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Abstract

In this paper we determine the integral quandle homology groups of Alexander quandles of prime order. As a special case, this settles the *delayed Fibonacci conjecture* by M. Niebrzydowski and J. H. Przytycki in [7]. Moreover, we determine the cohomology group of the Alexander quandle and obtain relatively simple presentations of all higher degree cocycles which generate the cohomology group. Furthermore, we prove that the integral quandle homology of a finite connected Alexander quandle is annihilated by the order of the quandle.

Keywords
rack, quandle, homology, cohomology, knot.

1 Introduction

The quandle (co)homology of a finite quandle X is introduced by J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford and M. Saito [1]. The $2, 3$ or 4-cocycles of the quandle cohomology give rise the quandle cocycle invariants for 1-knots or 2-knots (see [1] and [2] for details). In order to search the invariant it is important to determine quandle cohomology groups and to find those cocycles. T. Mochizuki listed all 2-cocycles for finite Alexander quandles over a finite field in [5] and all 3-cocycles for Alexander quandles on a finite field in [6]. R. A. Litherland and S. Nelson analyzed the free and torsion subgroup of the quandle homology group of a finite quandle [4]. For the quandle homology of higher degrees, M. Niebrzydowski and J. H. Przytycki constructed some quandle homological operations and estimated the torsion subgroup of the integral quandle homology groups of some quandles. J. S. Carter, S. Kamada and M. Saito investigated the correspondence between some higher dimensional X-colored link diagrams and some cycles of quandle homology [2]. The pairings between these higher degree cycles and higher degree cocycles are expected to be invariants of higher dimensional links of codimension two.

In this paper we determine the integral quandle homology groups of Alexander quandles of prime order p. The simplest non-trivial class among quandles is the Alexander quandle of prime order. More precisely, it is known [3] that any connected quandle of order p is isomorphic to an Alexander quandle. An Alexander quandle of order p is defined to be $\mathbb{Z}_p$ with a binary operation given by $x * y = \omega x + (1-\omega)y$, where $\mathbb{Z}_p$ means a cyclic group of order p and $\omega \in \mathbb{Z}_p$ is neither 0 nor 1 [1]. We denote by e the order of ω: in other word, $e > 0$ is the minimal number satisfying $\omega^e = 1$. We denote the integral quandle homology by $H_n^Q(X;\mathbb{Z})$. We will determine the integral quandle homology of the Alexander quandle as follows.

Theorem 1.1. *Let X be the Alexander quandle structure of order p with $\omega \neq 0, 1$. Let e be the order of ω. Then the integral quandle homology groups are $H_1^Q(X;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_p^{b_1}$ and $H_n^Q(X;\mathbb{Z}) \cong \mathbb{Z}_p^{b_n}$ for $n \geq 2$, where b_n is determined by*

$$b_{n+2e} = b_n + b_{n+1} + b_{n+2}, \quad b_1 = b_2 = \cdots = b_{2e-2} = 0, \quad \text{and} \quad b_{2e-1} = b_{2e} = 1.$$

As a corollary, Theorem 1.1 shows that $H_n^Q(X;\mathbb{Z}) \cong 0$ for the Alexander quandle X with $\omega \neq -1, 0, 1$ and $2 \leq n \leq 4$ (Corollary 2.3), since $e > 2$ and $2e - 1 > 4$. Therefore Corollary 2.3 tells us that only the dihedral quandle is useful in Alexander quandles of prime order for the study of quandle cocycle invariants of 1-knots and 2-knots.

Next, as a special case, we are interested in the case $\omega = -1$. When $\omega = -1$, the Alexander quandle X is said to be a dihedral quandle. By Theorem 1.1 we obtain

Corollary 1.2. *([7, Conjecture 5]) Let X be the dihedral quandle of order p. Then the integral quandle homology groups are $H_1^Q(X;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_p^{b_1}$ and $H_n^Q(X;\mathbb{Z}) \cong \mathbb{Z}_p^{b_n}$ for $n \geq 2$, where b_n is determined by $b_{n+3} = b_{n+2} + b_n$, $b_1 = b_2 = 0$, and $b_3 = 1$.*

This is the *delayed Fibonacci conjecture* by M. Niebrzydowski and J. H. Przytycki [7].

Moreover for the Alexander quandle X of order p, we also calculate the quandle *cohomology* group $H_Q^n(X;\mathbb{Z}_p)$ with $\mathbb{Z}_p$-coefficient. Namely, we show that $H_Q^n(X;\mathbb{Z}_p) \cong \mathbb{Z}_p^{c_n}$, where c_n determined by

$$c_{n+2e} = c_n + c_{n+1} + c_{n+2}, \quad c_1 = c_2 = \cdots = c_{2e-2} = 0, \quad c_{2e-1} = 1, \quad \text{and} \quad c_{2e} = 2,$$

(Theorem 3.3 and Remark 3.4). In order to prove $c_{n+2e} = c_n + c_{n+1} + c_{n+2}$, we construct cohomological operations

$$\Omega_n : H_Q^n(X;\mathbb{Z}_p) \oplus H_Q^{n+1}(X;\mathbb{Z}_p) \oplus H_Q^{n+2}(X;\mathbb{Z}_p) \longrightarrow H_Q^{n+2e}(X;\mathbb{Z}_p).$$

We will show that the operations are isomorphisms (Corollary 5.7); Further, we obtain relatively simple presentations of all higher degree cocycles; we will show that $H_Q^n(X;\mathbb{Z}_p)$ is spanned by n-cocycles introduced in Examples 5.1, 5.2 and 5.3 (Corollary 5.9). The n-cocycles are composed of the four polynomials introduced in (17), (27), (33) and (34).

[1] We ignore the case of $\omega = 0$ and 1 throughout this paper. If $\omega = 0$, then the binary operation is forbidden by quandle axioms. When $\omega = 1$, the Alexander quandle is a trivial quandle. Hence we can easily obtain the quandle homology: $H_n^Q(X;\mathbb{Z}) \cong \mathbb{Z}^{p(p-1)^{n-1}}$. Furthermore, we thus also omit the case $p = 2$.

For example, we present the resulting representative 4-cocycles of the dihedral quandle; if $\omega = -1$, then $H^4_Q(X;\mathbb{Z}_p) \cong \mathbb{Z}_p^2$ is generated by

$$\psi_{4,0}(x,y,z,w) := (x-y)\cdot\big(2(z-w)^p - (2z-w-y)^p - (y-w)^p\big)/p,$$
$$\psi_{4,1}(x,y,z,w) := \big((x-2y+z)^p + (x-z)^p - 2(x-y)^p\big)\cdot\big(2w^p - (2z-w)^p - z^p\big)/p^2.$$

Also, we discuss the torsion subgroup of $H^Q_n(M;\mathbb{Z})$ for a finite connected Alexander quandle M. We prove that for $n \geq 2$ $H^Q_n(M;\mathbb{Z})$ is annihilated by $|M|$ (Corollary 6.2). As a special case, if X is the Alexander quandle of order p, then $H^Q_n(X;\mathbb{Z})$ is annihilated by p (Corollary 6.4), proving [7, Conjecture 16]. It is known [4, Theorem.1] that $H^Q_n(X;\mathbb{Z})$ is annihilated by $|X|^n$ for a connected quandle X and each $n \geq 1$. Then Corollary 6.2 is a stronger estimate for Alexander quandles, while it does not hold for a connected quandles; for example, there exists a connected non-Alexander quandle $QS(6)$ whose third quandle homology is not annihilated by $|QS(6)|$ (Remark 6.3).

This paper is organized as follows. In Section 2 we review quandle homology and reformulate Theorem 1.1. In Section 2.3 we outline the proof of Theorem 1.1. In Section 3 we review quandle cohomology and give a decomposition of quandle cochain groups. In Section 4 we introduce an isomorphism $\overline{\Theta}_i$ and prove that $c_1 = c_2 = \cdots = c_{2e-2} = 0,\ \ c_{2e-1} = 1$ (Theorem 3.3 (I)). In Section 5 we explicitly present several cocycles and determine the quandle cohomology, leading to a proof of Theorem 1.1. In Section 6 we show that the integral quandle homology group of a finite connected Alexander quandle M is annihilated by $|M|$.

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2 Results

2.1 Preliminaries: rack and quandle homology groups

Throughout this paper we fix an odd prime number p.

Here we will review the rack and quandle homology groups introduced in [1]. For a given quandle $(X,*)$ and an abelian group A, let $C^R_n(X;A)$ be the free abelian group generated by n-tuples $(x_1, x_2, \ldots, x_n)$ of elements of X; in other words $C^R_n(X;A) = A\langle X^n\rangle$. Define a boundary homomorphism $\partial_n : C_n^R(X;A) \to C_{n-1}^R(X;A)$ for $n \geq 2$ to be

$$\partial_n(x_1, x_2, \dots, x_n) =$$
$$\sum_{i=2}^{n}(-1)^i\big((x_1 * x_i, \dots, x_{i-1} * x_i, x_{i+1}, \dots, x_n) - (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)\big),$$

and ∂_1 to zero map. We can check that the composite $\partial_{n-1} \circ \partial_n$ is zero. A pair of (C_*^R, ∂) is said to be the *rack chain complex* of X. Since $x * x = x$ for any $x \in X$, we have a subchain complex $C_n^D(X;A) \subset C_n^R(X;A)$, generated by n-tuples $(x_1, \dots, x_n)$ with $x_i = x_{i-1}$ for some i. Then we call the quotient chain complex $C_n^Q(X;A) = C_n^R(X;A)/C_n^D(X;A)$ *the quandle chain complex* of X. We denote the homology groups of those chain complexes by $H_n^R(X;A)$, $H_n^D(X;A)$, and $H_n^Q(X;A)$, respectively. $H_n^R(X;A)$ is said to be *rack homology* and $H_n^Q(X;A)$ is said to be *quandle homology*.

We will review Alexander quandles. In this paper, we are mainly interested in finite Alexander quandles as a class of quandles. An *Alexander quandle* is defined to be a $\mathbb{Z}[T, T^{-1}]$-module with a binary operation given by $x * y = Tx + (1-T)y$. It is known [3] that any connected quandle of prime order is isomorphic to an Alexander quandle $\mathbb{Z}[T, T^{-1}]/(p, T-\omega)$ for some $\omega \in \mathbb{Z}_p$, where ω is neither 0 nor 1 (see also [8, Section 5.1]). In particular, it is known that any Alexander quandle of prime order is of the type $\mathbb{Z}[T, T^{-1}]/(p, T-\omega)$ for some $\omega \in \mathbb{Z}_p$. As it were, this type is the simplest non-trivial quandle among quandles. If $\omega = -1$, the Alexander quandle is said to be *dihedral quandle*.

For calculations of quandle (co)homology groups of Alexander quandles it is convenient to change another "coordinate" as [6]. For an Alexander quandle M, we define $C_n^{R_U}(M;A)$ to be the free abelian group generated by n-tuples $(U_1, \dots, U_n)$ of elements of M^n and for ≥ 2 the boundary map to be

$$\begin{aligned}\partial_n(U_1, \cdots U_n) = &\sum_{i=1}^{n-1}(-1)^i(T\cdot U_1, \dots, T\cdot U_{i-1}, T\cdot U_i + U_{i+1}, U_{i+2}, \dots, U_n) \\ &- \sum_{i=1}^{n-1}(-1)^i(U_1, \dots, U_{i-1}, U_i + U_{i+1}, U_{i+2}, \dots, U_n).\end{aligned} \tag{1}$$

We define ∂_1 to be zero map. We can check that $\partial_{n-1} \circ \partial_n = 0$. Further we have a subchain complex generated by n-tuples $(U_1, \dots, U_n)$ with $U_i = 0$ for some $1 \leq i \leq n-1$. Then we have the quotient complex denoted by $C_n^{Q_U}(M;A)$. From the dual complexes, we can define the cochain groups denoted by $C^n_{R_U}(M;A)$ and $C^n_{Q_U}(M;A)$, respectively.

We will give a canonical correspondence between these two complexes of Alexander quandles. Let us consider a bijection from M^n to M^n given by

$$M^n \ni (x_1, \dots, x_n) \mapsto (x_1 - x_2, x_2 - x_3, \dots, x_{n-1} - x_n, x_n) \in M^n.$$

The map induces a chain map from $C_n^R(M;A)$ (resp. $C_n^Q(M;A)$) to $C_n^{R_U}(M;A)$ (resp. $C_n^{Q_U}(M;A)$). Then it can be verified that these chain maps are chain isomorphisms. We will mainly deal with the complexes $C_n^{R_U}(M;A)$ and $C_n^{Q_U}(M;A)$ later.

Let X be an Alexander quandle with ω of order p. By direct calculation it can be verified that $H_1(X;\mathbb{Z}) \cong \mathbb{Z}$ (see [4]). It is known [5, Corollary 2.2] that $H_2^Q(X;\mathbb{Z}) \cong 0$. It is shown [5, Theorem 3.1] that $H_3^Q(X;\mathbb{Z}) \cong 0$ if $\omega \neq -1, 0, 1$. It is known [7] that $H_3^Q(X;\mathbb{Z}) \cong \mathbb{Z}_p$ and $\mathbb{Z}_p \subset H_4^Q(X;\mathbb{Z})$ in the case $\omega = -1$. It is also shown [7, Corollary 10] that $H_n^Q(X;\mathbb{Z})$ is annihilated by p^{n-2}.

2.2 The main theorem and some examples

We will state our main theorem and give some corollaries and examples. We determine the integral quandle homology group of every Alexander quandle of order p as follows.

Theorem 2.1. *Let ω be an element of $\mathbb{Z}_p$ such that $\omega \neq 0, 1$. Let $X = \mathbb{Z}[T]/(p, T-\omega)$ be an Alexander quandle with the quandle structure. Let e be the order of ω: in other word, e is the minimal number satisfying $\omega^e = 1$. Then the integral quandle homology groups are $H_n^Q(X;\mathbb{Z}) \cong \mathbb{Z}_p^{b_n}$ for $n \geq 2$ and $H_1(X;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_p^{b_1}$, where b_n is determined by $b_n = b_{n-2e} + b_{n-2e+1} + b_{n-2e+2}$, $b_1 = b_2 = \cdots = b_{2e-2} = 0$, and $b_{2e-1} = b_{2e} = 1$.*

After deter-ming the quandle cohomology groups of X we will prove Theorem 2.1. In the next subsection we describe an outline of the proof.

As a special case, we obtain the homology in the case $\omega = -1$. This settles the *delayed Fibonacci conjecture* by M. Niebrzydowski and J. H. Przytycki [7, Conjecture 5]:

Corollary 2.2. *([7, Conjecture 5].) Let X be the dihedral quandle of order p. Then $H_n^{Q_U}(X;\mathbb{Z}) \cong \mathbb{Z}_p^{b_n}$, where b_n is determined by $b_{n+3} = b_{n+2} + b_n$, $b_1 = b_2 = 0$, and $b_3 = 1$.*

Proof. Note that $e = 2$. Put $F_b(x) = \sum_{i \geq 1} b_i x^i \in \mathbb{Z}[[x]]$. By Theorem 2.1 we have

$$F_b(x) = \frac{b_1x^1 + b_2x^2 + b_3x^3 + b_4x^4}{1 - x^2 - x^3 - x^4} = \frac{x^3 + x^4}{1 - x^2 - x^3 - x^4} = \frac{x^3}{1 - x - x^3}.$$

Therefore the generating function leads to the condition as required. □

For the study of quandle cocycle invariants of 1-knots and 2-knots, it is important to determine $H_n^{Q_U}(X;\mathbb{Z})$ for $n = 2, 3$ or 4 (see [1] and [2] for details). It follows from the following corollary 2.3 that the dihedral quandle is only useful in Alexander quandles of prime order for the invariants.

Corollary 2.3. *Let X be the Alexander quandle with $\omega \neq -1, 0, 1$. Then $H_n^{Q_U}(X;\mathbb{Z}) \cong 0$ for $n = 2, 3$ or 4.*

Proof. Since $\omega \neq -1$, $2e - 1 > 4$. $\square$

By corollary 2.2 and 2.3, the homology $H_n^Q(X;\mathbb{Z})$ of $n \leq 4$ and the quandle homology of the dihedral quandle do not depend on the odd prime p. On the other hand, the higher degree homology groups with $\omega \neq -1$ does depend on p and ω. Let us present some examples.

Example 2.4. We consider the case $p = 5$ and $\omega \neq 1, 0, -1$, that is, $\omega = 2$ or $\omega = 3$. The order of ω is 4. By Theorem 2.1 we list the non-vanishing terms of degree ≤ 23 as follows:

$$H_7^Q(X;\mathbb{Z}) \cong H_8^Q(X;\mathbb{Z}) \cong \mathbb{Z}_5 \cong H_{13}^Q(X;\mathbb{Z}),\ H_{14}^Q(X;\mathbb{Z}) \cong H_{15}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_5^2,\ \ H_{16}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_5,$$

$$H_{19}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_5,\ \ H_{20}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_5^3,\quad H_{21}^Q(X;\mathbb{Z}) \cong H_{22}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_5^5,\ \ H_{23}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_5^3.$$

Example 2.5. We assume $p = 7$ and consider the case where $\omega = 2$ or $\omega = 4$. The order of ω is 3. By Theorem 2.1 we list the alive terms of degree ≤ 21 as follows:

$$H_5^Q(X;\mathbb{Z}) \cong H_6^Q(X;\mathbb{Z}) \cong \mathbb{Z}_7 \cong H_9^Q(X;\mathbb{Z}),\ H_{10}^Q(X;\mathbb{Z}) \cong H_{11}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_7^2,\ H_{12}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_7,$$

$$H_{13}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_7,\ \ H_{14}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_7^3,\quad H_{15}^Q(X;\mathbb{Z}) \cong H_{16}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_7^5,\ \ H_{17}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_7^4,$$
$$H_{18}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_7^5,\quad H_{19}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_7^9,\ \ H_{20}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_7^{13},\ \ H_{21}^Q(X;\mathbb{Z}) \cong H_{22}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_7^{14}.$$

Note that $H_{17}^Q(X;\mathbb{Z}) \not\cong \mathbb{Z}_7^3$ (see $H_{23}^Q(X;\mathbb{Z})$ in Example 2.4 and $H_{35}^Q(X;\mathbb{Z})$ in Example 2.6).

Example 2.6. As the last case of $p = 7$, we here fix $\omega = 3$. The order is 6. The non-vanishing terms of degree ≤ 43 are as follows:

$$H_{11}^Q(X;\mathbb{Z}) \cong H_{12}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_7,\ \ H_{21}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_7,\quad H_{22}^Q(X;\mathbb{Z}) \cong H_{23}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_7^2,$$

$$H_{24}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_7,\ H_{31}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_7,\ \ H_{32}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_7^3,\ \ H_{33}^Q(X;\mathbb{Z}) \cong H_{34}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_7^5,$$
$$H_{35}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_7^3,\ \ H_{36}^Q(X;\mathbb{Z}) \cong H_{41}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_7\ \ H_{42}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_7^4\ \ H_{43}^Q(X;\mathbb{Z}) \cong \mathbb{Z}_7^9.$$

2.3 Outline of the proof

We here outline the proof of Theorem 2.1. Let X be an Alexander quandle of order p. It is known [5, Theorem 1.1] that the order of $H_n^Q(X;\mathbb{Z})$ is a finite power of p. In Section 6 we will show that $H_n^Q(X;\mathbb{Z})$ is annihilated by p (Corollary 6.4). This implies that $H_n^Q(X;\mathbb{Z})$ is a finite dimensional $\mathbb{Z}_p$-vector space; $H_n^Q(X;\mathbb{Z}) \cong \mathbb{Z}_p^{b_n}$ for some b_n. Hence, if we know the dimension of $H_n^Q(X;\mathbb{Z})$, then the proof of theorem would complete. For this we will calculate the quandle "cohomology" group with $\mathbb{Z}_p$-coefficient.

In Section 3 we will give a decomposition of the quandle cohomology group $H^n_Q(X;\mathbb{Z}_p)$. We will define another cohomology $H^{n(0)}(X)$ given by (5). In Section 3.4 we will show that $H^n_Q(X;\mathbb{Z}_p) \cong H^{n(0)}(X) \oplus H^{n-1(0)}(X)$ for each $n \geq 2$ (Proposition 3.2). Let us denote $\dim\bigl(H^{n(0)}(X)\bigr)$ by $c^{(0)}_n$. As a corollary, we will show $b_n = c^{(0)}_n$ for $n \geq 1$ by the universal coefficient theorem (Lemma 5.6). Therefore we shall calculate the dimension of $H^{n(0)}(X)$. In Section 3.5 we will construct an isomorphism $\bar{\phi}$ from $H^{n(0)}(X)$ to a certain quotient space using a differential operation (Proposition 3.8). This quotient space is a modification of a space introduced in [6, Section 3.2.4]. Then we will deal with the quotient space later.

For $n \geq 2e$, in Section 4 and Section 5 we will construct a some homomorphisms and consider their composition as follows:

$$\bar{\phi}^{-1}\circ(\overline{\Theta}_1)^{-1}\circ\cdots\circ(\overline{\Theta}_{e-1})^{-1}\circ\overline{\Psi}_m : H^{m-4(0)}(X)\oplus H^{m-3(0)}(X)\oplus H^{m-2(0)}(X) \longrightarrow H^{n(0)}(X),$$

where $m = n - 2e + 4$. In Section 4.1 we will construct $\overline{\Theta}_i$ given by (19) and show that $\overline{\Theta}_i$ is an isomorphism (Proposition 4.2). Moreover, in Section 5.1 we will construct $\overline{\Psi}_m$ given by (36). In Section 5.2 and 5.3, we will show that the map $\overline{\Psi}_m$ is an isomorphism. Therefore the above isomorphisms tells us that $c^{(0)}_n = c^{(0)}_{n-2e} + c^{(0)}_{n-2e+1} + c^{(0)}_{n-2e+2}$. On the other hand, we show that $c^{(0)}_1 = c^{(0)}_2 = \cdots = c^{(0)}_{2e-2} = 0$, $c^{(0)}_{2e-1} = c^{(0)}_{2e} = 1$ in Section 4.3. Since $b_n = c^{(0)}_n$ for $n \geq 1$, this completes the proof of Theorem 2.1.

Here is an additional remark on presentations of the n-cocycles of $H^{n(0)}(X)$. The above isomorphisms are constructed by some polynomials with the concrete forms (see Proposition 3.11, Corollary 4.4 and Proposition 5.4). Moreover, for simplicity, in Section 5.5 we reformulate the composite of the above isomorphisms, we denote it by Ω_{n-2e+4} (Corollary 5.7). Then we obtain relatively simple presentations of all higher degree cocycles which generate $H^{n(0)}(X)$ (Corollary 5.9).

3 Quandle cohomology groups of Alexander quandles of order p

In Section 3.1, we review quandle cochain groups and decompose the groups. In Section 3.2 we state Proposition 3.2 and Theorem 3.3. In Section 3.3 we prepare a differential operation and a integral operation on the cochain group. These operations are important methods in this paper. In Section 3.4 we show Proposition 3.2. As a result, we obtain a decomposition of the quandle cohomology as $H^n(X) \cong H^{n(0)}(X) \oplus H^{n-1(0)}(X)$. In Section 3.5 for the search of $H^{n(0)}(X)$ we will construct an isomorphism from $H^{n(0)}(X)$ to a quotient space (Proposition 3.8).

3.1 Preliminaries: quandle cochain groups

We will review the simplicity of the quandle cochain groups for Alexander quandles used in [6]. There is not anything new in this subsection. Let X be an Alexander quandle of order p. Then we define a complex as follows: for $n \geq 1$,

$$C^n(X) = \{\sum a_{i_1,\dots,i_n} \cdot U_1^{i_1} \cdots U_n^{i_n} \in \mathbb{Z}_p[U_1,\dots,U_n] |\ 1 \leq i_j \leq p-1\ (j \leq n-1), i_n \leq p-1\},$$

and $C^0(X) = \mathbb{Z}_p$. The coboundary is defined as follows: for $f \in C^n(X)$ and $n \geq 1$,

$$\delta_n(f)(U_1, U_2, \cdots, U_{n+1}) := \sum_{i=1}^{n} (-1)^{i-1} f(\omega \cdot U_1, \dots, \omega \cdot U_{i-1}, \omega \cdot U_i + U_{i+1}, U_{i+2}, \dots, U_{n+1})$$

$$- \sum_{i=1}^{n} (-1)^{i-1} f(U_1, \dots, U_{i-1}, U_i + U_{i+1}, U_{i+2}, \dots, U_{n+1}), \qquad (2)$$

and δ_0 is zero map. We can check that $\delta_n(C^n(X)) \subset C^{n+1}(X)$ and $\delta_{n+1} \circ \delta_n = 0$ for any n. We denote the cohomology group by $H^n(X)$. The complex $C^n(X)$ is isomorphic to the cochain group $C^n_{Q_U}(X; \mathbb{Z}_p)$ presented in Section 2. Hence it follows from the universal coefficient theorem that

$$H^n(X) \cong \mathrm{Hom}(H_n^{Q_U}(X;\mathbb{Z}), \mathbb{Z}_p) \oplus \mathrm{Ext}^1(H_{n-1}^{Q_U}(X;\mathbb{Z}), \mathbb{Z}_p). \qquad (3)$$

We will decompose the complex $C^n(X)$ by the homogenous degree; for any $d \geq n-1$ we define

$$C^n_d(X) = \{\sum a_{i_1,\dots,i_n} \cdot U_1^{i_1} \cdots U_n^{i_n} \in C^n(X) |\ \sum_{1 \leq h \leq n} i_h = d\}.$$

Since $\delta_n(C^n_d(X)) \subset C^{n+1}_d(X)$, we obtain a direct sum decomposition of the complex as $(C^n(X), \delta_n) = (\bigoplus_d C^n_d(X), \delta_n)$. Let f be an element of $C^n_d(X)$. We decompose $f = \sum_{0 \leq a \leq p-1} f_a(U_1, \dots, U_{n-1}) \cdot T_n^a$, where we denote the n-th variable by T_n instead of U_n. By definition and direct calculation we have the following fundamental formula to calculate the cocycles (see [6, Lemma 3.2]):

$$\delta_n(f)(U_1, \dots, U_n, T_{n+1}) = \sum_{0 \leq a \leq p-1} \delta_{n-1}(f_a)(U_1, \dots, U_n) \cdot T_{n+1}^a$$

$$+(-1)^{n-1} \sum_{0 \leq a \leq p-1} f_a(U_1, \dots, U_{n-1}) \cdot \big(\omega^d \cdot (U_n + \omega^{-1} T_{n+1})^a - (U_n + T_{n+1})^a\big). \qquad (4)$$

By the following lemma shown in [5, 6] we may consider only the case of $\omega^d = 1$.

Lemma 3.1. *([5, lemma 3.1]) If $\omega^d \neq 1$, then the complex $C^n_d(X)$ is acyclic.*

Proof. Let $f \in C^n_d(X)$ be an n-cocycle. Substituting $T_{n+1} = 0$ to (4), we obtain

$$0 = \delta_{n-1}(f_0)(U_1, \ldots, U_n) + (-1)^{n-1}(\omega^d - 1) \sum_{0 \le a \le p-1} f_a(U_1, \ldots, U_{n-1}) \cdot U_n^a.$$

We thus have $f = (-1)^n(\omega^d - 1)^{-1} \cdot \delta_{n-1}(f_0)$, which implies f is coboundary. $\square$

Next, we consider the following submodule of $C^n(X)$: for $n \ge 1$

$$C^{n(1)}(X) := \{f_0(U_1, \ldots, U_{n-1}) \in C^n(X) |\ f_0 \in \mathbb{Z}_p[U_1, \ldots, U_{n-1}]\},$$

and $C^{0(1)}(X) := \{0\}$. Put $C^{n(1)}_d(X) := C^{n(1)}(X) \cap C^n_d(X)$. Since $\delta_n(C^{n(1)}_d(X))$ is contained in $C^{n+1(1)}_d(X)$, we define

$$Z^n_d(X) := \mathrm{Ker}(\delta_n) \cap C^n_d(X), \quad B^n_d(X) := \delta_{n-1}(C^{n-1}_d(X)), \quad H^n_d(X) := Z^n_d(X)/B^n_d(X),$$
$$Z^{n(1)}_d(X) := \mathrm{Ker}(\delta_n) \cap C^{n(1)}_d(X), \quad B^{n(1)}_d(X) := \delta_{n-1}(C^{n-1(1)}_d(X)),$$
$$H^{n(1)}_d(X) := Z^{n(1)}_d(X)/B^{n(1)}_d(X), \quad H^{n(0)}_d(X) := Z^n_d(X)/(B^n_d(X) + Z^{n(1)}_d(X)). \tag{5}$$

We denote $\oplus_d H^{n(1)}_d(X)$ and $\oplus_d H^{n(0)}_d(X)$ by $H^{n(1)}(X)$ and $H^{n(0)}(X)$, respectively, where the direct sums are over all d satisfying $\omega^d = 1$. Further $c^{(1)}_n$ and $c^{(0)}_n$ denote $\dim\big(H^{n(1)}(X)\big)$ and $\dim\big(H^{n(0)}(X)\big)$ respectively. Note that by definition $c^{(0)}_0 = c^{(1)}_1 = 1$ is clear. It is shown [5, 6] that $c^{(0)}_1 = c^{(0)}_2 = c^{(1)}_2 = c^{(1)}_3 = 0$. It is also shown [5] that $c^{(0)}_3 = 1$ if $\omega = -1$, and that $c^{(0)}_3 = 1$ if $\omega \ne -1$.

3.2 Quandle cohomology groups of Alexander quandles of order p

In this subsection, we state a decomposition of and the dimension of the quandle cohomology group for an Alexander quandle of order p.

We first consider the short exact sequence as follows:

$$0 \longrightarrow C^{n(1)}_d(X) \xrightarrow{i^n_d} C^n_d(X) \xrightarrow{p^n_d} C^n_d(X)/C^{n(1)}_d(X) \longrightarrow 0.$$

This canonically induces the long exact sequence

$$\cdots \longrightarrow H^{n-1(0)}_d(X) \longrightarrow H^{n(1)}_d(X) \xrightarrow{(i^n_d)_*} H^n_d(X) \xrightarrow{(p^n_d)_*} H^{n(0)}_d(X) \longrightarrow H^{n+1(1)}_d(X) \longrightarrow \cdots.$$

Then there is a canonical decomposition of $H^n_d(X)$ as follows:

Proposition 3.2. *Let X be an Alexander quandle of order p.*
(I) *If $n \ge 2$ and $\omega^d = 1$, then $(i^n_d)_*$ is a splitting injection and $(p^n_d)_*$ is a splitting surjection. In particular $H^n_d(X) \cong H^{n(0)}_d(X) \oplus H^{n(1)}_d(X)$.*
(II) *If $n \ge 2$ and $\omega^d = 1$, then the canonical inclusion $C^{n(1)}_d(X) \hookrightarrow C^{n-1}_d(X)$ induces an isomorphism $H^{n-1(0)}_d(X) \cong H^{n(1)}_d(X)$. As a result, $H^n(X) \cong H^{n-1(0)}(X) \oplus H^{n(0)}(X)$ and $\dim(H^n(X)) = c^{(0)}_{n-1} + c^{(0)}_n$.*

We will show Proposition 3.2 in Section 3.4. By Proposition 3.2 in order to estimate $H^n(X)$ we will search $H^{n(0)}(X)$. In Section 5.3 we will show the following theorem which is the key to prove Theorem 2.2:

Theorem 3.3. *Let X be the Alexander quandle of order p with $\omega \neq 0,1$. Let e be the order of ω. Let $c_n^{(0)}$ be the dimension of $H^{n(0)}(X)$. Then*
(I) $c_0^{(0)} = c_{2e-1}^{(0)} = 1$, *and* $c_n^{(0)} = 0$ *for* $1 \leq n \leq 2e-2$.
(II) $c_n^{(0)} = c_{n-2e+2}^{(0)} + c_{n-2e+1}^{(0)} + c_{n-2e}^{(0)}$ *for* $n \geq 2e$.

Remark 3.4. By Theorem 3.3, $\dim(H^{n(0)})$ is determined by the above conditions (I) and (II). Moreover, we determine $H^n(X)$ as follows. Put $F_c(x) := \sum_{i\geq 0} \dim(H^i(X))x^i \in \mathbb{Z}[[x]]$. Since by Proposition 3.2 we have $\dim(H^n(X)) = c_{n-1}^{(0)} + c_n^{(0)}$, Theorem 3.3 follows $F_c(x) = (1 - x^{2e-2} + x^{2e}) \cdot (1 - x^{2e-2} - x^{2e-1} - x^{2e})^{-1}$.

Furthermore, in Section 5.5 we will give concrete presentations of all n-cocycles which span $H^{n(0)}(X)$ (Corollary 5.9). Therefore we can determine $\dim(H_d^n(X))$, and $\dim(H_d^{n(0)}(X))$ for any d and n, although we omit the explicit formulae.

3.3 Calculus on the quandle cochain groups

We will prepare a differential operation and an integral operation on the quandle cochain group. The calculus on the quandle cochain groups is a key method in this paper. We define the degree -1 homomorphism $D_d^n : C_d^n(X) \to C_{d-1}^n(X)$ given by

$$D_d^n\Big(\sum_{0\leq a\leq p-1} f_a(U_1, \cdots, U_{n-1}) \cdot T_n^a\Big) = \sum_{0\leq a\leq p-1} a \cdot f_a(U_1, \cdots, U_{n-1}) \cdot T_n^{a-1}.$$

Note that $\mathrm{Ker}(D_d^n) = C_d^{n(1)}(X)$ and that any elements of the form $f_{p-1}(U_1, \ldots, U_{n-1}) \cdot T_n^{p-1} \in C_{d-1}^n(X)$ are not contained in the image of D_d^n. Further we can check that $\delta_n \circ D_d^n = D_d^{n+1} \circ \delta_n$. Moreover for simplicity we denote by $D_{d-j\leq d}^n$ the composite $D_{d-j+1}^n \circ D_{d-j+2}^n \cdots \circ D_d^n : C_d^n(X) \to C_{d-j}^n(X)$. It goes without saying that $D_{d-j\leq d}^n$ is a chain homomorphism : $\delta_n \circ D_{d-j\leq d}^n = D_{d-j\leq d}^{n+1} \circ \delta_n$. Also note that $D_{d-p+1\leq d}^n(f)$ means the coefficient of T_n^{p-1} in $-f$ and that $D_{d-p\leq d}^n(f) = 0$. Further, if we regard $D_{d-p\leq d-1}^n$ as a map from $C_{d-1}^n(X)$ to $C_{d-p}^{n(1)}(X)$ and regard $C_{d-p}^{n(1)}(X)$ as a $\mathbb{Z}_p$-vector subspace of $C_{d-p}^{n-1}(X)$, then we may consider the composite $D_{d-p-j\leq d-p}^{n-1} \circ D_{d-p\leq d-1}^n : C_{d-1}^n(X) \to C_{d-p-j}^{n-1}(X)$ for any j.

On the other hand, we will introduce an integral operation. Note that the operation D_d^n induces a vector isomorphism from $C_d^n(X)/C_d^{n(1)}(X)$ to $\mathrm{Im}(D_d^n)$. Put a canonical crossed section $s : C_d^n(X)/C_d^{n(1)}(X) \to C_d^n(X)$. Then we define the integration $\int_n : \mathrm{Im}(D_d^n) \to C_d^n(X)$ to be the composite $s \circ (D_d^n)^{-1}$. More precisely, for $f = \sum_{0\leq a\leq p-2} f_a(U_1, \ldots, U_{n-1}) \cdot$

$U_n^a \in \mathrm{Im}(D_d^n)$ the integration is given by

$$\int_n (f)(U_1, \ldots, U_n) = \sum_{0 \le a \le p-2} (a+1)^{-1} \cdot f_a(U_1, \ldots, U_{n-1}) \cdot U_n^{a+1}. \tag{6}$$

Note that for $g = \sum_{0 \le a \le p-1} g_a(U_1, \ldots, U_{n-1}) \cdot U_n^a \in C_d^n(X)$, $(\int_n \circ D_d^n)(g) = g - g_0$. Further, by direct calculation we can show the relation between the integration and δ_n as follows.

Lemma 3.5. *For $f \in \mathrm{Im}(D_d^{n-1})$, let us regard $\int_{n-1}(f)$ as an element of $C_d^{n(1)}(X)$. Then*

$$(\delta_{n-1} \circ \int_{n-1})(f) = (\int_n \circ\, \delta_{n-1})(f) + (-1)^{n-1}(1 - \omega^d) \int_{n-1} (f) \quad \in C_d^n(X). \tag{7}$$

Remark that if $\omega^d = 1$, then the integral operation commutes with the boundary map δ_n.

3.4 The proof of Proposition 3.2

Proof. The proof of (I) proceeds as follows. We first construct a crossed section of $(i_d^n)_*$. By Lemma 3.6 (II) bellow we may put a map $p' : Z_d^n(X) \longrightarrow Z_d^{n(1)}(X)$ given by $p'(f) = f_0$ for $f = \sum f_a(U_1, \ldots, U_{n-2}) \cdot U_{n-1}^a \in Z_d^n(X)$. It follows from Lemma 3.6 (I) that the map p' induces $(p')_* : H_d^n(X) \longrightarrow H_d^{n(1)}(X)$. By construction we have $(p')_* \circ (i_d^n)_* = (p' \circ i_d^n)_* = (\mathrm{id}_{C_d^{n(1)}(X)})_* = \mathrm{id}_{H_d^{n(1)}(X)}$.

Next, we will construct a crossed section of $(p_d^n)_*$. By Lemma 3.6 (II) we may identify $Z_d^n(X)/Z_d^{n(1)}(X)$ with $Z_d^n(X) \cap \big(C_{d-1}^n(X) \cdot U_n\big)$. Hence we have a canonical inclusion $i' : Z_d^n(X)/Z_d^{n(1)}(X) \longrightarrow Z_d^n(X)$. Since this map i' does not depend on the coboundary, we obtain the map $(i')_* : H_d^{n(0)}(X) \longrightarrow H_d^n(X)$. It can be verified that $(p_d^n)_* \circ (i')_*$ is the identity map.

We will show (II). From the definition of $C_d^n(X)$ and $C_d^{n(1)}(X)$ we have $C_d^{n-1}(X) = C_d^{n-1(1)}(X) \oplus C_d^{n(1)}(X)$. Then

$$\begin{aligned} B_d^{n-1}(X) &= B_d^{n-1}(X) \cap \big(C_d^{n-1(1)}(X) \oplus C_d^{n(1)}(X)\big) = B_d^{n-1(1)}(X) \oplus \delta_{n-2}\big(C_d^{n-1(1)}(X)\big) \\ &= B_d^{n-1(1)}(X) \oplus \delta_{n-1}\big(C_d^{n-1(1)}(X)\big) = B_d^{n-1(1)}(X) \oplus B_d^{n(1)}(X), \end{aligned}$$

where the second equality is obtained from Lemma 3.6 (I), and the third equality is obtained from $\delta_{n-1}(f) = \delta_{n-2}(f) + (-1)^n(\omega^d - 1) \cdot f = \delta_{n-2}(f)$ for any $f \in C_d^{n-1(1)}(X)$ by the equality (4) and $\omega^d = 1$. On the other hand, by Lemma 3.6 (II) we have $Z_d^{n-1}(X) = Z_d^{n-1(1)}(X) \oplus Z_d^{n(1)}(X)$. Therefore from the definition of $H_d^{n(0)}(X)$ we have

$$\begin{aligned} H_d^{n-1(0)}(X) &= \Big(Z_d^{n(1)}(X) \oplus Z_d^{n-1(1)}(X)\Big) / \Big(\big(B_d^{n(1)}(X) \oplus B_d^{n-1(1)}(X)\big) + Z_d^{n-1(1)}(X)\Big) \\ &= \big(Z_d^{n(1)}(X) \oplus Z_d^{n-1(1)}(X)\big) / \big(B_d^{n(1)}(X) \oplus Z_d^{n-1(1)}(X)\big) \\ &\cong Z_d^{n(1)}(X)/B_d^{n(1)}(X) = H_d^{n(1)}(X), \end{aligned}$$

where the second equality is obtained from $B_d^{n-1(1)}(X) \subset Z_d^{n-1(1)}(X)$. Further it can be verified that the third isomorphism is derived from the canonical inclusion $C_d^{n(1)}(X) \hookrightarrow C_d^{n-1}(X)$. $\square$

Lemma 3.6. *Let X be an Alexander quandle of order p.*
(I) If $\omega^d = 1$ and $n \geq 2$, then $B_d^n(X) \cap C_d^{n(1)}(X) = B_d^{n(1)}(X)$.
(II) If $\omega^d = 1$ and $n \geq 2$, then $Z_d^n(X) = Z_d^{n(1)}(X) \oplus (Z_d^n(X) \cap C_{d-1}^n(X) \cdot U_n)$.

Proof. Put $f \in C_d^{n-1}(X)$. We decompose $f = \sum f_i(U_1, \dots, U_{n-2}) \cdot U_{n-1}^i$.

We will prove (I). " $\supseteq$ " is clear. Conversely we assume $\delta_{n-1}(f) \in C_d^{n(1)}(X)$. By (4) we have

$$\delta_{n-1}(f)(U_1, \dots, U_{n-1}, T_n) = \sum \delta_{n-2}(f_a)(U_1, \dots, U_{n-1}) \cdot T_n^a$$
$$+(-1)^n \sum f_a(U_1, \dots, U_{n-2}) \cdot \big((U_{n-1} + \omega^{-1}T_n)^a - (U_{n-1} + T_n)^a\big) \in C_d^{n(1)}(X). \tag{8}$$

By comparing the coefficient of T_n^1 in the both hand sides we have

$$\delta_{n-2}(f_1) = (-1)^{n-1}(1 - \omega^{-1}) \cdot D_d^{n-1}(f). \tag{9}$$

We integrate this equality by U_{n-1}, and obtain

$$(-1)^{n-1}(1 - \omega^{-1}) \cdot (f - f_0) = (\int_{n-1} \circ\, \delta_{n-2})(f_1).$$

Since $\delta_{n-2}(f_1)$ is contained in $\mathrm{Im}(D_d^{n-1})$ by (9), we apply δ_{n-1} to the equality, and obtain

$$(-1)^{n-1}(1-\omega^{-1}) \cdot \delta_{n-1}(f - f_0) = \delta_{n-1}\big((\int_{n-1} \circ\, \delta_{n-2})(f_1)\big) = (\int_{n} \circ\, \delta_{n-1} \circ\, \delta_{n-2})(f_1) = 0,$$

where the second equality is obtained from (7) and $\omega^d = 1$. Consequently we obtain $\delta_{n-1}(f) = \delta_{n-1}(f_0) \in B_d^{n(1)}(X)$, which completes the proof of (I).

To prove (II), let f be an $(n-1)$-cocycle. By comparing the coefficient of T_n^0 in (8), we have $\delta_{n-1}(f_0) = 0$. Therefore we obtain the required decomposition; $f = f_0 + (f - f_0)$. $\square$

3.5 Mochizuki's techniques in general case of n

In [5, 6] T. Mochizuki discovered all of 2- and 3-cocycles of the quandle cohomology groups of certain Alexander quandles. In general case of n we will follow his techniques in [6, Section 3.2.4] to order to calculate quandle n-cocycles of Alexander quandles of order p.

We will construct an isomorphism from $H^{n(0)}(X)$ to a quotient space given by (11) below. Assume that $f \in C_d^n(X)$ is an n-cocycle and $\omega^d = 1$. Then by (4) we obtain

$$0 = \sum_{0 \leq a \leq p-1} \delta_{n-1}(f_a)(U_1, \dots, U_n) \cdot T_{n+1}^a - (-1)^n f_a(U_1, \dots, U_{n-1}) \cdot \big((U_n + \omega^{-1}T_{n+1})^a - (U_n + T_{n+1})^a\big).$$

By comparing with the coefficients of T_{n+1}^1 in the both hand sides, we have

$$\delta_{n-1}(f_1)(U_1,\ldots,U_n) = (-1)^n(\omega^{-1}-1)\sum_{0\le a\le p-1} f_a(U_1,\ldots,U_{n-1})\cdot a\cdot U_n^{a-1}. \tag{10}$$

Namely, $\delta_{n-1}(f_1) = (-1)^n(\omega^{-1}-1)\cdot D_d^n(f)$. Hence $f_1 \in \delta_{n-1}^{-1}(\mathrm{Im}(D_d^n))$. We thus obtain a homomorphism $\phi : Z_d^n(X) \to \delta_{n-1}^{-1}(\mathrm{Im}(D_d^n))$ given by $\phi(f) = f_1$.

Lemma 3.7. *Let g be an element of $C_d^{n-1}(X)$. We decompose $g = \sum_a g_a(U_1,\ldots,U_{n-2})\cdot T_{n-1}^a$. If $\omega^d = 1$, then*

$$\phi(\delta_{n-1}(g)) = \delta_{n-2}(g_1) + (-1)^n(\omega^{-1}-1)\cdot D_d^{n-1}(g).$$

Proof. Straightforward. □

It is also clear that $\phi(Z_d^{n(1)}(X)) = \{0\}$. Hence the homomorphism ϕ induces

$$\bar{\phi} : \frac{\mathrm{Z}_d^n(X)}{B_d^n(X)+\mathrm{Z}_d^{n(1)}(X)} \longrightarrow \frac{\delta_{n-1}^{-1}(\mathrm{Im}(D_d^n))}{\mathrm{Im}(D_d^{n-1})+B_{d-1}^{n-1}(X)}. \tag{11}$$

T. Mochizuki showed that the homomorphism $\bar{\phi}$ is an isomorphism in the case of $n = 3$ [6, Lemma 3.17]. In general case of n it holds for Alexander quandles of order p as follows:

Proposition 3.8. *If $\omega^d = 1$, then the homomorphism $\bar{\phi}$ is an isomorphism. Further, for a representative of the form $g_{p-1}(U_1,\ldots,U_{n-2})\cdot U_{n-1}^{p-1} \in \delta_{n-1}^{-1}(\mathrm{Im}(D_d^n))$, the inverse element is represented by (14).*

Proof. We first show that the homomorphism $\bar{\phi}$ is injective. Let $f \in C_d^n(X)$ be an n-cocycle such that $\bar{\phi}(f) = 0$. Then we have $f_1 = \delta_{n-2}(h) + D_d^{n-1}(g)$ for some $h \in C_{d-1}^{n-2}(X)$ and $g \in C_d^{n-1}(X)$. Put an n-cocycle $\bar{f}$ given by $\bar{f} = f+(-1)^n(1-\omega^{-1})^{-1}\cdot\delta_{n-1}(g) \in C_d^n(X)$. It follows from (10) that $D_d^n(\bar{f}) = D_d^n(f) + (-1)^n(1-\omega^{-1})^{-1}\cdot\delta_{n-1}(D_d^{n-1}(g)) = 0$. Since $\mathrm{Ker}(D_d^n) = C_d^{n(1)}(X)$, it implies $f \in B_d^n(X) + Z_d^{n(1)}(X)$.

Next, we will show its surjection. Assume $g \in C_{d-1}^{n-1}(X)$ satisfies $\delta_{n-1}(g) \in \mathrm{Im}(D_d^n)$. Since we consider g modulo $\mathrm{Im}(D_d^{n-1})$, we may deal with only g of the form given by

$$g(U_1,\ldots,U_{n-2},T_{n-1}) = g_{p-1}(U_1,\ldots,U_{n-2})\cdot T_{n-1}^{p-1}, \tag{12}$$

where $g_{p-1} \in C_{d-p}^{n-2}(X)$. By Lemma 3.10 below, the g_{p-1} satisfies the equality (15), that is,

$$\delta_{n-2}(g_{p-1})(U_1,\ldots,U_{n-1}) = (-1)^n(1-\omega^{-1})\cdot g_{p-1}(U_1,\ldots,U_{n-2}) \in C_{d-p}^{n-1(1)}(X). \tag{13}$$

Then we claim that an inverse element of g is represented by

$$\hat{g}(U_1,\ldots,U_{n-1},T_n) = g_{p-1}(U_1,\ldots,U_{n-2})\cdot\frac{(1-\omega^{-1})T_n^p - (U_{n-1}+T_n)^p + (U_{n-1}+\omega^{-1}T_n)^p}{(1-\omega^{-1})p}.$$

(14)

We have to check that $\hat{g}$ is an n-cocycle. Indeed it follows from (2) that $(-1)^{n-1} \cdot (1 - \omega^{-1}) \cdot \delta_n(\hat{g})$ is given by

$$\begin{aligned}
p^{-1} \cdot \Big(& (-1)^{n-1} \cdot \delta_{n-2}(g_{p-1}) \cdot \big((1-\omega^{-1})T_{n+1}^p - (U_n + T_{n+1})^p + (U_n + \omega^{-1}T_{n+1})^p\big) \\
& + \omega^{d-p} \cdot g_{p-1} \cdot \big((1-\omega^{-1})T_{n+1}^p - (\omega U_{n-1} + U_n + T_{n+1})^p + (\omega U_{n-1} + U_n + \omega^{-1}T_{n+1})^p\big) \\
& - g_{p-1} \cdot \big((1-\omega^{-1})T_{n+1}^p - (U_{n-1} + U_n + T_{n+1})^p + (U_{n-1} + U_n + \omega^{-1}T_{n+1})^p\big) \\
& - \omega^{d-p} \cdot g_{p-1} \cdot \big((1-\omega^{-1})(\omega U_n + T_{n+1})^p - (\omega U_{n-1} + \omega U_n + T_{n+1})^p + (\omega U_{n-1} + U_n + \omega^{-1}T_{n+1})^p\big) \\
& + g_{p-1} \cdot \big((1-\omega^{-1})(U_n + T_{n+1})^p - (U_{n-1} + U_n + T_{n+1})^p + (U_{n-1} + \omega^{-1}U_n + \omega^{-1}T_{n+1})^p\big)\Big).
\end{aligned}$$

Using (13), it is not hard to see that all of the above terms cancel. Finally, it can be verified that $\phi(\hat{g}) = g_{p-1} \cdot T_{n-1}^{p-1}$. □

Remark 3.9. We can obtain the above form of $\hat{g}$ as follows. T. Mochizuki showed the integral formula to formulate n-cocycles of Alexander quandles on $\mathbb{R}$ (see [5, Proposition 3.1]). By pulling back the formula to $\mathbb{Z}_p$ he found a non-trivial 3-cocycle of the dihedral quandle of order p [5, Theorem 3.1]. Similarly for any n we can induce the above form of $\hat{g}$ from the integral formula given by (14) above.

Lemma 3.10. *Let g be an element of $C_{d-1}^{n-1}(X)$ with no term of $C_{d-1}^{n-1(1)}(X)$. Let us decompose g as $g = \sum_{1 \le a \le p-1} g_a(U_1, \ldots, U_{n-2}) \cdot U_{n-1}^a$. If $\delta_{n-1}(g) \in \mathrm{Im}(D_d^n)$, then it satisfies*

$$\delta_{n-2}(g_{p-1})(U_1, \ldots, U_{n-1}) = (-1)^n(1 - \omega^{d-1}) \cdot g_{p-1}(U_1, \ldots, U_{n-2}) \in C_{d-p}^{n-1(1)}(X). \quad (15)$$

Proof. By (4) we thus have

$$\begin{aligned}
\delta_{n-1}(g)(U_1, \ldots, U_{n-1}, T_n) = & \sum_{1 \le a \le p-1} \delta_{n-2}(g_a)(U_1, \ldots, U_{n-1}) \cdot T_n^a \\
& + (-1)^n g_a(U_1, \ldots, U_{n-2})\big(\omega^{d-1} \cdot (U_{n-1} + \omega^{-1}T_n)^a - (U_{n-1} + T_n)^a\big). \quad (16)
\end{aligned}$$

Since $\delta_{n-1}(g) \in \mathrm{Im}(D_d^n)$, by comparing with the coefficients of T_n^{p-1} in the right hand side we obtain the equality (15). □

Before going into the next section, we give a proposition for the forms of the cocycles. We define the following polynomial $E_{n-1}^{\omega} \in \mathbb{Z}_p[U_{n-1}, U_n]$:

$$\begin{aligned}
E_{n-1}^{\omega}(U_{n-1}, U_n) &= \big((U_{n-1} + U_n)^p + (1 - \omega^{-1})U_n^p - (U_{n-1} + \omega^{-1}U_n)^p\big)/p \\
&\equiv \sum_{1 \le j \le p-2} j^{-1} \cdot (1 - \omega^{-j}) \cdot U_{n-1}^{p-j} \cdot U_n^j \qquad (\mathrm{mod}\ p). \quad (17)
\end{aligned}$$

Then we will show the following proposition.

Proposition 3.11. *Let X be an Alexander quandle of order p. Then $H^{n(0)}(X)$ is generated by some elements of the form $f'_{n-2} \cdot E^{\omega}_{n-1}$, where f'_{n-2} is contained in $C^{n-2}_{d-p}(X)$ for some d and satisfies $\delta_{n-2}(f'_{n-2}) = (-1)^n(1-\omega^{-1}) \cdot f'_{n-2} \in C^{n-1(1)}_{d-p}(X)$.*

Proof. According to (14), if $f'_{n-2} \in C^{n-2}_{d-p}(X)$ satisfies $\delta_{n-2}(f'_{n-2}) = (-1)^n(1-\omega^{-1}) \cdot f'_{n-2}$ and $f'_{n-2} \cdot U^{p-1}_{n-1} \in C^{n-1}_{d-1}(X)$, then $f'_{n-2} \cdot E^{\omega}_{n-1} \in C^n_d(X)$ is an n-cocycle. Conversely it follows from Proposition 3.8 that any representatives of $H^{n(0)}(X)$ are derived from such f'_{n-2} through the isomorphism $\bar{\phi}$. □

4 The proof of Theorem 3.3 (I)

Throughout this section we fix the order e of ω, in other ward, e is the minimal number of satisfying $\omega^e = 1$. Also, we assume that $\omega^d = 1$. In Section 4.1, we will search the quotient space of (11) in the case $\omega \neq -1$. We will construct a homomorphism $\overline{\Theta}_i$ given by (19) and show that the map is an isomorphism (Proposition 4.2). Roughly speaking, the isomorphism makes the degree of the quotient space lower degree. In Section 4.3, as a corollary, we will prove Theorem 3.3 (I), i.e., $c^{(0)}_1 = \cdots = c^{(0)}_{2e-2} = 0$ and $c^{(0)}_0 = c^{(0)}_{2e-1} = 1$.

4.1 The isomorphism $\overline{\Theta}_i$

We will construct a map presented by (19) below. In this subsection we assume that $\omega \neq -1$ and we fix $1 \leq i \leq e-2$ such that $n \geq 2i+2$. To begin with, we put $g = \sum_j g_j(U_1, \ldots, U_{n-2i}) \cdot U^j_{n-2i+1}$ contained in $\delta^{-1}_{n-2i+1}(\mathrm{Im}(D^{n-2i+2}_{d-ip+p}))$. By Lemma 3.10 we have

$$\delta_{n-2i}(g_{p-1}) = (-1)^{n-2i+2}(1-\omega^{d-ip+p-1}) \cdot g_{p-1} = (-1)^n(1-\omega^{-i}) \cdot g_{p-1}.$$

Moreover, we decompose $g_{p-1}(U_1, \ldots, U_{n-2i+1}) = \sum_{1 \leq k \leq p-1} g_{k,p-1}(U_1, \ldots, U_{n-2i}) \cdot U^k_{n-2i+1}$. Then the above equalities mean

$$(-1)^n(1-\omega^{-i}) \cdot g_{p-1}(U_1, \ldots, U_{n-2i}) = \sum_{1 \leq k \leq p-1} \delta_{n-2i-1}(g_{k,p-1}(U_1, \ldots, U_{n-2i}) \cdot T^k_{n-2i+1}$$
$$+(-1)^{n-1} \sum_{1 \leq k \leq p-1} g_{k,p-1}(U_1, \ldots, U_{n-2i-1}) \cdot \big(\omega^{d-ip}(U_{n-2i} + \omega^{-1} T_{n-2i+1})^k - (U_{n-2i} - T_{n-2i+1})^k\big).$$

By comparing the coefficients of T^1_{n-2i+1} in the both hand sides we have

$$\delta_{n-2i-1}(g_{1,p-1}) = (-1)^n(\omega^{-i-1} - 1) \cdot D^{n-2i}_{d-ip}(g_{p-1}). \tag{18}$$

Therefore we obtain a homomorphism

$$\Theta_i : \delta^{-1}_{n-2i+1}(\mathrm{Im}(D^{n-2i+2}_{d-ip+p})) \longrightarrow \delta^{-1}_{n-2i-1}(\mathrm{Im}(D^{n-2i}_{d-ip})), \tag{19}$$

given by $\Theta_i(g) = g_{1,p-1}$. We notice the following lemma.

Lemma 4.1. *Let h be an element of $C^{n-2i}_{d-ip+p-1}(X)$. We decompose $h = \sum_{a,b} h_{a,b}(U_1, \ldots, U_{n-2i-2}) \cdot U^a_{n-2i-1} \cdot T^b_{n-2i}$. If $\omega^d = 1$, then*

$$\Theta_i(\delta_{n-2i}(h)) = \delta_{n-2i-2}(h_{1,p-1}) + (-1)^n(\omega^{-i-1}-1) \cdot (\sum_{1 \le a \le p-1} a \cdot h_{a,p-1}(U_1, \ldots, U_{n-2i-2}) \cdot U^{a-1}_{n-2i-1}).$$

Proof. By direct calculation. □

It is clear that $\Theta_i\big(\mathrm{Im}(D^{n-2i+1}_{d-ip+p})\big) = \{0\}$. Hence, the homomorphism Θ_i induces

$$\overline{\Theta}_i : \frac{\delta^{-1}_{n-2i+1}(\mathrm{Im}(D^{n-2i+2}_{d-ip+p}))}{\mathrm{Im}(D^{n-2i+1}_{d-ip+p}) + B^{n-2i+1}_{d-ip+p-1}(X)} \longrightarrow \frac{\delta^{-1}_{n-2i-1}(\mathrm{Im}(D^{n-2i}_{d-ip}))}{\mathrm{Im}(D^{n-2i-1}_{d-ip}) + B^{n-2i-1}_{d-ip-1}(X)}. \tag{20}$$

Note that if $i = 1$, then the left hand side is exactly the right hand side in (11).

Proposition 4.2. *If $\omega^d = 1$, $n - 2i - 2 \ge 0$ and $0 < i \le e - 2$, then $\overline{\Theta}_i$ is an isomorphism. Further, for a representative of the form $g'_{p-1}(U_1, \ldots, U_{n-2i-2}) \cdot U^{p-1}_{n-2i-1} \in \delta^{-1}_{n-2i-1}(\mathrm{Im}(D^{n-2i}_{d-ip}))$, the inverse element is represented by the formula (25) below.*

Proof. For the proof, we give some remarks . Since $\omega^d = 1$, we have $\omega^{d-ip-p-1} = \omega^{-i}$. Also, note that $1 - \omega^{-i}, 1 - \omega^{-i-1} \in \mathbb{Z}_p$ are nonzero elements from the definition of e. It goes without saying $(-1)^{n-2i+1} = (-1)^{n+1}$.

First, we will show the injection. Assume $\overline{\Theta}_i(g) = 0$, that is, $g_{1,p-1} \in \mathrm{Im}(D^{n-2i-1}_{d-ip}) + B^{n-2i-1}_{d-ip-1}(X)$. We will show the g can be killed as (22) below. We have $g_{1,p-1} = D^{n-2i-1}_{d-ip}(g') + \delta_{n-2i-2}(h)$ for some $g' \in C^{n-2i-1}_{d-ip}(X)$, and $h \in C^{n-2i-2}_{d-ip-1}(X)$. We may assume that the coefficient of U^0_{n-2i-1} in g' is zero. Then we integrate (18) by T_{n-2i}, and obtain

$$(-1)^n(\omega^{-i-1} - 1) \cdot g_{p-1} = (\int_{n-2i} \circ \delta_{n-2i-1} \circ D^{n-2i-1}_{d-ip})(g')$$

$$= (\int_{n-2i} \circ D^{n-2i}_{d-ip})\big(\delta_{n-2i-1}(g')\big) = \delta_{n-2i-1}(g') - (-1)^n(\omega^{-i} - 1) \cdot g', \tag{21}$$

where the last equality is obtained from that the coefficient of U^0_{n-2i-1} in $\delta_{n-2i-1}(g')$ is $(-1)^{n-2i-2}(\omega^{d-ip} - 1) \cdot g'$ by (4). We put $\check{g}(U_1, \ldots, U_{n-2i}) := g'(U_1, \ldots, U_{n-2i-1}) \cdot U^{p-1}_{n-2i} \in C^{n-2i}_{d-ip+p-1}(X)$. Then we have

$$\delta_{n-2i}(\check{g})(U_1, \ldots, U_{n-2i}, T_{n-2i+1}) = \delta_{n-2i-1}(g')(U_1, \ldots, U_{n-2i}) \cdot T^{p-1}_{n-2i+1}$$

$$+(-1)^{n+1} g'(U_1, \ldots, U_{n-2i-1}) \cdot \big(\omega^{-i}(U_{n-2i} + \omega^{-1}T_{n-2i+1})^{p-1} - (U_{n-2i} + T_{n-2i+1})^{p-1}\big)$$

$$\begin{aligned} &\equiv \big((-1)^n(\omega^{-i-1} - 1) \cdot g_{p-1} + (-1)^n(\omega^{-i} - 1) \cdot g'\big)(U_1, \ldots, U_{n-2i}) \cdot T^{p-1}_{n-2i+1} \\ &\quad + (-1)^{n+1}(\omega^{-i} - 1) \cdot g'(U_1, \ldots, U_{n-2i-1}) \cdot T^{p-1}_{n-2i+1} \\ &= (-1)^n(\omega^{-i-1} - 1) \cdot g_{p-1}(U_1, \ldots, U_{n-2i}) \cdot T^{p-1}_{n-2i+1} \end{aligned}$$

$$\equiv (-1)^n(\omega^{-i-1}-1)\cdot g(U_1,\dots,U_{n-2i},T_{n-2i+1}) \qquad \mod(\mathrm{Im}(D^{n-2i+1}_{d-ip+p})), \tag{22}$$

where the second equality is derived from (21). Provided that $\omega^{i+1}\neq 1$, the above equality implies that $g\in C^{n-2i+1}_{d-ip+p-1}(X)$ is killed as required.

Next, we will show the surjection. We put $g'\in \delta^{-1}_{n-2i-1}(\mathrm{Im}(D^{n-2i}_{d-ip}))$. Since we consider modulo $\mathrm{Im}(D^{n-2i-1}_{d-ip})$, we may assume that g' is of the form

$$g'(U_1,\dots,U_{n-2i-1}) = g'_{p-1}(U_1,\dots,U_{n-2i-2})\cdot U^{p-1}_{n-2i-1}.$$

Note that by Lemma 3.10 we obtain

$$\delta_{n-2i-2}(g'_{p-1}) = (-1)^n(1-\omega^{-i-1})\cdot g'_{p-1}\in C^{n-2i-1(1)}_{d-ip-p}(X). \tag{23}$$

We claim that the following polynomial is the inverse element of g':

$$\breve{g}'(U_1,\dots,U_{n-2i},T_{n-2i+1}) := (\omega^{-i-1}-1)^{-1}\cdot(\int_{n-2i}\circ\delta_{n-2i-1})(g')(U_1,\dots,U_{n-2i})\cdot T^{p-1}_{n-2i+1}. \tag{24}$$

By (23) we can check that $\breve{g}'$ is also presented by

$$(-1)^n(\omega^{-i-1}-1)^{-1}\cdot g'_{p-1}(U_1,\dots,U_{n-2i-2})\cdot E^{\omega,i}_{n-2i-1}(U_{n-2i-1},U_{n-2i})\cdot T^{p-1}_{n-2i+1}, \tag{25}$$

where the polynomial $E^{\omega,i}_{n-2i-1}(U_{n-2i-1},U_{n-2i})$ is introduced by the formula (26) below. Finally, we have to check that $\delta_{n-2i+1}(\breve{g}')\in\mathrm{Im}(D^{n-2i+2}_{d-ip+p})$. Indeed, the coefficient of T^{p-1}_{n-2i+2} in $\delta_{n-2i+1}(\breve{g}')$ vanishes, since $(-1)^n(\omega^{-i-1}-1)\cdot\delta_{n-2i+1}(\breve{g}')$ is given by

$$\begin{aligned}
&(-1)^n\delta_{n-2i+1}\Big(((\int_{n-2i}\circ\delta_{n-2i-1})(g'))\cdot U^{p-1}_{n-2i+1}\Big)\\
&=(-1)^n\delta_{n-2i}\big((\int_{n-2i}\circ\delta_{n-2i-1})(g')\big)\cdot T^{p-1}_{n-2i+2}\\
&\quad+\big((\int_{n-2i}\circ\delta_{n-2i-1})(g')\big)\cdot\big((\omega^{d-ip+p-1}(U_{n-2i+1}+\omega^{-1}T_{n-2i+2})^{p-1}-(U_{n-2i+1}+T_{n-2i+2})^{p-1}\big)\\
&=(-1)^n\big((\int_{n-2i+1}\circ\delta_{n-2i}\circ\delta_{n-2i-1})(g')+(-1)^n(1-\omega^{d-ip+p-1})(\int_{n-2i}\circ\delta_{n-2i-1})(g')\big)\cdot T^{p-1}_{n-2i+2}\\
&\quad+\int_{n-2i}\big(\delta_{n-2i-1}(g')\big)\cdot\big(\omega^{-i}(\omega U_{n-2i+1}+T_{n-2i+2})^{p-1}-(U_{n-2i+1}+T_{n-2i+2})^{p-1}\big)\\
&=\int_{n-2i}\big(\delta_{n-2i-1}(g')\big)\cdot\big((1-\omega^{-i})T^{p-1}_{n-2i+2}+\omega^{-i}(\omega U_{n-2i+1}+T_{n-2i+2})^{p-1}-(U_{n-2i+1}+T_{n-2i+2})^{p-1}\big),
\end{aligned}$$

where the second equality is obtained from (7). We easily check that $\Theta_i(\breve{g}')=g'$ by the formula (25). □

4.2 The composition of $\overline{\Theta}_i$ s

We will consider the composition of $\overline{\Theta}_i$s by Corollary 4.3. For this we prepare some polynomials. We now introduce the polynomial $E^{\omega,i}_{n-2i-1}(U_{n-2i-1}, U_{n-2i}) \in \mathbb{Z}_p[U_{n-2i-1}, U_{n-2i}]$ given by

$$\big((1-\omega^{-i-1})U^p_{n-2i} + \omega^{-i}(U_{n-2i-1} + \omega^{-1}U_{n-2i})^p - (U_{n-2i-1} + U_{n-2i})^p + (1-\omega^{-i})U^p_{n-2i-1}\big)/p$$
$$\equiv \sum_{1\le j\le p-1} j^{-1}\cdot(\omega^{-i-j}-1)\cdot U^{p-j}_{n-2i-1}\cdot U^j_{n-2i} \pmod p. \tag{26}$$

Moreover, when $\omega \ne -1$, we define the following product of the polynomials $E^{\omega,i}_{n-2i}$:

$$E^{\omega}_{n-2e+3\le n-2}(U_{n-2e+3},\dots,U_{n-2}) := \prod_{1\le k\le e-2} E^{\omega,k}_{n-2k-1}(U_{n-2k-1}, U_{n-2k}). \tag{27}$$

If $\omega = -1$, then we define $E^{\omega}_{n-2e+3\le n-2}$ to be $1 \in \mathbb{Z}_p$. We obtain the composition of the isomorphisms presented in Proposition 4.2 as follows.

Corollary 4.3. . *If $n \ge 2e-1$ and $\omega \ne -1$, then the composite map $\overline{\Theta}_{e-2}\circ\cdots\circ\overline{\Theta}_1$ gives rise an isomorphism*

$$\frac{\delta^{-1}_{n-1}(\mathrm{Im}(D^n_d))}{\mathrm{Im}(D^{n-1}_d) + B^{n-1}_d(X)} \longrightarrow \frac{\delta^{-1}_{n-2e+3}(\mathrm{Im}(D^{n-2e+4}_{d-ep+2p}))}{\mathrm{Im}(D^{n-2e+3}_{d-ep+2p}) + B^{n-2e+3}_{d-ep+2p-1}(X)}. \tag{28}$$

Further, for a representative of the form $g(U_1,\dots,U_{n-2e+2})\cdot U^{p-1}_{n-2e+3} \in \delta^{-1}_{n-2e+3}(\mathrm{Im}(D^{n-2e+4}_{d-ep+2p}))$, the inverse element is represented by $g(U_1,\dots,U_{n-2e+2})\cdot E^{\omega}_{n-2e+3\le n-2}(U_{n-2e+3},\dots,U_{n-2})\cdot T^{p-1}_{n-1}$.

Proof. According to Proposition 4.2 and the presentation of (24). □

In Section 5, we deal with the right hand side of (28).

Moreover, we consider the composition of $\overline{\Theta}_i$ and $\bar\phi$, where $\bar\phi$ is given by (11) and will give the presentation of n-cocycles as follows. Note that the statement holds for $\omega = -1$.

Corollary 4.4. *Assume $n \ge 2e-1$ and $\omega^d = 1$. Let g be an element of $C^{n-2e+2}_{d-ep+p}(X)$ which is divisible by U_{n-2e+2}. If $\omega \ne 0, 1$ and $\delta_{n-2e+2}(g) = (-1)^n(1-\omega)\cdot g \in C^{n-2e+3(1)}_{d-ep+p}(X)$, then $g(U_1,\dots,U_{n-2e+2})\cdot U^{p-1}_{n-2e+3}$ is contained in $\delta^{-1}_{n-2e+3}(\mathrm{Im}(D^{n-2e+4}_{d-ep+2p}))$.*

Moreover, for such g the following polynomial is an n-cocycle of degree d:

$$g(U_1,\dots,U_{n-2e+2})\cdot E^{\omega}_{n-2e+3\le n-2}(U_{n-2e+3},\dots,U_{n-2})\cdot E^{\omega}_{n-1}(U_{n-1},U_n). \tag{29}$$

Proof. The former is obtained by direct calculation similar to the proof in Lemma 3.10.

We will show the later part. Note that if $\omega = -1$, i.e., $e = 2$, then the above statement is the same as Proposition 3.8. For $\omega \ne -1$, since $g(U_1,\dots,U_{n-2e+2})\cdot U^{p-1}_{n-2e+3} \in$

$\delta^{-1}_{n-2e+3}(\mathrm{Im}(D^{n-2e+4}_{d-ep+2p}))$, then Corollary 4.3 yields $g(U_1, \ldots, U_{n-2e+2}) \cdot E^{\omega}_{n-2e+3 \leq n-2} \cdot T^{p-1}_{n-1} \in \delta^{-1}_{n-1}(\mathrm{Im}(D^n_{d-1}))$. Moreover according to Proposition 3.8 we conclude that the polynomial presented by (29) is an n-cocycle. $\square$

4.3 The proof of Theorem 3.3 (I)

Proof. We first consider the case of $n \leq 3$ or $\omega = -1$. Since $C^0(X) = \mathbb{Z}_p$, it is clear that $c^{(0)}_0 = 1$. By direct calculation we have $Z^{1(0)}(X) = 0$, which implies $c^{(0)}_1 = 0$. It is shown [5, Corollary 2.2] that $H^2_Q(X; \mathbb{Z}_p) \cong 0$. It is known [5, Theorem 3.1] that $H^{3(0)}(X) \cong \mathbb{Z}_p$ if $\omega = -1$, and that $H^{3(0)}_Q(X; \mathbb{Z}_p) \cong 0$ if $\omega \neq -1$. Note that this completes the proof in the case $\omega = -1$.

Then we may assume that $n \geq 4$ and $\omega \neq 1, 0, -1$. For the proof of $c^{(0)}_1 = \cdots = c^{(0)}_{2e-2} = 0$, by Proposition 3.8 it suffices to show that the right hand side of (11) vanishes for $n < 2e - 1$. For this, we consider the two cases where n is odd or even.

One considers the case where n is even. By Proposition 4.2, we have an isomorphism

$$\frac{\delta^{-1}_{n-1}(\mathrm{Im}(D^n_d))}{\mathrm{Im}(D^{n-1}_d) + B^{n-1}_{d-1}(X)} \xrightarrow{\simeq} \frac{\delta^{-1}_1(\mathrm{Im}(D^2_{d+p-np/2}))}{\mathrm{Im}(D^1_{d+p-np/2}) + B^1_{d-1+p-np/2}(X)},$$

given by the composite of $\overline{\Theta}_{n/2-1} \circ \cdots \circ \overline{\Theta}_1$. Any element of the right hand side is represented by $a_1 \cdot U^{p-1}_1$ for some $a_1 \in \mathbb{Z}_p$. Assume $a_1 \neq 0$. Therefore by comparing with the homogenous degree we have $p - 1 = d + p - np/2 - 1$, i.e., $d = pn/2$. Since $n/2 \leq e - 1$, we have $\omega^d = \omega^{n/2} \neq 1$, which contradicts $\omega^d = 1$. Hence, $a_1 = 0$, that is, the above right hand side does not have any non-trivial elements. Therefore $H^{n(0)}(X) \cong 0$.

On the other hand, we deal with the case where n is odd. Proposition 4.2 gives rise an isomorphism

$$\frac{\delta^{-1}_{n-1}(\mathrm{Im}(D^n_d))}{\mathrm{Im}(D^{n-1}_d) + B^{n-1}_{d-1}(X)} \xrightarrow{\simeq} \frac{\delta^{-1}_2(\mathrm{Im}(D^3_{d-p(n-3)/2}))}{\mathrm{Im}(D^2_{d-p(n-3)/2}) + B^2_{d-1-p(n-3)/2}(X)}, \tag{30}$$

obtained from the composite of $\overline{\Theta}_{(n-3)/2} \circ \cdots \circ \overline{\Theta}_1$. Any element of the right hand side is represented by $a_1 \cdot U^b_1 \cdot U^{p-1}_2 \in C^2_{d-1-p(n-3)/2}(X)$ for some $a_1 \in \mathbb{Z}_p$ and $b \neq 0$. By Lemma 3.10, it satisfies that

$$a_1(\omega^b - 1) \cdot U^b_1 = \delta_1(a_1 \cdot U^b_1) = a_1\big((\omega U_1 + U_2)^b - (U_1 + U_2)^b\big). \tag{31}$$

By elementary calculation we have $b = 1$ or $a_1 = 0$. If $a_1 \neq 0$, then by comparing with the homogenous degrees we have $1 + (p - 1) = d - 1 - p(n - 3)/2$, i.e., $d = p(n + 1)/2 + 1 - p$, which implies $\omega^d = \omega^{(n+1)/2} \neq 1$. Hence, $a_1 = 0$, that is, the right hand side in (30) vanishes.

Finally, we will show $c^{(0)}_{2e-1} = 1$. By Proposition 4.2, the isomorphism given by (30) also holds for the case of $n = 2e-1$. It suffices to show that the right hand side of (30) is one dimensional. Note that if $d = ep-p+1$, then $\omega^d = 1$. By the equality (31) and the above discussion, the right hand side of (30) is generated by $U_1 \cdot U_2^{p-1} \in C^2_p(X)$, which implies $c^{(0)}_{2e-1} \leq 1$. We claim that $U_1 \cdot U_2^{p-1} \in C^2_p(X)$ can not be killed. Indeed the homogenous degree of any element in $B^2_{d-1-p(n-1)/2}(X)$ is smaller than p by definition. In the sequel we obtain $c^{(0)}_{2e-1} = 1$. □

Remark 4.5. Since $\delta_1(U_1) = (\omega-1)\cdot U_1$, it follows from Corollary 4.4 that $H^{2e-1(0)}(X) \cong \mathbb{Z}_p$ is spanned by $U_1 \cdot E^{\omega}_{2\leq 2e-3}(U_2, \ldots, U_{2e-3}) \cdot E^{\omega}_{2e-2}(U_{2e-2}, U_{2e-1}) \in C^{2e-1}_{ep-p+1}(X)$.

5 Quandle cocycles and the proof of Theorem 3.3

In this section we deal with $H^{n(0)}(X)$ for $n \geq 2e$. For this, recall that by Corollary 4.3, $H^{n(0)}(X)$ is isomorphic to the quotient space in the right hand side of (28). In Section 5.1 we will present some examples of the cocycles and construct a linear map from $H^{n-2e(0)}(X) \oplus H^{n-2e+1(0)}(X) \oplus H^{n-2e+2(0)}(X)$ to the quotient space. In Section 5.2 we will show that the map is surjective (Proposition 5.4). In Section 5.3 we will show that the map is an isomorphism, leading to the required recurring formula in Theorem 3.3. In Section 5.4 we will prove Theorem 2.2. As a result, we will formulate presentations of all cocycles which span $H^n(X)$ (Corollary 5.9).

Throughout this section we fix $n \geq 2e$ and d satisfying $\omega^d = 1$. Further we denote by e the order of ω. To simply the exposition, we define $d' = d + (2-e)p$ and $m = n - 2e + 4$. Note that $m \geq 4$, $\omega^{d'-p-1} = \omega^{d'-2p} = 1$ and $(-1)^m = (-1)^n$. Also notice that $d = d'$ and $m = n$ if $\omega = -1$.

5.1 Examples of n-cocycles of the Alexander quandle of prime order

Example 5.1. Let $f_{m-3} \in C^{m-3}_{d'-p-1}(X)$ be an $(m-3)$-cocycle which is divisible by U_{m-3}. Put $g_{p-1}(U_1, \ldots, U_{m-2}) := f_{m-3}(U_1, \ldots, U_{m-3}) \cdot U_{m-2}$. Then we can check $\delta_{m-2}(g_{p-1}) = (-1)^n(1-\omega)\cdot g_{p-1}$. By corollary 4.4, we have $f_{m-3}\cdot U_{m-2}\cdot T^{p-1}_{m-1} \in \delta^{-1}_{m-1}(\mathrm{Im}(D^m_{d'}))$. Moreover by applying such g_{p-1} to (29), we have an n-cocycle given by

$$f_{m-3}(U_1, \ldots, U_{m-3}) \cdot U_{m-2} \cdot E^{\omega}_{m-1\leq n-2}(U_{m-1}, \ldots, U_{n-2}) \cdot E^{\omega}_{n-1}(U_{n-1}, T_n) \in C^n_d(X), \quad (32)$$

where E_{n-1} is the polynomial given by (17). In the case of $\omega = -1$, $n = 3$ and $f_0 = 1$, the resulting 3-cocycle $U_1 \cdot E^{-1}_2(U_2, U_3)$ is found in [5, Theorem 0.3] and [6, Example 2.4.1].

Example 5.2. We define the following polynomial $F^{\omega}_{m-3}(U_{m-3}, U_{m-2}) \in \mathbb{Z}_p[U_{m-3}, U_{m-2}]$:

$$F^{\omega}_{m-3}(U_{m-3}, U_{m-2}) = \big(\omega(U_{m-3} + \omega^{-1}U_{m-2})^p + (U_{m-3} - U_{m-2})^p - (1-\omega)U^p_{m-3}\big)/p$$

$$\equiv \sum_{2\le j\le p-1} j^{-1}\cdot(1-\omega^{1-j})\cdot U_{n-3}^{p-j}\cdot U_{n-2}^{j} \qquad (\mathrm{mod}\ p). \tag{33}$$

If $f_{m-4}\in C_{d'-2p}^{m-4}(X)$ is an $(m-4)$-cocycle with no term of $C_{d'-2p}^{m-4(1)}(X)$, then we put $g_{p-1}(U_1,\dots,U_{m-2}) := f_{m-4}(U_1,\dots,U_{m-4})\cdot F_{m-3}^{\omega}$. Hence by direct calculation we have $\delta_{m-2}(g_{p-1}) = (-1)^n(1-\omega)\cdot g_{p-1}$, which satisfies the condition in Corollary 4.4. Hence, $f_{m-4}(U_1,\dots,U_{m-4})\cdot F_{m-3}^{\omega}\cdot T_{m-1}^{p-1}\in\delta_{m-1}^{-1}(\mathrm{Im}(D_{d'}^{m}))$. By applying such g_{p-1} to (29), we obtain an n-cocycle given by

$$f_{m-4}(U_1,\dots,U_{m-4})\cdot F_{m-3}^{\omega}(U_{m-3},U_{m-2})\cdot E_{m-1\le n-2}^{\omega}(U_{m-1},\dots,U_{n-2})\cdot E_{n-1}^{\omega}(U_{n-1},T_n)\in C_d^n(X).$$

For example, when $\omega=-1$, $n=4$ and $f_0=1$, then we have a 4-cocycle of the form

$$\psi_{4,0}(U_1,U_2,U_3,T_4) = \big((U_1+U_2)^p+(U_1-U_2)^p-2U_1^p\big)\cdot\big((U_3+T_4)^p-(U_3-T_4)^p-2T_4^p\big)/p^2.$$

Example 5.3. We introduce the following polynomial $G_{m-3}^{\omega}(U_{m-3},U_{m-2})\in\mathbb{Z}_p[U_{m-3},U_{m-2}]$.

$$\begin{aligned} G_{m-3}^{\omega} &= \big((U_{m-3}+U_{m-2})^{p+1}-\omega(U_{m-3}+\omega^{-1}U_{m-2})^{p+1}-(1-\omega)U_{m-3}^{p+1}-(1-\omega^{-1})U_{m-2}^{p+1}\big)/p \\ &\equiv \sum_{1\le j\le p-2} j^{-1}(j+1)^{-1}\cdot(1-\omega^{-j})\cdot U_{n-3}^{p+1-j}\cdot U_{n-2}^{j} \qquad (\mathrm{mod}\ p). \end{aligned} \tag{34}$$

Let f_{m-2} be an $(m-2)$-cocycle of the form $f_{m-2}(U_1,\dots,U_{m-2}) = f'_{m-4}(U_1,\dots,U_{m-4})\cdot E_{m-3}(U_{m-3},U_{m-2})$. Then we put $g_{p-1} := f'_{m-4}\cdot G_{m-3}^{\omega}$. By direct calculations we get $\delta_{m-2}(g_{p-1}) = (-1)^n(1-\omega)\cdot g_{p-1}$. Therefore because of (29) we find an n-cocycle given by

$$f'_{m-4}(U_1,\dots,U_{m-4})\cdot G_{m-3}^{\omega}(U_{m-3},U_{m-2})\cdot E_{m-1\le n-2}^{\omega}(U_{m-1},\dots,U_{n-2})\cdot E_{n-1}^{\omega}(U_{n-1},T_n)\in C_d^n(X).$$

For example, when $\omega=-1$, $n=5$ and $f'_3 = U_1\cdot E_2^{-1}(U_2,U_3)$ is the above 3-cocycle presented in Example 5.1, we have a 5-cocycle given by

$$U_1\cdot\big((U_2+U_3)^{p+1}+(U_2-U_3)^{p+1}-2U_2^{p+1}-2U_3^{p+1}\big)\cdot\big((U_4+U_5)^p-(U_4-U_5)^p-2U_5^p\big)/p^2.$$

We will construct a map from $H^{m-2(0)}(X)\oplus H^{m-3(0)}(X)\oplus H^{m-4(0)}(X)$ given by (36) below. It follows from Example 5.1, 5.2 and 5.3 that we thus obtain the following homomorphism.

$$\begin{aligned} \Psi_m : \big(Z_{d'-2p}^{m-4}(X)/Z_{d'-2p}^{m-4(1)}(X)\big)\oplus\big(Z_{d'-p-1}^{m-3}(X)/Z_{d'-p-1}^{m-3(1)}(X)\big)\oplus\big(Z_{d'-p-1}^{m-2}(X)\cap C^{m-4}(X)\cdot E_{m-3}^{\omega}\big) \\ \longrightarrow \frac{\delta_{m-1}^{-1}(\mathrm{Im}(D_{d'}^{m}))}{\mathrm{Im}(D_{d'}^{m-1})+B_{d'-1}^{m-1}(X)}, \end{aligned} \tag{35}$$

$$\Psi_m(f_{m-4},f_{m-3},f'_{m-4}\cdot E_{m-3}^{\omega}) := f_{m-4}\cdot F_{m-3}^{\omega}\cdot T_{m-1}^{p-1}+f_{m-3}\cdot U_{m-2}\cdot T_{m-1}^{p-1}+f'_{m-4}\cdot G_{m-3}^{\omega}\cdot T_{m-1}^{p-1}.$$

By elementary calculation, we can check that the map does not depend on the coboundaries. We thus obtain the induced map as follows:

$$\overline{\Psi}_m : H^{m-4(0)}(X) \oplus H^{m-3(0)}(X) \oplus H^{m-2(0)}(X) \longrightarrow \bigoplus_{d'} \frac{\delta_{m-1}^{-1}(\mathrm{Im}(D_{d'}^m))}{\mathrm{Im}(D_{d'}^{m-1}) + B_{d'-1}^{m-1}(X)}, \quad (36)$$

where the direct sums are over all d' satisfying $\omega^{d'} = \omega^2$, i.e., $\omega^d = 1$. We will show

Proposition 5.4. *The above $\overline{\Psi}_m$ is an epimorphism.*

Note that as a result it follows from Proposition 3.8 and Corollary 4.3 that $c_n^{(0)} \leq c_{n-2e}^{(0)} + c_{n-2e+1}^{(0)} + c_{n-2e+2}^{(0)}$ where $n \geq 2e$.

5.2 The proof of Proposition 5.4

Proof. Let g be an element of the right hand side in (36). We may assume that $g \in C_{d'-1}^{m-1}(X)$ is of the form $g(U_1, \ldots, U_{m-2}, T_{m-1}) = g_{p-1}(U_1, \ldots, U_{m-2}) \cdot T_{m-1}^{p-1}$ satisfying $\delta_{m-1}(g) \in \mathrm{Im}(D_{d'}^m)$. Then g_{p-1} satisfies (15) by Lemma 3.10, which is rewritten,

$$\delta_{m-2}(g_{p-1})(U_1, \ldots, U_{m-1}) = (-1)^m(1 - \omega^{d'-1}) \cdot g_{p-1}(U_1, \ldots, U_{m-2}) \in C_{d'-p}^{m-1(1)}(X). \quad (37)$$

Let us decompose g_{p-1} as $g_{p-1}(U_1, \ldots, U_{m-2}) = \sum_{0 \leq j \leq p-1} g_{j,p-1}(U_1, \ldots, U_{m-3}) \cdot U_{m-2}^j$. Notice that $g_{0,p-1} = 0$.

First, we consider the case where $g_{j,p-1}$ vanishes for each $j \geq 2$. Namely, $g_{p-1}(U_1, \ldots, U_{m-2}) = g_{1,p-1}(U_1, \ldots, U_{m-3}) \cdot U_{m-2}$. We will reduce this case of g_{p-1} to an $(m-3)$-cocycle of type introduced in Example 5.1. Indeed, by applying the form to (37) we have

$$(-1)^m \cdot (1 - \omega) \cdot g_{1,p-1}(U_1, \ldots, U_{m-3}) \cdot U_{m-2} = \delta_{m-2}(g_{p-1})(U_1, \ldots, U_{m-2}, T_{m-1})$$

$$= \delta_{m-3}(g_{1,p-1})(U_1, \ldots, U_{m-2}) \cdot T_{m-1} + (-1)^{m-1} g_{1,p-1}(U_1, \ldots, U_{m-3}) \cdot \big((-1)^{d'-p}(1-\omega) \cdot U_{m-2}\big),$$

which implies $\delta_{m-3}(g_{1,p-1})(U_1, \ldots, U_{m-2}) = 0$, i.e., $g_{1,p-1} \in C_{d'-p-1}^{m-3}(X)$ is an $(m-3)$-cocycle. By construction of Φ_m, such g is deduced from $H^{m-3(0)}(X)$.

Next, we assume that $g_{1,p-1} = 0$. We will show the equality (38) below. Since by (37) we have $\delta_{m-2}(g_{p-1}) \in C_{d'-p}^{m-1(1)}(X)$, we apply $D_{d'-p}^{m-1}$ to (37), and obtain

$$0 = D_{d'-p}^{m-1}(\delta_{m-2}(g_{p-1})) = \delta_{m-2}(D_{d'-p}^{m-2}(g_{p-1})).$$

Note that the assumption of $g_{1,p-1} = 0$ is equivalent to that $D_{d'-p}^{m-2}(g_{p-1})$ has no term of $C_{d'-p-1}^{m-2(1)}(X)$. Hence, there exists a representative cocycle f_{m-2} of $H_{d'-p-1}^{m-2(0)}(X)$ and $h \in C_{d'-p-1}^{m-3}(X)$ satisfying

$$D_{d'-p}^{m-2}(g_{p-1}) = f_{m-2} + \delta_{m-3}(h). \quad (38)$$

Then it suffices to consider two cases of $f_{m-2} = 0$ or $\delta_{m-3}(h) = 0$.

(**Case I**) We assume that $\delta_{m-3}(h) = 0$. Then by Proposition 3.11 we have $f_{m-2} = f'_{m-4} \cdot E^{\omega}_{m-3}$ for some $f'_{m-4} \in \mathbb{Z}_p[U_1, \ldots, U_{m-4}]$. Note that

$$\int_0^{U_{m-2}} E^{\omega}_{m-3}(U_{m-3}, t)\ \mathrm{dt} = \int_0^{U_{m-2}} \frac{(U_{m-3}+t)^p - (U_{m-3}+\omega^{-1}t)^p - (1-\omega^{-1})t^p}{p}$$
$$= (p+1)^{-1} \cdot G^{\omega}_{m-3}(U_{m-3}, U_{m-2}), \tag{39}$$

where we temporarily deal with the above polynomials as polynomials over $\mathbb{R}$ and put the integral constant to be zero. Then by the integration of (38) we conclude $g_{p-1} = f'_{m-4} \cdot G^{\omega}_{m-3}$. After all from $D^{m-2}_{d'-p}(g_{p-1}) = f_{m-2}$ we arrive an $(m-2)$-cocycle and G^{ω}_{m-3} presented in Example 5.3.

(**Case II**) We assume $f_{m-2} = 0$, i.e., $D^{m-2}_{d'-p}(g_{p-1}) = \delta_{m-3}(h)$. Decompose $h \in C^{m-3}_{d'-p-1}(X)$ as $h(U_1, \ldots, U_{m-3}) = \sum_{0 \le k \le p-1} h_k(U_1, \ldots, U_{m-4}) \cdot U^k_{m-3}$.

We will show that g can be killed if $h_{p-1} = 0$. Then we may integrate (38), and obtain

$$g_{p-1}(U_1, \ldots, U_{m-2}) = \sum_{0 \le k \le p-2} (k+1)^{-1} \cdot \Big(\delta_{m-4}(h_k)(U_1, \ldots, U_{m-3}) \cdot T^{k+1}_{m-2}$$
$$+(-1)^m \cdot h_k(U_1, \ldots, U_{m-4}) \cdot \big(\omega(U_{m-3}+\omega^{-1}T_{m-2})^{k+1} - (U_{m-3}+T_{m-2})^{k+1} + (1-\omega)U^{k+1}_{m-3}\big)\Big), \tag{40}$$

noting $(-1)^{d'-p-1} = 1$ and $g_{0,p-1} = 0$. Then put

$$H_h(U_1, \ldots, U_{m-2}) := \sum_{0 \le k \le p-2} (k+1)^{-1} \cdot h_k(U_1, \ldots, U_{m-4}) \cdot U^{k+1}_{m-3} \cdot T^{p-1}_{m-2} \in C^{m-2}_{d'-1}(X).$$

Since we may handle the calculation modulo $\mathrm{Im}(D^{m-1}_d)$, $\delta_{m-2}(H_h)$ is given by

$$\delta_{m-2}(H_h)(U_1, \ldots, U_{m-2}, T_{m-1}) \equiv \sum_{0 \le k \le p-2} (k+1)^{-1} \cdot \Big(\delta_{m-4}(h_k)(U_1, \ldots, U_{m-3}) \cdot U^{k+1}_{m-2} \cdot T^{p-1}_{m-1}$$
$$+(-1)^m \cdot h_k(U_1, \ldots, U_{m-4}) \cdot \big(\omega(U_{m-3}+\omega^{-1}T_{m-2})^{k+1} - (U_{m-3}+T_{m-2})^{k+1} + (1-\omega)U^{k+1}_{m-3}\big)\Big) \cdot T^{p-1}_{m-1}.$$

The coefficient of T^{p-1}_{m-1} in the right hand side is the same as (40), which implies g is killed.

Therefore we may assume $h_k = 0$ for any $k \le p-2$. By (4) and (38) we obtain

$$D^{m-2}_{d'-p}(g_{p-1})(U_1, \ldots, U_{m-3}, T_{m-2}) = \delta_{m-4}(h_{p-1})(U_1, \ldots, U_{m-3}) \cdot T^{p-1}_{m-2}$$
$$+(-1)^m h_{p-1}(U_1, \ldots, U_{m-4}) \cdot \big((U_{m-3}+\omega^{-1}T_{m-2})^{p-1} - (U_{m-3}+T_{m-2})^{p-1}\big). \tag{41}$$

Since the equality is contained in the image of $D^{m-2}_{d'-p}$, the coefficient of T^{p-1}_{m-2} in (41) is zero. It implies that h_{p-1} is an $(m-4)$-cocycle, i.e., $\delta_{m-4}(h_{p-1}) = 0$. Further, since

$$\int_0^{T_{m-2}} (U_{m-3}+\omega^{-1} \cdot t)^{p-1} - (U_{m-3}+t)^{p-1}\ \mathrm{dt} = F^{\omega}_{m-3}(U_{m-3}, T_{m-2}),$$

then we can integrate the equality (41) by T_{m-2}, and conclude $g_{p-1} = h_{p-1}(U_1, \ldots, U_{m-4}) \cdot F^{\omega}_{m-3}(U_{m-3}, T_{m-2})$, which means the type of Example 5.2. □

In the next subsection we will show that $\overline{\Psi}_m$ is an isomorphism.

5.3 The proof of Theorem 3.3 (II)

Proof. According to the isomorphisms presented in Proposition 3.8 and Corollary 4.3, the right hand side in (36) is isomorphic to $H^{n(0)}(X)$. In order to prove the required recurring formula in Theorem 3.3, it suffices to show that $\overline{\Psi}_m$ presented in Proposition 5.4 is an isomorphism. For this we will construct an inverse map of $\overline{\Psi}_m$ step by step. The construction is inspired by the previous proof of Proposition 5.4.

Step 1 First, we will construct a homomorphism $\Phi^{d'}_{-2} : \delta^{-1}_{m-1}(\mathrm{Im}(D^m_{d'})) \longrightarrow H^{m-2}_{d'-p-1}(X)$ presented by (42). We claim that $(D^{m-2}_{d'-p} \circ D^{m-1}_{d'-p\leq d'-1})(g) \in C^{m-2}_{d'-p-1}(X)$ is an $(m-2)$-cocycle for $g \in \delta^{-1}_{m-1}(\mathrm{Im}(D^m_{d'}))$. Indeed, since g satisfies $\delta_{m-1}(g) \in \mathrm{Im}(D^m_{d'})$, we have

$$\begin{aligned}
&\delta_{m-2}\big((D^{m-2}_{d'-p} \circ D^{m-1}_{d'-p\leq d'-1})(g)\big)\\
&= D^{m-1}_{d'-p}\Big(\delta_{m-2}\big(D^{m-1}_{d'-p\leq d'-1}(g)\big)\Big)\\
&= D^{m-1}_{d'-p}\big(\delta_{m-1} \circ D^{m-1}_{d'-p\leq d'-1}(g)\big) + (-1)^{m-1}(\omega^{d'-p} - 1) \cdot (D^{m-1}_{d'-p} \circ D^{m-1}_{d'-p\leq d'-1})(g)\\
&= (D^{m-1}_{d'-p} \circ D^{m}_{d'-p\leq d'-1} \circ \delta_{m-1})(g) \ \in \mathrm{Im}(D^{m-1}_{d'-p} \circ D^{m}_{d'-p\leq d'}) = \{0\},
\end{aligned}$$

where the second equality is obtained from the equality (4) and $D^{m-1}_{d'-p\leq d'-1}(g) \in C^{m-1(1)}_{d'-p}(X)$. Therefore we obtain a homomorphism

$$\Phi^{d'}_{-2} : \delta^{-1}_{m-1}(\mathrm{Im}(D^m_{d'})) \longrightarrow H^{m-2}_{d'-p-1}(X), \tag{42}$$

given by $\Phi^{d'}_{-2}(g) = -[(D^{m-2}_{d'-p} \circ D^{m-1}_{d'-p\leq d'-1})(g)]$.

Step 2 We will show that the homomorphism $\Phi^{d'}_{-2}$ is independent of the coboundary. Indeed, for $h \in C^{m-2}_{d'-1}(X)$, $\Phi^{d'}_{-2}(\delta_{m-2}(h))$ is given by

$$\begin{aligned}
&-[(D^{m-2}_{d'-p} \circ D^{m-1}_{d'-p\leq d'-1} \circ \delta_{m-2})(h)]\\
&= -[(D^{m-2}_{d'-p}\Big(\delta_{m-2}\big(D^{m-2}_{d'-p\leq d'-1}(h)\big)\Big)]\\
&= [-D^{m-2}_{d'-p}\big((\delta_{m-3} \circ D^{m-2}_{d'-p\leq d'-1})(h)\big) + (-1)^{m-1}(\omega^{d'-p} - 1) \cdot (D^{m-2}_{d'-p} \circ D^{m-2}_{d'-p\leq d'-1})(h)]\\
&= -[(\delta_{m-3} \circ D^{m-3}_{d'-p} \circ D^{m-2}_{d'-p\leq d'-1})(h)] = 0,
\end{aligned}$$

where the third equality is obtained from the equality (4) and $D^{m-2}_{d'-p\leq d'-1}(h) \in C^{m-2(1)}_{d'-p}(X)$.

In conclusion the map $\Phi^{d'}_{-2}$ induces

$$\overline{\Phi}^{d'}_{-2(0)} \oplus \overline{\Phi}^{d'}_{-2(1)} : \frac{\delta^{-1}_{m-1}(\mathrm{Im}(D^m_{d'}))}{\mathrm{Im}(D^{m-1}_{d'}) + B^{m-1}_{d'-1}(X)} \longrightarrow H^{m-2(0)}_{d'-p-1}(X) \oplus H^{m-2(1)}_{d'-p-1}(X),$$

where we identity $H^{m-3}_{d'-p-1}(X)$ with $H^{m-2(0)}_{d'-p-1}(X) \oplus H^{m-2(1)}_{d'-p-1}(X)$ by Proposition 3.2 (I).

Step 3 Next, for the construction of a map from $\delta^{-1}_{m-1}(\mathrm{Im}(D^m_{d'}))$ to $C^{m-4}_{d'-2p+2}(X)$, we shall prepare four homomorphisms as follows. From $Z^{m-2}_{d'-p-1}(X) \cong H^{m-2}_{d'-p-1}(X) \oplus B^{m-2}_{d'-p-1}(X)$

we have a canonical projection $\pi_{m-2} : Z^{m-2}_{d'-p-1}(X) \to B^{m-2}_{d'-p-1}(X)$. Further, from $\delta_{m-3} : C^{m-3}_{d'-p-1}(X) \to B^{m-2}_{d'-p-1}(X)$, we have the isomorphism $\bar{\delta}_{m-3} : C^{m-3}_{d'-p-1}(X)/Z^{m-3}_{d'-p-1}(X) \to B^{m-2}_{d'-p-1}(X)$ by the homomorphism theorem. In addition we put a canonical crossed section $\iota^{m-3}_{d'-p-1} : C^{m-3}_{d'-p-1}(X)/Z^{m-3}_{d'-p-1}(X) \to C^{m-3}_{d'-p-1}(X)$. Finally we put $D^{m-3}_{d'-2p\le d'-p-1} : C^{m-3}_{d'-p-1}(X) \to C^{m-3(1)}_{d'-2p}(X)$. Let us identify $C^{m-3(1)}_{d'-2p}(X)$ with a subspace of $C^{m-4}_{d'-2p}(X)$. Then we regard $D^{m-3}_{d'-2p\le d'-p-1}$ as a map toward $C^{m-4}_{d'-2p}(X)$.

In summary we consider the composition of $D^{m-2}_{d'-p} \circ D^{m-1}_{d'-p\le d'-1}$ and the above four maps:

$$\delta^{-1}_{m-1}(\mathrm{Im}(D^m_{d'})) \xrightarrow{D^{m-2}_{d'-p}\circ D^{m-1}_{d'-p\le d'-1}} Z^{m-2}_{d'-p-1}(X) \xrightarrow{\pi_{m-2}} B^{m-2}_{d'-p-1}(X)$$

$$\xrightarrow{\bar{\delta}^{-1}_{m-3}} C^{m-3}_{d'-p-1}(X)/Z^{m-3}_{d'-p-1}(X) \xrightarrow{\iota^{m-3}_{d'-p-1}} C^{m-3}_{d'-p-1}(X) \xrightarrow{D^{m-3}_{d'-2p\le d'-p-1}} C^{m-4}_{d'-2p}(X),$$

noting that the image of $D^{m-2}_{d'-p} \circ D^{m-1}_{d'-p\le d'-1}$ is contained in $Z^{m-2}_{d'-p-1}(X)$ by the discussion of Step 1. We denote the composite homomorphism by $\Phi^{d'}_{-4} : \delta^{-1}_{m-1}(\mathrm{Im}(D^m_{d'})) \to C^{m-4}_{d'-2p}(X)$. In Step 4 and 5 we will check that $\Phi^{d'}_{-4}$ induces the homomorphism given by (43) below.

Step 4 For $g \in \delta^{-1}_{m-1}(\mathrm{Im}(D^m_{d'}))$ we will show that $\delta_{m-4}(\Phi^{d'}_{-4}(g)) = 0$. If we decompose $\iota^{m-3}_{d'-p-1} \circ \bar{\delta}^{-1}_{m-3} \circ \pi_{m-2} \circ D^{m-2}_{d'-p} \circ D^{m-1}_{d'-p\le d'-1}(g) \in C^{m-3}_{d'-p-1}(X)$ as $\sum_a h_a(U_1, \ldots, U_{m-4}) \cdot T^a_{m-3}$, then we have $(\Phi^{d'}_{-4})(g) = -h_{p-1}$. We have to show $\delta_{m-4}(h_{p-1}) = 0$. Notice that $\delta_{m-3}(\sum_a h_a \cdot T^a_{m-3}) \in \pi_{m-2}(\mathrm{Im}(D^{m-2}_{d'-p}))$. Hence it follows from Proposition 3.11 that $\delta_{m-3}(\sum_a h_a \cdot T^a_{m-3}) + f'_{m-4} \cdot E^{\omega}_{m-3} \in \mathrm{Im}(D^{m-2}_{d'-p})$ for some $f'_{m-4} \in C^{m-4}_{d'-2p}(X)$. Since from the definition of E^{ω}_{m-3} given by (17) we have $f'_{m-4} \cdot E^{\omega}_{m-3} \in \mathrm{Im}(D^{m-2}_{d'-p})$, by comparing the coefficient of T^{p-1}_{m-2} in $\delta_{m-3}(\sum_a h_a \cdot T^a_{m-3})$ we obtain $\delta_{m-4}(h_{p-1}) = 0$ (see (41)).

Step 5 Next, we will show that $\Phi^{d'}_{-4}(\delta_{m-2}(\iota^{m-2}_{d'-1}(h)))$ vanishes for any $h \in C^{m-2}_{d'-1}(X)/Z^{m-2}_{d'-1}(X)$. Note that $D^{m-2}_{d'-p}\circ D^{m-1}_{d'-p\le d'-1}$ induces a map from $C^{m-1}_{d'-1}(X)/Z^{m-1}_{d'-1}(X)$ to $C^{m-2}_{d'-p-1}(X)/Z^{m-2}_{d'-p-1}(X)$, since $D^{m-2}_{d'-p} \circ D^{m-1}_{d'-p\le d'-1}$ is a chain map. Also notice that $D^{m-2}_{d'-p\le d'-1} \circ \iota^{m-2}_{d'-1} = \iota^{m-3}_{d'-p} \circ D^{m-2}_{d'-p\le d'-1}$. Therefore we obtain the following commutative diagram.

$$\begin{tikzcd}
C^{m-4}_{d'-2p}(X)/Z^{m-4}_{d'-2p}(X) \arrow[r, "\iota^{m-4}_{d'-2p}"] & C^{m-4}_{d'-2p}(X) \arrow[r, "\delta_{m-4}"] & Z^{m-3}_{d'-2p}(X) & \\
C^{m-3}_{d'-p-1}(X)/Z^{m-3}_{d'-p-1}(X) \arrow[r, "\iota^{m-3}_{d'-p-1}"'] \arrow[u, "D^{m-3}_{d'-2p\le d'-p-1}"'] & C^{m-3}_{d'-p-1}(X) \arrow[r, "\delta_{m-3}"'] \arrow[u, "D^{m-3}_{d'-2p\le d'-p-1}"'] & Z^{m-2}_{d'-p-1}(X) \arrow[r, "\pi_{m-2}"] \arrow[u, "D^{m-2}_{d'-2p\le d'-p-1}"'] & B^{m-2}_{d'-p-1} \\
C^{m-2}_{d'-1}(X)/Z^{m-2}_{d'-1}(X) \arrow[r, "\iota^{m-2}_{d'-1}"'] \arrow[u, "D^{m-3}_{d'-p}\circ D^{m-2}_{d'-p\le d'-1}"'] & C^{m-2}_{d'-1}(X) \arrow[r, "\delta_{m-2}"'] \arrow[u, "D^{m-3}_{d'-p}\circ D^{m-2}_{d'-p\le d'-1}"'] & Z^{m-1}_{d'-1}(X) \arrow[u, "D^{m-2}_{d'-p}\circ D^{m-1}_{d'-p\le d'-1}"'] &
\end{tikzcd}$$

Hence the commutative diagram leads us that $\Phi^{d'}_{-4}(\delta_{m-2}(\iota^{m-2}_{d'-1}(h)))$ is equal to

$$\begin{aligned}
&\big(D^{m-3}_{d'-2p\leq d'-p-1}\circ\iota^{m-3}_{d'-p-1}\circ\bar{\delta}^{-1}_{m-3}\circ\pi_{m-2}\circ D^{m-2}_{d'-p}\circ D^{m-1}_{d'-p\leq d'-1}\circ\delta_{m-2}\big)\big(\iota^{m-2}_{d'-1}(h)\big)\\
&=\big(\iota^{m-4}_{d'-2p}\circ D^{m-3}_{d'-2p\leq d'-p-1}\circ\bar{\delta}^{-1}_{m-3}\circ\pi_{m-2}\circ\delta_{m-3}\circ D^{m-3}_{d'-p}\circ D^{m-2}_{d'-p\leq d'-1}\circ\iota^{m-2}_{d'-1}\big)(h)\\
&=\big(\iota^{m-4}_{d'-2p}\circ D^{m-3}_{d'-2p\leq d'-p-1}\circ\bar{\delta}^{-1}_{m-3}\circ\pi_{m-2}\circ\delta_{m-3}\circ\iota^{m-3}_{d'-p-1}\circ D^{m-3}_{d'-p}\circ D^{m-2}_{d'-p\leq d'-1}\big)(h)\\
&=\big(\iota^{m-4}_{d'-2p}\circ D^{m-3}_{d'-2p\leq d'-p-1}\circ D^{m-3}_{d'-p}\circ D^{m-2}_{d'-p\leq d'-1}\big)(h)\\
&=\big(\iota^{m-4}_{d'-2p}\circ D^{m-3}_{d'-2p\leq d'-p}\circ D^{m-2}_{d'-p\leq d'-1}\big)(h)=0,
\end{aligned}$$

where the third equality is obtained from that $\bar{\delta}^{-1}_{m-3}\circ\pi_{m-2}\circ\delta_{m-3}\circ\iota^{m-3}_{d'-p-1}$ is the identity map of $C^{m-2}_{d'-1}(X)/Z^{m-2}_{d'-1}(X)$ by the homomorphism theorem.

Consequently, $\Phi^{d'}_{-4}$ induces a homomorphism

$$\overline{\Phi}^{d'}_{-4}:\frac{\delta^{-1}_{m-1}(\mathrm{Im}(D^m_{d'}))}{\mathrm{Im}(D^{m-1}_{d'})+B^{m-1}_{d'-1}(X)}\longrightarrow H^{m-4(0)}_{d'-2p}(X). \tag{43}$$

Step 6 In summary by Step 2 and 5 we have the following homomorphism:

$$\overline{\Phi}^{d'}_{-2(0)}\oplus\overline{\Phi}^{d'}_{-2(1)}\oplus\overline{\Phi}^{d'}_{-4}:\frac{\delta^{-1}_{m-1}(\mathrm{Im}(D^m_{d'}))}{\mathrm{Im}(D^{m-1}_{d'})+B^{m-1}_{d'-1}(X)}\longrightarrow H^{m-2(0)}_{d'-p-1}(X)\oplus H^{m-3(0)}_{d'-p-1}(X)\oplus H^{m-4(0)}_{d'-2p}(X),$$

where we identify $H^{m-2(1)}_{d'-p-1}(X)$ with $H^{m-3(0)}_{d'-p-1}(X)$ by Proposition 3.11 (II). For the proof of Theorem 3.3, it suffices to show that $\oplus_{d'}(\overline{\Phi}^{d'}_{-2(0)}\oplus\overline{\Phi}^{d'}_{-2(1)}\oplus\overline{\Phi}^{d'}_{-4})\circ\overline{\Psi}_m$ is an identity map, where the direct sums are over all d' satisfying $\omega^{d'}=1$. Notice that by construction the composite map have a direct sum decomposition into three maps of the form $\big(\overline{\Phi}^{d'}_{-2(0)}\circ\overline{\Psi}_m\big)\oplus\big(\overline{\Phi}^{d'}_{-2(1)}\circ\overline{\Psi}_m\big)\oplus\big(\overline{\Phi}^{d'}_{-4}\circ\overline{\Psi}_m\big)$. Therefore we shall check the inverse on the each parts of $H^{m-2(0)}_{d'-p-1}(X)$, $H^{m-3(0)}_{d'-p-1}(X)$ and $H^{m-4(0)}_{d'-2p}(X)$ as follows.

Let $f_{m-3}\in C^{m-3}_{d'-p-1}(X)$ be an $(m-3)$-cocycle and let $f'_{m-4}\cdot E^{\omega}_{m-3}\in C^{m-2}_{d'-p-1}(X)$ be an $(m-2)$-cocycle. We may assume that f_{m-3} is divisible by U_{m-3}. Then

$$(\Phi^{d'}_{-2(1)}\circ\Psi_m)(f_{m-3})=\Phi^{d'}_{-2(1)}(f_{m-3}\cdot U_{m-2}\cdot T^{p-1}_{m-1})=D^{m-2}_{d'-p}(f_{m-3}\cdot U_{m-2})=f_{m-3},$$

$$(\Phi^{d'}_{-2(0)}\circ\Psi_m)(f'_{m-4}\cdot E^{\omega}_{m-3})=\Phi^{d'}_{-2(0)}(f'_{m-4}\cdot G^{\omega}_{m-3}\cdot T^{p-1}_{m-1})=D^{m-2}_{d'-p}(f'_{m-4}\cdot G^{\omega}_{m-3})=f'_{m-4}\cdot E^{\omega}_{m-3},$$

where the last equality is obtained from $D^{m-2}_{d'-p}(G^{\omega}_{m-3})=E^{\omega}_{m-3}$. Finally let $f_{m-4}\in C^{m-4}_{d'-2p}(X)$ be an $(m-4)$-cocycle which is divisible by U_{m-4}. Put a canonical crossed section $s^{m-3}_{d'-p-1}:C^{m-3}_{d'-p-1}(X)\longrightarrow C^{m-3}_{d'-p-1}(X)/Z^{m-3}_{d'-p-1}(X)$ of the map $\iota^{m-3}_{d'-p-1}$ above. Then

$$\begin{aligned}
&\big(\Phi^{d'}_{-4}\circ\Psi_m\big)(f_{m-4})=\Phi^{d'}_{-4}(f_{m-4}\cdot F^{\omega}_{m-3}\cdot T^{p-1}_{m-1})\\
&=\big(D^{m-3}_{d'-2p\leq d'-p-1}\circ\iota^{m-3}_{d'-p-1}\circ\bar{\delta}^{-1}_{m-3}\circ\pi_{m-2}\circ D^{m-2}_{d'-p}\circ D^{m-1}_{d'-p\leq d'-1}\big)\big(f_{m-4}\cdot F^{\omega}_{m-3}\cdot T^{p-1}_{m-1}\big)\\
&=-\big(D^{m-3}_{d'-2p\leq d'-p-1}\circ\iota^{m-3}_{d'-p-1}\circ\bar{\delta}^{-1}_{m-3}\circ\pi_{m-2}\big)\big(D^{m-2}_{d'-p}(f_{m-4}\cdot F^{\omega}_{m-3})\big)\\
&=-\big(D^{m-3}_{d'-2p\leq d'-p-1}\circ\iota^{m-3}_{d'-p-1}\circ\bar{\delta}^{-1}_{m-3}\circ\pi_{m-2}\circ\delta_{m-3}\big)\big(f_{m-4}\cdot T^{p-1}_{m-3}\big)
\end{aligned}$$

$$= -\big(D^{m-3}_{d'-2p\le d'-p-1} \circ \iota^{m-3}_{d'-p-1} \circ \bar{\delta}^{-1}_{m-3} \circ \pi_{m-2} \circ \delta_{m-3} \circ \iota^{m-3}_{d'-p-1} \circ s^{m-3}_{d'-p-1}\big)\big(f_{m-4} \cdot T^{p-1}_{m-3}\big)$$
$$= -\big(D^{m-3}_{d'-2p\le d'-p-1} \circ \iota^{m-3}_{d'-p-1} \circ s^{m-3}_{d'-p-1}\big)\big(f_{m-4} \cdot T^{p-1}_{m-3}\big),$$

where the forth equality is obtained from $D^{m-2}_{d'-p}(f_{m-4}\cdot F^{\omega}_{m-3}) = \delta_{m-3}(f_{m-4}\cdot T^{p-1}_{m-3})$ (see (41)) ,the fifth equality is derived from that $\delta_{m-3} = \delta_{m-3}\circ\iota^{m-3}_{d'-p-1}\circ s^{m-3}_{d'-p-1}$ from the definition of the section $s^{m-3}_{d'-p-1}$, the last equality is derived from that $\bar{\delta}^{-1}_{m-3}\circ\pi_{m-2}\circ\delta_{m-3}\circ\iota^{m-3}_{d'-p-1}$ is the identity map of $C^{m-2}_{d'-1}(X)/Z^{m-2}_{d'-1}(X)$. Moreover, note that $D^{m-3}_{d'-2p\le d'-p-1}\big(Z^{m-3}_{d'-p-1}(X)\big) \subset B^{m-4}_{d'-2p}(X)$ by Lemma 5.5 below. Hence,

$$\big(\overline{\Phi}^{d'}_{-4} \circ \overline{\Psi}_m\big)(f_{m-4}) = -[\big(D^{m-3}_{d'-2p\le d'-p-1} \circ \iota^{m-3}_{d'-p-1} \circ s^{m-3}_{d'-p-1}\big)\big(f_{m-4} \cdot T^{p-1}_{m-3}\big)]$$
$$= -[D^{m-3}_{d'-2p\le d'-p-1}\big(f_{m-4} \cdot T^{p-1}_{m-3}\big)] = [f_{m-4}] \in H^{m-4(0)}(X).$$

In summary $\oplus_{d'}(\overline{\Phi}^{d'}_{-4} \oplus \overline{\Phi}^{d'}_{-2(1)} \oplus \overline{\Phi}^{d'}_{-2(0)})$ turns out to be the inverse map of $\overline{\Psi}_m$. □

Lemma 5.5. *If $\omega^{d'} = \omega^2$, then $D^{m-3}_{d'-2p\le d'-p-1}\big(Z^{m-3}_{d'-p-1}(X)\big) \subset B^{m-4}_{d'-2p}(X)$.*

Proof. Let $f \in C^{m-3}_{d'-p-1}(X)$ be an $(m-3)$-cocycle which is divisible by U_{m-3}. Then there exists a representative cocycle ϕ_{m-3} of $H^{m-3(0)}_{d'-p-1}(X)$ and $h \in C^{m-4}_{d'-p-1}(X)$ satisfying $f = \phi_{m-3} + \delta_{m-4}(h)$. Recall that any representative of $H^{m-3(0)}_{d'-p-1}(X)$ is annihilated by $D^{m-3}_{d'-2p\le d'-p-1}$ by Proposition 3.11. Therefore

$$D^{m-3}_{d'-2p\le d'-p-1}(f) = \delta_{m-4}(D^{m-4}_{d'-2p\le d'-p-1}(h)\cdot U^{0}_{m-4}) = \delta_{m-5}(D^{m-4}_{d'-2p\le d'-p-1}(h)) \in B^{m-4}_{d'-2p}(X),$$

where the last equality is obtained form the equality (4) and $\omega^{d'-2p} = 1$. □

5.4 The proof of Theorem 2.2

Proof. It is known [5, Theorem 1.1] that $H^n_Q(X;\mathbb{Q}) \cong 0$ and $H^n_Q(X;\mathbb{Z}_q) \cong 0$ for any prime q $(\neq p)$. Hence it follows from the universal coefficient theorem that $H^Q_n(X;\mathbb{Z})$ is a finite abelian group whose order is a power of p. By Corollary 6.4 which we will show in the next section, $H^Q_n(X;\mathbb{Z})$ is annihilated by p, that is, $H^Q_n(X;\mathbb{Z})$ is a $\mathbb{Z}_p$-vector space. We thus have $H^{Q_U}_n(X;\mathbb{Z}) \cong \mathbb{Z}_p^{b_n}$ for some b_n.

By Lemma 5.6 below, $b_n = c^{(0)}_n$ for $n \geq 1$. Recall that by Theorem 3.3 $c^{(0)}_n$ is determined by $c^{(0)}_{n+2e} = c^{(0)}_{n+2} + c^{(0)}_{n+1} + c^{(0)}_n$, $c^{(0)}_1 = \cdots = c^{(0)}_{2e-2} = 0$, and $c^{(0)}_{2e-1} = c^{(0)}_{2e} = 1$. In conclusion, b_n is also determined by $b_{n+2e} = b_{n+2} + b_{n+1} + b_n$, $b_1 = \cdots = b_{2e-2} = 0$, and $b_{2e-1} = b_{2e} = 1$. □

Lemma 5.6. *The above b_n satisfies $b_n = c^{(0)}_n$ for $n \geq 1$.*

Proof. We will show this lemma by induction on n. By definition and direct calculation we can check that $H^{Q_U}_1(X;\mathbb{Z}) \cong \mathbb{Z}$ and $H^{1(0)}(X) \cong 0$. Hence $b_1 = c^{(0)}_1 = 0$. If $n = 2$, It is shown [5, Corollary 2.2] that $H^2_Q(X;\mathbb{Z}_p) \cong 0$. By (3) we obtain $H_2^{Q_U}(X;\mathbb{Z}) \cong 0$. Hence $b_2 = c_2^{(0)} = 0$.

Assume Lemma 5.6 holds for $n \leq k-1$. By the universal coefficient theorem and Proposition 3.2 we have

$$b_k + b_{k-1} = \dim(H^k(X)) = c_k^{(0)} + c_{k-1}^{(0)}.$$

By the assumption we have $b_k = c_k^{(0)}$. $\square$

5.5 Cohomological operations and presentations of n-cocycles of $H^{n(0)}(X)$

As a result of Theorem 3.3, we will construct a cohomological operation. Furthermore, we will present all cocycles which span the cohomology $H^n(X)$. The presentations are composed of the four polynomials E^ω_{n-1}, $E^\omega_{n-2e+3\leq n-2}$, F^ω_{n-2e+2} and G^ω_{n-2e+1} introduced in (17), (27), (33) and (34), respectively.

In [7] there are studies on homological operations for some quandles (see Section 2 and 6 in [7]). Here we will construct a operation on the quandle *cohomology* group. We define

$$\Omega_m : \big(Z^{m-4}_{d'-2p}(X)/Z^{m-4(1)}_{d'-2p}(X)\big) \oplus \big(Z^{m-3}_{d'-p-1}(X)/Z^{m-3(1)}_{d'-p-1}(X)\big) \oplus \big(Z^{m-2}_{d'-p-1}(X) \cap C^{m-4}(X)\cdot E^\omega_{m-3}\big)$$

$$\longrightarrow H^{n(0)}_d(X),$$

$$\Omega_m(f_{m-4}, f_{m-3}, f'_{m-4}\cdot E^\omega_{m-3}) := (f_{m-4}\cdot F^\omega_{m-3} + f_{m-3}\cdot U_{m-2} + f'_{m-4}\cdot G^\omega_{m-3})\cdot E^\omega_{m+1\leq n-2}\cdot E^\omega_{n-1},$$

where $m = n-2e+4$. Since this does not depend on the coboundaries, then Ω_m induces

$$\overline{\Omega}_{n-2e+4} : H^{n-2e(0)}(X) \oplus H^{n-2e+1(0)}(X) \oplus H^{n-2e+2(0)}(X) \longrightarrow H^{n(0)}(X).$$

From the construction, we can check that by Proposition 3.8 and Corollary 4.4, the above $\overline{\Omega}_{n-2e+4}$ is equal to the composite $\bar{\phi}^{-1} \circ (\overline{\Theta}_1)^{-1} \circ \cdots \circ (\overline{\Theta}_{e-1})^{-1} \circ \overline{\Psi}_m$. Since each homomorphisms are isomorphisms, we thus obtain

Corollary 5.7. *$\overline{\Omega}_{n-2e}$ is an isomorphism for $n \geq 2e$.*

Remark 5.8. Recall that $H^n(X) \cong H^{n(0)}(X) \oplus H^{n-1(0)}(X)$ by Proposition 3.2. Then, the direct sum $\overline{\Omega}_{n-2e+4} \oplus \overline{\Omega}_{n-2e+3}$ induces an isomorphic cohomological operation on the quandle cohomology $H^n(X)$.

Next, we will present all cocycles which span the cohomology $H^n(X)$. For the purpose we will define a set of some n-cocycles denoted by $\mathrm{Coc}^{\omega,(0)}_n$. We first define $\mathrm{Coc}^{\omega,(0)}_0$ to be $1 \in \mathbb{Z}_p$. Define $\mathrm{Coc}^{\omega,(0)}_i$ by the empty sets for $0 < i < 2e-1$. Put $\mathrm{Coc}^{\omega,(0)}_{2e-1} :=$

$\{U_1 \cdot E^{\omega}_{2\leq 2e-1} \cdot E^{\omega}_{2e-2} \in C^{2e-1}(X)\}$. By induction on n we define the following set of some n-cocycles:

$$\begin{aligned}\mathrm{Coc}^{\omega,(0)}_n := &\{f_{n-2e} \cdot F^{\omega}_{n-2e+2} \cdot E^{\omega}_{n-2e+3\leq n-2} \cdot E^{\omega}_{n-1} | f_{n-2e} \in \mathrm{Coc}^{\omega,(0)}_{n-2e}\} \\ &\cup \{f_{n-2e+1} \cdot U_{n-2e+2} \cdot E^{\omega}_{n-2e+3\leq n-2} \cdot E^{\omega}_{n-1} | f_{n-2e+1} \in \mathrm{Coc}^{\omega,(0)}_{n-2e+1}\} \\ &\cup \{f_{n-2e+2} \cdot G^{\omega}_{n-2e+1} \cdot E^{\omega}_{n-2e+3\leq n-2} \cdot E^{\omega}_{n-1}/E^{\omega}_{n-2e+1} | f_{n-2e+2} \in \mathrm{Coc}^{\omega,(0)}_{n-2e+2}\},\end{aligned}$$

noting that $f_{n-2e+2}/E^{\omega}_{n-2e+1} \in C^{n-2e}(X)$ from the definition of $\mathrm{Coc}^{\omega,(0)}_{n-2e+2}$. Also, by definition notice that any element of $\mathrm{Coc}^{\omega,(0)}_n$ has no term of $C^{n(1)}(X)$.

Corollary 5.9. *Let X be the Alexander quandle of order p with $\omega \neq 0, 1$. Then $H^{n(0)}(X)$ is independently generated by $\mathrm{Coc}^{\omega,(0)}_n$. Further, the quandle cohomology $H^n(X)$ is independently generated by $\mathrm{Coc}^{\omega,(0)}_n \cup \mathrm{Coc}^{\omega,(0)}_{n-1}$, where we regard the cocycles in $\mathrm{Coc}^{\omega,(0)}_{n-1}$ as elements of $C^{n(1)}(X)$.*

Proof. By definition $H^0(X) \cong \mathbb{Z}_p$ is generated by $1 \in \mathbb{Z}_p$. By Theorem 3.3 we have $H^{i(0)}(X) \cong 0$ for $0 < i < 2e-1$. By Remark 4.5, $H^{2e-1(0)}(X) \cong \mathbb{Z}_p$ is spanned by $\mathrm{Coc}^{\omega,(0)}_{2e-1}$. Let us consider the case $n \geq 2e$. Assume that for $n \leq k-1$, $H^{n(0)}(X)$ is independently generated by $\mathrm{Coc}^{\omega,(0)}_n$. Note that the construction of $\mathrm{Coc}^{\omega,(0)}_k$ is compatible with the isomorphism $\overline{\Omega}_{k-2e+4}$. Therefore $H^{k(0)}(X)$ is independently generated by $\mathrm{Coc}^{\omega,(0)}_k$.

To show the later part recall that by Proposition 3.2 a canonical inclusion $C^{n(1)}(X) \subset C^{n-1}(X)$ induces $H^{n(1)}(X) \cong H^{n-1(0)}(X)$. Therefore $H^{n(1)}(X)$ is spanned by $\mathrm{Coc}^{\omega,(0)}_{n-1}$. Moreover by the canonical direct sum decomposition $H^n(X) \cong H^{n(0)}(X) \oplus H^{n(1)}(X)$, $H^n(X)$ is independently generated by $\mathrm{Coc}^{\omega,(0)}_n \cup \mathrm{Coc}^{\omega,(0)}_{n-1}$. □

6 The torsion subgroup of $H^{R_U}_n(M;\mathbb{Z})$ of finite connected Alexander quandles.

The goal in this section is to show Theorem 6.1. As a corollary, for an Alexander quandle X of order p, the quandle homology is annihilated by p (Corollary 6.4).

We will review finite connected Alexander quandles and the torsion subgroups of the quandle homology groups. Let M be an Alexander quandle, that is, M is a $\mathbb{Z}[T, T^{-1}]$-module with an binary operation given by $x * y = Tx + (1-T)y$. If the quotient module $M/(1-T)M$ is zero, the quandle is said to be *connected*. It is known [4, Proposition.1] that this definition is equivalent to the standard definition of the connectedness. It is known [4, Theorem.1] that if M is finite and connected, then the free subgroup of $H^R_n(M;\mathbb{Z})$ is $\mathbb{Z}$ and the generator is represented by $(0, 0, \ldots, 0) \in C^R_n(M;\mathbb{Z})$ and that the torsion subgroup is annihilated by $|M|^n$ for each $n \geq 1$. It is shown [7, Theorem 17] that for the dihedral quandle X of order 3 the torsion part of $H^R_n(X;\mathbb{Z})$ is annihilated by 3.

More generally, in this paper we will show a stronger estimate for finite connected Alexander quandles.

Theorem 6.1. *Let M be a finite connected Alexander quandle. Then for each n the torsion part of $H_n^R(M;\mathbb{Z})$ is annihilated by $|M|$.*

Before the proof we will present some corollaries and remarks.

Corollary 6.2. *Let M be as above. Then $H_n^Q(M;\mathbb{Z})$ is annihilated by $|M|$ for $n \geq 2$.*

Proof. It is shown [4, Theorem 4] that $H_n^Q(M;\mathbb{Z})$ is a direct divisor of $H_n^R(M;\mathbb{Z})$. Since $(0,\dots,0)$ is zero in $C_n^Q(M;\mathbb{Z})$, $H_n^Q(M;\mathbb{Z})$ has no free part. $\square$

Remark 6.3. It is false that in general for a finite connected quandle $H_n^Q(X;\mathbb{Z})$ is annihilated by $|X|$. For example, let $QS(6)$ be the connected quandle of order 6 presented by [2, Example 2.2]. $QS(6)$ is not isomorphic to any Alexander quandle (see [8, Section 5.1]). Then it is known [2, Example 2.5] that $H_3^Q(QS(6);\mathbb{Z}) \cong \mathbb{Z}/24\mathbb{Z} \cong \mathbb{Z}/2^3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.

We immediately obtain the following corollaries, which we state without proof.

Corollary 6.4. *Let X be an Alexander quandle of order p, that is, $X = \mathbb{Z}_p[T]/(T-\omega)$ with the Alexander quandle operation for some $\omega \neq 0, 1$. Then $H_n^Q(X;\mathbb{Z})$ is annihilated by p.*

Corollary 6.5. *If $X = \mathbb{Z}_2[T]/(T^2+T+1)$ with the Alexander quandle operation, then $H_n^Q(X;\mathbb{Z})$ is annihilated by 4.*

Remark 6.6. Corollary 6.4 is conjectured in [7, Conjecture 16].

Proof of Theorem 6.1. We will deal with our coordinate of the rack chain group presented by (1). We now introduce the chain maps $f_+^j,\ f_0^j : C_n^{R_U}(M;\mathbb{Z}) \to C_n^{R_U}(M;\mathbb{Z})$ given by

$$f_+^j(U_1,\dots,U_n) = \begin{cases} \displaystyle\sum_{y\in M} \big(0,\dots,0,y,U_{j+1},\dots,U_n\big) & \text{for } \ 0<j<n \\ \displaystyle\sum_{y\in M} \big(0,\dots,0,y\big) & \text{for } \ j=n. \end{cases}$$

$$f_0^j(U_1,\dots,U_n) = \begin{cases} |M|\cdot\big(0,\dots,0,U_j,U_{j+1},\dots,U_n\big) & \text{for } \ 0<j<n \\ |M|\cdot\big(0,\dots,0\big) & \text{for } \ j=n. \end{cases}$$

Lemma 6.7. *$f_0^j - f_+^j$, $f_0^j - f_+^{j-1}$ are null homotopic for any $j \leq n$.*

We will show this below. If we assume the lemma, then $f_0^1 = |M|\cdot \mathrm{id}_{C_n^{R_U}(M;\mathbb{Z})}$ is chain homotopic to f_0^n. Therefore we have the induced maps $(f_0^1)_* = (f_0^n)_* : H_n^{R_U}(M;\mathbb{Z}) \to$

$H_n^{R_U}(M;\mathbb{Z})$. However recall that the free subgroup of $H_n^{R_U}(M;\mathbb{Z})$ is generated by $(0,\dots,0)$. Hence the image of $|M|\cdot\mathrm{id}_{H_n^{R_U}(M;\mathbb{Z})}$ is contained in the free part of $H_n^{R_U}(M;\mathbb{Z})$. Therefore the torsion part of $H_n^{R_U}(M;\mathbb{Z})$ is annihilated by $|M|$. $\square$

Proof of Lemma 6.7. Since M is finite, then we equip M with a $\mathbb{Z}[T,T^{-1}]$-algebra structure with unit. Therefore we may regard $1,T,T^{-1}$ and $1-T$ as elements of M. By the connectedness we have $M=(1-T)M$ as a $\mathbb{Z}[T,T^{-1}]$-module. Therefore $1-T$ is an invertible element of the ring M.

We claim that chain homotopies $D^j_{n,0}:C_n^{R_U}(M;\mathbb{Z})\to C_{n+1}^{R_U}(M;\mathbb{Z})$ between f_0^j and f_+^j are given by the formula

$$D^j_{n,0}(U_1,\dots,U_n)=\sum_{y\in M}\big(0,\dots,0,(T-1)^{-1}y,U_j+(1-T)^{-1}y,U_{j+1},\dots,U_n\big).$$

Indeed, by direct calculation we can check that $\partial_{n+1}D^j_{n,0}+D^j_{n-1,0}\partial_n=(-1)^j\big(f_+^j-f_0^j\big)$.

Next, we put homomorphisms $D^j_{n,+}:C_n^{R_U}(M;\mathbb{Z})\to C_{n+1}^{R_U}(M;\mathbb{Z})$ given by the formula

$$D^j_{n,+}(U_1,\dots,U_n)=\begin{cases}\displaystyle\sum_{y\in M}\big(0,\dots,0,(1-T)^{-1}y,(T-1)^{-1}y,U_j,U_{j+1},\dots,U_n\big) & \text{for } j<n\\ \displaystyle\sum_{y\in M}\big(0,\dots,0,(1-T)^{-1}y,(T-1)^{-1}y\big) & \text{for } j=n.\end{cases}$$

By direct calculation we obtain $\partial_{n+1}D^j_{n,+}+D^j_{n-1,+}\partial_n=(-1)^j\big(f_+^{j-1}-f_0^j\big)$ for $j>1$. Therefore $D^j_{n,+}$ are chain homotopies between f_+^j and f_0^{j+1}. This completes the proof. $\square$

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