The maximum multi-flow problems with bounded fractionality

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Abstract

We consider the weighted maximum multi-flow problem with respect to terminal weight \( \mu \). We show that if the dimension of the tight span associated with \( \mu \) is at most 2, then this problem has a \( 1/12 \)-integral optimal multi-flow for every Eulerian supply graph. This result solves a weighted generalization of Karzanov’s conjecture for classifying commodity graphs with finite fractionality. In addition, our proof technique proves the existence of an integral or half-integrality optimal multi-flow for a large class of multi-flow maximization problems, and gives a polynomial time algorithm.

1 Introduction

Let \( G = (V, E) \) be an undirected graph with integral edge capacity \( c : E \to \mathbb{Z}_+ \). Let \( S \subseteq V \) be a set of terminals. Let \( H \) be a simple undirected graph on \( S \), called commodity graph. A multi-flow (multicommodity flow) \( f \) is a pair \( (\mathcal{P}, \lambda) \) of a set \( \mathcal{P} \) of (simple) paths connecting the ends of some edge of \( H \) and a nonnegative flow-value function \( \lambda : \mathcal{P} \to \mathbb{R}_+ \) satisfying capacity constraint \( \sum_{P \in \mathcal{P}, e \in P} \lambda(P) \leq c(e) \) for \( e \in E \). The total flow-value \( \|f\| \) of a multi-flow \( f = (\mathcal{P}, \lambda) \) is defined as \( \sum_{P \in \mathcal{P}} \lambda(P) \). The maximum multi-flow problem with respect to \((G, H)\) is formulated as:

\[
\text{MFP: Maximize } \|f\| \text{ over all multi-flows } f \text{ for } (G, H).
\]

In the case of \( H = K_2 \), consisting of one edge, MFP is the ordinary (single-commodity) maximum flow problem. The max-flow min-cut theorem, due to Ford-Fulkerson [6], says that there exists an integral maximum flow. In the case of \( H = K_2 + K_2 \), consisting of two vertex-disjoint edges, MFP is the maximum 2-commodity flow problem. Hu [13] showed that there exists a half-integral maximum flow. However, no analogous theorem holds for the 3-commodity flow problem. It is known that there is no positive integer \( k \) such that all 3-commodity flow problems have a \( 1/k \)-integral maximum flow. On the other hand, for \( H = K_{|S|} \), the complete graph on \( S \), Lovász [26] and Cherkassky [3] independently showed that there exists a half-integral maximum flow.

In this way, the integrality (or half-integrality) property depends crucially on the structure of the commodity graph \( H \). Motivated by this fact, Karzanov [16] defined the fractionality, denoted by \( \text{frac}(H) \), of a commodity graph \( H \) as the least positive integer
$k$ such that there exists a $1/k$-integral maximum flow in MFP for every capacitated graph $G$ having $H$ as the commodity graph. If no such positive integer $k$ exists, then $\frac{H}{k}$ is defined to be $+\infty$. The above-mentioned examples show that $\frac{K_2}{2} = 1$, $\frac{K_2 + K_2}{2} = 2$, $\frac{K_n}{2} = 2$, and $\frac{K_2 + K_2 + K_2}{2} = +\infty$. Karzanov [16, 17] posed the following fundamental problem:

Classify the commodity graphs having finite fractionality.

The linear program dual to MFP gives a lower bound of the fractionality $\frac{H}{k}$. The dual fractionality $\frac{H}{k}$ is defined to be the least positive integer $k$ such that there exists a $1/k$-integral optimum in the LP-dual to MFP for every capacitated graph $G$ having $H$ as the commodity graph. Then the standard TDI argument implies $\frac{H}{k} \geq \frac{H}{k}$ [16]. Therefore the finiteness of the dual fractionality is a necessary condition for the finiteness of the (primal) fractionality.

Karzanov [16] gave a necessary and sufficient condition for the finiteness of the dual fractionality, and determined its possible values as follows. A commodity graph $H$ is said to have property $P$ if it satisfies the following condition:

(P) For any triple $A, B, C$ of pairwise intersecting maximal stable sets of $H$, we have $A \cap B = B \cap C = C \cap A$.

Theorem 1.1 ([16]). For a commodity graph $H$, we have the following:

1. If $H$ has property $P$, then $\frac{H}{k} \in \{1, 2, 4\}$.
2. If $H$ does not have property $P$, then $\frac{H}{k} = +\infty$ and hence $\frac{H}{k} = +\infty$.

See also [27, Section 73.3b]. Karzanov conjectured that property $P$ is also sufficient for the finiteness of primal fractionality, and, more strongly, that the possible values are also $1, 2, 4, +\infty$, as follows.

Conjecture 1.2 ([17]). Suppose that a commodity graph $H$ has property $P$. Then the following hold:

1. $\frac{H}{k} < +\infty$,
2. $\frac{H}{k} \in \{1, 2, 4\}$,

where (1) is the weaker form of the conjecture.

Recently, Theorem 1.1 and Conjecture 1.2 have been extended to a more general setting of the weighted maximum multflow problem. Instead of a commodity graph $H$, we are given a nonnegative integral terminal weight $\mu : \binom{S}{2} \rightarrow \mathbb{Z}_+$, where $\binom{S}{2}$ denotes the set of unordered pairs of elements in $S$. Then a multflow $f$ is a pair $(\mathcal{P}, \lambda)$ of a set $\mathcal{P}$ of paths connecting distinct terminals in $S$ and a nonnegative flow-value function $\lambda : \mathcal{P} \rightarrow \mathbb{R}_+$ satisfying the capacity constraint. The total flow-value $\|f\|_\mu$ is defined as

$$\|f\|_\mu := \sum_{P \in \mathcal{P}} \mu(s_P, t_P)\lambda(P),$$

where $s_P$ and $t_P$ denote the ends of $P$. The $\mu$-weighted maximum multflow problem is formulated as:

$\mu$-MFP: Maximize $\|f\|_\mu$ over all multflows $f$ for $(G, S)$.
If $\mu$ is 0-1 valued, then $\mu$-MFP coincides with MFP for the commodity graph $H$ that has an edge $st$ if and only if $\mu(s,t) = 1$.

The fractionality $\text{frac}(\mu)$ of a terminal weight $\mu$ is defined as the least positive integer $k$ such that $\mu$-MFP has a $1/k$-integral optimal multflow for every graph, and the dual fractionality $\text{frac}^*(\mu)$ is the least positive integer $k$ such that the LP-dual to $\mu$-MFP has a $1/k$-integral optimal solution for every capacitated graph. Again $\text{frac}(\mu) \geq \text{frac}^*(\mu)$ holds.

Karzanov [19] extended Theorem 1.1 concerning commodity graph $H$ to a similar statement for metric-weights, and it was extended further in [10] for general weights. For a terminal weight $\mu : (S^2) \rightarrow \mathbb{Z}_+$, define a polyhedral set $T_\mu$ in $\mathbb{R}^S_+$ as

$$T_\mu := \{ p \in \mathbb{R}^S \mid p(s) = \max_{t \in S} \{ \mu(s,t) - p(t) \} \},$$

where we let $\mu(s,s) = 0$. This polyhedral set $T_\mu$ is called the injective envelope or the tight span. Intuitively introduced independently by Isbell [14] and Dress [4] for metrics, and considered by [9] for general weights. The dimension $\dim T_\mu$ is defined to be the largest dimension of a face of $T_\mu$.

Theorem 1.3 ([19] for metrics and [10] for general weights). For a terminal weight $\mu$ on $S$, we have the following:

1. If $\dim T_\mu \leq 2$, then $\text{frac}^*(\mu) \in \{1, 2, 4\}$.
2. If $\dim T_\mu \geq 3$, then $\text{frac}^*(\mu) = \text{frac}(\mu) = +\infty$.

The property P of $H$ is equivalent to the 2-dimensionality of the tight span of the corresponding 0-1 weight $\mu$, as is observed in [10, Section 7]. Thus Conjecture 1.2 for primal fractionality is naturally generalized to the following:

Conjecture 1.4. Suppose that a terminal weight $\mu$ satisfies $\dim T_\mu \leq 2$. Then the following hold:

1. $\text{frac}(\mu) < +\infty$.
2. $\text{frac}(\mu) \in \{1, 2, 4\}$.

The main result of this paper is an affirmative solution of the weaker statement (1) of this generalized conjecture.

Theorem 1.5. For a terminal weight $\mu$ on $S$, if $\dim T_\mu \leq 2$, then $\mu$-MFP has a $1/12$-integral optimal multflow for every Eulerian graph.

This theorem implies the weaker statement (1) of Conjecture 1.2, and thus completes the classification of terminal weights and commodity graphs having finite fractionality as follows.

Corollary 1.6. A terminal weight $\mu$ has finite fractionality if and only if $\dim T_\mu \leq 2$. A commodity graph $H$ has finite fractionality if and only if $H$ has property P.

As a consequence of Theorem 1.5, the possible values of the fractionality are restricted to $1, 2, 3, 4, 6, 8, 12, 24,$ and $+\infty$. However we know no example of terminal weights having fractionality other than $1, 2, 4, +\infty$.

Our proof is constructive, and gives a strongly polynomial time to find a $1/12$-integral optimal multflow under some assumption.

Theorem 1.7. For a commodity graph $H$ with property P, there exists a strongly polynomial time algorithm to find a $1/12$-integral optimal multflow in every inner Eulerian graph.
For a metric $d$ is defined as $G$ graph. We denote $d = (V, E)$ as an edge-length $d : E \to R_+$ by $d(e) := d(x, y)$ for $e = xy$. For a path or a cycle $P$, $d(P)$ denotes the sum of $d(e)$ over all edges $e$ in $P$.

We use the notion of a cell complex; see [1, Chapter I.7] for a precise definition. For a cell complex $K$, a 1-dimensional cell of segment $[p, q]$ is also called an edge, denoted by $pq$. A 0-dimensional cell is called a vertex; the set of vertices is denoted by $V(K)$.

**Organization.** The rest of this paper is divided into three parts. In the first part (Sections 2 and 3), we introduce a duality framework using folder complexes (F-complexes for short), developed in the previous paper [12], and describe the proof outline of Theorem 1.5. An F-complex is a 2-dimensional cell complex obtained by gluing folders, which appeared in Karzanov [18, 19], and was introduced formally by Chepoi [2, Section 7]. If $\dim T_n \leq 2$, then $\mu$ can be embedded into some F-complex $K$, and the maximum value of $\mu$-MFP is equal to the minimum value of a discrete location problem on $K$. In Section 2, we introduce the concept of F-complex and its relation to the multiflow duality. Our proof is based on a fractional version of the splitting-off method combined with the dual update, called SPUP standing for Splitting-off with Potential Update, which is an effective framework for proving the existence of a 1/k-integral optimal multiflow for a bounded integer $k$, devised originally in the previous paper [11] for a special case. In Section 3, we describe the SPUP framework together with the proof outline of Theorem 1.5.

The second part (Sections 4 and 5) is the technical part. In Section 4, we analyze SPUP from the complementary slackness and the geometry of F-complexes. In Section 5, we complete the proof of Theorem 1.5 by showing that the SPUP framework actually works. This also gives a polynomial time algorithm to find a 1/12-integral optimal solution provided the size of F-complex is fixed.

In the third part (Sections 6 and 7), we describe consequences and implications; these sections can be read without the full knowledge of the second part. Our framework not only brings a unified understanding to previously known results but also a powerful algorithmic tool for proving the existence of an integral or half-integral optimal multiflow for Eulerian graphs. In Section 6, we introduce a powerful geometric criterion to show that $\mu$-MFP has an integral optimal multiflow for every Eulerian graph. In Section 7, we concentrate on $\mu_H$-MFP for a commodity graph $H$ with property $P$. We explicitly construct F-complexes for $H$, and prove the half-integrality theorem for a large class of commodity graphs, unifying the previous known results [15, 20, 22, 24, 25].

**Notation.** Let $R$, $R_+$, $Z$, and $Z_+$ denote the sets of reals, nonnegative reals, integers, and nonnegative integers, respectively. For a set $X$, let $R^X$ and $R^X_+$ denote the sets of functions from $X$ to $R$ and $X$ to $R_+$, respectively.

For a graph $G = (V, E)$ with terminal set $S \subseteq V$, each nonterminal node $x \in V \setminus S$ is called an inner node. $G$ is endowed with edge-capacity $c$. The degree of node $x \in V$ is the sum of $c(e)$ over all edges $e$ incident to $x$. By a path we mean a simple path, i.e., it has no repeated nodes. $G$ is said to be inner Eulerian if $c$ is integer-valued and each inner node has an even degree. For a positive integer $k$, $kG$ is the graph $(V, E)$ with edge-capacity $kc$.

A function $d : X \times X \to R_+$ on a set $X$ is called a metric if it satisfies $d(s, t) = d(t, s) \geq d(s, s) = 0$ and the triangle inequalities $d(s, t) + d(t, u) \geq d(s, u)$ for $s, t, u \in S$. For a metric $d$ on $X$ and two subsets $A, B \subseteq X$, the distance $d(A, B)$ between $A$ and $B$ is defined as

$$d(A, B) = \inf\{d(s, t) \mid s \in A, t \in B\}.$$  

We denote $d(A, \{p\})$ simply by $d(A, p)$. We often regard a metric $d$ on node set $V$ of graph $G = (V, E)$ as an edge-length $d : E \to R_+$ by $d(e) := d(x, y)$ for $e = xy$. For a path or a cycle $P$, $d(P)$ denotes the sum of $d(e)$ over all edges $e$ in $P$.

We use the notion of a cell complex; see [1, Chapter I.7] for a precise definition. For a cell complex $K$, a 1-dimensional cell of segment $[p, q]$ is also called an edge, denoted by $pq$. A 0-dimensional cell is called a vertex; the set of vertices is denoted by $V(K)$.
2 Basics on multiflow combinatorial dualities

As is well-known in the multiflow theory [24], an LP-dual of $\mu$-MFP is an optimization problem over metrics:

\[
\text{Minimize } \sum_{e=xy \in E} c(e)d(x, y) \\
\text{subject to } d : \text{metric on } V \text{ with } d(s, t) \geq \mu(s, t) \text{ for } s, t \in S.
\]

In the case of $\dim T_{\mu} \leq 2$, $\mu$ can be embedded into some folder complex $K$, and this embedding gives a combinatorial expression to LP (2.1). A folder complex is a 2-dimensional cell complex obtained by gluing folders (under some axiom) as depicted in Figure 1. Folder complex $K$ is endowed with a metric $d_K$. If a terminal weight $\mu$ is represented as the distances $d_K(R_s, R_t)$ between certain regions $R_s$ in $K$ indexed by $s \in S$, then a combinatorial dual problem for $\mu$-MFP takes the form of a discrete location problem on $K$.

In Section 2.1, we introduce F-complexes and summarize their basic geometric properties. In Section 2.2, we explain a combinatorial duality relation for $\mu$-MFP by F-complexes, and summarize basic facts, including optimality criteria.

2.1 Folder complex

We consider a 2-dimensional cell complex obtained by the following construction. Fix a positive real $\delta > 0$. A cell having an isometry into an isosceles right triangle $\{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_1 \leq x_2 \leq \delta\}$ in the Euclidean plane will be called a triangle, whereas a cell having an isometry to a square $\{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_1, x_2 \leq \delta\}$ is a square.
By a folder we mean a square or a cell complex obtained by gluing triangles along the common longer edge. See Figure 2. A square is particularly called a square-folder. A folder \( F \) is called a \( K_{2,m} \)-folder if \( F \) consists of \( m \) triangles, and also called a \( K_{2,*} \)-folder if \( F \) is a \( K_{2,m} \) for some \( m \). A \( K_{2,*} \)-folder has two types of edges: the (unique) longer edge and shorter edges. Following [2], we call the longer edge the hypotenuse, and a shorter edge a leg. Any edge of a square-folder is called a leg. A scale parameter \( \delta \) is called the leg-length.

Next we consider a cell complex \( K \) obtained by gluing folders and edges (1-dimensional cell) isometric to segment \([0, \delta]\), which we also call a leg, in such a way that any two of the folders are glued along one leg or at one vertex. Then \( K \) is called a folder complex (an F-complex for short) [2, Section 7] if it is simply-connected, and satisfies:

**Flag condition:** there exist no vertex \( p \) and three legs \( e_1, e_2, e_3 \) incident to \( p \) such that \( e_i \) and \( e_j \) belong to a common folder for \( 1 \leq i < j \leq 3 \).

This condition means that folders should be glued without a corner of 3-cube as in Figure 3. A metric on \( K \) is defined as follows. Each 2-dimensional cell (a triangle or a square) has a natural \( l_1 \)-metric by the isometry to \( \mathbb{R}^2 \). Then the \( l_1 \)-length of a path \( P \) in \( K \) is the sum, over all cells \( \sigma \), of the \( l_1 \)-length of \( \sigma \cap P \) measured by the \( l_1 \)-metric on \( \sigma \), where \( \sigma^\circ \) denotes the relative interior of \( \sigma \). The \( l_1 \)-length metric \( d_K(p,q) \) between \( p \) and \( q \) in \( K \) is defined to be the infimum of the lengths of all paths connecting \( p \) and \( q \) in \( K \).

We next introduce a certain class of regions in a folder complex \( K \); we will represent \( \mu \) as the distance between these regions in \( K \). A connected subcomplex \( R \) of \( K \) is called normal if it satisfies the following axiom:

**Boundary axiom:** the boundary of \( R \) (relative to \( K \)) consists of hypotenuses, i.e., if a leg \( e \) belongs to \( R \), then every cell containing \( e \) belongs to \( R \).

**Local convexity:** there exists no pair of triangles \( \sigma, \sigma' \) sharing a leg and a right angle such that \((\sigma \cup \sigma') \cap R\) coincides with the union of the hypotenuses of \( \sigma \) and \( \sigma' \).

Any normal set is a closed connected set. See Figure 4 for the violation of local convexity. We list several basic concepts of F-complex below.

### 2.1.1 Admissible orientations and orbits

An F-complex \( K \) is said to be orientable if the edge set of \( K \) has an orientation with the property that, for each folder \( F \) in \( K \), there is a pair \( p,q \) of vertices of \( F \) such that
each edge (leg or hypotenuse) of $F$ enters $p$ or leaves $q$; see Figure 5. This orientation is called an admissible orientation; in fact, an admissible orientation is acyclic. Vertices $p$ and $q$ are particularly called the source and the sink of $F$, respectively, with respect to this orientation.

An orbit is an equivalence class with respect to the equivalence relation obtained as the transitive closure of the relation $\sim$ on all edges (legs and hypotenuses) of $\mathcal{K}$ defined by $e \sim e'$ if $e$ and $e'$ are nonadjacent legs in some square-folder, or belong to a common $K_{2,*}$-folder. An admissible orientation is obtained by orienting orbits independently. Such orientation of an orbit is also said to be admissible. See Figure 6. Each orbit has exactly two admissible orientations; one is the reverse of the other.

For an admissible orientation $\overrightarrow{\mathcal{K}}$ of $\mathcal{K}$ and vertices $p, q \in V(\mathcal{K})$, we write $p \succeq_{\overrightarrow{\mathcal{K}}} q$ if $p = q$, $\overrightarrow{pq}$ is an oriented leg in $\overrightarrow{\mathcal{K}}$, or $(p, q)$ is the source-sink pair of some folder with respect to $\overrightarrow{\mathcal{K}}$. Let $O$ be an orbit and let $\overrightarrow{O}$ be an admissible orientation of $O$. If $O$ contains all edges of a folder $F$, then $\overrightarrow{O}$ determines the source and the sink of $F$, as in Figure 5. Similarly, we write $p \succeq_{\overleftarrow{O}} q$ if $p = q$, $\overleftarrow{pq}$ is an oriented leg in $\overleftarrow{O}$, or $(p, q)$ is the source-sink pair of some folder with respect to $\overleftarrow{O}$. Note that relations $\succeq_{\overrightarrow{\mathcal{K}}}$ and $\succeq_{\overleftarrow{O}}$ are not transitive.
2.1.2 Leg-graph

The \textit{leg-graph} \( \Gamma \) is the graph on \( V(\mathcal{K}) \) consisting of all legs (not including hypotenuses). The leg-graph is precisely a \textit{frame} in the sense of \cite{18} (although F-complexes and frames are essentially equivalent, F-complexes are suitable to represent normal regions). We often use the following elementary properties of \( \Gamma \), which can easily be verified \cite{12}.

(2.2) (1) The leg-graph \( \Gamma \) is bipartite.

(2) For normal sets \( N, M \), we have \( d_{\mathcal{K}}(N, M) = d_{\Gamma, \delta}(N \cap V(\mathcal{K}), M \cap V(\mathcal{K})) \),

where \( d_{\Gamma, \delta} \) denotes the shortest path metric on the leg-graph with respect to uniform edge-length \( \delta \).

2.1.3 Subdivisions

An F-complex \( \mathcal{K} \) has a natural subdivision operation. For a positive integer \( m \), subdivide each leg into \( m \) legs of length \( \delta/m \). Accordingly, subdivide each square into \( m \times m \) squares of leg-length \( \delta/m \), each triangle into \( m \) triangles and \( m(m-1)/2 \) squares of leg-length \( \delta/m \); see \cite[Figure 5]{12}. The resulting complex is denoted by \( \mathcal{K}^m \), called the \textit{m-subdivision} of \( \mathcal{K} \). One can easily see the following facts:

(2.3) \( \mathcal{K}^m \) is also an F-complex, and \( \mathcal{K}^2 \) is always orientable.

See Figure 12 (in Section 3) for verifying the orientability of \( \mathcal{K}^2 \).

2.1.4 Star-shaped F-complex and neighborhood

An F-complex \( \mathcal{K} \) is said to be \textit{star-shaped} if there exists a vertex \( p \) such that every maximal cell contains \( p \) and no triangle has \( p \) as its right angled corner. A star-shaped F-complex will be used in investigating the local structure around vertex \( p \). The \textit{neighborhood} \( \mathcal{K}_p \) of \( p \) consists of all cells containing \( p \) and their faces. Neighborhood \( \mathcal{K}_p \) is also an F-complex, and a geodesic subspace of \( \mathcal{K} \) with diameter at most 4\( \delta \). In particular,

\[(2.4) \quad d_{\mathcal{K}}(p, q) = d_{\mathcal{K}_p}(p, q) \in \{0, 1, 2, 3, 4\} \delta \quad (p, q \in V(\mathcal{K}_p)).\]

(The geodesic property \( d_{\mathcal{K}} = d_{\mathcal{K}_p} \) is implicit in \cite{12}. One can verify this property by using the properties of the leg-graph: every 4-cycle belongs to a unique folder \cite[(3.6)]{12} and every 6-cycle has a chord; see \cite{12, 18}).

Although \( \mathcal{K}_p \) may not be star-shaped, \( (\mathcal{K}^m)_p \) for \( m \geq 2 \) is always star-shaped. Let \( \Pi_p \) be the graph obtained by deleting \( p \) from the leg-graph of \( (\mathcal{K}^m)_p \) for some \( m \geq 2 \), where \( \Pi_p \) is independent of \( m \). See Figure 7. Then the flag condition can be rephrased by the following:

(2.5) \( \Pi_p \) has girth at least 8,

where the girth means the shortest length of a (simple) cycle. \( \Pi_p \) is a bipartite graph with bipartition \( \{L_p, Q_p\} \), where \( Q_p \) denotes the set of vertices incident to \( p \) by legs in \( (\mathcal{K}^m)_p \) and \( L_p \) denotes the set of the other vertices.

Even if \( \mathcal{K}_p \) is not star-shaped, the leg-graph of \( \mathcal{K}_p \) is a subgraph of that of \( (\mathcal{K}^m)_p \). Therefore we can naturally regard \( V(\mathcal{K}_p) \setminus \{p\} \) as a subset of \( L_p \cup Q_p \).
2.1.5 Orbits and summands

For a (disjoint) union $U$ of several orbits, we can construct a new complex $\mathcal{K}/U$ from $\mathcal{K}$ by contracting each edge not in $U$; see [12] for a more precise construction. Again $\mathcal{K}/U$ consists of folders, and indeed is an F-complex by the next proposition. We call $\mathcal{K}/U$ a summand of $\mathcal{K}$. See Figure 8. The contraction naturally induces a map $(\cdot)/U : V(\mathcal{K}) \to V(\mathcal{K}/U)$ by defining $p/U$ to be the contracted vertex. By extending linearly, we obtain a map $(\cdot)/U : \mathcal{K} \to \mathcal{K}/U$. Also define $\mathcal{K}\setminus U$ as the summand $\mathcal{K}/\overline{U}$ for the complement $\overline{U}$ of $U$, and define map $(\cdot)/U$ as $(\cdot)/\overline{U}$.

**Proposition 2.1** ([12, Proposition 3.15]). Let $U$ be the union of several orbits.

1. $\mathcal{K}/U$ is an F-complex.
2. For a normal set $R$ in $\mathcal{K}$, $R/U$ is also normal in $\mathcal{K}/U$.
3. For normal sets $M, N$ in $\mathcal{K}$, $d_\mathcal{K}(M, N) = d_{\mathcal{K}/U}(M/U, N/U) + d_{\mathcal{K}\setminus U}(M\setminus U, N\setminus U)$.

2.2 F-complex realization and multiflow combinatorial duality

Here we describe a combinatorial duality relation for $\mu$-MFP by an F-complex. For a weight $\mu$ on terminal set $S$, an F-complex realization (a realization for short) of $\mu$ is a pair $(\mathcal{K}; \{R_s\}_{s \in S})$ of an F-complex $\mathcal{K}$ and a family $\{R_s\}_{s \in S}$ of normal sets satisfying

$$\mu(s, t) = d_\mathcal{K}(R_s, R_t) \quad (s, t \in S).$$

Namely $\mu$ is realized as the distances among normal sets $R_s$. Figure 9 illustrates an example, where $s_7$ and $s_8$ are embedded into regions (of $R_{s_8}$ is the shaded region), and the
others are embedded into vertices. It is known that an existence of a realization of $\mu$ is characterized by the dimension of the tight span $T_\mu$ [12].

**Theorem 2.2** ([12, Theorem 4.5]). The following two conditions are equivalent:

1. $\dim T_\mu \leq 2$.

2. $\mu$ has an F-complex realization.

In fact, a realization of $\mu$ can be obtained by subdividing 2-dimensional polyhedral complex $T_\mu$ into folders with $\delta = 1/4$ [10].

An F-complex realization enables us to define a combinatorial problem dual to $\mu$-MFP, sharpening LP-dual (2.1). Suppose that a weight $\mu$ on $S$ has an F-complex realization $(K; \{R_s\}_{s \in S})$. We consider the following discrete location problem associated with $(K; \{R_s\}_{s \in S})$:

$$\text{DLP}(K; \{R_s\}_{s \in S}): \ \text{Minimize} \ \sum_{xy \in E} c(xy) d_K(\rho(x), \rho(y))$$

subject to $\rho: V \to V(K), \ \rho(s) \in R_s \ (s \in S)$. \\

Here $\rho$ represents an embedding of the node set $V$ of $G$ into that of $K$. Our previous paper established the following duality relation, extending a result in [18].

**Theorem 2.3** ([12, Theorem 2.1]). Suppose that $\mu$ has an orientable F-complex realization $(K; \{R_s\}_{s \in S})$. Then the maximum value of $\mu$-MFP for $(G, S)$ is equal to the minimum value of $\text{DLP}(K; \{R_s\}_{s \in S})$.

This theorem guarantees the existence of an optimal metric $d$ in (2.1) represented as $d(x, y) = d_K(\rho(x), \rho(y))$ for a map $\rho$ in $\text{DLP}(K; \{R_s\}_{s \in S})$; see Section 5.4 for a more detailed account of the relationship between DLP and LP-dual. The orientability requirement is not restrictive. By the subdivision operation (Section 2.1.3), we can always make a given F-complex realization orientable. Hence we tacitly assume that an F-complex is always orientable.

A map $\rho$ feasible to $\text{DLP}(K; \{R_s\}_{s \in S})$ is called a potential. For a potential $\rho$, let $d^\rho$ denote the metric on $V$ defined by $d^\rho(x, y) := d_K(\rho(x), \rho(y))$, and let $d^\rho(G)$ denote the objective value $\sum_{e \in E} c(e) d^\rho(e)$ of $\text{DLP}(K; \{R_s\}_{s \in S})$. Let $\text{opt}(\mu; G)$ denote the optimal value of $\mu$-MFP, which is equal to the optimal value of $\text{DLP}(K; \{R_s\}_{s \in S})$ by Theorem 2.3. We list several basic properties of $\mu$-MFP and $\text{DLP}(K; \{R_s\}_{s \in S})$ below.
2.2.1 Optimality criterion of primal-dual type

For a multiflow $f = (\mathcal{P}, \lambda)$ and a potential $\rho$, the duality gap $d^\rho(G) - \|f\|_\rho$ is given as

$$d^\rho(G) - \|f\|_\rho = \sum_{e \in E} c(e)d^\rho(e) - \sum_{P \in \mathcal{P}} \mu(s_P, t_P)\lambda(P)$$

$$= \sum_{e \in E} d^\rho(e)(c(e) - f^e) + \sum_{P \in \mathcal{P}} \lambda(P)(d^\rho(P) - d_K(R_{s_P}, R_{t_P})).$$

Here $f^e$ denotes the total amount of flows on $e$, i.e., $f^e = \sum_{P \in \mathcal{P}, e \subseteq P} \lambda(P)$, and note that $\mu(s_P, t_P) = d_K(R_{s_P}, R_{t_P})$ by the definition of the F-complex realization. Hence an optimality criterion of primal-dual type is given as follows.

**Lemma 2.4.** A multiflow $f = (\mathcal{P}, \lambda)$ and a potential $\rho$ are both optimal if and only if they satisfy:

**Saturation condition:** for each $e \in E$, $d^\rho(e) > 0$ implies $f^e = c(e)$.

**Geodesic condition:** for each $P \in \mathcal{P}$, $\lambda(P) > 0$ implies $d^\rho(P) = d_K(R_{s_P}, R_{t_P})$.

The geodesic condition says that paths in $f$ are embedded as shortest paths between terminal regions $R_s$ in $K$ by $\rho$. This view is most fundamental in every place of this paper.

2.2.2 Optimality criterion by neighbors

Next we describe an optimal criterion for $DLP(K; \{R_s\}_{s \in S})$ to the effect that: local optimality implies global optimality.

A potential $\rho'$ is called a neighbor of $\rho$ with respect to an oriented orbit $\vec{O}$ if $\rho(x) \geq \rho'(x)$ for all $x \in V$. See Section 2.1.1 for the notation. Namely $\rho'$ is obtained by moving part of $\rho$ along the direction $\vec{O}$. The following theorem is a basis for the SPUP framework in the next section. By a neighbor of $\rho$ we mean a neighbor with respect to some oriented orbit.

**Theorem 2.5** ([12, Theorem 4.1]). A potential $\rho$ is optimal to $DLP(K; \{R_s\}_{s \in S})$ if and only if $d^\rho(G) \leq d^{\rho'}(G)$ holds for every neighbor $\rho'$ of $\rho$.

A more relaxed neighbor concept, which will turn out to be useful, can be defined as follows. For an admissible orientation $\vec{K}$ of $K$, a potential $\rho'$ is called a semi-neighbor of $\rho'$ with respect to $\vec{K}$ if $\rho(x) \geq \rho'(x)$ for all $x \in V$. $\vec{K}$ induces an admissible orientation $\vec{O}_i$ of each orbit $O_i (i = 1, 2, \ldots, m)$ (by restriction). Thus, by definition, a neighbor with respect to $\vec{O}_i$ is a semi-neighbor with respect to $\vec{K}$. It is shown in [12, Section 4.1] that for a semi-neighbor $\rho'$ of $\rho$ with respect to $\vec{K}$, there exist neighbors $\rho_i$ of $\rho$ with respect to $\vec{O}_i$ such that $d^{\rho'} - d^\rho = \sum_i (d^{\rho_i} - d^\rho)$. By this property, we can use semi-neighbors instead of neighbors in many places.

2.2.3 Summands and locking property

For a union $U$ of several orbits, let $\mu/\mu$ be the weight on $S$ defined as $\mu/\mu(s, t) := d_K(U)((R_s)/U, (R_t)/U)$ for $s, t \in S$. Recall Section 2.1.5 for notations. $\mu/\mu$ is called a summand of $\mu$ with respect to $U$. By construction and Proposition 2.1, $(K/\mu; \{(R_s)/U\}_{s \in S})$ is a realization of $\mu/\mu$. Similarly, define $\mu\backslash\mu$ as $\mu\backslash\mu(s, t) := d_K(\backslash\mu)((R_s)\backslash U, (R_t)\backslash U)$ for $s, t \in S$. Then $(K\backslash\mu; \{(R_s)\backslash U\}_{s \in S})$ is a realization of $\mu\backslash\mu$. 

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Proposition 2.6. Let \( f \) be an optimal multi-flow and \( \rho \) an optimal potential. For a union \( U \) of several orbits, we have the following:

1. \( f \) is optimal to \( \mu\setminus U \)-MFP and \( \rho\setminus U \)-MFP.
2. \( \rho\setminus U \) and \( \rho\setminus U \) are optimal to DLP\( (K\setminus U; \{(R_s)\setminus U\}_{s \in S}) \) and DLP\( (K\setminus U; \{(R_s)\setminus U\}_{s \in S}) \), respectively.

Proof. \( \rho\setminus U \) and \( \rho\setminus U \) are feasible to DLP\( (K\setminus U; \{(R_s)\setminus U\}_{s \in S}) \) and DLP\( (K\setminus U; \{(R_s)\setminus U\}_{s \in S}) \), respectively. By Proposition 2.1 (3), we have

\[
\|f\|_\mu = \|f\|_\mu U + \|f\|_\mu U, \quad \text{and} \quad d^\rho = d^\rho U + d^\rho U. \]

Thus we have

\[
\|f\|_\mu = \|f\|_\mu U + \|f\|_\mu U \leq d^\rho U(G) + d^\rho U(G) = d^\rho(G) = \|f\|_\mu. \]

This explains the locking property of multflows, which means the existence of a multflow simultaneously optimal to several \( \mu \)-MFPs.

3 Proof outline: SPUP framework

In this section, we explain the proof outline of Theorem 1.5, which is a kind of a primal-dual algorithm by a fractional version of the splitting-off and the dual update. We call it SPUP, standing for Splitting-off with Potential Update.

3.1 SPUP (Splitting-off with Potential Update)

We begin with the splitting-off operation. Let \( G \) be a graph. For two consecutive edges \( e = xy \) and \( e' = yz \) of unit capacity incident to node \( y \), a triple \( \tau = (e, y, e') \) is called a fork. The splitting-off operation is to delete edges \( e, e' \) and to add a new edge of unit capacity connecting \( x \) and \( z \) if \( x \neq z \). If the splitting-off operation does not decrease the optimal flow-value \( \text{opt}(\mu; G) \), then a \((1/k)-\)integral optimal multflow in the original graph can be recovered from any \((1/k)-\)integral optimal multflow in the new graph. Such a fork is called splittable. If a fork \( \tau \) is not splittable, then \( \tau \) is called unsplittable.

We next introduce the fractional splitting-off operation. For a fork \( \tau = (e, y, e') \) and \( \alpha \in [0, 2] \), the graph \( G^{\tau, \alpha} \) is obtained by adding a new node \( y^\tau \), reconnecting \( e \) and \( e' \) to \( y^\tau \), and joining \( y \) and \( y^\tau \) by a new edge \( e^\tau = yy^\tau \) of capacity \( c(e^\tau) = 2 - \alpha \); see Figure 10. The resulting graph is denoted by \( G^{\tau, \alpha} \). In the case of \( \alpha = 0 \), the problems on \( G \) and on \( G^{\tau, 0} \) are equivalent, and in particular \( \text{opt}(\mu; G) = \text{opt}(\mu; G^{\tau, 0}) \). Any multflow in \( G \) is naturally extended to a multflow in \( G^{\tau, 0} \) by adding \( e^\tau \) for each path containing either \( e \) or \( e' \). So we regard a multflow in \( G \) as a multflow in \( G^{\tau, 0} \).
We consider increasing $\alpha$ from 0 without changing the optimal value. The maximum possible value is denoted by $\alpha_\tau$ or $\alpha_\tau(G)$, i.e.,

$$\alpha_\tau := \max\{\alpha \in [0, 2] \mid \text{opt}(\mu; G) = \text{opt}(\mu; G^{\tau,\alpha})\}.$$

The modification of $G$ to $G^{\tau,\alpha}$ is named here a fractional splitting-off operation. By reversing this operation, i.e., by contracting edge $e^\tau$, any $1/k$-integral optimal multiflow in $G^{\tau,\alpha}$ becomes a $1/k$-integral optimal multiflow in $G$. The case $\alpha_\tau = 2$ is nothing but the (ordinary) splitting-off operation.

We give here one fundamental relation between $\alpha_\tau$ for a fork $\tau = (e, y, e')$ and an optimal multiflow $f$, where $f^e$ (resp., $f^{e, e'}$) denotes the total amount of flows using $e$ (resp., $e$ and $e'$) in $f$.

**Lemma 3.1.** $\alpha_\tau \geq 2 - f^{e^\tau} \geq 2 f^{e,e'}$.

**Proof.** Since $f$ is also a multiflow in $G^{\tau,\alpha}$ for $\alpha = 2 - f^{e^\tau}$, we have $\|f\|_{\mu} \leq \text{opt}(\mu; G^{\tau,\alpha}) \leq \text{opt}(\mu; G) = \|f\|_{\mu}$, which implies the first inequality $\alpha_\tau \geq \alpha$. The second inequality follows from $2 - f^{e^\tau} = 2 - (f^e + f^{e'} - 2 f^{e,e'}) = (1 - f^e) + (1 - f^{e'}) + 2 f^{e,e'} \geq 2 f^{e,e'}$. \qed

Suppose that we are given a realization $(K; \{R_s\}_{s \in S})$ of $\mu$ with unit leg-length $\delta = 1$ and an optimal potential $\rho : V \rightarrow V(K)$ for DLP$(K; \{R_s\}_{s \in S})$. There is another formula for $\alpha_\tau$ involving $\rho$ and its neighbors. Any potential $\rho$ for $G$ is extended to a potential for $G^{\tau,\alpha}$ by defining $\rho(y^\tau) := \rho(y)$. An important observation here is:

(3.1) If $\rho$ is optimal to $G$, then $\rho$ is also optimal to $G^{\tau,\alpha}$ for $0 \leq \alpha \leq \alpha_\tau$.

Indeed, by $d^\rho(e^\tau) = 0$ we have $\text{opt}(\mu; G) = d^\rho(G) = d^\rho(G^{\tau,\alpha}) \geq \text{opt}(\mu; G^{\tau,\alpha}) = \text{opt}(\mu; G)$. This brings about a formula of $\alpha_\tau$ in terms of neighbors as follows:

**Proposition 3.2.** Let $\rho$ be an optimal potential, and $\tau$ an unsplittable fork. We have

$$\alpha_\tau = \min_{\rho'} \frac{d^\rho(G^{\tau,0}) - d^\rho(G^{\tau,0})}{d^\rho(e^\tau)},$$

where the minimum is taken over all neighbors $\rho'$ of $\rho$ with $d^\rho(e^\tau) > 0$.

**Proof.** We see the equivalence among the conditions (1) to (4) for $\alpha$ below:

(1) $0 \leq \alpha \leq \alpha_\tau$.

(2) $\text{opt}(\mu; G^{\tau,\alpha}) = \text{opt}(\mu; G)$.

(3) $\rho$ is optimal for $G^{\tau,\alpha}$.

(4) For every neighbor $\rho'$ of $\rho$, we have $d^\rho(G^{\tau,\alpha}) \geq d^\rho(G^{\tau,\alpha})$.

(1) $\Leftrightarrow$ (2) follows from the definition. (2) $\Leftrightarrow$ (3) follows from $\text{opt}(\mu; G) = d^\rho(G^{\tau,\alpha})$ by $d^\rho(e^\tau) = 0$. (3) $\Leftrightarrow$ (4) follows from Theorem 2.5. To obtain the desired formula, substitute $d^\rho(G^{\tau,\alpha}) = d^\rho(G^{\tau,0}) - \alpha d^\rho(e^\tau)$ and $d^\rho(G^{\tau,\alpha}) = d^\rho(G^{\tau,0})$ to (4). \qed

The minimization over neighbors in (3.2) can be replaced by that over semi-neighbors. A (semi-)neighbor $\rho'$ attaining the minimum in the formula of $\alpha_\tau$ is said be critical. Note that both $\rho$ and $\rho'$ are optimal to $G^{\tau,\alpha}$.

For an optimal potential $\rho$, an unsplittable fork $\tau$, and a critical neighbor $\rho'$ of $\rho$ respect to $\tau$, we consider the update $(G; \rho) \leftarrow (G^{\tau,\alpha}; \rho')$, which we call SPUP and specifically $\alpha$-SPUP when $\alpha = \alpha_\tau$ ($\alpha$ is a rational in $[0, 2]$).

Our proof of Theorem 1.5 basically goes along the following procedure starting from an inner Eulerian graph $G$ (without splittable forks) and an optimal potential $\rho$.

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SPUP procedure: Let \((G_0; \rho_0) := (G; \rho)\) and \(j := 1\). We repeat the following:

**step 1:** Take a fork \(\tau_j\) at a node \(y_j \in V(G_{j-1}) \setminus \{y_1, y_1^{\tau_1}, y_2, y_2^{\tau_2}, \ldots, y_{j-1}, y_{j-1}^{\tau_{j-1}}\}\) and a critical neighbor \(\rho'_j\) of \(\rho_{j-1}\) with respect to \(\tau_j\) in \(G_{j-1}\).

**step 2:** Do SPUP: \((G_j; \rho_j) \leftarrow (G_{j-1}^{\tau_j}; \rho'_j)\), and let \(K_j\) be the smallest positive integer such that \(K_jG_j\) is inner Eulerian.

**step 3:** If \(K_jG_j\) is guaranteed to have an integral optimal multiflow, then stop. Otherwise let \(j := j + 1\) and go to step 1.

We will prove Theorem 1.5 by showing: By appropriate choices of \(\tau_j, \rho'_j\) in step 1,

(a) for some \(j \leq |V|\), the algorithm terminates in step 3, and

(b) \(K_j\) is bounded by a constant, say 12, independent of \(|V|\).

If this is proved, then by reversing the operations (i.e., by contracting \(e^{\tau_j}\)) we can construct a 1/\(K_j\)-integral optimal multiflow in the original graph \(G\).

For (a), we will show that if \(\rho_j\) is an embedding to \(K\) with a certain special property, \(K_jG_j\) is guaranteed to have an integral optimal multiflow and the algorithm stops in step 3. To realize such an embedding, we will choose \((\tau_j, \rho'_j)\) in step 1 appropriately. For (b), we will bound \(K_j\) throughout the procedure. Each step creates edge \(e^{\tau_j}\) of (possibly fractional) capacity \(2 - \alpha_j = 2 - \{d^{\rho}(G^{\tau_j}; 0) - d^{\rho}(G^{\tau_j}; 0)\}/d^{\rho}(e^{\tau_j})\). Here \(d^{\rho}(e^{\tau_j})\) is one of \([1, 2, 3, 4]\) since \(\rho'(y)\) and \(\rho'(y^{\tau_j})\) belong to the neighborhood of \(\rho(y)\) in \(K\); see (2.4) in Section 2.1.4. So we will bound the denominator of \(d^{\rho}(G^{\tau_j}; 0) - d^{\rho}(G^{\tau_j}; 0)\).

We explain a concrete strategy of achieving this idea in the rest of this section, which is structured as follows. In Section 3.2, we classify terminals with a view to studying the parity of \(d^{\rho}(G^{\tau_j}; 0) - d^{\rho}(G^{\tau_j}; 0)\). In Section 3.3, we describe reductions of making each node have small degree, which simplifies our analysis in every place. In Section 3.4, we describe the whole proof outline of Theorem 1.5.

### 3.2 Proper/essential terminals and the parity of \(d^{\rho}(G^{\tau_j}; 0) - d^{\rho}(G^{\tau_j}; 0)\)

A terminal \(s\) is said to be **proper** (with respect to realization \((K; \{R_s\}_{s \in S})\)) if \(R_s\) contains no legs, i.e., if \(R_s\) has no interior. A terminal that is not proper is said to be **improper**. In Figure 9, \(s_8\) is improper and the other terminals are proper. A terminal \(s\) is said to be **essential** if every optimal multiflow \(f = (P, \lambda)\) has a path \(P \in P\) connecting \(s\) and another terminal \(t\) with \(\lambda(P) > 0\) and \(\mu(s, t) > 0\).

**Lemma 3.3.** For two optimal potentials \(\rho\) and \(\rho'\), if terminal \(s\) is proper or essential, then \(\rho'(s)\) and \(\rho(s)\) belong to the same connected component of the boundary of \(R_s\), and hence belong to the same color class of the leg-graph.

**Proof.** It suffices to consider the case where \(s\) is improper and essential. Take an optimal multiflow \(f = (P, \lambda)\), which has a path \(P\) connecting \(s, t\) with \(\mu(s, t) > 0\) and \(\lambda(P) > 0\). Both \(\rho(s)\) and \(\rho'(s)\) must be on the boundary of \(R_s\). Otherwise, it is impossible to satisfy the geodesic condition for \(P\). Necessarily \(R_s\) and \(R_t\) are disjoint by \(d_K(R_s, R_t) = \mu(s, t) > 0\). Delete the interior of \(R_s\) from \(K\). Let \(K'\) be the resulting connected component including \(R_t\). Since \(K\) is simply-connected and \(R_s\) is connected, \(R_s \cap K'\) is connected. Both \(\rho(s)\) and \(\rho'(s)\) belong to \(R_s \cap K'\) since any shortest path joining \(R_s\) and \(R_t\) must belong to \(K'\).

This fact has a consequence on the parity of \(d^{\rho}(G^{\tau_j}; 0) - d^{\rho}(G^{\tau_j}; 0)\) as follows, where \(\rho\) is an optimal potential, \(\tau\) is an unsplitting fork, and \(\rho'\) is a critical neighbor of \(\rho\) with respect to \(\tau\).
Lemma 3.4. Suppose that $G$ is an inner Eulerian graph such that each terminal is proper, essential, or has an even degree. Then $d^0(G^{\tau,0}) - d^0(G^{\tau,0})$ is an even integer, and hence $\alpha_T \in \{0, 1/2, 2/3, 1, 4/3, 3/2\}$.

Proof. Let $T \subseteq S$ be the set of proper or essential terminals. Then $G^{\tau,0}$ is inner Eulerian with respect to $T$. By edge-multiplication, we may assume that $G$ has unit capacity. Hence the edge set $E(G^{\tau,0})$ is a disjoint union of $T$-paths $P_j$ and (nonsimple) cycles $C_i$. Hence we have $d^0(G^{\tau,0}) - d^0(G^{\tau,0}) = \sum_i \{d^0(C_i) - d^0(C_i)\} + \sum_j \{d^0(P_j) - d^0(P_j)\}$.

Since the leg-graph is bipartite (Section 2.1.2), both $d^0(C_i)$ and $d^0(C_i)$ are even. Each essential terminal remains essential in $G^{\tau,0}$, and both $\rho$ and $\rho'$ are optimal to $G^{\tau,0}$. By Lemma 3.3, $\rho(s)$ and $\rho'(s)$ for $s \in T$ belong to the same color class. Thus $d^0(P_j) - d^0(P_j)$ is also even. Consequently $d^0(G^{\tau,0}) - d^0(G^{\tau,0})$ is even. As was noted, $d^0(e^\tau)$ is one of $\{1, 2, 3, 4\}$ since both $\rho'(y)$ and $\rho'(y^*)$ belong to the neighborhood of $\rho(y)$.

How to bound the denominator of $d^0(G^{\tau,0}) - d^0(G^{\tau,0})$. In the SPUP procedure, each step $j$ creates an edge $e^\tau_j$ of (possibly fractional) capacity $2 - \alpha_T$. Hence the inner Eulerian condition for $G_j$ does not hold even if $G_0$ is inner Eulerian, and Lemma 3.4 is not applicable. However, if created edge $e^\tau_j$ remains to have the same length, i.e., $d^0(e^\tau_k) = d^0(e^\tau_k)$ for $k = 1, 2, \ldots, j - 1$, then each (fractional) term $c(e^\tau_k)\{d^0(e^\tau_k) - d^0(e^\tau_k)\}$ in $d^0(G^{\tau,0}) - d^0(G^{\tau,0})$ vanishes, and Lemma 3.4 is applicable. In this way, we will bound $K_j$ by keeping the length $d^0(e^\tau)$ of the created edge $e^\tau_k$ as far as possible.

3.3 Degree reductions

We will mainly work on an inner Eulerian graph such that each inner node has degree 4 and each terminal has degree 1 or 2. A standard reduction is known to make the graph have degree at most 4: see [7, p. 50]. Let $G$ be an inner Eulerian graph, i.e., capacity $c$ is integer-valued and each inner node has an even degree. By edge-multiplication, we can make each edge have unit capacity.

Degree-4 reduction of an inner node. For an inner node $x$ of degree greater than four, we can reduce the degree by changing the incidence at $x$ as in Figure 11 (a). Then the problem does not change.

Degree-1 reduction of a terminal. For a terminal $s$ of degree $m$, we can reduce its degree to one as follows. Consider the case where $s$ is incident to a unique node $x$. Replace $s$ by new terminals $s_1, s_2, \ldots, s_m$, connect $x$ and each $s_i$ by an edge (of unit capacity), and define weight $\mu$ on $s_i$ by $\mu(s_i, t) = \mu(s, t)$ for $t \in S \setminus s$ and by $\mu(s_i, s_j) = 0$. Obviously the problem does not change. See Figure 11 (b). A realization of $(S', \mu')$ is
obtained by setting $R_{s_i} := R_s$ for each $i$. In the case where $s$ is incident to several nodes $x_1, x_2, \ldots, x_l$, add a new inner node $x$, and replace each edge $x_is$ by two edges $x_ix$ and $xs$. Then $s$ has a unique neighbor $x$. Apply the reduction above.

**Degree-2 reduction of a terminal with even degree.** For a terminal $s$ of even degree $m$, we can reduce its degree to two, as in the degree-1 reduction above, by adding new $m/2$ terminals $s_1, s_2, \ldots, s_{m/2}$ and connect $s_i$ by two parallel edges.

**Extending an optimal potential to the new problem.** In the reductions above, if we are given an optimal potential $\rho$ for the original problem, we can extend $\rho$ to an optimal potential for the new problem by setting $\rho(x') := \rho(x)$ for each new added node $x'$. This is a simple consequence of the optimality criterion (Lemma 2.4).

**Keeping a terminal essential.** The degree-1 and -2 reductions may create a nonessential terminal. In the degree-2 reduction, we can split off a unique fork for a nonessential terminal of degree 2, while keeping the inner Eulerian condition. In the degree-1 reduction, to guarantee that each new (improper) terminal is essential, we will use the following fact, where an optimal potential $\rho$ is assumed to be given.

\[(3.3)\] For a terminal $s$ incident to a unique node $x$ with $\rho(s) \neq \rho(x)$, the degree-1 reduction at $s$ keeps each new terminal $s_i$ essential.

Indeed, by the optimality criterion for $(f, \rho)$, every optimal multiflow $f$ must have paths connecting $s$ of the flow-value equal to the degree of $s$, i.e., $f^{sx} = c(sx)$. Obviously this flow property, stronger than the essentialness, is kept in the degree-1 reduction.

**Edge-subdivision.** We will also create an inner node of degree 2 by the subdivision of an edge $e = xy$, which is to add a new node $z$ and replace $xy$ by two edges $xz, zy$. The capacity is defined by $c(xz) = c(zy) := c(xy)$. This operation obviously does not change the problem. If we are given an optimal potential $\rho$ for the original problem, we can extend $\rho$ to an optimal potential to the new problem by defining $\rho(z) = d_K(\rho(x), \rho(y))$ so that $d_K(\rho(x), \rho(y)) = d_K(\rho(x), \rho) + d_K(p, \rho(y))$ for any $p \in V(K)$. This fact is also an easy consequence of the optimality criterion (Lemma 2.4).

### 3.4 Proof outline

Here we describe the outline of the proof of Theorem 1.5, which we prove under a weaker condition. Recall Theorem 2.2 that the 2-dimensionality of $T_\mu$ of $\mu$ is equivalent to the existence of an F-complex realization $(K; \{R_s\}_{s \in S})$. A graph $G$ is said to be **properly-inner Eulerian** with respect to a realization $(K; \{R_s\}_{s \in S})$ if the capacity is integral and each node other than proper terminals has an even degree.

**Theorem 3.5.** Suppose that $\mu$ has an F-complex realization $(K; \{R_s\}_{s \in S})$. There exists a 1/12-integral optimal multiflow in every properly-inner Eulerian graph.

The proof is based on the SPUP procedure and three claims (A), (B), and (C) below, which we will prove in Sections 4 and 5. To state and motivate three claims, we first introduce an overall framework, and then give the proof of Theorem 3.5.

Suppose that $\mu$ has a realization $(K; \{R_s\}_{s \in S})$ of leg-length $\delta = 2$ (by scaling). Let $G$ be a properly-inner Eulerian graph. We may assume the condition:

\[(3.4)\] Each terminal is proper or essential.
Indeed, we can make a nonessential (improper) terminal an inner node, while keeping the inner Eulerian condition. We will maintain this condition throughout the SPUP procedure.

The 2-subdivision \((K^2; \{R_s\}_{s \in S})\) is also a realization of \(\mu\), with unit leg-length. We consider \(\text{DLP}(K^2; \{R_s\}_{s \in S})\). Note that \(K^2\) has the following orbit property:

\[(3.5) \quad \text{If edge } e \text{ in } K \text{ is divided into } e_1 \text{ and } e_2 \text{ in } K^2, \text{ then } e_1 \text{ and } e_2 \text{ belong to different orbits in } K^2.\]

One can easily verify this property from the orientability of \(K\).

For an optimal potential \(\rho: V \to V(K^2)\), we define a partition of \(V\) into three sets,

\[
\begin{align*}
S_\rho &= \{ x \in V \mid \rho(x) \text{ is the midpoint of a folder in } K \}, \\
M_\rho &= \{ x \in V \mid \rho(x) \text{ is the midpoint of a leg in } K \}, \\
C_\rho &= \{ x \in V \mid \rho(x) \text{ is a vertex of } K \}.
\end{align*}
\]

See Figure 12 (b). The first claim says that inner nodes in \(S_\rho\) have a particularly nice property.

**(A)** Let \(G\) be an inner Eulerian graph, and \(\rho\) an optimal potential for \(\text{DLP}(K^2; \{R_s\}_{s \in S})\). If an inner node \(y\) belongs to \(S_\rho\), then \(y\) has a splittable fork.

Motivated by this claim, the number of inner nodes in \(M_{\rho_j} \cup C_{\rho_j}\) is decreased with the aid of the SPUP procedure. If \(G_j\) has no inner nodes in \(M_{\rho_j} \cup C_{\rho_j}\), then all inner nodes in \(K_jG_j\) are splittable by (A) in step 2. In addition, if the degree-1 reduction to \(K_jG_j\) keeps (3.4) and creates no new inner nodes in \(M_{\rho_j} \cup C_{\rho_j}\), then we can apply the splitting-off to obtain a graph consisting only of terminals of degree one. In this graph, an integral optimal multiflow obviously exists, and hence in \(K_jG_j\). Thus the SPUP procedure terminates in step 3, and a \(1/K_j\)-integral optimal multiflow is obtained in the original graph. Our goal is this situation.

We will choose a fork \(\tau_j\) and a critical neighbor \(\rho'\) in step 1 such that \(S_{\rho'} \supseteq S_{\rho_{j-1}}\) and \(M_{\rho'} \cup C_{\rho'} \subseteq M_{\rho_{j-1}} \cup S_{\rho_{j-1}}\). Consider an admissible orientation of \(K^2\) such that each vertex of \(K\) is a source and the midpoint of each folder in \(K\) is a sink; see Figure 12 (c). This orientation is admissible, and is called the forward orientation. Restricting the forward orientation to each orbit, we get an admissible orientation of an orbit, which is also called the forward orientation.

Then two types of neighbors can be distinguished. A neighbor is said to be forward if it is a neighbor with respect to the forward orientation, and backward if it is a neighbor of the opposite orientation. We use this terminology also for semi-neighbors. In the following argument, we can replace forward neighbors by forward semi-neighbors. An SPUP is said to be forward if the critical neighbor \(\rho'\) is forward, and backward if \(\rho'\) is backward.
Proof. (1). If \( y \in C_\rho \), then \( \rho' \) is forward, and if \( y \in S_\rho \), then \( \rho' \) is backward.

(2) If \( \rho' \) is forward, then \( d^{\rho'}(G^{\tau,0}) - d^{\rho}(G^{\tau,0}) \) is equal to
\[
\sum \{c(e)\{d^{\rho'}(e) - d^{\rho}(e)\} \mid e \text{ is incident to } M_\rho \cup C_\rho\}.
\]

(3) \( d^{\rho'}(e^{\tau}) \) is given as
\[
d^{\rho'}(e^{\tau}) \in \begin{cases} 
\{1, 2, 3, 4\} & \text{if } y \in C_\rho, \\
\{1, 2\} & \text{if } y \in M_\rho, \rho' : \text{forward}, \\
\{1\} & \text{if } y \in M_\rho, \rho' : \text{backward}, \\
\{1, 2\} & \text{if } y \in S_\rho.
\end{cases}
\]

See Figure 13 for the orbit structure around the midpoint \( p \) of a \( K_{2,\star} \)-folder and of a leg (in \( K \)), where the black and white arrows indicate distinct orbits (by (3.5)).

**Lemma 3.6.** For an optimal potential \( \rho \), an unsplittable fork \( \tau \) on a node \( y \), and a critical neighbor \( \rho' \) of \( \rho \) with respect to \( \tau \), we have the following:

(1) If \( y \in C_\rho \), then \( \rho(y) \) is a source of the orientation, and hence there is no backward neighbor \( \rho' \) with \( d^{\rho'}(e^{\tau}) > 0 \). The case of \( y \in S_\rho \) is similar.

(2). Use the following fact: for an edge \( e = xy \), if both ends belong to \( S_\rho \), then \( (\rho(x), \rho(y)) = (\rho'(x), \rho'(y)) \) implies \( d^{\rho}(e) = d^{\rho'}(e) \); see the paragraph after Lemma 3.4.

(3). Both \( \rho'(y) \) and \( \rho'(y^\ast) \) belong to neighborhood \( (K^2)_p \) for \( p = \rho(y) \) (Section 2.1.4). This implies \( d^{\rho'}(e^{\tau}) \in \{1, 2, 3, 4\} \). Suppose \( y \in S_\rho; p = \rho(y) \) is the midpoint of a folder in \( K \), as in Figure 13 (a). By (3.5), \( p \) touches (at least) two distinct orbits as in Figure 13 (a). Then \( \{\rho'(y), \rho'(y^\ast)\} \) belongs to one of \( F \) and \( F' \) in Figure 13 (a), implying \( d^{\rho'}(e^{\tau}) \in \{1, 2\} \). Suppose \( y \in M_\rho; p = \rho(y) \) is the midpoint of an edge \( qq' \) of \( K \) as in Figure 13 (b). If \( \rho' \) is backward, then \( \{\rho'(y), \rho'(y^\ast)\} \subseteq \{p, q, q'\} \). By (3.5), legs \( pq \) and \( pq' \) belong to different orbits. So \( \{\rho'(y), \rho'(y^\ast)\} = \{p, q\} \) or \( \{q, p'\} \), implying \( d^{\rho'}(e^{\tau}) = 1 \). If \( \rho' \) is forward, then \( \rho'(y) = q \) or a vertex adjacent to \( q \) by leg, and so is \( \rho'(y^\ast) \), implying \( d^{\rho'}(e^{\tau}) \in \{1, 2\} \).

\( \square \)
See Figures 14 and 15 for the behavior of critical neighbors. In particular, Lemma 3.6 (2) implies: As far as we apply forward SPUP, the (possibly fractional) capacities of edges within $S_\rho$ does not affect $d^{0'}(G^{\tau,0'}) - d^{0'}(G^{\tau,0})$. This is a key to bound $K_j$; see the paragraph after Lemma 3.4. We will keep the numerator in the formula (3.2) of even as much as possible. In this case, the possible values of $\alpha_\tau$ for a fork $\tau$ at node $y$ are given by

$$\alpha_\tau = \begin{cases} 
0, \frac{1}{2}, \frac{2}{3}, 1, \frac{4}{3}, \frac{3}{2}, 2 & \text{if } y \in C_\rho, \\
0, 1, 2 & \text{if } y \in M_\rho.
\end{cases}$$

In the forward SPUP procedure, it suffices to maintain this evenness between $\rho$ and its forward neighbor $\rho'$. Motivated by this, $(G; \rho)$ is said to be restricted Eulerian if every edge of $G$ has an integer capacity and every inner node in $M_\rho \cup C_\rho$ has an even degree; inner nodes in $S_\rho$ may have an odd degree. In this case, by Lemma 3.4 with the paragraph after the lemma, $d^{0'}(G^{\tau,0'}) - d^{0'}(G^{\tau,0})$ is an even integer as long as $\rho'$ is a forward neighbor of $\rho$.

As mentioned already, we will sweep out inner nodes first from $C_\rho$ and then from $M_\rho$. The forward SPUP at $M_\rho$ works well under the restricted Eulerian condition. By the degree-4 reduction, we may assume that each inner node in $M_\rho \cup C_\rho$ has degree four.

Take a fork $\tau$ at $y \in M_\rho$, and a critical neighbor $\rho'$ of $\rho$. Then, by Lemma 3.6 (3), we have $\alpha_\tau \in \{0, 1\}$ (even if $\rho'$ is backward). Suppose that $\rho'$ is forward. If $\alpha_\tau = 1$, then necessarily $d^{0'}(e^\tau) = 2$. Although both $y$ and $y^\tau$ have degree 3 in $G^{\tau,0}$, they fall into $S_{\rho'}$ (Figure 14). Therefore $(G^{\tau,0}; \rho')$ is restricted Eulerian. If $\alpha_\tau = 0$, then one of $y$ and $y^\tau$, say $y$, falls into $S_{\rho'}$. Contract $e^\tau$ to $y$ in $G^{\tau,0}$. Then $\rho'$ is optimal for the resulting graph, i.e., the original graph $G$; see (4.3). So $\rho'$ is an optimal forward neighbor of $\rho$ for $G$. In the both cases, we can update $(G; \rho)$ to sweep out $y$ from $M_\rho$ into $S_\rho$ while
Suppose that \( C \) and critical neighbors bound the denominator of the capacity of created edges, we will carefully choose forks. Let \( G \) we can make \((G; \rho)\) so that \((G; \rho)\) is restricted Eulerian and \( M_\rho \) have no inner nodes.

Our final goal ensures to make \( C_\rho \) have no inner node. SPUP at \( C_\rho \) is always forward by Lemma 3.6 (1). Therefore successive SPUP at \( C_\rho \) does not increase the number of the nodes in \( C_\rho \); see Figure 15. However \( \alpha_\tau \) can take fractional values \( 2/3, 4/3, 1/2, 3/2 \). To bound the denominator of the capacity of created edges, we will carefully choose forks and critical neighbors.

\((B)\) Suppose that \((G; \rho)\) is restricted Eulerian. Let \( y \) be an inner node in \( M_\rho \) and let \( \rho_i \) be a critical neighbor of \( \rho \) with respect to \( \tau_i \) at \( y \) \((i = 1, 2, 3)\). Then at least one of \( \rho_1, \rho_2, \rho_3 \) is forward. Hence, by SPUP operations at \( M_\rho \), we can modify \((G; \rho)\) so that \((G; \rho)\) is restricted Eulerian and \( M_\rho \) have no inner nodes.

\((C)\) Suppose that \( G \) is a properly inner-Eulerian graph and \( \rho \) is an optimal potential. By the degree-reductions keeping (3.4), the splitting-off, and SPUP operations at \( C_\rho \), we can modify \((G; \rho)\) so that \((6G; \rho)\) is restricted Eulerian, \( C_\rho \) has no inner nodes, and each terminal in \( C_\rho \) is incident to (at most) one node.

The proof of Theorem 3.5 assuming \((A), (B), \) and \((C)\). Our remaining task is to maintain (3.4) in the degree-1 reduction. We may assume that the condition (3.4) holds. We also note that \( M_\rho \) cannot have terminals under (3.4). Otherwise, for such a terminal \( s \), \( R_s \) includes the midpoint \( \rho(s) \) of a leg \( pq \). This means that \( R_s \) includes leg \( pq \) by normality of \( R_e \) in \( K \), and that \( \rho(s) \) is in the interior of \( R_s \). Therefore \( s \) is neither proper nor essential.

Take an optimal potential \( \rho \); obviously \((G; \rho)\) is restricted Eulerian. By claim \((C)\), we can make \((G; \rho)\) so that \((6G; \rho)\) is restricted Eulerian, \( C_\rho \) has no inner nodes, and each terminal in \( C_\rho \) is incident to a unique node.

Let \((G; \rho) \leftarrow (6G; \rho)\). Here \( C_\rho \) may have terminals. Such a terminal \( s \) is incident to a unique node \( x \). If \( x \in C_\rho \), then \( x \) is also a terminal with unique neighbor \( s \), and therefore we can fix (integral) flow between \( x \) and \( s \) and delete them. Suppose \( x \notin C_\rho \); in particular \( \rho(x) \neq \rho(s) \). Apply the degree-1 reduction to \( s \); this creates no inner nodes in \( C_\rho \). The added new terminals remain to be essential by (3.3). In this way, apply degree-1 reduction (or deletion) to all terminals in \( C_\rho \). Next apply the degree-4 reduction to inner nodes in \( M_\rho \). By claim \((B)\), we can repeat SPUP at inner nodes in \( M_\rho \) to make \((G; \rho)\) so that \((G; \rho)\) is restricted Eulerian and \( M_\rho \cup C_\rho \) has no inner nodes.

Let \((G; \rho) \leftarrow (2G; \rho)\). Still \( C_\rho \) may have terminals. Apply, again, the degree-1 reduction to all terminals in \( C_\rho \); new terminals are all essential by (3.3). Apply the degree-1 reduction to proper terminals in \( S_\rho \), and the degree-2 reduction to improper terminals in \( S_\rho \). Some new improper terminal \( s \) (of degree two) may fail to be essential. In this case, the unique fork \( \tau \) at \( s \) is splittable; split off \( \tau \) and delete \( s \). In this way, we can make all improper terminals essential.

Here, in fact, the degree-1 reduction keeps each improper terminal essential. To verify this, take an improper terminal \( s \) and consider the unique fork \( \tau = (e, s, e') \) at \( s \). Then \( \alpha_\tau = 0 \) holds; the proof is given at the end. By Lemma 3.1, every optimal multiflow \( f \) in \( G = G^{\tau, 0} \) satisfies \( f_{e'} = 2 = c(e') \) at \( e' \). Hence the degree-2 terminal \( s \) always has paths connecting \( s \) of flow-value 2. Thus the degree-1 reduction keeps \( s \) essential (see the argument after (3.3)). Apply the degree-1 reduction to all improper terminals \( s \) in \( S_\rho \).

Now \( G \) is an inner Eulerian graph such that \( M_\rho \cup C_\rho \) has no inner nodes, and each terminal, of degree one, is proper or essential. This is our goal. As mentioned already,
by claim (A), there exists an integral optimal multflow. Reversing these operations, we get an 1/12-integral optimal multflow in the original graph. The proof is done.

Proof of $\alpha_\tau = 0$. By Lemma 3.6 (3), we have $\alpha_\tau = 0$ or 1. Take a critical neighbor $\rho'$ of $\rho$ with respect to $\tau$, which is backward. Suppose (to the contrary) that $\alpha_\tau = 1$. Then $d'(e^*) = 2$, and $\{\rho'(s), \rho'(s^*)\} = \{p, l\}$ or $\{p, l'\}$ in Figure 13 (a) with $p = \rho(s)$ since $\rho'(s)$ and $\rho(s)$ must be in the same color class on the boundary of $R_s$ by Lemma 3.3. Furthermore $\rho'(s^*)$ is not in $R_s$ (otherwise the induced path connecting $R_s$ through $\rho'(s^*)$ is never shortest). This is impossible since $R_s$ must include hypotenuses $pl$ and $pl'$ by the normality in $K$.

4 Analysis of SPUP by multflows

To prove the existence of forks or critical neighbors with required properties (claims (A), (B), and (C)), we analyze the behavior of optimal multflows under an optimal potential. In Section 4.1, we study optimality-keeping rearrangements of an optimal multflow, and introduce the local geodesic condition as a criterion of such rearrangements.

The goal of this section is Theorem 4.3 in Section 4.2, which states interrelations among an optimal multflow, an optimal potential, critical neighbors, and the shape of $K_p$. The claims (A) and (B) are its immediate consequences. Also Theorem 4.3 brings a powerful splittability criterion in Section 6. The main proof tool is a combination of the first optimal criterion (Lemma 2.4), the second optimality criterion (Theorem 2.5), and the local geodesic condition.

Throughout this section, $G$ is a graph with terminal set $S$ and rational edge-capacity $c$, and $\mu$ is a terminal weight having a realization $(K; \{R_s\}_{s \in S})$ with unit leg-length. By rationality, we can always take an optimal multflow $f = (P, \lambda)$ with a rational-valued flow-value function $\lambda$. Therefore, by allowing $P$ to be a multiset, we can represent $f = (P, \lambda)$ by a pair of a multiset $P$ of $S$-paths and a uniform flow-value function $\lambda = 1/\kappa$ for some positive integer $\kappa$ (called the fractionality of $f$). We use this expression, and denote it by $f = (P; \kappa)$. For an edge $e$, the subset of paths in $P$ containing $e$ is denoted by $P(e)$. Its total flow-value $|P(e)|/\kappa$ is denoted by $f^e$. For consecutive two edges $e, e'$, the subset of paths passing $e$ and $e'$ is denoted by $P(e, e')$, and its flow value is denoted by $f^{e, e'}$. A path $P$ is called an $(A, y_1y_2 \cdots y_m, B)$-path if $P$ connects terminal subsets $A$ and $B$ by passing through nodes $y_1, y_2, \ldots, y_m$ in the order of $A \rightarrow y_1 \rightarrow y_2 \rightarrow \cdots \rightarrow y_m \rightarrow B$. An set $P'$ of paths is called an $(A, y_1y_2 \cdots y_m, B)$-set if $P'$ consists of all $(A, y_1y_2 \cdots y_m, B)$-paths. When $B$ (resp., $A$) is not specified, $B$ (resp., $A$) is replaced by * (e.g.: $P$ is an $(A, xy, *)$-path).

4.1 Local multflow rearrangement

The local multflow rearrangement plays a central role in our analysis. Let $f = (P; \kappa)$ be an optimal multflow and let $y$ be a node. Consider the following problem:

Split some of the paths in $P$ at $y$, and reconnect them while keeping optimality.

Suppose that we are given an optimal potential $\rho$ with $p := \rho(y)$. Then the split paths induce shortest paths connecting $p$ and normal regions. Therefore, to keep the optimality, it suffices to reconnect these paths so that the resulting induced paths are all shortest (by the geodesic condition in Lemma 2.4). See Figure 16.

This motivates us to consider the following geometric problem on $K$: For normal sets $M$ and $N$, suppose that we are given two shortest paths $P$ and $P'$ such that $P$ connects $p$ and $M$, and $P'$ connects $p$ and $N$. 

\[21\]
Figure 16: Local multiflow rearrangement to keep optimality

Is the concatenation $P + P'$ at $p$ shortest between $M$ and $N$?

The shortness is determined by the position of $(M, N)$ relative to the neighborhood $K_p$; recall the notions in Section 2.1.4. Suppose that $K_p$ is star-shaped (consider the 2-subdivision if necessary). Then the leg-graph of the boundary of $K_p$ is identified with $\Pi_p$, which is a bipartite graph of bipartition $\{L_p, Q_p\}$. For $u, v \in Q_p \cup L_p$, we write $u \sim v$ if $u = v$, $u$ is incident to $v$, or $u$ and $v$ have a common neighbor in $\Pi_p$, i.e., if $d_{K_p}(u, v) \leq 2$. Also we write $u \sim_1 v$ if $u = v$ or $u$ is incident to $v$ in $\Pi_p$, i.e., if $d_{K_p}(u, v) = 1$.

For a normal set $R$ not containing $p$, the vertex $g$ of (the boundary of) $K_p$ with $d_K(g, R) = d_K(K_p, R)$ is uniquely determined [12, Lemma 3.8]. We call this vertex $g$ the gate of $R$ in $K_p$, denoted by $R^p$; this concept comes from [5]. We can regard $R^p$ as a member of $L_p \cup Q_p$. For a normal set $R$ containing $p$, define $R^p$ to be the intersection $R \cap K_p$, which is normal in $K_p$. Hence we get a map $R \mapsto R^p$ from the set of normal sets in $K$ to that in $K_p$. Then $P + P'$ forms a shortest path between $M$ and $N$ if and only if $P + P'$ induces a shortest path between $M^p$ and $N^p$ in $K_p$ (Figure 16).

**Lemma 4.1** ([12, Lemmas 3.6 and 3.9]). For two normal sets $M$ and $N$, the following conditions are equivalent:

1. $d_K(M, N) = d_K(M, p) + d_K(p, N)$.
2. $d_{K_p}(M^p, N^p) = d_{K_p}(M^p, p) + d_{K_p}(p, N^p)$.
3. If $p \notin M$ and $p \notin N$, then there exists no $q \in Q_p$ with $M^p \sim_1 q \sim_1 N^p$. If $p \notin M$ and $p \in N$, then $M^p \notin N^p$.

Although a shortest path from $R$ to $p$ enters $K_p$ via $u \in Q_p \cup L_p$, the vertex $u$ may not be the gate $R^p$. But the vertex $u$ is at least adjacent to $R^p$ by leg, as follows.

**Lemma 4.2.** For a normal set $R$ and a vertex $u \in Q_p \cup L_p$, suppose $d_K(R, p) = d_K(R, u) + d_K(u, p)$. Then $R^p = u$ if $u \in L_p$, and $R^p \sim_1 u$ if $u \in Q_p$.

Note that this lemma is valid even if $K_p$ is not star-shaped.

**Proof.** By condition, $d_K(R, u) < d_K(R, p) + d_K(p, u)$ holds. Apply the previous lemma for $(M, N) = (R, u)$. Then there exists $q \in Q_p$ with $R^p \sim_1 q \sim_1 u$. If $u \in Q_p$, then necessarily $u = q \sim_1 R^p$ (by bipartiteness of $\Pi_p$). Otherwise $u \in L_p$, implying
For each operation at condition is also a necessary condition for In particular, we can rearrange the exchange operation of every pair of paths in them.

For an edge exchange/anti-exchange operations.

Let us return to the multi-flow rearrangement. For \( u \in Q_p \cup L_p \), let \([u]_1\) denote the set of terminals \( s \in S \) such that gate \((R_s)^p\) is \( u \), i.e.,

\[
[u] := \{ s \in S \mid p \notin R_s, u = (R_s)^p \}.
\]

For \( q \in Q_p \), let \([q]_1\) denote the set of terminals \( s \) with \((R_s)^p \sim q\), i.e.,

\[
[q]_1 := \bigcup_{u \sim q} [u].
\]

By Lemma 4.1 (3), under an optimal potential \( \rho \), the following local geodesic condition is sufficient to keep the optimality in the multi-flow rearrangement at a node \( y \) with \( \rho(y) = p \).

**Local geodesic condition:** A multi-flow \( f \) has no \(([q]_1, y, [q]_1)\)-paths for any \( q \in Q_p \), and has no \((s, y, [u])\)-paths for any \( u \in Q_p \cup L_p \) and \( s \in S \) with \( \{p, u\} \subseteq R_s \).

In particular, we can rearrange \( f \) at \( y \) as if \([u]_1\) is a single terminal. The local geodesic condition is also a necessary condition for \( f \) to be optimal.

Two basic flow-operations for an optimal multi-flow \( f = (P; \kappa) \) are given.

**Exchange/anti-exchange operations.** For an edge \( e = xy \), take two paths \( P_1 \) and \( P_2 \) from \( \mathcal{P}(e) \). The exchange operation of \( P_1 \) and \( P_2 \) at \( e \) is the following: For \( i = 1, 2 \), split \( P_i \) at \( x \) into two paths \( P_i^1 \) and \( P_i^2 \) so that \( P_i^2 \) contains \( y \). Reconnect \( P_i^1 \) and \( P_i^2 \) at \( x \), and reconnect \( P_1^2 \) ant \( P_2^2 \) at \( x \). If the resulting paths are not simple, then simplify them.

If the exchange operation of \( P_1 \) and \( P_2 \) keeps the optimal value \( \|f\|_\mu \), then \( P_1 \) and \( P_2 \) are said to be exchangeable at \( e \). A subset \( \mathcal{P}' \subseteq \mathcal{P}(e) \) is said to be exchangeable if the exchange operation of every pair of paths in \( \mathcal{P}' \) at \( e \) keeps the value of \( \|f\|_\mu \). If \( \mathcal{P}(e) \) itself is exchangeable, then \( f \) is exchangeable at \( e \). We will often use the exchange operation at \( e^c \) as in Figure 17.

The anti-exchange operation is the reverse way of exchanging \( P_1 \) and \( P_2 \). Namely, for each \( i = 1, 2 \), by deleting \( xy \), split \( P_i \) into two paths \( P_i^1 \) and \( P_i^2 \) so that \( P_i^2 \) contains \( y \). Reconnect \( P_1^1 \) and \( P_2^1 \) at \( x \), reconnect \( P_2^2 \) ant \( P_1^2 \) at \( y \), and simplify them if necessary.
4.2 Analysis

Here we analyze SPUP at an inner node with degree 4. Suppose that an inner node \( y \) is incident to four edges \( e = xy, e_1 = x_1y, e_2 = x_2y, \) and \( e_3 = x_3y \) with unit capacity.

(4.2) If \( y \) has multiple edges \( e, e_1 (x = x_1) \), then fork \((e, y, e_2)\) is splittable.

Indeed, let \( G' \) be the graph obtained from \( G \) by contracting edges \( e \) and \( e_1 \). Then \( \text{opt}(\mu; G') \geq \text{opt}(\mu; G) \). Let \( G'' \) be the graph obtained from \( G \) by splitting off forks \((e, y, e_2)\) and \((e_1, y, e_3)\). Then \( \text{opt}(\mu; G'') \leq \text{opt}(\mu; G) \). Here \( G' = G'' \). This means that \((e, y, e_2)\) is splittable.

Our interest lies in the case where there is no splittable fork. By symmetry, it suffices to consider three forks \((e, y, e_1), (e, y, e_2), (e, y, e_3)\). Fork \((e, y, e_1)\) is particularly denoted by \( \tau_i \), and \( \alpha_i \) is simply denoted by \( \alpha_i \).

Let \( \rho \) be an optimal potential, and let \( p := \rho(y) \); see Figure 18. Let \( \rho_i \) be a critical neighbor of \( \rho \) with respect to \( \tau_i \) for \( i = 1, 2, 3 \). We note an extremal case:

(4.3) If \( \alpha_i = 0 \), then the restriction of \( \rho_i \) to \( V \) is optimal for \( G \),

where \( V = V(G^{\alpha,0}) \setminus \{y^{\alpha}\} \). Indeed, \( \rho_i \) is optimal for \( G^{\alpha,0} \). Replace \( y^{\alpha} \) by two multiple edges \( e', e'' \) of unit capacity. Then \((e, y, e')\) is splittable by (4.2). Split it off. The resulting graph is the same as the original \( G \). This means that \( \rho_i \) is optimal to \( G \).

The positions \( (\rho_i(y), \rho_i(y^{\alpha})) \) \((i = 1, 2, 3)\) are interrelated, which often determine the local multiflow configuration at \( y \), or give the information of the local structure \( K_p \). The main statement in this section is the following:

**Theorem 4.3.** Suppose that each terminal is proper or essential, and \( \alpha_i \leq 1 \) for \( i = 1, 2, 3 \).

1. If \( \rho_i(y) \) and \( \rho_i(y^{\alpha}) \) are not adjacent by a leg, and belong to a common folder in \( K_p \) for \( i = 1, 2, 3 \), then there exist distinct \( l_1, l_2, l_3 \in L_p \) such that, by an appropriate relabeling of \( e, e_1, e_2, e_3 \),
   (i) \( (\rho_i(y), \rho_i(y^{\alpha})) = (p, l_i) \) \((i = 1, 2, 3)\), and
   (ii) for every optimal multiflow \( f = (P; \kappa), P(e_i, e_j) \) is an \((l_i, x_iy_j, [l_j])\)-set with \( f_{e_i, e_j} = 1/2 \) \((1 \leq i < j \leq 3)\).

2. For some legs \( pq \) and \( pq' \), if \( \{\rho_i(y), \rho_i(y^{\alpha})\} = \{p, q\} \) or \( \{p, q'\} \) \((i = 1, 2, 3)\), then \( q \neq q' \) and there exists a common folder containing \( pq \) and \( pq' \).

Figure 19 illustrates the situation of (1): necessarily \( l_i \sim l_j \) for the local geodesic condition. The rest of this subsection is devoted to proving Theorem 4.3. The proof
P is similar; apply the exchange operations between \( P \) and \( f \) created nonsimple paths (if exist), we obtain an optimal multi-flow. Take an optimal multi-flow \( G \) be an optimal multi-flow in \( G \) derived from \( f \) derived from \( P \) is exchangeable at \( e \), derived from \( P \) is exchangeable with a path \( (1) \) \( f_{e;e} = f_{e;e} + f_{e;e} \), if \( e' \in \{e, e_1\} \), if \( e' \in \{e_j, e_k\} \), (distinct \( i, j, k \in \{1, 2, 3\} \)). We will often use the following obvious relations:

\[
\begin{align*}
fe_i &= fe_i + fe_i, \\
fe' &= \begin{cases} 
fe'_{e} + fe'_{e} & \text{if } e' \in \{e, e_i\}, \\
fe'_{e} + fe'_{e} & \text{if } e' \in \{e_j, e_k\},
\end{cases} \\
(4.4) & \quad \text{(distinct } i, j, k \in \{1, 2, 3\})
\end{align*}
\]

**Lemma 4.4.**  
(1) \( \alpha_1 + \alpha_2 + \alpha_3 \geq 2 \).  
(2) \( \alpha_j \geq 2 - \alpha_1 \) \( (j = 2, 3) \) if there exists an optimal multi-flow \( f \) exchangeable at \( e_i \).

**Proof.** Take an optimal multi-flow \( f \) in \( G \). By Lemma 3.1, symmetry, and (4.4), we have \( \alpha_2 + \alpha_3 \geq \max\{2f_{e;e_1}, 2f_{e;e_2}\} + \max\{2f_{e;e_1}, 2f_{e;e_2}\} \geq (f_{e;e_2} + f_{e;e_3}) \geq f_{e;i} \geq 2 \), and hence \( \alpha_2 \geq 2 - \alpha_1 \). Thus we have (1).

Suppose that \( f \) is exchangeable at \( e_i \). By the exchange operations between \( P(e, e_3) \) and \( P(e_1, e_2) \) at \( e_i \), in as Figure 17, we can make \( f_{e;e_3} = 0 \) or \( f_{e;e_2} = 0 \). If \( f_{e;e_3} = 0 \), then \( f_{e;e_2} = f_{e;e_2} = f_{e;e_2} = 2 - f_{e;e_1} \), and hence \( \alpha_2 \geq 2 - f_{e;e_2} \geq 2 - \alpha_1 \). If \( f_{e;e_2} = 0 \), then \( f_{e;e_2} = f_{e;e_2} = f_{e;e_2} = 2 - f_{e;e_1} \), and hence \( \alpha_2 \geq 2 - \alpha_1 \). The case of \( j = 3 \) is similar; apply the exchange operations between \( P(e, e_2) \) and \( P(e_1, e_3) \) above.

Take an optimal multi-flow \( f' \) in \( G \). By contracting edge \( e_i \) and simplifying created nonsimple paths (if exist), we obtain an optimal multi-flow \( f \) in \( G \). In this case, we say that \( f \) is derived from \( f' \) or \( f \) is an optimal multi-flow in \( G \) derived from \( G \).

Note that \( P(e') \) may increase, which is caused by a path in \( f' \) passing through \( y \) and \( y' \) not using \( e_i \). The position \( (\rho_i(y), \rho_i(y')) \) in \( K_p \) gives information of \( P(e') \) as follows. See Figure 20 for an intuition of the lemma.

**Lemma 4.5.** Suppose \( d_{K_p}(\rho_i(y), p) + d_{K_p}(p, \rho_i(y')) \) with \( \rho_i(y') \neq p \). Let \( f = (P; \kappa) \) be an optimal multi-flow in \( G \) derived from \( G \), and let \( u := \rho_i(y') \in L_p \cup Q_p \).

1. \( P(e_i) \) is a \((*, yy, [u])\)-set if \( u \in L_p \), and a \((*, yy, [u])\)-set if \( u \in Q_p \).

2. If \( P \) in \( P(e, e_1) \) is exchangeable with a path \( P' \) in \( P(e') \) at \( e_i \), then \( P \) is a \((*, xyx_i, [u])\)-path if \( u \in L_p \), and a \((*, xyx_i, [u])\)-path if \( u \in Q_p \).
Proof. (1). Suppose that $f$ is derived from an optimal multiflow $f'$ in $G^{\tau, \alpha_i}$. Take an $(s, yy^\tau, t)$-path $P \in \mathcal{P}(\tau)$, which is contracted from an $(s, yy^\tau, t)$-path $\hat{P}$. Therefore, by geodesic condition for $(\rho_i, f')$, we have

$$d(e_i) = d(\rho_i),$$

by geodesic condition for $(\tau)$. Its flow value is denoted by $f$. Hence, by the assumption with the triangle inequality, $P$ satisfies $d_k(p, R_i) = d_k(p, u) + d_k(u, R_i)$. Therefore, by Lemma 4.2, if $u \in L_P$, then $(R_i)^0 = u$, implying $t \in [u]$, and if $u \in Q_P$, then $(R_i)^0 \sim u$, implying $t \in [u]$.

(2). Suppose that $P$ and $P'$ are obtained by contracting $e_i$ from an $(s, yy^\tau, t, *)$-path $\hat{P}$ and a $(s, xy^\tau, x_t, t, *)$-path $\hat{P}'$. Obviously, $\hat{P}$ is exchangeable with $\hat{P}'$ at $e_i$. Do the exchange operation. If a simplification occurs, then $f^e$ decreases on a created cycle, and hence the vertices in this cycle have the same potential (by the saturation condition). Thus the image of the resulting $(s, t)$-path passes through $R_s \to \rho_i(y) \to \rho_i(y^\tau) \to R_t$, i.e., (4.5) holds. Therefore, by the same argument, we have $t \in [u]$ if $u \in L_P$ and $t \in [u]$ if $u \in Q_P$. 

Next we analyze $\mathcal{P}(\tau)$ for an arbitrary optimal multiflow $f = (P; \kappa)$ in $G$. Let $\mathcal{P}(\tau; \rho_i)$ be the set of $(s, yy^\tau, t)$-paths $P$ in $\mathcal{P}(\tau)$ satisfying

$$d_k(R_s, R_t) = d_k(R_s, \rho_i(y)) + d_k(\rho_i(y), \rho_i(y^\tau)) + d_k(\rho_i(y^\tau), R_t).$$

Its flow value is denoted by $f^{\tau; \rho_i}$. Then Lemma 4.5 (1) also holds with replaced by $\mathcal{P}(\tau)$ by $\mathcal{P}(\tau; \rho_i)$. We can estimate $f^{\tau; \rho_i}(\leq f^{\tau, \rho_i})$ by the following lemma.

**Lemma 4.6.** Suppose that each terminal is proper or essential.

1. $d_\rho(\rho_i) f^{\tau; \rho_i} + (d_\rho(\tau) - 2)(f^{\tau, \rho_i} - f^{\tau, \rho_i}) \geq d_\rho(\tau)(2 - \alpha_i)$.

2. If $d_\rho(\tau) \geq 2$, then $f^{\tau; \rho_i} \geq 2 + d_\rho(\tau) - 2 f^{\tau, \rho_i}$.

*Proof.* We use the formula (2.6) of the duality gap. By definition, we have $\text{opt}(\mu; G) = \text{opt}(\mu; G^{\tau, \alpha_i}) = d_\rho(G^{\tau, \alpha_i})$. 

Figure 20: Perturbing $\rho$ to a critical neighbor $\rho_1$. 

![Diagram](image-url)
Let $f'$ be the multflow for $G^{r_i, \alpha_i}$ obtained by deleting all paths in $P(e^{r_i})$ from $f$. Then the duality gap for $(f', \rho_i)$ in $G^{r_i, \alpha_i}$ is

$$d^{\rho}(G^{r_i, \alpha_i}) - \|f'\|_\mu = \sum_{P \in P(e^{r_i})} \mu(s_P, t_P)/\kappa.$$  

We next estimate the first term $\delta_1 := \sum_{e \in E(G^{r_i, \alpha_i})} d^{\rho_i}(e)(c(e) - (f')^e)$ in (2.6), which means the unsaturation of edges. Since $e^{r_i}$ has no flow in $G^{r_i, \alpha_i}$, this contributes $d^{\rho_i}(e^{r_i})(2 - \alpha_i)$ for $\delta_1$.

For $s \in S$, let $\bar{R}_s$ denote the connected component of the boundary of $R_s$ containing $\rho(s)$. By the essentialness assumption and Lemma 3.3, $\bar{R}_s$ also contains $\rho_i(s)$. Therefore, the deletion of an $(s, y^{r_i}, t_P)$-path $P$ contributes at least $\{d_K(\bar{R}_{s_P}, \rho_i(y)) + d_K(\rho_i(y^{r_i}), \bar{R}_{t_P})\}/\kappa$ for the unsaturation of edges except $e^{r_i}$. Thus we have

$$\delta_1 \geq d^{\rho_i}(e^{r_i})(2 - \alpha_i) + \sum_{P \in P(e^{r_i})} \{d_K(\bar{R}_{s_P}, \rho_i(y)) + d_K(\rho_i(y^{r_i}), \bar{R}_{t_P})\}/\kappa.$$  

Since the duality gap (4.7) is at least $\delta_1$, we have

$$\sum_{P \in P(e^{r_i})} \{d^{\rho}(e^{r_i}) - \Delta_P\}/\kappa \geq d^{\rho_i}(e^{r_i})(2 - \alpha_i),$$

where $\Delta_P := d_K(\bar{R}_{s_P}, \rho_i(y)) + d^{\rho}(e^{r_i}) + d_K(\rho_i(y^{r_i}), \bar{R}_{t_P}) - d_K(\bar{R}_{s_P}, \bar{R}_{t_P})$.

We show:

$$\Delta_P = \text{a nonnegative even integer, and is zero if and only if } P \in P(e^{r_i}; \rho_i).$$

Suppose that this is true. Then the LHS of (4.8) is at most $d^{\rho_i}(e^{r_i})f^{e^{r_i}; \rho_i} + (d^{\rho_i}(e^{r_i}) - 2)(f^{e^{r_i}} - f^{e^{r_i}; \rho_i})$. Then we obtain (1). (2) follows from substituting $f^{e^{r_i}} = f^e + f^{e'} - 2f^{e, e'} \leq 2 - 2f^{e, e'}$ to (1).

We show now (4.9). Since $\rho(P)$ connects $\bar{R}_{s_P}$ and $\bar{R}_{t_P}$, with length $d_K(\bar{R}_{s_P}, \bar{R}_{t_P})$, we have $d_K(\bar{R}_{s_P}, R_{t_P}) = d_K(\bar{R}_{s_P}, \bar{R}_{t_P})$. Moreover the vertices in $\bar{R}_s$ belong to the same color class of the leg-graph. Thus we get the first statement. For the second statement, the if part follows from $d_K(\cdot, \bar{R}_s) \geq d_K(\cdot, R_s)$ and the first statement. For the only-if part, we show $d(R_{s_P}, \rho_i(y)) = d(\bar{R}_{s_P}, \rho_i(y))$ for $P \in P(e^{r_i}; \rho_i)$. This follows from the facts that $R_{s_P}$ cannot contain $\rho_i(y)$ in the interior, and that $\rho_i(y)$ and $\rho(y)$ belong to the same connected component obtained by deleting the interior of $R_{s_P}$ from $K$ (see the proof of Lemma 3.3).

For $i \in 1, 2, 3$, $\{\rho_i(y), \rho_i(y^{r_i})\}$ belongs to $K_p$. We classify the position $\{\rho_i(y), \rho_i(y^{r_i})\}$ into eight cases (1a), (1b), (2a), (2b), (2c), (2d), (3), (4) in Figure 21. For the six cases, Lemma 4.5 (1) is applicable, which determines a type of $P(e^{r_i}; \rho_i)$ (and $P(e^{r_i})$) if $f$ is an optimal multflow derived from $G^{r_i, \alpha_i}$ as summarized in Table 1. The third column indicates the exchangeability of $P(e^{r_i}; \rho_i)$ at $e^{r_i}$. By the local geodesic condition, a $(u_i, y^{r_i}, s_i)$-set is always exchangeable. To see the exchangeability of (2b), consider the 2-subdivision $K^2$ and consider $(K^2)^p_{l'}$ for the midpoint $l'$ of a folder in $K_p$ containing $p, q, q', l$; then $(q, q')$ is in case (4) in $(K^2)^p_{l'}$.

For distinct $i, j, k$, $P(e^{r_i})$ is a disjoint union of $P(e, e^{r_i}), P(e, e^{r_j})$, and $P(e, e^{r_k})$. We denote $P(e^{r_i}; \rho_i) \cap P(e, e^{r_i}), P(e^{r_i}; \rho_i) \cap P(e, e^{r_j})$, and $P(e^{r_i}; \rho_i) \cap P(e, e^{r_k})$ by $P(e, e^{r_i}; \rho_i), P(e, e^{r_j}; \rho_i), \rho_i, \rho_i, \rho_i, $ and $P(e, e^{r_k}; \rho_i)$, respectively. The corresponding flow-values are denoted by $f^{e, e^{r_i}; \rho_i}, f^{e, e^{r_j}; \rho_i}, f^{e, e^{r_k}; \rho_i}$, respectively. Obviously,

$$f^{e^{r_i}; \rho_i} = f^{e, e^{r_i}; \rho_i} + f^{e, e^{r_j}; \rho_i} + f^{e, e^{r_k}; \rho_i}.$$

We will use this notation and decomposition.
Figure 21: Possible patterns of \( \rho_i(y), \rho_i(y') \), where \( q, q' \in Q_p \) and \( l, l' \in L_p \)

Table 1: Type of \( P(e^i; \rho_i) \) (and \( P(e^i) \) if \( f \) is an optimal multiflow derived from \( G^{\alpha_1} \))

<table>
<thead>
<tr>
<th>case</th>
<th>( (\rho_i(y), \rho_i(y')) )</th>
<th>( P(e^i; \rho_i) )</th>
<th>exchangeability</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1b)</td>
<td>( (q, p) )</td>
<td>( (q)_1, yy^{i}, \ast )</td>
<td>( \bigcirc )</td>
</tr>
<tr>
<td>(2a)</td>
<td>( (l, p) )</td>
<td>( ([l], yy^{i}, \ast) )</td>
<td>( \bigcirc )</td>
</tr>
<tr>
<td>(2b)</td>
<td>( (q, q'), q \sim_1 l \sim_1 q' )</td>
<td>( (q)_1 \setminus [l], yy^{i}, [q'_1] \setminus [l] )</td>
<td>( \bigcirc )</td>
</tr>
<tr>
<td>(2c)</td>
<td>( (q, q'), q \neq q' )</td>
<td>( (q)_1, yy^{i}, [q'1], i )</td>
<td>( \bigcirc )</td>
</tr>
<tr>
<td>(3)</td>
<td>( (l, q), l \neq q )</td>
<td>( ([l], yy^{i}, [q_1]) )</td>
<td>( \bigcirc )</td>
</tr>
<tr>
<td>(4)</td>
<td>( (l, l'), l \neq l' )</td>
<td>( ([l], yy^{i}, [l']) )</td>
<td>( \bigcirc )</td>
</tr>
</tbody>
</table>

Proof of Theorem 4.3 (1). We first show that for every optimal multiflow \( f \), there is \( e' \in \{ e, e_1, e_2, e_3 \} \) with \( f^{e'} = 0 \). Suppose that this is true. Then \( e' \) is independent of \( f \), say, \( e' = e \) (after relabeling). Hence \( P(e^i') = P(e_i) \), and \( i \geq f^{e} = f^{e^i} \geq 2 - \alpha_i \geq 1 \).

Necessarily \( \alpha_i = 1 \), \( f^{e^i} = f^{e^i} = 1 \) and \( f^{e_{1,2}} = 1/2 \).

By Lemma 4.6 (2) with \( \alpha_1 = 1 \) and \( d^h(e^i) = 2 \), we have \( f^{e^i, \rho_i} \geq 1 \). Here \( \rho_1 \) is in case (2a) or (2b) in Figure 21. In particular, \( P(e^i; \rho_i) \) is exchangeable at \( e^i \) (Table 1).

By the exchange operations between \( P(e, e_j) \) and \( P(e, e_k) \), and between \( P(e, e_j) \) and \( P(e, e_j) \) (as in Figure 17), we can make \( f \) satisfy \( f^{e, e'^i} \geq 1/2 \) for some distinct \( e', e'' \in \{ e, e_1, e_2, e_3 \} \). Note that any optimality-keeping exchange operation keeps \( f^{e^i; \rho_i} \geq 1 \).

We may assume \( (e', e'') = (e_2, e_3) \) (by relabeling). Then the equality holds in \( 1 \geq 2 - f^{e_1} \geq 2f_{e_2, e_3} \geq 1 \). Hence, \( f_{e_2, e_3} = 1/2, f^{e_1} = 1 \), is optimal for \( G^{\alpha_1} \) (and is derived from an optimal multiflow in \( G^{\alpha_1} \)). We may assume \( f^{e, e_1} \leq 1/2 \leq f^{e_1, e_1} \).

Since \( P(e^i) \) is exchangeable, by the exchange operation between \( P(e, e_3) \) and \( P(e, e_2) \) at \( e^i \), we can make \( f \) satisfy \( f^{e, e_3} = 0, f^{e_1, e_3} = 1/2, f^{e_2, e_2} = 1/2 \). If \( f^{e_2, e_2} = 0 \), then \( f^{e} = 0 \), and this necessarily holds from the beginning and the exchange operations have not been applied above (in the exchange operations above the simplification of paths could not occur since such a nonsimple path uses two edges incident to \( y \)).

Suppose (indirectly) \( f^{e_2, e_2} > 0 \) (and hence \( f^{e_1, e_2} < 1/2 \)). By \( f^{e_1, e_2} = 1/2 \), we have \( \alpha_2 = 1 \) and \( f^{e_2} = f^{e_2, e_3} + f^{e_1, e_2} + f^{e, e_1} = 1 \). By \( f^{e_2, e_3} = 1/2 \) and \( f^{e_1, e_2} < 1/2 \), we have \( f^{e, e_1} > 0 \). Since \( P(e^i) \) is exchangeable, by the exchange operation between \( P(e, e_2) \) and \( P(e, e_3) \), we can make \( f \) satisfy \( f^{e_1, e_2} = 1/2 \). Since \( P(e^i) \) is also exchangeable, the exchange operation at \( e^i \) for two paths, one from \( P(e_2, e_3) \) and the other one from \( P(e, e_1) \neq \emptyset \), keeps the optimality and results in \( f^{e, e_2} > 1/2 \). A contradiction to \( 1 \geq \alpha_2 \geq 2f^{e_1, e_2} \). Therefore \( f^{e^i} = f^{e_1} = 1, f^{e} = 0, f^{e_1, e_2} = 1/2, \) and \( \alpha_i = 1 \).
particular, \( \mathcal{P}(e_i) (= \mathcal{P}(e_i)) \) is exchangeable at \( e_i \).

Next consider the position \((\rho_1(y), \rho_1(y))\) in \( K_\rho \); (2a) or (2b). Then \( d^{0}(e_i) = d_{K_\rho}(\rho_1(y), p) + d_{K_\rho}(\rho_1(y), p) \) holds. Since \( \mathcal{P}(e_i) \) is exchangeable at \( e_i \), any path in \( \mathcal{P}(e_2, e_3) \) is exchangeable with any path in \( \mathcal{P}(e_i) \) at \( e_2 \) and at \( e_3 \). By Lemma 4.5 (2), if \( \rho_1(y) \neq p \), then \( \mathcal{P}(e_2, e_3) \) is a \( ([q_1, x_2y_3, x_1]) \)-set for some \( q \in Q_\rho \); a contradiction to the local geodesic condition. Thus \( \rho_1(y) = p \) must hold. Consequently \((\rho_1(y), \rho_1(y)) = (p, l_1) \) for \( l_1 \in L_\rho \). By the same argument, we have \((\rho_2(y), \rho_2(y)) = (p, l_2) \) and \((\rho_3(y), \rho_3(y)) = (p, l_3) \) for \( l_2, l_3 \in L_\rho \). By Lemma 4.5 (1), \( \mathcal{P}(e_i, e_j) \) is an \( ([l_i, x_1y_3, [l_j]]) \)-set; the vertices \( l_1, l_2, l_3 \) are distinct by the local geodesic condition. □

**Proof of Theorem 4.3 (2).** Suppose to the contrary that \( q = q' \), or \( q \neq q' \) and there is no common folder containing \( pq \) and \( pq' \). By relabeling and symmetry, we may assume

\[
(4.11) \quad (\rho_i(y^{qi}), \rho_i(y)) = \begin{cases} (p, q) & \text{if } q = q', \\ (q', p) & \text{if } q \neq q', \end{cases} \quad (i = 1, 2, 3).
\]

They are in case (1b) in Figure 21. Let \( \bar{f}^{e_{i}} := f^{e_{i}} - f^{e_{i}; p} \). By Lemma 4.6 (1) for \( d^{0}(e_i) = 1 \), we have

\[
(4.12) \quad f^{e_{i}; p} - \bar{f}^{e_{i}} \geq 2 - \alpha_i \quad (i = 1, 2, 3).
\]

**Claim 4.7.** \( \mathcal{P}(e_i, e_j; p_i) \cap \mathcal{P}(e_i, e_j; p_j) = \emptyset \) for \( 1 \leq i < j \leq 3 \).

**Proof.** Take \( P \) from \( \mathcal{P}(e_i, e_j; p_i) \cap \mathcal{P}(e_i, e_j; p_j) \). Suppose (say) that \((\rho_i(y^{qi}), \rho_i(y)) = (p, q)\) according to Table 1. \( P \) is a \( ([q_1, y^{q_1}, *]) \)-path, and hence is a \( ([q_1, x_1y_3, *)) \)-path. If \((\rho_j(y^{qj}), \rho_j(y)) = (p, q)\), then \( P \) is a \( ([q_1, x_2y_3, [q_1]]) \)-path; a contradiction to the local geodesic condition. If \((\rho_j(y^{qj}), \rho_j(y)) = (q', p)\), then \( P \) is an \( ([q_1, x_1y_3, *]) \)-path, and hence is a \( ([q_1, x_2y_3, [q_1]]) \)-path; a contradiction to the local geodesic condition. \( [q_1] \cap [q_1] \neq 0 \), and the existence of \( u \) in \( \Pi_\rho \) with \( q' \sim u \sim q \). This in turn implies the existence of a folder containing \( pq \) and \( pq' \); a contradiction to the assumption. □

Hence \( \bar{f}^{e_{i}} \geq f^{e_{i}; e_{i}; p_{j}} + f^{e_{i}; e_{j}; p_{k}} \). By substituting this and (4.10) to (4.12), we get

\[
(4.13) \quad f^{e_{i}} + f^{e_{i}; e_{j}; p_{i}} + f^{e_{i}; e_{j}; p_{j}} - (f^{e_{i}; e_{i}; p_{j}} + f^{e_{i}; e_{i}; p_{k}}) \geq 2 - \alpha_i \quad (\text{for distinct } i, j, k \in \{1, 2, 3\}).
\]

Addition of these three inequalities yields

\[
f^{e_{i}} + f^{e_{i}; e_{j}; p_{i}} + f^{e_{i}; e_{j}; p_{j}} \geq 6 - \alpha_1 - \alpha_2 - \alpha_3.
\]

Since \( f^{e_{i}} + f^{e_{i}; e_{j}} \), we have \( 2f^{e} = 2(f^{e_{i}} + f^{e_{i}; e_{j}} + f^{e_{i}; e_{k}}) \geq f^{e_{i}} + f^{e_{j}; p} + f^{e_{j}; p} + f^{e_{k}; p} \). From \( f^{e} \leq c(e) = 1 \), we have \( \alpha_1 + \alpha_2 + \alpha_3 \geq 4 \). However this contradicts \( \alpha_i \leq 1 \) for \( i = 1, 2, 3 \). □

5 Proof of (A), (B), and (C) and algorithmic implication

In this section, we complete the proof of Theorem 1.5 by proving three claims (A), (B), and (C) in Section 3, which are given in Sections 5.1 and 5.2.

In a key step of the proof of claim (C), we will make use of the following lemma, called the uncrossing lemma. Recall the notions of the forward orientation of \( K^2 \), partition \( S_\rho, M_\rho, C_\rho \), and forward semi-neighbors. The proof of Lemma 5.1 is given in Section 5.3.

**Lemma 5.1.** For two optimal potentials \( \rho, \rho' \), there exists a forward semi-neighbor \( \rho^* \) of \( \rho \) that is optimal with \( C_\rho \rho^* = \{ x \in C_\rho \mid \rho(x) = \rho'(x) \} \).

In Section 5.4, we show that our proof indeed gives a polynomial time algorithm to find a 1/12-integral optimal multiflow provided the size of a realization of \( \mu \) is fixed.
Table 2: Classification of $\{\rho'(y), \rho'(y')\}$, where $q, q' \in Q_p$ and $l, l' \in L_p$

<table>
<thead>
<tr>
<th>case</th>
<th>${\rho'(y), \rho'(y')}$</th>
<th>$d^\rho(e^\tau)$</th>
<th>$\alpha$, $G$ admissible</th>
<th>$\alpha$, $3G$ admissible</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1a)</td>
<td>${q, l}$, $q \sim l$</td>
<td>1</td>
<td>0</td>
<td>0, 2/3, 4/3</td>
</tr>
<tr>
<td>(1b)</td>
<td>${p, q}$</td>
<td>1</td>
<td>0</td>
<td>0, 2/3, 4/3</td>
</tr>
<tr>
<td>(2a)</td>
<td>${p, l}$</td>
<td>2</td>
<td>0, 1</td>
<td>0, 1/3, 2/3, 1, 4/3, 5/3</td>
</tr>
<tr>
<td>(2b)</td>
<td>${q, q', l}$, $q \sim q'$</td>
<td>2</td>
<td>0, 1</td>
<td>0, 1/3, 2/3, 1, 4/3, 5/3</td>
</tr>
<tr>
<td>(2c)</td>
<td>${q, q', q \not= q'}$</td>
<td>2</td>
<td>0, 1</td>
<td>0, 1/3, 2/3, 1, 4/3, 5/3</td>
</tr>
<tr>
<td>(2d)</td>
<td>${l, l', l \sim l'}$</td>
<td>2</td>
<td>0, 1</td>
<td>0, 1/3, 2/3, 1, 4/3, 5/3</td>
</tr>
<tr>
<td>(3)</td>
<td>${q, l}$, $q \not= l$</td>
<td>3</td>
<td>0, 2/3, 4/3</td>
<td>2m/9 (0 \leq m \leq 8)</td>
</tr>
<tr>
<td>(4)</td>
<td>${l, l', l \not= l'}$</td>
<td>4</td>
<td>0, 1/2, 1, 3/2</td>
<td>6m/6 (0 \leq m \leq 11)</td>
</tr>
</tbody>
</table>

(*) means that every optimal multiflow derived from $G^{\tau, \alpha}$ is exchangeable at $e^\tau$

5.1 Proof of (A) and (B)

Claims (A) and (B) are easy consequences of Theorem 4.3.

(A). We may assume that $y$ has degree four. Suppose (to the contrary) that all three forks at $y$ are unsplittable. Consider critical neighbors $\rho_1, \rho_2, \rho_3$ of $\rho$ for three forks $\tau_1, \tau_2, \tau_3$. As was seen in the proof of Lemma 3.6, for $i = 1, 2, 3$, $\alpha_i \in \{0, 1\}$ and $\{\rho_i(y), \rho_i(y^\tau)\}$ belongs to folder $F$ or $F'$ in Figure 13 (a). By Lemma 4.4 (1), $\alpha_j = 1$ for some $j$, and then $d^\rho(e^\tau) = 2$ (since the numerator of (3.2) is even). This means that $\rho_j(y)$ and $\rho_j(y^\tau)$ are not adjacent by a leg ((2a) or (2b) in Table 1). Then any optimal multiflow derived from $G^{\tau, \alpha}$ is exchangeable at $e^\tau$. By Lemma 4.4 (2), we have $\alpha_i = 1$ for all $i \in \{1, 2, 3\}$. Hence $\rho_i(y)$ and $\rho_i(y^\tau)$ are not adjacent by a leg, and belong to a common folder ($F$ or $F'$) for $i \in \{1, 2, 3\}$. So Theorem 4.3 (1) is applicable. However, the configuration of (i) (Figure 19) is impossible. Therefore $y$ must have a splittable fork. □

(B). Suppose that $\rho(y)$ is the midpoint $p$ of a leg $qq'$ in $K$, and all $\rho_i$ are backward. Then $\{\rho_i(y), \rho_i(y^\tau)\} = \{p, q\}$ or $\{p, q'\}$ for all $i$ (see Figure 13 (b)). Since $G$ has an integer capacity and $d^\rho(e^\tau) = 1$, we have $\alpha_i \in \{0, 1\}$: the numerator of (3.2) is integral. By Theorem 4.3 (2), there is a folder containing $pq$ and $pq'$. However, such a folder does not exist. A contradiction. This means that at least one of $\rho_i$ is forward. □

5.2 Proof of (C)

We will repeat SPUP at inner nodes in $C_p$, which is always forward (Lemma 3.6 (1)). Then the number of inner nodes in $C_p$ is nonincreasing. To bound the denominator of created fractional edges, we introduce a sharper degree condition than the restricted Eulerian condition. $(G; \rho)$ is called admissible if the deletion of edges between $S_p$ makes $(G; \rho)$ restricted Eulerian. Namely, we allow edges between $S_p$ to have a fractional capacity. In view of the paragraph after Lemma 3.4, if $(G; \rho)$ is admissible and $\tau$ is a fork at $C_p$, then the numerator of formula (3.2) of $\alpha_\tau$ is even. Thus, for a critical neighbor $\rho'$ of $\rho$, the possible cases of $\{\rho'(y), \rho'(y')\}$ with $(d^\rho(e^\tau), \alpha_\tau)$ are summarized as in Table 2; see also Figure 21. Here, for $p \in V(K)$, $(K^2)_p$ is star-shaped, and the leg-graph of the boundary of $(K^2)_p$ is identified with $H_p$.

Our goal is to sweep out inner nodes from $C_p$. We will use the following fact for this purpose.

30
(5.1) For an edge \( e = xy \) with \( x, y \in C_\rho \) and \( \rho(x) = \rho(y) \), if \( c(e) = f^e \) for every optimal multiflow \( f \), then there exists a forward neighbor \( \rho' \) of \( \rho \) such that \( \rho'(x) \neq \rho'(y) \) and \( \rho' \) is optimal.

**Proof.** Decrease \( c(e) \) by \( \beta \geq 0 \). The resulting graph is denoted by \( G^{e, \beta} \). Obviously \( \text{opt}(\mu; G^{e, \beta}) \leq \text{opt}(\mu; G) \). By the same argument as in the proof of Proposition 3.2, the maximum possible \( \beta \geq 0 \) with \( \text{opt}(\mu; G^{e, \beta}) = \text{opt}(\mu; G) \) is the minimum of \( \{d^\rho(G) - d^{\rho'}(G)\}/d^\rho(e) \) over all neighbors \( \rho' \) of \( \rho \) with \( d^\rho(e) > 0 \). By \( c(e) = f^e \), this must be zero. Any neighbor \( \rho' \) attaining the maximum \( \beta \) is an optimal forward neighbor as required. \( \Box \)

In successive SPUP, the value of \( \alpha_r \) is monotone nonincreasing.

(5.2) For two forks \( \tau \) and \( \tau' \) on distinct nodes, we have \( \alpha_{r^\tau}(G^{r^\tau, r^{\tau'}}) \leq \alpha_{r^{\tau'}}(G) \).

**Proof.** For \( \alpha_r := \alpha_r(G) \) and \( \alpha' := \alpha_r(G^{r^\tau, r^{\tau'}}) \), \( (G^{r^\tau, r^{\tau'}})^{r^\tau, r'^{\tau'}} \) is well-defined, and \( \text{opt}(\mu; G^{r^\tau, r^{\tau'}}) = \text{opt}(\mu; G^{r^\tau, r^{\tau'}})^{r^\tau, r'^{\tau'}} \) by definition. Since \( \text{opt}(\mu; G^{r^\tau, r^{\tau'}}) \geq \text{opt}(\mu; G^{r^\tau, r^{\tau'}}) \geq \text{opt}(\mu; (G^{r^\tau, r^{\tau'}})^{r^\tau, r'^{\tau'}}) \), we have \( \text{opt}(\mu; G^{r^\tau, r^{\tau'}}) = \text{opt}(\mu; G) \). This means \( \alpha_{r^\tau}(G) \leq \alpha_{r^{\tau'}}(G) \). \( \Box \)

Let us start the proof of claim (C). In the initial step, \( G \) is properly-inner Eulerian. For any optimal potential \( \rho \) (for DLP\((K^2; \{R_s\}_{s \in S})\), \( (G; \rho) \) is restricted Eulerian and admissible. By the degree-reductions (Section 3.3), we can make \( G \) so that each inner node in \( C_\rho \) has degree four, each proper terminal in \( C_\rho \) has degree one, and each improper terminal in \( C_\rho \) has degree two. We may assume that there is no splittable fork at \( C_\rho \) and all improper terminals are essential (see (3.4)). By edge-subdivisions, we can further assume:

(5.3) For every edge \( xy \) with \( y \in C_\rho \), we have \( \rho(x) \in V((K^2)_{\rho(y)}). \)

After the preprocessing (Section 5.2.1), at first three stages, we apply SPUP at a fork having maximum \( \alpha_r \) so that split nodes go out \( C_\rho \) (Sections 5.2.2 and 5.2.3). Then the number of inner nodes in \( C_\rho \) decreases, and also the maximum \( \alpha_r \) decreases by (5.2). When the maximum \( \alpha_r \) becomes close to 1, the estimate by Lemmas 4.4 and 4.6 becomes effective, and finally we can apply 1-SPUP to reach the goal (Section 5.2.4).

### 5.2.1 Preprocessing

We first modify \( (G; \rho) \) so that \( (G; \rho) \) is restricted Eulerian and each terminal in \( C_\rho \) is incident to a unique node (while keeping (3.4)). Take an improper terminal \( s \) in \( C_\rho \) of degree two, incident to two nodes \( x, y \). For a fork \( \tau := (xs, s, sy) \) we have \( \alpha_r < 2 \) (since \( s \) is essential). If \( \alpha_r = 0 \), then every optimal multiflow \( f \) has paths connecting \( s \) with the flow-value 2 (by Lemma 3.1), and hence we can apply the degree-1 reduction to \( s \); the new terminals remain essential; see (3.3). So consider the case where \( 0 < \alpha_r < 2 \). Take a critical neighbor \( \rho' \) of \( \rho \). By \( 0 < \alpha_r < 2 \), we have \( d^{\rho'}(e^\tau) > 1 \) (the numerator of (3.2) is even). Also, every optimal multiflow in \( G^{r^\tau, r^{\tau'}} \) must have paths connecting \( s \) passing through \( e^\tau \). This means that \( \rho'(s) \) must lie in the boundary of \( R_s \), and \( \rho'(s') \) is not in \( R_s \) with \( d_K(R_s, \rho'(s')) = d_K(\rho'(s), \rho'(s')). \) By Lemma 3.3, \( \rho'(s) \) and \( \rho'(s) \) must lie in the same connected component of the boundary of \( R_s \). So the possible positions of \( \{\rho'(s), \rho'(s')\} \) are \( \rho'(s) = p \) and \( \rho'(s') \in L_p \) (case (2a)) or \( \rho'(s, \rho'(s')) \in L_p \times L_p \) with \( \rho'(s) \sim \rho'(s') \) (case (2d)), where \( p := \rho(s) \); see Figure 21. In both cases, we have \( d^{\rho'}(e^\tau) = 2 \) and hence \( \alpha_r = 1 \). Apply the corresponding 1-SPUP. Then \( s' \) falls into \( S_\rho \), and hence \( (G; \rho) \) keeps the restricted Eulerian condition. Furthermore \( s \) has degree one
and is essential (by (3.3)). Repeat this process to improper terminals until each terminal in $C_\rho$ has degree one and a unique neighbor.

### 5.2.2 3/2-SPUP

From here, we consider SPUP at inner nodes in $C_\rho$. By searching all forks at $C_\rho$, take a fork $\tau$ at inner node $y \in C_\rho$ with (maximum) $\alpha_\tau = 3/2$. Let $p := \rho(y)$. Take a critical neighbor $\rho'$ of $\rho$ with respect to $\tau$. Then $d^{\alpha}(e^\tau) = 4$, and thus $\{\rho'(y), \rho'(y^\tau)\}$ is in case (4) in Table 2; both $y$ and $y^\tau$ fall into $S_{\rho'}$. Apply 3/2-SPUP: $(G; \rho) \leftarrow (G^\tau, \alpha_\tau; \rho')$.

Then $(G; \rho)$ is admissible and $(2G; \rho)$ is restricted Eulerian. Repeat this process until there is no fork $\tau$ at $C_\rho$ with $\alpha_\tau = 3/2$. After that, the possible values of $\alpha_\tau$ of forks $\tau$ at $C_\rho$ are 0, 2/3, 1, 4/3. Here $\alpha_\tau = 1/2$ (case (4)) never occurs since this implies the existence of another fork $\tau'$ with $\alpha_{\tau'} = 3/2$ by Lemma 4.4 (2) and the exchangeability in case (4).

### 5.2.3 4/3-SPUP and 7/6-SPUP

By searching all forks at $C_\rho$, take a fork $\tau$ at an inner node $y \in C_\rho$ with (maximum) $\alpha_\tau = 4/3$. Then a critical neighbor $\rho'$ is in case (3) in Table 2. Apply 4/3-SPUP: $(G; \rho) \leftarrow (G^\tau, \alpha_\tau; \rho')$. Then one of $y$ and $y^\tau$ falls into $S_{\rho'}$, and the other falls into $M_{\rho'}$ and has degree 8/3. Therefore

$$(5.4) \quad (3G; \rho) \text{ is admissible and } (6G; \rho) \text{ is restricted Eulerian.}$$

From now on, we keep this condition (5.4). In the next SPUP, $\alpha_{\tau}$ belongs to $1/3(2\mathbb{Z} + 3 \cup \mathbb{Z} + 2)$; see the fifth column in Table 2. Note that $\alpha_{\tau} > 4/3$ is impossible by (5.2). By this fact together with Lemma 4.4 (2), $\alpha_{\tau} \in \{1/6, 2/9, 4/9\}$ is also impossible. So the possible values of $\alpha_{\tau}$ are 0, 1/3, 2/3, 5/6, 8/9, 1, 10/9, 7/6, 4/3.

Apply SPUP for a fork $\tau$ at an inner node in $C_\rho$ with $\alpha_\tau = 4/3$. Here, if $\alpha_{\tau} = 4/3$ in (1b, 2a) occurs, then $\rho'(y)$ or $\rho'(y^\tau)$ does not move, and the number of inner nodes in $C_\rho$ does not decrease. However, by the uncrossing lemma (Lemma 5.1), we can take a forward critical semi-neighbor $\rho^*$ of $\rho$ with $y, y^\tau \not\in C_{\rho^*}$ as follows.

Let $(G; \rho)$ be the graph with the optimal potential at just after the final 3/2-SPUP. By (5.2), necessarily $\alpha_\tau(G) = 4/3$ holds. This means that we could have chosen this fork $\tau$ in the first 4/3-SPUP. Consider a critical neighbor $\rho''$ of $\rho$ with respect to $\tau$ in $G$. $\rho''$ is necessarily in case (3), and can be regarded as an optimal potential for the current graph $G^\tau, \alpha_\tau$ by $\rho''(y^\tau) := \rho''(y)$ for processed forks $\hat{\tau}$ at $y$. By the uncrossing lemma for $(\rho', \rho'')$, there is another optimal forward semi-neighbor $\rho^*$ of $\rho'$ with $C_{\rho^*} = \{y \in C_{\rho'} \mid \rho'(y) = \rho''(y)\}$. Both $\rho''(y)$ and $\rho''(y^\tau)$ are in $V((K^2)_p) \setminus \{p\}$. Hence $y, y^\tau \not\in C_{\rho^*}$. Let $(G; \rho) \leftarrow (G^\tau, \alpha_\tau; \rho^*)$; the number of inner nodes in $C_{\rho}$ strictly decreases. In this way, repeat 4/3-SPUP. After the procedure, the possible values of $\alpha_{\tau}$ are 0, 1/3, 2/3, 5/6, 8/9, 1, 10/9, 7/6.

Next apply SPUP for a fork $\tau$ at inner node $y \in C_\rho$ with $\alpha_{\tau} = 7/6$. In this case, its critical neighbor $\rho'$ is in case (4). Thus 7/6-SPUP keeps (5.4), and the number of inner nodes in $C_\rho$ decreases. After the procedure, the possible values of $\alpha_{\tau}$ are 0, 1/3, 2/3, 8/9, 1, 10/9; note that $\alpha_{\tau} < 7/6$ excludes 5/6 (case (4)) by Lemma 4.4 (2).

### 5.2.4 1-SPUP

Take any inner node $y \in C_\rho$, a fork $\tau$, and a critical neighbor $\rho'$ of $\rho$ with respect to $\tau$. Let $p := \rho(y)$. The possible cases of $(\alpha_{\tau}, \rho')$ are $\alpha_{\tau} = 1/3$ in (2c, 2d), $\alpha_{\tau} = 2/3$ in (1a,
1b, 2c, 2d), $\alpha_\tau = 8/9$ in (3), $\alpha_\tau = 1$ in (2a, 2b, 2c, 2d, 4), and $\alpha_\tau = 10/9$ in (3). Note that $\alpha < 4/3$ excludes $\alpha_\tau \in \{1/3, 2/3\}$ in (2a, 2b, 3, 4) by Lemma 4.4 (2).

The main obstructions to keeping (5.4) are the occurrences of $\alpha_\tau = 10/9$ in (3) and $\alpha_\tau = 1$ in (2c). Sometimes we can proceed SPUP when the latter case occurs. Suppose $\alpha_\tau = 1$ in (2c) with $(\rho'(y), \rho'(y^\tau)) = (q, q')$. Then, for every optimal multifoil $f = (P; \kappa)$ in $G$ derived from $G^{r, \alpha_\tau}$, $P(e^\tau)$ is a $([q], yy^\tau, [q']_1)$-set (see Lemma 4.5 (1) with Table 1). Here consider the following condition:

(5.5) There is no optimal multifoil $f = (P; \kappa)$ derived from $G^{r, \alpha_\tau}$ such that

(i) $P(e^\tau)$ is an $([l], yy^\tau, [q']_1)$-set for some $l \in L_p$ with $l \sim_1 q$, or

(ii) $P(e^\tau)$ is a $([q], yy^\tau, [l']_1)$-set for some $l' \in L_p$ with $l' \sim_1 q'$.

Suppose that this condition is met. Apply SPUP: $(G; \rho) \leftarrow (G^{r, \alpha_\tau}; \rho')$. Continue SPUP at $C_{\rho}$. Then $d^{\rho'}(e^\tau)$ keeps 2, and hence $\alpha_\tau$ remains in $1/3(2Z_3 + 3 \cup Z_3)/2$ (see the paragraph after Lemma 3.4).

Indeed, suppose that one of $\rho(y)$ and $\rho(y^\tau)$, say $\rho(y)$, moves at some SPUP. Then $\rho(y) = l \in L_p$ with $l \sim_1 q$ (since SPUP is forward). Consider an optimal multifoil $f$ in the current graph. Then any $(s, yy^\tau, t)$-path in $f$ induces by $\rho$ a path passing through $R_s \rightarrow l \rightarrow \rho(y^\tau) \rightarrow R_t$. As in Lemma 4.5, the type of $(s, t)$ is determined by the position of $(l, \rho(y^\tau))$. Also $\rho(y^\tau)$ is $q'$ or $l' \in L_p$ with $l' \sim_1 q'$. If $\rho(y^\tau) = q'$ or $l' \in L_p$ with $l' \sim_1 q'$, then any $(s, yy^\tau, t)$-path is an $([l], yy^\tau, [q']_1)$-path (see Table 1). By contracting edges $e^\tau$ for processed $\tau'$ (after $\tau$), we get an optimal multifoil $f$ in $G^{r, \alpha_\tau}$ so that any $(s, yy^\tau, t)$-path is an $([l], yy^\tau, [q']_1)$-path. Then the optimal multifoil in $G$ derived from $f$ violates (5.5); a contradiction. Hence $\rho(y^\tau) = l'$ with $l \sim_1 l'$; in particular both $y$ and $y^\tau$ fall into $S_{\rho}$. In this way, we can continue SPUP without an increase in the denominator of $\alpha_\tau$.

1-SPUP with (2c) is called mixed if it satisfies (5.5), and called unmixed otherwise.

We can avoid 10/9-SPUP and unmixed 1-SPUP by examining all three forks $\tau_1, \tau_2, \tau_3$ at $y$ and their critical neighbors $\rho_1, \rho_2, \rho_3$. The main technical statement here is the following.

Proposition 5.2. Suppose that $\rho_j$ is in case neither (2d) nor (4) for $j = 2, 3$.

(1) If $\alpha_1 = 10/9$, then, for $j = 2$ or 3, $\rho_j$ is in case (2c) with $\alpha_j = 1$.

(2) If $\rho_1$ is in case (2c) with $\alpha_1 = 1$ such that the corresponding 1-SPUP is unmixed, then both $\rho_2$ and $\rho_3$ are in case (2c) with $\alpha_2 = \alpha_3 = 1$, and by a relabeling fixing $\{\{e, e_1\}, \{e_2, e_3\}\}$, one of the following holds:

(A) $\rho_3(y) \sim \rho_2(y) \sim \rho_3(y^{\tau_3}) \sim \rho_2(y^{\tau_2}) \sim \rho_3(y)$.

(B) $\rho_3(y) \sim \rho_2(y) \sim \rho_1(y^{\tau_1}) \sim \rho_3(y^{\tau_3}) \sim \rho_2(y^{\tau_2}) \sim \rho_1(y) \sim \rho_3(y)$ and $\rho_2(y^{\tau_2}) \sim \rho_3(y)$.

(C) $\rho_3(y) \sim \rho_2(y) \sim \rho_1(y^{\tau_1}) \sim \rho_3(y^{\tau_3}) \sim \rho_2(y^{\tau_2}) \sim \rho_1(y) \sim \rho_3(y)$ and $\rho_3(y^{\tau_3}) \sim \rho_2(y)$.

See Figure 22 for the positions of $\{\rho_i(y), \rho_i(y^{\tau_i})\}$ in (2). The proof of Proposition 5.2 is rather technical. Before the proof, let us proceed, assuming Proposition 5.2. Take an inner node $y \in C_{\rho}$ having a fork $\tau$ with maximum $\alpha_\tau \leq 10/9$. Consider three critical neighbors $\rho_i$ for $\tau_1$ ($i = 1, 2, 3$). If some $\rho_i$ is in case (2d) or (4), then both $y$ and $y^{\tau_1}$ fall into $S_{\rho_i}$ and apply 1-SPUP for $(\tau_1, \rho_i)$, which keeps (5.4). So suppose that neither (2d) nor (4) occurs. Suppose $\alpha_i = 10/9$. By Proposition 5.2 (1), for $j \neq i$, $\rho_j$ is in case
Suppose (to contrary) that all \( \alpha_i \) satisfy Proposition 5.2 (2). Then \( \alpha_i = 1 \) with (2c). If \( (\rho_1, \rho, \rho_k) \) violates the configuration of Proposition 5.2 (2), then \( (G; \rho) \sim (G^{(i)}; \rho_j) \) is guaranteed to be mixed 1-SPUP.

Here, it is impossible that all \( \rho_i \) satisfy Proposition 5.2 (2). To verify this fact, suppose (to contrary) that all \( \rho_i \) satisfy (2). Then all \( \rho_1(y), \rho_1(y^{\tau_3}) \) \((i = 1, 2, 3)\) are distinct. To derive a contradiction, we utilize the girth condition (2.5) for \( \Pi_p \). Suppose (first) that \( \rho_1 \) satisfies (2-1) as in Figure 22. By (2) for \( \rho_2 \), we have \( \rho_1(y) \sim \rho_3(y^{\tau_3}) \) or \( \rho_1(y^{\tau_1}) \sim \rho_3(y) \). The first case finds a 6-cycle (using \( \rho_3(y^{\tau_3}), u, \rho_2(y^{\tau_2}), l, \rho_1(y) \)) in \( \Pi_p \); a contradiction to (2.5). Consider the second case. Then \( \rho_1(y^{\tau_1}) \) must be incident to \( \tilde{u} \); otherwise, by \( \rho_1(y^{\tau_1}) \sim \rho_3(y) \), we find a 6-cycle (using \( \rho_1(y^{\tau_1}), \rho_3(y), \tilde{u}, \rho_2(y) \)). Again, by (2) for \( \rho_3 \), we have \( \rho_1(y) \sim \rho_3(y) \) or \( \rho(y^{\tau_1}) \sim \rho_2(y^{\tau_2}) \). Similarly we have \( \rho_1(y^{\tau_1}) \sim u \). Then \( \Pi_p \) has a 6-cycle \( (u, \rho_2(y^{\tau_2}), l, \rho_3(y), \tilde{u}, \rho_1(y^{\tau_1})) \). The case (2-2) is similar. Also, if all \( \rho_i \) satisfy (2-0), we can find a 6-cycle, as above, more easily.

Apply such mixed 1-SPUP as far as possible. Suppose that \( \alpha_1 = 1 \) with (2a) or (2b) occurs. Then necessarily \( \alpha_1 = \alpha_k = 1 \) (by Lemma 4.4 (2)). Then both \( \rho_j \) and \( \rho_k \) are also in case (2a) or (2b). By Theorem 4.3 (1), all \( \rho_i \) are necessarily in case (2a). So every multiflow configuration around \( y \) is given as in Figure 19 (after relabeling):

\[
(5.6) \quad f^{e_1,e_2} = f^{e_2,e_3} = f^{e_1,e_3} = 1/2, \quad f^e = 0.
\]

In particular, \( f^{e_1} = f^{e_2} = f^{e_3} = 1 \). If \( \rho(y) = \rho(x') \) for \( x' \in \{x_1, x_2, x_3\} \), then replace \( \rho \) by an optimal forward neighbor \( \rho' \) with \( \rho'(y) \neq \rho'(x') \), according to (5.1). Here \( \rho(x) = \rho(y) \) by \( f^e = 0 \) and the saturation condition. If \( x \) is a terminal, then \( x \) has degree one and has no flow. If \( x \) is an inner node, then \( x \) has the same configuration (5.6) as \( y \).

The remaining case is \( \alpha_1 = 1/3 \) or \( 2/3 \). By Lemma 4.4 (1), we have \( \alpha_1 = \alpha_2 = \alpha_3 = 2/3 \). By Lemma 3.1, every optimal multiflow \( f \) satisfies

\[
(5.7) \quad f^{e_1} = f^{e_2} = f^{e_3} = 1/3.
\]

Also \( f^{e} = f^{e_1} = f^{e_2} = f^{e_3} = 1 \). If \( \rho(y) = \rho(x') \) for \( x' \in \{x_1, x_2, x_3\} \), then replace \( \rho \) by an optimal forward neighbor \( \rho' \) with \( \rho'(y) \neq \rho'(x') \) as above. By (5.3), each \( x' \) above belongs to \( M_p \cup S_p \). By (5.6) and (5.7), we can split off all inner nodes in \( C_p \) in \( 6G \).

Split them off. Then \( (6G; \rho) \) is restricted Eulerian, there is no inner node in \( C_p \), and each terminal in \( C_p \) has a unique neighbor (Section 5.2.1). Then the proof of claim (C) is done. \( \Box \)
Figure 23: A-configuration and B-configuration

Proof of Proposition 5.2. Consider the case where \( \alpha_1 = 10/9 \) in case (3) or \( \alpha_1 = 1 \) in case (2c) such that the corresponding 1-SPUP for \((\tau_1, \rho_1)\) is unmixed. In either cases, we can take an optimal multiflow \( f = (P; \kappa) \) in \( G \) derived from \( G^{\tau_1; \alpha_1} \) such that

\[
\text{(5.8)} \quad \begin{align*}
\text{(i)} & \quad \mathcal{P}(e^{\tau_1}) \text{ is an } ([l], yy^{\tau_1}, [q'])_1 \text{-set for some } (l, q') \in L_p \times Q_p, \text{ or} \\
\text{(ii)} & \quad \mathcal{P}(e^{\tau_1}) \text{ is a } ([q], yy^{\tau_1}, [l'])_1 \text{-set for some } (q, l') \in Q_p \times L_p.
\end{align*}
\]

(Necessarily \( l \not\sim q' \) for (i), \( q \not\sim l' \) for (ii) by the local geodesic condition). Take such an optimal multiflow \( f \) of fractionality \( \kappa \) with minimum total support \( \sum_{e \in E} f_e \). Then every exchange operation keeping the optimality and (5.8) does not decrease the support; note that any optimality-keeping (anti-)exchange operation at edges except \( e^{\tau_2}, e^{\tau_3} \) keeps the property that \( f \) is derived from \( G^{\tau_1; \alpha_1} \).

Claim 5.3. If \( \mathcal{P}(e, e_1; \rho_2) \cap \mathcal{P}(e, e_1; \rho_3) = \emptyset \), then

\[
f^{e^{\tau_2}; \rho_2} + f^{e^{\tau_3}; \rho_3} \leq \frac{3}{2} \alpha_1,
\]

Furthermore \( f^{e^{\tau_2}; \rho_2} + f^{e^{\tau_3}; \rho_3} \) has the following upper bounds.
and, if, in addition, \( P(e_2, e_3; \rho_2) \cap P(e_2, e_3; \rho_3) = \emptyset \), then
\[
f^{e_2; \rho_2} + f^{e_3; \rho_3} \leq 1 + \frac{\alpha_1}{2} - \epsilon - f^{e_3, e_1}.
\]

**Proof.** This follows from substituting
\[
f^{e_1; \rho_2} + f^{e_1; \rho_3} \leq \begin{cases} 
\frac{\alpha_1}{2} - \epsilon & \text{if } P(e, e_1; \rho_2) \cap P(e, e_1; \rho_3) = \emptyset, \\
\alpha_1 - 2\epsilon & \text{otherwise},
\end{cases}
\]
\[
f^{e_2, e_3; \rho_2} + f^{e_2, e_3; \rho_3} \leq \begin{cases} 
1 - f^{e_1} + f^{e_3, e_1} & \text{if } P(e_2, e_3; \rho_2) \cap P(e_2, e_3; \rho_3) = \emptyset, \\
2(1 - f^{e_1} + f^{e_3, e_1}) & \text{otherwise},
\end{cases}
\]

into \( f^{e_1; \rho_2} + f^{e_1; \rho_3} + f^{e_2, e_3; \rho_2} + f^{e_2, e_3; \rho_3} \geq f^{e_2; \rho_2} + f^{e_3; \rho_3} - f^{e_1} + 2f^{e_3, e_1} \). Then use \( f^{e_1} = 2 - \alpha_1 + \epsilon - \epsilon' \) for \( \epsilon \geq \epsilon' \).

The first and second inequalities follow from \( f^{e_1; \rho_1} \leq 1 - f^{e_1} = \alpha_1/2 - \epsilon \) and \( f^{e_2, e_3; \rho_1} \leq 1 - f^{e_2, e_3} = 1 - f^{e_1} + f^{e_3, e_1} \), respectively. The third follows from adding \( f^{e_2; \rho_2} \leq f^{e_1; \rho_2} + f^{e_1} - f^{e_3, e_1} + f^{e_2, e_3; \rho_2} \) and \( f^{e_3; \rho_3} \leq f^{e_1; \rho_3} + f^{e_3, e_1} - f^{e_2, e_3; \rho_1} \), and using \( f^{e_1; \rho_1} + f^{e_3, e_1} = f^{e_1} \). \( \square \)

1. We first show the following, which includes Proposition 5.2 (1):
\[
(5.10) \quad \text{For } j = 2 \text{ or } 3, \rho_j \text{ is in case (2c) with } \alpha_j = 1.
\]
To prove this, suppose not. For all cases (2a, 2b, 3, 4) and \( j = 2, 3 \), \( P(e^j; \rho_j) \) is exchangeable at \( e^j \).
\[
(5.11) \quad \text{In A-configuration, } P(e_1, e_2; \rho_2) \neq \emptyset.
\]
Otherwise, \( P(e_1, e_2; \rho_2) = \emptyset \) would imply \( f^{e_1; \rho_2} + f^{e_2, e_3; \rho_2} = f^{e_2; \rho_2} \geq 7/9 + \epsilon \). Since \( \max\{f^{e_1; \rho_2}, f^{e_2, e_3; \rho_2}\} \leq 1 - f^{e_1} = 1 - \alpha_1/2 - \epsilon \leq 5/9 - \epsilon \), both \( P(e, e_1; \rho_2) \) and \( P(e_2, e_3; \rho_2) \) have the flow-value at least \( 2/9 + 2\epsilon \) (in each case). In B-configuration, by the exchange operation (without simplification) between \( P(e, e_1; \rho_2) \) and \( P(e_2, e_3; \rho_2) \) at \( e^{e_2} \) we can make \( f \) satisfy \( f^{e_1, e_2} \geq 4/9 - \epsilon + (2/9 + 2\epsilon) > 5/9 \) (while keeping optimality). Since \( f^{e_3} \leq 2 - 2f^{e_1, e_2} \), we have \( f^{e_3} < 8/9 \) (this holds after the simplification). Then \( \alpha_3 \geq 2 - f^{e_3} > 10/9 \); a contradiction to \( \alpha_3 \leq 10/9 \).

Similarly,
\[
(5.12) \quad \text{In B-configuration, } P(e, e_2; \rho_3) \neq \emptyset.
\]
Otherwise, both \( P(e, e_1; \rho_3) \) and \( P(e_2, e_3; \rho_3) \) would have the flow-value at least \( 2/9 + (\epsilon - \epsilon') \), and in A-configuration we can make \( f \) satisfy \( f^{e, e_2} \geq 4/9 + \epsilon + 2/9 + \epsilon - \epsilon' > 5/9 \), implying \( \alpha_2 > 10/9 \); a contradiction, as above. Then we have the following:
\[
(5.13) \quad P(e, e_1; \rho_2) \cap P(e, e_1; \rho_3) \text{ is empty.}
\]
Indeed, take \( P \) from \( P(e, e_1; \rho_2) \cap P(e, e_1; \rho_3) \) if exists. Then, by (5.11) and the fact that \( P(e^{e_2}; \rho_2) \) is exchangeable, \( P \) is exchangeable with a path in \( P(e^{e_1}) \) at \( e_1 \). Also, by (5.12) and the fact that \( P(e^{e_3}; \rho_3) \) is exchangeable, \( P \) is exchangeable with a path in \( P(e^{e_1}) \) at \( e_1 \). By Lemma 4.5 (2), \( P \) is a \([q_1, xyx_1, [q_1]-\text{path for some } q \in Q \); a contradiction to the local geodesic condition.

Therefore the first inequality in Claim 5.3 holds. Then the case \( \alpha_1 = \alpha_2 = \alpha_3 = 10/9 \) (case (3)) is the only possibility; the other cases yield \( \text{LHS} < \text{RHS} \). In particular, \((\rho_1(y), \rho_1(y^{[q]})) = (l, q') \) or \((q, l') \) holds. Suppose that the latter case holds. Here \( P(e_1, e_2) \)
Finally we show that with a path in \( (l', x_1y_2, *) \)-set, and \( \mathcal{P}(e, e_2) \) is an \( ([l'], xy_2, *) \)-set (by Table 1). \( \mathcal{P}(e, e_1; \rho_2) \) is also nonempty; otherwise \( 7/9 \leq f^{e_2; \rho_2} = f^{e_2; e^{*}; \rho_2} \leq 1 - f^{e_1; \rho_1} \leq 5/9 \). Then a path \( P \in \mathcal{P}(e, e_1; \rho_2) \) is exchangeable with a path \( P' \in \mathcal{P}(e_1, e_2; \rho_2) \) (\( \neq \emptyset \) by (5.11)) at \( e \), since \( \mathcal{P}(e^{*}; \rho_2) \) is exchangeable at \( e^{2}. \) Therefore \( \mathcal{P} \) is an \( ([l'], x_1y_2, *) \)-path. Then \( f \) includes an \( ([l'], yx, *) \)-path and a \( (*, yx, [l']) \)-path at \( e \). Then the anti-exchange operation for these two paths at \( e \) decreases the total support, while keeping optimality. A contradiction to the minimality of the support. Therefore \( (\rho_1(y), \rho_1(y^{*})) = (l, q') \) holds.

Consider \( (p_2(y), p_2'(y^{*})) \), which is also in case (3); \( p_2(y^{*}) = l_1' \in L_p \) or \( p_2(y) = l_1 \in L_p \). Take a path \( P \) from \( \mathcal{P}(e_1, e_2; \rho_2) \) (by (5.11)) in \( \mathcal{A} \)-configuration, which is a \( (*, x_1y_2, [l_1']) \)-path or an \( ([l_1], x_1y_2, *) \)-path. Since \( P \in \mathcal{P}(e^{*}) \), \( P \) is also a \( ([q'], x_1y_2, [l]) \)-path. Therefore \( p_2(y^{*}) = l_1' = l \) or \( p_2(y) = l_1 \) with \( l_1 \sim q' \). Suppose that the former case occurs. Then \( \mathcal{P}(e, e_1; \rho_2) \) is a \( ([l], xy_2, *) \)-set. Also \( \mathcal{P}(e, e_2) \) is a \( (x, xy_2, *) \)-set. Then the anti-exchange operation at \( e = xy \) works, as above; a contradiction to the minimality of the support. Hence the latter case \( (p_2(y) = l_1) \) holds. Similarly \( \rho_3(y^{*}) = l_2 \in L_p \) with \( l_2 \sim q' \). Then \( l_1 \neq l_2 \) necessarily holds; this means \( [l_1] \cap [l_2] = \emptyset \). Otherwise the anti-exchange operation at \( e \), which has both an \( ([l_2], xy, *) \)-path and a \( (*, xy, [l_1]) \)-path, works. In particular, \( \mathcal{P}(e_2, e_3; \rho_2) \cap \mathcal{P}(e_2, e_3; \rho_3) \) has no path; otherwise such a path is an \( ([l_1] \cap [l_2], x_2y_3x_3, *) \)-path.

Hence the second inequity in Claim 5.3 also holds. Then this completely determines the multifold configuration at \( y \) as \( \epsilon = \epsilon' = 0 \), \( f^{e_3; e^{*}} = 0 \), \( f^{e; e^{*}} = f^{e_1; e^{*}} = f^{e_2; e^{*}} = f^{e_3; e^{*}} = f^{e_2; e_3; e^{*}} \leq 4/9 \), and \( f^{e_2; e_3; e^{*}} = 3/7 \). In particular, both equalities hold in (5.9). Since \( f^{e_2; e_3} + f^{e_1; e_3} + f^{e_2; e_3} = 3/7 \) and \( f^{e_3; e_3} \leq 5/9 \), we may assume that both \( \mathcal{P}(e, e_1; \rho_3) \) and \( \mathcal{P}(e_2, e_3; \rho_3) \) are nonempty. By the exchange operation at \( e^{*} \) for two paths, one from \( \mathcal{P}(e_1, e_1; \rho_3) \) and another from \( \mathcal{P}(e_2, e_3; \rho_3) \), we can make \( f \) satisfy \( f^{e_1; e_3} = f^{e_1; e^{*}} > 4/9 \), while keeping the optimality and \( f^{e_2; e_3; e^{*}} = 7/9 \). This means that the inequality in Lemma 4.6 (2) fails; a contradiction. Thus we have (5.10), and hence Proposition 5.2 (1).

2. Next we show: if the condition of Proposition 5.2 (2) holds, i.e., \( \alpha_1 = 1 \) with unmixed \( (2c) \), then \( \rho_2 \) is in case \( (2c) \) with \( \alpha_2 = 1 \). If this is true, then necessarily \( f^{e_1; e^{*}} = f^{e_1; e^{*}} = 1 \) (\( \epsilon = \epsilon' = 0 \), and \( \rho_3 \) is also in case \( (2c) \) with \( \alpha_3 = 1 \) since we can interchange the roles of \( x \) and \( x_1 \).

Suppose (indirectly) that \( \rho_2 \) is not in case \( (2c) \). Then \( f^{e_2; e_3} \geq 5/6 + \epsilon \), (5.11) holds by the same argument, and (5.13) does not hold. By (5.10), \( \rho_3 \) is necessarily in case \( (2c) \) with \( \alpha_3 = 1 \) and \( (\rho_3(y), \rho_3(y^{*})) = ([q], [q']) \). Consider \( f \) in \( B \)-configuration. Then \( f^{e^{*}} = f^{e_1} + (f^{e_2} + f^{e_3; e^{*}}) \leq (1 - f^{e_1; e^{*}}) + (1 - f^{e_1; e^{*}}) = 2 - f^{e_1; e^{*}} \leq \alpha_1 = 1 \), and \( 1 = 2 - \alpha_3 \leq f^{e^{*}} \leq 1 \). Therefore \( f^{e^{*}} = 1 \), and \( f \) is an optimum for \( G^{*; \alpha_1} \). We may assume \( f^{e_2; e_3} > 0 \). Otherwise \( f^{e_1; e^{*}} = f^{e_1; e^{*}} = 1 \); we can change the role of \( x \) and \( x_1 \).

Take a path \( P \) in \( \mathcal{P}(e_1, e_2; \rho_2) \neq \emptyset \). Since \( \mathcal{P}(e^{*}) \) is exchangeable at \( e^{*} \), \( P \) is exchangeable with a path in \( \mathcal{P}(e, e_2) \neq \emptyset \). Since \( \mathcal{P}(e_2; \rho_2) \) is exchangeable at \( e^{*} \), \( P \) is exchangeable with a path in \( \mathcal{P}(e, e_1; \rho_2) \subseteq \mathcal{P}(e, e_1) \). By Lemma 4.5 (2), \( P \) is a \( ([q], y, [q']) \)-path; a contradiction to the local geodesic condition.

3. Finally we show that \( \{\rho_i(y), \rho_i(y^{*})\} \) (\( i = 1, 2, 3 \)) satisfy (2-0), (2-1), or (2-2). Now \( f^{e_1; e^{*}} = f^{e_1; e^{*}} = 1/2 \), and \( f \) is an optimum for \( G^{*; \alpha_2} \) in \( A \)-configuration \( (f^{e_2; e^{*}} = 1) \), and is an optimum for \( G^{*; \alpha_3} \) in \( B \)-configuration \( (f^{e_3; e^{*}} = 1) \). In particular, \( f^{e_1; e^{*}} = 1/2 \). Take a path \( P \) from \( \mathcal{P}(e, e_1) \), and suppose that \( P \) is a \( ([u], yx_1, [u]) \)-path. Here \( P \) is a \( ([u], y^{*}; y, [u]) \)-path and a \( ([u], y^{*}; y, [u]) \)-path. By Lemma 4.5 (1), \( \rho_2(y^{*}) \sim u \sim \rho_3(y^{*}) \) and \( \rho_2(y) \sim 1 \sim \rho_3(y) \). If \( f^{e_2; e_3} > 0 \), then we can apply the same argument for \( \mathcal{P}(e_2, e_3) \) and we get \( \rho_2(y^{*}) \sim \rho_3(y) \) and \( \rho_2(y) \sim \rho_3(y^{*}) \), i.e., (2-0) holds. Suppose
that \( f^{e_2,e_3} = 0 \); necessarily \( f^{e,e_2} = f^{e,e_1} = f^{e_1,e_2} = 1/2 \). Suppose that (5.8) (i) holds. Take a path \( P' \) from \( \mathcal{P}(e_1,e_2) \), which is a \( ([q'_1],x_1,y_2,[l]) \)-path for \( q' = \rho_1(y'^1) \) and \( l \sim \rho_1(y) \). Here \( P' \) is an \( ([l],y'^2,y',[q'_1]) \)-path. By Lemma 4.5 (1), we get \( \rho_2(y'^2) \sim l \) and \( q' \sim \rho_2(y) \). By the same argument for \( \mathcal{P}(e_2,e_3) \), we get \( \rho_3(y'^3) \sim l \) and \( q' \sim \rho_3(y'^3) \). Hence (2-1) holds; see Figure 22 (2-1). Similarly, for the case of (5.8) (ii), (2-2) holds; see Figure 22 (2-2). We are done. \( \Box \)

5.3 Proof of the uncrossing lemma

Here we prove Lemma 5.1; the proof technique is due to [11]. We use the relation between DLP and LP-dual, which is revealed in [12] and is summarized by Section 5.3.1. Our argument is algorithmic, and will be used in the next Section 5.4. For an \( F \)-complex \( \mathcal{K} \) with unit leg-length \( \delta = 1 \), let \( \text{diam} \mathcal{K} \) denote the diameter of \( \mathcal{K} \).

5.3.1 Relation between LP-dual and DLP

Consider the following continuous relaxation of DLP:

\[
\text{CLP}(\mathcal{K}; \{R_s\}_{s \in S}): \quad \text{Minimize } \sum_{xy \in E} c(xy)d_{\mathcal{K}}(\rho(x), \rho(y)) \quad \text{subject to } \rho : V \rightarrow \mathcal{K}, \quad \rho(s) \in R_s (s \in S).
\]

We also call a feasible map \( \rho \) in CLP a potential. For a potential \( \rho \) to CLP, metric \( d^\rho \) is feasible to LP-dual (2.1) with the same objective value. Conversely, for any metric \( d \) feasible to LP-dual (2.1), we can greedily construct a potential \( \rho \) in CLP with \( d^\rho \leq d \) as follows.

Let \( V = \{x_1, x_2, \ldots, x_n\} \) (\( \supseteq S \)). For \( k = 1, 2, \ldots, n \), define \( \rho(x_k) \) to be an arbitrary point in

\[
(5.14) \quad \bigcap_{s \in S} B(R_s, d(s, x_k)) \cap \bigcap_{i=1}^{k-1} B(\rho(x_i), d(x_i, x_k)) \quad (k = 1, 2, \ldots, n),
\]

where \( B(R, r) \) is the set of points \( p \) with \( d_{\mathcal{K}}(R, p) \leq r \). This construction is well-defined, since (5.14) is nonempty for every \( k \) [12]. Then, by construction, \( \rho \) is a potential in CLP with \( d^\rho \leq d \) (since \( \rho(x_i) \in B(\rho(x_i), d(x_i, x_k)) \)). Hence, if \( d \) is optimal to LP-dual, then \( \rho \) is optimal to CLP. Therefore, from an optimal metric \( d \), we can construct an optimal potential \( \rho \) in CLP in polynomial time.

Next we round a potential in CLP to a potential in DLP. Fix an admissible orientation \( \overrightarrow{\mathcal{K}} \) of \( \mathcal{K} \). This orientation determines an orientation of the local coordinate of every cell.

A leg \( uv \) oriented as \( \overrightarrow{uv} \) is identified with a segment in \( \mathbb{R} \) with ends \( u = 0, v = 1 \). A triangle \( \sigma \) with oriented legs \( \overrightarrow{uv}, \overrightarrow{vw} \) and hypotenuse \( \overrightarrow{uw} \) is identified with a triangle in \( \mathbb{R}^2 \) with vertices \((u, v, w) = ((0, 0), (1, 0), (1, 1))\). For simplicity, we regard a square-folder \( F \) as a \( K_{2,2} \)-folder with the hypotenuse joining the sink and the source in \( F \).

For \( a \in [0, 1] \), we can define a rounding map \( \phi^a : \mathcal{K} \rightarrow V(\mathcal{K}) \) as follows. For a point \( p \in \mathcal{K} \), we can take a cell \( \sigma \) containing \( p \). In the case where \( \sigma \) is a triangle with vertices \( u, v, w \) oriented as above, \( p \) is locally represented as a point \((x, y) \in \mathbb{R}^2 \) with \( 0 \leq y \leq x \leq 1 \). Define \( \phi^a \) by

\[
\phi^a(p) := \begin{cases} 
  u & \text{if } y \leq x \leq a, \\
  v & \text{if } y \leq a < x, \\
  w & \text{if } a < x \leq y.
\end{cases}
\]
in the case where σ is a leg \( \overline{uv} \), \( p \) is locally represented as a point \( x \) with \( 0 \leq x \leq 1 \).

Define \( \phi^a(p) := u \) if \( x \leq a \) and \( \phi^a(p) := v \) if \( a < x \). See Figure 24. This map \( \phi^a \) is well-defined. This rounding is due to [18], and it is known in [12, 18] that

\begin{equation}
\text{If } \rho \text{ is optimal to CLP, so is } \phi^a \circ \rho.
\end{equation}

Consequently an optimal potential can be obtained from any optimal metric in polynomial time.

### 5.3.2 Constructing a semi-neighbor from two potentials

We can consider an analogue of a convex combination of two distinct potentials \( \rho \) and \( \rho' \). Here we assume the following condition:

\begin{equation}
\text{For every } \rho \in V(K), \text{ there is a terminal } s \in S \text{ with } R_s = \{p\}.
\end{equation}

This is achieved by adding at most \( |V(K)| \) dummy terminals. Take \( \sigma \in [0,1] \). Let \( d_\sigma = (1 - \sigma)d^0 + \sigma d^0' \). Then \( d_\sigma \) is feasible to LP-dual (2.1). According to (5.14), we can construct a potential \( \rho_\sigma \) in CLP with \( d^0_\sigma \leq d \). In particular, if both \( \rho \) and \( \rho' \) are optimal to CLP, then \( \rho_\sigma \) is also optimal to CLP.

\begin{equation}
\text{Under assumption (5.16), we have } d_K(\rho(x), \rho_\sigma(x)) = \sigma d_K(\rho(x), \rho'(x)) \text{ and } d_K(\rho_\sigma(x), \rho'(x)) = (1 - \sigma) d_K(\rho(x), \rho'(x)) \text{ for } x \in V.
\end{equation}

Indeed, for each \( x \in V \) there are terminals \( s, t \) with \( R_s = \{\rho(x)\} \) and \( R_t = \{\rho'(x)\} \); so \( \rho(x) = \rho(s) = \rho'(s) \) and \( \rho'(x) = \rho(t) = \rho'(t) \). Thus, by construction (5.14), we have

\[
\rho_\sigma(x) \in B(R_s, d_\sigma(s,x)) \cap B(R_t, d_\sigma(t,x)) = B(\rho(x), \sigma d_K(\rho(x), \rho'(x))) \cap B(\rho'(x), (1 - \sigma) d_K(\rho(x), \rho'(x))).
\]

Hence \( d_K(\rho(x), \rho_\sigma(x)) \leq \sigma d_K(\rho(x), \rho'(x)) \) and \( d_K(\rho_\sigma(x), \rho'(x)) \leq (1 - \sigma) d_K(\rho(x), \rho'(x)). \)

By the triangle inequality \( d_K(\rho(x), \rho'(x)) \leq d_K(\rho(x), \rho_\sigma(x)) + d_K(\rho_\sigma(x), \rho'(x)) \), we obtain the equalities in (5.17).

Let \( \rho \) be a potential in DLP. If we are given a potential \( \rho' \) in CLP close to \( \rho \), we can construct a semi-neighbor of \( \rho \) from \( \rho' \). A semi-neighbor is called forward if it is a semi-neighbor with respect to \( K^f \) and is called backward if it is a semi-neighbor with respect to the opposite orientation of \( K \).

\begin{equation}
\text{For a potential } \rho \text{ to DLP and a potential } \rho' \text{ to CLP, if } d_K(\rho(x), \rho'(x)) < 1/2 \text{ for } x \in V, \text{ then } \phi^a \circ \rho' \text{ is a forward (resp., backward) semi-neighbor of } \rho \text{ for } a \in [0, 1/2) \text{ (resp., } a \in [1/2, 1)).
\end{equation}

This property can be easily seen from the right of Figure 24, where shaded regions depict...
disjoint balls around vertices in $K$ with radius less than $1/2$; if $p = \rho(x)$, then $\rho'(x)$ is contained by the ball around $p$, and is rounded to $\phi^\sigma \circ \rho'(x)$ along the arrows. The uncrossing lemma is now immediate.

**Proof of the uncrossing lemma.** Take positive $\sigma \leq 1/(2 \text{diam } K^2 + 1)$. Consider $d_\sigma := (1 - \sigma)d^\rho + \sigma d^\rho'$. Then $d_\sigma$ is optimal. Next, according to (5.14), we construct a potential $\rho_\sigma$ to CLP with $d^\rho_\sigma \leq d_\sigma$, which is also optimal. By (5.17), $d_{K^2}(\rho(x), \rho_\sigma(x)) < 1/2$, and $\rho(x) \neq \rho'(x)$ implies $\rho(x) \neq \rho_\sigma(x)$. In the forward orientation, round $\rho_\sigma$ to $\rho^* := \phi^\sigma \circ \rho_\sigma$. Here take $a = 0$. By (5.18), $\rho^*$ is a forward semi-neighbor and is a desired one. $\square$

### 5.4 Algorithmic implication

The proof of Theorem 1.5, we have shown above, is constructive. Each step searches all forks for one having required properties, and applies SPUP or splitting-off to decrease the number of nodes in question. Once the problem becomes trivial to have an integral optimum, we obtain a $1/k$-integral optimum for the original problem by reversing the operations.

Here we verify that our proof indeed yields a (strongly) polynomial time algorithm under the assumption that the size (the number of cells) of a realization is fixed.

**Theorem 5.4.** Suppose that a realization of $\mu$ is given and its size is fixed. Then there exists a strongly polynomial time algorithm to find a $1/12$-integral optimal multflow in $\mu$-MFP for every property-inner Eulerian graph.

The size of a realization is not polynomially bounded by the bit size of $\mu$ in general; see the 2-commodity F-complex in Section 6. In the case of 0-1 weight, there is a realization of $O(|S|^2)$ size; see (7.2) in Section 7.1.

In the case where the edge capacity is not so large, our proof gives a strongly polynomial time algorithm, assuming the oracles of finding an optimal potential, the splitting capacity $\alpha_\tau$, a critical neighbor, and a forward semi-neighbor in the uncrossing lemma. We note that our proof goes on without any explicit multflow calculation; the mixed 1-SPUP (in Section 5.2.4) can be done without checking all optimal multflows by Proposition 5.2. Also a critical neighbor can be relaxed to a critical semi-neighbor.

These computations can be done (in a combinatorial way) if we get an optimal metric in LP-dual (2.1). Since LP (2.1) is given by a $\{-1, 0, 1\}$ coefficient matrix of polynomial size, we can evaluate the optimal value and find an optimal metric in strongly polynomial time by the method of Tardos [28]. In Section 5.3 we mentioned polynomial time constructions of an optimal potential from an optimal metric and of a forward semi-neighbor in the uncrossing lemma. Hence, in the rest of this subsection, we explain how to compute $\alpha_\tau$ and a critical semi-neighbor, and how to reduce edge-capacities.

#### 5.4.1 Computing $\alpha_\tau$ and a critical semi-neighbor

The computation of $\alpha_\tau$ is a fractional programming. Let $\rho$ be an optimal potential to $G$ and let $\tau$ be a fork. Let $h(\alpha) := \text{opt}(\mu; G) - \text{opt}(\mu; G^{\tau, \alpha})$. Then $\alpha_\tau := \max\{\alpha \mid h(\alpha) = 0\}$. The gradient of $h$ at $\alpha$ is given by $d^\rho(e^\tau)$ for some optimal potential $\rho$ of $G^{\tau, \alpha}$. So the possible values of the gradients are $0, 1, 2, \ldots, \text{diam } K$. Here $h$ is a monotone nondecreasing piecewise linear convex function. Hence, by the discrete Newton method, we can determine $\alpha_\tau$ by solving (2.1) at most $\text{diam } K$ time.

Next suppose $\alpha_\tau \leq 2$, and consider a critical semi-neighbor. A semi-neighbor $\rho'$ of $\rho$ is critical with respect to $\tau$ if and only if it satisfies $d^\rho'(e^\tau) > 0$ and it is optimal.
to $G^{\tau,\alpha\tau}$. We can construct a critical neighbor from any feasible metric $d$ such that it satisfies $d(e^*) > 0$ and it is optimal for $G^{\tau,\alpha\tau}$. Such a metric, also called critical, can be naturally obtained at the computation of $\alpha_{\tau}$ above.

Consider $d_{\sigma} = (1 - \sigma)d^0 + \sigma d$ for positive $\sigma(\leq 1/(2 \text{diam} K + 1))$; obviously $d_{\sigma}$ is also critical. Next take a potential $\hat{\rho}$ to CLP with $d^\hat{\rho} \leq d_{\sigma}$, according to (5.14). Again $\hat{\rho}$ is optimal to $G^{\tau,\alpha\tau}$. Also $d^\hat{\rho}(e^*) > 0$. Indeed, if $d^\hat{\rho}(e^*) = 0$, then for small positive $\epsilon > 0$ we have $d^\hat{\rho}(G^{\tau,\alpha\tau}) = d^\hat{\rho}(G^{\tau,\alpha\tau+\epsilon}) \leq d_{\sigma}(G^{\tau,\alpha\tau+\epsilon}) < d_{\sigma}(G^{\tau,\alpha\tau}) = d^\hat{\rho}(G^{\tau,\alpha\tau})$. A contradiction. Thus $d^\hat{\rho}$ is also critical. Fix an admissible orientation. Take $a$ and round $\hat{\rho}$ to $\phi^a \circ \hat{\rho}$, which is a semi-neighbor of $\rho$. Since $\hat{\rho}(y) \neq \hat{\rho}(y^*)$ we can choose $a$ so that $\phi^a \circ \hat{\rho}(y) \neq \phi^a \circ \hat{\rho}(y^*)$. Then $\phi^a \circ \hat{\rho}$ is a critical semi-neighbor of $\rho$ as required. This construction can be done in strongly polynomial time.

5.4.2 Reducing edge-capacities

Finally, we explain a preprocess to reduce the edge-capacities. This can be done in splitting-off. We may assume that $G = (V, E)$ is a complete graph. Let $n = |V|$. We use a capacitated version of the splitting-off. For a fork $\tau = (xy, y, yz)$ and a nonnegative integer $\beta \leq \min\{c(xy), c(yz)\}$, decrease $c(xy)$ and $c(yz)$ by $\beta$ and increase $c(xz)$ by $\beta$. The splitting-off operation is to decrease the maximum possible value $\beta_{\tau}$ keeping the optimal value. We also consider the degenerate fork $(xy, y, yx)$. In this case the splitting-off operation is to decrease $c(xy)$ by the maximum possible even integer $\beta_{\tau}$ keeping the optimal value. We can recover an optimal multifold in the original graph from any optimal multifold in the graph obtained by a splitting-off. Again $\beta_{\tau}$ is also computed in the same manner as in the previous section.

By repeating the splitting-off $O(n^3)$ times, we can make $(G, c)$ so that $\beta_{\tau} = 0$ for every fork $\tau$. Indeed, take a node $x$, and apply the splitting-off for all forks at $x$ in an arbitrary order. Then $\beta_{\tau} = 0$ for every fork $\tau$ at $x$. If we apply the splitting-off to a fork at another node $x'$, then this does not increase the degree of $x$, and also does not produce a new splittable fork at $x$. Apply this procedure to all nodes. Then $\beta_{\tau} = 0$ for all forks in $G$. At this moment,

\begin{equation}
(5.19) \quad \text{each inner node } y \text{ has } O(n^2) \text{ degree.}
\end{equation}

Indeed, consider an optimal multifold $f$. Then (5.19) follows from:

\[ \sum_{x \in V \setminus \{y\}} c(xy) = \sum_{x \in V \setminus \{y\}} (c(xy) - f^{xy}) + 2 \sum_{x, z \in V \setminus \{y\}} f^{xy,yz}. \]

Then $f^{xy,yz} \leq 1$; otherwise the fork $(xy, y, yz)$ is splittable (Lemma 3.1). Also $c(xy) - f^{xy} \leq 2$; otherwise the degenerate fork $(xy, y, yx)$ is splittable. Thus the degree of $y$ is at most $2(n - 1) + 2 \binom{n-2}{2} = O(n^2).

Terminals may have a large degree. Next compute an optimal multifold $f = (P, \lambda)$ by solving LP; we can use a compact representation for multflows. For each pair $(s, t)$ of terminals, check the flow-value $\lambda(P)$ of the path $P$ of a single edge $st$, and decrease the edge capacity $c(st)$ by the maximum even integer $l_{st}$ not exceeding $\lambda(P)$. Again we can recover an optimum in the original problem from any optimum in the new problem by adding the path of a single edge $st$ of flow-value $l_{st}$. Then

\begin{equation}
(5.20) \quad \text{each terminal } s \text{ has } O(n^2) \text{ degree.}
\end{equation}
Indeed we have

\[
\sum_{x \in V \setminus \{s\}} c(sx) = \sum_{x \in V \setminus \{s\}} (c(sx) - f^{sx}) + \sum_{x,y \in V \setminus \{s\}} 2f^{xs,xy} + \sum_{x,y \in V \setminus \{s\}} f^{sx,xy} + \sum_{t \in S \setminus \{s\}} f^{st},
\]

\[
\leq 2(n - 1) + 2\left(\frac{n - 1}{2}\right) + (n - 1)(n - 2) + 2(|S| - 1),
\]

where \(f^{sx,xy}\) denotes the total amount of \((s, sxy, *)\)-flow, and \(f^{st}\) denotes the total amount of \((s, st, t)\)-flow.

At this moment, if the existence of an integral optimal solution is guaranteed, then the degree of every inner node is zero, and the multflow of one-edge paths is optimal.

By edge multiplication, make each edge have unit capacity, and apply the degree reduction in Section 3.3. Then we obtain a graph \(G^*\) with degree at most four, \(O(n^5)\) vertices, and unit capacity. Consequently, our proof of Theorem 1.5 to \(G^*\) finds a 1/12-integral optimal multflow for \(G^*\) in strongly polynomial time. By reversing the process above, we get a 1/12-integral optimal multflow for the original graph \(G\) in strongly polynomial time.

6  Sparsity and integrality

In this section, we give a powerful geometric criterion of the splittability/integrality. We introduce the concept of a sparse vertex in an orientable F-complex, and show that if an inner node \(y\) is mapped to a sparse vertex by some optimal potential \(\rho\), then \(y\) has a splittable fork (under Eulerian condition). This generalizes claim (A), and enables us to prove the integrality theorem for a large class of \(\mu\)-MFP.

6.1  Sparsity

A vertex \(p\) in an orientable F-complex \(K\) is said to be sparse if, for every oriented orbit \(\overrightarrow{O}\), every pair of vertices \(q, q'\) with \(p \supseteq q\) and \(p \supseteq q'\) belongs to a common folder in \(K_p\). This concept generalizes and localizes the one due to Karzanov [20, Definition 1.3], who introduced the sparseness concept for a different purpose. The main result in this section connects the geometric notion of the sparseness and the splittability/integrality in \(\mu\)-MFP.

Let \(G\) be a graph with terminal set \(S\), and let \(\mu\) be a terminal weight having a realization \((K; \{R_s\}_{s \in S})\). We consider \(\mu\)-MFP and DLP\((K; \{R_s\}_{s \in S})\). A terminal \(s\) is said to be strong if \(R_s\) is a path of hypotenuses or a single vertex. Note that any strong terminal is proper. \(G\) is said to be strongly-inner Eulerian (with respect to \((K; \{R_s\}_{s \in S})\)) if each node other than strong terminals has an even degree.

**Theorem 6.1.** Suppose that \(G\) is strongly-inner Eulerian. If there exists an optimal potential \(\rho\) such that \(\rho(x)\) is sparse for every node \(x\), then there exists an integral optimal multflow.

The large part of this theorem follows from the following splittability criterion. Recall that \(G\) is said to be properly-inner Eulerian (with respect to \((K; \{R_s\}_{s \in S})\)) if each node other than proper terminals has an even degree (see Section 3.2 for the definition of proper terminals).

**Theorem 6.2.** Suppose that \(G\) is properly-inner Eulerian. For an optimal potential \(\rho\), an inner node \(y\) has a splittable fork if
(i) \( \rho(y) \) is sparse, and

(ii) there exists no odd-degree terminal \( s \) such that \( \rho(s) = \rho(y) \) and \( R_s \) has three hypotenuses incident to \( \rho(y) \).

The proofs of Theorems 6.1 and 6.2 are given in Section 6.3. We first describe consequences of Theorem 6.1.

### 6.2 Locally sparse F-complex and blow-up

The integrality of \( \mu\)-MFP is closely related to the embeddability of \( \mu \) into a nice F-complex. An orientable F-complex \( K \) is said to be locally sparse if each vertex is sparse. An immediate, but powerful, consequence of Theorem 6.1 is the following.

**Theorem 6.3.** Suppose that \( \mu \) has a realization \( (K;\{R_s\}_{s\in S}) \) with a locally sparse F-complex \( K \). Then \( \mu\)-MFP has an integral optimal multiflow for every strongly-inner Eulerian graph with respect to \( (K;\{R_s\}_{s\in S}) \).

The local sparsity is an easily checkable property. The following F-complexes are all locally sparse:

- a folder itself.
- a subdivision of a locally sparse F-complex.
- an F-complex without \( K_{2,*} \)-folders.
- a star-shaped F-complex without a pair of \( K_{2,*} \)-folders having a common leg.
- an F-complex each of whose summands is a single leg or a single folder.

By Theorem 6.3, for any weight \( \mu \) realized by these F-complexes, \( \mu\)-MFP admits an integral optimal multiflow for every Eulerian graph. For example, take \( \mu \) as the graph metric \( d_{K_2,r} \) of \( K_2,r \). Then \( \mu \) is realized by a single folder. Hence we obtain the integrality theorem due to Karzanov-Manoussakis \([23]\): *There exists an integral optimal multiflow in \( d_{K_2,r}\)-MFP for every inner Eulerian graph*. Consider an F-complex \( K \) without \( K_{2,*} \)-folders, i.e., \( K \) is a cubical complex. The corresponding integrality theorem is nothing but the *multiflow locking theorem* due to Karzanov-Lomonosov \([22]\); see for \([19, \text{Section 5}]\) for the detail of this relation. Theorem 6.3 includes many other integrality instances. For example, consider \( \mu \) in Figure 9. Then the F-complex in the right is locally sparse, and hence the integrality theorem holds for this weight.

Interestingly, even if \( K \) is not locally sparse, sometimes we can represent \( K \) as a summand of a locally sparse one; see \( K/U \) in Figure 8. By combining Theorem 6.1 with the locking property (Proposition 2.6 in Section 2.2.3), we can prove the integrality theorem for such \( \mu \) that is a summand of another weight \( \mu^* \) having a locally sparse realization.

**Theorem 6.4.** Suppose that \( \mu \) is a summand of \( \mu^* \) having a realization \( (K^*;\{R^*_s\}_{s\in S}) \) with a locally sparse F-complex \( K^* \). Then \( \mu\)-MFP has an integral optimal multiflow for every strongly-inner Eulerian graph with respect to \( (K^*;\{R^*_s\}_{s\in S}) \).

A sparse (resp., nonsparse) vertex is an analogue of a *nonsingular* (resp., *singular*) point in an algebraic variety. We call the process of constructing an F-complex \( K^* \) having \( K \) as a summand a *blow-up*.

An illustrative application of Theorem 6.4 is shown.
Multiterminal weighted 2-commodity flows. Suppose that $S$ is partitioned into four sets $\{T, T', U, U'\}$. For relatively prime positive integers $a$ and $b$, let $\mu$ be the weight on $S$ such that $\mu(t, u) = a$ for $(t, u) \in T \times U$, $\mu(t', u') = b$ for $(t', u') \in T' \times U'$, and $\mu$ vanishes for other pairs. Then the corresponding $\mu$-MFP is a weighted version of the multiterminal 2-commodity flow maximization problem; note that the standard super-terminal technique does not work to reduce this problem to a single-terminal problem.

Theorem 6.5. The multiterminal weighted 2-commodity flow problem has an integral optimal flow for every inner Eulerian graph.

We prove this from Theorem 6.4. We first construct a realization of $\mu$. Consider a rectangle in the $l_1$-plane $\mathbb{R}^2$ such that the edge-directions are $(1, 1)$ and $(1, -1)$, and the edge-lengths are $a$ and $b$; see Figure 25. Subdivide this rectangle into squares and right isosceles triangles along lines parallel to coordinate axes as in the left of Figure 25. Set the leg-length to be $1/2$. The resulting complex $\mathcal{K}$ is clearly an (orientable) F-complex. Let $(R_t, R_u)$ and $(R_{t'}, R_{u'})$ be opposite pairs of edges of length $b$ and $a$, respectively. Then we obtain a realization of $\mu$. Although $\mathcal{K}$ is not locally sparse (with all edges belonging to a common orbit), we can blow up $\mathcal{K}$ to a locally sparse F-complex as follows. Delete all legs from $\mathcal{K}$, and insert squares and triangles along deleted legs as in the middle of Figure 25. The inserted edges form two orbits, different from the orbit to which the original edges belong. From this, one can see that the resulting F-complex $\mathcal{K}^*$ is locally sparse, and has $\mathcal{K}$ as a summand. Each $R_s$ is naturally extended to series of hypotenuses $R_s^*$; each terminal is strong. Thus by Theorem 6.4 we get Theorem 6.5.

6.3 Proof

We first prove Theorem 6.2 and then Theorem 6.1. Theorem 6.2 is a consequence of Theorem 4.3 (1).

Proof of Theorem 6.2. Let $p := \rho(y)$ and $X_p := \rho^{-1}(p)$. By applying the degree reductions (Section 3.3) at $X_p$, we may assume that each inner node in $X_p$ has degree four, each proper terminal having no three hypotenuses at $p$ has degree one, and the other terminals have degree two. We may assume that all improper terminals are essential. We consider $\mathcal{K}^2$; the sparsity of $p$ is kept. We regard $\rho$ as $V \rightarrow V(\mathcal{K}^2)$.

It suffices to show that some inner node in $X_p$ has a splittable fork; then so does each inner node in $X_p$. Let $y$ be an inner node in $X_p$, incident to four edges $e = xy$, $e_i = x_i y (i = 1, 2, 3)$. Suppose to the contrary that all three forks $\tau_i = (xy, y, yx_i)$ are unsplittable. Consider a critical neighbor $\rho_i$ with respect to $\tau_i$ for $i = 1, 2, 3$. We have
Figure 26: Flow rearrangement

Figure 27: Forbidden folder structures around $p$

$p \succeq \rho_i(y)$ and $p \succeq \rho_i(y^*)$ for some oriented orbit $O_i$. By the sparsity condition, $\rho_i(y)$ and $\rho_i(y^*)$ belong to a common folder in $K_p$, i.e., $d^{\rho_i}(e^*) \in \{1, 2\}$. By the properly-inner Eulerian condition and Lemma 3.4, the numerator of formula (3.2) of $\alpha_\tau$ is even. Therefore $\alpha_i > 0$ implies $\alpha_i = 1$ and $d^{\rho_i}(e^*) = 2$, i.e., $\rho_i(y)$ and $\rho_i(y^*)$ are not adjacent by a leg; (2a, 2b) in Figure 21. Thus Theorem 4.3 (1) is applicable. There is a triple $l_1, l_2, l_3 \in L_p$ with properties (i) and (ii) (in Theorem 4.3 (1)). Take an optimal multiflow $f$. Then $\mathcal{P}(e_i, e_j)$ is an $(\{l_i, x, y, y_j, [l_j]\})$-set with $f^{e_i, e_j} = 1/2$ for $1 \leq i < j \leq 3$. Edge $xy$ has no flow, and thus $\rho(x) = \rho(y) = p$ (by the saturation condition in Lemma 2.4).

Consider next the splittability property at $x \in X_p$, which is an inner node or a terminal. Suppose first that $x$ is an inner node incident to $y, y_1, y_2, y_3$. The edge $y_i x$ is denoted by $\tilde{e}_i$ for $i = 1, 2, 3$. The fork $(e, x, \tilde{e}_i)$ is denoted by $\tilde{\tau}_i$ for $i = 1, 2, 3$. If $x$ has a splittable fork, this is a desired node. Suppose not. Again, by Theorem 4.3 (1), there is a triple $l_1', l_2', l_3' \in L_p$ such that $\mathcal{P}(\tilde{e}_i, \tilde{e}_j)$ is an $(\{l_i', y_i y_j, [l_j]\})$-set with $f^{\tilde{e}_i, \tilde{e}_j} = 1/2$ for $1 \leq i < j \leq 3$. See the left of Figure 26. Suppose $l_1 \not\sim l_1'$. Then we can rearrange $f$ as in Figure 26. By the local geodesic condition, the resulting multiflow $f$ is also optimal, and $f^{e_i, e_j} > 1/2$, which contradicts $\alpha_i = 1$. Therefore $l_1 \not\sim l_j, l_1' \not\sim l_j'$, and $l_i \sim l_j'$ for any $i, j$. Then $\Pi_p$ contains the subdivision of $K_{3,3}$, and all edges incident to $l_i, l_j'$ in $\Pi_p$ belong to a common orbit. See the left of Figure 27. Therefore $p$ cannot be sparse; a contradiction.

Suppose that $x$ is a terminal (of degree one or two). Since $f^{\alpha_\tau} = 1$, $f$ is optimal for $G^{\alpha_\tau}$, and $f^{e} = 0$, we have $\rho(x) = \rho(y)$ and $\rho_i(x) = \rho_i(y)$ $(i = 1, 2, 3)$. Hence we have $p, l_1, l_2, l_3 \in R_x$. Here $l_i$ is the midpoint of a folder in $K$. By the normality of $R_x$ in $K$, for $i = 1, 2, 3$, $R_x$ has a hypotenuse $pl_i$ or a square-folder including $p$ and $l_i$. By the properly-inner-Eulerian condition and the condition (ii), $x$ must have degree

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two, incident to $y$ and $z$. Since $s$ is essential, $\tau' := (zx, xy)$ is unsplittable. By $f^e = 0$, we have $\alpha_{\tau'} = 1$ and $f^{xx} = 1$ (Lemma 3.1). In particular, $f^{\tau'} = 1$, and $f$ is also optimal for $G^{\tau', \alpha_{\tau'}}$. Consider a critical neighbor $\rho'$ of $\rho$ with respect to $\tau'$, and consider the position $(\rho'(x), \rho'(x^*))$. Necessarily $\rho'(x)$ is on the boundary of $R_s$, and $\rho'(x^*)$ is not in $R_s$. By the sparsity, $\rho'(x)$ and $\rho'(x^*)$ belong to a common folder. So $(\rho'(x), \rho'(x^*)) = (p, l)$ for some $l \in L_p$ (case (2a) in Figure 21) is the only possibility. Since $f$ is also optimal for $G^{\tau', \alpha_{\tau'}}$, $\mathcal{P}(zx) = \mathcal{P}(e^*)$ is an $(\|l\|, zz, \ast)$-set (Table 1). If $l \neq l_1$, then by a rearrangement similar to Figure 26, we have $f^{e_2, e_3} > 1/2$: a contradiction to $1 = \alpha_1 \geq 2f^{e_2, e_3}$. So suppose $l \sim l_i$ for $i = 1, 2, 3$, i.e., $l$ and $l_i$ have a common neighbor $q_i$ as in the right of Figure 27. Necessarily $q_p$ and $q_i$ belong to a common orbit $O_1$ (otherwise $(\rho'(y), \rho'(y^*)) = (p, l_i)$ does not occur). Since there is a $K_{2, \ast}$-folder including $p, l, q_1, q_2, q_3$, we have $O_1 = O_2 = O_3$. So $p$ cannot be sparse. A contradiction. □

**Proof of Theorem 6.1.** By the degree reduction (Section 3.3), we can assume that each inner node has degree four, each strong terminal has degree one, and the other terminals have degree two. All inner nodes are splittable by Theorem 6.2. So we may assume that there is no inner node. We may assume that each terminal of degree two is unsplittable; otherwise we can split it off. We show that the multiflow consisting of one-edge paths is optimal. By the optimality criterion, it suffices to show that $d_K(\rho(s), \rho(t)) = d_K(R_s, R_t)$ for each edge $st$. Again we consider $K^2$ and regard $\rho$ as $V \rightarrow V(K^2)$.

Take an edge $st$ with $\rho(s) \neq \rho(t)$. Let $p := \rho(s)$. Take an optimal multiflow $f = (\mathcal{P}; \kappa)$; necessarily $f^{st} = 1$. Since $\mathcal{P}(st)$ contains a $(s, st, t)$-path (otherwise $t$ is unsplittable), we have $d_K(\rho(s), R_t) = d_K(\rho(s), \rho(t))$. Consider the gate $g$ of $R_t$ at $p$. If $g \not\in R_s$, then, by Lemma 4.1, we have $d_K(R_s, R_t) = d_K(\rho(s), R_t) = d_K(\rho(s), \rho(t))$ as required.

Suppose (to the contrary) that $g \in R_s$, i.e., $d_K(\rho(s), R_t) > d_K(R_s, R_t)$. Take $u \in Q_p$ with $d_K(p, \rho(t)) = d_{K_s}(p, u) + d_K(u, \rho(t))$. Then $d_K(p, R_t) = d_K(\rho(s), \rho(t))$. By Lemma 4.2, we have $u \sim_1 g$. Since $\mathcal{P}(st)$ also contains an $(s, st, \ast)$-path (otherwise $s$ is unsplittable), we have $d_K(\rho(s), \rho(t)) = d_K(R_s, \rho(t)) = d_K(R_s, u) + d_K(u, \rho(t))$. In particular, $d_K(R_s, u) = d_{K_s}(p, u)(> 0)$, and $u \not\in R_s$. So $p, g \in R_s \not\sim u$, and $g$ is incident to $u$. In particular, $g$ belongs to $L_p$ and is the midpoint of a folder in $K$; recall that we are working on $K^2$. By the normality, $p$, $g$, and $u$ form a triangle $\sigma$ in some $K_{2, \ast}$-folder $F$ with hypotenuse $pq = \sigma \cap R_s$. Consequently, $F$ belongs to a common orbit $O$. See Figure 28.

The terminal $s$ must have degree two and a unique (unsplittable) fork $\tau$ with $\alpha_{\tau} > 0$; otherwise $f$ has a path connecting $s$ and $t$ in $st$, implying $d_K(\rho(s), \rho(t)) = d_K(R_s, R_t)$. Consider a critical neighbor $\rho'$ of $\rho$ with respect to $\tau$. Since the numerator of formula (3.2) of $\alpha_{\tau}$ is even (Lemma 3.4), we have $d^{\rho'(e^*)} \geq 2$. Moreover $\rho'(s)$ belongs to the boundary of $R_s$, and $\rho'(s^*)$ is not in $R_s$. By the sparsity, $\rho'(s)$ and $\rho'(s^*)$ belong to a common folder. Therefore $(\rho'(s), \rho'(s^*)) = (p, l)$ for $l \in L_p$ (case (2a) in Figure 21) and $\alpha_{\tau} = 1$. Necessarily $g \not\in l(\not\in R_s)$. Consider an optimal multiflow $f = (\mathcal{P}; \kappa)$ for $G_{\tau', \alpha_{\tau'}}$ and take a path $P \in \mathcal{P}(e^*, s^*t)(\not\in \emptyset)$, which connects $s$ and some terminal $t'$ through $e^*$. By the geodesic condition for $(f, \rho')$, $d_K(p, R_{t'}) = d_{K_s}(p, l) + d_K(l, R_{t'})$ holds, and hence the gate of $R_{t'}$ is $l$ by Lemma 4.2. Since $(f, \rho)$ is also optimal for $G(= G_{\tau', 0})$, $d_K(p, R_{t'}) = d_{K_s}(p, u) + d_K(u, R_{t'})$ holds. By Lemma 4.2, $u$ and $l$ are adjacent, and hence $pu$ and $ul$ belong to a common folder $F'(\not\in F)$ and a common orbit $O'$. Consequently $F$ and $F'$ belong to common orbit $O = O'$. This is impossible by the sparsity. □
Here we focus on $H$-MFP for a commodity graph $H$ with property P, where $H$ is the 0-1 weight corresponding to $H$ by the relation: $\mu_H(s, t) = 1 \iff st \in E(H)$. Application of our results in the previous sections reveals an interesting hierarchy of problem classes admitting integrality or half-integrality theorems. This gives a unified understanding to previously known results, as well as to new half-integrality results.

In Section 7.1 we introduce three F-complexes $K_H$, $K_s^H$, and $K_e^H$ with the properties that $K_H$ realizes $H$, $K_s^H$ is star-shaped, and $K_e^H$ has both $K_H$ and $K_s^H$ as summands. From $K_H$ and $K_s^H$, we define weights $\mu_s^H$ and $\mu_e^H$ such that $\mu_s^H$ is a metric and $\mu_e^H$ has both $\mu_s^H$ and $\mu_e^H$ as summands. Recall Sections 2.1.5 and 2.2.3 for summands.

In Section 7.2, we show that the local sparsity of $K_e^H$ is equivalent to the ant clique-bipartite condition on $H$. This fact and Theorem 6.4 immediately imply the classical Karzanov-Lomonosov integrality theorem [22].

In Section 7.3, the fractionality relation $\frac{\text{frac}(H)}{\text{frac}(\mu_s^H)} \leq 2$ is stated. This relation reduces the fractionality study for $\mu_H$-MFP to that for $\mu_s^H$-MFP. Since $K_s^H$ is star-shaped, $\mu_s^H$-MFP has a much simpler structure than $\mu_H$-MFP has. Applying the result to $K_s^H$ in the previous section, we prove the half-integrality theorem for a large class of commodity graphs including the previously known. In Section 7.4, we prove, algorithmically, the fractionality relation.

In this section we assume that commodity graph $H$ has no isolated nodes. A maximal stable set of $H$ is called an ant clique. In constructions of F-complexes, a square-folder with legs $pp', pq', q'q, qp$ is denoted by $pq'$, and a triangle with hypotenuse $pp'$ and legs $pq$ and $qp'$ is denoted by $pq'p$.

### 7.1 F-complexes for a commodity graph with the property P

Let $A$ be the set of ant cliques of $H$, and $D$ be the set of nonempty subsets $D \subseteq S$ represented as the intersection of (at least) two distinct ant cliques. By property P, we have $D = \bigcap \{ A \in A \mid D \subseteq A \}$ for any $D \in D$. Let $A_0 \subseteq A$ be the set of ant cliques $A$ with $A' \cap A = \emptyset$ for every $A' \in A \setminus \{ A \}$.

Let $\Pi_H$ be the bipartite graph with bipartition $\{ D, A \}$ and edge set $\{ DA \mid D \subseteq A \}$. By property P, we easily see:

$$\Pi_H \text{ has girth at least 8.}$$

Indeed, a 6-cycle corresponds to an intersecting triple of ant cliques with distinct intersections.
**The first F-complex $K_H$.** The first F-complex $K_H$ is constructed as follows. The vertices of $K_H$ are $p^D, q^A (D \in \mathcal{D}), p^A, q^A (A \in \mathcal{A})$. For $D \in \mathcal{D}$, consider $K_{2*}$-folder $F_D$ consisting of triangles $p^Dq^Ap^D$ over all anticliques $A$ including $D$. If $F_D$ is a $K_{2*}$-folder, then replace $F_D$ by a square-folder (on the same vertices). Such a $K_{2*}$-folder corresponds to a member of $\mathcal{D}$ which is the intersection of exactly two anticliques. Next, for each anticlique $A$ including $D$, attach triangles $p^Dq^Ap^A$ to $F_D$. Let $K_D$ be the resulting complex. Glue $K_D$ over all $D \in \mathcal{D}$. Finally, for each $A \in \mathcal{A}_0$, attach series of two legs $p^Dq^A, q^Ap^A$ to $p^D$. Let $K_H$ be the resulting complex. The leg-length is defined to be 1/4. See Figure 29 (a).

$K_H$ is an F-complex. Indeed, it is contractible, and hence simply-connected. It suffices to verify the flag condition at $p^D$. Observe $\Pi_{p^D} = \Pi_H$. Thus $\Pi_{p^D}$ has girth at least 8. Furthermore, $K_H$ is orientable; we can orient $K_H$ so that $p^D$ and $p^A$ are sources.

To realize $\mu_H$, normal sets $R_s (s \in S)$ are defined as follows. By property $P$, each $s \in S$ belongs to either a unique $D \in \mathcal{D}$ or a unique $A \in \mathcal{A}$. In the former case, define $R_s$ as the union of hypotenuses $p^Dp^A$ over all anticliques $A$ including $D$. In the latter case, define $R_s$ as the single vertex $p^A$. Then each $R_s$ is clearly normal; $R_s$ is a star of hypotenuses or a single vertex. Then $\mu_H(s, t) = d_{K_H}(R_s, R_t)$. Indeed, $R_s \cap R_t \neq \emptyset \iff s$ and $t$ belong to a common anticlique $\leftrightarrow \mu_H(s, t) = 0$. Conversely, $R_s \cap R_t = \emptyset$ implies $d_{K_H}(R_s, R_t) = 1$. Therefore $(K_H, \{R_s\}_{s \in S})$ is a realization of $\mu_H$. The corresponding combinatorial duality relation (Theorem 2.3) coincides with that given in [16].

**The second F-complex $K'_H$.** The second F-complex $K'_H$ is the neighborhood of $p^D$ in $K_H$, i.e., $K'_H := (K_H)_{p^D}$. For each terminal $s$, let $p_s := p^D$ if $s$ belongs to a unique $D \in \mathcal{D}$, and let $p_s := p^A$ if $s$ belongs to a unique $A \in \mathcal{A}$. Define $\mu'_H$ by $\mu'_H(s, t) := d_{K'_H}(p_s, p_t)$ for $s, t \in S$. Then $(K'_H, \{p_s\}_{s \in S})$ is a realization of $\mu'_H$.

**The third F-complex $K''_H$.** In the construction of $K_D$ in $K_H$ above, relabel $(p^A, q^A, p^D)$ by $(p^A_D, q^A_D, p^A_D)$. For each anticlique $A$ including $D$, attach squares $p^A_Dq^A_Dp^A_D$ and $p^D_Aq^A_Dp^D_A$ to $K_D$. Also attach to $K_D$ the $K_{2*}$-folder, denoted by $F'_D$, consisting of triangles $p^D_Ap^A_Dp^A_D$ over all anticliques $A$ including $D$. Replace $F'_D$ by a square-folder if $F'_D$ is a $K_{2*}$-folder. Glue $K_D$ over all $D \in \mathcal{D}$. For each $A \in \mathcal{A}_0$, attach series of three legs $p^A_Dq^A_A, q^A_Ap^A_A, p^A_A$ to $p^D$. Let $K''_H$ be the resulting complex. See Figure 29 (b). Clearly $K''_H$ is also an orientable F-complex; see Figure 29 (c) for an admissible orientation.

Let $R^*_s$ be the union of hypotenuses $p^Dp^A_D$ if $s$ belongs to a unique $D \in \mathcal{D}$ and let $R^*_s$ be the vertex $p^A$ if $s$ belongs to a unique $A \in \mathcal{A}$. Then define $\mu''_H$ on $S$ as $\mu''_H(s, t) := d_{K''_H}(R^*_s, R^*_t)$ for $s, t \in S$. Again $(K''_H, \{R^*_s\}_{s \in S})$ is a realization of $\mu''_H$.
Example. We consider complement-triangle-free commodity graphs as a class of commodity graphs having a simpler construction. A commodity graph $H$ is called complement-triangle-free if the complement $\overline{H}$ has no triangle $K_3$. Such a commodity graph has property P since every anticlique has cardinality at most 2. In this case, the construction of $K^e_H$ is quite simple; $H$ is the subdivision of $H$. Figure 30 illustrates three F-complexes $K^e_H$, $K^s_H$, and $K^c_H$ for a complement-triangle-free commodity graph $H$.

**Summand relation between $K_H$, $K^s_H$ and $K^c_H$.** These F-complexes and the corresponding weights are in a relation of summands (Sections 2.1.5 and 2.2.3). Observe that $(K^c_H)_p$ and $(K^c_H)_p$ belong to distinct orbits; see Figure 29, where the black and the white arrows indicate different orbits. Let $U$ be the union of the orbits meeting $(K^c_H)_p$. Then $(K^c_H)/U = K^s_H$ and $(K^c_H)\setminus U = K^c_H$. Also $(R^s_p)/U = \{p_s\}$ and $(R^s_p)\setminus U = R^s_p$. Thus Proposition 2.6 implies the following.

**Theorem 7.1.** Both $\mu_H$ and $\mu^s_H$ are summands of $\mu^c_H$, and thus any optimal multifold to $\mu^c_H$-MFP is also optimal to both $\mu_H$-MFP and $\mu^s_H$-MFP.

This locking property is known in Lomonosov [24] for special commodity graphs.

**Algorithmic implication: Proof of Theorem 1.7.** The sizes of these F-complexes are bounded:

$$(7.2)\quad K_H, K^s_H \text{ and } K^c_H \text{ have } O(|S|^2) \text{ cells.}$$

Indeed, $A_0 \cup D$ is a subpartition of $S$. This implies $|A_0| + |D| = O(|S|)$. Also $\{A \setminus D \mid D \subseteq A \in A\}$ for $D \in D$ is a disjoint family. This implies that there exist at most $O(|S|)$ anticliques containing $D \in D$. Thus the number of cells in $K_D$ is $O(|S|)$, and consequently we have (7.2).

By the general theorem, Theorem 5.4, there exists a strongly polynomial time to find a $1/12$-integral optimal multifold in every inner Eulerian graph for $\mu_H$, $\mu^c_H$, and $\mu^c_H$-MFP. Note that there is no improper terminal in this case. This implies Theorem 1.7 in Introduction.

7.2 Local sparsity of $K^c_H$ and anticlique-bipartite commodity graphs

We show that the local sparsity of $K^c_H$ is equivalent to the classical anticlique-bipartite condition [22]. We first note that the following sparse/nonsparse properties of $K^c_H$:
(7.3) (0) \( r^A, q^A, \) and \( r^A \) are sparse if \( A \in \mathcal{A}_0. \)

(1) \( p^O_D \) and \( p^A_D \) are sparse.

(2) \( p^D \) is not sparse if \( D \) is the intersection of at least three anticliques.

(0) is obvious. (1) can be seen from Figure 29 (c), where the black and the white arrows indicate distinct orbits. (2) follows from the fact that all edges incident to \( p^D \) belong to a common orbit (by \( K_{2,s} \)-folders around \( p^D \)).

A commodity graph \( H \) is said to be loose if it satisfies:

\begin{equation}
(7.4) \quad \text{for every triple } A, B, C \in \mathcal{A}, \text{ at least one of } A \cap B, B \cap C, C \cap A \text{ is empty.}
\end{equation}

This condition, due to [22], is stronger than property P. So a loose commodity graph has property P. This condition is equivalent to that each \( D \in \mathcal{D} \) is the intersection of two anticliques. Geometrically, this condition says that there is no \( K_{2,m} \)-folder for \( m \geq 2 \) in the three F-complexes. In particular, \( (K^r_H)_{D^O} (= K^s_H) \) consists of square-folders, each of which meets two distinct orbits. Hence \( p^O_D \) is sparse in \( K^r_H \), and consequently \( K^s_H \) is locally sparse.

A loose commodity graph \( H \) is called anticlique-bipartite if the intersection graph of \( \mathcal{A} \) is bipartite, and otherwise it is called anticlique-nonbipartite. The complement \( \overline{C_n} \) of \( n \)-cycle \( C_n \) \((n \geq 4)\) is loose, and \( \overline{C_n} \) is anticlique-bipartite if and only if \( n \) is even. Figure 31 illustrates \( K^r_{C_n} \) for the case \( n = 5, 6. \)

Each \( p^D \) may be sparse or nonsparse. Trace the orbit starting from \( p^A q^A \), as in Figure 31. If \( H \) is anticlique-bipartite, then this orbit never meets \( q^A r^A \) (by the bipartiteness), and hence \( p^D, q^A_D, \) and \( q^A \) are sparse; all vertices are sparse. On the other hand, if \( H \) is anticlique-nonbipartite, then for some \( D \) the orbit returns to \( q^A r^A \), and hence \( p^D, q^A_D, \) and \( q^A \) are not sparse.

**Theorem 7.2.** \( H \) is anticlique-bipartite if and only if \( K^s_H \) is locally sparse.

By virtue of this characterization, we can derive, as a corollary of Theorem 6.4, the following fundamental result; see also [8]. Here each \( R_s \) is a path of hypotenuses or a single vertex; each terminal is strong in the sense of the previous section.

**Theorem 7.3 ([15, 22, 24]).** If \( H \) is anticlique-bipartite, then \( \mu_H \)-MFP has an integral optimal multiflow for every inner Eulerian graph.

It is known that the integrality theorem fails for anticlique-nonbipartite commodity graphs. Nevertheless Karzanov-Lomonosov [22] proved that the half-integrality theorem still holds.

**Theorem 7.4 ([15, 22, 24]).** If \( H \) is anticlique-nonbipartite, then \( \mu_H \)-MFP has a half-integral optimal multiflow for every inner Eulerian graph.

We will prove this theorem as an immediate consequence of the fractionality relation between \( \mu^s_H \) and \( \mu_H \) in the next section.

### 7.3 Fractionality relation and its consequences

The *fractionality relation*, which is the main result in this section, says that \( 1/k \)-integrality of \( \mu^s_H \)-MFP guarantees \( 1/(2k) \)-integrality of \( \mu_H \)-MFP.

**Theorem 7.5.** Let \( H \) be a commodity graph with property P. Suppose that \( \mu^s_H \)-MFP has a \( 1/k \)-integral optimal multiflow for every inner Eulerian graph.
Figure 31: $K^*_H$ for $H = \overline{C_5}$ and $\overline{C_6}$

(1) If $k$ is even, then $\mu_H$-MFP has a $1/k$-integral optimal multflow for every inner Eulerian graph.

(2) If $k$ is odd, then $\mu_H$-MFP has a $1/(2k)$-integral optimal multflow for every inner Eulerian graph.

In particular, $\frac{H}{2} \leq \frac{H^*}{\mu_H}$ holds.

The proof is given in Section 7.4. Here we describe consequences. Theorem 7.4 immediately follows from this theorem and the integrality of $\mu^*_H$-MFP. The integrality of $\mu^*_H$-MFP follows from the multflow locking theorem [22] or, in our framework, by Theorem 6.4 and the local sparsity of $K^*_H$, which consists of square-folders.

Consider $H = K_2 + K_r$, i.e., the vertex-disjoint union of a single edge and complete graph $K_r$ ($r \geq 3$), which is complement-triangle-free. Then $\Pi_H$ is the subdivision of $K_{2,r}$ and, $\mu^*_H$ is the graph metric $d_{K_{2,r}}$ of $K_{2,r}$. Thus $\mu^*_H$ admits an integral optimal multflow by Karzanov-Manoussakis integrality theorem [23] or, in our framework, by Theorem 6.4 and the local sparsity of $K^*_H$, which is the subdivision of a single folder. Hence Theorem 7.5 implies the following.

Theorem 7.6 ([20] for $r = 3$ and [25] for $r > 3$). If $H = K_2 + K_r$, then $\mu_H$-MFP has a half-integral optimal multflow for every inner Eulerian graph.

A commodity graph $H$ with property $P$ is called sparse if $K^*_H$ is locally sparse. By Theorems 6.3 and 7.5, we have the following, which includes the two theorems above.

Theorem 7.7. If $H$ is sparse, then $\mu_H$-MFP has a half-integral optimal multflow for every inner Eulerian graph.

The commodity graph $H$ in Figure 30 is sparse, and hence the half-integrality result holds for this $H$. A sparse commodity graph can be easily characterized by the following observation: $K^*_H$ is locally sparse if and only if it has no adjacent pair of $K_{2,3}$-folders. This characterization can be rephrased in terms of $A$ as follows.

Proposition 7.8. A commodity graph $H$ with property $P$ is sparse if and only if $H$ has no five anticliques $A_1, A_2, B, C_1, C_2$ with $\emptyset \neq A_1 \cap A_2 = A_2 \cap B = A_3 \cap B \neq C_1 \cap C_2 = C_2 \cap B = C_3 \cap B \neq \emptyset$.

Again Theorem 6.4 enlarges the class of commodity graphs admitting the half-integrality property. A commodity graph $H$ with property $P$ is called sparsible if $K^*_H$ is a summand of a locally sparse $F$-complex.
Theorem 7.9. If $H$ is sparsible, then $\mu_H$-MFP has a half-integral optimal multflow for every inner Eulerian graph.

We give an example of sparsible commodity graph $H$ together with a blow-up of $K^*_{\overline{H}}$ in Figure 32; the half-integrality theorem holds for this commodity graph. However, we do not know any nice characterization of a sparsible commodity graph.

A commodity graph $H$ with property P is called weakly-integral if $s_{\overline{H}}$-MFP has an integral optimal multflow for every inner Eulerian graph. Obviously, by Theorem 7.5, the half-integral theorem holds for weakly-integral commodity graphs. Thus we have the following hierarchy:

- loose \subset sparse \subset sparsible \subset weakly-integral \subset property P.

The vertex-disjoint union of two triangles $H_{3,3} := K_3 + K_3$ is a typical nonintegral example. This implies that $H_{3,3}$ is not sparsible. One can directly see the nonsparsibility from $\Pi_{H_{3,3}}$, which is the subdivision of $K_{3,3}$ (Figure 27). We do not know whether sparsible = weakly-integral holds or not.

Let us rephrase these results by using the notion of the fractionality $\text{frac}(H)$. The commodity graphs of fractionality 1 or 2 have already been classified by Karzanov [16, 21] as follows:

1. $\text{frac}(H) = 1$ if and only if $H$ is a complete bipartite graph.
2. $\text{frac}(H) = 2$ if and only if $H$ is $K_2 + K_3$ or anticlique-bipartite (not complete bipartite).

Other commodity graphs have fractionality at least 4. Combining this classification with our results, we get the following.

Corollary 7.10. A sparse/sparsible/weakly-integral commodity graph that is neither anticlique-bipartite nor $K_2 + K_3$ has fractionality 4.

Our proof of Theorem 7.5 is based on SPUP framework, and constructs algorithmically a half-integral optimum in $\mu^e_{\overline{H}}$-MFP from an integral optimum in $\mu^i_{\overline{H}}$-MFP. An integral optimum of $\mu^i_{\overline{H}}$-MFP is obtained by splitting-off if its existence is guaranteed (Section 5.4.2). As a by-product we obtain the following.

Theorem 7.11. Suppose that $H$ is sparse/sparsible/weakly-integral. Then there exists a strongly polynomial time algorithm to find a half-integral optimal multflow in $\mu_{\overline{H}}$-MFP for every inner Eulerian graph.
7.4 Proof of the fractionality relation

Here we give an algorithmic proof of the fractionality relation (Theorem 7.5) according to the SPUP framework (Section 3).

7.4.1 Preliminary: terminal creations

As a preliminary, we introduce terminal creation techniques under an optimal potential \( \rho \). This technique works for general DLP\((K; \{R_s\}_{s \in S})\). We assume that \( G \) has unit capacity.

**Terminal creation I.** Suppose that there are an edge \( e = xy \) and a folder \( F \) such that \((x) \) and \((y) \) are distinct vertices in \( F \), nonadjacent by a leg; see Figure 33. In this case, we can make the following change on \( \mu, S, G, \rho \).

Delete edge \( xy \), add new terminals \( s_x, s_y \), and add new edges \( xs_x, ys_y \). Set \( R_{s_x} := \{p\} \) and \( R_{s_y} := \{p'\} \), and extend \( \mu \) to \( S \cup \{s_x, s_y\} \) by \( \mu(s_x, t) := d(R_{s_x}, R_t) \) and \( \mu(s_y, t) := d(R_{s_y}, R_t) \). Extend \( \rho \) by \( (\rho(s_x), \rho(s_y)) := (p, p') \).

Take an optimal multiflow \( f = (P; \kappa) \) for the original problem. For each path in \( P(xy) \), delete \( xy \) to split it into two paths, add edge \( xs_y \) to one of the two paths having \( x \), and add edge \( ys_x \) to the other path. Then we obtain a multiflow and a potential for the new problem. Both are optimal. Indeed, the saturation condition holds, and the new paths are all geodesic by Lemmas 4.1 and 4.2; consider \((K^2)_p\) for the center \( p^* \) of folder \( F \).

Conversely, take an arbitrary optimal multiflow \( f = (P; \kappa) \) to the new problem. Take a pair \( (P', P'') \in P(xs_y) \times P(ys_x) \), and concatenate \( P' \) and \( P'' \) by deleting edges \( xs_y \) and \( ys_x \) and by adding edge \( xy \) to get a path in \( P(xy) \). Repeating this concatenation, we obtain a multiflow \( f' \) in the original graph of fractionality \( \kappa \). Again all the new paths satisfy the geodesic condition by Lemmas 4.1 and 4.2. Hence \( f' \) is optimal.

**Terminal creation II.** Let \( p \) be a vertex in \( K \) such that \( p \) is incident to four vertices \( q_1, q_2, r_1, r_2 \) by legs and \( \Pi_p \) is an 8-cycle \( (q_1, l_{11}, r_1, l_{21}, q_2, l_{22}, r_2, l_{12}) \), as in Figure 34. Suppose that there is an edge \( e = xy \in E \) with \( (\rho(x), \rho(y)) = (q_1, q_2) \). In this case, we can make the following change on \( \mu, S, G, \rho \).

Subdivide \( e \) into two edges \( xz \) and \( zy \), add new terminals \( s \) and \( t \), and join them to \( z \). Set \( R_s := \{r_1\} \) and \( R_t := \{r_2\} \), and extend \( \mu \) to \( S \cup \{s, t\} \). Also extend \( \rho \) by \( (\rho(s), \rho(z), \rho(t)) := (r_1, p, r_2) \).
Take an optimal multiflow \( f = (\mathcal{P}; \kappa) \) for the original problem. We can extend \( f \) for the new graph by subdividing each path in \( \mathcal{P}(e) \) at \( z \) and adding \((s, t)\)-paths of two edges \( sz \) and \( zt \) so that \( f^{sz,zt} = 1 \). Then the resulting \( g \) and \( f \) are both optimal.

Conversely, take an arbitrary optimal multiflow \( f = (\mathcal{P}; \kappa) \) in the new problem. We can construct an optimal multiflow for the original problem by the following way. Since four edges incident to \( z \) are all saturated, we have \((f^{xz,zs}, f^{xz,zy}, f^{xz,zt}) = (f^{tz,zy}, f^{zs,zs}, f^{sz,zy})\). If \( f^{xz,zs} = f^{xz,zy} = f^{xz,zt} = f^{sz,zy} = 0 \), then the deletion of \( \mathcal{P}(sz,zt) \) gives an optimal multiflow (of fractionality \( \kappa \)) in the original problem. Suppose that \( f^{xz,zs} = f^{xz,zy} > 0 \). Then \( \mathcal{P}(xz, zs) \) is a \(([q_1] \cup [l_1], xzs, [r_1])\)-set, and \( \mathcal{P}(xz, zs) \) is an \(([r_2], xzs, [q_2] \cup [l_2])\)-set. Reconnect paths from \( s \) and paths from \( t \), and reconnect paths from \( x \) and paths from \( y \). Then the local geodesic condition (Section 4.1) is kept, and thus we can make \( f \) satisfy \( f^{xz,zs} = f^{xz,zy} = f^{xz,zt} = f^{sz,zy} = 0 \) (while keeping the optimality), and we get an optimal multiflow of fractionality \( \kappa \) in the original graph.

7.4.2 Proof of Theorems 7.5 and 7.11

We reduce \( \mu_{H} \)-MFP and DLP\( (\mathcal{K}^g_{r}: \{R^r_e\}_{e \in S}) \) to \( \mu_{H} \)-MFP and DLP\( (\mathcal{K}^g_{r}: \{p_s\}_{s \in S}) \) (thanks to Theorem 7.1). Let \( G \) be an inner Eulerian graph with terminal set \( S \). There is no improper terminal. We may assume that \( G \) has unit capacity, also in the algorithmic sense explained in Section 5.4.2.

0. Let us construct the SPUP scheme for \( \mathcal{K}^g_{r} \), as in Section 3.4. The forward orientation is a unique orientation such that \( p^O, p^D, \) and \( q^A \) are sources; see Figure 29 (c). Then \( p^O_H \) and \( p^D_H \) are sinks, and are sparse by (7.3) (1). For an optimal potential \( \rho \), partition \( V \) into three subsets \( S_\rho, M_\rho, \) and \( C_\rho \):

\[
S_\rho := \{x \in V \mid \rho(x) = p^A_D, \text{or } p^D_D\}, \\
M_\rho := \{x \in V \mid \rho(x) = r^A_D, q^A_D, \text{or } p^A\}, \\
C_\rho := \{x \in V \mid \rho(x) = p^D, q^A, \text{or } p^O\}.
\]

Recall the restricted Eulerian condition; each inner node not in \( S_\rho \) has an even degree. Then all properties for \( S_\rho, M_\rho, C_\rho \) in Section 3.4 hold. For example, by Theorem 6.2,
claim (A) holds. Also claim (B) holds since \( p^D, q^A_D \) and \( p^A \) have the same local orbit structure as the midpoint of \( K^2 \) (Theorem 4.3 (2) is applicable).

Let \( C^D_\rho := \rho^{-1}(p^D) \), \( C^A_\rho := \rho^{-1}(q^A) \), and \( C^D_\rho := \rho^{-1}(p^D) \). Then \( C_\rho \) is the union of \( C^D_\rho \), \( C^A_\rho \), and \( C^D_\rho \) over all \( D, A \). Since the folder structures around \( p^D \) and \( q^A \) are rather special, we do not need claim (C) to make both \( C^D_\rho \) and \( C^A_\rho \) empty while keeping the restricted Eulerian condition, which we will show below.

By the degree reduction (Section 3.3), we modify \( G \) so that each inner node has degree four and each terminal has degree one. We may assume that there exists no splittable fork. Then there is no inner node \( x \) with \( \rho(x) = p^A, q^A \) or \( r^A \) for \( A \in A_0 \) by the sparsity (7.3) (2) and Theorem 6.2.

1. First we repeat applying the forward 1-SPUP until \( C^D_\rho \) is empty for all \( D \in \mathcal{D} \).

Take \( D \in \mathcal{D} \) with \( C^D_\rho \neq \emptyset \). We may assume that there is an edge \( xy \) with \( y \in C^D_\rho \) and \( x \notin C^D_\rho \). Consider the gate of \( \rho(x) \) at \((K^2)_{\rho^D} \), which is (i) \( p^D_\rho \), (ii) \( q^D_\rho \), or (iii) \( p^D_\rho \).

First consider case (i). Suppose that \( \rho(x) = q^A_D \), i.e., \( x \in M_\rho \). According to claim (B) there is a fork \( \tau \) at \( x \) such that its critical neighbor \( \rho^{\prime} \) is forward. If \( \alpha_\tau = 1 \), then \( \{\rho^{\prime}(y'), \rho^{\prime}(y)\} = \{p^D_\rho, p^D_\rho\} \), and thus 1-SPUP for \((\tau, \rho^{\prime})\) succeeds. If \( \alpha_\tau = 0 \), then we can replace \( \rho \) by its optimal forward neighbor \( \rho^{\prime} \) with \( \rho^{\prime}(x) = p^D_\rho \) or \( p^A \) by (4.3)).

Therefore we can decrease the number of edges in case (ii). So suppose \( \rho(x) \neq q^D_\rho \).

By the edge-subdivision (Section 2.2.1), we may assume that \( \rho(x) = q^A \). Here we use the terminal creation II. Subdivide \( xy \) into \( xz \) and \( yz \). Add two new terminals \( s \) and \( t \) joined to \( z \). Set \( R_s := \{p^A_s\} \) and \( R_t := \{p^A_t\} \). Extend potential \( \rho \) for the new problem by \( (\rho(s), \rho(z), \rho(t)) := (p^A_s, q^A, p^A_t) \). For a fork \( \tau := (yz, z, zs) \), consider \( \alpha_\tau \). If \( \alpha_\tau = 2 \), then split off \( \tau \). Consider the case of \( \alpha_\tau < 2 \). Take a critical neighbor \( \rho^{\prime} \) of \( \rho \) with respect to \( \tau \). We show that \( \rho^{\prime} \) is forward. Consider an optimal multiflow \( f \) for \( G^{\tau, \alpha_\tau} \).

Since \( f^{zz} = f^{tt} = 1 \) (by the saturation condition), \( \mathcal{P}(e^{\tau}) \) contains paths \( P_s \) and \( P_t \) such that \( P_s \) connects \( s \) and \( P_t \) connects \( t \). If \( \rho^{\prime} \) is backward, then \( \{\rho^{\prime}(y), \rho^{\prime}(y')\} = \{p^D_\rho, q^A\} \) or \( q^A \) or \( q^A \). Since \( (\rho^{\prime}(s), \rho^{\prime}(t)) := (p^D_\rho, q^A) \), \( \rho^{\prime}(P_s) \) passes through \( q^A \) to \( u \to p^A_D \) and \( \rho^{\prime}(P_t) \) passes through \( q^A \) to \( u \to p^A_D \). Then one of \( \rho^{\prime}(P_s) \) and \( \rho^{\prime}(P_t) \) is not geodesic. A contradiction to the optimality. Thus \( \rho^{\prime} \) is necessarily forward. As in claim (B), if \( \alpha_\tau = 1 \), then \( \{\rho(z), \rho(z^{\tau})\} = \{p^A_D, p^A_D\} \), and apply 1-SPUP for \((\tau, \rho^{\prime})\). If \( \alpha_\tau = 0 \), then we can replace \( \rho \) by a forward neighbor \( \rho^{\prime} \) with \( \rho^{\prime}(z) \in \{p^A_D, p^A_D\} \). In this way, we can decrease the number of edges in case (ii).

Consider case (iii). We may assume \( \rho(x) = p^A_D \). Otherwise, subdivide \( xy \) and extend \( \rho \) by defining the potential of the new node as \( p^A_D \); see (1.1). If \( y \) is a terminal (of degree one), then replace \( \rho(y) \) by \( p^A_D \); this keeps the feasibility and the optimality. So suppose that \( y \) is an inner node. Take a fork \( \tau \) at \( y \) with \( 0 < \alpha_\tau < 2 \) (by Lemma 4.4 (1)), and consider a critical neighbor \( \rho^{\prime} \) of \( \rho \) with respect to \( \tau \). Then \( \rho^{\prime} \) is necessarily forward.

So \( \rho^{\prime}(x) = p^A_D \). Consider an optimal multiflow \( f = (\mathcal{P}; \kappa) \) for \( G^{\tau, \alpha_\tau} \) and take a path \( P \) from \( \mathcal{P}(xy^\tau, y^\tau y) \neq \emptyset \). Then \( \rho^{\prime}(P) \) passes through \( p^A_D \to \rho^{\prime}(y^\tau) \to \rho^{\prime}(y) \), which must be geodesic. Also \( d^D(e^{\tau}) > 1 \) is necessary (otherwise \( \alpha_\tau = 2 \)). Thus the possible configurations of \( \rho^{\prime} \) are (2a) \( (\rho^{\prime}(y'), \rho^{\prime}(y)) = (p^D_\rho, p^D_\rho) \) and (2d) \( (\rho^{\prime}(y'), \rho^{\prime}(y)) = (p^D_\rho, p^D_\rho) \). If all three forks at \( y \) have a critical neighbor in case (2a), this contradicts Theorem 4.3 (1).

Thus there exists a fork \( \tau \) having a critical neighbor in case (2d). Then \( \alpha_\tau = 1 \), and both \( \rho^{\prime}(y') \) and \( \rho^{\prime}(y) \) fall into \( S_\rho \). Apply 1-SPUP. In this way, we can decrease the number of edges in (iii).

So suppose that all edges entering \( C^D_\rho \) from the outside are in case (i). Then there is no flow connecting a terminal \( s \) in \( C^D_\rho \). Indeed, take an arbitrary edge \( xy \) with \( y \in C^D_\rho \neq x \). By edge-subdivision with (1.1), we may assume that \( \rho(x) = p^A_D \) for some
\(A \in \mathcal{A}\). Since \(p_D^s \in R_s\) for each terminal \(s\) in \(C^D_\rho\), the flow on \(xy\) cannot connect \(s\) (by the geodesic condition), and goes out \(C^D_\rho\) through another edge \(x'y'\) with \(y' \in C^D_\rho \neq x'\) and \(\rho(x) = p_D^y\).

Delete all terminals in \(C^D_\rho\) and edges connecting them; at this moment, inner nodes in \(C^D_\rho\) may have an odd degree. Next replace \(\rho(x)\) by \(p_D^D\) for all \(x \in C^D_\rho\). This change keeps the saturation condition and the geodesic condition (for any optimal multiflow in the original graph). Thus the resulting \(\rho\) is also optimal, \((G; \rho)\) is restricted Eulerian, and \(C^D_\rho\) is empty. Apply this procedure until \(C^D_\rho = \emptyset\) for all \(D \in \mathcal{D}\). Next, according to claim (B), apply the forward SPUP to make \(M_\rho\) empty.

2. Second, by using terminal creation \(I\), we decompose the current primal-dual pair of MFP and DLP into two primal-dual pairs; one admitting a half-integral optimal multiflow, and the other realized by a subcomplex \((K_H^e)'\) of \(K_H^e\); see Figure 35.

Consider an edge \(e = xy\) connecting \(y \in C_\rho\) and an inner node \(x \notin C_\rho\). Then \(\rho(y) = p_D^D\) or \(q^A\) for some \(A \in \mathcal{A}\). Consider the gate \(g\) of \(\rho(y)\) at \((K_H^e)_y\rho(y)\), and, by edge-subdivision with (4.1), we may assume that \(\rho(x) = g\). Since \(M_\rho\) is empty, \((\rho(x), \rho(y))\) is a nonadjacent (by leg) pair of some folder. Hence we can apply the terminal creation \(I\) at \((\rho(x), \rho(y))\). Apply this procedure for all such edges. Then \(G\) is separated into two disjoint graphs \(G_0\) and \(G_1\), with terminal sets \(S_0\) and \(S_1\), respectively. Here \(G_1\) consists of edges joining an inner node in \(C^D_\rho\) or in \(C^A_\rho\) for some \(A \in \mathcal{A}\). \(G_0\) consists of the other edges. Recall that each terminal has degree one. So we can consider the multiflow problem for \(G_0\) and for \(G_1\) separately. All inner nodes of \(G_0\) belong to \(S_\rho\).

Each (new) terminal \(s\) in \(C_\rho\) is incident to a single node \(x\) with \(\rho(x) \neq \rho(s)\). Consider \(2G_0\) and apply the degree-1 reduction (Section 3.3) to terminals; this does not produce inner nodes in \(C_\rho\). All inner nodes are splittable by claim (A), and \(2G_0\) has an integral optimal multiflow. Hence \(G_0\) has a half-integral optimal multiflow. Therefore if \(G_1\) has a 1/k-integral optimal multiflow, then the original graph has a 1/k-integral optimal multiflow if \(k\) is even and a 1/(2k)-integral optimal multiflow if \(k\) is odd.

Here terminal region \(R_s\) for \(s \in S_1\) is \(p_D^s\), \(p^A_\rho\), or \(p^A\). Therefore, as in Figure 35, we can delete all cells containing \(p^D\) from \(K_H^e\) for all \(D \in \mathcal{D}\). Then the resulting F-complex \((K_H^e)'\) together with \(\{R_s\}_{s \in S_1}\) is still a realization of \(\mu\) restricted to \(S_1\). Therefore we may consider DLP on \((K_H^e)'\). In \((K_H^e)'\), legs \(p^Aq^A\) and \(q^Aq^A\) belong to distinct orbits, and hence \(q^A\) is now sparse. So include \(\rho^{-1}(q^A)\) in \(S_\rho\); claim (A) holds. Moreover \(q^A\) has the same local orbit structure as that of the midpoint of legs in \(K^2\). We can apply claim (B) to sweep out inner nodes from \(C^A_\rho\) to \(S_\rho\), while keeping restricted Eulerian condition (for new \(S_\rho\)).
3. Finally, we decompose the current MFP for $G_1$ into two MFPs, one admitting a half-integral optimal multflow, and the other being $\mu_H^*-\text{MFP}$. This completes the reduction from $\mu_H^*-\text{MFP}$ to $\mu_H^*-\text{MFP}$.

Take an edge $e = xy$ with $x \not\in C^O_\rho$ and $y \in C^O_\rho$. Then $\rho(x)$ is $p^O_\rho$, $q^A_\rho$, $p^A_\rho$, or $p^A$ for some $A, D$. For the first three cases, we can apply the terminal creation I as above. If $\rho(x) = p^A$, then $x$ is necessarily a terminal $s$ with $R_s = \{p^A\}$. Replace $R_s$ by $\{r^A\}$ and $\rho(s)$ by $r^A$, and, accordingly modify $\mu$; this does not change the problem. Again the graph $G_1$ is separated into two disjoint graphs $G'_1$ and $G''_1$. Here $G'_1$ consists of edges joining an inner node having a potential $p^O$, and $G''_1$ consists of the other edges. By the same argument as above, $G''_1$ has a half-integral optimal multflow. As above, we can consider MFP/DLP for $G'_1$ by deleting all cells except $(K_H^\rho)_s$ from $(K_H^\rho)'$. Then the MFP for $G'_1$ is nothing but $\mu_H^*-\text{MFP}$, and has a $1/k$-integral optimal multflow by the assumption. Thus we obtain an optimal multflow in the original graph, which is $1/k$-integral if $k$ is even and $1/2k$-integral if $k$ is odd. The proof of Theorem 7.5 is completed. \(\square\)

It is worth noting that this reduction can be done in strongly polynomial time. If $\mu_H^*-\text{MFP}$ has an integral optimal multflow $f$ (for $G'_1$), then $f$ is obtained in strongly polynomial time (see Section 5.4). Thus the half-integral optimal multflow in the original problem can also be found in strongly polynomial time. This implies Theorem 7.11.

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