem is an important technical step in Habiro's construction of the unified Witten-Reshetikhin-Turaev invariants for integral homology spheres. Habiro [5] also proved that Conjecture 1.1 would imply the following Theorem 1.5, *with a ribbon link replaced by a boundary link.* Thus, Theorem 1.5 follows from Theorem 1.2 and Habiro's argument in [5].

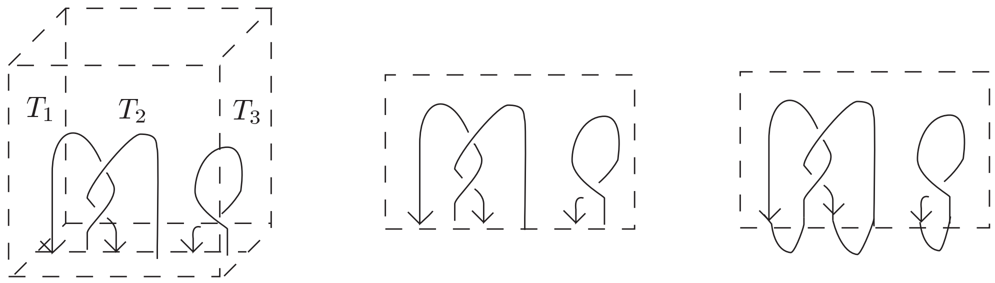
Theorem 1.5. *Let L be an n-component ribbon link with 0-framing. We have*

$$J_{L;\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}} \in \frac{\{2l_j+1\}_{q,l_j+1}}{\{1\}_q} I_{l_1}\cdots \hat{I}_{l_j}\cdots I_{l_n},$$

for $l_1,\dots,l_n \geq 0$, where j is a number such that $l_j = \max\{l_i\}_{1\leq i\leq n}$. Here, for $l \geq 0$, I_l is the ideal in $\mathbb{Z}[q,q^{-1}]$ generated by the elements $\{l-k\}_q!\{k\}_q!$ for $k = 0,\dots,l$, and $\hat{I}_{l_j}$ denotes omission of I_{l_j}.

Remark 1.6. For $m \geq 1$, let $\Phi_m(q) \in \mathbb{Z}[q]$ denote the mth cyclotomic polynomial. It is not difficult to prove that $I_l, l \geq 0$, is contained in the principle ideal generated by $\prod_m \Phi_m(q)^{f(l,m)}$, where $f(l,m) = \max\{0, \lfloor \frac{l+1}{m} \rfloor - 1\}$. Here for $r \in \mathbb{Q}$, we denote by $\lfloor r \rfloor$ the largest integer smaller than or equal to r.

Remark 1.7. As we have mentioned, Eisermann [2] proved that the Jones polynomial $V(L) \in \mathbb{Z}[v,v^{-1}]$ of an n-component ribbon link L is divisible by the Jones polynomial $V(O^n) = (v+v^{-1})^n$ of the n-component unlink O^n. This result does not follow directly from Theorem 1.5. However, we give another proof of it in [15] by proving a refinement of Theorem 1.2 involving a subalgebra of $U_h^{\hat{\otimes} n}$ smaller than $(\bar{U}_q^{ev})^{\hat{\otimes} n}$. We do not describe it in the present paper since the proof in [15] is quite complicated and also since we expect further refinements.



(a) (b) (c)

Figure 1: (a) A 3-component bottom tangle $T = T_1 \cup T_2 \cup T_3$. (b) A diagram of T in a rectangle. (c) The closure of T.

1.3 Organization of the paper.

The rest of the paper is organized as follows. In Section 2, we define bottom tangles and ribbon bottom tangles. In Section 3, we define the quantized enveloping algebra U_h, and its subalgebras. In Section 4, we consider the universal sl_2 invariant of bottom tangles and ribbon bottom tangles. In Sections 5, we prove Theorem 1.2. In Section 6, we consider the cases of the Borromean tangle and the Borromean rings.

2 Bottom tangles and ribbon bottom tangles.

In this section, we recall from [4] the notion of *bottom tangles*. We also define the notion of *ribbon bottom tangles*, which is implicit in [4].

2.1 Bottom tangles.

An n-component *bottom tangle* $T = T_1 \cup \cdots \cup T_n$ is an oriented, ordered, framed tangle in a cube $[0,1]^3$ consisting of n arcs $T_1, \dots, T_n$, whose boundary points are on the bottom line $[0,1] \times \{\frac{1}{2}\} \times \{0\}$, such that for each $i = 1, \dots, n$, the component T_i runs from the $2i$th boundary point to the $(2i-1)$th boundary point, where the boundary points are ordered by the first coordinate. As usual, we draw a bottom tangle as a diagram in a rectangle, see Figure 1 (a),(b). For each $n \geq 0$, let BT_n denote the set of the isotopy classes of n-component bottom tangles, and set $BT = \bigcup_{n \geq 0} BT_n$.

The *closure* of T is the link obtained from T by pasting a "$\cup$-shaped tangle" to each component of T, as depicted in Figure 1 (c). For any link L, there is a bottom tangle whose closure is isotopic to L.

The *linking matrix* $\mathrm{Lk}(T)$ of a bottom tangle $T = T_1 \cup \cdots \cup T_n$ is defined as that of the closure of T. Thus, for $1 \leq i \neq j \leq n$, the linking number of T_i and T_j is defined as the linking number of the corresponding components in the closure of T, and, for $1 \leq i \leq n$, the framing of T_i is defined as the framing of the closure of T_i.

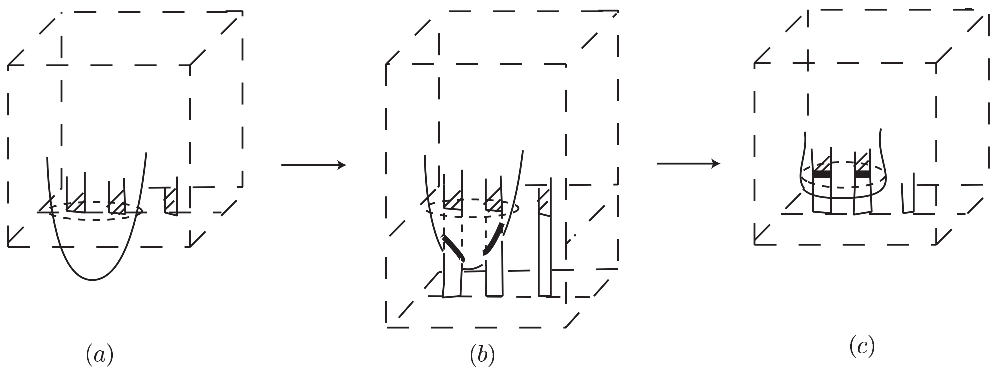


Figure 2: (a) A system of ribbon disks for the closure link $\tilde{T}$. (b) A system of ribbon disks for a link isotopic to $\tilde{T}$. (c) A system of ribbon disks for bottom tangle T.

Two bottom tangle $T, T' \in BT_n$ are *concordant* if there is an proper embedding;

$$f\colon \coprod^{n}[0,1]\times[0,1] \hookrightarrow [0,1]^3\times[0,1],$$

such that $f(\coprod^n[0,1]\times\{0\}) = T\times\{0\}$, $f(\coprod^n[0,1]\times\{1\}) = T'\times\{1\}$, and

$$f(\coprod^{n}\partial[0,1]\times[0,1]) = \partial T\times[0,1] = \partial T'\times[0,1].$$

2.2 Ribbon bottom tangles.

Definition 2.1. A bottom tangle $T\in BT$ is called a *ribbon bottom tangle* if and only if the closure of T is a ribbon link.

A system of ribbon disks for an n-component bottom tangle $T = T_1\cup\ldots\cup T_n$ is a immersed surface with ribbon singularities in $[0,1]^3$ consisting of n disks bounded by the link $\tilde{T} = (T_1\cup\gamma_1)\cup\ldots\cup(T_n\cup\gamma_n)$, where $\gamma_i\subset[0,1]\times\{\frac{1}{2}\}\times\{0\}$ is the line segment such that $\partial\gamma_i = \partial T_i$ for $1\le i\le n$.

Proposition 2.2. *A bottom tangle $T\in BT_n$ is a ribbon bottom tangle if and only if it admits a system of ribbon disks.*

Proof. Let $X\subset S^3$ be a system of ribbon disks for the link $\tilde{T}$. Up to isotopy in S^3 fixed on the link $\tilde{T}$, we can assume that $X\subset[0,1]^2\times[-1,1]$. If we admit introducing new ribbon singularities, we can transform X into a system of ribbon disks for the bottom tangle T by pulling the segment part $\gamma_i\subset[0,1]\times\{\frac{1}{2}\}\times\{0\}$ straight down to the $[0,1]\times\{\frac{1}{2}\}\times\{-1\}$, and transforming $[0,1]^2\times[-1,1]$ into $[0,1]^3$ by isotopy of S^3. For example, see Figure 2. $\square$

3 The quantized enveloping algebra U_h and its subalgebras.

We mostly follow the notations in [5].

3.1 The quantized enveloping algebra U_h.

Recall that $v = \exp\frac{h}{2}$, and $q = v^2$. We denote by U_h the h-adically complete $\mathbb{Q}[[h]]$-algebra, topologically generated by the elements H, E, and F, satisfying the relations

$$HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = \frac{K - K^{-1}}{v - v^{-1}},$$

where we set

$$K = v^H = \exp\frac{hH}{2}.$$

We equip U_h with a topological $\mathbb{Z}$-graded algebra structure with $\deg F = -1$, $\deg E = 1$, and $\deg H = 0$. For a homogeneous element x of U_h, the degree of x is denoted by $|x|$.

There is a unique complete ribbon Hopf algebra structure on U_h such that

$$\begin{aligned} \Delta(H) &= H \otimes 1 + 1 \otimes H, \quad \varepsilon(H) = 0, \quad S(H) = -H, \\ \Delta(E) &= E \otimes 1 + K \otimes E, \quad \varepsilon(E) = 0, \quad S(E) = -K^{-1}E, \\ \Delta(F) &= F \otimes K^{-1} + 1 \otimes F, \quad \varepsilon(F) = 0, \quad S(F) = -FK. \end{aligned}$$

The universal R-matrix and its inverse are given by

$$R = D\Big(\sum_{n\geq 0} v^{\frac{1}{2}n(n-1)} \frac{(v - v^{-1})^n}{[n]!} F^n \otimes E^n\Big), \tag{1}$$

$$R^{-1} = D^{-1}\Big(\sum_{n\geq 0} (-1)^n v^{-\frac{1}{2}n(n-1)} \frac{(v - v^{-1})^n}{[n]!} F^n K^n \otimes K^{-n} E^n\Big), \tag{2}$$

where $D = v^{\frac{1}{2}H\otimes H} = \exp\left(\frac{h}{4} H \otimes H\right) \in U_h^{\hat{\otimes} 2}$. The ribbon element and its inverse are given by

$$r = \sum \bar{\alpha} K^{-1} \bar{\beta} = \sum \bar{\beta} K \bar{\alpha}, \quad r^{-1} = \sum \alpha K \beta = \sum \beta K^{-1} \alpha,$$

where $R = \sum \alpha \otimes \beta$, and $R^{-1} = (S \otimes 1)R = \sum \bar{\alpha} \otimes \bar{\beta}$.

We use notations $D = \sum D^+_{[1]} \otimes D^+_{[2]}$, and $D^{-1} = \sum D^-_{[1]} \otimes D^-_{[2]}$. We shall use the following formulas.

$$\sum D^+_{[2]} \otimes D^+_{[1]} = D, \quad (\Delta \otimes 1)D = D_{13}D_{23}, \tag{3}$$

$$(\varepsilon \otimes 1)(D) = 1, \quad (1 \otimes S)D = (S \otimes 1)D = D^{-1}, \tag{4}$$

$$D(1 \otimes x) = (K^{|x|} \otimes x)D, \tag{5}$$

where $D_{13} = \sum D^+_{[1]} \otimes 1 \otimes D^+_{[2]}$, $D_{23} = 1 \otimes D$, and x is a homogeneous element of U_h.

3.2 Subalgebras $U_{\mathbb{Z},q}$ and $U_{\mathbb{Z},q}^{ev}$ of U_h.

For $i \in \mathbb{Z}, n \geq 0$, set

$$[i] = \frac{v^i - v^{-i}}{v - v^{-1}}, \quad [n]! = [n][n-1]\cdots[1].$$

Let $U_{\mathbb{Z}}$ denote Lusztig's integral form of U_h (cf. [10]), which is defined to be the $\mathbb{Z}[v, v^{-1}]$-subalgebra of U_h generated by K, K^{-1}, $E^{(n)} = E^n/[n]!$, and $F^{(n)} = F^n/[n]!$ for $n \geq 1$.

Set

$$\tilde{E}^{(n)} = (v^{-1}E)^n/[n]_q! = v^{-\frac{1}{2}n(n+1)}E^{(n)},$$
$$\tilde{F}^{(n)} = F^nK^n/[n]_q! = v^{-\frac{1}{2}n(n-1)}F^{(n)}K^n,$$

for $n \geq 0$. Let $U_{\mathbb{Z},q}$ denote the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_{\mathbb{Z}}$ generated by $K, K^{-1}, \tilde{E}^{(n)}$, and $\tilde{F}^{(n)}$ for $n \geq 1$. Note that

$$U_{\mathbb{Z}} = U_{\mathbb{Z},q} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}[v, v^{-1}].$$

Let $U_{\mathbb{Z},q}^{ev}$ denote the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_{\mathbb{Z},q}$ generated by $K^2, K^{-2}, \tilde{E}^{(n)}$, and $\tilde{F}^{(n)}$ for $n \geq 1$. $U_{\mathbb{Z},q}$ is equipped with a $(\mathbb{Z}/2\mathbb{Z})$-graded $\mathbb{Z}[q, q^{-1}]$-algebra structure

$$U_{\mathbb{Z},q} = U_{\mathbb{Z},q}^{ev} \oplus KU_{\mathbb{Z},q}^{ev}.$$

There is a Hopf $\mathbb{Z}[q, q^{-1}]$-algebra structure on $U_{\mathbb{Z},q}$ inherited from U_h such that

$$\Delta(K^i) = K^i \otimes K^i, \quad S^{\pm 1}(K^i) = K^{-i}, \tag{6}$$

$$\Delta(\tilde{E}^{(n)}) = \sum_{j=0}^{n} \tilde{E}^{(n-j)}K^j \otimes \tilde{E}^{(j)}, \quad \Delta(\tilde{F}^{(n)}) = \sum_{j=0}^{n} \tilde{F}^{(n-j)}K^j \otimes \tilde{F}^{(j)}, \tag{7}$$

$$S^{\pm 1}(\tilde{E}^{(n)}) = (-1)^n q^{\frac{1}{2}n(n\mp 1)}K^{-n}\tilde{E}^{(n)}, \quad S^{\pm 1}(\tilde{F}^{(n)}) = (-1)^n q^{-\frac{1}{2}n(n\mp 1)}K^{-n}\tilde{F}^{(n)}, \tag{8}$$

$$\varepsilon(K^i) = 1, \quad \varepsilon(\tilde{E}^{(n)}) = \varepsilon(\tilde{F}^{(n)}) = \delta_{n,0}, \tag{9}$$

for $i \in \mathbb{Z}, n \geq 0$.

3.3 Subalgebras $\bar{U}_q$ and $\bar{U}_q^{ev}$ of U_h.

Let $\bar{U}$ denote the $\mathbb{Z}[v, v^{-1}]$-subalgebra of U_h generated by the elements K, K^{-1}, $(v - v^{-1})E$, and $(v - v^{-1})F$ (cf. [1]).

Set

$$e = v^{-1}(q-1)E, \quad f = (q-1)FK.$$

Let $\bar{U}_q$ denote the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_{\mathbb{Z},q}$ generated by the elements K, K^{-1}, e and f. Note that

$$\bar{U} = \bar{U}_q \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}[v, v^{-1}].$$

Let $\bar{U}_q^{ev}$ denote the $\mathbb{Z}[q,q^{-1}]$-subalgebra of $U_{\mathbb{Z},q}^{ev}$ generated by the elements K^2, K^{-2}, e and f. We have

$$\bar{U}_q^{ev} = \bar{U}_q \cap U_{\mathbb{Z},q}^{ev}, \quad \bar{U}_q = \bar{U}_q^{ev} \oplus K\bar{U}_q^{ev}.$$

There is a Hopf $\mathbb{Z}[q,q^{-1}]$-algebra structure on $\bar{U}_q$ inherited from U_h such that

$$\Delta(e^n) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q e^{n-j}K^j \otimes e^j, \quad \Delta(f^n) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{-j(n-j)} f^{n-j}K^j \otimes f^j, \tag{10}$$

$$S^{\pm 1}(e^n) = (-1)^n q^{\frac{1}{2}n(n\mp 1)} K^{-n}e^n, \quad S^{\pm 1}(f^n) = (-1)^n q^{-\frac{1}{2}n(n\mp 1)} K^{-n} f^n, \tag{11}$$

$$\varepsilon(e^n) = \varepsilon(f^n) = \delta_{n,0}, \tag{12}$$

for $n \geq 0$.

We have

$$e^m f^n = \sum_{p=0}^{\min(m,n)} q^{\frac{1}{2}p(p+1)-nm} \{p\}_q! \begin{bmatrix} m \\ p \end{bmatrix}_q \begin{bmatrix} n \\ p \end{bmatrix}_q f^{n-p} \{H-m-n+2p\}_{q,p} e^{m-p}, \tag{13}$$

for $m, n \geq 0$. Here, for $i \in \mathbb{Z}$ and $p \geq 0$, we set

$$\{H+i\}_{q,p} = \{H+i\}_q \{H+i-1\}_q \cdots \{H+i-p+1\}_q,$$

where

$$\{H+j\}_q = q^{H+j} - 1 = q^j K^2 - 1,$$

for $j \in \mathbb{Z}$.

The following lemma, which is a $\mathbb{Z}[q,q^{-1}]$-version of a well known result for $\bar{U}$ by De Concini and Procesi [1], can be proved by using the formula (13).

Lemma 3.1. *$\bar{U}_q$ (resp. $\bar{U}_q^{ev}$) is freely $\mathbb{Z}[q,q^{-1}]$-spanned by the elements $f^iK^je^k$ (resp. $f^iK^{2j}e^k$) with $i,k \geq 0$ and $j \in \mathbb{Z}$.*

3.4 Adjoint action.

We use the left adjoint action of U_h defined by

$$\mathrm{ad}(a \otimes b) := \sum a' b S(a''),$$

where $\Delta(a) = \sum a' \otimes a''$. We also use the notation $a \triangleright b := \mathrm{ad}(a \otimes b)$.

The following proposition is suggested by Habiro. In fact, Habiro and Le [7] prove a generalization of a $\mathbb{Z}[v,v^{-1}]$-version of the following proposition with $i=0$ to quantized enveloping algebras for all simple Lie algebras.

Proposition 3.2. *For $i = 0, 1$, we have*

$$U_{\mathbb{Z},q} \triangleright K^i \bar{U}_q^{ev} \subset K^i \bar{U}_q^{ev}.$$

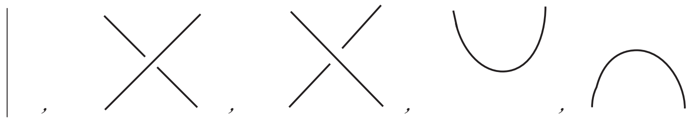


Figure 3: Fundamental tangles. The orientations of the strands are arbitrary.

Proof. In view of Lemma 3.1, it is enough to prove that $x \triangleright f^{i_1}K^{i_2}e^{i_3} \in K^{i_2}\bar{U}_q^{ev}$ for every $x \in \{K, K^{-1}, \tilde{E}^{(n)}, \tilde{F}^{(n)} \mid n \geq 0\}$ and $i_1, i_3 \geq 0$, $i_2 \in \mathbb{Z}$. By computation, we have

$$K^{\pm 1} \triangleright f^{i_1}K^{i_2}e^{i_3} = q^{\pm(i_3-i_1)}f^{i_1}K^{i_2}e^{i_3}, \tag{14}$$

$$\begin{aligned}\tilde{E}^{(n)} &\triangleright f^{i_1}K^{i_2}e^{i_3} \\ &= \sum_{p=0}^{\min(i_1,n)} (-1)^n q^{\frac{1}{2}p(p+1)-n(i_1+i_2)+2i_2p} \begin{bmatrix} i_1 \\ p \end{bmatrix}_q f^{i_1-p}K^{i_2}g(i_1,i_2,i_3,n,p)e^{i_3+n-p},\end{aligned} \tag{15}$$

$$\begin{aligned}\tilde{F}^{(n)} &\triangleright f^{i_1}K^{i_2}e^{i_3} \\ &= \sum_{p=0}^{\min(i_3,n)} q^{\frac{1}{2}p(p+1)-n(i_1+i_2)+2i_2p} \begin{bmatrix} i_3 \\ p \end{bmatrix}_q f^{n+i_1-p}K^{i_2}g(i_3,i_2,i_1,n,p)e^{i_3-p},\end{aligned} \tag{16}$$

where

$$g(i_1,i_2,i_3,n,p) = \sum_{s=0}^{p} (-1)^s q^{\frac{1}{2}s(s+1)-s(n-p+i_1)} \begin{bmatrix} p \\ s \end{bmatrix}_q \begin{bmatrix} n-p+i_2+i_3+s-1 \\ n-p \end{bmatrix}_q K^{2s}.$$

The right hand sides of (14)–(16) are all contained in $K^{i_2}\bar{U}_q^{ev}$, hence we have the assertion. □

4 The universal sl_2 invariant of bottom tangles.

In this section, we define the universal sl_2 invariant of bottom tangles [4], and study the values of it. Then we discuss the case of ribbon bottom tangles.

4.1 Decorated diagrams.

We use diagrams of tangles obtained from copies of the fundamental tangles, as depicted in Figure 3, by pasting horizontally and vertically. A *decorated diagram* of a bottom tangle $T \in BT$ is a diagram P of T together with finitely many dots on strands, each labeled by an element of U_h. We also allow pairs of dots, each connected by an oriented dashed line which is labeled by an element of $U_h^{\hat{\otimes}2}$ so that the first tensorand is attached to the start point of the line, and the second tensorand to the end point, see Figure 4 (a). If the element $y \in U_h^{\hat{\otimes}2}$ on it is symmetric, we do not have to specify the orientation of a dashed line.

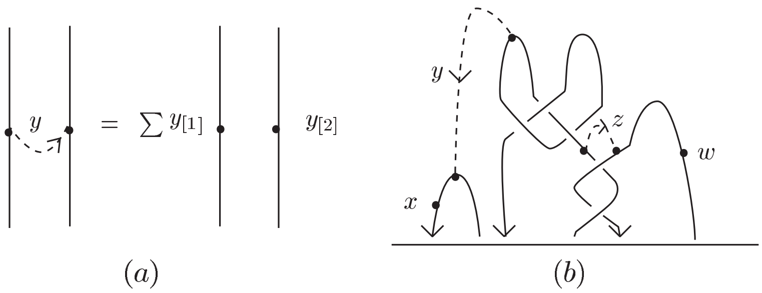


Figure 4: (a) How to label an element $y = \sum y_{[1]} \otimes y_{[2]}$ to the connected dots. (b) A decorated diagram P.

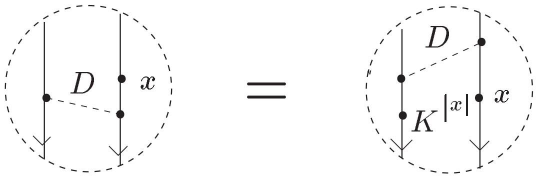


Figure 5: A graphical version of (5). By the two pictures above, we mean two decorated diagrams of a bottom tangle which are identical outside the dotted circles.

For every decorated diagram P for an n-component bottom tangle $T = T_1 \cup \cdots \cup T_n \in BT_n$, we define an element $J(P) \in U_h^{\hat{\otimes} n}$ as follows. The ith component of $J(P)$ is defined to be the product of the elements put on the component corresponding to T_i, where the elements are read off along each component reversing the orientation of P, and written from left to right. For example, for the decorated diagram P depicted in Figure 4 (b), we have

$$J(P) = \sum x y_{[2]} \otimes y_{[1]} z_{[1]} \otimes z_{[2]} w,$$

where $y = \sum y_{[1]} \otimes y_{[2]}$ and $z = \sum z_{[1]} \otimes z_{[2]}$. In what follows, we sometimes identify a decorated diagram and its image by J. For example, the picture depicted in Figure 5 represents the formula (5).

4.2 The universal sl_2 invariant of bottom tangles.

For $T = T_1 \cup \cdots \cup T_n \in BT_n$, we define the universal sl_2 invariant $J_T \in U_h^{\hat{\otimes} n}$ of T as follows. We choose a diagram P of T. We denote by $C(P)$ the set of the crossings of P. We call a map

$$s\colon\ C(P) \ \to\ \{0, 1, 2, \ldots\}$$

a *state*. We denote by $\mathcal{S}(P)$ the set of states for P. For each state $s \in \mathcal{S}(P)$, we define a decorated diagram (P, s) (by abusing the notation) as follows.

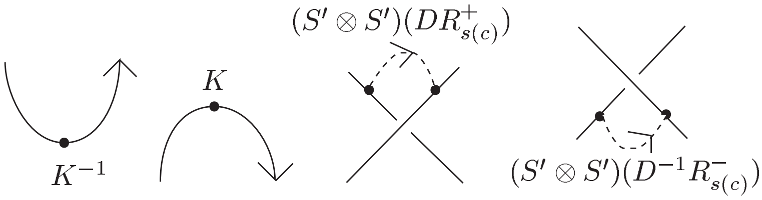


Figure 6: How to place elements on the fundamental tangles.

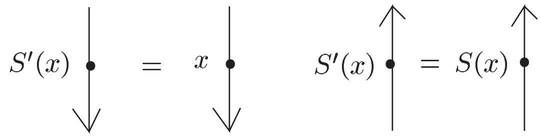


Figure 7: The definition of S'.

We rewrite the R-matrix (1) and its inverse (2) as

$$R^{\pm 1} = D^{\pm 1} \sum_{n \geq 0} R^{\pm}_{n}, \tag{17}$$

$$R^{+}_{n} = q^{\frac{1}{2}n(n-1)} \tilde{F}^{(n)} K^{-n} \otimes e^{n}, \quad R^{-}_{n} = (-1)^{n} \tilde{F}^{(n)} \otimes K^{-n} e^{n}. \tag{18}$$

We use the notations $R^{+}_{n} = \sum R^{+}_{n[1]} \otimes R^{+}_{n[2]}$ and $R^{-}_{n} = \sum R^{-}_{n[1]} \otimes R^{-}_{n[2]}$.

For each fundamental tangle in P, we attach elements following the rule described in Figure 6, where "S'" should be replaced with id if the string is oriented downward, and with S otherwise, see Figure 7. Thus we have an element $J(P,s) \in U_h^{\hat{\otimes} n}$ as the image of the decorated diagram (P,s) by J.

Set

$$J_T = \sum_{s \in \mathcal{S}(P)} J(P,s).$$

As is well known [13], J_T does not depend on the choice of the diagram P, and defines an isotopy invariant of bottom tangles.

For example, let us compute the universal sl_2 invariant J_C of a bottom tangle C with a diagram P as depicted in Figure 8 (a), where c_1 (resp. c_2) denotes the upper (resp. lower) crossing of P. The decorated diagram (P,s) for the state $s \in \mathcal{S}(P)$ is depicted in Figure 8 (b), where we set $m = s(c_1), n = s(c_2)$. We have

$$\begin{aligned}
J_C &= \sum_{s \in \mathcal{S}(P)} J(P,s) \\
&= \sum_{s \in \mathcal{S}(P)} \sum S(D^{+}_{[1]} R^{+}_{m[1]}) S(D'^{+}_{[2]} R^{+}_{n[2]}) \otimes D'^{+}_{[1]} R^{+}_{n[1]} D^{+}_{[2]} R^{+}_{m[2]} \\
&= \sum_{m,n \geq 0} (-1)^{m+n} q^{-n+2mn} D^{-2} (\tilde{F}^{(m)} K^{-2n} e^{n} \otimes \tilde{F}^{(n)} K^{-2m} e^{m}).
\end{aligned}$$

where $D^{\pm 1} = \sum D^{\pm}_{[1]} \otimes D^{\pm}_{[2]} = \sum D'^{\pm}_{[1]} \otimes D'^{\pm}_{[2]}$.

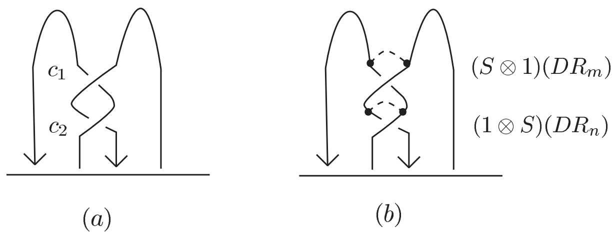


Figure 8: (a) A diagram P of $C \in BT_2$. (b) The decorated diagram (P, s).

4.3 The colored Jones polynomial.

If V is a finite dimensional representation of U_h, then the quantum trace $\mathrm{tr}_q^V(x)$ in V of an element $x \in U_h$ is defined by

$$\mathrm{tr}_q^V(x) = \mathrm{tr}^V(\rho_V(K^{-1}x)) \in \mathbb{Q}[[h]],$$

where $\rho_V\colon\ U_h \to \mathrm{End}(V)$ denotes the left action of U_h on V, and $\mathrm{tr}^V\colon\ \mathrm{End}(V) \to \mathbb{Q}[[h]]$ denotes the trace in V. For every element $y = \sum_n a_n V_n \in \mathcal{R}$, $a_n \in \mathbb{Q}(v)$, we set

$$\mathrm{tr}_q^y(x) = \sum_n a_n \,\mathrm{tr}_q^{V_n}(x) \in \mathbb{Q}(v)$$

for $x \in U_h$.

The universal sl_2 invariant of bottom tangles has a universality property to the colored Jones polynomials of links as the following.

Proposition 4.1 (Habiro [5]). *Let $L = L_1 \cup \cdots \cup L_n$ be an n-component, ordered, oriented, framed link in S^3. Choose an n-component bottom tangle T whose closure is isotopic to L. For $y_1, \ldots, y_n \in \mathcal{R}$, the colored Jones polynomial $J_{L;y_1,\ldots,y_n}$ of L can be obtained from J_T by*

$$J_{L;y_1,\ldots,y_n} = (\mathrm{tr}_q^{y_1} \otimes \cdots \otimes \mathrm{tr}_q^{y_n})(J_T).$$

4.4 Values of the universal sl_2 invariant of bottom tangles.

In this subsection we consider the value of $J(P, s)$ for a decorated diagram (P, s). Let us prepare some notations.

For $n \geq 1, 1 \leq i \leq n$, and for $X \in U_h$, we define $X_i \in U_h^{\hat{\otimes} n}$ by

$$X_i = 1 \otimes \cdots \otimes X \otimes \cdots \otimes 1,$$

where X is at the ith position.

For $1 \leq i, j \leq n$, and for $Y = \sum y_1 \otimes y_2 \in U_h^{\hat{\otimes} 2}$, we define $Y_{ij} \in U_h^{\hat{\otimes} n}$ by

$$Y_{ij} = \sum (y_1)_i (y_2)_j.$$

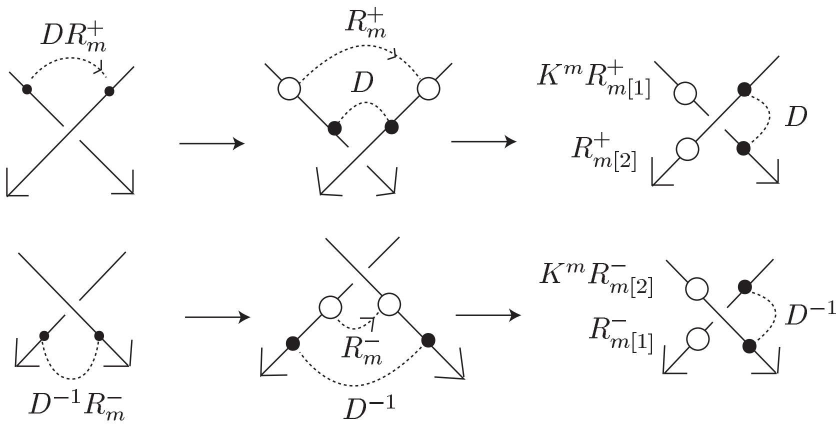


Figure 9: The modification process of (P, s) on positive and negative crossings.

For every symmetric integer matrix $M = (m_{ij})_{1\le i,j\le n}$ of size $n \ge 1$, we define two invertible elements $D^M, \tilde{D}^M \in U_h^{\hat{\otimes} n}$ by

$$D^M = \prod_{1\le i,j\le n} D_{ij}^{m_{ij}} = \prod_{1\le i<j\le n} D_{ij}^{2m_{ij}} \prod_{1\le i\le n} (v^{H^2/2})_i^{m_{ii}},$$
$$\tilde{D}^M = D^M \prod_{1\le i\le n} K_i^{m_{ii}} = \prod_{1\le i<j\le n} D_{ij}^{2m_{ij}} \prod_{1\le i\le n} (v^{H^2/2}K)_i^{m_{ii}}.$$

Later, we shall use the following proposition.

Proposition 4.2. *Let $T = T_1 \cup \cdots \cup T_n$ be an n-component bottom tangle. For every diagram P of T and every state $s \in \mathcal{S}(P)$, we have*

$$J(P, s) \in \tilde{D}^{\mathrm{Lk}(T)} (U_{\mathbb{Z},q}^{ev})^{\otimes n}.$$

Before proving Proposition 4.2, we modify the dots of the decorated diagram (P, s). Then we define three decorated diagrams $(P, s)^\circ, (P, s)^\bullet$, and $(P, s)^\diamond$, which we use in the proof of Proposition 4.2.

In what follows, we can work up to the equivalence relation $\sim$ on $(U_{\mathbb{Z},q})^{\otimes n}$ generated by multiplication on any tensorands by $\pm q^j, K^{2j} (j \in \mathbb{Z})$. The modification process goes as follows. Let c be a crossing of (P, s) with strands oriented downward, and set $m = s(c)$. As depicted in Figure 9, we replace the two dots labeled by $D^{\pm 1}R_m^\pm$ with two black dots labeled by $D^{\pm 1}$ and two white dots labeled by $R_m^\pm$. Then we slide the black (resp. white) dots to the right hand side (resp. the left hand side) of the crossings, and put the produced element K^m into the same dot of $R_m^\pm$. Here the transformation follows from the formulas

$$\begin{aligned} DR_m^+ &= \sum D_{[1]} R_{m[1]}^+ \otimes D_{[2]} R_{m[2]}^+ \\ &= \sum D_{[1]} K^m R_{m[1]}^+ \otimes R_{m[2]}^+ D_{[2]}. \end{aligned}$$

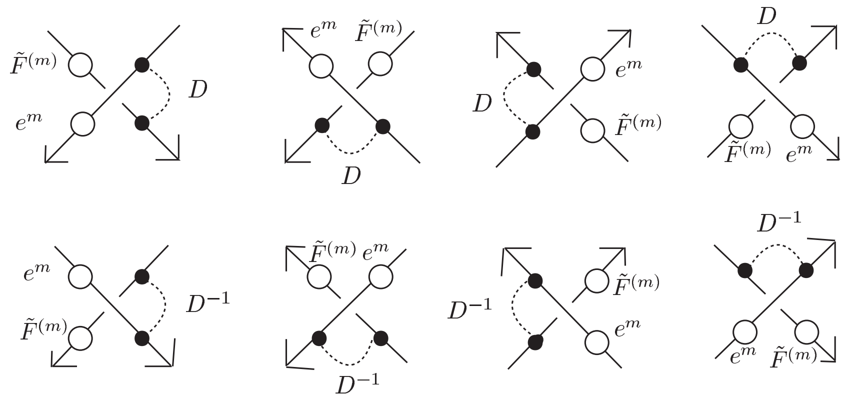


Figure 10: Crossings of the decorated diagram (P, s) after the modification.

and

$$\begin{aligned} D^{-1}R^-_m &= \sum D^-_{[1]} R^-_{m[1]} \otimes D^-_{[2]} R^-_{m[2]} \\ &= \sum R^-_{m[1]} D^-_{[1]} \otimes D^-_{[2]} K^m R^-_{m[2]}. \end{aligned}$$

Note that

$$K^m R^+_{m[1]} \otimes R^+_{m[2]} \sim \tilde{F}^{(m)} \otimes e^m, \tag{19}$$

$$R^-_{m[1]} \otimes K^m R^-_{m[2]} \sim \tilde{F}^{(m)} \otimes e^m. \tag{20}$$

Similarly, we modify the dots on the other crossings as depicted in Figure 10. We have completed the modification. By abusing the notation, we denote by (P, s) the decorated diagram obtained from the modification.

We define the decorated diagrams $(P, s)^\circ$, $(P, s)^\bullet$, and $(P, s)^\diamond$ as follows.

(1) Let $(P, s)^\circ$ denote the diagram P together with the white dots on crossings of (P, s). Note that

$$J(P, s)^\circ \in (U^{ev}_{\mathbb{Z},q})^{\otimes n}. \tag{21}$$

Let $\vec{\cap}$, $\vec{\cup}$ and $\cap$ denote the fundamental tangles defined by

$\vec{\cap} =$  , $\vec{\cup} =$  , $\cap =$ .

(2) Let $(P, s)^\bullet$ denote the diagram P with the black dots labeled by $D^{\pm 1}$ on crossings of (P, s), and dots on $\vec{\cap}$ and $\vec{\cup}$ of (P, s).

(3) For $i = 1, \ldots, n$, let P_i denote the part of P corresponding to T_i. We call the $2i$th (resp. $(2i - 1)$th) boundary point of P the *start point* (resp. *end point*) of P_i. On (P, s), we slide all white dots to the start points of the strands of P. When we

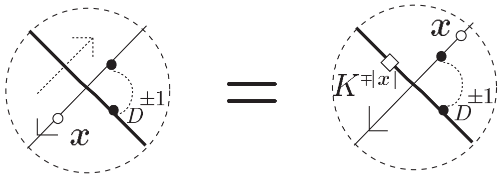


Figure 11: The picture when we slide a homogeneous x through a dot labeled by $D^{\pm 1}$. This is essentially the same with the picture in Figure 5.

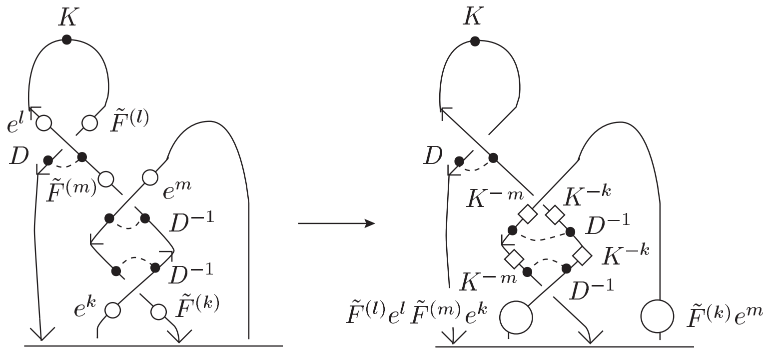


Figure 12: The sliding process for a decorated diagram (P, s), where we set $s(c_1) = l, s(c_2) = m$, and $s(c_3) = k$ for the upper, the middle, and the lower crossings c_1, c_2, and c_3, respectively. We work up to multiplication by $\pm q^j, K^{2j} (j \in \mathbb{Z})$.

slide a white dot through a dot on $\vec{\cap}$ or $\vec{\cup}$, a scalar $q^j (j \in \mathbb{Z})$ appears, which we can ignore. When we slide a white dot through a dot labeled by $D^{\pm}$, a power of K appears, see Figure 11. We attach such element to a new white diamond. Let $(P_i, s)^{\diamondsuit}$ be the diagram P_i with the white diamonds on P_i. Set

$$J(P, s)^{\diamondsuit} = J(P_1, s)^{\diamondsuit} \otimes \cdots \otimes J(P_n, s)^{\diamondsuit}.$$

For example, for the decorated diagram (P, s) in Figure 12, we have

$$\begin{aligned}
&\mathrm{Lk}(T) = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}, \\
&\tilde{D}^{\mathrm{Lk}(T)} = D^{-2}(v^{H^2/2}K \otimes 1), \\
&J(P, s)^{\circ} \sim \tilde{F}^{(l)}e^l\tilde{F}^{(m)}e^k \otimes \tilde{F}^{(k)}e^m, \\
&J(P, s)^{\bullet} \sim D^{-2}(v^{H^2/2}K \otimes 1), \\
&J(P_1, s)^{\diamondsuit} \sim K^{-2k} \sim 1, \\
&J(P_2, s)^{\diamondsuit} \sim K^{-2m} \sim 1.
\end{aligned}$$

We reduce Proposition 4.2 to the following two lemmas.

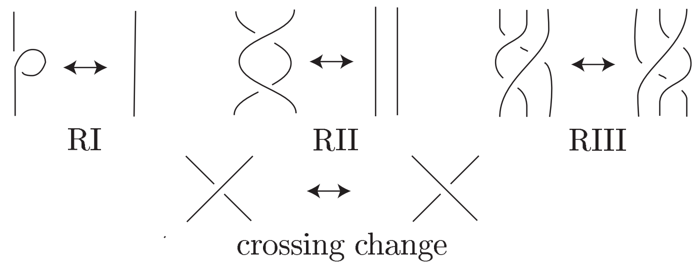


Figure 13: The Reidemeister moves RI, RII, RIII, and the crossing change.

Lemma 4.3. *For every diagram P of a bottom tangle $K \in BT_1$ with framing $r(K) \in \mathbb{Z}$, let $u(P) \in \mathbb{Z}_{\geq 0}$ be the total number of the copies of $\vec{\cap}$ and $\vec{\cup}$ which are contained in P. Then, the sum $u(P) + r(K)$ is even.*

Proof. Note that the parity of $u(P) + r(K)$ does not change by the Reidemeister moves RI, RII, RIII, and crossing changes as depicted in Figure 13. Since P is equal to the bottom tangle $\cap$ up to those moves, we have

$$u(P) + r(K) \equiv u(\cap) + r(\cap) = 0 \qquad (\mathrm{mod}\ 2).$$

This completes the proof. □

Let U_h^0 denote the $\mathbb{Q}[[h]]$-subalgebra of U_h generated by K, K^{-1}. Set

$$\bar{U}_q^{ev0} = \bar{U}_q^{ev} \cap U_h^0,$$

which is the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $\bar{U}_q^{ev}$ generated by K^2, K^{-2}.

Lemma 4.4. *We have*

$$J(P,s)^{\bullet} \in \tilde{D}^{\mathrm{Lk}(T)}(\bar{U}_q^{ev0})^{\otimes n}.$$

Proof. For each $i = 1, \ldots, n$, we denote by κ_i the product of the $K^{\pm 1}$s on the copies of $\vec{\cap}$ and $\vec{\cup}$ of P_i. We have

$$\begin{aligned} J(P,s)^{\bullet} =& D^{\mathrm{Lk}(T)}(\kappa_1 \otimes \cdots \otimes \kappa_n) \\ =& \tilde{D}^{\mathrm{Lk}(T)}(K^{-m_{1,1}}\kappa_1 \otimes \cdots \otimes K^{-m_{n,n}}\kappa_n). \end{aligned}$$

Since we have $K^{-m_{i,i}}\kappa_i \in \bar{U}_q^{ev0}$ by Lemma 4.3, the right hand side is contained in $\tilde{D}^{\mathrm{Lk}(T)}(\bar{U}_q^{ev0})^{\otimes n}$. This completes the proof. □

Lemma 4.5. *For every $i = 1, \ldots, n$, we have*

$$J(P_i, s)^{\diamondsuit} \sim 1.$$

If we assume Lemma 4.5, then Proposition 4.2 follows from

$$J(P,s) \sim J(P,s)^{\bullet} J(P,s)^{\diamondsuit} J(P,s)^{\circ} \in \tilde{D}^{\mathrm{Lk}(T)}(\bar{U}_q^{ev0})^{\otimes n} \cdot (U_{\mathbb{Z},q}^{ev})^{\otimes n} \subset \tilde{D}^{\mathrm{Lk}(T)}(U_{\mathbb{Z},q}^{ev})^{\otimes n},$$

by (21) and Lemma 4.4.

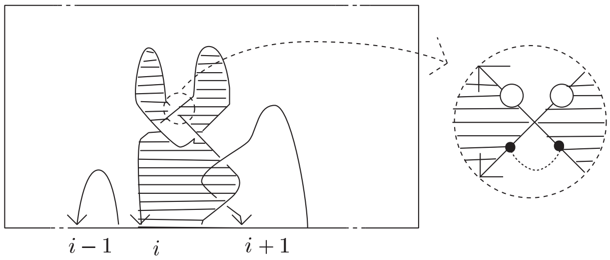


Figure 14: A diagram $P = P_1 \cup \cdots \cup P_n$ colored by chessboard fashion associated to P_i. We depict only the $(i-1)$, i, and $(i+1)$th component.

Proof of Lemma 4.5. For a crossing c of (P, s), we denote by E_c (resp. F_c) the white dot on the over (resp. under) strand labeled by $e^{s(c)}$ (resp. $\tilde{F}^{(s(c))}$). We slide those white dots to the start points of strands of P, and count the powers of K labeled to the white diamonds on each strands.

Note that each time we exchange E_c with one of the two dots connected by dashed line, labeled by $D^{\pm 1}$, a white diamond labeled by $K^{\mp s(c)}$ appears next to the other dot, see Figure 11 again. Similarly, if we exchange F_c with one of the two dots labeled by $D^{\pm 1}$, then a white diamond labeled by $K^{\pm s(c)}$ appears next to the other dot.

Let $p_i(E_c)$ denotes the number of times E_c traverses the strand P_i during the sliding process. Define $p_i(F_c)$ similarly. Then we have $J(P_i, s)^{\diamondsuit} = K^{d_i}$, where

$$d_i \equiv \sum_{c \in C(P)} s(c)(p_i(E_c) + p_i(F_c)) \pmod 2.$$

Hence it is enough to prove that $p_i(E_c) + p_i(F_c)$ is even for each crossing c. We prove the assertion with three types of crossings as follows.

(i) Self crossings of P_i.

(ii) Crossings of P_j with P_l for $j \neq i, l \neq i$.

(iii) Crossings of P_i with P_j for $j \neq i$.

Color black or white, in chessboard fashion, the regions of the complements of P_i in the rectangle so that the outermost region is colored white. For example, see Figure 14. Divide the strand P_i into two parts B_i and W_i, each consisting of segments bounded by self crossing points or the boundary points of P_i, such that if one goes along a segment in W_i (resp. B_i) to the start point of P_i, then one sees a white (resp. black) region on the left.

Note that the boundary points of the strand P_l, $i \neq l$, are contained in the white region, and those of P_i are contained in W_i.

(i) For a self crossing c of P_i.

Note that when we trace along P_i from the end point to the start point, every time we traverse the self crossing of P_i, B_P and W_P appear one after the other.

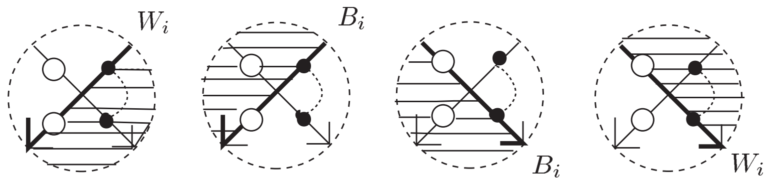


Figure 15: The four types of crossings.

For every self crossing $c \in P_i$, both E_c and F_c are either in B_P or in W_P. Hence if we slide E_c and F_c to the start point, then the parities of $p_i(E_c)$ and $p_i(F_c)$ are the same. Thus, $p_i(E_c) + p_i(F_c)$ is even.

(ii) For a crossing c of P_j and P_l with $j \neq i, l \neq i$.

If the crossing c is in the white region, then both $p_i(E_c)$ and $p_i(F_c)$ are even. If c is in the black region, then both $p_i(E_c)$ and $p_i(F_c)$ are odd. Hence $p_i(E_c)+p_i(F_c)$ is even in both cases.

(iii) For a crossing c of P_i and P_j with $j \neq i$.

See Figure 15. There are four types of crossings such that whether the white dot on P_i is in W_i or in B_i, and whether the white dot on P_j is in the white region or in the black region. We assume P_i is the over strand, i.e., E_c is attached on P_i. The other case is almost the same. For (a), since E_c starts and ends in W_i, $p_i(E_c)$ is even. Similarly, since F_c starts and ends in the white region, $p_i(F_c)$ is even. Thus, $p_i(E_c) + p_i(F_c)$ is even. For the other three cases, in a similar way, we can prove that the parities of $p_i(E_c)$ and $p_i(F_c)$ are the same. Hence $p_i(E_c) + p_i(F_c)$ is even.

Therefore we have $J(P_i, s)^{\diamondsuit} \sim 1$ for $i = 1, \ldots, n$, this completes the proof. $\square$

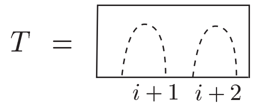
Remark 4.6. As defined in [5], let $\mathcal{U}_q^{ev} \subset U_{\mathbb{Z},q}$ denote the subalgebra of U_h freely generated over $\mathbb{Z}[q, q^{-1}]$ by the elements $\tilde{F}^{(i)} K^{2j} e^k$ for $i, k \geq 0, j \in \mathbb{Z}$. Note that the right hand sides of (19) and (20) are in $(\mathcal{U}_q^{ev})^{\otimes 2}$. This implies a result stronger than Proposition 4.2 ;

$$J(P, s) \in \tilde{D}^{\mathrm{Lk}(T)} (\mathcal{U}_q^{ev})^{\otimes n}.$$

This implies the following, which is proved by Habiro when $\mathrm{Lk}(T) = 0$ in the other way.

$$J_T \in \tilde{D}^{\mathrm{Lk}(T)} (\tilde{\mathcal{U}}_q^{ev})^{\tilde{\otimes} n},$$

where $(\tilde{\mathcal{U}_q}^{ev})^{\tilde{\otimes} n}$ is the Habiro's completion of $(\mathcal{U}_q^{ev})^{\otimes n}$ in [5].



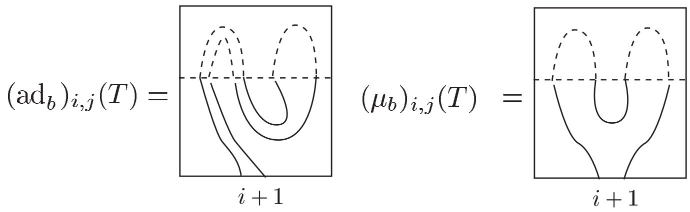


Figure 16: A bottom tangle $T \in BT_{i+j+2}$ and the bottom tangles $(\mathrm{ad}_b)_{(i,j)}(T)$, $(\mu_b)_{(i,j)}(T) \in BT_{i+j+1}$. We depict only the $(i+1),(i+2)$th components of T, and the $(i+1)$th components of $(\mathrm{ad}_b)_{(i,j)}(T)$, $(\mu_b)_{(i,j)}(T)$.

4.5 The universal sl_2 invariant of ribbon bottom tangles.

Habiro [5] studied the universal sl_2 invariant of 1-component ribbon bottom tangles. We generalize those to n-component ribbon bottom tangles for $n \geq 1$.

For $T \in BT_{i+j+2}$, $i,j \geq 0$, let $(\mathrm{ad}_b)_{i,j}(T) \in BT_{i+j+1}$ and $(\mu_b)_{(i,j)}(T) \in BT_{i+j+1}$ denote the bottom tangles as depicted in Figure 16. We use the following lemma.

Lemma 4.7 (Habiro [4]). *For every bottom tangle $T \in BT_{i+j+2}$, $i,j \geq 0$, we have*

$$J_{(\mathrm{ad}_b)_{i,j}(T)} = \mathrm{ad}_{i,j}(J_T),$$
$$J_{(\mu_b)_{i,j}(T)} = \mu_{i,j}(J_T),$$

where we set

$$\mathrm{ad}_{i,j} = \mathrm{id}^{\otimes i} \otimes \mathrm{ad} \otimes \mathrm{id}^{\otimes j} \colon\ U_h^{\hat{\otimes} i+j+2} \to U_h^{\hat{\otimes} i+j+1},$$
$$\mu_{i,j} = \mathrm{id}^{\otimes i} \otimes \mu \otimes \mathrm{id}^{\otimes j} \colon\ U_h^{\hat{\otimes} i+j+2} \to U_h^{\hat{\otimes} i+j+1}.$$

Here $\mu\colon U_h \hat{\otimes} U_h \to U_h$ is the multiplication of U_h.

For a $2k$-component bottom tangle $W = W_1 \cup \cdots \cup W_{2k} \in BT_{2k}, k \geq 0$, set

$$W^{ev} = \bigcup_{i=1}^{k} W_{2i} \in BT_k, \quad \text{and} \quad W^{odd} = \bigcup_{i=1}^{k} W_{2i-1} \in BT_k.$$

For a diagram P of W, let P^{ev} (resp. P^{odd}) denote the part of the diagram P corresponding to W^{ev} (resp. W^{odd}). We say a bottom tangle $W \in BT_{2k}$ is *even-trivial* if W^{ev} is a trivial bottom tangle. For example, see Figure 17. We also say a diagram P of W is *even-trivial* if and only if P^{ev} has no self crossings. Note that a bottom tangle W has an even-trivial diagram if and only if W is even-trivial.

The following lemma is almost the same as [4, Theorem 11.5].

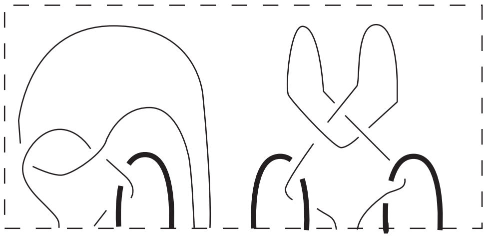
$W =$ 

Figure 17: An even-trivial bottom tangle $W \in BT_6$. Here W^{ev} is depicted with thick lines.

Proposition 4.8. *For any bottom tangle $T \in BT_n$, the following conditions are equivalent.*

(1) T is a ribbon bottom tangle.

(2) There is an even-trivial bottom tangle $W \in BT_{2k}, k \geq 0$, and there are integers $N_1, \ldots, N_n \geq 0$ satisfying $N_1 + \cdots + N_n = k$, such that

$$T = \mu_b^{[N_1,\ldots,N_n]} \operatorname{ad}_b^{\otimes k}(W), \tag{22}$$

where

$$\operatorname{ad}_b^{\otimes k} \colon\ BT_{2k} \to BT_k$$

is as depicted in Figure 18, and

$$\mu_b^{[N_1,\ldots,N_n]} \colon\ BT_{N_1+\cdots+N_n} \to BT_n$$

is as depicted in Figure 19.

If (22) holds, then we call $(W; N_1, \ldots, N_n)$ a *ribbon data* for T. For example, the ribbon bottom tangle $\mu^{[1,2,0]}(ad_b)^{\otimes 3}(W) \in BT_3$ with the ribbon data $(W \in BT_3; 1, 2, 0)$, where W is the bottom tangle in Figure 17, is as depicted in Figure 20.

Proof of Proposition 4.8. In view of Proposition 2.2, the proof is almost the same as that of Theorem 11.5 in [4]. $\square$

For $n \geq 1$, let

$$\mu^{[n]} \colon\ U_h^{\hat{\otimes} n} \to U_h, \quad x_1 \otimes \cdots \otimes x_n \mapsto x_1 x_2 \cdots x_n$$

denote the n-input multiplication. For integers $N_1, \ldots, N_n \geq 0$, $N_1 + \cdots + N_n = k$, set

$$\mu^{[N_1,\ldots,N_n]} = \mu^{[N_1]} \otimes \cdots \otimes \mu^{[N_n]} \colon\ U_h^{\hat{\otimes} k} \to U_h^{\hat{\otimes} n}.$$

Proposition 4.9. *Let $T \in BT_n$ be a ribbon bottom tangle and $(W \in BT_{2k}; N_1, \ldots, N_n)$ a ribbon data for T. Then we have*

$$J_T = \mu^{[N_1,\ldots,N_n]} \operatorname{ad}^{\otimes k}(J_W).$$

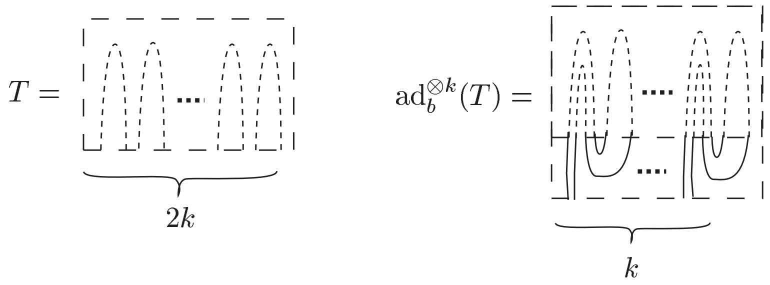


Figure 18: A bottom tangle $T \in BT_{2k}$ and the bottom tangle $\mathrm{ad}_b^{\otimes k}(T) \in BT_k$.

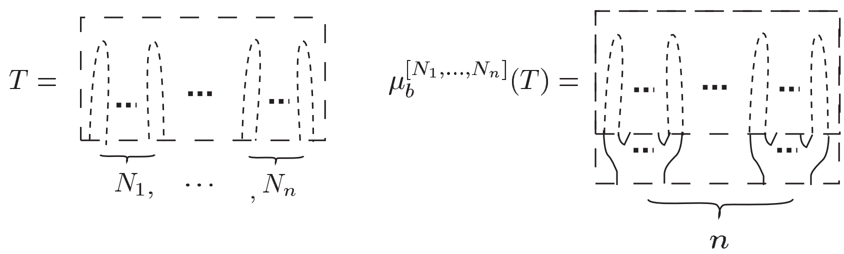


Figure 19: A bottom tangle $T \in BT_k$ and the bottom tangle $\mu_b^{[N_1,\dots,N_n]}(T) \in BT_n$.

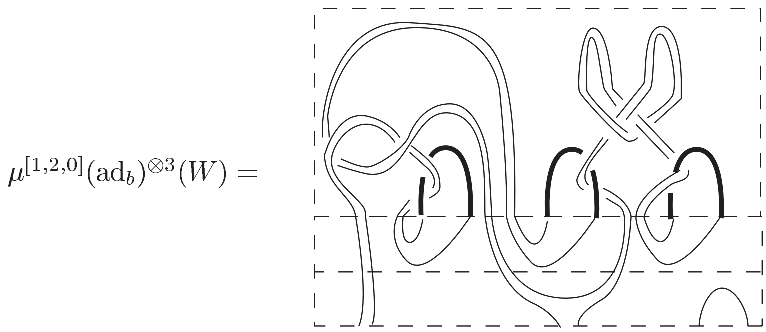


Figure 20: The ribbon bottom tangle $\mu^{[1,2,0]}(ad_b)^{\otimes 3}(W) \in BT_3$ for the even-trivial bottom tangle $W \in BT_3$ in Figure 17.

Proof. By Lemma 4.7, we have

$$J_{\mathrm{ad}_b^{\otimes k}(T)} = \mathrm{ad}^{\otimes k}(J_T),$$

for $T \in BT_{2k}$, and

$$J_{\mu_b^{[N_1,\dots,N_n]}(T)} = \mu^{[N_1,\dots,N_n]}(J_T),$$

for $T \in BT_k$. This implies the assertion. $\square$

5 Proof of Theorem 1.2.

In this section, we prove Theorems 1.2. Let $T \in BT_n$, $n \geq 0$, be a ribbon bottom tangle, and $(W \in BT_{2k}; N_1, \dots, N_n)$, $k \geq 0$, a ribbon data for T. Let P_W be an even-trivial diagram of W, and $s \in \mathcal{S}(P_W)$ a state. We use this setting throughout this section. The proof of Theorem 1.2 is outlined as follows.

First, we prove the following proposition.

Proposition 5.1. *We have*

$$J(P_W, s) \in \tilde{D}^{\mathrm{Lk}(T)}(U_{\mathbb{Z},q}^{ev} \otimes \bar{U}_q^{ev})^{\otimes k}.$$

Then we consider the contribution of $\tilde{D}^{\mathrm{Lk}(T)}$ to the adjoint action, and we construct an element $\tilde{J}(P_W, s) \in (U_{\mathbb{Z},q}^{ev} \otimes \bar{U}_q^{ev})^{\otimes k}$ such that

$$\mathrm{ad}^{\otimes k}(J(P_W, s)) = \mathrm{ad}^{\otimes k}(\tilde{J}(P_W, s)). \tag{23}$$

Thus, by Proposition 3.2, we have

$$\mathrm{ad}^{\otimes k}(J(P_W, s)) \in (\bar{U}_q^{ev})^{\otimes k}. \tag{24}$$

Finally, we define a completion $(\bar{U}_q^{ev})^{\hat{}\hat{\otimes} k}$ of $(\bar{U}_q^{ev})^{\otimes k}$ and prove Theorem 1.2, i.e., we prove

$$J_T = \mu^{[N_1,\dots,N_n]} \sum_{s \in \mathcal{S}(P_W)} \mathrm{ad}^{\otimes k}(J(P_W, s)) \in (\bar{U}_q^{ev})^{\hat{}\hat{\otimes} n}.$$

5.1 Proof of Proposition 5.1.

We modify the proof of Proposition 4.2. The key to the proof is the fact

$$K^m R^+_{m[1]} \otimes R^+_{m[2]}, R^-_{m[1]} \otimes K^m R^-_{m[2]} \in (U_{\mathbb{Z},q}^{ev} \otimes \bar{U}_q^{ev}) \cap (\bar{U}_q^{ev} \otimes U_{\mathbb{Z},q}^{ev}),$$

which follows from (19) and (20). Since P_W is even-trivial, the set $C(P_W)$ of the crossings of P_W is the disjoint union of two subsets

$$C^{eo} = \{ \text{ crossings of } P_W^{ev} \text{ with } P_W^{odd} \},$$

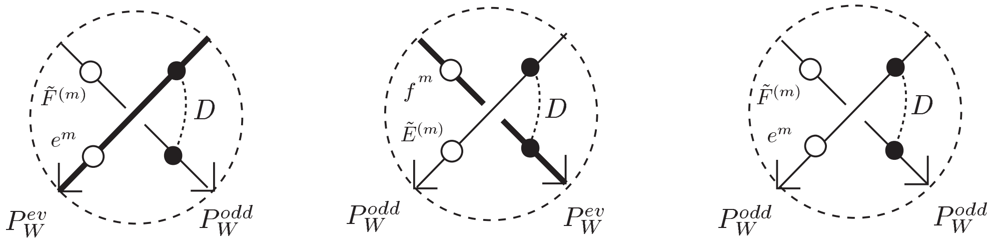


Figure 21: The three types of positive crossings. We work up to multiplication by $\pm q^j, K^{2j} (j \in \mathbb{Z})$.

and

$$C^{oo} = \{ \text{ crossings of } P_W^{odd} \text{ with } P_W^{odd} \}.$$

Thus, on the decorated diagram (P_W, s), we can assume that the element attached to the white dot on P_W^{ev} (resp. P_W^{odd}) is contained in $\bar{U}_q^{ev}$ (resp. $U_{\mathbb{Z},q}^{ev}$). For example, we attach elements to positive crossings as depicted in Figure 21. Then for the decorated diagram $(P_W, s)^\circ$, we have

$$J(P_W, s)^\circ \in (U_{\mathbb{Z},q}^{ev} \otimes \bar{U}_q^{ev})^{\otimes k}. \tag{25}$$

The rest is analogous to the proof of Proposition 4.2.

5.2 The element $\tilde{J}(P_W, s)$.

In this subsection, we construct the element $\tilde{J}(P_W, s) \in (U_{\mathbb{Z},q}^{ev} \otimes \bar{U}_q^{ev})^{\otimes k}$ satisfying (23).

Lemma 5.2. *For homogeneous elements $x, y \in U_h$, we have*

(i) $\sum (D_{[1]}^{\pm} \triangleright x) \otimes D_{[2]}^{\pm} = x \otimes K^{\pm|x|}$,

(ii) $\sum (D_{[1]}^{\pm} \triangleright x) \otimes (D_{[2]}^{\pm} \triangleright y) = q^{\pm|x||y|} x \otimes y$, *and*

(iii) $(v^{H^2/2}K)^{\pm 1} \triangleright x = q^{\pm|x|(|x|+1)} x$.

Proof. We prove the formulas for the positive signs. Then the other cases are similar. By the formulas (3)–(5), we have

(i) $\sum (D_{[1]}^{+} \triangleright x) \otimes D_{[2]}^{+} = \sum (D_{[1]}^{+} x D_{[1]}^{-}) \otimes (D_{[2]}^{+} D_{[2]}^{-}) = x \otimes K^{|x|}$.

Using (i), we obtain

(ii) $\sum (D_{[1]}^{+} \triangleright x) \otimes (D_{[2]}^{+} \triangleright y) = \sum x \otimes (K^{|x|} \triangleright y) = q^{|x||y|} x \otimes y$, and

(iii) $(v^{H^2/2}K) \triangleright x = \sum (D_{[1]}^{+} D_{[2]}^{+} K) \triangleright x = q^{|x|}(D_{[1]}^{+} D_{[2]}^{+} \triangleright x) = q^{|x|}(K^{|x|} \triangleright x) = q^{|x|(|x|+1)} x$.

□

Lemma 5.3. *For $k \geq 0$, let $M = (m_{i,j})_{1 \leq i,j \leq 2k}$ be a symmetric integer matrix of size $2k$, satisfying $m_{2i,2j} = 0$ for $1 \leq i,j \leq k$. Let $X = x_1 \otimes \cdots \otimes x_{2k} \in U_h^{\otimes 2k}$ be the tensor product of homogeneous elements $x_1, \ldots, x_{2k} \in U_h$. We have*

$$\mathrm{ad}^{\otimes k}(\tilde{D}^M X) = q^{N(M,X)} \, \mathrm{ad}^{\otimes k} \big((1 \otimes K^{2a_1(M,X)} \otimes \cdots \otimes 1 \otimes K^{2a_m(M,X)}) X \big),$$

where if we set $X_i = x_{2i-1} \triangleright x_{2i}$, then

$$\begin{aligned} a_i(M,X) &= \sum_{1 \leq j \leq k} m_{2i,2j-1} |X_j|, \\ N(M,X) &= \sum_{1 \leq i < j \leq k} 2m_{2i-1,2j-1} |X_i||X_j| + \sum_{1 \leq i \leq k} m_{2i-1,2i-1} |X_i|(|X_i|+1). \end{aligned}$$

Here $|X_i| = |x_{2i-1}| + |x_{2i}|$ is the degree of X_i defined in Section 3.

Proof. We use induction on $\sum_{1 \leq i,j \leq 2k} |m_{ij}|$. If $\sum_{1 \leq i,j \leq 2k} |m_{ij}| = 0$, i.e., $M = 0$, then the claim is clear. Let us assume $M \neq 0$. Then there is a matrix M' satisfying the assertion, and either

$$\begin{aligned} &M = M' \pm (1_{2i,2j-1} + 1_{2j-1,2i}), \quad \text{for } 1 \leq i \neq j \leq k, \text{ or} \\ &M = M' \pm (1_{2i-1,2j-1} + 1_{2j-1,2i-1}), \quad \text{for } 1 \leq i \neq j \leq k, \text{ or} \\ &M = M' \pm 1_{2i-1,2i-1}, \quad \text{for } 1 \leq i \leq k, \end{aligned}$$

where $1_{i,j}$ is the matrix of size $2k$ such that the (i,j)-component is 1 and the others are 0. Note that

$$\begin{aligned} &\tilde{D}^{M' \pm (1_{i,j} + 1_{j,i})} = \tilde{D}^{M'} D_{i,j}^{\pm 2}, \quad \text{for } \ 1 \leq i \neq j \leq 2k, \text{ and} \\ &\tilde{D}^{M' \pm 1_{ii}} = \tilde{D}^{M'} (v^{H^2/2} K)_i^{\pm 1}, \quad \text{for } \ 1 \leq i \leq 2k. \end{aligned}$$

Then the following formulas using Lemma 5.2 imply the assertion.

$$\begin{aligned} \mathrm{ad}^{\otimes k}(D_{2i,2j-1}^{\pm 1} X) &= X_1 \otimes \cdots \otimes (x_{2i-1} \triangleright D_{[1]}^{\pm} x_{2i}) \otimes \cdots \otimes (D_{[2]}^{\pm} \triangleright X_j) \otimes \cdots \otimes X_k \\ &= X_1 \otimes \cdots \otimes (x_{2i-1} \triangleright K^{\pm |X_j|} x_{2i}) \otimes \cdots \otimes X_j \otimes \cdots \otimes X_k, \\ \mathrm{ad}^{\otimes k}(D_{2i,2i-1}^{\pm 1} X) &= X_1 \otimes \cdots \otimes (D_{[2]}^{\pm} x_{2i-1} \triangleright D_{[1]}^{\pm} x_{2i}) \otimes \cdots \otimes \cdots \otimes X_k \\ &= X_1 \otimes \cdots \otimes (x_{2i-1} \triangleright K^{\pm |X_i|} x_{2i}) \otimes \cdots \otimes X_j \otimes \cdots \otimes X_k, \\ \mathrm{ad}^{\otimes k}(D_{2i-1,2j-1}^{\pm 1} X) &= X_1 \otimes \cdots \otimes (D_{[1]}^{\pm} \triangleright X_i) \otimes \cdots \otimes (D_{[2]}^{\pm} \triangleright X_j) \otimes \cdots \otimes X_k \\ &= q^{\pm |X_i||X_j|} X_1 \otimes \cdots \otimes X_i \otimes \cdots \otimes X_j \otimes \cdots \otimes X_k, \\ \mathrm{ad}^{\otimes k} \big((v^{H^2/2} K)_{2i-1}^{\pm 1} X \big) &= X_1 \otimes \cdots \otimes ((v^{H^2/2} K)^{\pm 1} \triangleright X_i) \otimes \cdots \otimes X_k \\ &= q^{\pm |X_i|(|X_i|+1)} X_1 \otimes \cdots \otimes X_i \otimes \cdots \otimes X_k, \end{aligned}$$

for $1 \leq i \neq j \leq k$. ∎

By Proposition 5.1, we have

$$X := (\tilde{D}^{\mathrm{Lk}(W)})^{-1} J(P_W, s) \in (U_{\mathbb{Z},q}^{ev} \otimes \bar{U}_q^{ev})^{\otimes k}.$$

Since the linking matrix $\mathrm{Lk}(W)$ of W satisfies the assumption of Lemma 5.3, we obtain the element $\tilde{J}(P_W, s) \in (U_{\mathbb{Z},q}^{ev} \otimes \bar{U}_q^{ev})^{\otimes k}$ satisfying (23), such that

$$\tilde{J}(P_W, s) := q^N (1 \otimes K^{2a_1} \otimes \cdots \otimes 1 \otimes K^{2a_k}) X,$$

where we set

$$N = N(\mathrm{Lk}(W), X),$$

and

$$a_i = a_i(\mathrm{Lk}(W), X),$$

for $i = 1, \ldots, k$, as in Lemma 5.3.

5.3 Filtrations of $\bar{U}_q^{ev}$.

In this subsection, we define two filtrations $\{A_p\}_{p\geq 0}$ and $\{C_p\}_{p\geq 0}$ of $\bar{U}_q^{ev}$, which are cofinal with each other. We give four equivalent definitions for $\{A_p\}_{p\geq 0}$, and two for $\{C_p\}_{p\geq 0}$.

For a subset $X \subset \bar{U}_q^{ev}$, let $\langle X\rangle_{ideal}$ denote the two-sided ideal of $\bar{U}_q^{ev}$ generated by X. For $p \geq 0$, set

$$\begin{aligned}
A_p &= \langle U_{\mathbb{Z},q} \triangleright e^p \rangle_{ideal},\\
A'_p &= \langle U_{\mathbb{Z},q} \triangleright f^p \rangle_{ideal},\\
B_p &= \langle K^p (U_{\mathbb{Z},q} \triangleright K^{-p} e^p) \rangle_{ideal},\\
B'_p &= \langle K^p (U_{\mathbb{Z},q} \triangleright f^p K^{-p}) \rangle_{ideal},\\
C_p &= \langle \sum_{p'\geq p} (U_{\mathbb{Z},q} \tilde{E}^{(p')} \triangleright \bar{U}_q^{ev}) \rangle_{ideal},\\
C'_p &= \langle \sum_{p'\geq p} (U_{\mathbb{Z},q} \tilde{F}^{(p')} \triangleright \bar{U}_q^{ev}) \rangle_{ideal}.
\end{aligned}$$

Proposition 5.4. *For $p \geq 0$, we have*

$$A_p = A'_p = B_p = B'_p.$$

Proof. By the formulas

$$f^p K^{-p} = (-1)^p q^{-p^2} \tilde{F}^{(2p)} \triangleright K^{-p} e^p \in U_{\mathbb{Z},q} \triangleright K^{-p} e^p, \tag{26}$$

$$K^{-p} e^p = (-1)^p q^{p^2} \tilde{E}^{(2p)} \triangleright f^p K^{-p} \in U_{\mathbb{Z},q} \triangleright f^p K^{-p}, \tag{27}$$

we have $B_p = B'_p$. We prove $A_p = B_p$, then $A'_p = B'_p$ is similar. By Proposition 3.2, we have

$$\begin{aligned}
K^p (U_{\mathbb{Z},q} \triangleright K^{-p} e^p) &\subset K^p (U_{\mathbb{Z},q} \triangleright K^{-p}) \cdot (U_{\mathbb{Z},q} \triangleright e^p)\\
&\subset \bar{U}_q^{ev} (U_{\mathbb{Z},q} \triangleright e^p) \subset A_p.
\end{aligned}$$

Hence we have $B_p \subset A_p$. Conversely, we have

$$\begin{aligned} U_{\mathbb{Z},q} \triangleright e^p = U_{\mathbb{Z},q} \triangleright K^p K^{-p} e^p &\subset (U_{\mathbb{Z},q} \triangleright K^p) \cdot (U_{\mathbb{Z},q} \triangleright K^{-p} e^p) \\ &\subset \bar{U}_q^{ev} K^p (U_{\mathbb{Z},q} \triangleright K^{-p} e^p) \subset B_p. \end{aligned}$$

Hence we have $A_p \subset B_p$, this completes the proof. $\square$

Proposition 5.5. *(i) For $p \geq 0$, we have $C_p = C'_p$.*

(ii) For $p \geq 0$, we have $C_{2p} \subset A_p$.

(iii) If $p \geq 0$ is even, then we have $C_{2p} = A_p$.

Proof. (i) We prove $C_p \subset C'_p$, then $C_p \supset C'_p$ is similar. Using the formula

$$\tilde{E}^{(2p)} \triangleright \tilde{F}^{(p)} K^{-p} = (-1)^p q^{-\frac{1}{2}p(p+1)} K^{-p} \tilde{E}^{(p)},$$

we have

$$\begin{aligned} U_{\mathbb{Z},q} \tilde{E}^{(p)} &\subset U_{\mathbb{Z},q} \big(\tilde{E}^{(2p)} \triangleright \tilde{F}^{(p)} K^{-p} \big) \\ &\subset U_{\mathbb{Z},q} \tilde{F}^{(p)} U_{\mathbb{Z},q}. \end{aligned}$$

Hence we have

$$\begin{aligned} U_{\mathbb{Z},q} \tilde{E}^{(p)} \triangleright \bar{U}_q^{ev} &\subset U_{\mathbb{Z},q} \tilde{F}^{(p)} U_{\mathbb{Z},q} \triangleright \bar{U}_q^{ev} \\ &\subset U_{\mathbb{Z},q} \tilde{F}^{(p)} \triangleright \bar{U}_q^{ev}. \end{aligned}$$

This completes the proof.

(ii) In view of Lemma 3.1, it is enough to prove that

$$\tilde{E}^{(p')} \triangleright f^{i_1} K^{2i_2} e^{i_3} \subset A_p,$$

for $p' \geq 2p$. If $i_1 \geq p' \geq p$, then the assertion follows from

$$U_{\mathbb{Z},q} \triangleright f^{i_1} K^{2i_2} e^{i_3} \subset (U_{\mathbb{Z},q} \triangleright f^{i_1}) \bar{U}_q^{ev} \subset A'_p = A_p.$$

If $i_1 < p'$, then we have

$$\begin{aligned} \tilde{E}^{(p')} \triangleright f^{i_1} K^{2i_2} e^{i_3} &\in \langle U_{\mathbb{Z},q} \triangleright f^{i_1} \rangle_{ideal} \cap \langle e^{i_3 + p' - i_1} \rangle_{ideal}, \\ &\subset A'_{i_1} \cap A_{i_3 + p' - i_1} \\ &\subset A_{\max\{i_1, i_3 + p' - i_1\}}, \end{aligned}$$

where the $\in$ follows from the formula (15), and the last $\subset$ follows from Proposition 5.4. Hence the assertion follows from

$$\max\{i_1, i_3 + p' - i_1\} \geq \frac{i_3 + p'}{2} \geq p.$$

(iii) If $p \geq 0$ is even, then we have

$$\begin{aligned} K^p (U_{\mathbb{Z},q} \triangleright K^{-p} e^p) &= (-1)^p q^{p^2} K^p (U_{\mathbb{Z},q} \triangleright (\tilde{E}^{(2p)} \triangleright f^p K^{-p})) \\ &\subset \langle U_{\mathbb{Z},q} \tilde{E}^{(2p)} \triangleright \bar{U}_q^{ev} \rangle_{ideal} \subset C_{2p}, \end{aligned}$$

from (26). Hence we have $C_{2p} \supset B_p (= A_p)$, this completes the proof. $\square$

Corollary 5.6. *For $p \geq 0$, we have*

$$C_{2p} \subset h^p U_h.$$

Proof. Since $e^p \subset h^p U_h$, we have $C_{2p} \subset A_p \subset h^p U_h$ by Proposition 5.5. $\square$

5.4 The completion $(\bar{U}_q^{ev})^{\hat{\otimes} n}$ of $(\bar{U}_q^{ev})^{\otimes n}$.

In this subsection we define the completion $(\bar{U}_q^{ev})^{\hat{\otimes} n}$ of $(\bar{U}_q^{ev})^{\otimes n}$, and prove Theorem 1.2. Let $(\bar{U}_q^{ev})\hat{}$ denote the completion in U_h of $\bar{U}_q^{ev}$ with respect to the decreasing filtration $\{C_p\}_{p\geq 0}$, i.e., $(\bar{U}_q^{ev})\hat{}$ is the image of the homomorphism

$$\varprojlim_p \bar{U}_q^{ev}/C_p \to U_h.$$

induced by the inclusion $\bar{U}_q^{ev} \subset U_h$, which is well defined since $C_{2p} \subset h^p U_h$ for $p \geq 0$. For $n \geq 1$, we define a filtration $\{C_p^{(n)}\}_{p\geq 0}$ for $(\bar{U}_q^{ev})^{\otimes n}$ by

$$C_p^{(n)} = \sum_{j=1}^{n} \bar{U}_q^{ev} \otimes \cdots \otimes \bar{U}_q^{ev} \otimes C_p \otimes \bar{U}_q^{ev} \otimes \cdots \otimes \bar{U}_q^{ev},$$

where C_p is at the jth position. Define the completion $(\bar{U}_q^{ev})^{\hat{\otimes} n}$ of $(\bar{U}_q^{ev})^{\otimes n}$ as the image of the homomorphism

$$\varprojlim_p \big((\bar{U}_q^{ev})^{\otimes n}/C_p^{(n)}\big) \to U_h^{\hat{\otimes} n}.$$

For $n = 0$, it is natural to set

$$C_p^{(0)} = \begin{cases} \mathbb{Z}[q, q^{-1}] & \text{if } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have

$$(\bar{U}_q^{ev})^{\hat{\otimes} 0} = \mathbb{Z}[q, q^{-1}].$$

Recall the setting mentioned at the beginning of this section. For $i = 1, \ldots, 2k$, let P_i denote the part of P_W corresponding to the ith component of $W = W_1 \cup \cdots \cup W_{2k}$, and $C(P_i)$ the set of the crossings on the component P_i. For $p \geq 0$, we denote by $\mathcal{I}_p$ the two-sided ideal of $U_{\mathbb{Z},q}$ generated by $\tilde{E}^{(p)}, \tilde{F}^{(p)} \in U_{\mathbb{Z},q}$. For $s \in \mathcal{S}(P_W)$, set $|s|_i = \max\{s(c) \mid c \in C(P_i)\}$.

Lemma 5.7. *For each $s \in \mathcal{S}(P_W)$, there are elements $w_{2i-1} \in U_{\mathbb{Z},q}^{ev} \cap \mathcal{I}_{|s|_{2i-1}}$ and $w_{2i} \in \bar{U}_q^{ev} \cap \mathcal{I}_{|s|_{2i}}$ for $i = 1, \ldots k$, such that*

$$\tilde{J}(P_W, s) = w_1 \otimes \cdots \otimes w_{2k}.$$

Proof. Let $(P_i, s)^\circ$ denote the decorated diagram with P_i and white dots of $(P_W, s)^\circ$ on P_i (see p14 for the definition of $(P_W, s)^\circ$). Recall that one of the elements $\tilde{E}^{(s(c))}$, $\tilde{F}^{(s(c))}$, $e^{s(c)}$, $f^{s(c)}$ is labeled on a white dot on a crossings c of the decorated diagram $(P_W, s)^\circ$. Since each of those elements is contained in $\mathcal{I}_{s(c)}$, we have

$$J(P_i, s)^\circ \in \mathcal{I}_{|s|_i}.$$

Note that

$$\begin{aligned}\tilde{J}(P_W, s) &\sim (\tilde{D}^{\mathrm{Lk}(W)})^{-1} J(P_W, s)\\ &\sim J(P_1, s)^\circ \otimes \cdots \otimes J(P_{2k}, s)^\circ,\end{aligned}$$

where $\sim$ means equality up to multiplication by $\pm q^j, K^{2j} (j \in \mathbb{Z})$ on any tensorands. This and Proposition 5.1 complete the proof. □

Proof of Theorem 1.2. Let $|s| = \max\{s(c) \mid c \in C(P_W)\}$ denote the maximal integer of the image of s. Since every crossing of P_W has at least one strand in P_W^{odd}, we can assume $s(c) = |s|$ for a crossing c that has a strand of P_{2j-1}, $1 \leq j \leq k$. Take elements $w_{2i-1} \in U_{\mathbb{Z},q}^{ev} \cap \mathcal{I}_{|s|_{2i-1}}$ and $w_{2i} \in \bar{U}_q^{ev} \cap \mathcal{I}_{|s|_{2i}}$, $i = 1, \ldots, k$, as in Lemma 5.7. We have

$$w_{2j-1} \in \mathcal{I}_{|s|}.$$

Since $\mathcal{I}_{|s|} \triangleright \bar{U}_q^{ev} \subset C_{|s|}$, we have

$$w_{2j-1} \triangleright w_{2j} \in C_{|s|}.$$

In view of Proposition 3.2, we have

$$\mathrm{ad}^{\otimes k}(\tilde{J}(P_W, s)) = \mathrm{ad}^{\otimes k}(w_1 \otimes \cdots \otimes w_{2k}) \in C_{|s|}^{(k)}.$$

Thus by Proposition 4.9, we have

$$\begin{aligned}J_T &= \mu^{[N_1, \ldots, N_n]} \,\mathrm{ad}^{\otimes k}(J_W)\\ &= \sum_{l \geq 0} \sum_{s \in \mathcal{S}(P_W), |s| = l} \mu^{[N_1, \ldots, N_n]} \,\mathrm{ad}^{\otimes k}(\tilde{J}(P_W, s)) \in (\bar{U}_q^{ev})^{\hat{}\hat{\otimes} n}.\end{aligned}$$

This completes the proof. □

Remark 5.8. Recall from [5] the $\mathbb{Z}[q, q^{-1}]$-subalgebra $(\bar{U}_q^{ev})^{\tilde{}\hat{\otimes} n}$ of $U_h^{\hat{\otimes} n}$. We can prove the inclusion $(\bar{U}_q^{ev})^{\hat{}\hat{\otimes} n} \subset (\bar{U}_q^{ev})^{\tilde{}\hat{\otimes} n}$ as follows. We have only to prove $C_{2p} \subset \mathcal{F}_p(\mathcal{U}_q^{ev})$, for $p \geq 0$, where $\mathcal{F}_p(\mathcal{U}_q^{ev})$ denote the two-sided ideal of $\mathcal{U}_q^{ev}$ generated by e^p. In view of Proposition 5.5, we have only to prove $A_p \subset \mathcal{F}_p(\mathcal{U}_q^{ev})$.

Set

$$\begin{bmatrix} H+i \\ p \end{bmatrix}_q = \{H+i\}_{q,p}/\{p\}_q!,$$

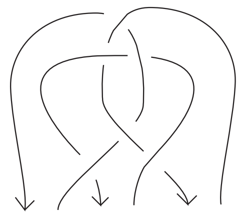


Figure 22: The Borromean tangle $B \in BT_3$.

for $i \in \mathbb{Z}$, $p \geq 0$. One can show that

$$U^{ev}_{\mathbb{Z},q} = \bigoplus_{i,j\geq 0} \tilde{F}^{(i)} U^{0ev}_{\mathbb{Z},q} \tilde{E}^{(j)},$$

where $U^{0ev}_{\mathbb{Z},q}$ is the $\mathbb{Z}[q,q^{-1}]$-subalgebra of $U^{ev}_{\mathbb{Z},q}$ generated by the elements K^2, K^{-2}, and $\begin{bmatrix} H+i \\ p \end{bmatrix}_q$ for $i \in \mathbb{Z}$, $p \geq 0$ (This fact is a variant of a well known fact on Lusztig's integral form $U_{\mathbb{Z}}$ [10]). Thus it is enough to prove that

$$\tilde{F}^{(i)} g \tilde{E}^{(j)} \triangleright e^p \subset \mathcal{F}_p(\mathcal{U}^{ev}_q),$$

for $i,j \geq 0$ and $g \in U^{0ev}_{\mathbb{Z},q}$. For a homogeneous element $x \in U_h$, we have $U^{0ev}_{\mathbb{Z},q} \triangleright x \subset \mathbb{Z}[q,q^{-1}]x$ since

$$K \triangleright x = q^{|x|}x, \quad \begin{bmatrix} H+k \\ l \end{bmatrix}_q \triangleright x = \begin{bmatrix} 2|x|+k \\ l \end{bmatrix}_q x,$$

for $k \in \mathbb{Z}$, $l \geq 0$. Then the claim follows from

$$\tilde{E}^{(j)} \triangleright e^p = (-1)^j \begin{bmatrix} j+p-1 \\ j \end{bmatrix}_q e^{p+j},$$

$$\tilde{F}^{(i)} \triangleright e^{p+j} = \sum_{j=0}^{n} (-1)^j q^{-\frac{1}{2}j(j-1)+j(p+j)} \tilde{F}^{(n-j)} e^{p+j} \tilde{F}^{(j)} \subset \mathcal{F}_p(\mathcal{U}^{ev}_q).$$

6 Examples.

The Borromean tangle $B \in BT_3$ is the bottom tangle depicted in Figure 22. Note that B is a 3-component, algebraically-split, 0-framed bottom tangle, and the closure of B is the Borromean rings L_B. It is well known that L_B is not a ribbon link. In [5], the formulas of the universal sl_2 invariant of B is observed;

$$\begin{aligned} J_B = & \sum_{m_1,m_2,m_3,n_1,n_2,n_3\geq 0} q^{m_3+n_3}(-1)^{n_1+n_2+n_3} q^{\sum_{i=1}^{3}\left(-\frac{1}{2}m_i(m_i+1)-n_i+m_im_{i+1}-2m_in_{i-1}\right)} \\ & \tilde{F}^{(n_3)} e^{m_1} \tilde{F}^{(m_3)} e^{n_1} K^{-2m_2} \otimes \tilde{F}^{(n_1)} e^{m_2} \tilde{F}^{(m_1)} e^{n_2} K^{-2m_3} \otimes \tilde{F}^{(n_2)} e^{m_3} \tilde{F}^{(m_2)} e^{n_3} K^{-2m_1} \\ & \notin (\bar{U}^{ev}_q)^{\hat{\otimes} 3}, \end{aligned} \tag{28}$$

where the index i should be considered modulo 3. The following is also observed in [5];

$$J_{L_B;\tilde{P}'_i,\tilde{P}'_j,\tilde{P}'_k} = \begin{cases} (-1)^i q^{-i(3i-1)}\{2i+1\}_{q,i+1}/\{1\}_q & \text{if} \quad i=j=k, \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

Since $\frac{\{2i+1\}_{q,i+1}}{\{1\}_q} \notin \frac{\{2i+1\}_{q,i+1}}{\{1\}_q} I_i I_i$ for $i \geq 1$, each of (28) and (29) implies that the Borromean rings L_B is not a ribbon link.

Remark 6.1. Let L_K be the 2-component link obtained from a knot K by duplicating the component. Indeed, L_K is a boundary link. In particular, if K is a ribbon knot, then L_K is a ribbon link. We can prove

$$J_{L_K;\tilde{P}'_m,\tilde{P}'_n} \in \frac{\{2m+1\}_{q,m+1}}{\{1\}_q} I_n$$

as follows. By the formulas in Section 8 in [5], we have

$$\begin{aligned} \tilde{P}'_m \tilde{P}'_n &= \sum_{k=0}^{\min(m,n)} q^{-kl} \frac{\{m+n\}_q!}{\{k\}_q!\{m-k\}_q!\{n-k\}_q!} \tilde{P}'_l \\ &= \sum_{k=0}^{\min(m,n)} q^{-l(k+l+1)} C_{k,m,n}(q) P''_l, \end{aligned}$$

where $l = m+n-k$, $P''_l = \frac{\{1\}_q}{\{2l+1\}_{q,l+1}} q^{l(l+1)} \tilde{P}'_l$, and

$$\begin{aligned} C_{k,m,n}(q) &= \frac{\{2m+1\}_{q,m+1}}{\{1\}_q} \{k\}_q!\{n-k\}_q! \begin{bmatrix} 2l+1 \\ 2m+1 \end{bmatrix}_q \begin{bmatrix} 2(n-k) \\ n-k \end{bmatrix}_q \begin{bmatrix} m+n \\ k \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q \\ &\in \frac{\{2m+1\}_{q,m+1}}{\{1\}_q} I_n. \end{aligned}$$

Theorem 6.4 in [5] implies that $J_{K;P''_l} \in \mathbb{Z}[q,q^{-1}]$ for $l \geq 0$, hence we have

$$\begin{aligned} J_{L_K;\tilde{P}'_m,\tilde{P}'_n} &= J_{K;\tilde{P}'_m \cdot \tilde{P}'_n} \\ &= \sum_{k=0}^{\min(m,n)} q^{-l(k+l+1)} C_{k,m,n}(q) J_{K;P''_l} \\ &\in \frac{\{2m+1\}_{q,m+1}}{\{1\}_q} I_n. \end{aligned}$$

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