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WKB analysis of higher order Painlevé equations with a large parameter. II. — Structure theorem for instanton-type solutions of $(P_J)_m$ ($J = \mathrm{I}, 34, \mathrm{II}\text{-}2$ or IV) near a simple P-turning point of the first kind

Dedicated to Professor Mikio SATO on his eightieth birthday

By

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0 Introduction

This paper is the third of a series of articles on the exact WKB analysis of higher order Painlevé equations; the first of the series is [KKoNT], and the second one is [KT5]. In [KKoNT] we studied basic properties of higher order Painlevé equations $(P_J)_m$ with a large parameter η (J = I, II-1, II-2; $m = 1, 2, \ldots$); we first constructed a particular formal solution called a 0-parameter solution, and we then clarified the relationship between

(i) the Stokes geometry of the linearization $(\Delta P_J)_m$ of $(P_J)_m$ at the 0-parameter solution (often called the Fréchet derivative),

and

(ii) the Stokes geometry of (one of) the underlying pair $(L_J)_m$ of linear differential equations (Lax pair) with the 0-parameter solution substituted into the coefficients.

To avoid possible confusions of the reader we used in [KT5] the terminologies "P-turning points" and "P-Stokes curves" (following the suggestion of the referee) to mean "turning points of the Fréchet derivative" and "Stokes curves of the Fréchet derivative", and in this paper we follow [KT5] in using this wording. The main subject of [KT5] was to establish a structure theorem for 0-parameter solutions of $(P_J)_m$ (J = I, II-1, II-2); any 0-parameter solution can be formally and locally transformed near a simple P-turning point of the first kind to a 0-parameter solution of the second order Painlevé-I equation with a large parameter η:

$$\frac{d^2\lambda_{\mathrm{I}}}{dt^2} = \eta^2(6\lambda_{\mathrm{I}}^2 + t). \tag{0.1}$$

In proving this result we made essential use of the geometric results obtained in [KKoNT]. The above structure theorem is a generalization of a result for the second order Painlevé equations ([KT1, Theorem 2.3]) to that applicable to an arbitrarily higher order equation $(P_J)_m$. It is worth emphasizing that [KT1] covers only 6 equations, the classical Painlevé equations $(P_{\rm I})$, $(P_{\rm II})$, ..., $(P_{\rm VI})$, and that the results in [KT5] are applied to infinitely many equations. The purpose of this paper is to further generalize the results in [KT5] by replacing 0-parameter solutions with instanton-type $(2m)$-parameter solutions ([T1],[T2]) of $(P_J)_m$; our main result (Theorem 5.1.1) means that Part 5 of the Toulouse Project ([KT3]) has been completed near a simple P-turning point of the first kind. We note that in this paper we basically follow [Ko2] concerning notational issues; this means that we use in this paper symbols that are slightly different from those used in [KKoNT] and [KT5]. This is a nuissance, but it removes some clumsiness from the presentation of [KT5]. The point is that [Ko2] presents three different ways of expressing the same higher order Painlevé equations, $(P_J)_m$, $(\tilde{P}_J)_m$ and $(G_J)_m$ ($J =$ I, 34, II-2 and IV). The first one is given in terms of polynomials of unknown functions and their derivatives, the second one is a system of first order non-linear differential equations, and the third one is given by choosing some suitable Garnier system and restricting it to an appropriate complex line; symbol $(G_J)_m$ is not used in the literature, but for the sake of convenience we use this symbol in this paper. Thus $(P_{\rm I})_m$ in [KKoNT] and [KT5] is designated as $(\tilde{P}_{\rm I})_m$ in this paper. The Lax pair that underlies $(P_J)_m$ or $(\tilde{P}_J)_m$ is respectively denoted by $(L_J)_m$ or $(\tilde{L}_J)_m$; we arrange the two equations in $(L_J)_m$ and $(\tilde{L}_J)_m$ so that the first one of them is deformed by the second one that contains the differentiation with respect to the deformation parameter t, which is the independent variable of $(P_J)_m$ and $(\tilde{P}_J)_m$ in question. We emphasize that each of these three expres-

sions of a higher order Painlevé equation has its own advantage. For example, $(P_J)_m$ and $(L_J)_m$ are amenable to the concrete computation because of its concise form and $(\tilde{P}_J)_m$ and $(G_J)_m$ most neatly explain the intrinsic meaning of the change of unknown functions from "u" to "λ" that is used in [KT5]. In [KT5] the meaning of the transformation was not explained well for $(P_{\text{II-1}})_m$ or $(P_{\text{II-2}})_m$; with the introduction of $(\tilde{P}_J)_m$ we clearly see that the unknown function u_j $(j = 1, 2, \dots, m)$ of $(\tilde{P}_J)_m$ is the j-th elementary symmetric polynomial of the unknown functions λ_k's of $(G_J)_m$. The important role that $(G_J)_m$ plays in our paper is basically due to its Hamiltonian structure on which the construction of instanton-type solutions is based. (See [T1] and [T2].) For the convenience of the reader, we list up in Appendix the symbols and equations used in this paper, following the presentation of [Ko2].

The plan of this paper is as follows. In Section 1 we first rewrite the Lax pair $(L_J)_m$ as a pair of a Schrödinger equation $(SL_J)_m$ and its deformation equation $(D_J)_m$. As the derivation procedure of this system of scalar equations is essentially the same for all J ($J =$ I, 34, II-2, IV), we present the explicit computation only for $J =$ IV. (Cf. [KT5], [KT7].) In Section 2 we summarize basic properties of $(2m)$-parameter solutions of $(P_J)_m$, which have been constructed and called instanton-type solutions in [T2]. These solutions are the main target of our study in this paper. We note that in studying the effect of substituting an instanton-type solution into the coefficients of $Q_{(J,m)}$, the potential of the Schrödinger equation $(SL_J)_m$, we make use of the third order equation (2.2.2) that $Q_{(J,m)}$ satisfies together with the function $\mathfrak{a}_{(J,m)}$ that appears in $(D_J)_m$ (Subsection 2.2). Although this equation is known to be a basic one in the theory of deformations of linear differential equations (cf., e.g., [KT4, (4.44)]), this is the first time that we have used this equation as an essential ingredient in the study of $(SL_J)_m$. The equation (2.2.2) plays an important role also in Section 3. Using

the results in Section 2, we establish in Section 3 a WKB theoretic theorem (Theorem 3.1) to the effect that $(SL_J)_m$ with instanton-type solutions substituted into its coefficients can be brought to a canonical equation called (Can) near its double turning point $x = \lambda_{j_0,0}(t)$; in particular, we describe how an instanton-type solution λ_{j_0} of $(P_J)_m$ is related to the invariants $\rho^{(j_0)}$ and $\sigma^{(j_0)}$ that appear in the canonical equation. (See Theorem 3.1 and 3.2 for the precise statements.) In Section 4 we investigate the instanton structure of the invariants by making use of the Hamiltonian structure of $(G_J)_m$. The results on the instanton structure of the invariants are used in an essential manner in proving our main result (Theorem 5.1.1). In Appendix A we list up the symbols and notations used in this paper; we follow [Ko2] as possible as we can. Subsections A.1 $\sim$ A.4 are concerned with P_{I}-hierarchy with a large parameter η, Subsections A.5 $\sim$ A.8 are concerned with P_{34}-hierarchy with a large parameter η, and so on. Finally in Appendix B we explain the parity structure of instanton-type solutions which is used in Section 5.

1 Derivation of a Schrödinger equation $(SL_J)_m$ and its deformation equation $(D_J)_m$

The purpose of this section is to rewrite $(L_J)_m$ (or $(\tilde{L}_J)_m$) as a pair of a Schrödinger equation $(SL_J)_m$ and its deformation equation $(D_J)_m$ so that the Lax pair may be analyzed in the framework of [KT1], [AKT] and [KT2]. Although we study only $(L_{\mathrm{IV}})_m$ in a detailed manner, our procedure is uniformly applicable to any of $(L_J)_m$ or $(\tilde{L}_J)_m$ ($J =$ I, 34, II-2, IV). To emphasize this fact we rewrite (A.15.1) in a

somewhat abstract style:

$$(1.1)\qquad \begin{cases} \dfrac{\partial}{\partial x}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \eta\begin{pmatrix}p & q\\ r & -p\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} & (1.1.a)\\ \dfrac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \eta\begin{pmatrix}\delta & 1\\ \varepsilon & -\delta\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} & (1.1.b)\end{cases}$$

Here all the coefficients are those given by (A.15.1) with a solution (u, v) of $(P_{\mathrm{IV}})_m$ substituted. One can immediately see that any of $(L_J)_m$ or $(\tilde{L}_J)_m$ has this form with the exceptions of $(\tilde{L}_{\mathrm{I}})_m$ and $(\tilde{L}_{34})_m$; in $(\tilde{L}_{\mathrm{I}})_m$ and $(\tilde{L}_{34})_m$ the (1,2) component of the matrix in (1.1.b) is 2, not 1. (Cf. Subsections A.3, A.7, A.11 and A.15.) We try to find a system of scalar differential equations that ψ_1 satisfies. It follows from (1.1.a) that

$$(1.2)\qquad \frac{\partial^2\psi_1}{\partial x^2} - \frac{q_x}{q}\frac{\partial\psi_1}{\partial x} - (\eta^2(p^2+qr) + \eta(p_x - \frac{pq_x}{q}))\psi_1 = 0.$$

Here and in what follows, q_x etc. and q_t etc. respectively stand for $\partial q/\partial x$ etc. and $\partial q/\partial t$ etc. To rewrite (1.2) in a form of a Schrödinger-type equation, we introduce

$$(1.3)\qquad \psi = \exp\left(\frac{1}{2}\int^x(-\frac{q_x}{q})dx\right)\psi_1 = q^{-1/2}\psi_1.$$

Then ψ satisfies

$$(1.4)\qquad \frac{\partial^2\psi}{\partial x^2} = \eta^2 Q_{(\mathrm{IV},m)}\psi$$

with

$$(1.5)\qquad Q_{(\mathrm{IV},m)} = p^2 + qr + \eta^{-1}(p_x - \frac{pq_x}{q}) + \eta^{-2}\left(\frac{3q_x^2}{4q^2} - \frac{q_{xx}}{2q}\right).$$

The equation (1.4) corresponds to (SL_J) in [KT1], and we use a symbol $(SL_{\mathrm{IV}})_m$ to denote the Schrödinger equation. The next thing to do is

to find its deformation equation. For this purpose we note that (1.1.b) entails the following:

$$\frac{\partial \psi_1}{\partial t} = \eta\delta\psi_1 + \eta\psi_2. \tag{1.6}$$

Combining (1.6) with the first row of (1.1.a), we find

$$\frac{\partial \psi_1}{\partial x} = \eta p\psi_1 + q\left(\frac{\partial \psi_1}{\partial t} - \eta\delta\psi_1\right). \tag{1.7}$$

Using (1.3) we obtain the following relation (1.8) from (1.7):

$$\frac{\partial}{\partial x}(q^{1/2}\psi) = q\frac{\partial}{\partial t}(q^{1/2}\psi) + \eta q^{1/2}(p - q\delta)\psi. \tag{1.8}$$

Then we find

$$q\frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial x} + \frac{1}{2}q^{-1}q_x\psi - (\frac{1}{2}q_t + \eta p - \eta q\delta)\psi. \tag{1.9}$$

We now substitute the following explicit values of p, q and δ into (1.9):

$$p = \frac{1}{4\gamma x}(-(2x-u)(\mathcal{K} + 2\gamma t) - \eta^{-1}\frac{d\mathcal{K}}{dt} - 2\eta^{-1}\gamma), \tag{1.10}$$

$$q = \frac{1}{2\gamma x}(\mathcal{K} + 2\gamma t), \tag{1.11}$$

and

$$\delta = -x + \frac{u}{2}. \tag{1.12}$$

Then we find

$$\begin{aligned} &\frac{1}{2}q_t + \eta(p - q\delta) \\ &= \frac{1}{4\gamma x}\Big[\mathcal{K}_t + 2\gamma - \eta(2x-u)(\mathcal{K} + 2\gamma t) \\ &- \mathcal{K}_t - 2\gamma - 2\eta(\mathcal{K} + 2\gamma t)(-x + \frac{u}{2})\Big] = 0. \end{aligned} \tag{1.13}$$

Thus (1.9) assumes the following form:

$$\frac{\partial\psi}{\partial t} = q^{-1}\frac{\partial\psi}{\partial x} - \frac{1}{2}\left(\frac{\partial}{\partial x}q^{-1}\right)\psi. \tag{1.14}$$

Hence, if we choose

$$\mathfrak{a}_{(\mathrm{IV},m)} = q^{-1} = \frac{2\gamma x}{(\mathcal{K}+2\gamma t)}, \tag{1.15}$$

we obtain the required deformation equation:

$$(D_{\mathrm{IV}})_m : \frac{\partial\psi}{\partial t} = \mathfrak{a}_{(\mathrm{IV},m)}\frac{\partial\psi}{\partial x} - \frac{1}{2}\frac{\partial\mathfrak{a}_{(\mathrm{IV},m)}}{\partial x}\psi. \tag{1.16}$$

We note that the most peculiar part of $\eta^2 Q_{(\mathrm{IV},m)}$, i.e.,

$$Q_2 = \frac{3{q_x}^2}{4q^2} - \frac{q_{xx}}{2q}, \tag{1.17}$$

satisfies

$$\mathfrak{a}_{(\mathrm{IV},m)}Q_{2,x} + 2\mathfrak{a}_{(\mathrm{IV},m),x}Q_2 = \frac{1}{2}(\mathfrak{a}_{(\mathrm{IV},m)})_{xxx}. \tag{1.18}$$

In fact, one can readily see that both sides of (1.18) are equal to

$$\frac{1}{2}\left(\frac{-6q_x^3}{q^4} + \frac{6q_xq_{xx}}{q^3} - \frac{q_{xxx}}{q^2}\right), \tag{1.19}$$

without using any specific feature of q.

We also note that, if we choose $\gamma = 2$,

$$q = \frac{1}{2x}(U + C + 2t) = \frac{1}{2x}\prod_{j=1}^{m}(x - \lambda_j) \tag{1.20}$$

holds. (Cf. (A.15.8) and (A.16.1); similar relations hold also for other J's.)

Relations (1.18) and (1.20) play important roles in our WKB-theoretic study of $(SL_J)_m$ in the subsequent sections.

Remark 1.1. The simultaneous equations $(SL_{\mathrm{IV}})_m$ and $(D_{\mathrm{IV}})_m$ share with the pair of equations $(SL_J)_m$ and $(D_J)_m$ ($J =$ I, II-2) the following property: the singular point $x = \lambda_{j,0}$ of $(D_{\mathrm{IV}})_m$ is a double turning point of $(SL_{\mathrm{IV}})_m$. Making use of this property, we can confirm all the results in [KT5] also for $J =$ IV, that is, we can prove the regularity near $x = \lambda_{j,0}$ of S_{odd} for $(SL_{\mathrm{IV}})_m$ with a 0-parameter solution substituted into its coefficients (cf. [KT5, Theorem 2.4]) and we can further prove the reduction theorem for a 0-parameter solution λ_j (cf. [KT5, Theorem 3.2]) not only for $J =$ I, II-2 but also for $J =$ IV (and also for $J = 34$).

2 Basic properties of instanton-type solutions

In Subsection 2.1 we recall basic properties of a $(2m)$-parameter solution of $(P_J)_m$ constructed by Takei ([T1],[T2]). Such a solution is usually called an instanton-type solution. As is noted in Section 0, the argument of [T2] applies to all $J =$ I, 34, II-2 and IV thanks to the existence of Hamilton-Jacobi system $(G_J)_m$ that is equivalent to $(P_J)_m$ ([Ko1], [Ko2]). By its definition an instanton-type solution contains a term of order $(-1/2)$ in η, and when substituted into the coefficients of $Q_{(J,m)}$ it may provoke the appearance of a term of the form $Q_{1/2}\eta^{-1/2}$ in the resulting potential Q. Fortunately we can confirm in Subsection 2.2 that $Q_{1/2}$ actually vanishes thanks to the compatibility of $(SL_J)_m$ and $(D_J)_m$, and hence we can develop WKB analysis of $(SL_J)_m$ with an instanton-type solution substituted into its coefficients.

2.1 Structure of an instanton-type solution ([T2, Theorem 1])

Here, and in what follows, we use the symbol $\nu_j(t)$ to denote a root of the characteristic equation of the Fréchet derivative $(\Delta P_J)_m$ of $(P_J)_m$

at some 0-parameter solution. As is confirmed in [KKoNT] and [KoN], we may, and do, label ν_j's so that

$$\nu_{j+m} = -\nu_j \quad (1 \leq j \leq m) \tag{2.1.1}$$

may hold. To construct a $(2m)$-parameter solution, we first fix a point t_0 for which the following conditions are satisfied:

$$t_0 \text{ is not a } P\text{-turning point of } (P_J)_m, \tag{2.1.2}$$

(2.1.3)
$\sum_{j=1}^{m} n_j \nu_j(t)$ does not vanish identically for any $(n_1, \dots, n_m) \in \mathbb{Z}^m \backslash \{0\}$.

Then, on a neighborhood of t_0, we can construct an instanton-type solution $(u_j, v_j)_{1 \leq j \leq m}$ of $(\tilde{P}_J)_m$ which has the following form:

$$\begin{aligned} u_j(t, \eta; \alpha) &= u_{j,0}(t) + \eta^{-1/2} u_{j,1/2}(t, \Psi, \Phi) \\ &\quad + \eta^{-1} u_{j,1}(t, \Psi, \Phi) + \cdots \end{aligned} \tag{2.1.4}$$

$$\begin{aligned} v_j(t, \eta; \alpha) &= v_{j,0}(t) + \eta^{-1/2} v_{j,1/2}(t, \Psi, \Phi) \\ &\quad + \eta^{-1} v_{j,1}(t, \Psi, \Phi) + \cdots , \end{aligned} \tag{2.1.5}$$

where $u_{j,l/2}(t, \Psi, \Phi)$ and $v_{j,l/2}(t, \Psi, \Phi) (l = 1, 2, \dots)$ are polynomials of (Ψ, Φ) of degree at most l which depend analytically on t. Here $\Psi = (\Psi_1, \dots, \Psi_m)$ and $\Phi = (\Phi_1, \dots, \Phi_m)$ are "instantons", that is, formal series of exponential type of the form

$$\Psi_j = \alpha_j \exp\Big\{ \eta \int^t \Big(\sum_{k=0}^{\infty} \eta^{-k} \sum_{|\mu|=k} (\mu_j + 1) g_{\mu + e_j}(t, \eta) \sigma^{\mu} \Big) dt \Big\}, \tag{2.1.6}$$

(2.1.7)

$$\Phi_j = \alpha_{j+m} \exp\Big\{ -\eta \int^t \Big(\sum_{k=0}^{\infty} \eta^{-k} \sum_{|\mu|=k} (\mu_j + 1) g_{\mu + e_j}(t, \eta) \sigma^{\mu} \Big) dt \Big\},$$

where $j \in \{1, \dots, m\}$, α_j $(1 \leq j \leq 2m)$ are free complex numbers, σ stands for $(\sigma_1, \dots, \sigma_m)$ with $\sigma_j = \alpha_j \alpha_{j+m}$, $\mu = (\mu_1, \dots, \mu_m)$ $(\mu_j \in \mathbb{Z}, \mu_j \geq 0)$, $e_j = (0, \dots, 0, \overset{j}{\check{1}}, 0, \dots, 0)$ are multi-indices, and for each multi-index $\nu = (\nu_1, \dots, \nu_m)$ $g_\nu(t, \eta)$ is a formal power series of $\eta^{-1/2}$ with analytic coefficients of the following form:

$$g_\nu(t, \eta) = \sum_{l=0}^{\infty} \eta^{-l/2} g_{\nu, l/2}(t). \tag{2.1.8}$$

Furthermore we obtain the following result concerning their structure.

Theorem 2.1.1. ([T2, Theorem 1 and Remark 1])
(*i*) *The top order part* $(u_{j,0}, v_{j,0})$ *of* $(u_j(t, \eta; \alpha, \beta), v_j(t, \eta; \alpha, \beta))$ *coincides with the top order part* $(\hat{u}_{j,0}, \hat{v}_{j,0})$ *of the* 0*-parameter solution* $(\hat{u}_j, \hat{v}_j)$.
(*ii*) *The top order part of* $g_{e_j}(t, \eta)$, *i.e.,* $g_{e_j,0}(t)$, *coincides with* $\nu_j(t)$.

Remark 2.1.1. Although we have given the statement for a solution $(u_j, v_j)_{l \leq j \leq m}$ of $(\tilde{P}_J)_m$, the instanton structure of $\{\lambda_j\}_{j=1}^m$ is seen to be the same as that of $(u_j, v_j)_{1 \leq j \leq m}$ by the fact that λ_j $(j = 1, 2, \cdots, m)$ are solutions of

$$U(x) + \widetilde{C}(x, t) = 0, \tag{2.1.9}$$

where $U(x) = x^m - \sum_{j=1}^{m} u_j x^{m-1}$ and $\widetilde{C}(x, t)$ is 0 for $J = \mathrm{I}$, $t/2$ for $J = 34$, $C(x) = \sum_{j=1}^{m} c_j x^{m-j}$ for $J = \text{II-2}$ and $C(x) + 2t$ for $J = \mathrm{IV}$.

2.2 Vanishing of $Q_{1/2}$

In view of the definition of the Borel transformation, wave functions discussed in the exact WKB analysis should have the form

$$\exp(\eta r_{-1}(x))(1+o(\eta^0)). \tag{2.2.1}$$

On the other hand, the term of the degree $(-1/2)$ in η in an instanton type solution may provoke the appearance of a term of degree $(-1/2)$ in η in the potential Q, i.e., $Q_{(J,m)}$ with an instanton-type solution substituted into its coefficients. If it were the case, we could not expect (2.2.1) in view of the way of constructing a WKB solution via the associated Riccati equation. Fortunately the compatibility of $(SL_J)_m$ and $(D_J)_m$ forces such a term to vanish. In fact, one expression of the compatibility condition is

$$\frac{\partial Q_{(J,m)}}{\partial t} = \mathfrak{a}_{(J,m)}\frac{\partial Q_{(J,m)}}{\partial x} + 2\frac{\partial \mathfrak{a}_{(J,m)}}{\partial x}Q_{(J,m)} - \frac{1}{2}\eta^{-2}\frac{\partial^3 \mathfrak{a}_{(J,m)}}{\partial x^3}. \tag{2.2.2}$$

(See [KT4, (4.44)] for example.) In view of Theorem 2.1.1 (ii), $Q_{1/2}$ should be of the form

$$\sum_j a_j(x,t)\exp(\phi_j(t)\eta) + \sum_k b_k(x,t)\exp(-\phi_k(t)\eta), \tag{2.2.3}$$

with

$$\phi_j(t) = \int^t \nu_j(s)ds. \tag{2.2.4}$$

If it were not 0, the left-hand side of (2.2.2) should contain a non-zero term which is of degree $(1/2)$ in η. But the right-hand side of (2.2.2) cannot contain such a term, as it contains the differentiation only with respect to x. Therefore we find

$$Q_{1/2} = 0. \tag{2.2.5}$$

We will use this result frequently in the sequel without explicitly mentioning so.

3 Local reduction of $(SL_J)_m$ to (Can) near a double turning point

Hereinafter we always assume that an instanton-type solution (u_j, v_j) $(1 \leq j \leq m)$ of $(\tilde{P}_J)_m$ (or (λ_j, μ_j) $(1 \leq j \leq m)$ of $(G_J)_m$) is substituted into the coefficients of the potential $Q_{(J,m)}$ of $(SL_J)_m$. Then, if we let τ be a simple P-turning point of the first kind of $(P_J)_m$ that does not coincide with any other P-turning points of $(P_J)_m$, there exists a pair of a double turning point $x = \lambda_{j_0,0}(t)$ and a simple turning point $x = a(t)$ of $(SL_J)_m$ which merge at $t = \tau$ ([KKoNT], [KoN]). Let t_* be a point sufficiently close to τ that lies in a P-Stokes curve emanating from τ, and let V be a sufficiently small neighborhood of t_*. Furthermore we suppose

$$\lambda_{j,0}(t_*) \neq \lambda_{k,0}(t_*) \quad (j \neq k) \tag{3.1}$$

for any (j, k). Then we have the following

Theorem 3.1. *In the situation described above, we can find a neighborhood U of $x = \lambda_{j_0,0}(t)$, a formal series*

$$z(x,t,\eta) = z_0(x,t,\eta) + \eta^{-1/2} z_{1/2}(x,t,\eta) + \eta^{-1} z_1(x,t,\eta) + \cdots , \tag{3.2}$$

whose coefficients $z_{l/2}(x,t,\eta)$ are holomorphic on $U \times V$, and formal series

$$E^{(j_0)}(t,\eta) = E_0^{(j_0)}(t,\eta) + E_{1/2}^{(j_0)}(t,\eta)\eta^{-1/2} + E_1^{(j_0)}(t,\eta)\eta^{-1} + \cdots \tag{3.3}$$

and

$$\rho^{(j_0)}(t,\eta) = \rho_0^{(j_0)}(t,\eta) + \rho_{1/2}^{(j_0)}(t,\eta)\eta^{-1/2} + \rho_1^{(j_0)}(t,\eta)\eta^{-1} + \cdots , \tag{3.4}$$

whose coefficients are holomorphic on V, so that the following con-

ditions (3.5) $\sim$ (3.10) *may hold*:

(3.5) z_0 *is free from* η,

(3.6) $\dfrac{\partial z_0}{\partial x}$ *never vanishes on* $U \times V$,

(3.7) $z_0(\lambda_{j_0,0}(t), t) = 0$,

(3.8) $z_{1/2}$ *identically vanishes*,

(3.9)

$$
\begin{aligned}
&Q_{(J,m)}(x,t,\eta) \\
&= \left(\frac{\partial z}{\partial x}\right)^2 \Bigg[4z(x,t,\eta)^2 + \eta^{-1} E^{(j_0)}(t,\eta) \\
&\quad + \frac{\eta^{-3/2}\rho^{(j_0)}(t,\eta)}{z(x,t,\eta) - z(\lambda_{j_0}(t,\eta),t,\eta)} + \frac{3\eta^{-2}}{4(z(x,t,\eta) - z(\lambda_{j_0}(t,\eta),t,\eta))^2} \Bigg] \\
&\quad - \frac{1}{2}\eta^{-2}\{z(x,t,\eta); x\}
\end{aligned}
$$

holds on $U \times V$, *where* $\{z; x\}$ *stands for the Schwarzian derivative,*

(3.10) *the* η*-dependence of* $z_{l/2}(x,t,\eta), E^{(j_0)}_{l/2}(t,\eta)$ *and* $\rho^{(j_0)}_{l/2}(t,\eta)$ *is through the instanton terms that* $\lambda_{j_0}(t,\eta)$ *contains.*

Theorem 3.2. *The series* $E^{(j_0)}(t,\eta)$ *and* $\rho^{(j_0)}(t,\eta)$ *in the preceding theorem are written down in term of* $\{\lambda_j\}_{j=1}^m$ *and* $z(x,t,\eta)$ *in* (3.2)

in the following manner:

$$\rho^{(j_0)}(t,\eta) = -\,\eta^{-1/2}\left(\frac{\partial z}{\partial x}(\lambda_{j_0}(t,\eta),t,\eta)\right)^{-1} \times\left[\frac{1}{2}\left(\frac{\partial}{\partial t}\lambda_{j_0}(t,\eta)\right)\left(\frac{1}{(x-\lambda_{j_0}(t,\eta))\mathfrak{a}_{(J,m)}}\right) + \frac{1}{2}\left(\frac{\partial\mathfrak{a}_{(J,m)}/\partial x}{\mathfrak{a}_{(J,m)}} + \frac{1}{(x-\lambda_{j_0}(t,\eta))}\right) + \frac{3}{4}\,\frac{\partial^2 z/\partial x^2}{\partial z/\partial x}\right]\Bigg|_{x=\lambda_{j_0}(t,\eta)}, \tag{3.11}$$

$$E^{(j_0)}(t,\eta) = (\rho^{(j_0)}(t,\eta))^2 - 4(\eta^{1/2}z(\lambda_{j_0}(t,\eta),t,\eta))^2. \tag{3.12}$$

Remark 3.1. In what follows we use the symbol $\sigma^{(j_0)}(t,\eta)$ to denote

$$\eta^{1/2}z(\lambda_{j_0}(t,\eta),t,\eta), \tag{3.13}$$

Note that (3.7) implies that the degree of $\sigma^{(j_0)}(t,\eta)$ with respect to η is at most 0 despite the multiplication by $\eta^{1/2}$ (if we count the degree of instanton terms to be 0, as usual).

Definition 3.1. *The equation* (Can) *is, by definition, the following Schrödinger equation:*

$$\left(-\frac{\partial^2}{\partial z^2} + \eta^2 Q_{\mathrm{can}}(z,E,\rho,\sigma,\eta)\right)\varphi = 0, \tag{3.14}$$

where

$$Q_{\mathrm{can}} = 4z^2 + \eta^{-1}E + \frac{\eta^{-3/2}\rho}{z-\eta^{-1/2}\sigma} + \frac{3\eta^{-2}}{4(z-\eta^{-1/2}\sigma)^2} \tag{3.15}$$

with

$$E = \rho^2 - 4\sigma^2. \tag{3.16}$$

To prove Theorem 3.1 and Theorem 3.2, we need the following

Lemma 3.3. *Let $c_l(t,\eta)$ $(l=-2,-1,0,1,2,\cdots)$ denote the coefficient of $(x-\lambda_{j_0}(t,\eta))^l$ in the expansion of $Q_{(J,m)}$ ($J=$ I, 34, II-2, IV) in powers of $(x-\lambda_{j_0}(t,\eta))$ with t being sufficiently close to t_*. Then we find*

$$c_0=\eta^2c_{-1}^2. \tag{3.17}$$

Proof. For the sake of definiteness we discuss the case $J=$ IV. This is the situation that seems to be most complicated in its appearance. Actually the computation in other cases is slightly simpler than that given below, and the logical structure of the proof is the same in all cases. Throughout the proof of this lemma we let Q denote $Q_{(\mathrm{IV},m)}$ with an instanton-type solution being substituted into its coefficients. As in Section 1, we let Q_2 denote

$$\frac{3q_x^2}{4q^2}-\frac{q_{xx}}{2q}, \tag{3.18}$$

with q being given in (1.11) and we define $\widetilde{Q}$ by

$$\eta^2Q-Q_2 \tag{3.19}$$

(cf. (1.17)). For the sake of the notational simplicity we assume $j_0=1$ in what follows. We also set

$$X_j=x-\lambda_j(t,\eta). \tag{3.20}$$

Then for $J=$ IV with $\gamma=2$, we see by (1.20) and (1.15)

$$q=\frac{1}{2x}\prod_{j=1}^{m}X_j \tag{3.21}$$

and

$$\mathfrak{a}(=\mathfrak{a}_{(\mathrm{IV},m)})=q^{-1}=\frac{2x}{\prod_{j=1}^{m}X_j}. \tag{3.22}$$

Note that the factor x^{-1} in q does not appear for $J = \mathrm{I}$ or II-2; it appears only for $J = 34$ or IV.

Our strategy of the proof is to write down the relation (2.2.2) in power series of X_1 (including negative degrees). We start with the following relation (3.23) that is obtained by the substitution of (1.18) into (2.2.2):

$$\eta^2 Q_t = Q_{2,t} + \widetilde{Q}_t = \mathfrak{a}\widetilde{Q}_x + 2\mathfrak{a}_x\widetilde{Q}, \tag{3.23}$$

where Q_t etc. stand for $\partial Q/\partial t$ etc. First we note

$$\frac{q_x}{q} = \sum_{j=1}^{m} \frac{1}{X_j} - \frac{1}{x}. \tag{3.24}$$

Since

$$\left(\frac{q_x}{q}\right)_x = -\frac{q_x^2}{q^2} + \frac{q_{xx}}{q},$$

we use (3.24) to find

$$\begin{aligned} Q_2 =& \frac{3q_x^2}{4q^2} - \frac{q_{xx}}{2q} \\ =& \frac{1}{4}\frac{q_x^2}{q^2} - \frac{1}{2}\left(\frac{q_x}{q}\right)_x \\ =& \frac{1}{4}\left(\sum_{j=1}^{m}\frac{1}{X_j}\right)^2 - \frac{1}{2x}\left(\sum_{j=1}^{m}\frac{1}{X_j}\right) \\ &+ \frac{1}{2}\left(\sum_{j=1}^{m}\frac{1}{X_j^2}\right) - \frac{1}{4x^2}. \end{aligned} \tag{3.25}$$

Hence we obtain

$$Q_{2,t} = \frac{1}{2}\left(\sum_{j=1}^{m}\frac{1}{X_j}\right)\left(\sum_{j=1}^{m}\frac{\lambda_j'}{X_j^2}\right) - \frac{1}{2x}\left(\sum_{j=1}^{m}\frac{\lambda_j'}{X_j^2}\right) + \sum_{j=1}^{m}\frac{\lambda_j'}{X_j^3}, \tag{3.26}$$

where λ_j' etc. stand for $d\lambda_j/dt$ etc. On the other hand, in view of the explicit form (1.5) of Q, we see that $\tilde{Q}$ has the form

$$\frac{\alpha(t,\eta)}{X_1} + \beta(t,\eta) + O(X_1) \tag{3.27}$$

when expanded in powers of X_1, which is regarded as a small quantity. We now compute the coefficients of X_1^{-l} $(l = 3, 2)$ in (3.23).

Let Λ_j $(j \geq 2)$ denote

$$\Lambda_j = \lambda_1 - \lambda_j. \tag{3.28}$$

We first compute the expansion of $\mathfrak{a}$ and $\mathfrak{a}_x$ in X_1. If we write $\mathfrak{a}$ as

$$\mathfrak{a} = \frac{1}{X_1}f(x) \quad \text{with} \quad f(x) = \frac{2x}{\prod_{j=2}^{m} X_j}, \tag{3.29}$$

we readily find

$$\begin{aligned}\mathfrak{a} &= \frac{1}{X_1}\left(f(\lambda_1) + \frac{df}{dx}(\lambda_1)X_1 + O(X_1^2)\right) \\ &= \frac{f(\lambda_1)}{X_1}\left(1 + (\frac{d}{dx}\log f)\Big|_{x=\lambda_1} X_1 + O(X_1^2)\right)\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}\frac{d}{dx}\log\mathfrak{a} &= -\frac{1}{X_1} + (\frac{d}{dx}\log f)\Big|_{x=\lambda_1} + O(X_1) \\ &= -\frac{1}{X_1}\left(1 - (\frac{d}{dx}\log f)\Big|_{x=\lambda_1} X_1 + O(X_1^2)\right).\end{aligned} \tag{3.31}$$

Hence we obtain

$$\mathfrak{a}_x = \mathfrak{a}\frac{d}{dx}\log\mathfrak{a} \tag{3.32}$$

$$= -\frac{f(\lambda_1)}{X_1^2}\left(1 + (\frac{d}{dx}\log f)\Big|_{x=\lambda_1} X_1 + O(X_1^2)\right)$$

$$\times \left(1 - (\frac{d}{dx}\log f)\Big|_{x=\lambda_1} X_1 + O(X_1^2)\right)$$

$$= -\frac{f(\lambda_1)}{X_1^2} + O(1).$$

Here the symbol $O(1)$ means that the part consists of terms which contain a factor of the form X_1^p $(p \geq 0)$. We note that the absence of terms of order $O(X_1^{-1})$ in $\mathfrak{a}_x$ is observed for J's other than IV; the existence of the extra factor x in $\mathfrak{a}$ has nothing to do with this fact. Since

$$f(\lambda_1) = \frac{2\lambda_1}{\prod_{j=2}^{m}\Lambda_j} \tag{3.33}$$

and

$$(\frac{d}{dx}\log f)\Big|_{x=\lambda_1} = \frac{1}{\lambda_1} - \sum_{j=2}^{m}\frac{1}{\Lambda_j}, \tag{3.34}$$

it then follows from (3.27), (3.30) and (3.32) that

(3.35)

$$\mathfrak{a}\tilde{Q}_x + 2\mathfrak{a}_x\tilde{Q}$$

$$= \left(\frac{f(\lambda_1)}{X_1} + f(\lambda_1)(\frac{d}{dx}\log f)\Big|_{x=\lambda_1} + O(X_1)\right)\left(-\frac{\alpha}{X_1^2} + O(1)\right)$$

$$
\begin{aligned}
&+2\left(-\frac{f(\lambda_1)}{X_1^2}+O(1)\right)\left(\frac{\alpha}{X_1}+\beta+O(X_1)\right)\\
=&\Big[-\frac{\alpha f(\lambda_1)}{X_1^3}-\alpha f(\lambda_1)(\frac{d}{dx}\log f)\Big|_{x=\lambda_1}\frac{1}{X_1^2}+O(X_1^{-1})\Big]\\
&+\Big[-\frac{2\alpha f(\lambda_1)}{X_1^3}-\frac{2\beta f(\lambda_1)}{X_1^2}+O(X_1^{-1})\Big]\\
=&-\frac{3\alpha f(\lambda_1)}{X_1^3}-f(\lambda_1)\Big[\alpha(\frac{d}{dx}\log f)\Big|_{x=\lambda_1}+2\beta\Big]\frac{1}{X_1^2}+O(X_1^{-1})\\
=&-\frac{6\alpha\lambda_1}{\prod_{j=2}^{m}\Lambda_j}\frac{1}{X_1^3}-\frac{2\lambda_1}{\prod_{j=2}^{m}\Lambda_j}\Big[\alpha\Big(\frac{1}{\lambda_1}-\sum_{j=2}^{m}\frac{1}{\Lambda_j}\Big)+2\beta\Big]\frac{1}{X_1^2}+O(X_1^{-1}).
\end{aligned}
$$

On the other hand, (3.26) and (3.27) entail that the left-hand side of (3.23), i.e.,

$$
\eta^2 Q_t = Q_{2,t}+\tilde{Q}_t, \tag{3.36}
$$

has the form

(3.37)

$$
\begin{aligned}
&\left[\frac{3\lambda_1^{'}}{2X_1^3}+\frac{\lambda_1^{'}}{2}\left(\sum_{j=2}^{m}\frac{1}{X_j}-\frac{1}{x}\right)\frac{1}{X_1^2}+O(X_1^{-1})\right]+\left[\frac{\alpha\lambda_1^{'}}{X_1^2}+\frac{\alpha^{'}}{X_1}+O(1)\right]\\
=&\frac{3\lambda_1^{'}}{2X_1^3}+\frac{\lambda_1^{'}}{2}\left(\sum_{j=2}^{m}\frac{1}{\Lambda_j}-\frac{1}{\lambda_1}+2\alpha\right)\frac{1}{X_1^2}+O(X_1^{-1}).
\end{aligned}
$$

By comparing (3.35) and (3.37), we find

$$-6\alpha\lambda_1 = \frac{3}{2}\lambda_1' \prod_{j=2}^{m} \Lambda_j \tag{3.38}$$

and

$$\begin{aligned} &\frac{\lambda_1'}{2}\left(\sum_{j=2}^{m}\frac{1}{\Lambda_j} - \frac{1}{\lambda_1} + 2\alpha\right) \\ &\quad = \left[2\alpha\lambda_1\left(\sum_{j=2}^{m}\frac{1}{\Lambda_j}\right) - 2\alpha - 4\beta\lambda_1\right]\frac{1}{\prod_{j=2}^{m}\Lambda_j}. \end{aligned} \tag{3.39}$$

Thus we obtain

$$\alpha = -\frac{\lambda_1'}{4\lambda_1}\prod_{j=2}^{m}\Lambda_j \tag{3.40}$$

and

$$-\lambda_1'\left(\sum_{j=2}^{m}\frac{1}{\Lambda_j} - \frac{1}{\lambda_1}\right) + \frac{\lambda_1'^2}{4\lambda_1}\prod_{j=2}^{m}\Lambda_j = \frac{4\beta\lambda_1}{\prod_{j=2}^{m}\Lambda_j}, \tag{3.41}$$

i.e.,

$$\beta = \left(\frac{\lambda_1'}{4\lambda_1}\prod_{j=2}^{m}\Lambda_j\right)^2 - \frac{\lambda_1'}{4\lambda_1}\left(\sum_{j=2}^{m}\frac{1}{\Lambda_j} - \frac{1}{\lambda_1}\right)\prod_{j=2}^{m}\Lambda_j. \tag{3.42}$$

Next let us compute the contribution from Q_2 to $\eta^2 Q$, i.e., δ_{-1} and δ_0 given below:

$$\delta_{-1} = \eta^2 c_{-1} - \alpha \tag{3.43}$$

and

$$\delta_0 = \eta^2 c_0 - \beta. \tag{3.44}$$

To find these quantities we rewrite Q_2 in (3.25) in powers of X_1 as follows:

(3.45)

$$\begin{aligned}
Q_2 =& \frac{1}{4X_1^2} + \frac{1}{2X_1}\left(\sum_{j=2}^{m}\frac{1}{X_j}\right) + \frac{1}{4}\left(\sum_{j=2}^{m}\frac{1}{X_j}\right)^2 - \frac{1}{2xX_1} \\
& - \frac{1}{2x}\left(\sum_{j=2}^{m}\frac{1}{X_j}\right) + \frac{1}{2X_1^2} + \frac{1}{2}\left(\sum_{j=2}^{m}\frac{1}{X_j^2}\right) - \frac{1}{4x^2} \\
=& \frac{3}{4X_1^2} + \frac{1}{2}\sum_{j=2}^{m}\frac{1}{\Lambda_j}\left(\frac{1}{X_1} - \frac{1}{X_j}\right) + \frac{1}{4}\left(\sum_{j=2}^{m}\frac{1}{X_j}\right)^2 \\
& - \frac{1}{2\lambda_1}\left(\frac{1}{X_1} - \frac{1}{x}\right) - \frac{1}{2x}\left(\sum_{j=2}^{m}\frac{1}{X_j}\right) + \frac{1}{2}\left(\sum_{j=2}^{m}\frac{1}{X_j^2}\right) - \frac{1}{4x^2} \\
=& \frac{3}{4X_1^2} + \frac{1}{2}\left(\sum_{j=2}^{m}\frac{1}{\Lambda_j} - \frac{1}{\lambda_1}\right)\frac{1}{X_1} - \frac{1}{2}\sum_{j=2}^{m}\frac{1}{\Lambda_j^2} \\
& + \frac{1}{4}\left(\sum_{j=2}^{m}\frac{1}{\Lambda_j}\right)^2 + \frac{1}{2\lambda_1^2} - \frac{1}{2\lambda_1}\left(\sum_{j=2}^{m}\frac{1}{\Lambda_j}\right)
\end{aligned}$$

$$+\frac{1}{2}\sum_{j=2}^{m}\frac{1}{\Lambda_j^2}-\frac{1}{4\lambda_1^2}+O(X_1)$$

$$=\frac{3}{4X_1^2}+\frac{1}{2}\left(\sum_{j=2}^{m}\frac{1}{\Lambda_j}-\frac{1}{\lambda_1}\right)\frac{1}{X_1}$$

$$+\frac{1}{4}\left(\sum_{j=2}^{m}\frac{1}{\Lambda_j}\right)^2-\frac{1}{2\lambda_1}\left(\sum_{j=2}^{m}\frac{1}{\Lambda_j}\right)+\frac{1}{4\lambda_1^2}+O(X_1).$$

Thus we find

$$\delta_{-1}=\frac{1}{2}\left(\sum_{j=2}^{m}\frac{1}{\Lambda_j}-\frac{1}{\lambda_1}\right) \tag{3.46}$$

and

$$\delta_0=\frac{1}{4}\left(\sum_{j=2}^{m}\frac{1}{\Lambda_j}\right)^2-\frac{1}{2\lambda_1}\left(\sum_{j=2}^{m}\frac{1}{\Lambda_j}\right)+\frac{1}{4\lambda_1^2} \tag{3.47}$$

$$=\left[\frac{1}{2}\left(\sum_{j=2}^{m}\frac{1}{\Lambda_j}-\frac{1}{\lambda_1}\right)\right]^2.$$

Combining (3.40), (3.42), (3.46) and (3.47), we find

$$\eta^2 c_{-1}=-\frac{\lambda_1^{'}}{4\lambda_1}\prod_{j=2}^{m}\Lambda_j+\frac{1}{2}\left(\sum_{j=2}^{m}\frac{1}{\Lambda_j}-\frac{1}{\lambda_1}\right) \tag{3.48}$$

and

$$\eta^2 c_0=\left(\frac{\lambda_1^{'}}{4\lambda_1}\prod_{j=2}^{m}\Lambda_j\right)^2-\frac{\lambda_1^{'}}{4\lambda_1}\left(\sum_{j=2}^{m}\frac{1}{\Lambda_j}-\frac{1}{\lambda_1}\right)\prod_{j=2}^{m}\Lambda_j \tag{3.49}$$

$$+\left[\frac{1}{2}\left(\sum_{j=2}^{m}\frac{1}{\Lambda_j}-\frac{1}{\lambda_1}\right)\right]^2$$

$$=\left[\frac{\lambda_1'}{4\lambda_1}\prod_{j=2}^{m}\Lambda_j-\frac{1}{2}\left(\sum_{j=2}^{m}\frac{1}{\Lambda_j}-\frac{1}{\lambda_1}\right)\right]^2$$

Therefore we obtain the required relation:

$$(\eta^2 c_{-1})^2=\eta^2 c_0, \tag{3.50}$$

i.e.,

$$c_0=\eta^2 c_{-1}^2. \tag{3.51}$$

This completes the proof of Lemma 3.3. □

Proof of Theorem 3.1 and Theorem 3.2. In proving Theorem 3.1 we construct the series $z(x,t,\eta)$ by using the induction on the degree of η. To explain how Lemma 3.3 is used in the induction procedure, we first examine the structure of the right-hand side of (3.9) assuming that the series z is given, regardless of the validity of the equality (3.9). In what follows we assume $j_0=1$ for the notational simplicity, as in the proof of Lemma 3.3. In view of the relations

$$\left(\frac{\partial z}{\partial x}\right)^2\frac{1}{z-z(\lambda_1,t,\eta)}=\frac{z'(\lambda_1)}{x-\lambda_1}+\frac{3}{2}z''(\lambda_1)+\cdots, \tag{3.52}$$

(3.53)

$$\left(\frac{\partial z}{\partial x}\right)^2\frac{1}{(z-z(\lambda_1,t,\eta))^2}=\frac{1}{(x-\lambda_1)^2}+\frac{z''(\lambda_1)}{z''(\lambda_1)}\,\frac{1}{x-\lambda_1}$$

$$+\left\{\frac{2}{3}\,\frac{z'''(\lambda_1)}{z'(\lambda_1)}-\frac{1}{4}\left(\frac{z''(\lambda_1)}{z'(\lambda_1)}\right)^2\right\}+\cdots,$$

where $z'(\lambda_1)$ etc. stand for the derivatives of $z(x,t,\eta)$ with respect to x that is evaluated at $x=\lambda_1$, we find the right-hand side of (3.9) is of the following form:

(3.54)

$$\frac{3\eta^{-2}}{4(x-\lambda_1)^2}+\eta^{-3/2}\left\{\rho^{(1)}z'(\lambda_1)+\frac{3}{4}\,\eta^{-1/2}\,\frac{z''(\lambda_1)}{z'(\lambda_1)}\right\}\frac{1}{x-\lambda_1}$$

$$+\eta^{-1}\left\{z'(\lambda_1)^2E^{(1)}+\frac{3}{2}\,\eta^{-1/2}\rho^{(1)}z''(\lambda_1)+\frac{9}{16}\,\eta^{-1}\left(\frac{z''(\lambda_1)}{z'(\lambda_1)}\right)^2\right\}$$

$$+4z'(\lambda_1)^2z(\lambda_1)^2+r_1,$$

where r_1 is a sum of terms of order $O(x-\lambda_1)$. If we further assume (3.12), we find the coefficients $\tilde{c}_l$ $(l=-1,0)$ of $(x-\lambda_1)^l$ in (3.54) satisfy the following:

$$\tilde{c}_0=\eta^2\tilde{c}_{-1}^2. \tag{3.55}$$

Thus Lemma 3.3 lets us expect that we can construct the required series z by first adjusting the coefficients of $(x-\lambda_1)^{-1}$ in both sides of (3.9) and then defining the constant $E^{(1)}$ by (3.12). Note that the most singular part, i.e., the double pole part, is the same in both sides of (3.9). To put this expectation into practice, we use the induction on the degree of η. In what follows we choose

$$z_0(x,t)=\left(\int_{\lambda_{1,0}}^{x}\sqrt{Q_{(J,m),0}}\,dx\right)^{1/2}. \tag{3.56}$$

We also note that the relation (2.2.5) enables us to choose

$$z_{1/2}=0. \tag{3.57}$$

As a convention we choose

$$\rho^{(1)}_{-1/2}=E^{(1)}_{-1/2}=0. \tag{3.58}$$

Our task is to construct the series $z(x,t,\eta)$ so that (3.9) may hold. In view of Lemma 3.3 and (3.54), the series should eventually satisfy

$$\rho^{(1)} = \eta^{3/2}\frac{c_{-1}}{z'(\lambda_1)} - \frac{3}{4}\,\eta^{-1/2}\frac{z''(\lambda_1)}{(z'(\lambda_1))^2}, \tag{3.59}$$

$$E^{(1)} = \rho^{(1)2} - 4\eta z(\lambda_1,t,\eta)^2. \tag{3.60}$$

To construct the required series, we let $\Delta = \Delta_0 + \Delta_{1/2}\,\eta^{-1/2} + \Delta_1\,\eta^{-1} + \Delta_{3/2}\,\eta^{-3/2} + \cdots$ (with $\Delta_{1/2} = 0$) denote the left-hand side of (3.9) minus its right-hand side. We then prove the following assertion $(\mathcal{A})_n$ by the induction on n, starting with $n = -1$.

$(\mathcal{A})_n$ *We can construct* $z_{(j+2)/2}$, $\rho^{(1)}_{j/2}$ *and* $E^{(1)}_{j/2}$ $(j = 0,1,\ldots,n)$ *so that the following relations* $(3.61)_n$ *and* $(3.62)_n$ *are satisfied:*

$(3.61)_n$ $\Delta_{j/2} = 0$ *for* $j = 0,1,\ldots,n+2$

$(3.62)_n$ (3.59) *and* (3.60) *hold modulo terms of order at most or equal to* $\eta^{-(n+1)/2}$.

It is clear that $(\mathcal{A})_{-1}$ holds by (3.56), (3.57) and (3.58). Let us suppose $(\mathcal{A})_{n-1}$ to hold. We can then construct $\rho^{(1)}_{n/2}$ (resp., $E^{(1)}_{n/2}$) as the homogeneous part of degree $n/2$ (with respect to η^{-1}) of the right-hand side of (3.59) (resp., (3.60)). Note that in constructing $\rho^{(1)}_{n/2}$ through (3.59) we only need $z_{j/2}$ up to $j = n+1$ since $c_{-1} = O(\eta^{-1})$ holds thanks to Lemma 3.3. Thus $(3.62)_n$ is attained. On the other hand, $\Delta_{(n+2)/2}$ has the following form:

$$\begin{aligned}\Delta_{(n+2)/2} =& 8z_0^2\left(\frac{\partial z_0}{\partial x}\right)\frac{\partial z_{(n+2)/2}}{\partial x}\\ &+ 8z_0\left(\frac{\partial z_0}{\partial x}\right)^2 z_{(n+2)/2} + R_{(n+2)/2},\end{aligned} \tag{3.63}$$

where $R_{(n+2)/2}$ is a function defined by $\{z_{j/2}\}_{j\leq n+1}$, $\{\rho_{j/2}\}_{j\leq n-1}$ and $\{E_{j/2}\}_{j\leq n}$. Furthermore $(3.62)_n$ attained above guarantees that

$$\sum_{j=0}^{n+2} \Delta_{j/2}\eta^{-j/2} \tag{3.64}$$

has no singularity at $x=\lambda_1$ and that it vanishes there modulo terms of order at most or equal to $\eta^{-(n+3)/2}$, while the induction hypothesis entails

$$\Delta_{j/2}=0 \quad (j=0,1,\dots,n+1). \tag{3.65}$$

Therefore we find

$$\Delta_{(n+2)/2}=0 \tag{3.66}$$

at $x=\lambda_{1,0}$. Since

$$z_0(\lambda_{1,0}(t),t)=0 \tag{3.67}$$

by its definition, we conclude

$$R_{(n+2)/2}(\lambda_{1,0}(t),t)=0. \tag{3.68}$$

This means that we can divide the equation

$$\Delta_{(n+2)/2}=0 \tag{3.69}$$

by $z_0(x,t)$ to find an ordinary differential equation for $z_{(n+2)/2}$ that is with regular singularity at $x=\lambda_{1,0}(t)$ with the characteristic index -1. Thus we can find a holomorphic solution $z_{(n+2)/2}$ of (3.69), as is required by $(3.61)_n$. Hence the induction proceeds, completing the proof of Theorem 3.1. In particular we have obtained (3.59) and (3.60).

Then (3.48) and (3.59) entail that we have, for $J = \mathrm{IV}$,

(3.70)

$$\rho^{(1)} = -\eta^{-1/2}\left(\frac{\partial z}{\partial x}(\lambda_1(t,\eta),t,\eta)\right)^{-1} \times \left[\frac{\lambda_1'}{4\lambda_1}\prod_{j=2}^{m}\Lambda_j - \frac{1}{2}\left(\sum_{j=2}^{m}\frac{1}{\Lambda_j} - \frac{1}{\lambda_1}\right) + \frac{3}{4}\left(\left.\frac{\partial^2 z/\partial x^2}{\partial z/\partial x}\right|_{x=\lambda_1(t,\eta)}\right)\right].$$

On the other hand the explicit form (3.22) of $\mathfrak{a}_{(\mathrm{IV},m)}$ readily implies

$$\left.\frac{1}{X_1\mathfrak{a}_{(\mathrm{IV},m)}}\right|_{x=\lambda_1(t,\eta)} = \frac{\prod_{j=2}^{m}\Lambda_j}{2\lambda_1} \tag{3.71}$$

and

$$\left.\left(\frac{\partial\mathfrak{a}_{(\mathrm{IV},m)}/\partial x}{\mathfrak{a}_{(\mathrm{IV},m)}} + \frac{1}{X_1}\right)\right|_{x=\lambda_1(t,\eta)} = -\left(\sum_{j=2}^{m}\frac{1}{\Lambda_j} - \frac{1}{\lambda_1}\right). \tag{3.72}$$

Combining (3.70), (3.71) and (3.72) we obtain (3.11) for $J = \mathrm{IV}$. The computation can be done in the same way for other J's. This completes the proof of Theorem 3.2. □

4 Splitting of the top order part of $(\Delta G_J)_m$

Once we obtain Theorems 3.1 and 3.2, the next thing to do would be to try to extend the domain of definition of the series $z(x,t,\eta)$ so that it may contain the simple turning point $x = a(t)$ of $(SL_J)_m$ that merges with $x = \lambda_{j_0,0}(t)$ at $t = \tau$. Such an extension is done in [KT2] when $m = 1$. As we will see in Section 5, we need to confirm some particular instanton structure of $\rho^{(j_0)}$ and $\sigma^{(j_0)}$ in attaining such an extention when m is greater than 1; our reasoning there requires that

the top degree part $\rho_0^{(j_0)}$ and $\sigma_0^{(j_0)}$ of $\rho^{(j_0)}$ and $\sigma^{(j_0)}$ contain instanton terms whose phase functions are "related to" the P-turning point in question. Here a phase function related to the P-turning point in question is, by definition,

$$\int_\tau^t \nu_{j_0}(t)dt \quad \text{or} \quad \int_\tau^t \nu_{j_0+m}(t)dt \tag{4.1}$$

in the labeling (2.1.1). To confirm (4.1) we use in this paper Theorem 4.1 below. Our proof of (4.1) in [KT6] is somewhat more complicated but more elementary in the sense that it does not use the Hamiltonian form of $(P_J)_m$.

Theorem 4.1. *The top degree part of the Fréchet derivative* $(\Delta G_J)_m$ *of* $(G_J)_m$ $(J = \text{I, 34, II-2, IV})$ *at a* 0*-parameter solution splits into a direct sum of* 2×2 *systems.*

Proof. Let K denote the Hamiltonian of $(G_J)_m$ and let $(\lambda^{(0)}, \mu^{(0)})$ denote a 0-parameter solution of $(G_J)_m$. An explicit way of the presentation of Theorem 4.1 is then given as follows:

$$\left(\frac{\partial^2 K}{\partial \lambda_j \partial \lambda_k}\bigg|_{(\lambda,\mu)=(\lambda^{(0)},\mu^{(0)})} \right)_0 = 0 \quad (j \neq k), \tag{4.2}$$

$$\left(\frac{\partial^2 K}{\partial \lambda_j \partial \mu_k}\bigg|_{(\lambda,\mu)=(\lambda^{(0)},\mu^{(0)})} \right)_0 = 0 \quad (j \neq k), \tag{4.3}$$

and

$$\left(\frac{\partial^2 K}{\partial \mu_j \partial \mu_k}\bigg|_{(\lambda,\mu)=(\lambda^{(0)},\mu^{(0)})} \right)_0 = 0 \quad (j \neq k). \tag{4.4}$$

Here

$$\left(\frac{\partial^2 K}{\partial \lambda_j \partial \lambda_k}\bigg|_{(\lambda,\mu)=(\lambda^{(0)},\mu^{(0)})} \right)_0 = 0 \tag{4.5}$$

etc. denote the 0-th degree (in η) part of $\partial^2 K/\partial\lambda_j\partial\lambda_k$ etc. evaluated at the 0-parameter solution. In what follows let the symbol

$$\left[\frac{\partial^2 K}{\partial\lambda_j\partial\lambda_k}\right]_0 \tag{4.6}$$

stand for (4.5). We also use the symbol N_j to denote

$$\prod_{k\neq j}(\lambda_j-\lambda_k)^{-1}. \tag{4.7}$$

To begin with we observe that the results in [Ko1] and [Ko2] (cf. Appendix) imply that

$$K=\sum_{j=1}^{m} N_j F(\lambda_j,\mu_j,t) \tag{4.8}$$

for some polynomial $F(\lambda,\mu,t)$. Hence we find

$$\frac{\partial K}{\partial\mu_j}=N_j\left(\frac{\partial F}{\partial\mu}\right)(\lambda_j,\mu_j,t). \tag{4.9}$$

Therefore we have

$$\frac{\partial^2 K}{\partial\mu_j\partial\mu_k}=0 \quad (j\neq k), \tag{4.10}$$

which immediately entails (4.4). It also follows from (4.9) that

$$\frac{\partial^2 K}{\partial\lambda_k\partial\mu_j}=\frac{\partial N_j}{\partial\lambda_k}\left(\frac{\partial F}{\partial\mu}\right)(\lambda_j,\mu_j,t) \tag{4.11}$$

holds if $j\neq k$. On the other hand, looking at the highest degree part of $(G_J)_m$ in η, we find

$$\left[\frac{\partial K}{\partial\mu_j}\right]_0=0, \quad j=1,2,\ldots,m. \tag{4.12}$$

Then (4.9) entails

$$\left[\frac{\partial F}{\partial \mu}(\lambda_j, \mu_j, t)\right]_0 = 0, \quad j = 1, 2, \ldots, m. \tag{4.13}$$

Therefore (4.11) proves (4.3). Thus what remains to be proved is (4.2). In proving (4.2) we may assume without loss of generality that $(j, k) = (2, 1)$. In order to prove (4.2), we first show

$$\frac{\partial^2 K}{\partial \lambda_1 \partial \lambda_2} = (\lambda_1 - \lambda_2)^{-1} \frac{\partial K}{\partial \lambda_1} + (\lambda_2 - \lambda_1)^{-1} \frac{\partial K}{\partial \lambda_2}. \tag{4.14}$$

Since we find

$$\left[\frac{\partial K}{\partial \lambda_j}\right]_0 = 0, \quad j = 1, 2, \ldots, m \tag{4.15}$$

by observing the highest degree part of $(G_J)_m$ in η, we can deduce (4.2) from (4.14). In what follows we use symbols F_j and F_j' to denote respectively

$$F(\lambda_j, \mu_j, t) \tag{4.16}$$

and

$$\left(\frac{\partial F}{\partial \lambda}\right)(\lambda_j, \mu_j, t); \tag{4.17}$$

for example we have

$$\frac{\partial K}{\partial \lambda_1} = \sum_{j=1}^{m} \frac{\partial N_j}{\partial \lambda_1} F_j + N_1 F_1'. \tag{4.18}$$

Concerning $\partial N_j / \partial \lambda_1$ etc., we can readily confirm the following relations:

$$\frac{\partial N_1}{\partial \lambda_1} = -\left(\sum_{k \geq 2} (\lambda_1 - \lambda_k)^{-1}\right) N_1, \tag{4.19}$$

$$\frac{\partial N_j}{\partial \lambda_1} = (\lambda_j - \lambda_1)^{-1} N_j \quad (j \geq 2), \tag{4.20}$$

$$\frac{\partial N_1}{\partial \lambda_j} = (\lambda_1 - \lambda_j)^{-1} N_1 \quad (j \geq 2), \tag{4.21}$$

$$\frac{\partial N_2}{\partial \lambda_2} = -\left(\sum_{k \neq 2} (\lambda_2 - \lambda_k)^{-1} \right) N_2. \tag{4.22}$$

Hence we find

$$\frac{\partial^2 N_1}{\partial \lambda_1 \partial \lambda_2} = -2(\lambda_1 - \lambda_2)^{-2} N_1 - (\lambda_1 - \lambda_2)^{-1} \left(\sum_{k \geq 3} (\lambda_1 - \lambda_k)^{-1} \right) N_1 \tag{4.23}$$

and

$$\frac{\partial^2 N_2}{\partial \lambda_1 \partial \lambda_2} = -2(\lambda_2 - \lambda_1)^{-2} N_2 - (\lambda_2 - \lambda_1)^{-1} \left(\sum_{k \geq 3} (\lambda_2 - \lambda_k)^{-1} \right) N_2. \tag{4.24}$$

Combining these relations, we obtain

$$\frac{\partial K}{\partial \lambda_1} = N_1 F_1' - \left(\sum_{k \geq 2} (\lambda_1 - \lambda_k)^{-1} \right) N_1 F_1 + \sum_{j \geq 2} (\lambda_j - \lambda_1)^{-1} N_j F_j, \tag{4.25}$$

$$\begin{aligned} \frac{\partial K}{\partial \lambda_2} =& N_2 F_2' - (\lambda_2 - \lambda_1)^{-1} N_2 F_2 - \left(\sum_{k \geq 3} (\lambda_2 - \lambda_k)^{-1} \right) N_2 F_2 \\ &+ (\lambda_1 - \lambda_2)^{-1} N_1 F_1 + \sum_{j \geq 3} (\lambda_j - \lambda_2)^{-1} N_j F_j, \end{aligned} \tag{4.26}$$

(4.27)

$$\frac{\partial^2 K}{\partial\lambda_1\partial\lambda_2} = (\lambda_2-\lambda_1)^{-1}N_2F_2' - 2(\lambda_2-\lambda_1)^{-2}N_2F_2$$

$$- (\lambda_2-\lambda_1)^{-1}\left(\sum_{k\geq 3}(\lambda_2-\lambda_k)^{-1}\right)N_2F_2$$

$$- 2(\lambda_1-\lambda_2)^{-2}N_1F_1 - (\lambda_1-\lambda_2)^{-1}\left(\sum_{k\geq 3}(\lambda_1-\lambda_k)^{-1}\right)N_1F_1$$

$$+ (\lambda_1-\lambda_2)^{-1}N_1F_1' + \sum_{j\geq 3}(\lambda_j-\lambda_2)^{-1}(\lambda_j-\lambda_1)^{-1}N_jF_j.$$

Then (4.25) and (4.26) imply

(4.28) $$(\lambda_1-\lambda_2)^{-1}\frac{\partial K}{\partial\lambda_1} + (\lambda_2-\lambda_1)^{-1}\frac{\partial K}{\partial\lambda_2}$$

$$=(\lambda_1-\lambda_2)^{-1}N_1F_1' - (\lambda_1-\lambda_2)^{-2}N_1F_1$$

$$- (\lambda_1-\lambda_2)^{-1}\left(\sum_{k\geq 3}(\lambda_1-\lambda_k)^{-1}\right)N_1F_1$$

$$- (\lambda_1-\lambda_2)^{-2}N_2F_2 + (\lambda_1-\lambda_2)^{-1}\left(\sum_{j\geq 3}(\lambda_j-\lambda_1)^{-1}N_jF_j\right)$$

$$+ (\lambda_2-\lambda_1)^{-1}N_2F_2' - (\lambda_2-\lambda_1)^{-2}N_2F_2$$

$$- (\lambda_2-\lambda_1)^{-1}\left(\sum_{k\geq 3}(\lambda_2-\lambda_k)^{-1}\right)N_2F_2$$

$$- (\lambda_1 - \lambda_2)^{-2} N_1 F_1 + (\lambda_2 - \lambda_1)^{-1} \left(\sum_{j \geq 3} (\lambda_j - \lambda_2)^{-1} N_j F_j \right).$$

Let us now compare (4.27) and (4.28). First, the coefficient of $N_1 F_1'$ (resp., $N_2 F_2'$) is $(\lambda_1 - \lambda_2)^{-1}$ (resp., $(\lambda_2 - \lambda_1)^{-1}$) in either case. Secondly the coefficient of $N_1 F_1$ is

$$-2(\lambda_1 - \lambda_2)^{-2} - (\lambda_1 - \lambda_2)^{-1} \left(\sum_{k \geq 3} (\lambda_1 - \lambda_k)^{-1} \right) \tag{4.29}$$

in either case. Note that $-(\lambda_1 - \lambda_2)^{-2} N_1 F_1$ originates from both $\partial K / \partial \lambda_1$ and $\partial K / \partial \lambda_2$ in (4.28), giving the factor $-2(\lambda_1 - \lambda_2)^{-2}$. The situation is the same for the coefficient of $N_2 F_2$. Finally let us compare the coefficients of $N_j F_j$ $(j \geq 3)$. It is

$$(\lambda_j - \lambda_2)^{-1} (\lambda_j - \lambda_1)^{-1} \tag{4.30}$$

in (4.27), while in (4.28) it is

$$\begin{aligned} &(\lambda_1 - \lambda_2)^{-1} (\lambda_j - \lambda_1)^{-1} + (\lambda_2 - \lambda_1)^{-1} (\lambda_j - \lambda_2)^{-1} \\ &= (\lambda_1 - \lambda_2)^{-1} ((\lambda_j - \lambda_1)^{-1} - (\lambda_j - \lambda_2)^{-1}) \\ &= (\lambda_j - \lambda_1)^{-1} (\lambda_j - \lambda_2)^{-1}. \end{aligned} \tag{4.31}$$

Thus they coincide. Summing up all these comparisons, we have confirmed (4.14). This completes the proof of Theorem 4.1. □

5 Structure theorem for instanton-type solutions of $(P_J)_m$ near a simple P-turning point of the first kind

The purpose of this section is to prove our main result (Theorem 5.1.1) which shows that near a simple P-turning point of $(P_J)_m$ of the first

kind we can transform an instanton-type solution λ_J of $(G_J)_m$ associated with the P-turning point to an appropriate 2-parameter solution of $(P_{\rm I})_1$, the classical (i.e., second order) Painlevé-I equation. In Subsection 5.1 we first fix our notations and then we present our main result. In Subsection 5.2 we recall the definition of the system $(\mathcal{DC}an)$, i.e., the simultaneous equations (Can) and its deformation equation $(D_{\rm can})$, which was introduced in [KT2]. Then in Subsection 5.3 we show the local equivalence near the double turning point $x = \lambda_{j_0,0}(t)$ between $(\mathcal{DC}an)$ and the simultaneous equations $(SL_J)_m$ and $(D_J)_m$, which shall be denoted by $(\mathcal{DSL}_J)_m$ in what follows. In Subsection 5.4 the local equivalence is further ameliorated to become a semi-global one covering not only the double turning point but also a simple turning point $x = a(t)$ of $(SL_J)_m$ that is found (Subsection 5.1) in conjuction with the P-turning point τ in question. The resulting semi-global equivalence plays a key role in Theorem 5.1.1.

5.1 The geometric setting for the main result

In order to state our main result in a precise manner, let us first clarify the geometric setting which we use in our subsequent discussion. It is basically the same as the situation we encountered in Section 3. See also [KT5, Section 3]. Let us start with a simple P-turning point τ of the first kind of $(P_J)_m$ ($J =$ I, 34, II-2, IV) that does not coincide with any other P-turning points of $(P_J)_m$. As was noted in Section 3, there exists a pair of turning points of $(SL_J)_m$, one a double turning point $x = \lambda_{j_0,0}(t)$ and the other a simple turning point $x = a(t)$, which merge at $t = \tau$. These two turning points of $(SL_J)_m$ will play a central role in our analysis in Subsections 5.3 and 5.4. Next we fix a point σ ($\neq \tau$) that is sufficiently close to τ and that lies in a P-Stokes curve emanating from τ. A characteristic feature of σ is that the double turning point $x = \lambda_{j_0,0}(\sigma)$ and the simple turning point $x = a(\sigma)$ are connected by

a Stokes curve γ (i.e., a Stokes "segment") of $(SL_J)_m$. Actually the mathematical definition of the "sufficient closeness of σ and τ" is given through the appearance of this degeneration of the Stokes geometry of $(SL_J)_m$. (See [KT5, Appendix B] for the proof.) Since τ is supposed to be of the first kind, we can find a pair of characteristic roots, say (ν_{j_0}, ν_{j_0+m}), of the Fréchet derivative $(\Delta G_J)_m$ so that

$$\nu_{j_0+m} = -\nu_{j_0}, \tag{5.1.1}$$

$$\nu_{j_0}(\tau) = \nu_{j_0+m}(\tau) = 0, \tag{5.1.2}$$

and

$$\int_\tau^t \nu_{j_0}(s)ds = 2\int_{a(t)}^{\lambda_{j_0,0}(t)} \sqrt{Q_{(J,m),0}(x,t)}dx \tag{5.1.3}$$

hold. We let $\phi_{j_0}(t)$ denote

$$\int_\tau^t \nu_{j_0}(s)ds. \tag{5.1.4}$$

Note that the P-Stokes curve in which σ lies is given by

$$\text{Im } \phi_{j_0}(t) = 0. \tag{5.1.5}$$

Note also that Theorem 4.1 implies that the degree $(-1/2)$ part (in η) $\lambda_{j_0,1/2}$ of the instanton-type solution λ_{j_0} of $(G_J)_m$ is of the form

$$\alpha_{j_0,0}a(t)\exp(\eta\phi_{j_0}(t)) + \alpha_{j_0+m,0}b(t)\exp(-\eta\phi_{j_0}(t)) \tag{5.1.6}$$

with some constants $\alpha_{j_0,0}$ and $\alpha_{j_0+m,0}$ and some analytic functions $a(t)$ and $b(t)$ (cf. (2.1.4) and (2.1.5) in Subsection 2.1).

Using the setting so far described, we now present our main result which asserts that the solution $\lambda_{j_0}(t,\eta)$ of $(G_J)_m$ can be locally transformed near σ to an appropriate 2-parameter solution of the classical Painlevé-I equation.

Theorem 5.1.1. *Suppose*

$$E_0^{(j_0)} \neq 0. \tag{5.1.7}$$

Then there exist a 2-parameter solution $\tilde{\lambda}_{\mathrm{I}}(\tilde{t}, \eta; \tilde{\beta}_1(\eta), \tilde{\beta}_2(\eta))$ *of the equation*

$$\frac{d^2\tilde{\lambda}_{\mathrm{I}}}{d\tilde{t}^2} = \eta^2(6\tilde{\lambda}_{\mathrm{I}}^2 + \tilde{t}\,), \tag{5.1.8}$$

where $\tilde{\beta}_i(\eta) = \sum_{l\geq 0} \tilde{\beta}_{i,l}\eta^{-l}$ $(i = 1, 2)$ *with* $\tilde{\beta}_{i,l}$ *being a constant, a neighborhood* ω *of the point* σ*, a neighborhood* Ω *of the Stokes segment* γ*,* $\tilde{x}(x, t, \eta) = \sum_{l\geq 0} \tilde{x}_{l/2}(x, t, \eta)\eta^{-l/2}$ *with* $\tilde{x}_{l/2}$ *being holomorphic on* $\Omega \times \omega$ *and* $\tilde{t}(t, \eta) = \sum_{l\geq 0} \tilde{t}_{l/2}\eta^{-l/2}$ *with* $\tilde{t}_{l/2}$ *being holomorphic on* ω *for which the following hold:*

$$\tilde{x}(\lambda_{j_0}(t, \eta), t, \eta) = \tilde{\lambda}_{\mathrm{I}}(\tilde{t}(t, \eta), \eta; \tilde{\beta}), \tag{5.1.9}$$

$$\alpha_{j_0,0} = 2c\tilde{\beta}_{1,0} \text{ and } \alpha_{j_0+m,0} = 2c^{-1}\tilde{\beta}_{2,0} \text{ holds for a constant } c \text{ that depends only on } E_0^{(j_0)}, \tag{5.1.10}$$

$$\tilde{x}_{1/2} \text{ and } \tilde{t}_{1/2} \text{ vanish identically,} \tag{5.1.11}$$

$$\text{the } \eta\text{-dependence of } \tilde{x}_{l/2} \text{ and } \tilde{t}_{l/2} \text{ is only through instanton terms that they contain, and } \tilde{x}_0,\ \tilde{x}_1,\ \tilde{t}_0 \text{ and } \tilde{t}_1 \text{ are free from instanton terms.} \tag{5.1.12}$$

5.2 Systems $(\mathcal{DC}an)$ and $(\mathcal{DSL}_J)_m$

As is shown in [KT2, Proposition 2.1], the system (Can) in Definition 3.1 is compatible with another equation (deformation equation)

$$(D_{\mathrm{can}}) \quad \frac{\partial\varphi}{\partial s} = A_{\mathrm{can}}\frac{\partial\varphi}{\partial z} - \frac{1}{2}\frac{\partial A_{\mathrm{can}}}{\partial z}\varphi \tag{5.2.1}$$

with

$$A_{\mathrm{can}} = \frac{1}{2(z - \eta^{-1/2}\sigma_{\mathrm{can}})}, \tag{5.2.2}$$

on the condition that $(\rho_{\rm can}(s,\eta),\sigma_{\rm can}(s,\eta))$ satisfies the following Hamiltonian system:

$$(H_{\rm can}):\begin{cases}\dfrac{d\rho_{\rm can}}{ds}=-4\eta\sigma_{\rm can}\\[2ex]\dfrac{d\sigma_{\rm can}}{ds}=-\eta\rho_{\rm can}.\end{cases}\tag{5.2.3}$$

In what follows we use the symbol $(\mathcal{DC}an)$ to denote the simultaneous system of equations (Can) and $(D_{\rm can})$:

$$\begin{cases}\left(-\dfrac{\partial^2}{\partial z^2}+\eta^2Q_{\rm can}(z,E_{\rm can}(s,\eta),\rho_{\rm can}(s,\eta),\sigma_{\rm can}(s,\eta),\eta)\right)\varphi=0,\\[2ex]\dfrac{\partial}{\partial s}\varphi=A_{\rm can}\dfrac{\partial\varphi}{\partial z}-\dfrac{1}{2}\dfrac{\partial A_{\rm can}}{\partial z}\varphi.\end{cases}\tag{5.2.4}$$

In parallel with this notation we use the symbol $(\mathcal{DSL}_J)_m$ to denote the simultaneous system of equations $(SL_J)_m$ and $(D_J)_m$, that is,

$$\begin{cases}\left(-\dfrac{\partial^2}{\partial x^2}+\eta^2Q_{(J,m)}\right)\psi=0,\\[2ex]\dfrac{\partial}{\partial t}\psi=\mathfrak{a}_{(J,m)}\dfrac{\partial\psi}{\partial x}-\dfrac{1}{2}\dfrac{\partial\mathfrak{a}_{(J,m)}}{\partial x}\psi.\end{cases}\tag{5.2.5}$$

Although Theorem 3.1 guarantees that

$$\psi(x,t,\eta)=\left(\frac{\partial z}{\partial x}\right)^{-1/2}\varphi(z(x,t,\eta),t,\eta)\tag{5.2.6}$$

solves $(SL_J)_m$ near $x=\lambda_{j_0,0}(t)$ if φ is a solution of (Can) with $(\rho_{\rm can},\sigma_{\rm can})=(\rho^{(j_0)},\sigma^{(j_0)})$, ψ given by (5.2.6) does not satisfy $(D_J)_m$ in general; we have to find an appropriate correspondence between s and t besides the change of variables z and x. The results in [KT2] indicate that we should be able to find such a correspondence by requiring the

existence of infinite series

$$s(t,\eta)=\sum_{l\geq 0}s_{l/2}(t)\eta^{-l/2} \tag{5.2.7}$$

which satisfies

$$\rho^{(j_0)}(t,\eta)=\rho_{\rm can}(s(t,\eta),\eta) \tag{5.2.8}$$

and

$$\sigma^{(j_0)}(t,\eta)=\sigma_{\rm can}(s(t,\eta),\eta). \tag{5.2.9}$$

In the subsequent subsections we first construct the series $s(t,\eta)$ that contains some free parameters and then adjust the constants so that the series $z(x,t,\eta)$ and $s(t,\eta)$ thus constructed may satisfy (5.1.9). At the first step we show in Subsection 5.3 that

$$\psi(x,t,\eta)=\left(\frac{\partial z}{\partial x}\right)^{-1/2}\varphi(z(x,t,\eta),s(t,\eta),\eta) \tag{5.2.10}$$

is a solution of $(\mathcal{DSL}_J)_m$ near $x=\lambda_{j_0,0}(t)$ if $\varphi(z,s,\eta)$ is a solution of $(\mathcal{DC}an)$. In Subsection 5.4 we construct a semi-global equivalence between $(\mathcal{DSL}_J)_m$ and $(\mathcal{DSL}_{\rm I})_1$ on a neighborhood of the Stokes segment γ of $(SL_J)_m$ by appropriately combining the transformations constructed in Subsection 5.3, and then we prove that the constructed equivalence gives the required relation (5.1.9).

5.3 Correspondence between $(\mathcal{DSL}_J)_m$ and $(\mathcal{DC}an)$

The purpose of this section is to establish a local correspondence near $x=\lambda_{j_0,0}(t)$ between a solution of $(\mathcal{DC}an)$ and that of $(\mathcal{DSL}_J)_m$ by finding an appropriate transformation $s=s(t,\eta)$. Our first task is to construct $s(t,\eta)$ so that s satisfies (5.2.8) and (5.2.9). To undertake this task we first note that the compatibility of $(SL_J)_m$ with the deformation equation $(D_J)_m$ entails the following invariance property of the constant $E^{(j_0)}(t,\eta)$ in Theorems 3.1 and 3.2.

Lemma 5.3.1. *The series $E^{(j_0)}(t,\eta)$ is independent of t.*

Proof. Let S_{odd} denote the odd part of $S_{(J,m)}$, that is,

$$\frac{1}{2}\left(S^{+}_{(J,m)} - S^{-}_{(J,m)}\right), \tag{5.3.1}$$

where $S^{\pm}_{(J,m)}$ respectively denotes the solution of the Riccati equation associated with $(SL_J)_m$, namely

$$S^2 + \frac{dS}{dx} = \eta^2 Q_{(J,m)}, \tag{5.3.2}$$

whose highest degree part in η is $\pm\eta\sqrt{Q_{(J,m),0}}$. An important property of S_{odd}, or, as is often the case, denoted by $S_{(J,m),\mathrm{odd}}$, is that it satisfies

$$\frac{\partial S_{(J,m),\mathrm{odd}}}{\partial t} = \frac{\partial}{\partial x}(\mathfrak{a}_{(J,m)} S_{(J,m),\mathrm{odd}}), \tag{5.3.3}$$

as a consequence of the deformation equation $(D_J)_m$ that the wave function ψ satisfies (cf. [AKT, Section 2]). It is also well-known (e.g., [KT4, Corollary 2.1.7]) that (3.9) entails

$$S_{\mathrm{odd}}(x,t,\eta) = \frac{dz}{dx} S_{\mathrm{can,odd}}(z(x,t;\eta),t,\eta), \tag{5.3.4}$$

where we define $S_{\mathrm{can,odd}}$ in the same way as S_{odd} by using Q_{can} instead of $Q_{(J,m)}$ in (5.3.2). As a consequence of these properties we obtain

$$\begin{aligned} &\oint_{|x-\lambda_{j_0,0}|=\delta} S_{\mathrm{odd}} dx \\ &= \oint_{|x-\lambda_{j_0,0}|=\delta} S_{\mathrm{can,odd}}(z(x,t,\eta),t,\eta)\frac{\partial z}{\partial x} dx \\ &= \oint_{|z|=\delta'} S_{\mathrm{can,odd}} dz = \frac{\pi i}{2} E^{(j_0)} \end{aligned} \tag{5.3.5}$$

for sufficiently small positive numbers δ and δ'. On the other hand, (5.3.3) and the definition of $S_{\rm odd}$ entail

$$\frac{\partial}{\partial t}\oint S_{\rm odd}dx = \oint \frac{\partial}{\partial x}(\mathfrak{a}_{(J,m)}S_{\rm odd})dx = 0. \tag{5.3.6}$$

This completes the proof of the lemma. □

We also note that, if we define $E_{\rm can}(s,\eta)$ by

$$E_{\rm can} = \rho_{\rm can}^2 - 4\sigma_{\rm can}^2, \tag{5.3.7}$$

the series $E_{\rm can}$ is also independent of s; in fact (H_{can}) implies

$$\begin{aligned}\frac{d}{ds}E_{\rm can} &= 2\rho_{\rm can}\frac{d\rho_{\rm can}}{ds} - 8\sigma_{\rm can}\frac{d\sigma_{\rm can}}{ds} \\ &= -8\eta\rho_{\rm can}\sigma_{\rm can} + 8\eta\sigma_{\rm can}\rho_{\rm can} \\ &= 0.\end{aligned} \tag{5.3.8}$$

Actually the series $E_{\rm can}$ can be explicitly expressed in terms of the constant defined by $(\sigma_{\rm can},\rho_{\rm can})$ as follows: $(\sigma_{\rm can},\rho_{\rm can})$ has the following form as a solution of (H_{can}):

$$\sigma_{\rm can}(s,\eta) = A(\eta)\exp(2\eta s) + B(\eta)\exp(-2\eta s), \tag{5.3.9}$$

$$\rho_{\rm can}(s,\eta) = -2A(\eta)\exp(2\eta s) + 2B(\eta)\exp(-2\eta s), \tag{5.3.10}$$

where $A(\eta) = \sum_{l\geq 0} A_{l/2}\eta^{-l/2}$ and $B(\eta) = \sum_{l\geq 0} B_{l/2}\eta^{-l/2}$ with $A_{l/2}$ and $B_{l/2}$ being constants. It then follows from (5.3.7) that

$$E_{\rm can} = -16A(\eta)B(\eta). \tag{5.3.11}$$

In particular, we find

$$E_{\rm can,0} = -16A_0B_0. \tag{5.3.12}$$

On the other hand, for the α-dependence of $E^{(j_0)}$ we have the following

Lemma 5.3.2. (i) *The highest degree part* $E_0^{(j_0)}$ *of* $E^{(j_0)} = \sum_{l\geq 0} E_{l/2}^{(j_0)} \eta^{-l/2}$ *satisfies*

$$E_0^{(j_0)} = C_0 \alpha_{j_0,0} \alpha_{j_0+m,0} \tag{5.3.13}$$

for some non-zero constant C_0 *which is independent of the free parameters* $\{\alpha_j\}_{1\leq j\leq 2m}$ *contained in an instanton-type solution.*
(ii) *For any odd integer* l, $E_{l/2}^{(j_0)}$ *vanishes.*

Proof. It follows from (3.11) and (5.1.6) that

$$\begin{aligned} \rho_0^{(j_0)} = &- \left(\frac{\partial z_0}{\partial x}(\lambda_{j_0,0})\right)^{-1} (\alpha_{j_0,0} a(t) \phi'_{j_0}(t) \exp(\eta \phi_{j_0}(t)) \\ &- \alpha_{j_0+m,0} b(t) \phi'_{j_0}(t) \exp(-\eta \phi_{j_0}(t))/2\mathfrak{b}_{(J,m),0}^{(j_0)}(\lambda_{j_0,0}) \end{aligned} \tag{5.3.14}$$

where

$$\mathfrak{b}_{(J,m)}^{(j_0)}(x,t,\eta) = (x - \lambda_{j_0}(t,\eta))\mathfrak{a}_{(J,m)}(x,t,\eta). \tag{5.3.15}$$

On the other hand, (3.13) implies

$$\begin{aligned} \sigma_0^{(j_0)} = &\frac{\partial z_0}{\partial x}(\lambda_{j_0,0}(t))(\alpha_{j_0,0} a(t) \exp(\eta \phi_{j_0}(t)) \\ &+ \alpha_{j_0+m,0} b(t) \exp(-\eta \phi_{j_0}(t)). \end{aligned} \tag{5.3.16}$$

In order to compute $E_0^{(j_0)}$, we now prepare the following

Sublemma 5.3.3. *For* $J =$ I, 34, II-2 *or* IV *we find*

$$\left(\frac{\partial z_0}{\partial x}\right)^4 \Bigg|_{x=\lambda_{j_0,0}} = \nu_{j_0}^2 \Big/ \left(4\mathfrak{b}_{(J,m),0}^{(j_0)}(\lambda_{j_0,0})\right)^2 \tag{5.3.17}$$

for z_0 *in* (3.2).

Proof of Sublemma 5.3.3. Let us first recall that

$$Q_{(J,m),0} = (\det B_0)/(\mathfrak{a}_{(J,m),0})^2 \tag{5.3.18}$$

holds for the matrix B used to define the Lax pair that underlies $(P_J)_m$ in the notation of Appendix A. (See [KKoNT] and [KoNT] for the proof of (5.3.18).) Then the Taylor expansion of the highest degree part of (3.9) shows with the help of (5.3.18)

$$\frac{\det B_0}{(x-\lambda_{j_0,0})^2(\mathfrak{a}_{(J,m),0})^2}\bigg|_{x=\lambda_{j_0,0}} = 4\left(\frac{\partial z_0}{\partial x}\right)^4\bigg|_{x=\lambda_{j_0,0}}. \tag{5.3.19}$$

Since we know ([KKoNT, Proposition 2.1.3 and (2.3.8)], [KoNT])

$$\det B_0\Big|_{x=\lambda_{j_0,0}} = \frac{\nu_{j_0}^2}{4}, \tag{5.3.20}$$

we conclude

$$\left(\frac{\partial z_0}{\partial x}\right)^4\bigg|_{x=\lambda_{j_0,0}} = \nu_{j_0}^2 \Big/ \left(4\mathfrak{b}^{(j_0)}_{(J,m),0}(\lambda_{j_0,0})\right)^2. \tag{5.3.21}$$

This completes the proof of the sublemma. □

We now resume the proof of Lemma 5.3.2. Since it follows from the definition of $E^{(j_0)}$ that

$$E_0^{(j_0)} = (\rho_0^{(j_0)})^2 - 4(\sigma_0^{(j_0)})^2 \tag{5.3.22}$$

holds, we deduce the following relation (5.3.23) from (5.3.14) and (5.3.16):

(5.3.23)

$$\begin{aligned} E_0^{(j_0)} &= \left(\alpha_{j_0,0}^2 a^2 \exp(2\eta\phi_{j_0}) + \alpha_{j_0+m,0}^2 b^2 \exp(-2\eta\phi_{j_0})\right) \\ &\times \left[\frac{1}{4}\left(\frac{\partial z_0}{\partial x}(\lambda_{j_0,0})\right)^{-2} \phi_{j_0}'^2 \left(\mathfrak{b}^{(j_0)}_{(J,m),0}(\lambda_{j_0,0})\right)^{-2} - 4\left(\frac{\partial z_0}{\partial x}(\lambda_{j_0,0})\right)^2\right] \\ &- \frac{1}{2}\alpha_{j_0,0}\alpha_{j_0+m,0}ab\left(\frac{\partial z_0}{\partial x}(\lambda_{j_0,0})\right)^{-2} \phi_{j_0}'^2 \left(\mathfrak{b}^{(j_0)}_{(J,m),0}(\lambda_{j_0,0})\right)^{-2} \end{aligned}$$

$$- 8\alpha_{j_0,0}\alpha_{j_0+m,0}ab\left(\frac{\partial z_0}{\partial x}(\lambda_{j_0,0})\right)^2.$$

As it follows from the definition that

$$\nu_{j_0} = \phi'_{j_0}, \tag{5.3.24}$$

(5.3.17) and (5.3.23) entail

$$E_0^{(j_0)} = -16\alpha_{j_0,0}\alpha_{j_0+m,0}ab\left(\frac{\partial z_0}{\partial x}(\lambda_{j_0,0})\right)^2. \tag{5.3.25}$$

Then we find by Lemma 5.3.1 that

$$C_0(t) \underset{\text{def}}{=} a(t)b(t)\left(\frac{\partial z_0}{\partial x}(\lambda_{j_0,0}(t),t)\right)^2 \tag{5.3.26}$$

is independent of t. Thus we obtain (5.3.13). This completes the proof of the assertion (i). To confirm the assertion (ii) we again note (3.12). Then by the "alternating parity" structure of instanton-type solutions (Appendix B), $E_{l/2}^{(j_0)}$ (l: odd) is a sum of monomials of instantons of odd degree. This means that it cannot be a constant unless it vanishes identically. Therefore Lemma 5.3.1 shows (ii). □

In view of our definition of instanton-type solutions (Appendix B), the assumption $\alpha_{j_0,0}\alpha_{j_0+m,0} \neq 0$ enables us to choose $(A(\eta), B(\eta))$ in (5.3.9) and (5.3.10) so that the following relations may hold:

$$E_{\text{can}} = E^{(j_0)}, \tag{5.3.27}$$

$$A_{l/2} = B_{l/2} = 0. \tag{5.3.28}$$

Fixing $(A(\eta), B(\eta))$ in this manner, we construct $s(t,\eta)$ so that it satisfies (5.2.8) and (5.2.9). To describe the precise structure of $s(t,\eta)$ we summarize its properties in Lemma 5.3.4 below. We call the attention of the reader to the fact that the series $s(t,\eta)$ relates the objects

attached to $(\mathcal{DSL}_J)_m$ with those attached to $(\mathcal{DC}an)$. Thus its role is substantially different from that of the series $\tilde{t}(t,\eta)$ used in Theorem 5.1.1 and to be explicitly constructed in Theorem 5.4.1. The series $\tilde{t}(t,\eta)$ relates the objects attached to $(\mathcal{DSL}_J)_m$ with those attached to $(\mathcal{DSL}_I)_1$.

Lemma 5.3.4. *Let us consider the problem in the setting described in Subsection 5.1. In particular, let ω denote a neighborhood of the point σ that is close to, but different from, the P-turning point τ in question. Then we can construct a series $s(t,\eta) = \sum_{l\geq 0} s_{l/2}(t,\eta)\eta^{-l/2}$ so that it satisfies the following conditions:*

$$\sigma_{\mathrm{can}}(s(t,\eta),\eta) = \sigma^{(j_0)}(t,\alpha,\eta), \tag{5.3.29}$$

$$\rho_{\mathrm{can}}(s(t,\eta),\eta) = \rho^{(j_0)}(t,\alpha,\eta), \tag{5.3.30}$$

$$\text{each } s_{l/2}(t,\eta) \text{ is holomorphic on } \omega, \tag{5.3.31}$$

$$s_0(t) = \frac{1}{2}\phi_{j_0}(t), \tag{5.3.32}$$

$$s_{1/2} = 0, \tag{5.3.33}$$

$$s_1(t) = \frac{1}{2}\log\left(A_0^{-1}\alpha_{j_0,0}a(t)\frac{\partial z_0}{\partial x}(\lambda_{j_0,0}(t),t)\right) \tag{5.3.34}$$

$$\left(= \frac{1}{2}\log\left(B_0\alpha_{j_0+m,0}^{-1}b(t)^{-1}\frac{\partial z}{\partial x}(\lambda_{j_0,0}(t),t)^{-1}\right)\right),$$

$$s_{l/2}(t,\eta) \ (l\geq 3) \text{ is a polynomial of instantons of degree } (l-2). \tag{5.3.35}$$

Proof. Here and in what follows we use the symbol

$$[\sigma_{\mathrm{can}}(s_0(t)+\eta^{-1}s_1(t),\eta)]_l \tag{5.3.36}$$

to denote the degree l part (in η^{-1}) of $\sigma_{\mathrm{can}}(s_0(t)+\eta^{-1}s_1(t),\eta)$ with counting the degree of an instanton to be 0 by convention. We first construct $(s_0, s_{1/2}(=0), s_1)$ by using

$$[\sigma_{\mathrm{can}}(s_0(t)+\eta^{-1}s_1(t),\eta)]_0 = \sigma^{(j_0)}(t,\eta), \tag{5.3.37}$$

and then confirm that it also satisfies

$$[\rho_{\mathrm{can}}(s_0(t)+\eta^{-1}s_1(t),\eta)]_0 = \rho^{(j_0)}(t,\eta). \tag{5.3.38}$$

We find by (5.3.16) that

$$A_0\exp(2\eta s_0+2s_1) = \alpha_{j_0,0}a(t)\frac{\partial z_0}{\partial x}(\lambda_{j_0,0}(t))\exp(\eta\phi_{j_0}(t)) \tag{5.3.39}$$

and

$$B_0\exp(-2\eta s_0-2s_1) = \alpha_{j_0+m,0}b(t)\frac{\partial z_0}{\partial x}(\lambda_{j_0,0}(t))\exp(-\eta\phi_{j_0}(t)) \tag{5.3.40}$$

should be satisfied. It is then clear that we should choose s_0 and s_1 so that they satisfy

$$s_0(t) = \frac{1}{2}\phi_{j_0}(t), \tag{5.3.41}$$

$$A_0\exp(2s_1(t)) = \alpha_{j_0,0}a(t)\frac{\partial z_0}{\partial x}(\lambda_{j_0,0}(t),t) \tag{5.3.42}$$

and

$$B_0\exp(-2s_1(t)) = \alpha_{j_0+m,0}b(t)\frac{\partial z_0}{\partial x}(\lambda_{j_0,0}(t),t). \tag{5.3.43}$$

On the other hand, (5.3.12) and (5.3.25) tell us that the condition (5.3.27) reads as

$$-16A_0B_0 = -16\alpha_{j_0,0}\alpha_{j_0+m,0}a(t)b(t)\left(\frac{\partial z_0}{\partial x}(\lambda_{j_0,0}(t),t)\right)^2. \tag{5.3.44}$$

Note that

$$C_0(t) = a(t)b(t)\left(\frac{\partial z_0}{\partial x}(\lambda_{j_0,0}(t),t)\right)^2 \tag{5.3.45}$$

is independent of t (cf. (5.3.26)). Thanks to (5.3.44), (5.3.42) and (5.3.43) are simultaneously solved if we choose $s_1(t)$ so that it satisfies

$$\exp(2s_1(t)) = A_0^{-1}\alpha_{j_0,0}a(t)\frac{\partial z_0}{\partial x}(\lambda_{j_0,0}(t),t). \tag{5.3.46}$$

Furthermore the relation (5.3.21) guarantees the functions $s_0(t)$ and $s_1(t)$ thus chosen also satisfy

$$[\rho_{\mathrm{can}}(s_0(t)+\eta^{-1}s_1(t)]_0 = \rho_0^{(j_0)}(t,\eta) \tag{5.3.47}$$

(with the interchange of indices of j_0 and j_0+m so that the appropriate sign of ν_{j_0} is chosen in the relation (5.3.21)). In fact, (5.3.47) holds if (5.3.41) and

(5.3.48)

$$2A_0\exp(2s_1)$$

$$= \frac{1}{2}\alpha_{j_0,0}a(t)\left(\frac{\partial z_0}{\partial x}(\lambda_{j_0,0}(t),t)\right)^{-1}\phi'_{j_0}(t)\left(\mathfrak{b}^{(j_0)}_{(J,m),0}(\lambda_{j_0,0}(t),t)\right)^{-1},$$

while (5.3.48) follows from (5.3.21), (5.3.24) and (5.3.42). Thus we have found s_0 and s_1 that satisfy (5.3.37) and (5.3.38).

Let us now embark on the construction of $s_{l/2}(t,\eta)$ ($l \geq 3$) by the induction on l; we construct $s_{l/2}$ by supposing that $s_{l'/2}$ ($l' \leq l-1$) have been given. The method is basically the same as that used in the proof of Lemma 3.1 of [KT2]. However, as $\rho^{(j_0)}$ and $\sigma^{(j_0)}$ may contain instanton terms other than $\exp(\pm n\eta\phi_{j_0}(t))$ ($n \in \mathbb{Z}$), we have to pay some attention in the construction procedure. To make our argument clearer we prepare the following

Sublemma 5.3.5. *Let T denote $\exp(\eta\phi_{j_0}(t))$ and let $f = \sum_{l=-p}^{p} a_l T^l$ and $g = \sum_{l=-p}^{p} b_l T^l$ be instanton-type solutions given by* (2.1.4) *and* (2.1.5)*. Assume that*

$$(\alpha T - \beta T^{-1})f = (\alpha T + \beta T^{-1})g \tag{5.3.49}$$

holds for some instanton-free series α and β whose top degree parts α_0 and β_0 satisfy

$$\alpha_0\beta_0 \neq 0. \tag{5.3.50}$$

Then there exists an instanton-type solution $h = \sum_{l=-p+1}^{p-1} c_l T^l$ which satisfies

$$f = (\alpha T + \beta T^{-1})h. \tag{5.3.51}$$

Furthermore c_l is a linear combination of a_k's $(-p \leq k \leq p)$ whose coefficients are described in terms of $\alpha, \beta, \alpha^{-1}$ and β^{-1}.

Proof of Sublemma 5.3.5. Let f_{even} (resp., f_{odd}, g_{even} and g_{odd}) denote the even degree (in T) part of f (resp., the odd degree part of f, the even degree part of g and the odd degree part of g). By rewriting (5.3.49) as

$$(\alpha T^2 - \beta)f = (\alpha T^2 + \beta)g, \tag{5.3.52}$$

we compare the even degree parts and the odd degree parts in (5.3.52) to find

$$(\alpha T^2 - \beta)f_{\text{even}} = (\alpha T^2 + \beta)g_{\text{even}} \tag{5.3.53}$$

and

$$(\alpha T^2 - \beta)f_{\text{odd}} = (\alpha T^2 + \beta)g_{\text{odd}}; \tag{5.3.54}$$

hence we obtain

$$(\alpha T - \beta T^{-1})f_{\text{even}} = (\alpha T + \beta T^{-1})g_{\text{even}} \tag{5.3.55}$$

and

$$(\alpha T - \beta T^{-1}) f_{\text{odd}} = (\alpha T + \beta T^{-1}) g_{\text{odd}}. \tag{5.3.56}$$

Therefore it suffices to show the existence of h by

(i) assuming f and g are both of even degree in T

and

(ii) assuming f and g are both of odd degree in T.

As the logical structure of the proof is the same either in case (i) or in case (ii), here we only consider the case (i), i.e., the even degree case. Let us suppose

$$f = \sum_{l=-n}^{n} a_{2l} T^{2l} \tag{5.3.57}$$

and

$$g = \sum_{l=-n}^{n} b_{2l} T^{2l}. \tag{5.3.58}$$

Then, multiplying both sides by $\alpha^{-1} T^{2n+1}$, we find (5.3.55) can be written as

$$(T^2 - \alpha^{-1}\beta)(T^{2n} f) = (T^2 + \alpha^{-1}\beta)(T^{2n} g). \tag{5.3.59}$$

Hence it follows from (5.3.59) that

$$(T^{2n} f)\Big|_{T^2 = -\alpha^{-1}\beta} = 0. \tag{5.3.60}$$

In what follows we use the following expression for $T^{2n} f$ and $T^{2n} g$:

$$T^{2n} f = \sum_{l=-n}^{n} a_{2l} T^{2(l+n)} = \sum_{l=0}^{2n} \tilde{a}_l T^{2(2n-l)}, \tag{5.3.61}$$

$$T^{2n}g = \sum_{l=-n}^{n} b_{2l}T^{2(l+n)} = \sum_{l=0}^{2n} \tilde{b}_l T^{2(2n-l)}, \tag{5.3.62}$$

that is, we let $\tilde{a}_j = a_{2(n-j)}$ and $\tilde{b}_j = b_{2(n-j)}$ $(j = 0, \dots, 2n)$. Under this notation we can readily confirm the following "division" formula:

$$T^{2n}f = \sum_{l=0}^{2n} \tilde{a}_l T^{2(2n-l)} = (T^2 + \alpha^{-1}\beta) \sum_{l=0}^{2n-1} \tilde{c}_l T^{2(2n-1-l)} + \tilde{c}_{2n}, \tag{5.3.63}$$

where $\tilde{c}_0 = \tilde{a}_0$ and

$$\tilde{c}_l = \tilde{a}_l + (-\alpha^{-1}\beta)\tilde{a}_{l-1} + \cdots + (-\alpha^{-1}\beta)^l \tilde{a}_0 \tag{5.3.64}$$

for $l = 1, \dots, 2n$. In particular, (5.3.60) implies

$$\tilde{c}_{2n} = (T^{2n}f)\Big|_{T^2 = -\alpha^{-1}\beta} = 0, \tag{5.3.65}$$

and hence we obtain

$$T^{2n}f = (T^2 + \alpha^{-1}\beta) \sum_{l=0}^{2n-1} \tilde{c}_l T^{2(2n-1-l)}, \tag{5.3.66}$$

that is,

$$f = \alpha^{-1}(\alpha T + \beta T^{-1}) \left(\sum_{l=0}^{2n-1} \tilde{c}_l T^{2n-1-2l} \right). \tag{5.3.67}$$

Thus, letting

$$h = \alpha^{-1} \sum_{l=0}^{2n-1} \tilde{c}_l T^{2n-1-2l}, \tag{5.3.68}$$

we obtain (5.3.51). This completes the proof of Sublemma 5.3.5. □

We now resume the proof of Lemma 5.3.4. By using the Taylor expansion of $\exp \pm 2(s_{3/2}\eta^{-1/2} + \cdots + s_{l/2}\eta^{-(l-2)/2})$, we deduce the

relation (5.3.70) below from the requirement

(5.3.69)

$$\Big[\sigma_{\mathrm{can}}(s_0(t)+s_1(t)\eta^{-1}+s_{3/2}(t,\eta)\eta^{-3/2}+\cdots+s_{l/2}(t,\eta)\eta^{-l/2},\eta)\Big]_{(l-2)/2}$$
$$=\sigma^{(j_0)}_{(l-2)/2}(t,\eta),$$

(5.3.70)

$$s_{l/2}(t,\eta)=\frac{X_{l/2}}{2(A_0\exp(2\eta s_0+2s_1)-B_0\exp(-(2\eta s_0+2s_1)))},$$

where $X_{l/2}$ is a polynomial of instantons of degree $(l-1)$ by the instanton structure of $\sigma^{(j_0)}$ (cf. Appendix B) together with the induction hypothesis, i.e., (5.3.35). Furthermore it follows from (5.3.69) and (5.3.27) that

(5.3.71)

$$\Big[\rho_{\mathrm{can}}(s_0(t)+\cdots+s_{l/2}(t,\eta)\eta^{-l/2},\eta)^2\Big]_{(l-2)/2}=\Big[\rho^{(j_0)}(t,\eta)^2\Big]_{(l-2)/2}$$

holds. Hence (5.3.38) entails

$$\Big[\rho_{\mathrm{can}}(s_0(t)+\cdots+s_{l/2}(t,\eta)\eta^{-l/2},\eta)\Big]_{(l-2)/2}=\rho^{(j_0)}_{(l-2)/2}(t,\eta). \tag{5.3.72}$$

Then, in parallel with the way of deducing (5.3.70) from (5.3.69), we obtain from (5.3.72) the following relation:

(5.3.73)

$$s_{l/2}(t,\eta)=\frac{Y_{l/2}}{-4(A_0\exp(2\eta s_0+2s_1)+B_0\exp(-(2\eta s_0+2s_1)))},$$

where $Y_{l/2}$ is a polynomial of instantons of degree $(l-1)$ by the instanton structure of $\rho^{(j_0)}$ together with the induction hypothesis. Combining (5.3.70) and (5.3.73) we now find

$$(\alpha T-\beta T^{-1})X_{l/2}=-\frac{1}{2}(\alpha T+\beta T^{-1})Y_{l/2} \tag{5.3.74}$$

by choosing

$$\alpha=A_0\exp(2s_1), \tag{5.3.75}$$

$$\beta = -B_0 \exp(-2s_1), \tag{5.3.76}$$

$$T = \exp(2\eta s_0). \tag{5.3.77}$$

Therefore Sublemma 5.3.5 guarantees that $X_{l/2}$ is divisible by $(\alpha T + \beta T^{-1})$ in the polynomial ring generated with T and T^{-1} with instantons in the coefficients. Hence (5.3.70) implies that $s_{l/2}$ is of the required instanton structure (5.3.35). Thus the induction proceeds, completing the proof of Lemma 5.3.4. □

Remark 5.3.1. Although we have imposed constraints (5.3.27) and (5.3.28), there still remains arbitrariness in the choice of either $A_{2l/2}$ or $B_{2l/2}$, say $A_{2l/2}$. This arbitrariness is inherited to $s_{l/2}$.

The series $s(t,\eta)$ constructed in Lemma 5.3.4 together with the series constructed in Theorem 3.1 brings the simultaneous equation $(\mathcal{DSL}_J)_m$ to $(\mathcal{DC}an)$; the precise statement is as follows.

Proposition 5.3.6. *Let us consider the problem in the setting of Subsection 5.1; in particular, we assume* (5.1.7). *Let $\varphi(z,s,\eta)$ be a WKB solution of (Can) that satisfies (D_{can}) also, and let $\psi(x,t,\eta)$ be given by the following:*

$$\psi(x,t,\eta) = \left(\frac{\partial z(x,t,\eta)}{\partial x}\right)^{-1/2} \varphi(z(x,t,\eta), s(t,\eta), \eta), \tag{5.3.78}$$

where $z = z(x,t,\eta)$ and $s = s(t,\eta)$ are the transformations given respectively by Theorem 3.1 and Lemma 5.3.4. Then $\psi(x,t,\eta)$ satisfies $(\mathcal{DSL}_J)_m$, i.e., the simultaneous equations (5.2.5) *near $x = \lambda_{j_0,0}(\sigma)$.*

Proof. First we note that the s-dependence of Q_{can} is through E_{can}, ρ_{can} and σ_{can}. Since we obtain S_{can} by recursively solving the Riccati equation

$$S_{\mathrm{can}}^2 + \frac{\partial S_{\mathrm{can}}}{\partial z} = \eta^2 Q_{\mathrm{can}} \tag{5.3.79}$$

in an algebraic way, the s-dependence of S_{can} is also only through E_{can} $(= \rho_{\mathrm{can}}^2 - 4\sigma_{\mathrm{can}}^2)$, ρ_{can} and σ_{can}, that is,

$$S_{\mathrm{can}}(z, s, \eta) = S_{\mathrm{can}}(z, \rho_{\mathrm{can}}(s, \eta), \sigma_{\mathrm{can}}(s, \eta), \eta). \tag{5.3.80}$$

Now, as is well-known (e.g. [KT4, Corollary 2.1.7]), the relation between $Q_{(J,m)}$ and Q_{can} given by (3.9) implies the following relation (5.3.81) between S_{odd} given by (5.3.1) and $S_{\mathrm{can,odd}}$:

(5.3.81)

$$S_{\mathrm{odd}}(x, t, \eta) = \frac{\partial z(x, t, \eta)}{\partial x} S_{\mathrm{can,odd}}(z(x, t, \eta), \rho^{(j_0)}(t, \eta), \sigma^{(j_0)}(t, \eta)).$$

On the other hand, (5.3.29) and (5.3.30) means that the right-hand side of (5.3.81) is identical with

$$\frac{\partial z}{\partial x} S_{\mathrm{can,odd}}(z(x, t, \eta), \rho_{\mathrm{can}}(s(t, \eta), \eta), \sigma_{\mathrm{can}}(s(t, \eta), \eta)). \tag{5.3.82}$$

Hence, by using (5.3.80), we find

$$S_{\mathrm{odd}}(x, t, \eta) = \frac{\partial z(x, t, \eta)}{\partial x} S_{\mathrm{can,odd}}(z(x, t, \eta), s(t, \eta), \eta). \tag{5.3.83}$$

Then, differentiating (5.3.83) by t, we obtain

$$\begin{aligned} \frac{\partial S_{\mathrm{odd}}}{\partial t} =& \frac{\partial^2 z}{\partial x \partial t} S_{\mathrm{can,odd}}(z(x, t, \eta), s(t, \eta), \eta) \\ &+ \frac{\partial z}{\partial x} \left(\frac{\partial S_{\mathrm{can,odd}}(z(x, t, \eta), s(t, \eta), \eta)}{\partial z} \frac{\partial z}{\partial t} \right. \\ &\left. + \frac{\partial S_{\mathrm{can,odd}}(z(x, t, \eta), s(t, \eta), \eta)}{\partial s} \frac{\partial s}{\partial t} \right). \end{aligned} \tag{5.3.84}$$

It then follows from (5.3.3) and (5.3.83) that

$$\frac{\partial}{\partial x}\left\{\mathfrak{a}_{(J,m)}\frac{\partial z}{\partial x}S_{\mathrm{can,odd}}(z(x,t,\eta),s(t,\eta),\eta)\right\} \tag{5.3.85}$$

$$=\frac{\partial}{\partial x}\left\{\frac{\partial z}{\partial t}S_{\mathrm{can,odd}}(z(x,t,\eta),s(t,\eta),\eta)\right\}$$

$$+\frac{\partial z}{\partial x}\frac{\partial s}{\partial t}\frac{\partial S_{\mathrm{can,odd}}(z(x,t,\eta),s(t,\eta),\eta)}{\partial s}.$$

Since we know

$$\frac{\partial S_{\mathrm{can,odd}}}{\partial s}=\frac{\partial}{\partial z}(A_{\mathrm{can}}S_{\mathrm{can,odd}}), \tag{5.3.86}$$

we can rewrite (5.3.85) as

$$\frac{\partial}{\partial x}\left\{\left(\mathfrak{a}_{(J,m)}\frac{\partial z}{\partial x}-\frac{\partial z}{\partial t}\right)S_{\mathrm{can,odd}}(z(x,t,\eta),s(t,\eta),\eta)\right\} \tag{5.3.87}$$

$$=\frac{\partial s}{\partial t}\frac{\partial z}{\partial x}\frac{\partial}{\partial z}\left\{(A_{\mathrm{can}}S_{\mathrm{can,odd}})(z(x,t,\eta),s(t,\eta),\eta)\right\}$$

$$=\frac{\partial s}{\partial t}\frac{\partial}{\partial x}\left\{(A_{\mathrm{can}}S_{\mathrm{can,odd}})(z(x,t,\eta),s(t,\eta),\eta)\right\},$$

that is,

$$\frac{\partial}{\partial x}\left\{\left(\mathfrak{a}_{(J,m)}\frac{\partial z}{\partial x}-\frac{\partial z}{\partial t}-A_{\mathrm{can}}\frac{\partial s}{\partial t}\right)S_{\mathrm{can,odd}}(z(x,t,\eta),s(t,\eta),\eta)\right\}=0. \tag{5.3.88}$$

Here we recall [KT1, Proposition 2.2]; its proof applies to the current situation without any changes and it shows the following:

$$\frac{\partial\psi(x,t,\eta)}{\partial t}=\mathfrak{a}_{(J,m)}\frac{\partial\psi}{\partial x}-\frac{1}{2}\frac{\partial\mathfrak{a}_{(J,m)}}{\partial x}\psi \tag{5.3.89}$$

follows from the relation

$$\mathfrak{a}_{(J,m)}\frac{\partial z}{\partial x}-\frac{\partial z}{\partial t}-A_{\mathrm{can}}\frac{\partial s}{\partial t}=0 \tag{5.3.90}$$

on the condition that φ solves $(\mathcal{DC}an)$. Thus our task is to deduce (5.3.90) from (5.3.88). To attain this task we follow the way of reasoning in [KT2, Section 3]; that is, we introduce the following two symbols $\mathcal{J}$ and $\mathcal{K}$ and we deduce $\mathcal{J} = 0$ from the relation (5.3.93) below by using the induction on the degree in $\eta^{-1/2}$ in $\mathcal{J}$:

$$\mathcal{J} = 2(z(x,t,\eta) - \eta^{-1/2}\sigma_{\mathrm{can}}(s(t,\eta),\eta)) \times \left(\mathfrak{a}_{(J,m)}\frac{\partial z(x,t,\eta)}{\partial x} - \frac{\partial z(x,t,\eta)}{\partial t} - A_{\mathrm{can}}\frac{\partial s(t,\eta)}{\partial t}\right), \tag{5.3.91}$$

$$\mathcal{K} = \frac{\eta^{-1}S_{\mathrm{can,odd}}(z(x,t,\eta),s(t,\eta),\eta)}{2(z(x,t,\eta) - \eta^{-1/2}\sigma_{\mathrm{can}}(s(t,\eta),\eta)}. \tag{5.3.92}$$

It is then clear that (5.3.88) can be rewritten as

$$\frac{\partial}{\partial x}(\mathcal{J}\mathcal{K}) = 0. \tag{5.3.93}$$

In order to make the induction argument run smoothly, we prepare the following sublemmas:

Sublemma 5.3.7. *In the current situation we find*

(5.3.94)

$$\rho^{(j_0)}(t,\eta)\left[\left(2\mathfrak{b}^{(j_0)}_{(J,m)}\left(\frac{\partial z}{\partial x}\right)^2\right)\Bigg|_{x=\lambda_{j_0}} - \frac{\partial s}{\partial t}\right]$$

$$= -\eta^{-1/2}\left[\left\{\frac{\partial}{\partial x}\left(\mathfrak{b}^{(j_0)}_{(J,m)}\right)\left(\frac{\partial z}{\partial x}\right) + \left(\frac{3}{2}\mathfrak{b}^{(j_0)}_{(J,m)}\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial t}\right)\right\}\Bigg|_{x=\lambda_{j_0}}\right].$$

Sublemma 5.3.8. *Let X denote $x - \lambda_{j_0}$ and let $O(X^l)$ denote a*

sum of terms containing a factor X^m $(m \geq l)$. *Then we find*

(5.3.95)

$$\mathcal{J} = \left(\left[2\mathfrak{b}^{(j_0)}_{(J,m)} \left(\frac{\partial z}{\partial x} \right)^2 \right] \Bigg|_{x=\lambda_{j_0}} - \frac{\partial s}{\partial t} \right)$$

$$+ \left[\left(3\mathfrak{b}^{(j_0)}_{(J,m)} \frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial}{\partial x} \left(\mathfrak{b}^{(j_0)}_{(J,m)} \right) \left(\frac{\partial z}{\partial x} \right) - 2 \frac{\partial z}{\partial t} \right) \frac{\partial z}{\partial x} \right] \Bigg|_{x=\lambda_{j_0}} X + O(X^2).$$

Proof of Sublemma 5.3.7. Using the definition of $\sigma^{(j_0)}$, we differentiate both sides of (5.3.29) with respect to t to find

(5.3.96)

$$\left. \frac{d\sigma_{\mathrm{can}}}{ds} \right|_{s=s(t,\eta)} \frac{\partial s}{\partial t} = \eta^{1/2} \left(\left. \frac{\partial z}{\partial x} \right|_{x=\lambda_{j_0}(t,\eta)} \frac{d\lambda_{j_0}}{dt} + \left. \frac{\partial z}{\partial t} \right|_{x=\lambda_{j_0}(t,\eta)} \right).$$

On the other hand, (3.11) entails

(5.3.97)

$$\frac{d\lambda_{j_0}}{dt} = - \Bigg[2\eta^{1/2} \rho^{(j_0)} \mathfrak{b}^{(j_0)}_{(J,m)} \frac{\partial z}{\partial x}$$

$$+ \frac{\partial}{\partial x} \left(\mathfrak{b}^{(j_0)}_{(J,m)} \right) + \frac{3}{2} \mathfrak{b}^{(j_0)}_{(J,m)} \frac{\partial^2 z / \partial x^2}{\partial z / \partial x} \Bigg] \Bigg|_{x=\lambda_{j_0}}.$$

Then by using (5.2.3), (5.3.30) and (5.3.97) we obtain

(5.3.98)

$$\eta^{1/2} \rho^{(j_0)} \left(2\mathfrak{b}^{(j_0)}_{(J,m)} \left(\frac{\partial z}{\partial x} \right)^2 \right) \Bigg|_{x=\lambda_{j_0}} - \eta^{1/2} \rho^{(j_0)} \frac{\partial s}{\partial t}$$

$$= - \left[\frac{\partial}{\partial x} \left(\mathfrak{b}^{(j_0)}_{(J,m)} \right) \left(\frac{\partial z}{\partial x} \right) + \frac{3}{2} \mathfrak{b}^{(j_0)}_{(J,m)} \frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial t} \right] \Bigg|_{x=\lambda_{j_0}}.$$

Thus we have confirmed (5.3.94). □

Proof of Sublemma 5.3.8. Using (5.3.29), the definition of $\sigma^{(j_0)}$ and the Taylor expansion in X, we obtain

$$
\begin{aligned}
(5.3.99) \qquad \mathcal{J} &= 2\left\{ \left.\frac{\partial z}{\partial x}\right|_{x=\lambda_{j_0}} X + \frac{1}{2}\left.\frac{\partial^2 z}{\partial x^2}\right|_{x=\lambda_{j_0}} X^2 \right. \\
&\quad \left. + O(X^3) \right\} \left\{ \frac{1}{X} \left(\left. \mathfrak{b}^{(j_0)}_{(J,m)} \right|_{x=\lambda_{j_0}} + \left. \frac{\partial \mathfrak{b}^{(j_0)}_{(J,m)}}{\partial x} \right|_{x=\lambda_{j_0}} X \right. \right. \\
&\quad \left. + O(X^2) \right) \left(\left. \frac{\partial z}{\partial x} \right|_{x=\lambda_{j_0}} + \left. \frac{\partial^2 z}{\partial x^2} \right|_{x=\lambda_{j_0}} X \right. \\
&\quad \left. \left. + O(X^2) \right) - \left(\left. \frac{\partial z}{\partial t} \right|_{x=\lambda_{j_0}} + O(X) \right) \right\} - \frac{\partial s}{\partial t} \\
&= 2\mathfrak{b}^{(j_0)}_{(J,m)} \left(\frac{\partial z}{\partial x} \right)^2 \Bigg|_{x=\lambda_{j_0}} - \frac{\partial s}{\partial t} \\
&\quad + \left[3\mathfrak{b}^{(j_0)}_{(J,m)} \frac{\partial z}{\partial x} \frac{\partial^2 z}{\partial x^2} + 2\frac{\partial}{\partial x}\left(\mathfrak{b}^{(j_0)}_{(J,m)} \right) \left(\frac{\partial z}{\partial x} \right)^2 \right. \\
&\quad \left. - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial t} \right] \Bigg|_{x=\lambda_{j_0}} X + O(X^2).
\end{aligned}
$$

Thus we have verified (5.3.95). □

Let us now resume the proof of Proposition 5.3.6. Our strategy is

to employ (5.3.93) to prove that

$$\mathcal{J} = \sum_{k\geq 0} \eta^{-k/2} \mathcal{J}_{k/2} \tag{5.3.100}$$

vanishes by using the induction on k. Let us first confirm $\mathcal{J}_0 = 0$. Since E_0 does not vanish by assumption (5.1.7),

$$\rho_0^{(j_0)} \neq 0. \tag{5.3.101}$$

Hence (5.3.94) shows

$$2\mathfrak{b}^{(j_0)}_{(J,m),0} \left(\frac{\partial z_0}{\partial x} \right)^2 \Big|_{x=\lambda_{j_0,0}} - \frac{\partial s_0}{\partial t} = 0, \tag{5.3.102}$$

and then Sublemma 5.3.8 implies

$$\mathcal{J}_0 \Big|_{x=\lambda_{j_0,0}} = 0. \tag{5.3.103}$$

Since $\mathcal{K}_0 = 1$, this means

$$\mathcal{J}_0 \mathcal{K}_0 \Big|_{x=\lambda_{j_0,0}} = 0. \tag{5.3.104}$$

Therefore (5.3.93) proves $\mathcal{J}_0\mathcal{K}_0$, and hence $\mathcal{J}_0$ also, vanishes identically.

Let us now suppose

$$\mathcal{J}_{k/2} = 0 \quad \text{for} \quad k \leq k_0. \tag{5.3.105}$$

Since

$$\frac{\partial z_0}{\partial x} \Big|_{x=\lambda_{j_0,0}} \neq 0, \tag{5.3.106}$$

$(\partial z/\partial x)|_{x=\lambda_{j_0}}$ is invertible and its inverse is of degree equal to or at

most 0 in η. Hence Sublemma 5.3.8 entails

$$\left(\frac{\partial \mathcal{J}}{\partial x}\Big|_{x=\lambda_{j_0}}\right) \Big/ \left(2\frac{\partial z}{\partial x}\Big|_{x=\lambda_{j_0}}\right) = -\left(\frac{\partial z}{\partial t} - \frac{3}{2}\mathfrak{b}^{(j_0)}_{(J,m)}\frac{\partial^2 z}{\partial x^2} - \frac{\partial \mathfrak{b}^{(j_0)}_{(J,m)}}{\partial x}\frac{\partial z}{\partial x}\right)\Big|_{x=\lambda_{j_0}}. \tag{5.3.107}$$

Then (5.3.105) implies that the left-hand side of (5.3.107), and hence its right-hand side also, is of degree equal to or at most $(-(k_0+1)/2)$ in η. As the right-hand side of (5.3.94) is $(-1)\eta^{-1/2}$ multiple of the right-hand side of (5.3.107), the left-hand side of (5.3.94) is of degree equal to or at most $(-(k_0+2)/2)$. Again using the assumption (5.1.7), we then find that

$$\left[\left(2\mathfrak{b}^{(j_0)}_{(J,m)}\left(\frac{\partial z}{\partial x}\right)^2\right)\Big|_{x=\lambda_{j_0}} - \frac{\partial s}{\partial t}\right]_{(k_0+1)/2}, \tag{5.3.108}$$

i.e., the degree $-((k_0+1)/2)$ part in η of the second factor of the left-hand side of (5.3.94), should vanish. Then Sublemma 5.3.8 implies

$$\mathcal{J}_{(k_0+1)/2}\Big|_{x=\lambda_{j_0,0}} = 0. \tag{5.3.109}$$

On the other hand the induction hypothesis (5.3.105) and (5.3.93) entail

$$\frac{\partial}{\partial x}\left(\mathcal{J}_{(k_0+1)/2}\mathcal{K}_0\right) = 0. \tag{5.3.110}$$

Combining (5.3.109) and (5.3.110), we find

$$\mathcal{J}_{(k_0+1)/2}\mathcal{K}_0 = 0. \tag{5.3.111}$$

Thus we have shown

$$\mathcal{J}_{(k_0+1)/2} = 0, \tag{5.3.112}$$

as $\mathcal{K}_0 = 1$. Therefore the induction proceeds, completing the proof of Proposition 5.3.6. □

The zig-zag reasoning between (5.3.94) and (5.3.95), which is the core part of the induction used in the proof of Proposition 5.3.6, has proved the following results as a by-product.

Proposition 5.3.9. *In the situation discussed in* Proposition 5.3.6 *we find the following relations:*

$$\frac{\partial s}{\partial t} = \left(2\mathfrak{b}^{(j_0)}_{(J,m)} \left(\frac{\partial z}{\partial x} \right)^2 \right) \Bigg|_{x=\lambda_{j_0}}, \tag{5.3.113}$$

$$\left(\frac{\partial z}{\partial t} - \frac{3}{2}\, \mathfrak{b}^{(j_0)}_{(J,m)} \frac{\partial^2 z}{\partial x^2} - \frac{\partial \mathfrak{b}^{(j_0)}_{(J,m)}}{\partial x} \frac{\partial z}{\partial x} \right) \Bigg|_{x=\lambda_{j_0}} = 0. \tag{5.3.114}$$

Combining (5.2.8), (5.2.9) and (5.3.113) with (H_{can}), we obtain

Corollary 5.3.10. *In the above situation we have*

$$\begin{cases} \dfrac{\partial \sigma^{(j_0)}}{\partial t} = -2\eta \left(\mathfrak{b}^{(j_0)}_{(J,m)} \left(\dfrac{\partial z}{\partial x} \right)^2 \right) \Bigg|_{x=\lambda_{j_0}} \rho^{(j_0)}, \\ \dfrac{\partial \rho^{(j_0)}}{\partial t} = -8\eta \left(\mathfrak{b}^{(j_0)}_{(J,m)} \left(\dfrac{\partial z}{\partial x} \right)^2 \right) \Bigg|_{x=\lambda_{j_0}} \sigma^{(j_0)}. \end{cases} \tag{5.3.115}$$

Remark 5.3.2. An important consequence of (5.3.113) is that the series $s(t, \eta)$ is uniquely fixed modulo an additive infinite series in η that is free from t.

5.4 Semi-global equivalence of $(\mathcal{DSL}_J)_m$ and $(\mathcal{DSL}_{\mathrm{I}})_1$

Proposition 5.3.6 enables us to construct a local correspondence between $(\mathcal{DSL}_J)_m$ and $(\mathcal{DSL}_{\mathrm{I}})_1$ near the double turning point $x =$

$\lambda_{j_0,0}(t)$ of $(SL_J)_m$ by using $(\mathcal{DCan})$ as an intermediator. Here $(\mathcal{DSL}_{\mathrm{I}})_1$ is the same as $(\mathcal{DSL}_{\mathrm{I}})$ in [KT2] by its definition. The local correspondence thus constructed is an almost unique one, but it is not really unique; it inherits the arbitrariness contained in $s(t,\eta)$ that is noted in Remark 5.3.2. By appropriately getting rid of the arbitrariness we can analytically extend the local equivalence so that it may be defined on a neighborhood of a simple turning point $x = a(t)$ of $(SL_J)_m$ in the setting described in Subsection 5.1. As we will show at the end of this subsection, the semi-global equivalence thus obtained gives us the transformation $(\tilde{x}(x,t,\eta), \tilde{t}(t,\eta))$ used in Theorem 5.1.1. As three systems of differential equations, $(\mathcal{DSL}_J)_m$, $(\mathcal{DSL}_{\mathrm{I}})$ $(= (\mathcal{DSL}_{\mathrm{I}})_1)$ and $(\mathcal{DCan})$ are involved in the construction of the transformation, we use the following symbols to facilitate the identification of the differential equation studied at that spot: (x,t) designates (resp., $(\tilde{x},\tilde{t})$ and (z,s)) the independent variable of $(\mathcal{DSL}_J)_m$ (resp., $(\mathcal{DSL}_{\mathrm{I}})$ and $(\mathcal{DCan})$), and we normally put the symbol $\tilde{}$ (resp., suffix "can") over (resp., to) the quantities relevant to $(\mathcal{DSL}_{\mathrm{I}})$ (resp., $(\mathcal{DCan})$), as possible as we can. For example, the symbol $\tilde{S}_{\mathrm{odd}}(\tilde{x},\tilde{t},\eta)$ means the odd part of a solution $\tilde{S}$ of the Riccati equation associated with (SL_{I}), and $S_{\mathrm{odd}}(x,t,\eta)$ (resp., $S_{\mathrm{can,odd}}(z,s,\eta)$) stands for a similar object which we encounter in analyzing $(\mathcal{DSL}_J)_m$ (resp., $(\mathcal{DCan})$). We believe that the usage of symbols employed here is systematic and reasonable by itself. In stating the following Theorem 5.4.1, we use the same symbols as those introduced in Subsection 5.1. The logical structure of the proof of Theorem 5.4.1 is essentially the same as that of Theorem 4.1 of [KT2], but for the sake of completeness we include it here without paring it down.

Theorem 5.4.1. *There exist a neighborhood V of σ, a neighborhood U of the Stokes segment γ that connects $\lambda_{j_0,0}(\sigma)$ and $a(\sigma)$, holomorphic functions $\tilde{x}_{l/2}(x,t,\eta)$ on $U \times V$ and $\tilde{t}_{l/2}(t,\eta)$ on V*

($l = 0, 1, 2, \cdots$) *for which the following conditions* (i) $\sim$ (vii) *are satisfied:*
(i) *The function* $\tilde{t}_0(t, \eta)$ *is independent of* η*, and it satisfies*

$$\phi_{j_0}(t) = \tilde{\phi}_{\mathrm{I}}(\tilde{t}_0(t)), \tag{5.4.1}$$

where $\tilde{\phi}_{\mathrm{I}}(\tilde{t})$ *designates the phase function that appears in an instanton-type solution* $\tilde{\lambda}_{\mathrm{I}}$ *of the classical Painlevé-*I *equation* (5.1.8).
(ii) $d\tilde{t}_0/dt$ *never vanishes on* V.
(iii) *The function* $\tilde{x}_0(x, t, \eta)$ *is also independent of* η*, and it satisfies*

$$\tilde{x}_0(\lambda_{j_0,0}(t), t) = \tilde{\lambda}_{\mathrm{I},0}(\tilde{t}_0(t)) \tag{5.4.2}$$

and

$$\tilde{x}_0(a(t), t) = -2\tilde{\lambda}_{\mathrm{I},0}(\tilde{t}_0(t)). \tag{5.4.3}$$

(iv) $\partial\tilde{x}_0/\partial x$ *never vanishes on* $U \times V$.
(v) $\tilde{x}_{1/2}$ *and* $\tilde{t}_{1/2}$ *vanish identically.*
(vi) *The* η*-dependence of* $\tilde{x}_{l/2}$ *and* $\tilde{t}_{l/2}$ ($l \geq 2$) *is solely through instanton terms originating from those in* $\lambda_{j_0}(t, \eta)$*, and* $\tilde{x}_{l/2}$ *and* $\tilde{t}_{l/2}$ *are polynomials of instantons of degree at most* $(l-2)$.
(vii) *For* $\tilde{x}(x, t, \eta) = \sum_{l\geq 0} \tilde{x}_{l/2}(x, t, \eta)\eta^{-l/2}$ *and* $\tilde{t}(t, \eta) = \sum_{l\geq 0} \tilde{t}_{l/2}(t, \eta)\eta^{-l/2}$, *the following relation holds:*

$$\begin{aligned} Q_{(J,m)}(x, t, \eta) = &\left(\frac{\partial\tilde{x}(x, t, \eta)}{\partial x}\right)^2 \tilde{Q}_{\mathrm{I}}(\tilde{x}(x, t, \eta), \tilde{t}(t, \eta), \eta) \\ &- \frac{1}{2}\eta^{-2}\{\tilde{x}(x, t, \eta); x\}, \end{aligned} \tag{5.4.4}$$

where $\tilde{Q}_{\mathrm{I}}$ *designates the potential that appears* (SL_{I}) *and* $\{\tilde{x}; x\}$ *stands for the Schwarzian derivative.*

Proof. Since $\tilde{x}_0(x, t)$ and $\tilde{t}_0(t)$, i.e., the top degree part of the transformation that satisfies (i) $\sim$ (iv), have already been constructed in

[KT5, Section 3.2], it suffices to construct higher degree parts of the transformation. Let us first fix a correspondence among parameters of $(\beta_1(\eta), \beta_2(\eta))$, $(A(\eta), B(\eta))$ and $(\alpha_1, \alpha_2, \ldots, \alpha_{2m})$. Using the assumption (5.1.7) and Lemma 5.3.2, we can fix $(A(\eta), B(\eta))$ and $(\beta_1(\eta), \beta_2(\eta))$ so that

$$\tilde{E}_{\mathrm{I}} = E_{\mathrm{can}} = E^{(j_0)} \tag{5.4.5}$$

holds. Then Lemma 5.3.4 enables us to find $t^{(j_0)}(t, \eta)$ and $t_{\mathrm{I}}(\tilde{t}, \eta)$ so that they satisfy the relations (5.4.6) $\sim$ (5.4.9). Here we note that the corresponding object is denoted by $s(t, \eta)$ in Lemma 5.3.4.

$$\sigma_{\mathrm{can}}(t^{(j_0)}(t, \eta), \eta) = \sigma^{(j_0)}(t, \eta), \tag{5.4.6}$$

$$\rho_{\mathrm{can}}(t^{(j_0)}(t, \eta), \eta) = \rho^{(j_0)}(t, \eta), \tag{5.4.7}$$

$$\sigma_{\mathrm{can}}(t_{\mathrm{I}}(\tilde{t}, \eta), \eta) = \tilde{\sigma}_{\mathrm{I}}(\tilde{t}, \eta), \tag{5.4.8}$$

$$\rho_{\mathrm{can}}(t_{\mathrm{I}}(\tilde{t}, \eta), \eta) = \tilde{\rho}_{\mathrm{I}}(\tilde{t}, \eta). \tag{5.4.9}$$

Here $(\tilde{\sigma}_{\mathrm{I}}, \tilde{\rho}_{\mathrm{I}})$ denotes the infinite series corresponding to $(\sigma^{(j_0)}, \rho^{(j_0)})$ for $(\mathcal{DSL}_{\mathrm{I}})$, where j_0 is uniquely fixed and usually not referred to.

In parallel with the above usage of symbols $t^{(j_0)}(t, \eta)$ and $t_{\mathrm{I}}(\tilde{t}, \eta)$ we let $(x^{(j_0)}(x, t, \eta), t^{(j_0)}(t, \eta))$ and $(x_{\mathrm{I}}(\tilde{x}, \tilde{t}, \eta), t_{\mathrm{I}}(\tilde{t}, \eta))$ denote respectively the transformation with which we find a WKB solution $\psi(x, t, \eta)$ of $(\mathcal{DSL}_J)_m$ from a WKB solution $\varphi_{\mathrm{can}}(z, s, \eta)$ of $(\mathcal{DC}an)$ by defining

$$\psi(x, t, \eta) = \left(\frac{\partial x^{(j_0)}(x, t, \eta)}{\partial x}\right)^{-1/2} \varphi_{\mathrm{can}}\left(x^{(j_0)}(x, t, \eta), t^{(j_0)}(t, \eta), \eta\right) \tag{5.4.10}$$

and the transformation with which we find a WKB solution $\tilde{\psi}_{\mathrm{I}}(\tilde{x}, \tilde{t}, \eta)$ of $(\mathcal{DSL}_{\mathrm{I}})$ from a WKB solution $\varphi_{\mathrm{can}}(z, s, \eta)$ of $(\mathcal{DC}an)$ by defining

$$\tilde{\psi}_{\mathrm{I}}(\tilde{x}, \tilde{t}, \eta) = \left(\frac{\partial x_{\mathrm{I}}(\tilde{x}, \tilde{t}, \eta)}{\partial \tilde{x}}\right)^{-1/2} \varphi_{\mathrm{can}}\left(x_{\mathrm{I}}(\tilde{x}, \tilde{t}, \eta), t_{\mathrm{I}}(\tilde{t}, \eta), \eta\right). \tag{5.4.11}$$

We note that both transformations were designated simply by $(z(x,t,\eta), s(t,\eta))$ in Proposition 5.3.6.

In order to construct the required series $\tilde{x}(x,t,\eta)$ and $\tilde{t}(t,\eta)$ using the above transformations, we first note that (5.3.32) together with (5.4.1) (or, originally, (3.2.6) of [KT5]) entails

$$t_0^{(j_0)}(t) = t_{\mathrm{I},0}(\tilde{t}_0(t)). \tag{5.4.12}$$

This means that $\tilde{t}_0(\sigma)$ is not a P-turning point of (P_{I}); that is,

$$\tilde{t}_0(\sigma) \neq 0. \tag{5.4.13}$$

On the other hand it follows from (5.3.32) that

$$\frac{dt_{\mathrm{I},0}(\tilde{t})}{d\tilde{t}} = \frac{1}{2}\frac{d\tilde{\phi}_{\mathrm{I}}(\tilde{t})}{d\tilde{t}} = \sqrt{3\tilde{\lambda}_{\mathrm{I},0}(\tilde{t})} = \left(-\frac{3\tilde{t}}{2}\right)^{1/4}, \tag{5.4.14}$$

and hence it does not vanish at $\tilde{t}_0(\sigma)$. Therefore $t_{\mathrm{I}}(\tilde{t},\eta)$ is invertible near $\tilde{t}_0(\sigma)$. Similarly (3.6) guarantees that $x_{\mathrm{I}}(\tilde{x},\tilde{t},\eta)$ is also invertible near $(\tilde{x}_0(\lambda_{j_0,0}(\sigma),\sigma),\tilde{t}_0(\sigma)) = (\tilde{\lambda}_{\mathrm{I},0}(\tilde{t}_0(\sigma)),\tilde{t}_0(\sigma))$. Thus the transformation

$$\begin{cases} z = x_{\mathrm{I}}(\tilde{x},\tilde{t},\eta) \\ s = t_{\mathrm{I}}(\tilde{t},\eta) \end{cases} \tag{5.4.15}$$

can be inverted as

$$\begin{cases} \tilde{x} = x_{\mathrm{I}}^{-1}(z,s,\eta) \\ \tilde{t} = t_{\mathrm{I}}^{-1}(s,\eta). \end{cases} \tag{5.4.16}$$

Using this inverse transformation, we now define

$$\begin{cases} \tilde{x}(x,t,\eta) = x_{\mathrm{I}}^{-1}(x^{(j_0)}(x,t,\eta), t^{(j_0)}(t,\eta),\eta) \\ \tilde{t}(t,\eta) = t_{\mathrm{I}}^{-1}(t^{(j_0)}(t,\eta),\eta) \end{cases} \tag{5.4.17}$$

near $(x,t) = (\lambda_{j_0,0}(\sigma),\sigma)$. The relation (5.4.17) may be also expressed

as

$$\begin{cases} x_{\mathrm{I}}(\tilde{x}(x,t,\eta),\tilde{t}(t,\eta),\eta) = x^{(j_0)}(x,t,\eta) \\ t_{\mathrm{I}}(\tilde{t}(t,\eta),\eta) = t^{(j_0)}(t,\eta). \end{cases} \tag{5.4.18}$$

Then (5.3.83) entails

(5.4.19)

$$\begin{aligned} S_{\mathrm{odd}}(x,t,\eta) &= \left(\frac{\partial x^{(j_0)}(x,t,\eta)}{\partial x}\right) S_{\mathrm{can,odd}}\left(x^{(j_0)}(x,t,\eta), t^{(j_0)}(t,\eta),\eta\right) \\ &= \frac{\partial x_{\mathrm{I}}(\tilde{x}(x,t,\eta),\tilde{t}(t,\eta),\eta)}{\partial x} \\ &\quad \times S_{\mathrm{can,odd}}\left(x_{\mathrm{I}}(\tilde{x}(x,t,\eta),\tilde{t}(t,\eta),\eta), t_{\mathrm{I}}(\tilde{t}(t,\eta),\eta),\eta\right) \\ &= \left(\frac{\partial x_{\mathrm{I}}}{\partial \tilde{x}}\right)\left(\tilde{x}(x,t,\eta),\tilde{t}(t,\eta),\eta\right)\frac{\partial \tilde{x}(x,t,\eta)}{\partial x} \\ &\quad \times S_{\mathrm{can,odd}}\left(x_{\mathrm{I}}(\tilde{x}(x,t,\eta),\tilde{t}(t,\eta),\eta), t_{\mathrm{I}}(\tilde{t}(t,\eta),\eta),\eta\right) \\ &= \frac{\partial \tilde{x}(x,t,\eta)}{\partial x}\tilde{S}_{\mathrm{I,odd}}\left(\tilde{x}(x,t,\eta),\tilde{t}(t,\eta),\eta\right). \end{aligned}$$

This relation means that the series $(\tilde{x}(x,t,\eta),\tilde{t}(t,\eta))$ enjoys the required properties near $(x,t) = (\lambda_{j_0,0}(\sigma),\sigma)$. However, it may be singular at $x = a(\sigma)$; our task is to adjust the free parameters that $\tilde{t}(t,\eta)$ contains so that $\tilde{x}_{l/2}(x,t)$ may be holomorphic near $(x,t) = (a(\sigma),\sigma)$. We also note that (5.4.18) enables us to deduce the following relations (5.4.20) and (5.4.21) from (5.4.6) $\sim$ (5.4.9):

$$\tilde{\sigma}_{\mathrm{I}}(\tilde{t}(t,\eta),\eta) = \sigma^{(j_0)}(t,\eta), \tag{5.4.20}$$

$$\tilde{\rho}_{\mathrm{I}}(\tilde{t}(t,\eta),\eta) = \rho^{(j_0)}(t,\eta). \tag{5.4.21}$$

To find an appropriate way of fixing the free parameters contained in $\tilde{t}(t,\eta)$ we first construct a transformation

$$\tilde{y}(x,t,\eta)=\sum_{l\geq 0}\tilde{y}_{l/2}(x,t,\eta)\eta^{-l/2} \tag{5.4.22}$$

that brings $(SL_J)_m$ to $(SL_{\mathrm{I}})_1$ near the simple turning point $x=a(\sigma)$. Contrary to the situation near $x=\lambda_{j_0,0}(\sigma)$, $Q_{(J,m)}$ is non-singular near $x=a(\sigma)$. Hence the reasoning in [KT4, Section 2] readily applies to our situation, and we can construct the series $\tilde{y}(x,t,\eta)$ that satisfies

$$S_{(J,m),\mathrm{odd}}(x,t,\eta)=\frac{\partial\tilde{y}(x,t,\eta)}{\partial x}\tilde{S}_{\mathrm{I,odd}}(\tilde{y}(x,t,\eta),\tilde{t}(t,\eta),\eta) \tag{5.4.23}$$

near $(x,t)=(a(\sigma),\sigma)$. In the course of the construction of $\tilde{y}$, one finds that $\tilde{y}_{l/2}$ is a polynomial of instantons of degree equal to or at most $l-2$ (cf. Appendix B).

For the computation required for the adjustment of the constants we prepare following series:

$$R(\tilde{x},\tilde{t},\eta)\underset{\mathrm{def}}{=}\int_{-2\tilde{\lambda}_{\mathrm{I},0}(\tilde{t})}^{\tilde{x}}\eta^{-1}\tilde{S}_{\mathrm{I,odd}}(w,\tilde{t},\eta)dw, \tag{5.4.24}$$

$$\mathcal{F}(x,t,\eta)\underset{\mathrm{def}}{=}R(\tilde{x}(x,t,\eta),\tilde{t}(t,\eta),\eta), \tag{5.4.25}$$

$$\mathcal{G}(x,t,\eta)\underset{\mathrm{def}}{=}R(\tilde{y}(x,t,\eta),\tilde{t}(t,\eta),\eta). \tag{5.4.26}$$

It then follows from (5.4.19) and (5.4.23) that

$$\frac{\partial\mathcal{F}}{\partial x}=\eta^{-1}S_{(J,m),\mathrm{odd}}(x,t,\eta) \tag{5.4.27}$$

and

$$\frac{\partial\mathcal{G}}{\partial x}=\eta^{-1}S_{(J,m),\mathrm{odd}}(x,t,\eta). \tag{5.4.28}$$

Hence we find

$$\frac{\partial}{\partial x}(\mathcal{F}-\mathcal{G})=0. \tag{5.4.29}$$

Next we try to prove

$$\frac{\partial}{\partial t}(\mathcal{F}-\mathcal{G})=0. \tag{5.4.30}$$

In order to prove (5.4.30), we prepare the following

Sublemma 5.4.2. *For the functions $\mathfrak{a}_{(J,m)}$ and $\tilde{\mathfrak{a}}_{(\mathrm{I},1)}$ that appear in the deformation equation, we find*

$$\frac{\partial\tilde{x}(x,t,\eta)}{\partial t} \tag{5.4.31}$$

$$=\mathfrak{a}_{(J,m)}(x,t)\frac{\partial\tilde{x}(x,t,\eta)}{\partial x}-\tilde{\mathfrak{a}}_{(\mathrm{I},1)}(\tilde{x}(x,t,\eta),\tilde{t}(t,\eta))\frac{\partial\tilde{t}(t,\eta)}{\partial t}.$$

Proof of Sublemma 5.4.2. Since we have confirmed (5.3.90) in the course of the proof of Proposition 5.3.6, we have

$$\mathfrak{a}_{(J,m)}(x,t)\frac{\partial x^{(j_0)}}{\partial x}-\frac{\partial x^{(j_0)}}{\partial t}-A_{\mathrm{can}}(x^{(j_0)}(x,t,\eta),t^{(j_0)}(t,\eta))\frac{\partial t^{(j_0)}}{\partial t}=0 \tag{5.4.32}$$

and

$$\tilde{\mathfrak{a}}_{(\mathrm{I}.1)}(\tilde{x},\tilde{t})\frac{\partial x_{\mathrm{I}}}{\partial\tilde{x}}-\frac{\partial x_{\mathrm{I}}}{\partial\tilde{t}}-A_{\mathrm{can}}(x_{\mathrm{I}}(\tilde{x},\tilde{t},\eta),t_{\mathrm{I}}(\tilde{t},\eta))\frac{\partial t_{\mathrm{I}}}{\partial\tilde{t}}=0. \tag{5.4.33}$$

On the other hand, we differentiate the relation (5.4.18) with respect to x and t to find

$$\left.\frac{\partial x_{\mathrm{I}}}{\partial\tilde{x}}\right|_{\substack{\tilde{x}=\tilde{x}(x,t,\eta)\\ \tilde{t}=\tilde{t}(t,\eta)}}\frac{\partial\tilde{x}}{\partial x}=\frac{\partial x^{(j_0)}}{\partial x}, \tag{5.4.34}$$

$$\left.\frac{\partial x_{\mathrm{I}}}{\partial\tilde{x}}\right|_{\substack{\tilde{x}=\tilde{x}(x,t,\eta)\\ \tilde{t}=\tilde{t}(t,\eta)}}\frac{\partial\tilde{x}}{\partial t}+\left.\frac{\partial x_{\mathrm{I}}}{\partial\tilde{t}}\right|_{\substack{\tilde{x}=\tilde{x}(x,t,\eta)\\ \tilde{t}=\tilde{t}(t,\eta)}}\frac{\partial\tilde{t}}{\partial t}=\frac{\partial x^{(j_0)}}{\partial t} \tag{5.4.35}$$

and

$$\left.\frac{\partial t_{\mathrm{I}}}{\partial \tilde{t}}\right|_{\tilde{t}=\tilde{t}(t,\eta)} \frac{\partial \tilde{t}}{\partial t} = \frac{\partial t^{(j_0)}}{\partial t}. \tag{5.4.36}$$

Substituting (5.4.32) and (5.4.33) into (5.4.35), we obtain

(5.4.37)

$$\begin{aligned}
&\left.\frac{\partial x_{\mathrm{I}}}{\partial \tilde{x}}\right|_{\substack{\tilde{x}=\tilde{x}(x,t,\eta)\\ \tilde{t}=\tilde{t}(t,\eta)}} \frac{\partial \tilde{x}}{\partial t} + \left[\tilde{\mathfrak{a}}_{(\mathrm{I}.1)}(\tilde{x},\tilde{t})\frac{\partial x_{\mathrm{I}}(\tilde{x},\tilde{t},\eta)}{\partial \tilde{x}}\right. \\
&\qquad \left. - A_{\mathrm{can}}(x_{\mathrm{I}}(\tilde{x},\tilde{t},\eta),t_{\mathrm{I}}(\tilde{t},\eta))\frac{\partial t_{\mathrm{I}}(\tilde{t},\eta)}{\partial \tilde{t}}\right]\Bigg|_{\substack{\tilde{x}=\tilde{x}(x,t,\eta)\\ \tilde{t}=\tilde{t}(t,\eta)}} \frac{\partial \tilde{t}(t,\eta)}{\partial t} \\
&= \mathfrak{a}_{(J,m)}(x,t)\frac{\partial x^{(j_0)}}{\partial x} - A_{\mathrm{can}}(x^{(j_0)}(x,t,\eta),t^{(j_0)}(t,\eta))\frac{\partial t^{(j_0)}(t,\eta)}{\partial t}.
\end{aligned}$$

Since (5.4.18) entails

(5.4.38)

$$\begin{aligned}
&A_{\mathrm{can}}(x_{\mathrm{I}}(\tilde{x}(x,t,\eta),\tilde{t}(t,\eta),\eta),t_{\mathrm{I}}(\tilde{t}(t,\eta),\eta))\frac{\partial t_{\mathrm{I}}(\tilde{t}(t,\eta),\eta)}{\partial \tilde{t}}\,\frac{\partial \tilde{t}(t,\eta)}{\partial t} \\
&= A_{\mathrm{can}}(x^{(j_0)}(x,t,\eta),t^{(j_0)}(t,\eta))\frac{\partial t^{(j_0)}(t,\eta)}{\partial t},
\end{aligned}$$

(5.4.36) and (5.4.37) imply

$$\frac{\partial x_{\mathrm{I}}}{\partial \tilde{x}}\left(\frac{\partial \tilde{x}}{\partial t} + \tilde{\mathfrak{a}}_{(\mathrm{I}.1)}(\tilde{x},\tilde{t})\frac{\partial \tilde{t}}{\partial t}\right)\Bigg|_{\substack{\tilde{x}=\tilde{x}(x,t,\eta)\\ \tilde{t}=\tilde{t}(t,\eta)}} = \mathfrak{a}_{(J,m)}(x,t)\frac{\partial x^{(j_0)}}{\partial x}. \tag{5.4.39}$$

It then follows from (5.4.34) that

(5.4.40)
$$\frac{\partial\tilde{x}}{\partial t}+\tilde{\mathfrak{a}}_{(\mathrm{I},1)}(\tilde{x}(x,t,\eta),\tilde{t}(t,\eta))\frac{\partial\tilde{t}(t,\eta)}{\partial t}=\mathfrak{a}_{(J,m)}(x,t)\frac{\partial\tilde{x}(x,t,\eta)}{\partial x}.$$

Thus we have obtained (5.4.31). □

Remark 5.4.1. An important point in Sublemma 5.4.2 is that we can confirm (5.4.31) despite the fact that $\tilde{x}(x,t,\eta)$ may be singular at the simple turning point $x=a(t)$. As [KT1, Proposition 2.2] shows, (5.4.31) rather straightforwardly follows from (5.3.3), the deformation equation for the odd part of a solution of the Riccati equation, if the transformation series involved is defined near a simple turning point.

We now resume the proof of Theorem 5.4.1. Using Sublemma 5.4.2, we find

(5.4.41)
$$\begin{aligned}\frac{\partial\mathcal{F}}{\partial t}&=\frac{\partial R}{\partial\tilde{x}}\frac{\partial\tilde{x}}{\partial t}+\frac{\partial R}{\partial\tilde{t}}\frac{\partial\tilde{t}}{\partial t}\\&=\eta^{-1}\tilde{S}_{\mathrm{I,odd}}\left(\mathfrak{a}_{(J,m)}\frac{\partial\tilde{x}}{\partial x}-\tilde{\mathfrak{a}}_{(\mathrm{I},1)}\frac{\partial\tilde{t}}{\partial t}\right)-\frac{\partial R}{\partial\tilde{t}}\frac{\partial\tilde{t}}{\partial t}.\end{aligned}$$

On the other hand, the deformation equation (5.3.3) applied to $\tilde{S}_{\mathrm{I,odd}}$ implies

(5.4.42)
$$\frac{\partial R}{\partial\tilde{t}}=\eta^{-1}\tilde{\mathfrak{a}}_{(\mathrm{I},1)}(\tilde{x},\tilde{t})\tilde{S}_{\mathrm{I,odd}}(\tilde{x},\tilde{t},\eta).$$

Hence it follows from (5.4.19) that

(5.4.43)
$$\frac{\partial\mathcal{F}}{\partial t}=\eta^{-1}\mathfrak{a}_{(J,m)}S_{(J,m),\mathrm{odd}}.$$

As we noted in Remark 5.4.1, (5.4.31) is valid if we replace $x(\tilde{x},\tilde{t},\eta)$ by $\tilde{y}(x,t,\eta)$. This means that the above computation of $\partial\mathcal{F}/\partial t$ is equally applicable to the computation of $\partial\mathcal{G}/\partial t$, that is, we find

(5.4.44)
$$\frac{\partial\mathcal{G}}{\partial t}=\eta^{-1}\mathfrak{a}_{(J,m)}S_{(J,m),\mathrm{odd}}.$$

Therefore we obtain (5.4.30).

It then follows from (5.4.29) and (5.4.30) that

$$\mathcal{F} - \mathcal{G} = \sum_{l \geq 0} C_{l/2} \eta^{-l/2} \tag{5.4.45}$$

holds with a genuine constant $C_{l/2}$.

Let us now prove the following assertion $(\mathcal{C})_l$ for any l by the induction on l.

$(\mathcal{C})_l$ *An appropriate choice of* $t_{l/2}$ *guarantees the vanishing of* $C_{l/2}$ *and the coincidence of* $x_{l/2}$ *and* $y_{l/2}$.

It follows from the definition of x_0 and t_0 that $(\mathcal{C})_0$ holds. Since $x_{1/2} = y_{1/2} = 0$ and $t_{1/2} = 0$, $(\mathcal{C})_1$ is trivially valid. Let us now suppose $(\mathcal{C})_k$ holds for every $k < l$ and that l is even. Let $\mathcal{H}(l/2)$ denote the part of $(\mathcal{F}_{l/2} - \mathcal{G}_{l/2})$ which is irrelevant to $\tilde{S}_{\mathrm{I},-1}$; that is,

$$\mathcal{H}(l/2) = \mathcal{F}_{l/2} - \mathcal{G}_{l/2} - \left[\int_{-2\tilde{\lambda}_{\mathrm{I},0}(\tilde{t}_0(t))}^{\tilde{x}(x,t,\eta)} \tilde{S}_{\mathrm{I},-1}(w, \tilde{t}(t,\eta)) dw \right. \tag{5.4.46}$$

$$\left. - \int_{-2\tilde{\lambda}_{\mathrm{I},0}(\tilde{t}_0(t))}^{\tilde{y}(x,t,\eta)} \tilde{S}_{\mathrm{I},-1}(w, \tilde{t}(t,\eta)) dw \right]_{l/2}.$$

Thus $\mathcal{H}(l/2)$ consists of terms originating from $\tilde{S}_{\mathrm{I},j}$ $(j \geq 0)$. On the other hand each term in $\mathcal{H}(l/2)$ is of degree $(-l/2)$ in η by its definition. Hence only $\tilde{x}_{l'/2}$ and $\tilde{y}_{l''/2}$ $(l', l'' \leq l-1)$ are relevant to $\mathcal{H}(l/2)$. Then the induction hypothesis implies that $\mathcal{H}(l/2)$ should vanish. Thus we concentrate our attention to the terms relevant to $\tilde{S}_{\mathrm{I},-1}$. Then by using the Taylor expansion we find

$$\int_{-2\tilde{\lambda}_{\mathrm{I},0}(\tilde{t})}^{\tilde{x}(x,t,\eta)} \tilde{S}_{\mathrm{I},-1}(w, \tilde{t}) dw - \int_{-2\tilde{\lambda}_{\mathrm{I},0}(\tilde{t})}^{\tilde{y}(x,t,\eta)} \tilde{S}_{\mathrm{I},-1}(w, \tilde{t}) dw \tag{5.4.47}$$

$$
\begin{aligned}
&= \tilde{S}_{\mathrm{I},-1}(\tilde{x}_0,\tilde{t})\big\{(\eta^{-1}\tilde{x}_1+\cdots+\eta^{-l/2}\tilde{x}_{l/2}+\cdots)\\
&\qquad -(\eta^{-1}\tilde{y}_1+\cdots+\eta^{-l/2}\tilde{y}_{l/2}+\cdots)\big\}\\
&\quad+\frac{1}{2!}\frac{\partial\tilde{S}_{\mathrm{I},-1}}{\partial\tilde{x}}(\tilde{x}_0,\tilde{t})\left\{(\eta^{-1}\tilde{x}_1+\cdots)^2-(\eta^{-1}\tilde{y}_1+\cdots)^2\right\}\\
&\quad+\cdots .
\end{aligned}
$$

Hence terms that contain $\tilde{x}_{l/2}$ or $\tilde{y}_{l/2}$ and that contributed to $\mathcal{F}_{l/2}-\mathcal{G}_{l/2}$ are

$$
\tilde{S}_{\mathrm{I},-1}(\tilde{x}_0,\tilde{t}_0)(\tilde{x}_{l/2}-\tilde{y}_{l/2})\eta^{-l/2}. \tag{5.4.48}
$$

Recalling the concrete form of $\tilde{S}_{\mathrm{I},-1}$, we obtain

(5.4.49)

$$
\begin{aligned}
&C_{l/2}\\
&=2\sqrt{\tilde{x}_0+2\tilde{\lambda}_{\mathrm{I},0}(\tilde{t}_0(t))}\,\big(\tilde{x}_0-\tilde{\lambda}_{\mathrm{I},0}(\tilde{t}_0(t))\big)\big(\tilde{x}_{l/2}(x,t,\eta)-\tilde{y}_{l/2}(x,t,\eta)\big).
\end{aligned}
$$

We now apply the same reasoning to (5.4.19). The degree $-(l-2)/2$ in η part in the last term in (5.4.19) is then seen to be of the following form:

$$
\begin{aligned}
&\tilde{S}_{\mathrm{I},-1}(\tilde{x}_0,\tilde{t}_0)\frac{\partial\tilde{x}_{l/2}}{\partial x}\\
&\quad+\frac{\partial\tilde{x}_0}{\partial x}\frac{\partial\tilde{S}_{\mathrm{I},-1}}{\partial\tilde{x}}(\tilde{x}_0(x,t),\tilde{t}_0(t))\tilde{x}_{l/2}\\
&\quad+\frac{\partial\tilde{x}_0}{\partial x}\frac{\partial\tilde{S}_{\mathrm{I},-1}}{\partial\tilde{t}}(\tilde{x}_0(x,t),\tilde{t}_0(t))\tilde{t}_{l/2}\\
&\quad+R_{l/2}\left(\tilde{x}_0,\cdots,\tilde{x}_{(l-1)/2},\tilde{t}_0,\cdots,\tilde{t}_{(l-1)/2}\right),
\end{aligned} \tag{5.4.50}
$$

where $R_{l/2}$ is determined by $\tilde{x}_{l'/2}$ and $\tilde{t}_{l''/2}$ with $l', l'' \leq l-1$. Then the top degree part of the deformation equation (5.3.3) applied to $\tilde{S}_{\mathrm{I,odd}}$ entails

$$\frac{\partial \tilde{S}_{\mathrm{I},-1}}{\partial \tilde{t}} = \frac{\partial}{\partial \tilde{x}} \left(\sqrt{\tilde{x} + 2\tilde{\lambda}_{\mathrm{I},0}(\tilde{t})} \right). \tag{5.4.51}$$

Thus it follows from (5.4.19) that we find the following:

$$\begin{aligned} &\frac{\partial}{\partial x} \left(2\sqrt{\tilde{x}_0(x,t) + 2\tilde{\lambda}_{\mathrm{I},0}(\tilde{t}_0(t))}(\tilde{x}_0(x,t) - \tilde{\lambda}_{\mathrm{I},0}(\tilde{t}_0(t)))\tilde{x}_{l/2} \right) \\ &+ \left(\frac{\partial}{\partial x} \left(\sqrt{\tilde{x}_0(x,t) + 2\tilde{\lambda}_{\mathrm{I},0}(\tilde{t}_0(t))} \right) \tilde{t}_{l/2} \right) + R_{l/2} \\ &= S_{(J,m),(l-2)/2}(x,t,\eta). \end{aligned} \tag{5.4.52}$$

Exactly the same reasoning applied to (5.4.23) entails the following:

$$\begin{aligned} &\frac{\partial}{\partial x} \left(2\sqrt{\tilde{y}_0(x,t) + 2\tilde{\lambda}_{\mathrm{I},0}(\tilde{t}_0(t))}(\tilde{y}_0(x,t) - \tilde{\lambda}_{\mathrm{I},0}(\tilde{t}_0(t)))\tilde{y}_{l/2} \right) \\ &+ \left(\frac{\partial}{\partial x} \left(\sqrt{\tilde{y}_0(x,t) + 2\tilde{\lambda}_{\mathrm{I},0}(\tilde{t}_0(t))} \right) \tilde{t}_{l/2} \right) \\ &+ R_{l/2}(\tilde{y}_0, \cdots, \tilde{y}_{(l-1)/2}, \tilde{t}_0, \cdots, \tilde{t}_{(l-1)/2}) \\ &= S_{(J,m),(l-2)/2}(x,t,\eta). \end{aligned} \tag{5.4.53}$$

Since $x_{l/2}$ is non-singular at $x = \lambda_{j_0,0}(t)$, comparison of (5.4.52) and (5.4.53) with the help of the induction hypothesis entails that $\tilde{y}_{l/2}$ has an at most simple pole near $x = \lambda_{j_0,0}(t)$. That is,

$$\begin{aligned} \tilde{y}_{l/2} =& \frac{d_{l/2}(t,\eta) - \tilde{t}_{l/2}(t,\eta)}{2(\tilde{x}_0(x.t) - \tilde{\lambda}_{\mathrm{I},0}(\tilde{t}_0(t)))} \\ &+ (\text{non-singular function near } x = \lambda_{j_0,0}(t)), \end{aligned} \tag{5.4.54}$$

where $d_{l/2}$ is determined by $\tilde{x}_{l'/2}$ and $\tilde{t}_{l''/2}$ with l', $l'' \leq (l-1)$. Substituting (5.4.54) into (5.4.49) and evaluating the resulting function at $x = \lambda_{j_0,0}(t)$, we find

$$\sqrt{3\tilde{\lambda}_{\mathrm{I},0}(\tilde{t}_0(t))}(\tilde{t}_{l/2}(t,\eta) - d_{l/2}(t,\eta)) = C_{l/2}. \tag{5.4.55}$$

Since $\tilde{t}_{l/2}(t,\eta)$ contains a free parameter originating from the arbitrary parameter $(A_{l/2}, B_{l/2})$ contained in $(\rho_{\mathrm{can}}, \sigma_{\mathrm{can}})$, we can choose some point t_* at which $C_{l/2}$ in (5.4.55) vanishes. Then it follows from (5.4.49) that $\tilde{x}_{l/2} = \tilde{y}_{l/2}$ holds. Since l is even by the assumption, $l+1$ is odd. This means $C_{(l+1)/2}$ is a sum of monomials of instantons of odd degree. But then it should vanish to become a constant. Therefore we find $\tilde{x}_{(l+1)/2} = \tilde{y}_{(l+1)/2}$. Thus the induction proceeds, completing the proof of Theorem 5.4.1. □

The semi-global transformation $(\tilde{x}(x,t,\eta), \tilde{t}(t,\eta))$ found in Theorem 5.4.1 is the required one in Theorem 5.1.1. Actually Sublemma 5.4.2 entails

$$\begin{aligned} &2(\tilde{x}(x,t,\eta) - \tilde{\lambda}_{\mathrm{I}}(\tilde{t}(t,\eta),\eta))\frac{\partial\tilde{x}(x,t,\eta)}{\partial t} + \frac{\partial\tilde{t}(t,\eta)}{\partial t} \\ &\quad = 2\mathfrak{b}^{(j_0)}_{(J,m)}(x,t)\left(\frac{\tilde{x}(x,t,\eta) - \tilde{\lambda}_{\mathrm{I}}(\tilde{t}(t,\eta),\eta)}{x - \lambda_{j_0}(t,\eta)}\right)\frac{\partial\tilde{x}(x,t,\eta)}{\partial x}, \end{aligned} \tag{5.4.56}$$

as

$$\tilde{\mathfrak{a}}_{(\mathrm{I},1)} = \frac{1}{2(\tilde{x} - \tilde{\lambda}_{\mathrm{I}}(\tilde{t},\eta))}. \tag{5.4.57}$$

Since the left-hand side of (5.4.56) is non-singular at $x = \lambda_{j_0}(t,\eta)$, we find

$$\tilde{x}(\lambda_{j_0}(t,\eta),t,\eta) = \tilde{\lambda}_{\mathrm{I}}(\tilde{t}(t,\eta),\eta). \tag{5.4.58}$$

This is the required relation (5.1.9). Thus Theorem 5.4.1 together with Sublemma 5.4.2 proves Theorem 5.1.1.

Appendix A: Basic properties of the (P_J)-hierarchy ($J = $ I, 34, II-2 or IV) with a large parameter η

For the convenience of the reader we list up the symbols and equations we use in this paper. We basically follow the notations used in [Ko2], and as its consequence the symbols used here sometimes differ slightly from those used in [KKoNT]. For example, the differential polynomial $\mathcal{F}_j(c)$ of u given by (A.1.2) below corresponds to $\mathcal{G}_j$ in [KKoNT], the constant 2γ in (A.1.1) below is designated by g in [KKoNT], and so on. In this appendix we confine our attention to the notational aspect of the problem, and we refer the reader to [Ko2] concerning the theoretical issues such as the equivalence of two expressions etc.

A.1 Definition of $(P_{\mathrm{I}})_m$ and $(\tilde{P}_{\mathrm{I}})_m$

As is discussed in [KKoNT, Appendix B], P_{I}-hierarchy can be expressed in two different but equivalent ways. Here, and in what follows, we use the symbol $(P_{\mathrm{I}})_m$ to denote the following equation:

$$\mathcal{F}_{m+1}(c) + 2\gamma t = 0, \tag{A.1.1}$$

where $\mathcal{F}_j(c)$ is, by definition,

$$\mathcal{F}_j(c) = \sum_{k=0}^{m} c_k F_{j-k} \quad (\text{with } c_0 = 1). \tag{A.1.2}$$

Here F_l's denote appropriately normalized Gel'fand-Dickey polynomials with a large parameter η; they are polynomials of the (unknown) function u and its derivatives, and they satisfy

$$\frac{dF_{l+1}}{dt} = \eta^{-2}\frac{d^3F_l}{dt^3} + 4u\frac{dF_l}{dt} + 2\frac{du}{dt}F_l \tag{A.1.3}$$

with

$$F_0 = 1/2. \tag{A.1.4}$$

We note that they are normalized as follows:

(A.1.5) $$F_1 = u,$$

(A.1.6) $$F_2 = 3u^2 + \eta^{-2}\frac{d^2u}{dt^2},$$

(A.1.7) $$F_3 = 10u^3 + \eta^{-2}\left(10u\frac{d^2u}{dt^2} + 5\left(\frac{du}{dt}\right)^2\right) + \eta^{-4}\frac{d^4u}{dt^4},$$

and so on. In the equation (A.1.1) we normally assume

(A.1.8) $$c_1 = c_{m+1} = 0$$

by adding appropriate constants to u and t respectively. In practice, we usually abbreviate $\mathcal{F}_{m+1}(c)$ to $\mathcal{F}_m$.

Another expression of P_{I}-hierarchy given below is denoted by $(\tilde{P}_{\mathrm{I}})_m$, which is denoted by $(P_{\mathrm{I}})_m$ in [KKoNT].

(A.1.9)

$$(\tilde{P}_{\mathrm{I}})_m : \quad \begin{cases} \eta^{-1}\dfrac{du_j}{dt} = 2v_j & (j = 1, 2, \ldots, m) \\ \eta^{-1}\dfrac{dv_j}{dt} = 2(u_{j+1} + u_1u_j + w_j) & (j = 1, 2, \ldots, m), \end{cases}$$

where

(A.1.10) $$u_{m+1} = \tilde{\gamma}t$$

and

(A.1.11) $$w_j = \frac{1}{2}\sum_{k=1}^{j} u_k u_{j-k+1} + \sum_{k=1}^{j-1} u_k w_{j-k} - \frac{1}{2}\sum_{k=1}^{j-1} v_k v_{j-k} + \tilde{c}_0(2u_j - \sum_{k=1}^{j-1} u_k u_{j-k}) + \tilde{c}_j.$$

A.2 Correspondence between $(P_{\rm I})_m$ and $(\tilde{P}_{\rm I})_m$

For a solution u of $(P_{\rm I})_m$, the equation $(\tilde{P}_{\rm I})_m$ is satisfied by $(u_j, v_j)_{1\le j\le m}$ defined by (A.2.1) through $\mathcal{F}_j$ given in (A.1.2) if constants $\tilde{\gamma}$ and $\tilde{c}_j$ $(0 \le j \le m)$ are chosen to satisfy (A.2.2) below.

$$u_j = -2^{-2j+1}\mathcal{F}_j,\ v_j = -2^{-2j}\eta^{-1}\frac{d\mathcal{F}_j}{dt} \quad (1 \le j \le m), \tag{A.2.1}$$

$$\tilde{\gamma} = 4^{-m}\gamma,\ \tilde{c}_j = 2^{-2j-3}\sum_{k=0}^{j+1} c_{j-k+1}c_k \quad (0 \le j \le m). \tag{A.2.2}$$

Conversely if $(u_j, v_j)_{1\le j\le m}$ is a solution of $(\tilde{P}_{\rm I})_m$,

$$u = -2(u_1 + \tilde{c}_0) \tag{A.2.3}$$

is a solution of $(P_{\rm I})_m$ on the condition that the relation (A.2.2) is satisfied.

A.3 Lax pairs $(L_{\rm I})_m$ for $(P_{\rm I})_m$ and $(\tilde{L}_{\rm I})_m$ for $(\tilde{P}_{\rm I})_m$

The m-th member of $(P_{\rm I})$-hierarchy is the compatibility condition of the following system $(L_{\rm I})_m$ of linear differential equations, which we call the Lax pair underlying $(P_{\rm I})_m$.

$$(L_{\rm I})_m : \begin{cases} 2\eta^{-1}\gamma\dfrac{\partial\vec{\psi}}{\partial x} = A\vec{\psi} & \text{(A.3.1.a)} \\ \eta^{-1}\dfrac{\partial\vec{\psi}}{\partial t} = B\vec{\psi} & \text{(A.3.1.b)} \end{cases} \tag{A.3.1}$$

with

$$A = \begin{pmatrix} -\eta^{-1}\dfrac{\partial\mathcal{F}}{\partial t} & 2\mathcal{F} \\ -\eta^{-2}\dfrac{\partial^2\mathcal{F}}{\partial t^2} + 2(x-u)\mathcal{F} & \eta^{-1}\frac{\partial\mathcal{F}}{\partial t} \end{pmatrix} + N \tag{A.3.2}$$

(A.3.3) $$B = \begin{pmatrix} 0 & 1 \\ x-u & 0 \end{pmatrix},$$

where

(A.3.4) $$\mathcal{F} = \sum_{j=0}^{m} (4x)^{m-j} \mathcal{F}_j$$

and

(A.3.5) $$N = \begin{pmatrix} 0 & 0 \\ -(\mathcal{F}_{m+1} + 2\gamma t) & 0 \end{pmatrix}.$$

Remark A.3.1. The Lax pair (A.3.1) is slightly different from that given in [GP]. (See [KKoNT, Appendix B] also.) We add a matrix N to A of [GP] in such a way that the compatibility condition of (A.3.1) exactly becomes $\mathcal{F}_{m+1} + 2\gamma t = 0$ instead of $(d/dt)(\mathcal{F}_{m+1} + 2\gamma t) = 0$. For the role of the matrix N see also Remark A.7.1 below.

On the other hand, the Lax pair $(\tilde{L}_{\mathrm{I}})_m$ that underlies $(\tilde{P}_{\mathrm{I}})_m$ is given as follows:

(A.3.6) $$(\tilde{L}_{\mathrm{I}})_m : \begin{cases} \eta^{-1}\tilde{\gamma}\dfrac{\partial \vec{\psi}}{\partial x} = \tilde{A}\vec{\psi} & \text{(A.3.6.a)} \\ \eta^{-1}\dfrac{\partial \vec{\psi}}{\partial t} = \tilde{B}\vec{\psi} & \text{(A.3.6.b),} \end{cases}$$

where

(A.3.7)
$$\tilde{A} = \begin{pmatrix} V(x)/2 & U(x) \\ \big(2x^{m+1} - (x - 2\tilde{c}_0)U(x) + 2W(x) + 2\tilde{\gamma}t\big)/4 & -V(x)/2 \end{pmatrix},$$

(A.3.8)
$$\tilde{B} = \begin{pmatrix} 0 & 2 \\ u_1 + x/2 + \tilde{c}_0 & 0 \end{pmatrix}$$

with

$$U(x) = x^m - \sum_{j=1}^{m} u_j x^{m-j}, \tag{A.3.9}$$

$$V(x) = \sum_{j=1}^{m} v_j x^{m-j}, \tag{A.3.10}$$

$$W(x) = \sum_{j=1}^{m} w_j x^{m-j}. \tag{A.3.11}$$

Note that the relation (A.2.1) entails that the $(1,2)$-component of the matrix A is the same as 4^m multiple of that of $\tilde{A}$.

To avoid some numerical complexity in the description of the associated Hamiltonian (to be given in Subsection A.4 below) we assume in this paper that $\tilde{\gamma} = 1/2$; this means that we can choose the constant Θ in [Ko2] to be 1. This choice of the constant $\tilde{\gamma}$ causes, however, a tiny difference between the Lax pair used here and that used in [KKoNT].

A.4 The Hamiltonian structure of $(\tilde{P}_{\mathrm{I}})_m$ ([Ko2, Theorem 1.10])

Let us choose $(\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_m)$ so that they satisfy

$$U(x) = \prod_{j=1}^{m} (x - \lambda_j) \tag{A.4.1}$$

and

$$\mu_j = V(\lambda_j) \quad (1 \leq j \leq m) \tag{A.4.2}$$

for $U(x)$ and $V(x)$ defined respectively by (A.3.9) and (A.3.10). By letting N_j $(1 \leq j \leq m)$ denote

$$\prod_{\substack{k=1,2,\ldots,m \\ k \neq j}} (\lambda_j - \lambda_k)^{-1}, \tag{A.4.3}$$

we define

$$H_1 = \sum_{j=1}^{m} N_j(\mu_j^2 - (\lambda_j^{2m+1} + \sum_{k=1}^{m} t_k \lambda_j^{m+k-1})) \tag{A.4.4}$$

for complex numbers $\{t_k\}_{k=1}^m$. It is known ([Ko2]) that H_1 is a Hamiltonian for the degenerate Garnier system $G(m+5/2; m)$.

Set

$$t_1 = t + 2\tilde{c}_m \tag{A.4.5}$$

and

$$t_j = 2\tilde{c}_{m-j+1} \quad (2 \leq j \leq m) \tag{A.4.6}$$

in H_1, and let K denote the resulting function of $(t; \lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_m)$. Then the system $(\tilde{P}_{\mathrm{I}})_m$ is equivalent to the following Hamiltonian system $(G_{\mathrm{I}})_m$:

$$(G_{\mathrm{I}})_m : \begin{cases} \dfrac{d\lambda_j}{dt} = \eta \dfrac{\partial K}{\partial \mu_j} & (j = 1, 2, \ldots, m) \\ \dfrac{d\mu_j}{dt} = -\eta \dfrac{\partial K}{\partial \lambda_j} & (j = 1, 2, \ldots, m). \end{cases} \tag{A.4.7}$$

A.5 Definition of $(P_{34})_m$ and $(\tilde{P}_{34})_m$

In view of the results in [CJP] we study (P_{34})-hierarchy as an equivalent substitute of $(P_{\text{II-1}})$-hierarchy in [KKoNT]. One advantage in studying (P_{34})-hierarchy is its intimate connection with (P_{I})-hierarchy; a similar connection is also observed between (P_{IV})-hierarchy and $(P_{\text{II-2}})$-hierarchy discussed in the subsequent subsections. We note that the awkward naming (P_{34})-hierarchy is due to its relevance to the equation numbered XXXIV that appears in the classical study of the Painlevé property ([I, p.340]).

The m-th member of (P_{34})-hierarchy is, by definition,

$$(A.5.1)\qquad (P_{34})_m: \quad 2\eta^{-2}(\mathcal{F}_m+2\gamma t)\frac{d^2\mathcal{F}_m}{dt^2}-\eta^{-2}\left(\frac{d\mathcal{F}_m}{dt}+2\gamma\right)^2 + 4u(\mathcal{F}_m+2\gamma t)^2+\kappa^2=0,$$

where η is a large parameter, γ ($\neq 0$) and κ are constants and $\mathcal{F}_m = \mathcal{F}_m(c)$ is a sum of Gel'fand-Dickey polynomials given in (A.1.2). In what follows we use the symbol I_m to denote the left-hand side of (A.5.1), i.e.,

$$(A.5.2)\qquad I_m = 2\eta^{-2}(\mathcal{F}_m+2\gamma t)\frac{d^2\mathcal{F}_m}{dt^2}-\eta^{-2}\left(\frac{d\mathcal{F}_m}{dt}+2\gamma\right)^2 + 4u(\mathcal{F}_m+2\gamma t)^2+\kappa^2.$$

Another expression of (P_{34})-hierarchy is given by $(\tilde{P}_{34})_m$ below ([Ko2, Theorem 2.3]).

(A.5.3)

$$(\tilde{P}_{34})_m: \begin{cases} \eta^{-1}\dfrac{du_j}{dt}=2v_j & (1\le j\le m)\\ \eta^{-1}\dfrac{dv_j}{dt}=2(u_1u_j+u_{j+1}+w_j) & (1\le j\le m)\\ u_{m+1}=-w_m+\tilde{c}_0u_m-\tilde{\gamma}t(u_1+\tilde{c}_0)+\dfrac{(v_m-\eta^{-1}\tilde{\gamma}/2)^2-\tilde{\kappa}^2}{2(u_m-\tilde{\gamma}t)}, \end{cases}$$

where w_j is a polynomial of $(u_l, v_{l'})_{1\le l,l'\le j}$ which is recursively determined through (A.1.11) containing constants $\tilde{c}_j$, and $\tilde{c}_0$, $\tilde{\gamma}$ and $\tilde{\kappa}$ are also constants.

A.6 Correspondence between $(P_{34})_m$ and $(\tilde{P}_{34})_m$ ([Ko2, Theorem 2.3])

For a solution u of $(P_{34})_m$, the equation $(\tilde{P}_{34})_m$ is satisfied by $(u_j, v_j)_{1\le j\le m}$ defined by (A.2.1) with the help of $\mathcal{F}_j$ if constants $\tilde{\gamma}$, $\tilde{\kappa}$ and $\tilde{c}_j$ ($0\le$

$j \leq m$) satisfy

$$\tilde{\gamma} = 4^{-m+1}\gamma, \quad \tilde{\kappa} = 4^{-m}\kappa, \quad \tilde{c}_j = 2^{-2j-3}\sum_{k=0}^{j+1} c_{j-k+1}c_k. \tag{A.6.1}$$

Conversely, if $(u_j, v_j)_{1\leq j\leq m}$ satisfies $(\tilde{P}_{34})_m$, $u = -2(u_1 + \tilde{c}_0)$ is a solution of $(P_{34})_m$. (Note that [Ko2, (2.3)] contains some typogrphical errors; $\tilde{\gamma} = 4^{m-1}\gamma$, $\tilde{\kappa} = 2^{-m}\kappa$ in [Ko2, (2.3)] should be replaced by the first two relations in (A.6.1). In what follows we correct such errors without mentioning when citing formulas in [Ko2]. We refer the reader to [Ko3] for a list of corrections of typogrphical errors in [Ko2].)

A.7 Lax pairs $(L_{34})_m$ and $(\tilde{L}_{34})_m$ ([GP], [Ko2])

The Lax pair $(L_{34})_m$ (resp., $(\tilde{L}_{34})_m$) that underlies $(P_{34})_m$ (resp., $(\tilde{P}_{34})_m$) is as follows:

$$(L_{34})_m : \begin{cases} 4\eta^{-1}\gamma x\dfrac{\partial\vec{\psi}}{\partial x} = A\vec{\psi} & \text{(A.7.1.a)} \\ \eta^{-1}\dfrac{\partial\vec{\psi}}{\partial t} = B\vec{\psi} & \text{(A.7.1.b)} \end{cases} \tag{A.7.1}$$

(A.7.2)

$$A = \frac{1}{2}\begin{pmatrix} -\eta^{-1}(\dfrac{\partial\mathcal{F}}{\partial t} + 2\gamma) & 2(\mathcal{F} + 2\gamma t) \\ -\eta^{-2}\dfrac{\partial^2\mathcal{F}}{\partial t^2} + 2(x-u)(\mathcal{F} + 2\gamma t) & \eta^{-1}(\dfrac{\partial\mathcal{F}}{\partial t} + 2\gamma) \end{pmatrix} + N$$

and

$$B = \begin{pmatrix} 0 & 1 \\ x-u & 0 \end{pmatrix}, \tag{A.7.3}$$

where

$$\mathcal{F} = \sum_{j=0}^{m} (4x)^{m-j} \mathcal{F}_j \tag{A.7.4}$$

and

$$N = \begin{pmatrix} 0 & 0 \\ I_m/(4(\mathcal{F}_m + 2\gamma t)) & 0 \end{pmatrix}. \tag{A.7.5}$$

Remark A.7.1. The purpose of adding the matrix N to the original Lax pair used in [GP] is two-fold: first it fixes the expression of the Lax pair by eliminating the $2m$-th derivative of u in A, and secondly it makes the compatibility condition of (A.7.1.a) and (A.7.1.b) coincides with $(P_{34})_m$ with the parameter κ^2 fixed. The first fact enabled Koike ([Ko2]) to smoothly find the corresponding Lax pair $(\tilde{L}_{34})_m$ of $(\tilde{P}_{34})_m$, and the second fact is effectively used in his reasoning to relate $(\tilde{P}_{34})_m$, and hence $(P_{34})_m$, with a Garnier system. Note that the original formulation of [GP] gives a family of $(P_{34})_m$ (parameterized by κ^2 in the notation of [Ko2]) as the compatibility condition of their Lax pair. In practice we always substitute a solution of $(P_{34})_m$ into the coefficients of $(L_{34})_m$ in this paper and hence we may ignore N in analyzing the Lax pair.

With the help of Remark A.7.1, Koike ([Ko2]) gives the Lax pair $(\tilde{L}_{34})_m$ of $(\tilde{P}_{34})_m$ as follows:

$$(\tilde{L}_{34})_m : \begin{cases} \eta^{-1}\tilde{\gamma} x \dfrac{\partial \vec{\psi}}{\partial x} = \tilde{A}\vec{\psi} & \text{(A.7.6.a)} \\ \eta^{-1} \dfrac{\partial \vec{\psi}}{\partial t} = \tilde{B}\vec{\psi} & \text{(A.7.6.b)} \end{cases} \tag{A.7.6}$$

where

(A.7.7)

$$\tilde{A} = \begin{pmatrix} \frac{1}{2}V(x) - \frac{1}{4}\eta^{-1}\tilde{\gamma} & U + \tilde{\gamma}t \\ \frac{1}{4}(2x^{m+1} - (x - 2\tilde{c}_0)U + 2W & -(\frac{1}{2}V(x) - \frac{1}{4}\eta^{-1}\tilde{\gamma}) \\ \quad +\tilde{\gamma}t(x + 2u_1 + 2\tilde{c}_0) + 2u_{m+1}) & \end{pmatrix}$$

and

$$\tilde{B} = \begin{pmatrix} 0 & 2 \\ \frac{1}{2}x + u_1 + \tilde{c}_0 & 0 \end{pmatrix}. \tag{A.7.8}$$

Here U, V and W are the polynomials given respectively by (A.3.9), (A.3.10) and (A.3.11). In view of (A.2.1) the $(1,2)$-component of A is again found to be the same as 2^{2m-1} multiple of that of $\tilde{A}$.

In parallel with the study of $(\tilde{P}_{\mathrm{I}})_m$, we choose $\tilde{\gamma}$ to be $1/2$ in what follows. This choice of $\tilde{\gamma}$ enables us to assume the constant Θ in [Ko2] to be 1.

A.8 The Hamiltonian structure of $(\tilde{P}_{34})_m$ ([Ko2, Theorem 2.19])

Let us choose $(\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_m)$ so that they satisfy

$$U(x) + \frac{1}{2}t = \prod_{j=1}^{m}(x - \lambda_j) \tag{A.8.1}$$

and

$$\mu_j = \frac{1}{\lambda_j}(V(\lambda_j) - \tilde{\kappa} - \frac{1}{4}\eta^{-1}) \quad (1 \leq j \leq m). \tag{A.8.2}$$

By letting N_j $(1 \leq j \leq m)$ denote

$$\prod_{\substack{k=1,2,\ldots,m \\ k \neq j}} (\lambda_j - \lambda_k)^{-1} \tag{A.8.3}$$

we define

$$H_1 = \sum_{j=1}^{m} N_j \left(\lambda_j \mu_j^2 - 2\tilde{\kappa}\mu_j - (\lambda_j^{2m} + \sum_{k=1}^{m} t_k \lambda_j^{m+k-1}) \right) \tag{A.8.4}$$

for complex numbers $\{t_k\}_{k=1}^m$. It is known ([Ko2]) that H_1 is a Hamiltonian for the degenerate Garnier system $G(1, m+3/2; m)$. Set

$$t_1 = 4t + 2\tilde{c}_{m-1} \tag{A.8.5}$$

and

$$t_j = 2\tilde{c}_{m-j} \quad (2 \leq j \leq m) \tag{A.8.6}$$

in H_1, and let

(A.8.7)

$$K(t; \lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_m) = 4H_1 \Big|_{t_1 = 4t + 2\tilde{c}_{m-1}, t_2 = 2\tilde{c}_{m-2}, \cdots, t_m = 2\tilde{c}_0}.$$

Then the system $(\tilde{P}_{34})_m$ is equivalent to the following Hamiltonian system $(G_{34})_m$:

$$(G_{34})_m : \begin{cases} \dfrac{d\lambda_j}{dt} = \eta \dfrac{\partial K}{\partial \mu_j} & (j = 1, 2, \ldots, m) \\ \dfrac{d\mu_j}{dt} = -\eta \dfrac{\partial K}{\partial \lambda_j} & (j = 1, 2, \ldots, m). \end{cases} \tag{A.8.8}$$

A.9 Definition of $(P_{\text{II-2}})_m$ and $(\tilde{P}_{\text{II-2}})_m$

The $(P_{\text{II-2}})$-hierarchy and the (P_{IV})-hierarchy are defined in [GJP1, GJP2] with the help of differential polynomials $(K_n, L_n)_{n \geq 0}$ of a pair of functions (u, v) and their derivatives, which are recursively determined

by the following relation:

(A.9.1)

$$\begin{cases} K_{n+1} = \dfrac{1}{2}(uK_n + 2L_n - \eta^{-1}\dfrac{dK_n}{dt}) & \text{(A.9.1.a)} \\ L_{n+1} = \dfrac{1}{4}\displaystyle\sum_{j=0}^{n}(vK_{n-j}K_j - L_{n-j}L_j + \eta^{-1}K_{n-j}\frac{dL_j}{dt}) & \text{(A.9.1.b)} \end{cases}$$

with

$$K_0 = 2, \quad L_0 = 0. \tag{A.9.2}$$

(See [N] for the background of the formula (A.9.1).) For the convenience of the notation we introduce

$$\begin{pmatrix} \mathcal{K}_n \\ \mathcal{L}_n \end{pmatrix} = c_0 \begin{pmatrix} K_n \\ L_n \end{pmatrix} + c_1 \begin{pmatrix} K_{n-1} \\ L_{n-1} \end{pmatrix} + \cdots + c_n \begin{pmatrix} K_0 \\ L_0 \end{pmatrix}, \tag{A.9.3}$$

where c_j's are constants. Unless otherwise stated explicitly, we choose c_0 to be 1. For example,

(A.9.4)

$$\begin{aligned} \begin{pmatrix} \mathcal{K}_3 \\ \mathcal{L}_3 \end{pmatrix} &= \frac{1}{4}\left[\begin{pmatrix} u^3 + 6uv \\ 3u^2v + 3v^2 \end{pmatrix} + 3u\eta^{-1}\frac{d}{dt}\begin{pmatrix} -u \\ v \end{pmatrix} + \eta^{-2}\frac{d^2}{dt^2}\begin{pmatrix} u \\ v \end{pmatrix}\right] \\ &\quad + \frac{c_1}{2}\left[\begin{pmatrix} u^2 + 2v \\ 2uv \end{pmatrix} + \eta^{-1}\frac{d}{dt}\begin{pmatrix} -u \\ v \end{pmatrix}\right] + c_2\begin{pmatrix} u \\ v \end{pmatrix} + c_3\begin{pmatrix} 2 \\ 0 \end{pmatrix}. \end{aligned}$$

(Cf. [KKoNT, Remark 1.3.1].) The m-th member of $(P_{\text{II-2}})$-hierarchy is, by definition, the following equation (A.9.5). ([KKoNT, Definition 1.3.1]; note, however, that we have reversed the order of labelling the constants c_j's so that our notation may become consistent with those used in [Ko2].)

$$(P_{\text{II-2}})_m : \quad \begin{cases} \mathcal{K}_{m+1} + 2\gamma t = 0 & \text{(A.9.5.a)} \\ \mathcal{L}_{m+1} = 2\kappa & \text{(A.9.5.b)} \end{cases} \tag{A.9.5}$$

with $\gamma(\neq 0)$ and κ being constants. Unless otherwise stated, we suppose $c_{m+1} = 0$ for simplicity.

Another expression (A.9.6) of $(P_{\text{II-2}})_m$ is given by Koike ([Ko1]), which is denoted by $(\tilde{P}_{\text{II-2}})_m$ in [Ko2].

(A.9.6)

$$(\tilde{P}_{\text{II-2}})_m : \begin{cases} \eta^{-1}\dfrac{du_j}{dt} = -2(u_1u_j + v_j + u_{j+1}) + 2c_ju_1 \quad (1 \le j \le m) & \text{(A.9.6.a)} \\ \eta^{-1}\dfrac{dv_j}{dt} = 2(v_1u_j + v_{j+1} + w_j) - 2c_jv_1 \qquad (1 \le j \le m) & \text{(A.9.6.b)} \\ \text{with} & \\ u_{m+1} = \gamma t,\ v_{m+1} = \kappa. & \text{(A.9.6.c)} \end{cases}$$

Here $\gamma(\neq 0)$, κ and c_j's are constants used in the definition of $(P_{\text{II-2}})_m$, and w_j is a polynomial of $(u_k, v_k)_{k\le j}$ that is recursively determined by the following relation:

$$\text{(A.9.7)}\quad w_j = \sum_{k=1}^{j-1} u_{j-k}w_k + \sum_{k=1}^{j} u_{j-k+1}v_k + \frac{1}{2}\sum_{k=1}^{j-1} v_{j-k}v_k - \sum_{k=1}^{j-1} c_{j-k}w_k.$$

A.10 Correspondence between $(P_{\text{II-2}})_m$ and $(\tilde{P}_{\text{II-2}})_m$ ([Ko2, Theorem 3.5])

For a solution (u, v) of $(P_{\text{II-2}})_m$, the equation $(\tilde{P}_{\text{II-2}})_m$ is satisfied by $(u_j, v_j)_{1\le j\le m}$ defined by (A.10.1) through $(\mathcal{K}_j, \mathcal{L}_j)_{1\le j\le m}$.

$$\text{(A.10.1)}\qquad \begin{cases} u_j = -\dfrac{1}{2}\mathcal{K}_j + c_j & (1 \le j \le m) \\ v_j = \dfrac{1}{2}\mathcal{L}_j & (1 \le j \le m). \end{cases}$$

Conversely, if $(u_j, v_j)_{1\le j\le m}$ is a solution of $(\tilde{P}_{\text{II-2}})_m$, then

$$\text{(A.10.2)}\qquad u = -2u_1, \quad v = 2v_1$$

is a solution of $(P_{\text{II-2}})_m$.

A.11 Lax pairs $(L_{\text{II-2}})_m$ and $(\tilde{L}_{\text{II-2}})_m$ ([Ko2, §3])

The Lax pair $(L_{\text{II-2}})_m$ (resp., $(\tilde{L}_{\text{II-2}})_m$) that underlies $(P_{\text{II-2}})_m$ (resp., $(P_{\text{II-2}})_m$) is as follows:

$$(\text{A.11.1}) \qquad (L_{\text{II-2}})_m : \begin{cases} \gamma\eta^{-1}\dfrac{\partial\vec{\psi}}{\partial x} = A\vec{\psi} & (\text{A.11.1.a}) \\ \eta^{-1}\dfrac{\partial\vec{\psi}}{\partial t} = B\vec{\psi} & (\text{A.11.1.b}) \end{cases}$$

with

(A.11.2)

$$A = \frac{1}{4}\begin{pmatrix} -(2x-u)\mathcal{K} - \eta^{-1}\dfrac{d\mathcal{K}}{dt} & 2\mathcal{K} \\ -2\eta^{-1}\dfrac{d\mathcal{L}}{dt} - 2v\mathcal{K} & (2x-u)\mathcal{K} + \eta^{-1}\dfrac{d\mathcal{K}}{dt} \end{pmatrix} + N$$

and

$$(\text{A.11.3}) \qquad B = \begin{pmatrix} -x + u/2 & 1 \\ v & x - u/2 \end{pmatrix},$$

where

$$(\text{A.11.4}) \qquad \mathcal{K} = \sum_{j=0}^{m} x^{m-j}\mathcal{K}_j, \quad \mathcal{L} = \sum_{j=0}^{m} x^{m-j}\mathcal{L}_j$$

and

$$(\text{A.11.5}) \qquad N = \frac{1}{2}\begin{pmatrix} -\mathcal{K}_{m+1} - 2\gamma t & 0 \\ 2(\mathcal{L}_{m+1} - 2\kappa) & \mathcal{K}_{m+1} + 2\gamma t \end{pmatrix}.$$

Remark A.11.1. Concerning the role of the additional matrix N, the situation is essentially the same as that for the case of $(L_{34})_m$.

$$(\text{A.11.6}) \qquad (\tilde{L}_{\text{II-2}})_m : \begin{cases} \gamma\eta^{-1}\dfrac{\partial\vec{\psi}}{\partial x} = \tilde{A}\vec{\psi} & (\text{A.11.6.a}) \\ \eta^{-1}\dfrac{\partial\vec{\psi}}{\partial t} = \tilde{B}\vec{\psi} & (\text{A.11.6.b}) \end{cases}$$

with

(A.11.7)

$$\tilde{A} = \begin{pmatrix} -(x^{m+1} + V + xC(x) + \gamma t) & U + C(x) \\ -2(xV + W + \kappa) & x^{m+1} + V + xC(x) + \gamma t \end{pmatrix}$$

and

$$(\text{A.11.8}) \qquad \tilde{B} = \begin{pmatrix} -(x + u_1) & 1 \\ -2v_1 & x + u_1 \end{pmatrix},$$

where

$$(\text{A.11.9}) \qquad C(x) = \sum_{j=1}^{m} c_j x^{m-j}$$

and U, V and W are polynomials of x given respectively by (A.3.9), (A.3.10) and (A.3.11). Note that the $(1,2)$-component of A exactly coincides with that of $\tilde{A}$ thanks to (A.10.1).

A.12 The Hamiltonian structure of $(\tilde{P}_{\text{II-2}})_m$ ([Ko1, Theorem 1.3])

Let us choose $(\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_m)$ so that they satisfy

$$(\text{A.12.1}) \qquad U(x) + C(x) = \prod_{j=1}^{m}(x - \lambda_j) \quad (j = 1, 2, \ldots, m)$$

and

$$(\text{A.12.2}) \qquad \mu_j = -V(\lambda_j) \quad (j = 1, 2, \ldots, m).$$

By letting N_j $(1 \leq j \leq m)$ denote

$$\prod_{\substack{k=1,2,\dots,m \\ k \neq j}} (\lambda_j - \lambda_k)^{-1} \tag{A.12.3}$$

and letting $A_m(z,t)$ denote

$$2z^{m+1} + \sum_{j=1}^{m} j t_j z^{j-1}, \tag{A.12.4}$$

we define

$$H_1 = \frac{1}{2} \sum_{j=1}^{m} N_j \left(\mu_j^2 - A_m(\lambda_j, t)\mu_j - (2\alpha+1)\lambda_j^m \right) \tag{A.12.5}$$

for a constant α. Then H_1 (for $\alpha \neq -1/2$) is a Hamiltonian of a degenerate Garnier system called A_m-system ([L]). (The Hamiltonian H_1 used in [Ko2] is the same as that of [L], which contains some additional terms depending only on t. As such terms independent of (λ_j, μ_j) are irrelevant in defining the Hamiltonian system, we eliminate these terms here.) Set

$$t_1 = 2t \tag{A.12.6}$$

and

$$t_j = 2c_{m-j}/j \quad (2 \leq j \leq m) \tag{A.12.7}$$

and let K denote the resulting function of $(t; \lambda_1, \cdots, \lambda_m, \mu_1, \cdots, \mu_m)$. Then $(\tilde{P}_{\text{II-2}})_m$ is equivalent to the following Hamiltonian system $(G_{\text{II-2}})_m$.

$$(G_{\text{II-2}})_m : \begin{cases} \dfrac{d\lambda_j}{dt} = \eta \dfrac{\partial K}{\partial \mu_j} & (1 \leq j \leq m) \\ \dfrac{d\mu_j}{dt} = -\eta \dfrac{\partial K}{\partial \lambda_j} & (1 \leq j \leq m), \end{cases} \tag{A.12.8}$$

if we choose

(A.12.9) $$\kappa = -(\alpha + 1/2).$$

We note κ may vanish in (A.12.8).

A.13 Definition of $(P_{\rm IV})_m$ and $(\tilde{P}_{\rm IV})_m$

With the help of differential polynomials $\mathcal{K}_m$ and $\mathcal{L}_m$ the m-th member of $(P_{\rm IV})$-hierarchy is given by the following:

(A.13.1)

$$(P_{\rm IV})_m : \begin{cases} \eta^{-1}\dfrac{d\mathcal{K}_m}{dt} = 2\mathcal{L}_m + u\mathcal{K}_m + 2\gamma tu - 4\theta_1 - 2\eta^{-1}\gamma & \text{(A.13.1.a)} \\ \eta^{-1}(\mathcal{K}_m + 2\gamma t)\dfrac{d\mathcal{L}_m}{dt} = -v(\mathcal{K}_m + 2\gamma t)^2 + (\mathcal{L}_m - 2\theta_1)^2 - 4\theta_2^2 & \text{(A.13.1.b)} \end{cases}$$

with $\gamma, \theta_1, \theta_2$ and c_j's $(1 \le j \le m)$ being constants.

Another expression (A.13.2) of $(P_{\rm IV})_m$ is found by Koike ([Ko1]), which is denoted by $(\tilde{P}_{\rm IV})_m$ in [Ko2].

(A.13.2)

$$(\tilde{P}_{\rm IV})_m : \begin{cases} \eta^{-1}\dfrac{du_j}{dt} = -2(u_1u_j + v_j + u_{j+1}) + 2c_ju_1 \quad (1 \le j \le m) & \text{(A.13.2.a)} \\ \eta^{-1}\dfrac{dv_j}{dt} = 2(v_1u_j + v_{j+1} + w_j) - 2c_jv_1 \qquad (1 \le j \le m) & \text{(A.13.2.b)} \\ v_{m+1} = -(\gamma tu_1 + \theta_1 + \frac{1}{2}\eta^{-1}\gamma) & \text{(A.13.2.c)} \\ v_{m+1} = -w_m - \gamma tv_1 - \dfrac{(v_m - \theta_1)^2 - \theta_2^2}{2(u_m - \gamma t - c_m)}, & \text{(A.13.2.d)} \end{cases}$$

where the constants are the same as those in $(P_{\rm IV})_m$ and w_j is the polynomial determined by (A.9.7).

A.14 Correspondence between $(P_{\mathrm{IV}})_m$ and $(\tilde{P}_{\mathrm{IV}})_m$ ([Ko2, Theorem 3.6])

For a solution (u, v) of $(P_{\mathrm{IV}})_m$ the equation $(\tilde{P}_{\mathrm{IV}})_m$ is satisfied by $(u_j, v_j)_{1\leq j\leq m}$ defined by (A.10.1), and for a solution $(u_j, v_j)_{1\leq j\leq m}$ of $(\tilde{P}_{\mathrm{IV}})_m$ the functions (u, v) given by (A.10.2) provide a solution of $(P_{\mathrm{IV}})_m$; the situation is exactly in parallel with the situation of the pair $(P_{\text{II-2}})_m$ and $(\tilde{P}_{\text{II-2}})_m$.

A.15 Lax pairs $(L_{\mathrm{IV}})_m$ and $(\tilde{L}_{\mathrm{IV}})_m$ ([Ko2, Theorem 3.8 and 3.9])

The Lax pair $(L_{\mathrm{IV}})_m$ (resp., $(\tilde{L}_{\mathrm{IV}})_m$) that underlies $(P_{\mathrm{IV}})_m$ (resp., $(\tilde{P}_{\mathrm{IV}})_m$) is as follows:

$$(\mathrm{A.15.1}) \quad (L_{\mathrm{IV}})_m : \begin{cases} \gamma x \eta^{-1} \dfrac{\partial \vec{\psi}}{\partial x} = A\vec{\psi} & (\mathrm{A.15.1.a}) \\ \eta^{-1} \dfrac{\partial \vec{\psi}}{\partial t} = B\vec{\psi} & (\mathrm{A.15.1.b}) \end{cases}$$

with

$$(\mathrm{A.15.2}) \qquad A = \frac{1}{4}\begin{pmatrix} -(2x-u)(\mathcal{K}+2\gamma t) - \eta^{-1}\dfrac{d\mathcal{K}}{dt} - 2\eta^{-1}\gamma & 2(\mathcal{K}+2\gamma t) \\ -2\eta^{-1}\dfrac{d\mathcal{L}}{dt} - 2v(\mathcal{K}+2\gamma t) & (2x-u)(\mathcal{K}+2\gamma t) + \eta^{-1}\dfrac{d\mathcal{K}}{dt} - 2\eta^{-1}\gamma \end{pmatrix} + N$$

and

$$(\mathrm{A.15.3}) \qquad B = \begin{pmatrix} -x + u/2 & 1 \\ v & x - u/2 \end{pmatrix}$$

Here $\mathcal{K}$ and $\mathcal{L}$ are the polynomials given in (A.11.4) and

$$N = \frac{1}{4}\begin{pmatrix} I_m & 0 \\ 2J_m/(\mathcal{K}_m + 2\gamma t) & -I_m \end{pmatrix}, \tag{A.15.4}$$

where

$$I_m = \eta^{-1}\frac{d\mathcal{K}_m}{dt} - u(\mathcal{K}_m + 2\gamma t) - 2\mathcal{L}_m + 4\theta_1 + 2\eta^{-1}\gamma \tag{A.15.5}$$

and

$$J_m = \eta^{-1}(\mathcal{K}_m+2\gamma t)\frac{d\mathcal{L}_m}{dt}+v(\mathcal{K}_m+2\gamma t)^2-(\mathcal{L}_m-2\theta_1)^2+4\theta_2^2. \tag{A.15.6}$$

Remark A.15.1. Concerning the role of the additional matrix N, the situation is essentially the same as that for the case of $(L_{34})_m$.

$$(\tilde{L}_{\mathrm{IV}})_m : \begin{cases} \gamma x\eta^{-1}\dfrac{\partial\vec{\psi}}{\partial x} = \tilde{A}\vec{\psi} & \text{(A.15.7.a)} \\ \eta^{-1}\dfrac{\partial\vec{\psi}}{\partial t} = \tilde{B}\vec{\psi} & \text{(A.15.7.b)} \end{cases} \tag{A.15.7}$$

with

(A.15.8)

$$\tilde{A} = \begin{pmatrix} -(x^{m+1} + V + xC(x) + \gamma xt - \theta_1) & U + C(x) + \gamma t \\ -2(xV + W + v_{m+1} + \gamma t v_1) & x^{m+1} + V + xC(x) + \gamma xt - \theta_1 \end{pmatrix}$$

and

$$\tilde{B} = \begin{pmatrix} -(x + u_1) & 1 \\ -2v_1 & x + u_1 \end{pmatrix}, \tag{A.15.9}$$

where $C(x), U, V$ and W are polynomials of x respectively given by (A.11.9), (A.3.9), (A.3.10) and (A.3.11). The $(1,2)$-component of A again coincides with that of $\tilde{A}$.

A.16 The Hamiltonian structure of $(P_{\mathrm{IV}})_m$ ([Ko1, Theorem 1.4])

Let us fix the constant γ to be 2 and choose $(\lambda_1, \cdots, \lambda_m, \mu_1, \cdots, \mu_m)$ so that they satisfy

$$U(x) + C(x) + 2t = \prod_{j=1}^{m}(x - \lambda_j) \tag{A.16.1}$$

and

$$\mu_j = -\frac{1}{2\lambda_j}(V(\lambda_j) - \theta_1 - \theta_2) \qquad (j = 1, 2, \cdots, m). \tag{A.16.2}$$

Set

$$\kappa_0 = \theta_2 \tag{A.16.3}$$

and

$$\kappa_\infty = (\theta_1 + \theta_2)/2. \tag{A.16.4}$$

In parallel with the case of $(\tilde{P}_{\text{II-2}})_m$, we let N_j $(1 \leq j \leq m)$ denote

$$\prod_{\substack{k=1,2,\cdots,m \\ k \neq j}} (\lambda_j - \lambda_k)^{-1} \tag{A.16.5}$$

and define

$$H_1 = \sum_{j=1}^{m} N_j \left(\lambda_j \mu_j^2 - (\sum_{k=1}^{m} t_k \lambda_j^k + \lambda_j^{m+1} + \kappa_0)\mu_j + \kappa_\infty \lambda_j^m \right), \tag{A.16.6}$$

where t_k's are complex parameters. Then H_1 is a Hamiltonian of a degenerate Garnier system called Kawamuko system ([Kwm]). Set

$$t_1 = 2t + c_m \tag{A.16.7}$$

and

$$t_k = c_{m-k+1} \qquad (2 \leq k \leq m), \tag{A.16.8}$$

and let K denote the resulting function of $(t; \lambda_1, \cdots, \lambda_m, \mu_1, \cdots, \mu_m)$. Then $(\tilde{P}_{\rm IV})_m$ is equivalent to the following Hamiltonian system $(G_{\rm IV})_m$.

$$(\text{A.16.9}) \qquad (G_{\rm IV})_m : \begin{cases} \dfrac{d\lambda_j}{dt} = \eta \dfrac{\partial K}{\partial \mu_j} & (1 \le j \le m) \\ \dfrac{d\mu_j}{dt} = -\eta \dfrac{\partial K}{\partial \lambda_j} & (1 \le j \le m). \end{cases}$$

Appendix B: Parity structure of instanton-type solutions

To prove Lemma 5.3.2 and Lemma 5.3.4 we have used the "alternating parity" structure of instanton-type solutions in Section 5. In Appendix B we explain this parity structure of instanton-type solutions of $(P_J)_m$ ($J =$ I, 34, II-2 or IV, $m = 1, 2, \ldots$).

Let us start with a brief review of the core part of [T2] for the construction of instanton-type solutions. Each member $(P_J)_m$ of the (P_J)-hierarchy can be expressed in the form of the Hamiltonian system

$$(\text{B.1}) \qquad \begin{cases} \dfrac{dq_j}{dt} = \eta \dfrac{\partial H}{\partial p_j} & (j = 1, \ldots, m) \\ \dfrac{dp_j}{dt} = -\eta \dfrac{\partial H}{\partial q_j} & (j = 1, \ldots, m) \end{cases}$$

with the Hamiltonian $H = H(t, q, p; \eta^{-1})$ by introducing an appropriate canonical variable $(q, p) = (q_j, p_j)_{1 \le j \le m}$ through a canonical transform

$$(\text{B.2}) \qquad u_j = u_j(t, q, p; \eta^{-1}), \quad v_j = v_j(t, q, p; \eta^{-1}).$$

For example, the variable (λ_j, μ_j) discussed in Appendix A (cf. Subsections A.4, A.8, A.12 and A.16) is one of such canonical variables. (See [T2] for another choice of canonical variables in the case of $(P_{\rm I})$-hierarchy.) To construct instanton-type solutions of $(P_J)_m$ we use the

Hamiltonian form (B.1). We first note that the system (B.1) admits a formal power series solution (with respect to η^{-1}) of the form

(B.3) $\hat{q}_j = \hat{q}_j^{(0)}(t)+\eta^{-1}\hat{q}_j^{(1)}(t)+\cdots, \quad \hat{p}_j = \hat{p}_j^{(0)}(t)+\eta^{-1}\hat{p}_j^{(1)}(t)+\cdots,$

which is called a 0-parameter solution. We then introduce a new canonical variable (ψ_j, φ_j) defined by

(B.4) $$q_j = \hat{q}_j + \eta^{-1/2}\psi_j, \quad p_j = \hat{p}_j + \eta^{-1/2}\varphi_j.$$

In the new variable (ψ_j, φ_j) (B.1) is expressed again in the Hamiltonian form as

(B.5) $$\begin{cases} \dfrac{d\psi_j}{dt} = \eta\dfrac{\partial K}{\partial\varphi_j} & (j=1,\dots,m) \\ \dfrac{d\varphi_j}{dt} = -\eta\dfrac{\partial K}{\partial\psi_j} & (j=1,\dots,m) \end{cases}$$

where the Hamiltonian $K = K(t,\psi,\varphi;\eta^{-1/2})$ is given by

(B.6) $$K = \sum_{|\mu+\nu|\geq 2} \frac{1}{\mu!\nu!}\eta^{-(|\mu+\nu|-2)/2}\frac{\partial^{|\mu+\nu|}H}{\partial q^\mu \partial p^\nu}(t,\hat{q},\hat{p})\psi^\mu\varphi^\nu$$

(cf. [T2, (21),(22)]). Note that, if

(B.7) $$K = \sum_{k=0}^{\infty}\eta^{-k/2}K^{(k)}(t,\psi,\varphi)$$

denotes the formal power series expansion of K in $\eta^{-1/2}$, each coefficient $K^{(k)}$ is a polynomial of (ψ,φ) of degree at most $k+2$ and has the following parity structure:

(B.8) When k is an odd (resp., even) integer, $K^{(k)}$ is a sum of monomials of odd (resp., even) degree.

As is shown in [T2, Theorem 2], if we assume (2.1.2) and (2.1.3), in a neighborhood of an arbitrarily given point $t = t_0$ we can find a

canonical transform

(B.9)

$$\psi_j = \sum_{k=0}^{\infty} \eta^{-k/2} \psi_j^{(k)}(t, \tilde{\psi}, \tilde{\varphi}; \eta^{-1/2}), \quad \varphi_j = \sum_{k=0}^{\infty} \eta^{-k/2} \varphi_j^{(k)}(t, \tilde{\psi}, \tilde{\varphi}; \eta^{-1/2}),$$

where $\psi_j^{(k)}$ and $\varphi_j^{(k)}$ are homogeneous polynomials of degree $(k+1)$ in $(\tilde{\psi}, \tilde{\varphi})$, that transforms (B.5) into its Birkhoff normal form

$$\text{(B.10)} \qquad \begin{cases} \dfrac{d\tilde{\psi}_j}{dt} = \eta \dfrac{\partial \tilde{K}}{\partial \tilde{\varphi}_j} & (j = 1, \ldots, m) \\ \dfrac{d\tilde{\varphi}_j}{dt} = -\eta \dfrac{\partial \tilde{K}}{\partial \tilde{\psi}_j} & (j = 1, \ldots, m) \end{cases}$$

with the Hamiltonian $\tilde{K}$ of the form

$$\text{(B.11)} \qquad \tilde{K} = \tilde{K}(t, \sigma_1, \ldots, \sigma_m; \eta^{-1/2})$$

(where $\sigma_j = \tilde{\psi}_j \tilde{\varphi}_j$). Since we can readily confirm that

$$\text{(B.12)} \qquad \begin{cases} \tilde{\Psi}_j = \alpha_j \exp\left(\eta \displaystyle\int^t \left. \frac{\partial \tilde{K}}{\partial \sigma_j} \right|_{\sigma_1 = \alpha_1 \alpha_{m+1}, \cdots, \sigma_m = \alpha_m \alpha_{2m}} dt \right) \\ \tilde{\Phi}_j = \alpha_{j+m} \exp\left(-\eta \displaystyle\int^t \left. \frac{\partial \tilde{K}}{\partial \sigma_j} \right|_{\sigma_1 = \alpha_1 \alpha_{m+1}, \cdots, \sigma_m = \alpha_m \alpha_{2m}} dt \right), \end{cases}$$

$(j = 1, \ldots, m)$ provides a solution of (B.10), we thus obtain a formal solution (2.1.4)-(2.1.5) of the original $(P_J)_m$ by substituting (B.12) successively into (B.9), (B.4) and (B.2). This is an outline of the construction of instanton-type solutions discussed in [T2].

In the course of the proof of [T2, Theorem 2] (cf. [T2, Section 4]) we see that

(B.13) the coefficients $\psi_j^{(k)}$ and $\varphi_j^{(k)}$ in (B.9) are formal power series of η^{-1} (not of $\eta^{-1/2}$), that is, $\psi_j^{(k)}$ and $\varphi_j^{(k)}$ contain no odd degree terms of $\eta^{-1/2}$.

Furthermore

(B.14) $\tilde{K}$ also contains no odd degree terms of $\eta^{-1/2}$.

Otherwise stated, writing $\tilde{K}$ as

$$\tilde{K} = \sum_{k=0}^{\infty} \eta^{-k} \sum_{|\nu|=k+1} g_\nu(t,\eta)\sigma^\nu \tag{B.15}$$

in accordance with the expressions (2.1.6) and (2.1.7) of instantons, we find that $g_\nu(t,\eta)$ is a formal power series of η^{-1}, that is, there are no odd degree terms of $\eta^{-1/2}$ in the expansion (2.1.8). Hence every instanton contains terms of integral powers of η^{-1} only. It then follows from this fact and (B.13) that the coefficients $u_{j,l/2}(t,\Psi,\Phi)$ and $v_{j,l/2}(t,\Psi,\Phi)$ of instanton-type solutions (2.1.4) and (2.1.5) have the following "alternating parity" structure:

(B.16) When l is an odd (resp., even) integer, $u_{j,l/2}$ and $v_{j,l/2}$ are sums of monomials of (Ψ,Φ) of odd (resp., even) degree (at most l).

Finally, combining this parity structure (B.16) of instanton-type solutions with the definitions (3.11) and (3.13) of $\rho^{(j_0)}(t,\eta) = \sum_{l\geq 0} \eta^{-l/2}\rho_{l/2}^{(j_0)}$ and $\sigma^{(j_0)}(t,\eta) = \sum_{l\geq 0} \eta^{-l/2}\sigma_{l/2}^{(j_0)}$, respectively, we can confirm the following:

(B.17) When l is an odd (resp., even) integer, $\rho_{l/2}^{(j_0)}$ and $\sigma_{l/2}^{(j_0)}$ are sums of monomials of (Ψ,Φ) of even (resp., odd) degree (at most $l+1$).

As a consequence of (B.17) and the definition (3.12) we also find the following parity structure for $E^{(j_0)} = \sum_{l\geq 0} \eta^{-l/2} E^{(j_0)}_{l/2}$:

(B.18) $E^{(j_0)}_{l/2}$ is a sum of monomials of instantons of odd (resp., even) degree for an odd (resp., even) integer l.

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