The maximum multiflow problems with bounded fractionality

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Abstract

We consider the weighted maximum multiflow problem with respect to terminal weight μ. We show that if the dimension of the tight span associated with μ is at most 2, then this problem has a 1/12-integral optimal multiflow for every Eulerian supply graph. This result solves a weighted generalization of Karzanov's conjecture for classifying commodity graphs with finite fractionality. In addition, our proof technique proves the existence of an integral or half-integrality optimal multiflow for a large class of multiflow maximization problems, and gives a polynomial time algorithm.

1 Introduction

Let $G = (V, E)$ be an undirected graph with integral edge capacity $c : E \to \mathbf{Z}_+$. Let $S \subseteq V$ be a set of terminals. Let H be a simple undirected graph on S, called *commodity graph*. A *multiflow* (*multicommodity flow*) f is a pair $(\mathcal{P}, \lambda)$ of a set $\mathcal{P}$ of (simple) paths connecting the ends of some edge of H and a nonnegative flow-value function $\lambda : \mathcal{P} \to \mathbf{R}_+$ satisfying capacity constraint $\sum_{P \in \mathcal{P}: e \in P} \lambda(P) \leq c(e)$ for $e \in E$. The total flow-value $\|f\|$ of a multiflow $f = (\mathcal{P}, \lambda)$ is defined as $\sum_{P \in \mathcal{P}} \lambda(P)$. The *maximum multiflow problem* with respect to (G, H) is formulated as:

MFP: Maximize $\|f\|$ over all multiflows f for (G, H).

In the case of $H = K_2$, consisting of one edge, MFP is the ordinary (single-commodity) maximum flow problem. The max-flow min-cut theorem, due to Ford-Fulkerson [6], says that there exists an integral maximum flow. In the case of $H = K_2 + K_2$, consisting of two vertex-disjoint edges, MFP is the maximum 2-commodity flow problem. Hu [13] showed that there exists a *half-integral* maximum flow. However, no analogous theorem holds for the 3-commodity flow problem. It is known that there is no positive integer k such that all 3-commodity flow problems have a $1/k$-integral maximum flow. On the other hand, for $H = K_{|S|}$, the complete graph on S, Lovász [26] and Cherkassky [3] independently showed that there exists a half-integer maximum flow.

In this way, the integrality (or half-integrality) property depends crucially on the structure of the commodity graph H. Motivated by this fact, Karzanov [16] defined the *fractionality*, denoted by frac(H), of a commodity graph H as the least positive integer

k such that there exists a $1/k$-integral maximum flow in MFP for every capacitated graph G having H as the commodity graph. If no such positive integer k exists, then $\mathrm{frac}(H)$ is defined to be $+\infty$. The above-mentioned examples show that $\mathrm{frac}(K_2) = 1$, $\mathrm{frac}(K_2 + K_2) = 2$, $\mathrm{frac}(K_n) = 2$, and $\mathrm{frac}(K_2 + K_2 + K_2) = +\infty$. Karzanov [16, 17] posed the following fundamental problem:

Classify the commodity graphs having finite fractionality.

The linear program dual to MFP gives a lower bound of the fractionality $\mathrm{frac}(H)$. The *dual fractionality* $\mathrm{frac}^*(H)$ is defined to be the least positive integer k such that there exists a $1/k$-integral optimum in the LP-dual to MFP for every capacitated graph G having H as the commodity graph. Then the standard TDI argument implies $\mathrm{frac}(H) \geq \mathrm{frac}^*(H)$ [16]. Therefore the finiteness of the dual fractionality is a necessary condition for the finiteness of the (primal) fractionality.

Karzanov [16] gave a necessary and sufficient condition for the finiteness of the dual fractionality, and determined its possible values as follows. A commodity graph H is said to have *property P* if it satisfies the following condition:

(P) For any triple A, B, C of pairwise intersecting maximal stable sets of H, we have $A \cap B = B \cap C = C \cap A$.

Theorem 1.1 ([16]). *For a commodity graph H, we have the following:*

(1) *If H has property P, then* $\mathrm{frac}^*(H) \in \{1, 2, 4\}$.

(2) *If H does not have property P, then* $\mathrm{frac}^*(H) = +\infty$ *and hence* $\mathrm{frac}(H) = +\infty$.

See also [27, Section 73.3b]. Karzanov conjectured that property P is also *sufficient* for the finiteness of primal fractionality, and, more strongly, that the possible values are also $1, 2, 4, +\infty$, as follows.

Conjecture 1.2 ([17]). Suppose that a commodity graph H has property P. Then the following hold:

(1) $\mathrm{frac}(H) < +\infty$,

(2) $\mathrm{frac}(H) \in \{1, 2, 4\}$,

where (1) is the weaker form of the conjecture.

Recently, Theorem 1.1 and Conjecture 1.2 have been extended to a more general setting of the *weighted* maximum multiflow problem. Instead of a commodity graph H, we are given a nonnegative integral terminal weight $\mu : \binom{S}{2} \to \mathbf{Z}_+$, where $\binom{S}{2}$ denotes the set of unordered pairs of elements in S. Then a multiflow f is a pair $(\mathcal{P}, \lambda)$ of a set $\mathcal{P}$ of paths connecting distinct terminals in S and a nonnegative flow-value function $\lambda : \mathcal{P} \to \mathbf{R}_+$ satisfying the capacity constraint. The total flow-value $\|f\|_\mu$ is defined as

$$\|f\|_\mu := \sum_{P \in \mathcal{P}} \mu(s_P, t_P)\lambda(P),$$

where s_P and t_P denote the ends of P. The *μ-weighted maximum multiflow problem* is formulated as:

μ-MFP: Maximize $\|f\|_\mu$ over all multiflows f for (G, S).

If μ is 0-1 valued, then μ-MFP coincides with MFP for the commodity graph H that has an edge st if and only if $\mu(s,t) = 1$.

The *fractionality* $\mathrm{frac}(\mu)$ of a terminal weight μ is defined as the least positive integer k such that μ-MFP has a $1/k$-integral optimal multiflow for every graph, and the *dual fractionality* $\mathrm{frac}^*(\mu)$ is the least positive integer k such that the LP-dual to μ-MFP has a $1/k$-integral optimal solution for every capacitated graph. Again $\mathrm{frac}(\mu) \geq \mathrm{frac}^*(\mu)$ holds.

Karzanov [19] extended Theorem 1.1 concerning commodity graph H to a similar statement for metric-weights, and it was extended further in [10] for general weights. For a terminal weight $\mu : \binom{S}{2} \to \mathbf{Z}_+$, define a polyhedral set T_μ in $\mathbf{R}^S_+$ as

$$T_\mu := \{p \in \mathbf{R}^S \mid p(s) = \max_{t \in S}\{\mu(s,t) - p(t)\}\}, \tag{1.1}$$

where we let $\mu(s,s) = 0$. This polyhedral set T_μ is called the *injective envelope* or the *tight span*, introduced independently by Isbell [14] and Dress [4] for metrics, and considered by [9] for general weights. The dimension $\dim T_\mu$ is defined to be the largest dimension of a face of T_μ.

Theorem 1.3 ([19] for metrics and [10] for general weights)**.** *For a terminal weight μ on S, we have the following:*

(1) *If $\dim T_\mu \leq 2$, then $\mathrm{frac}^*(\mu) \in \{1,2,4\}$.*

(2) *If $\dim T_\mu \geq 3$, then $\mathrm{frac}^*(\mu) = \mathrm{frac}(\mu) = +\infty$.*

The property P of H is equivalent to the 2-dimensionality of the tight span of the corresponding 0-1 weight μ, as is observed in [10, Section 7]. Thus Conjecture 1.2 for primal fractionality is naturally generalized to the following:

Conjecture 1.4. Suppose that a terminal weight μ satisfies $\dim T_\mu \leq 2$. Then the following hold:

(1) $\mathrm{frac}(\mu) < +\infty$.

(2) $\mathrm{frac}(\mu) \in \{1,2,4\}$.

The main result of this paper is an affirmative solution of the weaker statement (1) of this generalized conjecture.

Theorem 1.5. *For a terminal weight μ on S, if $\dim T_\mu \leq 2$, then μ-MFP has a $1/12$-integral optimal multiflow for every Eulerian graph.*

This theorem implies the weaker statement (1) of Conjecture 1.2, and thus completes the classification of terminal weights and commodity graphs having finite fractionality as follows.

Corollary 1.6. *A terminal weight μ has finite fractionality if and only if $\dim T_\mu \leq 2$. A commodity graph H has finite fractionality if and only if H has property P.*

As a consequence of Theorem 1.5, the possible values of the fractionality are restricted to $1, 2, 3, 4, 6, 8, 12, 24$, and $+\infty$. However we know no example of terminal weights having fractionality other than $1, 2, 4, +\infty$.

Our proof is constructive, and gives a strongly polynomial time to find a 1/12-integral optimal multiflow under some assumption.

Theorem 1.7. *For a commodity graph H with property P, there exists a strongly polynomial time algorithm to find a $1/12$-integral optimal multiflow in every inner Eulerian graph.*

Organization. The rest of this paper is divided into three parts. In the first part (Sections 2 and 3), we introduce a duality framework using *folder complexes* (*F-complexes* for short), developed in the previous paper [12], and describe the proof outline of Theorem 1.5. An F-complex is a 2-dimensional cell complex obtained by gluing folders, which appeared in Karzanov [18, 19], and was introduced formally by Chepoi [2, Section 7]. If $\dim T_\mu \leq 2$, then μ can be *embedded into* some F-complex $\mathcal{K}$, and the maximum value of μ-MFP is equal to the minimum value of a *discrete location problem* on $\mathcal{K}$. In Section 2, we introduce the concept of F-complex and its relation to the multiflow duality. Our proof is based on a fractional version of the splitting-off method combined with the dual update, called *SPUP* standing for Splitting-off with Potential Update, which is an effective framework for proving the existence of a $1/k$-integral optimal multiflow for a bounded integer k, devised originally in the previous paper [11] for a special case. In Section 3, we describe the SPUP framework together with the proof outline of Theorem 1.5.

The second part (Sections 4 and 5) is the technical part. In Section 4, we analyze SPUP from the complementary slackness and the geometry of F-complexes. In Section 5, we complete the proof of Theorem 1.5 by showing that the SPUP framework actually works. This also gives a polynomial time algorithm to find a 1/12-integral optimal solution provided the size of F-complex is fixed.

In the third part (Sections 6 and 7), we describe consequences and implications; these sections can be read without the full knowledge of the second part. Our framework not only brings a unified understanding to previously known results but also a powerful algorithmic tool for proving the existence of an integral or half-integral optimal multiflow for Eulerian graphs. In Section 6, we introduce a powerful geometric criterion to show that μ-MFP has an integral optimal multiflow for every Eulerian graph. In Section 7, we concentrate on μ_H-MFP for a commodity graph H with property P. We explicitly construct F-complexes for H, and prove the half-integrality theorem for a large class of commodity graphs, unifying the previous known results [15, 20, 22, 24, 25].

Notation. Let $\mathbf{R}$, $\mathbf{R}_+$, $\mathbf{Z}$, and $\mathbf{Z}_+$ denote the sets of reals, nonnegative reals, integers, and nonnegative integers, respectively. For a set X, let $\mathbf{R}^X$ and $\mathbf{R}_+^X$ denote the sets of functions from X to $\mathbf{R}$ and X to $\mathbf{R}_+$, respectively.

For a graph $G = (V, E)$ with terminal set $S \subseteq V$, each nonterminal node $x \in V \setminus S$ is called an *inner node*. G is endowed with edge-capacity c. The *degree* of node $x \in V$ is the sum of $c(e)$ over all edges e incident to x. By a path we mean a simple path, i.e., it has no repeated nodes. G is said to be *inner Eulerian* if c is integer-valued and each inner node has an even degree. For a positive integer k, kG is the graph (V, E) with edge-capacity kc

A function $d : X \times X \to \mathbf{R}_+$ on a set X is called a *metric* if it satisfies $d(s,t) = d(t,s) \geq d(s,s) = 0$ and the triangle inequalities $d(s,t) + d(t,u) \geq d(s,u)$ for $s,t,u \in S$. For a metric d on X and two subsets $A, B \subseteq X$, the distance $d(A, B)$ between A and B is defined as

$$d(A, B) = \inf\{d(s,t) \mid s \in A, t \in B\}.$$

We denote $d(A, \{p\})$ simply by $d(A, p)$. We often regard a metric d on node set V of graph $G = (V, E)$ as an edge-length $d : E \to \mathbf{R}_+$ by $d(e) := d(x, y)$ for $e = xy$. For a path or a cycle P, $d(P)$ denotes the sum of $d(e)$ over all edges e in P.

We use the notion of a cell complex; see [1, Chapter I.7] for a precise definition. For a cell complex $\mathcal{K}$, a 1-dimensional cell of segment $[p, q]$ is also called an *edge*, denoted by pq. A 0-dimensional cell is called a *vertex*; the set of vertices is denoted by $V(\mathcal{K})$

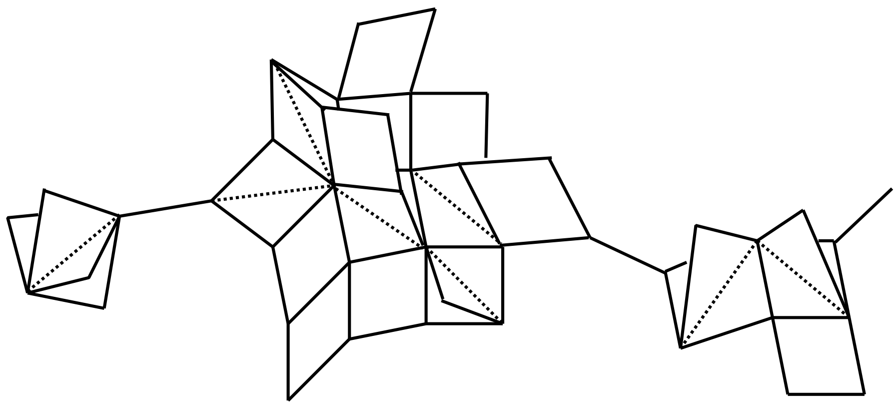


Figure 1: Folder complex

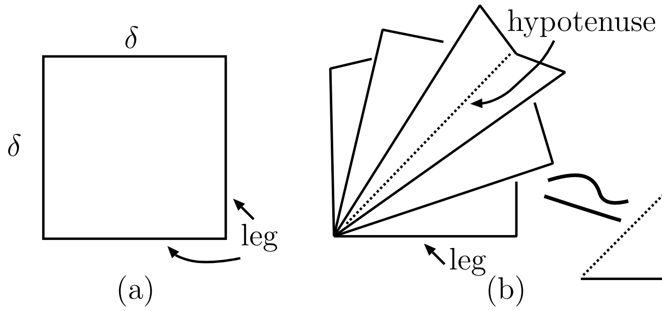


Figure 2: Folders: (a) square-folder and (b) $K_{2,6}$-folder

2 Basics on multiflow combinatorial dualities

As is well-known in the multiflow theory [24], an LP-dual of μ-MFP is an optimization problem over metrics:

$$\text{(2.1)} \qquad \begin{array}{ll} \text{Minimize} & \displaystyle\sum_{e=xy\in E} c(e)d(x,y) \\ \text{subject to} & d: \text{metric on } V \text{ with } d(s,t) \geq \mu(s,t) \text{ for } s,t \in S. \end{array}$$

In the case of $\dim T_\mu \leq 2$, μ can be embedded into some folder complex $\mathcal{K}$, and this embedding gives a combinatorial expression to LP (2.1). A folder complex is a 2-dimensional cell complex obtained by gluing *folders* (under some axiom) as depicted in Figure 1. Folder complex $\mathcal{K}$ is endowed with a metric $d_\mathcal{K}$. If a terminal weight μ is represented as the distances $d_\mathcal{K}(R_s, R_t)$ between certain regions R_s in $\mathcal{K}$ indexed by $s \in S$, then a combinatorial dual problem for μ-MFP takes the form of a discrete location problem on $\mathcal{K}$.

In Section 2.1, we introduce F-complexes and summarize their basic geometric properties. In Section 2.2, we explain a combinatorial duality relation for μ-MFP by F-complexes, and summarize basic facts, including optimality criteria.

2.1 Folder complex

We consider a 2-dimensional cell complex obtained by the following construction. Fix a positive real $\delta > 0$. A cell having an isometry into an isosceles right triangle $\{(x_1, x_2) \in \mathbf{R}^2 \mid 0 \leq x_1 \leq x_2 \leq \delta\}$ in the Euclidean plane will be called a *triangle*, whereas a cell having an isometry to a square $\{(x_1, x_2) \in \mathbf{R}^2 \mid 0 \leq x_1, x_2 \leq \delta\}$ is a *square*.

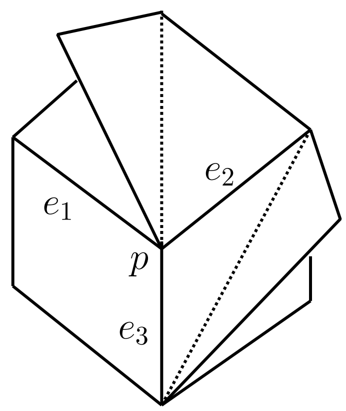


Figure 3: A corner of 3-cube

By a *folder* we mean a square or a cell complex obtained by gluing triangles along the common longer edge. See Figure 2. A square is particularly called a *square-folder*. A folder F is called a $K_{2,m}$-*folder* if F consists of m triangles, and also called a $K_{2,*}$-*folder* if F is a $K_{2,m}$ for some m. A $K_{2,*}$-folder has two types of edges: the (unique) longer edge and shorter edges. Following [2], we call the longer edge the *hypotenuse*, and a shorter edge a *leg*. Any edge of a square-folder is called a leg. A scale parameter δ is called the *leg-length*.

Next we consider a cell complex $\mathcal{K}$ obtained by gluing folders and edges (1-dimensional cell) isometric to segment $[0, \delta]$, which we also call a leg, in such a way that any two of the folders are glued along one leg or at one vertex. Then $\mathcal{K}$ is called a *folder complex* (an *F-complex* for short) [2, Section 7] if it is simply-connected, and satisfies:

Flag condition: there exist no vertex p and three legs e_1, e_2, e_3 incident to p such that e_i and e_j belong to a common folder for $1 \leq i < j \leq 3$.

This condition means that folders should be glued without a *corner of 3-cube* as in Figure 3. A metric on $\mathcal{K}$ is defined as follows. Each 2-dimensional cell (a triangle or a square) has a natural l_1-metric by the isometry to $\mathbf{R}^2$. Then the l_1-length of a path P in $\mathcal{K}$ is the sum, over all cells σ, of the l_1-length of $\sigma^\circ \cap P$ measured by the l_1-metric on σ, where σ° denotes the relative interior of σ. The l_1-*length metric* $d_{\mathcal{K}}(p, q)$ between p and q in $\mathcal{K}$ is defined to be the infimum of the lengths of all paths connecting p and q in $\mathcal{K}$.

We next introduce a certain class of regions in a folder complex $\mathcal{K}$; we will represent μ as the distance between these regions in $\mathcal{K}$. A connected subcomplex R of $\mathcal{K}$ is called *normal* if it satisfies the following axiom:

Boundary axiom: the boundary of R (relative to $\mathcal{K}$) consists of hypotenuses, i.e., if a leg e belongs to R, then every cell containing e belongs to R.

Local convexity: there exists no pair of triangles σ, σ' sharing a leg and a right angle such that $(\sigma \cup \sigma') \cap R$ coincides with the union of the hypotenuses of σ and σ'.

Any normal set is a closed connected set. See Figure 4 for the violation of local convexity. We list several basic concepts of F-complex below.

2.1.1 Admissible orientations and orbits

An F-complex $\mathcal{K}$ is said to be *orientable* if the edge set of $\mathcal{K}$ has an orientation with the property that, for each folder F in $\mathcal{K}$, there is a pair p, q of vertices of F such that

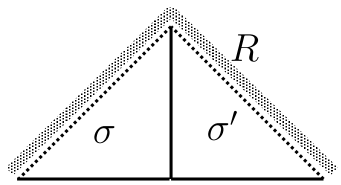


Figure 4: Violation of local convexity

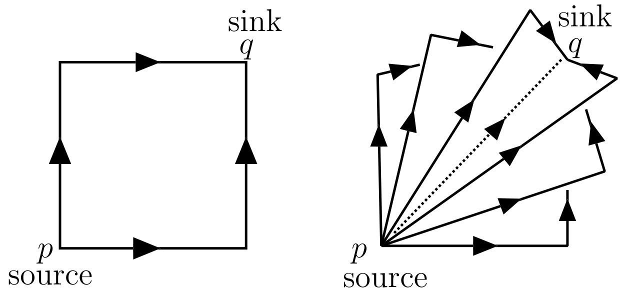


Figure 5: An admissible orientation (restricted to folders)

each edge (leg or hypotenuse) of F enters p or leaves q; see Figure 5. This orientation is called an *admissible orientation*; in fact, an admissible orientation is acyclic. Vertices p and q are particularly called the *source* and the *sink* of F, respectively, with respect to this orientation.

An *orbit* is an equivalence class with respect to the equivalence relation obtained as the transitive closure of the relation $\simeq$ on all edges (legs and hypotenuses) of $\mathcal{K}$ defined by $e \simeq e'$ if e and e' are nonadjacent legs in some square-folder, or belong to a common $K_{2,*}$-folder. An admissible orientation is obtained by orienting orbits independently. Such orientation of an orbit is also said to be *admissible*. See Figure 6. Each orbit has exactly two admissible orientations; one is the reverse of the other.

For an admissible orientation $\overrightarrow{\mathcal{K}}$ of $\mathcal{K}$ and vertices $p, q \in V(\mathcal{K})$, we write $p \succeq_{\overrightarrow{\mathcal{K}}} q$ if $p = q$, $\overrightarrow{pq}$ is an oriented leg in $\overrightarrow{\mathcal{K}}$, or (p, q) is the source-sink pair of some folder with respect to $\overrightarrow{\mathcal{K}}$. Let O be an orbit and let $\overrightarrow{O}$ be an admissible orientation of O. If O contains all edges of a folder F, then $\overrightarrow{O}$ determines the source and the sink of F, as in Figure 5. Similarly, we write $p \succeq_{\overrightarrow{O}} q$ if $p = q$, $\overrightarrow{pq}$ is an oriented leg in $\overrightarrow{O}$, or (p, q) is the source-sink pair of some folder with respect to $\overrightarrow{O}$. Note that relations $\succeq_{\overrightarrow{\mathcal{K}}}$ and $\succeq_{\overrightarrow{O}}$ are not transitive.

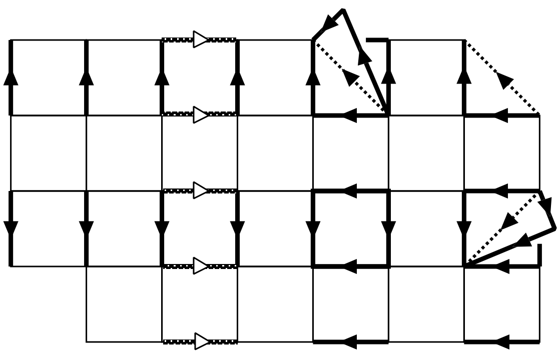


Figure 6: Oriented orbits

2.1.2 Leg-graph

The *leg-graph* $\mathit{\Gamma}$ is the graph on $V(\mathcal{K})$ consisting of all legs (not including hypotenuses). The leg-graph is precisely a *frame* in the sense of [18] (although F-complexes and frames are essentially equivalent, F-complexes are suitable to represent normal regions). We often use the following elementary properties of $\mathit{\Gamma}$, which can easily be verified [12].

(2.2) (1) The leg-graph $\mathit{\Gamma}$ is bipartite.

(2) For normal sets N, M, we have $d_{\mathcal{K}}(N, M) = d_{\mathit{\Gamma},\delta}(N \cap V(\mathcal{K}), M \cap V(\mathcal{K}))$,

where $d_{\mathit{\Gamma},\delta}$ denotes the shortest path metric on the leg-graph with respect to uniform edge-length δ.

2.1.3 Subdivisions

An F-complex $\mathcal{K}$ has a natural subdivision operation. For a positive integer m, subdivide each leg into m legs of length δ/m. Accordingly, subdivide each square into $m \times m$ squares of leg-length δ/m, each triangle into m triangles and $m(m-1)/2$ squares of leg-length δ/m; see [12, Figure 5]. The resulting complex is denoted by $\mathcal{K}^m$, called the *m-subdivision* of $\mathcal{K}$. One can easily see the following facts:

(2.3) $\mathcal{K}^m$ is also an F-complex, and $\mathcal{K}^2$ is always orientable.

See Figure 12 (in Section 3) for verifying the orientability of $\mathcal{K}^2$.

2.1.4 Star-shaped F-complex and neighborhood

An F-complex $\mathcal{K}$ is said to be *star-shaped* if there exists a vertex p such that every maximal cell contains p and no triangle has p as its right angled corner. A star-shaped F-complex will be used in investigating the local structure around vertex p. The *neighborhood* $\mathcal{K}_p$ of p consists of all cells containing p and their faces. Neighborhood $\mathcal{K}_p$ is also an F-complex, and a geodesic subspace of $\mathcal{K}$ with diameter at most 4δ. In particular,

$$d_{\mathcal{K}}(p, q) = d_{\mathcal{K}_p}(p, q) \in \{0, 1, 2, 3, 4\}\delta \quad (p, q \in V(\mathcal{K}_p)). \tag{2.4}$$

(The geodesic property $d_{\mathcal{K}} = d_{\mathcal{K}_p}$ is implicit in [12]. One can verify this property by using the properties of the leg-graph: every 4-cycle belongs to a unique folder [12, (3.6)] and every 6-cycle has a chord; see [12, 18]).

Although $\mathcal{K}_p$ may not be star-shaped, $(\mathcal{K}^m)_p$ for $m \geq 2$ is always star-shaped. Let $\mathit{\Pi}_p$ be the graph obtained by deleting p from the leg-graph of $(\mathcal{K}^m)_p$ for some $m \geq 2$, where $\mathit{\Pi}_p$ is independent of m. See Figure 7. Then the flag condition can be rephrased by the following:

(2.5) $\mathit{\Pi}_p$ has girth at least 8,

where the girth means the shortest length of a (simple) cycle. $\mathit{\Pi}_p$ is a bipartite graph with bipartition $\{L_p, Q_p\}$, where Q_p denotes the set of vertices incident to p by legs in $(\mathcal{K}^m)_p$ and L_p denotes the set of the other vertices.

Even if $\mathcal{K}_p$ is not star-shaped, the leg-graph of $\mathcal{K}_p$ is a subgraph of that of $(\mathcal{K}^m)_p$. Therefore we can naturally regard $V(\mathcal{K}_p) \setminus \{p\}$ as a subset of $L_p \cup Q_p$.

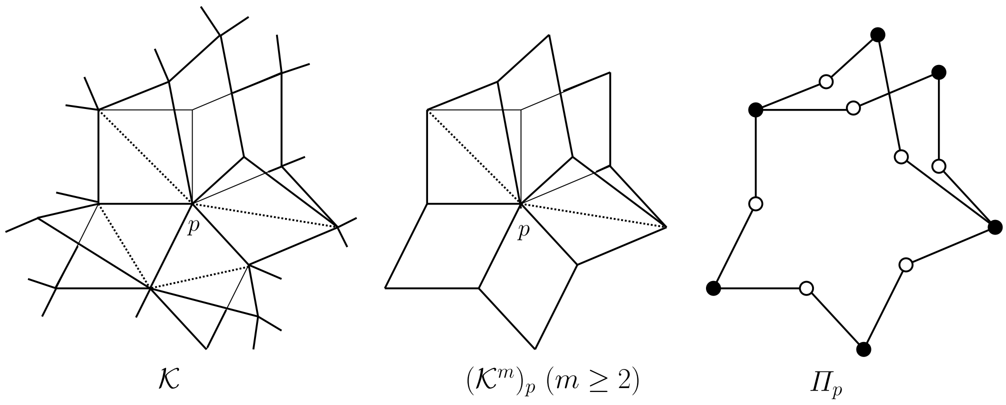


Figure 7: Neighborhood of p

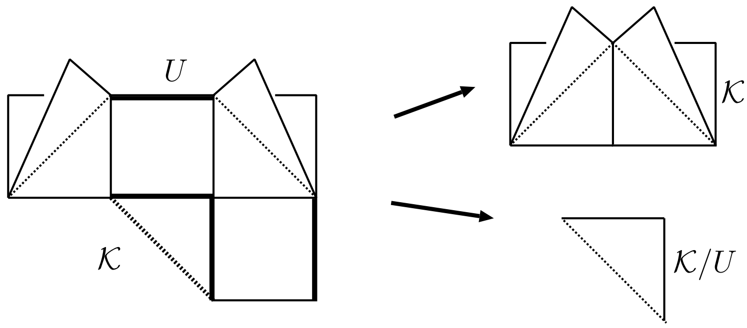


Figure 8: Summands

2.1.5 Orbits and summands

For a (disjoint) union U of several orbits, we can construct a new complex $\mathcal{K}/U$ from $\mathcal{K}$ by contracting each edge *not in* U; see [12] for a more precise construction. Again $\mathcal{K}/U$ consists of folders, and indeed is an F-complex by the next proposition. We call $\mathcal{K}/U$ a *summand* of $\mathcal{K}$. See Figure 8. The contraction naturally induces a map $(\cdot)/U : V(\mathcal{K}) \to V(\mathcal{K}/U)$ by defining p/U to be the contracted vertex. By extending linearly, we obtain a map $(\cdot)/U : \mathcal{K} \to \mathcal{K}/U$. Also define $\mathcal{K}\backslash U$ as the summand $\mathcal{K}/\overline{U}$ for the complement $\overline{U}$ of U, and define map $(\cdot)\backslash U$ as $(\cdot)/\overline{U}$.

Proposition 2.1 ([12, Proposition 3.15]). *Let U be the union of several orbits.*

(1) *$\mathcal{K}/U$ is an F-complex.*

(2) *For a normal set R in $\mathcal{K}$, R/U is also normal in $\mathcal{K}/U$.*

(3) *For normal sets M, N in $\mathcal{K}$, $d_{\mathcal{K}}(M, N) = d_{\mathcal{K}/U}(M/U, N/U) + d_{\mathcal{K}\backslash U}(M\backslash U, N\backslash U)$.*

2.2 F-complex realization and multiflow combinatorial duality

Here we describe a combinatorial duality relation for μ-MFP by an F-complex. For a weight μ on terminal set S, an *F-complex realization* (a *realization* for short) of μ is a pair $(\mathcal{K}; \{R_s\}_{s \in S})$ of an F-complex $\mathcal{K}$ and a family $\{R_s\}_{s \in S}$ of normal sets satisfying

$$\mu(s, t) = d_{\mathcal{K}}(R_s, R_t) \quad (s, t \in S).$$

Namely μ is realized as the distances among normal sets R_s. Figure 9 illustrates an example, where s_7 and s_8 are embedded into regions (R_{s_8} is the shaded region), and the

$\mu =$

	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	1	2	3	3	5	2	1
s_2		1	4	4	4	1	0
s_3			3	3	5	2	1
s_4				2	2	3	1
s_5					2	3	1
s_6						1	0
s_7							0

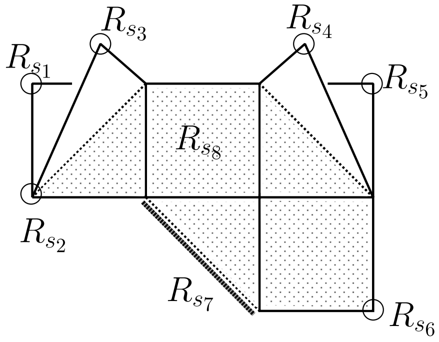


Figure 9: F-complex realization

others are embedded into vertices. It is known that an existence of a realization of μ is characterized by the dimension of the tight span T_μ [12].

Theorem 2.2 ([12, Theorem 4.5])**.** *The following two conditions are equivalent:*

(1) $\dim T_\mu \leq 2$.

(2) *μ has an F-complex realization.*

In fact, a realization of μ can be obtained by subdividing 2-dimensional polyhedral complex T_μ into folders with $\delta = 1/4$ [10].

An F-complex realization enables us to define a combinatorial problem dual to μ-MFP, sharpening LP-dual (2.1). Suppose that a weight μ on S has an F-complex realization $(\mathcal{K}; \{R_s\}_{s \in S})$. We consider the following *discrete location problem* associated with $(\mathcal{K}; \{R_s\}_{s \in S})$:

$$\text{DLP}(\mathcal{K}; \{R_s\}_{s\in S}): \qquad \text{Minimize} \quad \sum_{xy \in E} c(xy) d_{\mathcal{K}}(\rho(x), \rho(y))$$
$$\text{subject to} \quad \rho : V \to V(\mathcal{K}), \quad \rho(s) \in R_s \ (s \in S).$$

Here ρ represents an embedding of the node set V of G into that of $\mathcal{K}$. Our previous paper established the following duality relation, extending a result in [18].

Theorem 2.3 ([12, Theorem 2.1])**.** *Suppose that μ has an orientable F-complex realization $(\mathcal{K}; \{R_s\}_{s\in S})$. Then the maximum value of μ-MFP for (G, S) is equal to the minimum value of DLP$(\mathcal{K}; \{R_s\}_{s\in S})$.*

This theorem guarantees the existence of an optimal metric d in (2.1) represented as $d(x, y) = d_{\mathcal{K}}(\rho(x), \rho(y))$ for a map ρ in DLP$(\mathcal{K}; \{R_s\}_{s\in S})$; see Section 5.4 for a more detailed account of the relationship between DLP and LP-dual. The orientability requirement is not restrictive. By the subdivision operation (Section 2.1.3), we can always make a given F-complex realization orientable. Hence we tacitly assume that an F-complex is always orientable.

A map ρ feasible to DLP$(\mathcal{K}; \{R_s\}_{s\in S})$ is called a *potential*. For a potential ρ, let d^ρ denote the metric on V defined by $d^\rho(x, y) := d_{\mathcal{K}}(\rho(x), \rho(y))$, and let $d^\rho(G)$ denote the objective value $\sum_{e\in E} c(e) d^\rho(e)$ of DLP$(\mathcal{K}; \{R_s\}_{s\in S})$. Let opt$(\mu; G)$ denote the optimal value of μ-MFP, which is equal to the optimal value of DLP$(\mathcal{K}; \{R_s\}_{s\in S})$ by Theorem 2.3. We list several basic properties of μ-MFP and DLP$(\mathcal{K}; \{R_s\}_{s\in S})$ below.

2.2.1 Optimality criterion of primal-dual type

For a multiflow $f = (\mathcal{P}, \lambda)$ and a potential ρ, the duality gap $d^\rho(G) - \|f\|_\mu$ is given as

$$
(2.6)\quad \begin{aligned} d^\rho(G) - \|f\|_\mu &= \sum_{e \in E} c(e) d^\rho(e) - \sum_{P \in \mathcal{P}} \mu(s_P, t_P)\lambda(P) \\ &= \sum_{e \in E} d^\rho(e)(c(e) - f^e) + \sum_{P \in \mathcal{P}} \lambda(P)(d^\rho(P) - d_{\mathcal{K}}(R_{s_P}, R_{t_P})). \end{aligned}
$$

Here f^e denotes the total amount of flows on e, i.e., $f^e = \sum_{P \in \mathcal{P}: e \in P} \lambda(P)$, and note that $\mu(s_P, t_P) = d_{\mathcal{K}}(R_{s_P}, R_{t_P})$ by the definition of the F-complex realization. Hence an optimality criterion of primal-dual type is given as follows.

Lemma 2.4. *A multiflow $f = (\mathcal{P}, \lambda)$ and a potential ρ are both optimal if and only if they satisfy:*

Saturation condition: *for each $e \in E$, $d^\rho(e) > 0$ implies $f^e = c(e)$.*

Geodesic condition: *for each $P \in \mathcal{P}$, $\lambda(P) > 0$ implies $d^\rho(P) = d_{\mathcal{K}}(R_{s_P}, R_{t_P})$.*

The geodesic condition says that paths in f are embedded as shortest paths between terminal regions R_s in $\mathcal{K}$ by ρ. This view is most fundamental in every place of this paper.

2.2.2 Optimality criterion by neighbors

Next we describe an optimal criterion for DLP($\mathcal{K}; \{R_s\}_{s \in S}$) to the effect that: *local optimality implies global optimality.*

A potential ρ' is called a *neighbor* of ρ with respect to an oriented orbit $\overrightarrow{O}$ if $\rho(x) \succeq_{\overrightarrow{O}} \rho'(x)$ for all $x \in V$. See Section 2.1.1 for the notation. Namely ρ' is obtained by *moving part of ρ along the direction $\overrightarrow{O}$*. The following theorem is a basis for the SPUP framework in the next section. By a neighbor of ρ we mean a neighbor with respect to some oriented orbit.

Theorem 2.5 ([12, Theorem 4.1])**.** *A potential ρ is optimal to DLP($\mathcal{K}; \{R_s\}_{s \in S}$) if and only if $d^\rho(G) \leq d^{\rho'}(G)$ holds for every neighbor ρ' of ρ.*

A more relaxed neighbor concept, which will turn out to be useful, can be defined as follows. For an admissible orientation $\overrightarrow{\mathcal{K}}$ of $\mathcal{K}$, a potential ρ' is called a *semi-neighbor* of ρ' with respect to $\overrightarrow{\mathcal{K}}$ if $\rho(x) \succeq_{\overrightarrow{\mathcal{K}}} \rho'(x)$ for all $x \in V$. $\overrightarrow{\mathcal{K}}$ induces an admissible orientation $\overrightarrow{O_i}$ of each orbit O_i $(i = 1, 2, \ldots, m)$ (by restriction). Thus, by definition, a neighbor with respect to $\overrightarrow{O_i}$ is a semi-neighbor with respect to $\overrightarrow{\mathcal{K}}$. It is shown in [12, Section 4.1] that for a semi-neighbor ρ' of ρ with respect to $\overrightarrow{\mathcal{K}}$, there exist neighbors ρ_i of ρ with respect to $\overrightarrow{O_i}$ such that $d^{\rho'} - d^\rho = \sum_i \{d^{\rho_i} - d^\rho\}$. By this property, we can use semi-neighbors instead of neighbors in many places.

2.2.3 Summands and locking property

For a union U of several orbits, let $\mu_{/U}$ be the weight on S defined as $\mu_{/U}(s,t) := d_{\mathcal{K}/U}((R_s)/U, (R_t)/U)$ for $s, t \in S$. Recall Section 2.1.5 for notations. $\mu_{/U}$ is called a *summand* of μ with respect to U. By construction and Proposition 2.1, $(\mathcal{K}/U; \{(R_s)/U\}_{s \in S})$ is a realization of $\mu_{/U}$. Similarly, define $\mu_{\backslash U}$ as $\mu_{\backslash U}(s,t) := d_{\mathcal{K} \backslash U}((R_s) \backslash U, (R_t) \backslash U)$ for $s, t \in S$. Then $(\mathcal{K} \backslash U; \{(R_s) \backslash U\}_{s \in S})$ is a realization of $\mu_{\backslash U}$.

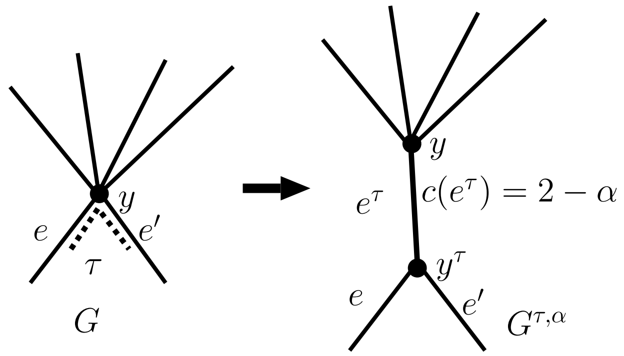


Figure 10: Construction of $G^{\tau,\alpha}$

Proposition 2.6. *Let f be an optimal multiflow and ρ an optimal potential. For a union U of several orbits, we have the following:*

(1) *f is optimal to $\mu_{/U}$-MFP and $\mu_{\backslash U}$-MFP.*

(2) *ρ/U and $\rho\backslash U$ are optimal to DLP$(\mathcal{K}/U; \{(R_s)/U\}_{s\in S})$ and DLP$(\mathcal{K}\backslash U; \{(R_s)\backslash U\}_{s\in S})$, respectively.*

Proof. ρ/U and $\rho\backslash U$ are feasible to DLP$(\mathcal{K}/U; \{(R_s)\backslash U\}_{s\in S})$ and DLP$(\mathcal{K}\backslash U; \{(R_s)\backslash U\}_{s\in S})$, respectively. By Proposition 2.1 (3), we have $\|f\|_\mu = \|f\|_{\mu_{/U}} + \|f\|_{\mu_{\backslash U}}$ and $d^\rho = d^{\rho/U} + d^{\rho\backslash U}$. Thus we have $\|f\|_\mu = \|f\|_{\mu_{/U}} + \|f\|_{\mu_{\backslash U}} \leq d^{\rho/U}(G) + d^{\rho\backslash U}(G) = d^\rho(G) = \|f\|_\mu$. □

This explains the *locking property* of multiflows, which means the existence of a multiflow simultaneously optimal to several μ-MFPs.

3 Proof outline: SPUP framework

In this section, we explain the proof outline of Theorem 1.5, which is a kind of a primal-dual algorithm by a fractional version of the splitting-off and the dual update. We call it *SPUP*, standing for *Splitting-off with Potential UPdate*.

3.1 SPUP (Splitting-off with Potential UPdate)

We begin with the splitting-off operation. Let G be a graph. For two consecutive edges $e = xy$ and $e' = yz$ of unit capacity incident to node y, a triple $\tau = (e, y, e')$ is called a *fork*. The *splitting-off operation* is to delete edges e, e' and to add a new edge of unit capacity connecting x and z if $x \neq z$. If the splitting-off operation does not decrease the optimal flow-value opt$(\mu; G)$, then a $(1/k$-)integral optimal multiflow in the original graph can be recovered from any $(1/k$-)integral optimal multiflow in the new graph. Such a fork is called *splittable*. If a fork τ is not splittable, then τ is called *unsplittable*.

We next introduce the fractional splitting-off operation. For a fork $\tau = (e, y, e')$ and $\alpha \in [0, 2]$, the graph $G^{\tau,\alpha}$ is obtained by adding a new node y^τ, reconnecting e and e' to y^τ, and joining y and y^τ by a new edge $e^\tau = yy^\tau$ of capacity $c(e^\tau) = 2-\alpha$; see Figure 10. The resulting graph is denoted by $G^{\tau,\alpha}$. In the case of $\alpha = 0$, the problems on G and on $G^{\tau,0}$ are equivalent, and in particular opt$(\mu; G) =$ opt$(\mu; G^{\tau,0})$. Any multiflow in G is naturally extended to a multiflow in $G^{\tau,0}$ by adding e^τ for each path containing either e or e'. So we regard a multiflow in G as a multiflow in $G^{\tau,0}$.

We consider increasing α from 0 without changing the optimal value. The maximum possible value is denoted by α_τ or $\alpha_\tau(G)$, i.e.,

$$\alpha_\tau := \max\{\alpha \in [0,2] \mid \mathrm{opt}(\mu; G) = \mathrm{opt}(\mu; G^{\tau,\alpha})\}.$$

The modification of G to G^{τ,α_τ} is named here a *fractional splitting-off operation*. By reversing this operation, i.e., by contracting edge e^τ, any $1/k$-integral optimal multiflow in G^{τ,α_τ} becomes a $1/k$-integral optimal multiflow in G. The case $\alpha_\tau = 2$ is nothing but the (ordinary) splitting-off operation.

We give here one fundamental relation between α_τ for a fork $\tau = (e, y, e')$ and an optimal multiflow f, where f^e (resp., $f^{e,e'}$) denotes the total amount of flows using e (resp., e and e') in f.

Lemma 3.1. $\alpha_\tau \geq 2 - f^{e^\tau} \geq 2f^{e,e'}$.

Proof. Since f is also a multiflow in $G^{\tau,\alpha}$ for $\alpha = 2 - f^{e^\tau}$, we have $\|f\|_\mu \leq \mathrm{opt}(\mu; G^{\tau,\alpha}) \leq \mathrm{opt}(\mu; G) = \|f\|_\mu$, which implies the first inequality $\alpha_\tau \geq \alpha$. The second inequality follows from $2 - f^{e^\tau} = 2 - (f^e + f^{e'} - 2f^{e,e'}) = (1 - f^e) + (1 - f^{e'}) + 2f^{e,e'} \geq 2f^{e,e'}$. □

Suppose that we are given a realization $(\mathcal{K}; \{R_s\}_{s\in S})$ of μ with unit leg-length $\delta = 1$ and an optimal potential $\rho : V \to V(\mathcal{K})$ for $\mathrm{DLP}(\mathcal{K}; \{R_s\}_{s\in S})$. There is another formula for α_τ involving ρ and its neighbors. Any potential ρ for G is extended to a potential for $G^{\tau,\alpha}$ by defining $\rho(y^\tau) := \rho(y)$. An important observation here is:

(3.1) If ρ is optimal to G, then ρ is also optimal to $G^{\tau,\alpha}$ for $0 \leq \alpha \leq \alpha_\tau$.

Indeed, by $d^\rho(e^\tau) = 0$ we have $\mathrm{opt}(\mu; G) = d^\rho(G) = d^\rho(G^{\tau,\alpha}) \geq \mathrm{opt}(\mu; G^{\tau,\alpha}) = \mathrm{opt}(\mu; G)$. This brings about a formula of α_τ in terms of neighbors as follows:

Proposition 3.2. *Let ρ be an optimal potential, and τ an unsplittable fork. We have*

$$\alpha_\tau = \min_{\rho'} \frac{d^{\rho'}(G^{\tau,0}) - d^\rho(G^{\tau,0})}{d^{\rho'}(e^\tau)}, \tag{3.2}$$

where the minimum is taken over all neighbors ρ' of ρ with $d^{\rho'}(e^\tau) > 0$.

Proof. We see the equivalence among the conditions (1) to (4) for α below:

(1) $0 \leq \alpha \leq \alpha_\tau$.

(2) $\mathrm{opt}(\mu; G^{\tau,\alpha}) = \mathrm{opt}(\mu; G)$.

(3) ρ is optimal for $G^{\tau,\alpha}$.

(4) For every neighbor ρ' of ρ, we have $d^{\rho'}(G^{\tau,\alpha}) \geq d^\rho(G^{\tau,\alpha})$.

(1) ⇔ (2) follows from the definition. (2) ⇔ (3) follows from $\mathrm{opt}(\mu; G) = d^\rho(G^{\tau,\alpha})$ by $d^\rho(e^\tau) = 0$. (3) ⇔ (4) follows from Theorem 2.5. To obtain the desired formula, substitute $d^{\rho'}(G^{\tau,\alpha}) = d^{\rho'}(G^{\tau,0}) - \alpha d^{\rho'}(e^\tau)$ and $d^\rho(G^{\tau,\alpha}) = d^\rho(G^{\tau,0})$ to (4). □

The minimization over neighbors in (3.2) can be replaced by that over semi-neighbors. A (semi-)neighbor ρ' attaining the minimum in the formula of α_τ is said be *critical*. Note that both ρ and ρ' are optimal to G^{τ,α_τ}.

For an optimal potential ρ, an unsplittable fork τ, and a critical neighbor ρ' of ρ respect to τ, we consider the update $(G; \rho) \leftarrow (G^{\tau,\alpha_\tau}; \rho')$, which we call *SPUP* and specifically *α-SPUP* when $\alpha = \alpha_\tau$ (α is a rational in $[0, 2)$).

Our proof of Theorem 1.5 basically goes along the following procedure starting from an inner Eulerian graph G (without splittable forks) and an optimal potential ρ.

SPUP procedure: Let $(G_0; \rho_0) := (G; \rho)$ and $j := 1$. We repeat the following:

- **step 1:** Take a fork τ_j at a node $y_j \in V(G_{j-1}) \setminus \{y_1, {y_1}^{\tau_1}, y_2, {y_2}^{\tau_2}, \dots, y_{j-1}, {y_{j-1}}^{\tau_{j-1}}\}$ and a critical neighbor ρ' of ρ_{j-1} with respect to τ_j in G_{j-1}.
- **step 2:** Do SPUP: $(G_j; \rho_j) \leftarrow ({G_{j-1}}^{\tau_j, \alpha_{\tau_j}}; \rho')$, and let K_j be the smallest positive integer such that $K_j G_j$ is inner Eulerian.
- **step 3:** If $K_j G_j$ is guaranteed to have an integral optimal multiflow, then stop. Otherwise let $j := j + 1$ and go to **step 1**.

We will prove Theorem 1.5 by showing: By appropriate choices of τ_j, ρ' in **step 1**,

(a) for some $j \leq |V|$, the algorithm terminates in **step 3**, and

(b) K_j is bounded by a constant, say 12, independent of $|V|$.

If this is proved, then by reversing the operations (i.e., by contracting e^{τ_j}) we can construct a $1/K_j$-integral optimal multiflow in the original graph G.

For (a), we will show that if ρ_j is an embedding to $\mathcal{K}$ with a certain special property, $K_j G_j$ is guaranteed to have an integral optimal multiflow and the algorithm stops in **step 3**. To realize such an embedding, we will choose (τ_j, ρ') in **step 1** appropriately. For (b), we will bound K_j throughout the procedure. Each step creates edge e^{τ_j} of (possibly fractional) capacity $2 - \alpha_j = 2 - \{d^{\rho'}(G^{\tau_j,0}) - d^{\rho_j}(G^{\tau_j,0})\}/d^{\rho'}(e^{\tau_j})$. Here $d^{\rho'}(e^{\tau_j})$ is one of $\{1, 2, 3, 4\}$ since $\rho'(y)$ and $\rho'(y^{\tau_j})$ belong to the neighborhood of $\rho(y)$ in $\mathcal{K}$; see (2.4) in Section 2.1.4. So we will bound the denominator of $d^{\rho'}(G^{\tau_j,0}) - d^{\rho_j}(G^{\tau_j,0})$.

We explain a concrete strategy of achieving this idea in the rest of this section, which is structured as follows. In Section 3.2, we classify terminals with a view to studying the parity of $d^{\rho'}(G^{\tau_j,0}) - d^{\rho_j}(G^{\tau_j,0})$. In Section 3.3, we describe reductions of making each node have small degree, which simplifies our analysis in every place. In Section 3.4, we describe the whole proof outline of Theorem 1.5.

3.2 Proper/essential terminals and the parity of $d^{\rho'}(G^{\tau,0}) - d^{\rho}(G^{\tau,0})$

A terminal s is said to be *proper* (with respect to realization $(\mathcal{K}; \{R_s\}_{s \in S})$) if R_s contains no legs, i.e., if R_s has no interior. A terminal that is not proper is said to be *improper*. In Figure 9, s_8 is improper and the other terminals are proper. A terminal s is said to be *essential* if every optimal multiflow $f = (\mathcal{P}, \lambda)$ has a path $P \in \mathcal{P}$ connecting s and another terminal t with $\lambda(P) > 0$ and $\mu(s,t) > 0$.

Lemma 3.3. *For two optimal potentials ρ and ρ', if terminal s is proper or essential, then $\rho'(s)$ and $\rho(s)$ belong to the same connected component of the boundary of R_s, and hence belong to the same color class of the leg-graph.*

Proof. It suffices to consider the case where s is improper and essential. Take an optimal multiflow $f = (\mathcal{P}, \lambda)$, which has a path P connecting s and t with $\mu(s,t) > 0$ and $\lambda(P) > 0$. Both $\rho(s)$ and $\rho'(s)$ must be on the boundary of R_s. Otherwise, it is impossible to satisfy the geodesic condition for P. Necessarily R_s and R_t are disjoint by $d_{\mathcal{K}}(R_s, R_t) = \mu(s,t) > 0$. Delete the interior of R_s from $\mathcal{K}$. Let $\mathcal{K}'$ be the resulting connected component including R_t. Since $\mathcal{K}$ is simply-connected and R_s is connected, $R_s \cap \mathcal{K}'$ is connected. Both $\rho(s)$ and $\rho'(s)$ belong to $R_s \cap \mathcal{K}'$ since any shortest path joining R_s and R_t must belong to $\mathcal{K}'$. □

This fact has a consequence on the parity of $d^{\rho'}(G^{\tau,0}) - d^{\rho}(G^{\tau,0})$ as follows, where ρ is an optimal potential, τ is an unsplittable fork, and ρ' is a critical neighbor of ρ with respect to τ.

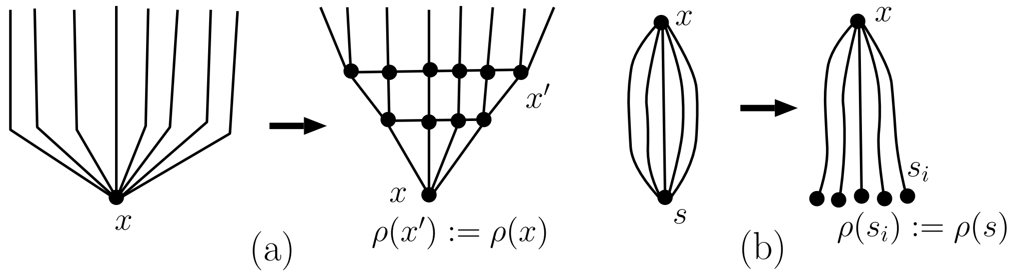


Figure 11: Degree reductions for (a) inner node x and (b) terminal s incident to a unique neighbor x

Lemma 3.4. *Suppose that G is an inner Eulerian graph such that each terminal is proper, essential, or has an even degree. Then $d^{\rho'}(G^{\tau,0}) - d^{\rho}(G^{\tau,0})$ is an even integer, and hence $\alpha_\tau \in \{0, 1/2, 2/3, 1, 4/3, 3/2\}$.*

Proof. Let $T \subseteq S$ be the set of proper or essential terminals. Then $G^{\tau,0}$ is inner Eulerian with respect to T. By edge-multiplication, we may assume that G has unit capacity. Hence the edge set $E(G^{\tau,0})$ is a disjoint union of T-paths P_j and (nonsimple) cycles C_i. Hence we have $d^{\rho'}(G^{\tau,0}) - d^{\rho}(G^{\tau,0}) = \sum_i \{d^{\rho'}(C_i) - d^{\rho}(C_i)\} + \sum_j \{d^{\rho'}(P_j) - d^{\rho}(P_j)\}$. Since the leg-graph is bipartite (Section 2.1.2), both $d^{\rho'}(C_i)$ and $d^{\rho}(C_i)$ are even. Each essential terminal remains essential in G^{τ,α_τ}, and both ρ and ρ' are optimal to G^{τ,α_τ}. By Lemma 3.3, $\rho(s)$ and $\rho'(s)$ for $s \in T$ belong to the same color class. Thus $d^{\rho'}(P_j) - d^{\rho}(P_j)$ is also even. Consequently $d^{\rho'}(G^{\tau,0}) - d^{\rho}(G^{\tau,0})$ is even. As was noted, $d^{\rho'}(e^{\tau})$ is one of $\{1, 2, 3, 4\}$ since both $\rho'(y)$ and $\rho'(y^{\tau})$ belong to the neighborhood of $\rho(y)$. □

How to bound the denominator of $d^{\rho'}(G^{\tau_j,0}) - d^{\rho_j}(G^{\tau_j,0})$. In the SPUP procedure, each step j creates an edge e^{τ_j} of (possibly fractional) capacity $2 - \alpha_{\tau_j}$. Hence the inner Eulerian condition for G_j does not hold even if G_0 is inner Eulerian, and Lemma 3.4 is not applicable. However, if created edge e^{τ_j} remains to have the same length, i.e., $d^{\rho'}(e^{\tau_k}) = d^{\rho_{j-1}}(e^{\tau_k})$ for $k = 1, 2, \ldots, j-1$, then each (fractional) term $c(e^{\tau_k})\{d^{\rho'}(e^{\tau_k}) - d^{\rho_{j-1}}(e^{\tau_k})\}$ in $d^{\rho'}(G^{\tau_j,0}) - d^{\rho_{j-1}}(G^{\tau_j,0})$ vanishes, and Lemma 3.4 is applicable. In this way, we will bound K_j by keeping the length $d^{\rho}(e^{\tau_k})$ of the created edge e^{τ_k} as far as possible.

3.3 Degree reductions

We will mainly work on an inner Eulerian graph such that each inner node has degree 4 and each terminal has degree 1 or 2. A standard reduction is known to make the graph have degree at most 4; see [7, p. 50]. Let G be an inner Eulerian graph, i.e., capacity c is integer-valued and each inner node has an even degree. By edge-multiplication, we can make each edge have unit capacity.

Degree-4 reduction of an inner node. For an inner node x of degree greater than four, we can reduce the degree by changing the incidence at x as in Figure 11 (a). Then the problem does not change.

Degree-1 reduction of a terminal. For a terminal s of degree m, we can reduce its degree to one as follows. Consider the case where s is incident to a unique node x. Replace s by new terminals $s_1, s_2, \ldots, s_m$, connect x and each s_i by an edge (of unit capacity), and define weight μ on s_i by $\mu(s_i, t) = \mu(s, t)$ for $t \in S \setminus s$ and by $\mu(s_i, s_j) = 0$. Obviously the problem does not change. See Figure 11 (b). A realization of (S', μ') is obtained by setting $R_{s_i} := R_s$ for each i. In the case where s is incident to several nodes $x_1, x_2, \ldots, x_l$, add a new inner node x, and replace each edge $x_i s$ by two edges $x_i x$ and xs. Then s has a unique neighbor x. Apply the reduction above.

Degree-2 reduction of a terminal with even degree. For a terminal s of even degree m, we can reduce its degree to two, as in the degree-1 reduction above, by adding new $m/2$ terminals $s_1, s_2, \ldots, s_{m/2}$ and connect s_i by two parallel edges.

Extending an optimal potential to the new problem. In the reductions above, if we are given an optimal potential ρ for the original problem, we can extend ρ to an optimal potential for the new problem by setting $\rho(x') := \rho(x)$ for each new added node x'. This is a simple consequence of the optimality criterion (Lemma 2.4).

Keeping a terminal essential. The degree-1 and -2 reductions may create a nonessential terminal. In the degree-2 reduction, we can split off a unique fork for a nonessential terminal of degree 2, while keeping the inner Eulerian condition. In the degree-1 reduction, to guarantee that each new (improper) terminal is essential, we will use the following fact, where an optimal potential ρ is assumed to be given.

(3.3) For a terminal s incident to a unique node x with $\rho(s) \neq \rho(x)$, the degree-1 reduction at s keeps each new terminal s_i essential.

Indeed, by the optimality criterion for (f, ρ), every optimal multiflow f must have paths connecting s of the flow-value equal to the degree of s, i.e., $f^{sx} = c(sx)$. Obviously this flow property, stronger than the essentialness, is kept in the degree-1 reduction.

Edge-subdivision. We will also create an inner node of degree 2 by the *subdivision* of an edge $e = xy$, which is to add a new node z and replace xy by two edges xz, zy. The capacity is defined by $c(xz) = c(zy) := c(xy)$. This operation obviously does not change the problem. If we are given an optimal potential ρ for the original problem, we can extend ρ to an optimal potential to the new problem by defining $\rho(z)$ so that $d_{\mathcal{K}}(\rho(x), \rho(y)) = d_{\mathcal{K}}(\rho(x), p) + d_{\mathcal{K}}(p, \rho(y))$ for any $p \in V(\mathcal{K})$. This fact is also an easy consequence of the optimality criterion (Lemma 2.4).

3.4 Proof outline

Here we describe the outline of the proof of Theorem 1.5, which we prove under a weaker condition. Recall Theorem 2.2 that the 2-dimensionality of T_μ of μ is equivalent to the existence of an F-complex realization $(\mathcal{K}; \{R_s\}_{s \in S})$. A graph G is said to be *properly-inner Eulerian* with respect to a realization $(\mathcal{K}; \{R_s\}_{s \in S})$ if the capacity is integral and each node other than proper terminals has an even degree.

Theorem 3.5. *Suppose that μ has an F-complex realization $(\mathcal{K}; \{R_s\}_{s \in S})$. There exists a $1/12$-integral optimal multiflow in every properly-inner Eulerian graph.*

The proof is based on the SPUP procedure and three claims (A), (B), and (C) below, which we will prove in Sections 4 and 5. To state and motivate three claims, we first introduce an overall framework, and then give the proof of Theorem 3.5.

Suppose that μ has a realization $(\mathcal{K}; \{R_s\}_{s \in S})$ of leg-length $\delta = 2$ (by scaling). Let G be a properly-inner Eulerian graph. We may assume the condition:

(3.4) Each terminal is proper or essential.

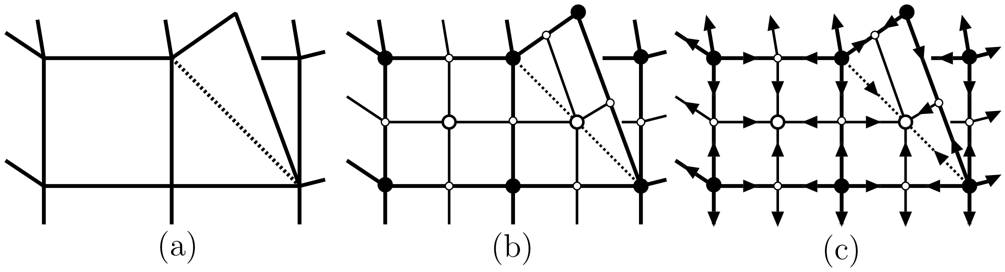


Figure 12: (a) $\mathcal{K}$, (b) $\mathcal{K}^2$, and (c) the orientation of $\mathcal{K}^2$

Indeed, we can make a nonessential (improper) terminal an inner node, while keeping the inner Eulerian condition. We will maintain this condition throughout the SPUP procedure.

The 2-subdivision $(\mathcal{K}^2; \{R_s\}_{s\in S})$ is also a realization of μ, with unit leg-length. We consider DLP$(\mathcal{K}^2; \{R_s\}_{s\in S})$. Note that $\mathcal{K}^2$ has the following orbit property:

(3.5) If edge e in $\mathcal{K}$ is divided into e_1 and e_2 in $\mathcal{K}^2$, then e_1 and e_2 belong to different orbits in $\mathcal{K}^2$.

One can easily verify this property from the orientability of $\mathcal{K}$.

For an optimal potential $\rho : V \to V(\mathcal{K}^2)$, we define a partition of V into three sets,

(3.6)
$$\begin{aligned} S_\rho &= \{x \in V \mid \rho(x) \text{ is the midpoint of a folder in } \mathcal{K}\}, \\ M_\rho &= \{x \in V \mid \rho(x) \text{ is the midpoint of a leg in } \mathcal{K}\}, \\ C_\rho &= \{x \in V \mid \rho(x) \text{ is a vertex of } \mathcal{K}\}. \end{aligned}$$

See Figure 12 (b). The first claim says that inner nodes in S_ρ have a particularly nice property.

(A) *Let G be an inner Eulerian graph, and ρ an optimal potential for DLP$(\mathcal{K}^2; \{R_s\}_{s\in S})$. If an inner node y belongs to S_ρ, then y has a splittable fork.*

Motivated by this claim, the number of inner nodes in $M_{\rho_j} \cup C_{\rho_j}$ is decreased with the aid of the SPUP procedure. If G_j has no inner nodes in $M_{\rho_j} \cup C_{\rho_j}$, then all inner nodes in K_jG_j are splittable by (A) in **step 2**. In addition, if the degree-1 reduction to K_jG_j keeps (3.4) and creates no new inner nodes in $M_{\rho_j} \cup C_{\rho_j}$, then we can apply the splitting-off to obtain a graph consisting only of terminals of degree one. In this graph, an integral optimal multiflow obviously exists, and hence in K_jG_j. Thus the SPUP procedure terminates in **step 3**, and a $1/K_j$-integral optimal multiflow is obtained in the original graph. Our goal is this situation.

We will choose a fork τ_j and a critical neighbor ρ' in **step 1** such that $S_{\rho'} \supseteq S_{\rho_{j-1}}$ and $M_{\rho'} \cup C_{\rho'} \subseteq M_{\rho_{j-1}} \cup S_{\rho_{j-1}}$. Consider an admissible orientation of $\mathcal{K}^2$ such that each vertex of $\mathcal{K}$ is a source and the midpoint of each folder in $\mathcal{K}$ is a sink; see Figure 12 (c). This orientation is admissible, and is called the *forward orientation*. Restricting the forward orientation to each orbit, we get an admissible orientation of an orbit, which is also called the forward orientation.

Then two types of neighbors can be distinguished. A neighbor is said to be *forward* if it is a neighbor with respect to the forward orientation, and *backward* if it is a neighbor of the opposite orientation. We use this terminology also for semi-neighbors. In the following argument, we can replace forward neighbors by forward semi-neighbors. An SPUP is said to be *forward* if the critical neighbor ρ' is forward, and *backward* if ρ' is backward.

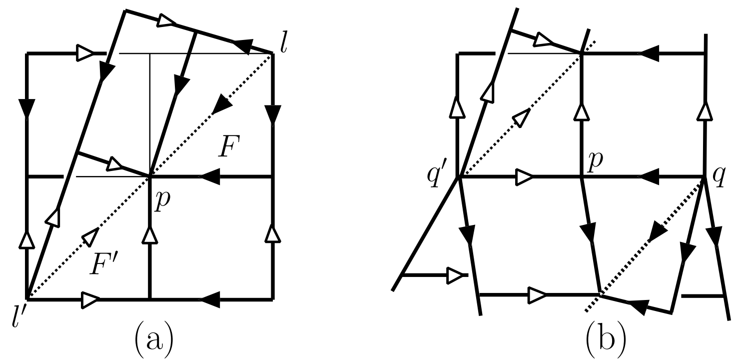


Figure 13: Orbit structure around the midpoint of (a) a folder and (b) a leg, where the black and white arrows indicate distinct orbits.

In **step 1**, we choose fork τ_j at an inner node in $M_{\rho_{j-1}} \cup C_{\rho_{j-1}}$ and do SPUP *only when ρ' is forward.* Then the image of potentials $\{\rho_j(x)\}$ moves toward the midpoint of folders in $\mathcal{K}$. Equivalently, the number of inner nodes in $M_{\rho_j} \cup C_{\rho_j}$ decreases. By forward SPUP, we will sweep out inner nodes first from C_ρ and then from M_ρ. To implement this scheme, the following properties are essential; the numerator and the denominator of formula (3.2) of α_τ crucially depend on the position $\rho(y)$ in $\mathcal{K}^2$.

Lemma 3.6. *For an optimal potential ρ, an unsplittable fork τ on a node y, and a critical neighbor ρ' of ρ with respect to τ, we have the following:*

(1) *If $y \in C_\rho$, then ρ' is forward, and if $y \in S_\rho$, then ρ' is backward.*

(2) *If ρ' is forward, then $d^{\rho'}(G^{\tau,0}) - d^{\rho}(G^{\tau,0})$ is equal to*

$$\sum \{c(e)\{d^{\rho'}(e) - d^{\rho}(e)\} \mid e \text{ is incident to } M_\rho \cup C_\rho\}.$$

(3) *$d^{\rho'}(e^\tau)$ is given as*

$$d^{\rho'}(e^\tau) \in \begin{cases} \{1,2,3,4\} & \text{if } y \in C_\rho, \\ \{1,2\} & \text{if } y \in M_\rho,\ \rho' : \textit{forward}, \\ \{1\} & \text{if } y \in M_\rho,\ \rho' : \textit{backward}, \\ \{1,2\} & \text{if } y \in S_\rho. \end{cases}$$

See Figure 13 for the orbit structure around the midpoint p of a $K_{2,*}$-folder and of a leg (in $\mathcal{K}$), where the black and white arrows indicate distinct orbits (by (3.5)).

Proof. (1). If $y \in C_\rho$, then $\rho(y)$ is a source of the orientation, and hence there is no backward neighbor ρ' with $d^{\rho'}(e^\tau) > 0$. The case of $y \in S_\rho$ is similar.

(2). Use the following fact: for an edge $e = xy$, if both ends belong to S_ρ, then $(\rho(x), \rho(y)) = (\rho'(x), \rho'(y))$ implies $d^\rho(e) = d^{\rho'}(e)$; see the paragraph after Lemma 3.4.

(3). Both $\rho'(y)$ and $\rho'(y^\tau)$ belong to neighborhood $(\mathcal{K}^2)_p$ for $p = \rho(y)$ (Section 2.1.4). This implies $d^{\rho'}(e^\tau) \in \{1,2,3,4\}$. Suppose $y \in S_\rho$; $p = \rho(y)$ is the midpoint of a folder in $\mathcal{K}$, as in Figure 13 (a). By (3.5), p touches (at least) two distinct orbits as in Figure 13 (a). Then $\{\rho'(y), \rho'(y^\tau)\}$ belongs to one of F and F' in Figure 13 (a), implying $d^{\rho'}(e^\tau) \in \{1,2\}$. Suppose $y \in M_\rho$; $p = \rho(y)$ is the midpoint of an edge qq' of $\mathcal{K}$ as in Figure 13 (b). If ρ' is backward, then $\{\rho'(y), \rho'(y^\tau)\} \subseteq \{p, q, q'\}$. By (3.5), legs pq and pq' belong to different orbits. So $\{\rho'(y), \rho'(y^\tau)\} = \{p, q\}$ or $\{q, p'\}$, implying $d^{\rho'}(e^\tau) = 1$. If ρ' is forward, then $\rho'(y)$ is q or a vertex adjacent to q by leg, and so is $\rho'(y^\tau)$, implying $d^{\rho'}(e^\tau) \in \{1,2\}$. $\square$

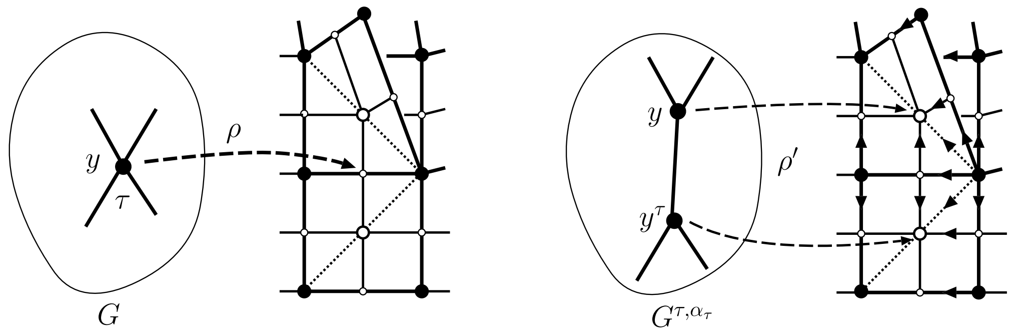


Figure 14: SPUP at M_ρ

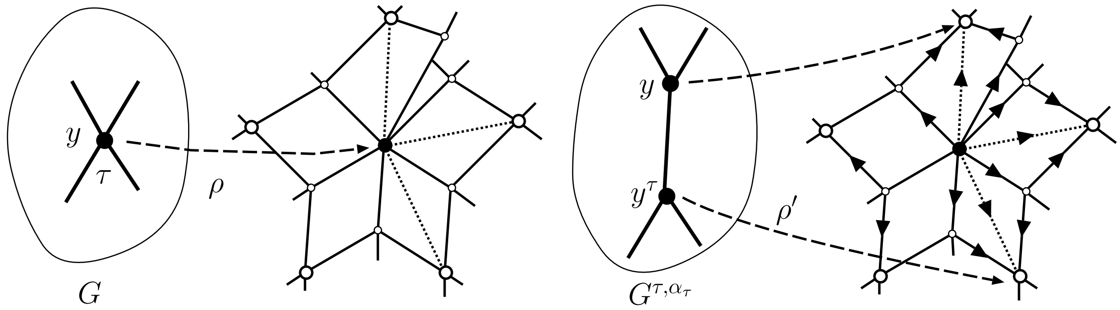


Figure 15: SPUP at C_ρ

See Figures 14 and 15 for the behavior of critical neighbors. In particular, Lemma 3.6 (2) implies: As far as we apply forward SPUP, the (possibly fractional) capacities of edges within S_ρ does not affect $d^{\rho'}(G^{\tau,0}) - d^{\rho}(G^{\tau,0})$. This is a key to bound K_j; see the paragraph after Lemma 3.4. We will keep the numerator in the formula (3.2) of α_τ even as much as possible. In this case, the possible values of α_τ for a fork τ at node y are given by

$$\alpha_\tau = \begin{cases} 0, \dfrac{1}{2}, \dfrac{2}{3}, 1, \dfrac{4}{3}, \dfrac{3}{2}, 2 & \text{if } y \in C_\rho, \\ 0, 1, 2 & \text{if } y \in M_\rho. \end{cases} \tag{3.7}$$

In the forward SPUP procedure, it suffices to maintain this evenness between ρ and its forward neighbor ρ'. Motivated by this, $(G;\rho)$ is said to be *restricted Eulerian* if every edge of G has an integer capacity and every inner node in $M_\rho \cup C_\rho$ has an even degree; inner nodes in S_ρ may have an odd degree. In this case, by Lemma 3.4 with the paragraph after the lemma, $d^{\rho'}(G^{\tau,0}) - d^{\rho}(G^{\tau,0})$ is an even integer as long as ρ' is a forward neighbor of ρ.

As mentioned already, we will sweep out inner nodes first from C_ρ and then from M_ρ. The forward SPUP at M_ρ works well under the restricted Eulerian condition. By the degree-4 reduction, we may assume that each inner node in $M_\rho \cup C_\rho$ has degree four.

Take a fork τ at $y \in M_\rho$, and a critical neighbor ρ' of ρ. Then, by Lemma 3.6 (3), we have $\alpha_\tau \in \{0,1\}$ (even if ρ' is backward). Suppose that ρ' is forward. If $\alpha_\tau = 1$, then necessarily $d^{\rho'}(e^\tau) = 2$. Although both y and y^τ have degree 3 in G^{τ,α_τ}, they fall into $S_{\rho'}$ (Figure 14). Therefore $(G^{\tau,\alpha_\tau};\rho')$ is restricted Eulerian. If $\alpha_\tau = 0$, then one of y and y^τ, say y, falls into $S_{\rho'}$. Contract e^τ to y (in $G^{\tau,0}$). Then ρ' is optimal for the resulting graph, i.e., the original graph G; see (4.3). So ρ' is an optimal forward neighbor of ρ for G. In the both cases, we can update $(G;\rho)$ to sweep out y from M_ρ into S_ρ while keeping the restricted Eulerian condition. However, if ρ' were backward, then this would crash our program. Fortunately we can avoid such a backward SPUP by examining all the three forks τ_1, τ_2, τ_3 at y, where a degree-four node has three forks up to symmetry.

(B) *Suppose that $(G;\rho)$ is restricted Eulerian. Let y be an inner node in M_ρ and let ρ_i be a critical neighbor of ρ with respect to τ_i at y $(i = 1, 2, 3)$. Then at least one of ρ_1, ρ_2, ρ_3 is forward. Hence, by SPUP operations at M_ρ, we can modify $(G;\rho)$ so that $(G;\rho)$ is restricted Eulerian and M_ρ have no inner nodes.*

Our final goal ensures to make C_ρ have no inner node. SPUP at C_ρ is always forward by Lemma 3.6 (1). Therefore successive SPUP at C_ρ does not increase the number of the nodes in C_ρ; see Figure 15. However α_τ can take fractional values $2/3, 4/3, 1/2, 3/2$. To bound the denominator of the capacity of created edges, we will carefully choose forks and critical neighbors.

(C) *Suppose that G is a properly inner-Eulerian graph and ρ is an optimal potential. By the degree-reductions keeping (3.4), the splitting-off, and SPUP operations at C_ρ, we can modify $(G;\rho)$ so that $(6G;\rho)$ is restricted Eulerian, C_ρ has no inner nodes, and each terminal in C_ρ is incident to (at most) one node.*

The proof of Theorem 3.5 assuming (A),(B), and (C). Our remaining task is to maintain (3.4) in the degree-1 reduction. We may assume that the condition (3.4) holds. We also note that M_ρ cannot have terminals under (3.4). Otherwise, for such a terminal s, R_s includes the midpoint $\rho(s)$ of a leg pq. This means that R_s includes leg pq by normality of R_s in $\mathcal{K}$, and that $\rho(s)$ is in the interior of R_s. Therefore s is neither proper nor essential.

Take an optimal potential ρ; obviously $(G;\rho)$ is restricted Eulerian. By claim (C), we can make $(G;\rho)$ so that $(6G;\rho)$ is restricted Eulerian, C_ρ has no inner nodes, and each terminal in C_ρ is incident to a unique node.

Let $(G;\rho) \leftarrow (6G;\rho)$. Here C_ρ may have terminals. Such a terminal s is incident to a unique node x. If $x \in C_\rho$, then x is also a terminal with unique neighbor s, and therefore we can fix (integral) flow between x and s and delete them. Suppose $x \notin C_\rho$; in particular $\rho(x) \neq \rho(s)$. Apply the degree-1 reduction to s; this creates no inner nodes in C_ρ. The added new terminals remain to be essential by (3.3). In this way, apply degree-1 reduction (or deletion) to all terminals in C_ρ. Next apply the degree-4 reduction to inner nodes in M_ρ. By claim (B), we can repeat SPUP at inner nodes in M_ρ to make $(G;\rho)$ so that $(G;\rho)$ is restricted Eulerian and $M_\rho \cup C_\rho$ has no inner nodes.

Let $(G;\rho) \leftarrow (2G;\rho)$. Still C_ρ may have terminals. Apply, again, the degree-1 reduction to all terminals in C_ρ; new terminals are all essential by (3.3). Apply the degree-1 reduction to proper terminals in S_ρ, and the degree-2 reduction to improper terminals in S_ρ. Some new improper terminal s (of degree two) may fail to be essential. In this case, the unique fork τ at s is splittable; split off τ and delete s. In this way, we can make all improper terminals essential.

Here, in fact, the degree-1 reduction keeps each improper terminal essential. To verify this, take an improper terminal s and consider the unique fork $\tau = (e, s, e')$ at s. Then $\alpha_\tau = 0$ holds; the proof is given at the end. By Lemma 3.1, every optimal multiflow f in $G = G^{\tau,0}$ satisfies $f^{e^\tau} = 2 = c(e^\tau)$ at e^τ. Hence the degree-2 terminal s always has paths connecting s of flow-value 2. Thus the degree-1 reduction keeps s essential (see the argument after (3.3)). Apply the degree-1 reduction to all improper terminals s in S_ρ.

Now G is an inner Eulerian graph such that $M_\rho \cup C_\rho$ has no inner nodes, and each terminal, of degree one, is proper or essential. This is our goal. As mentioned already, by claim (A), there exists an integral optimal multiflow. Reversing these operations, we get an 1/12-integral optimal multiflow in the original graph. The proof is done.

Proof of $\alpha_\tau = 0$. By Lemma 3.6 (3), we have $\alpha_\tau = 0$ or 1. Take a critical neighbor ρ' of ρ with respect to τ, which is backward. Suppose (to the contrary) that $\alpha_\tau = 1$. Then $d^{\rho'}(e^\tau) = 2$, and $\{\rho'(s), \rho'(s^\tau)\} = \{p, l\}$ or $\{p, l'\}$ in Figure 13 (a) with $p = \rho(s)$ since $\rho'(s)$ and $\rho(s)$ must be in the same color class on the boundary of R_s by Lemma 3.3. Furthermore $\rho'(s^\tau)$ is not in R_s (otherwise the induced path connecting R_s through $\rho'(s^\tau)$ is never shortest). This is impossible since R_s must include hypotenuses pl and pl' by the normality in $\mathcal{K}$. □

4 Analysis of SPUP by multiflows

To prove the existence of forks or critical neighbors with required properties (claims (A),(B),and (C)), we analyze the behavior of optimal multiflows under an optimal potential. In Section 4.1, we study optimality-keeping rearrangements of an optimal multiflow, and introduce the *local geodesic condition* as a criterion of such rearrangements. The goal of this section is Theorem 4.3 in Section 4.2, which states interrelations among an optimal multiflow, an optimal potential, critical neighbors, and the shape of $\mathcal{K}_p$. The claims (A) and (B) are its immediate consequences. Also Theorem 4.3 brings a powerful splittablity criterion in Section 6. The main proof tool is a combination of the first optimal criterion (Lemma 2.4), the second optimality criterion (Theorem 2.5), and the local geodesic condition.

Throughout this section, G is a graph with terminal set S and rational edge-capacity c, and μ is a terminal weight having a realization $(\mathcal{K}; \{R_s\}_{s \in S})$ with unit leg-length. By rationality, we can always take an optimal multiflow $f = (\mathcal{P}, \lambda)$ with a *rational-valued* flow-value function λ. Therefore, by allowing $\mathcal{P}$ to be a multiset, we can represent $f = (\mathcal{P}, \lambda)$ by a pair of a multiset $\mathcal{P}$ of S-paths and a *uniform* flow-value function $\lambda = 1/\kappa$ for some positive integer κ (called the *fractionality* of f). We use this expression, and denote it by $f = (\mathcal{P}; \kappa)$. For an edge e, the subset of paths in $\mathcal{P}$ containing e is denoted by $\mathcal{P}(e)$. Its total flow-value $|\mathcal{P}(e)|/\kappa$ is denoted by f^e. For consecutive two edges e, e', the subset of paths passing e and e' is denoted by $\mathcal{P}(e, e')$, and its flow value is denoted by $f^{e,e'}$. A path P is called *an* $(A, y_1y_2 \ldots y_m, B)$*-path* if P connects terminal subsets A and B by passing through nodes $y_1, y_2, \ldots, y_m$ in the order of $A \to y_1 \to y_2 \to \cdots \to y_m \to B$. A set $\mathcal{P}'$ of paths is called *an* $(A, y_1y_2 \ldots y_m, B)$*-set* if $\mathcal{P}'$ consists of all $(A, y_1y_2 \ldots y_m, B)$-paths. When B (resp., A) is not specified, B (resp., A) is replaced by $*$ (e.g.: P is an $(A, xy, *)$-path).

4.1 Local multiflow rearrangement

The local multiflow rearrangement plays a central role in our analysis. Let $f = (\mathcal{P}; \kappa)$ be an optimal multiflow and let y be a node. Consider the following problem:

Split some of the paths in $\mathcal{P}$ *at* y*, and reconnect them while keeping optimality.*

Suppose that we are given an optimal potential ρ with $p := \rho(y)$. Then the split paths induce shortest paths connecting p and normal regions. Therefore, to keep the optimality, it suffices to reconnect these paths so that the resulting induced paths are all shortest (by the geodesic condition in Lemma 2.4). See Figure 16.

This motivates us to consider the following geometric problem on $\mathcal{K}$: For normal sets M and N, suppose that we are given two shortest paths P and P' such that P connects p and M, and P' connects p and N.

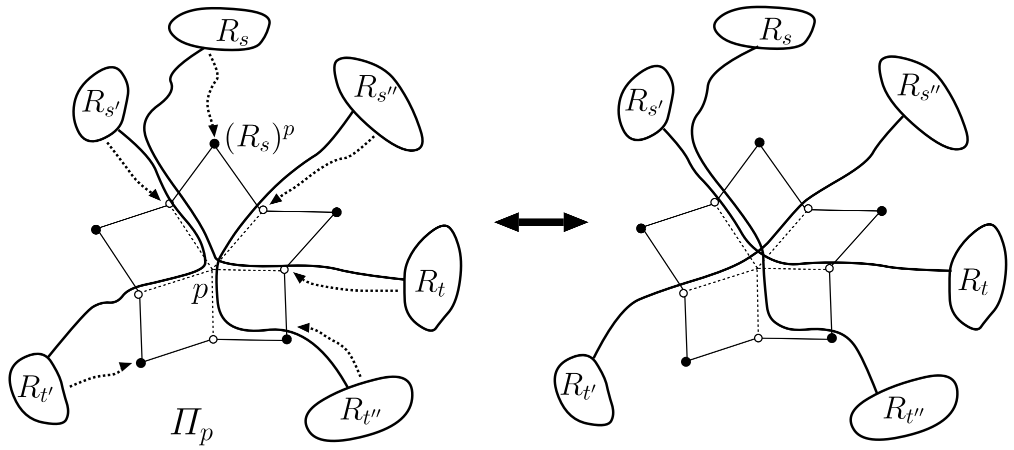


Figure 16: Local multiflow rearrangement to keep optimality

Is the concatenation $P + P'$ at p shortest between M and N ?

The shortestness is determined by the position of (M, N) relative to the neighborhood $\mathcal{K}_p$; recall the notions in Section 2.1.4. Suppose that $\mathcal{K}_p$ is star-shaped (consider the 2-subdivision if necessary). Then the leg-graph of the boundary of $\mathcal{K}_p$ is identified with $\varPi_p$, which is a bipartite graph of bipartition $\{L_p, Q_p\}$. For $u, v \in Q_p \cup L_p$, we write $u \sim v$ if $u = v$, u is incident to v, or u and v have a common neighbor in $\varPi_p$, i.e., if $d_{\mathcal{K}_p}(u, v) \leq 2$. Also we write $u \sim_1 v$ if $u = v$ or u is incident to v in $\varPi_p$, i.e., if $d_{\mathcal{K}_p}(u, v) \leq 1$.

For a normal set R not containing p, the vertex g of (the boundary of) $\mathcal{K}_p$ with $d_{\mathcal{K}}(g, R) = d_{\mathcal{K}}(\mathcal{K}_p, R)$ is uniquely determined [12, Lemma 3.8]. We call this vertex g the *gate* of R in $\mathcal{K}_p$, denoted by R^p; this concept comes from [5]. We can regard R^p as a member of $L_p \cup Q_p$. For a normal set R containing p, define R^p to be the intersection $R \cap \mathcal{K}_p$, which is normal in $\mathcal{K}_p$. Hence we get a map $R \mapsto R^p$ from the set of normal sets in $\mathcal{K}$ to that in $\mathcal{K}^p$. Then $P + P'$ forms a shortest path between M and N if and only if $P + P'$ induces a shortest path between M^p and N^p in $\mathcal{K}_p$ (Figure 16).

Lemma 4.1 ([12, Lemmas 3.6 and 3.9])**.** *For two normal sets M and N, the following conditions are equivalent:*

(1) $d_{\mathcal{K}}(M, N) = d_{\mathcal{K}}(M, p) + d_{\mathcal{K}}(p, N)$.

(2) $d_{\mathcal{K}_p}(M^p, N^p) = d_{\mathcal{K}_p}(M^p, p) + d_{\mathcal{K}_p}(p, N^p)$.

(3) *If $p \notin M$ and $p \notin N$, then there exists no $q \in Q_p$ with $M^p \sim_1 q \sim_1 N^p$. If $p \notin M$ and $p \in N$, then $M^p \notin N^p$.*

Although a shortest path from R to p enters $\mathcal{K}_p$ via $u \in Q_p \cup L_p$, the vertex u may not be the gate R^p. But the vertex u is at least adjacent to R^p by leg, as follows.

Lemma 4.2. *For a normal set R and a vertex $u \in Q_p \cup L_p$, suppose $d_{\mathcal{K}}(R, p) = d_{\mathcal{K}}(R, u) + d_{\mathcal{K}_p}(u, p)$. Then $R^p = u$ if $u \in L_p$, and $R^p \sim_1 u$ if $u \in Q_p$.*

Note that this lemma is valid even if $\mathcal{K}_p$ is not star-shaped.

Proof. By condition, $d_{\mathcal{K}}(R, u) < d_{\mathcal{K}}(R, p) + d_{\mathcal{K}_p}(p, u)$ holds. Apply the previous lemma for $(M, N) = (R, u)$. Then there exists $q \in Q_p$ with $R^p \sim_1 q \sim_1 u$. If $u \in Q_p$, then necessarily $u = q \sim_1 R^p$ (by bipartiteness of $\varPi_p$). Otherwise $u \in L_p$, implying

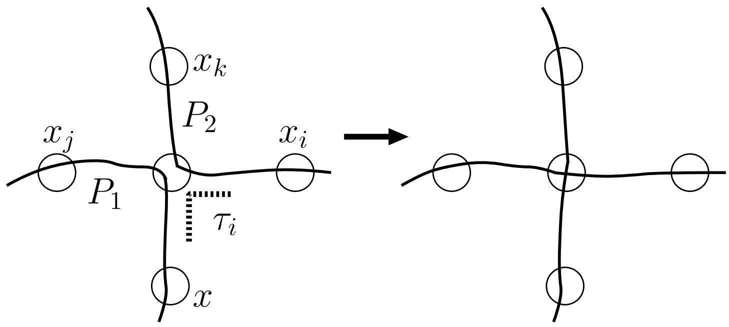


Figure 17: Exchange operation between $\mathcal{P}(e, e_j)$ and $\mathcal{P}(e_i, e_k)$ at e^{τ_i}

$d_{\mathcal{K}_p}(u, p) = 2 \geq d_{\mathcal{K}_p}(R^p, p) \in \{1, 2\}$. By $d_{\mathcal{K}_p}(p, R^p) + d_{\mathcal{K}}(R^p, R) \geq d_{\mathcal{K}}(p, R) = d_{\mathcal{K}_p}(p, u) + d_{\mathcal{K}}(u, R)$, we have $d_{\mathcal{K}}(R^p, R) \geq d_{\mathcal{K}}(u, R)$. By the definition and the uniqueness of the gate, we have $u = R^p$. □

We note the basic property of the gate, which is included in the proof above:

$$d_{\mathcal{K}}(p, R) = d_{\mathcal{K}_p}(p, R^p) + d_{\mathcal{K}}(R^p, R). \tag{4.1}$$

Let us return to the multiflow rearrangement. For $u \in Q_p \cup L_p$, let $[u]$ denote the set of terminals $s \in S$ such that gate $(R_s)^p$ is u, i.e.,

$$[u] := \{s \in S \mid p \notin R_s, u = (R_s)^p\}.$$

For $q \in Q_p$, let $[q]_1$ denote the set of terminals s with $(R_s)^p \sim_1 q$, i.e.,

$$[q]_1 := \bigcup_{u \sim_1 q} [u].$$

By Lemma 4.1 (3), under an optimal potential ρ, the following *local geodesic condition* is sufficient to keep the optimality in the multiflow rearrangement at a node y with $\rho(y) = p$.

Local geodesic condition: A multiflow f has no $([q]_1, y, [q]_1)$-paths for any $q \in Q_p$, and has no $(s, y, [u])$-paths for any $u \in Q_p \cup L_p$ and $s \in S$ with $\{p, u\} \subseteq R_s$.

In particular, we can rearrange f at y *as if* $[u]$ *is a single terminal.* The local geodesic condition is also a necessary condition for f to be optimal.

Two basic flow-operations for an optimal multiflow $f = (\mathcal{P}; \kappa)$ are given.

Exchange/anti-exchange operations. For an edge $e = xy$, take two paths P_1 and P_2 from $\mathcal{P}(e)$. The *exchange operation* of P_1 and P_2 at e is the following: For $i = 1, 2$, split P_i at x into two paths P_i^1 and P_i^2 so that P_i^2 contains y. Reconnect P_1^1 and P_2^2 at x, and reconnect P_2^1 ant P_1^2 at x. If the resulting paths are not simple, then simplify them.

If the exchange operation of P_1 and P_2 keeps the optimal value $\|f\|_\mu$, then P_1 and P_2 are said to be *exchangeable at* e. A subset $\mathcal{P}' \subseteq \mathcal{P}(e)$ is said to be *exchangeable* if the exchange operation of every pair of paths in $\mathcal{P}'$ at e keeps the value of $\|f\|_\mu$. If $\mathcal{P}(e)$ itself is exchangeable, then f is *exchangeable at* e. We will often use the exchange operation at e^τ as in Figure 17.

The *anti-exchange operation* is the reverse way of exchanging P_1 and P_2. Namely, for each $i = 1, 2$, by deleting xy, split P_i into two paths P_i^1 and P_i^2 so that P_i^2 contains y. Reconnect P_1^1 and P_2^1 at x, reconnect P_2^2 ant P_1^2 at y, and simplify them if necessary.

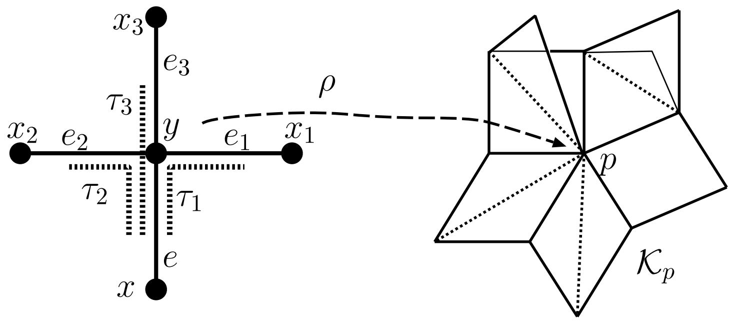


Figure 18: An inner node y of degree four mapped to p by ρ

4.2 Analysis

Here we analyze SPUP at an inner node with degree 4. Suppose that an inner node y is incident to four edges $e = xy$, $e_1 = x_1y$, $e_2 = x_2y$, and $e_3 = x_3y$ with unit capacity.

(4.2) If y has multiple edges e, e_1 ($x = x_1$), then fork (e, y, e_2) is splittable.

Indeed, let G' be the graph obtained from G by contracting edges e and e_1. Then $\mathrm{opt}(\mu; G') \geq \mathrm{opt}(\mu; G)$. Let G'' be the graph obtained from G by splitting off forks (e, y, e_2) and (e_1, y, e_3). Then $\mathrm{opt}(\mu; G'') \leq \mathrm{opt}(\mu; G)$. Here $G' = G''$. This means that (e, y, e_2) is splittable.

Our interest lies in the case where there is no splittable fork. By symmetry, it suffices to consider three forks $(e, y, e_1), (e, y, e_2), (e, y, e_3)$. Fork (e, y, e_i) is particularly denoted by τ_i, and α_{τ_i} is simply denoted by α_i.

Let ρ be an optimal potential, and let $p := \rho(y)$; see Figure 18. Let ρ_i be a critical neighbor of ρ with respect to τ_i for $i = 1, 2, 3$. We note an extremal case:

(4.3) If $\alpha_i = 0$, then the restriction of ρ_i to V is optimal for G,

where $V = V(G^{\tau_i,0}) \setminus \{y^{\tau_i}\}$. Indeed, ρ_i is optimal for $G^{\tau_i,0}$. Replace e^{τ_i} by two multiple edges e', e'' of unit capacity. Then (e, y, e') is splittable by (4.2). Split it off. The resulting graph is the same as the original G. This means that ρ_i is optimal to G.

The positions $(\rho_i(y), \rho_i(y^{\tau_i}))$ $(i = 1, 2, 3)$ are interrelated, which often determine the local multiflow configuration at y, or give the information of the local structure $\mathcal{K}_p$. The main statement in this section is the following:

Theorem 4.3. *Suppose that each terminal is proper or essential, and $\alpha_i \leq 1$ for $i = 1, 2, 3$.*

(1) *If $\rho_i(y)$ and $\rho_i(y^{\tau_i})$ are not adjacent by a leg, and belong to a common folder in $\mathcal{K}_p$ for $i = 1, 2, 3$, then there exist distinct $l_1, l_2, l_3 \in L_p$ such that, by an appropriate relabeling of e, e_1, e_2, e_3,*

   (i) $(\rho_i(y), \rho_i(y^{\tau_i})) = (p, l_i)$ $(i = 1, 2, 3)$, *and*

   (ii) *for every optimal multiflow $f = (\mathcal{P}; \kappa)$, $\mathcal{P}(e_i, e_j)$ is an $([l_i], x_iyx_j, [l_j])$-set with $f^{e_i,e_j} = 1/2$ $(1 \leq i < j \leq 3)$.*

(2) *For some legs pq and pq', if $\{\rho_i(y), \rho_i(y^{\tau_i})\} = \{p, q\}$ or $\{p, q'\}$ $(i = 1, 2, 3)$, then $q \neq q'$ and there exists a common folder containing pq and pq'.*

Figure 19 illustrates the situation of (1); necessarily $l_i \not\sim l_j$ for the local geodesic condition. The rest of this subsection is devoted to proving Theorem 4.3. The proof

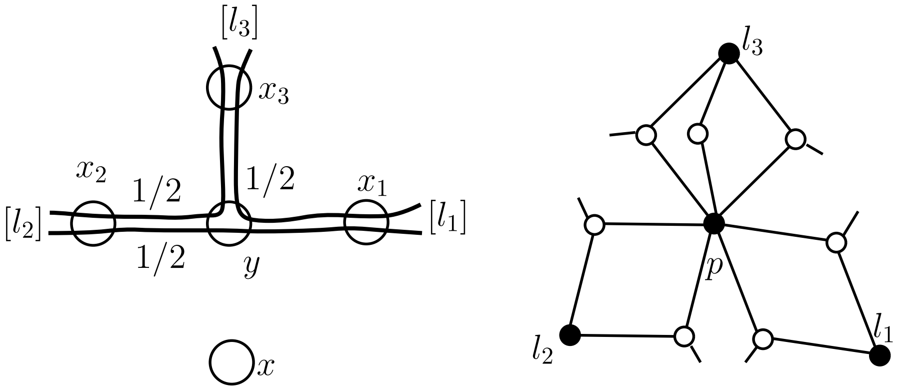


Figure 19: Flow-potential configuration in Theorem 4.3 (1)

technique is also used by proving claim (C) in the next section. As was noted, multiflows in G are identified with multiflows in $G^{\tau_i,0}$. In particular $\mathcal{P}(e^{\tau_i})$ is the disjoint union of $\mathcal{P}(e,e_j)$, $\mathcal{P}(e,e_k)$, $\mathcal{P}(e_i,e_j)$, and $\mathcal{P}(e_i,e_k)$. We will often use the following obvious relations:

$$
(4.4) \qquad \begin{aligned} f^{e^{\tau_i}} &= f^{e,e^{\tau_i}} + f^{e_i,e^{\tau_i}} = f^{e_j,e^{\tau_i}} + f^{e_k,e^{\tau_i}}, \\ f^{e',e^{\tau_i}} &= \begin{cases} f^{e',e_j} + f^{e',e_k} & \text{if } e' \in \{e, e_i\}, \\ f^{e',e} + f^{e',e_i} & \text{if } e' \in \{e_j, e_k\}, \end{cases} \quad (\text{distinct } i,j,k \in \{1,2,3\}). \end{aligned}
$$

Lemma 4.4. (1) $\alpha_1 + \alpha_2 + \alpha_3 \geq 2$.

(2) $\alpha_j \geq 2 - \alpha_1$ $(j = 2,3)$ *if there exists an optimal multiflow f exchangeable at e^{τ_1}.*

Proof. Take an optimal multiflow f in G. By Lemma 3.1, symmetry, and (4.4), we have $\alpha_2+\alpha_3 \geq \max\{2f^{e,e_2}, 2f^{e_1,e_3}\}+\max\{2f^{e,e_3}, 2f^{e_1,e_2}\} \geq (f^{e,e_2}+f^{e_1,e_3})+(f^{e,e_3}+f^{e_1,e_2}) = f^{e^{\tau_1}} \geq 2 - \alpha_1$. Thus we have (1).

Suppose that f is exchangeable at e^{τ_1}. By the exchange operations between $\mathcal{P}(e,e_3)$ and $\mathcal{P}(e_1,e_2)$ at e^{τ_1}, as in Figure 17, we can make f satisfy $f^{e,e_3} = 0$ or $f^{e_1,e_2} = 0$. If $f^{e,e_3} = 0$, then $f^{e^{\tau_2}} = (f^{e,e_1}+f^{e_1,e_2})+f^{e_2,e_3} \leq (1-f^{e_1,e_3})+(1-f^{e_1,e_2}-f^{e,e_2}) = 2-f^{e^{\tau_1}} \leq \alpha_1$, and hence $\alpha_2 \geq 2 - f^{e^{\tau_2}} \geq 2 - \alpha_1$. If $f^{e_1,e_2} = 0$, then $f^{e^{\tau_2}} = (f^{e,e_1} + f^{e,e_3}) + f^{e_2,e_3} \leq (1-f^{e,e_2})+(1-f^{e,e_3}-f^{e_1,e_3}) = 2-f^{e^{\tau_1}} \leq \alpha_1$, and hence $\alpha_2 \geq 2-\alpha_1$. The case of $j = 3$ is similar; apply the exchange operations between $\mathcal{P}(e,e_2)$ and $\mathcal{P}(e_1,e_3)$ above. □

Take an optimal multiflow f' in G^{τ_i,α_i}. By contracting edge e^{τ_i} and simplifying created nonsimple paths (if exist), we obtain an optimal multiflow f in G. In this case, we say that f is *derived from* f' or f is an optimal multiflow in G *derived from* G^{τ_i,α_i}. Note that $\mathcal{P}(e^{\tau_i})$ may increase, which is caused by a path in f' passing through y and y^{τ_i} not using e^{τ_i}. The position $(\rho_i(y), \rho_i(y^{\tau_i}))$ in $\mathcal{K}_p$ gives information of $\mathcal{P}(e^{\tau_i})$ as follows. See Figure 20 for an intuition of the lemma.

Lemma 4.5. *Suppose $d^{\rho_i}(e^{\tau_i}) = d_{\mathcal{K}_p}(\rho_i(y), p)+d_{\mathcal{K}_p}(p, \rho_i(y^{\tau_i}))$ with $\rho_i(y^{\tau_i}) \neq p$. Let $f = (\mathcal{P}; \kappa)$ be an optimal multiflow in G derived from G^{τ_i,α_i}, and let $u := \rho_i(y^{\tau_i}) \in L_p \cup Q_p$.*

(1) *$\mathcal{P}(e^{\tau_i})$ is a $(*, yy^{\tau_i}, [u])$-set if $u \in L_p$, and a $(*, yy^{\tau_i}, [u]_1)$-set if $u \in Q_p$.*

(2) *If P in $\mathcal{P}(e,e_i)$ is exchangeable with a path P' in $\mathcal{P}(e^{\tau_i})$ at e_i, then P is a $(*, xyx_i, [u])$-path if $u \in L_p$ and a $(*, xyx_i, [u]_1)$-path if $u \in Q_p$.*

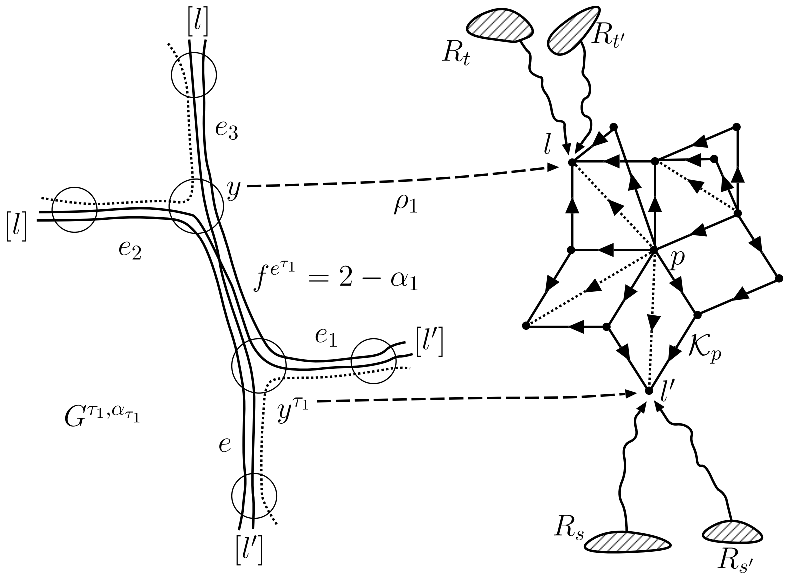


Figure 20: Perturbing ρ to a critical neighbor ρ_1

Proof. (1). Suppose that f is derived from an optimal multiflow f' in G^{τ_i,α_i}. Take an (s, yy^{τ_i}, t)-path $P \in \mathcal{P}(e^{\tau_i})$, which is contracted from an (s, yy^{τ_i}, t)-path $\bar{P}$. Therefore, by geodesic condition for (ρ_i, f'), we have

$$d_{\mathcal{K}}(R_s, R_t) = d_{\mathcal{K}}(R_s, \rho_i(y)) + d_{\mathcal{K}}(\rho_i(y), \rho_i(y^{\tau_i})) + d_{\mathcal{K}}(\rho_i(y^{\tau_i}), R_t). \tag{4.5}$$

Hence, by the assumption with the triangle inequality, R_t satisfies $d_{\mathcal{K}}(p, R_t) = d_{\mathcal{K}_p}(p, u) + d_{\mathcal{K}}(u, R_t)$. Therefore, by Lemma 4.2, if $u \in L_p$, then $(R_t)^p = u$, implying $t \in [u]$, and if $u \in Q_p$, then $(R_t)^p \sim_1 u$, implying $t \in [u]_1$.

(2). Suppose that P and P' are obtained by contracting e^{τ_i} from an $(s, yy^{\tau_i}x_i, *)$-path $\bar{P}$ and a $(*, xy^{\tau_i}x_i, t)$-path $\bar{P}'$. Obviously, $\bar{P}$ is exchangeable with $\bar{P}'$ at e_i. Do the exchange operation. If a simplification occurs, then f^e decreases on a created cycle, and hence the vertices in this cycle have the same potential (by the saturation condition). Thus the image of the resulting (s,t)-path passes through $R_s \to \rho_i(y) \to \rho_i(y^{\tau_i}) \to R_t$, i.e., (4.5) holds. Therefore, by the same argument, we have $t \in [u]$ if $u \in L_p$ and $t \in [u]_1$ if $u \in Q_p$. □

Next we analyze $\mathcal{P}(e^{\tau_i})$ for an arbitrary optimal multiflow $f = (\mathcal{P}; \kappa)$ in G. Let $\mathcal{P}(e^{\tau_i}; \rho_i)$ be the set of (s, yy^{τ_i}, t)-paths P in $\mathcal{P}(e^{\tau_i})$ satisfying

$$d_{\mathcal{K}}(R_s, R_t) = d_{\mathcal{K}}(R_s, \rho_i(y)) + d_{\mathcal{K}_p}(\rho_i(y), \rho_i(y^{\tau_i})) + d_{\mathcal{K}}(\rho_i(y^{\tau_i}), R_t). \tag{4.6}$$

Its flow-value is denoted by $f^{e^{\tau_i};\rho_i}$. Then Lemma 4.5 (1) also holds with replaced by $\mathcal{P}(e^{\tau_i})$ by $\mathcal{P}(e^{\tau_i}; \rho_i)$. We can estimate $f^{e^{\tau_i};\rho_i} (\leq f^{e^{\tau_i}})$ by the following lemma.

Lemma 4.6. *Suppose that each terminal is proper or essential.*

(1) $d^{\rho_i}(e^{\tau_i}) f^{e^{\tau_i};\rho_i} + (d^{\rho_i}(e^{\tau_i}) - 2)(f^{e^{\tau_i}} - f^{e^{\tau_i};\rho_i}) \geq d^{\rho_i}(e^{\tau_i})(2 - \alpha_i)$.

(2) *If* $d^{\rho_i}(e^{\tau_i}) \geq 2$, *then* $f^{e^{\tau_i};\rho_i} \geq 2 + (d^{\rho_i}(e^{\tau_i}) - 2) f^{e,e_i} - \dfrac{d^{\rho_i}(e^{\tau_i})\alpha_i}{2}$.

Proof. We use the formula (2.6) of the duality gap. By definition, we have

$$\mathrm{opt}(\mu; G) = \mathrm{opt}(\mu; G^{\tau_i,\alpha_i}) = d^{\rho_i}(G^{\tau_i,\alpha_i}).$$

Let f' be the multiflow for G^{τ_i,α_i} obtained by deleting all paths in $\mathcal{P}(e^{\tau_i})$ from f. Then the duality gap for (f', ρ_i) in G^{τ_i,α_i} is

$$d^{\rho_i}(G^{\tau_i,\alpha_i}) - \|f'\|_\mu = \sum_{P \in \mathcal{P}(e^{\tau_i})} \mu(s_P, t_P)/\kappa. \tag{4.7}$$

We next estimate the first term $\delta_1 := \sum_{e \in E(G^{\tau_i,\alpha_i})} d^{\rho_i}(e)(c(e) - (f')^e)$ in (2.6), which means the unsaturation of edges. Since e^{τ_i} has no flow in G^{τ_i,α_i}, this contributes $d^{\rho_i}(e^{\tau_i})(2 - \alpha_i)$ for δ_1.

For $s \in S$, let $\bar{R}_s$ denote the connected component of the boundary of R_s containing $\rho(s)$. By the essentialness assumption and Lemma 3.3, $\bar{R}_s$ also contains $\rho_i(s)$. Therefore, the deletion of an (s_P, yy^{τ_i}, t_P)-path P contributes at least $\{d_{\mathcal{K}}(\bar{R}_{s_P}, \rho_i(y)) + d_{\mathcal{K}}(\rho_i(y^{\tau_i}), \bar{R}_{t_P})\}/\kappa$ for the unsaturation of edges except e^{τ_i}. Thus we have

$$\delta_1 \geq d^{\rho_i}(e^{\tau_i})(2 - \alpha_i) + \sum_{P \in \mathcal{P}(e^{\tau_i})} \{d_{\mathcal{K}}(\bar{R}_{s_P}, \rho_i(y)) + d_{\mathcal{K}}(\rho_i(y^{\tau_i}), \bar{R}_{t_P})\}/\kappa.$$

Since the duality gap (4.7) is at least δ_1, we have

$$\sum_{P \in \mathcal{P}(e^{\tau_i})} \{d^{\rho_i}(e^{\tau_i}) - \Delta_P\}/\kappa \geq d^{\rho_i}(e^{\tau_i})(2 - \alpha_i), \tag{4.8}$$
$$\text{where } \Delta_P := d_{\mathcal{K}}(\bar{R}_{s_P}, \rho_i(y)) + d^{\rho_i}(e^{\tau_i}) + d_{\mathcal{K}}(\rho_i(y^{\tau_i}), \bar{R}_{t_P}) - d_{\mathcal{K}}(R_{s_P}, R_{t_P}).$$

We show:

(4.9) Δ_P is a nonnegative even integer, and is zero if and only if $P \in \mathcal{P}(e^{\tau_i}; \rho_i)$.

Suppose that this is true. Then the LHS of (4.8) is at most $d^{\rho_i}(e^{\tau_i}) f^{e^{\tau_i};\rho_i} + (d^{\rho_i}(e^{\tau_i}) - 2)(f^{e^{\tau_i}} - f^{e^{\tau_i};\rho_i})$. Then we obtain (1). (2) follows from substituting $f^{e^{\tau_i}} = f^e + f^{e'} - 2f^{e,e'} \leq 2 - 2f^{e,e_i}$ to (1).

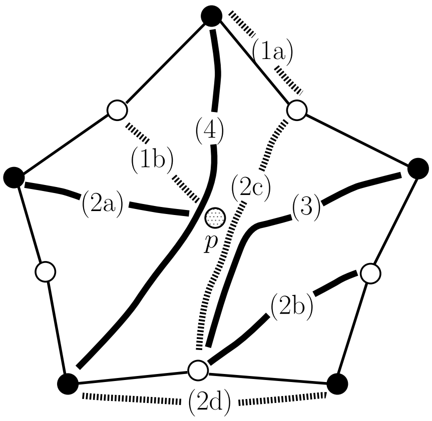
We show now (4.9). Since $\rho(P)$ connects $\bar{R}_{s_P}$ and $\bar{R}_{t_P}$ with length $d_{\mathcal{K}}(R_{s_P}, R_{t_P})$, we have $d_{\mathcal{K}}(R_{s_P}, R_{t_P}) = d_{\mathcal{K}}(\bar{R}_{s_P}, \bar{R}_{t_P})$. Moreover the vertices in $\bar{R}_s$ belong to the same color class of the leg-graph. Thus we get the first statement. For the second statement, the if part follows from $d_{\mathcal{K}}(\cdot, \bar{R}_s) \geq d_{\mathcal{K}}(\cdot, R_s)$ and the first statement. For the only-if part, we show $d(R_{s_P}, \rho_i(y)) = d(\bar{R}_{s_P}, \rho_i(y))$ for $P \in \mathcal{P}(e^{\tau_i}; \rho_i)$. This follows from the facts that R_{s_P} cannot contain $\rho_i(y)$ in the interior, and that $\rho_i(y)$ and $\rho(y)$ belong to the same connected component obtained by deleting the interior of R_{s_P} from $\mathcal{K}$ (see the proof of Lemma 3.3). □

For $i \in 1, 2, 3$, $\{\rho_i(y), \rho_i(y^{\tau_i})\}$ belongs to $\mathcal{K}_p$. We classify the position $\{\rho_i(y), \rho_i(y^{\tau_i})\}$ into eight cases (1a), (1b), (2a), (2b), (2c), (2d), (3), (4) in Figure 21. For the six cases, Lemma 4.5 (1) is applicable, which determines a type of $\mathcal{P}(e^{\tau_i}; \rho_i)$ (and $\mathcal{P}(e^{\tau_i})$ if f is an optimal multiflow derived from G^{τ_i,α_i}) as summarized in Table 1. The third column indicates the exchangeability of $\mathcal{P}(e^{\tau_i}; \rho_i)$ at e^{τ_i}. By the local geodesic condition, a $([u], yy^{\tau_i}, *)$-set is always exchangeable. To see the exchangeability of (2b), consider the 2-subdivision $\mathcal{K}^2$ and consider $(\mathcal{K}^2)_{p'}$ for the midpoint p' of a folder in $\mathcal{K}_p$ containing p, q, q', l; then (q, q') is in case (4) in $(\mathcal{K}^2)_{p'}$.

For distinct i, j, k, $\mathcal{P}(e^{\tau_i})$ is a disjoint union of $\mathcal{P}(e, e^{\tau_i})$, $\mathcal{P}(e_i, e_j)$, and $\mathcal{P}(e_i, e_k)$. We denote $\mathcal{P}(e^{\tau_i}; \rho_i) \cap \mathcal{P}(e, e^{\tau_i})$, $\mathcal{P}(e^{\tau_i}; \rho_i) \cap \mathcal{P}(e_i, e_j)$, and $\mathcal{P}(e^{\tau_i}; \rho_i) \cap \mathcal{P}(e_i, e_k)$ by $\mathcal{P}(e, e^{\tau_i}; \rho_i)$, $\mathcal{P}(e_i, e_j; \rho_i)$, and $\mathcal{P}(e_i, e_k; \rho_i)$, respectively. The corresponding flow-values are denoted by $f^{e,e^{\tau_i};\rho_i}$, $f^{e_i,e_j;\rho_i}$, and $f^{e_i,e_k;\rho_i}$, respectively. Obviously,

$$f^{e^{\tau_i};\rho_i} = f^{e,e^{\tau_i};\rho_i} + f^{e_i,e_j;\rho_i} + f^{e_i,e_k;\rho_i}. \tag{4.10}$$

We will use this notation and decomposition.



(1a) $\{l, q\}, l \sim q$
(1b) $\{p, q\}$
(2a) $\{p, l\}$
(2b) $\{q, q'\}, q \sim q'$
(2c) $\{q, q'\}, q \not\sim q'$
(2d) $\{l, l'\}, l \sim l'$
(3) $\{l, q\}, l \not\sim q$
(4) $\{l, l'\}, l \not\sim l'$

Figure 21: Possible patterns of $\{\rho_i(y), \rho_i(y^{\tau_i})\}$, where $q, q' \in Q_p$ and $l, l' \in L_p$

Table 1: Type of $\mathcal{P}(e^{\tau_i}; \rho_i)$ (and $\mathcal{P}(e^{\tau_i})$ if f is an optimal multiflow derived from G^{τ_i,α_i})

case	$(\rho_i(y), \rho_i(y^{\tau_i}))$	$\mathcal{P}(e^{\tau_i}; \rho_i)$	exchangeability
(1b)	(q, p)	$([q]_1, yy^{\tau_i}, *)$	
(2a)	(l, p)	$([l], yy^{\tau_i}, *)$	○
(2b)	(q, q'), $q \sim_1 l \sim_1 q'$	$([q]_1 \setminus [l], yy^{\tau_i}, [q']_1 \setminus [l])$	○
(2c)	(q, q'), $q \not\sim q'$	$([q]_1, yy^{\tau_i}, [q']_1)$	
(3)	(l, q), $l \not\sim q$	$([l], yy^{\tau_i}, [q]_1)$	○
(4)	(l, l'), $l \not\sim l'$	$([l], yy^{\tau_i}, [l'])$	○

Proof of Theorem 4.3 (1). We first show that for every optimal multiflow f, there is $e' \in \{e, e_1, e_2, e_3\}$ with $f^{e'} = 0$. Suppose that this is true. Then e' is independent of f, say, $e' = e$ (after relabeling). Hence $\mathcal{P}(e^{\tau_i}) = \mathcal{P}(e_i)$, and $1 \geq f^{e_i} = f^{e^{\tau_i}} \geq 2 - \alpha_i \geq 1$. Necessarily $\alpha_i = 1$, $f^{e^{\tau_i}} = f^{e_i} = 1$ and $f^{e_i,e_j} = 1/2$.

By Lemma 4.6 (2) with $\alpha_i \leq 1$ and $d^{\rho_i}(e^{\tau_i}) = 2$, we have $f^{e^{\tau_i};\rho_i} \geq 1$. Here ρ_i is in case (2a) or (2b) in Figure 21. In particular, $\mathcal{P}(e^{\tau_i}; \rho_i)$ is exchangeable at e^{τ_i} (Table 1). By the exchange operations between $\mathcal{P}(e, e_j)$ and $\mathcal{P}(e_i, e_k)$, and between $\mathcal{P}(e, e_k)$ and $\mathcal{P}(e_i, e_j)$ (as in Figure 17), we can make f satisfy $f^{e',e''} \geq 1/2$ for some distinct $e', e'' \in \{e, e_1, e_2, e_3\}$. Note that any optimality-keeping exchange operation keeps $f^{e^{\tau_i};\rho_i} \geq 1$. We may assume $(e', e'') = (e_2, e_3)$ (by relabeling). Then the equality holds in $1 \geq \alpha_1 \geq 2 - f^{e^{\tau_1}} \geq 2f^{e_2,e_3} \geq 1$. Hence, $f^{e_2,e_3} = 1/2$, $f^{e^{\tau_1}} = 1$, f is optimal for G^{τ_1,α_1} (and is derived from an optimal multiflow in G^{τ_1,α_1}). We may assume $f^{e,e^{\tau_1}} \leq 1/2 \leq f^{e_1,e^{\tau_1}}$. Since $\mathcal{P}(e^{\tau_1})$ is exchangeable, by the exchange operation between $\mathcal{P}(e, e_3)$ and $\mathcal{P}(e_1, e_2)$ at e^{τ_1}, we can make f satisfy $f^{e,e_3} = 0$, $f^{e_1,e_3} = 1/2$, and $f^{e,e_2} + f^{e_1,e_2} = 1/2$. If $f^{e,e_2} = 0$, then $f^e = 0$, and this necessarily holds from the beginning and the exchange operations have not been applied above (in the exchange operations above the simplification of paths could not occur since such a nonsimple path uses two edges incident to y).

Suppose (indirectly) $f^{e,e_2} > 0$ (and hence $f^{e_1,e_2} < 1/2$). By $f^{e_1,e_3} = 1/2$, we have $\alpha_2 = 1$ and $f^{e^{\tau_2}} = f^{e_2,e_3} + f^{e_1,e_2} + f^{e,e_1} = 1$. By $f^{e_2,e_3} = 1/2$ and $f^{e_1,e_2} < 1/2$, we have $f^{e,e_1} > 0$. Since $\mathcal{P}(e^{\tau_1})$ is exchangeable, by the exchange operation between $\mathcal{P}(e, e_2)$ and $\mathcal{P}(e_1, e_3)$ at e^{τ_1}, we can make f satisfy $f^{e_1,e_2} = 1/2$. Since $\mathcal{P}(e^{\tau_2})$ is also exchangeable, the exchange operation at e^{τ_2} for two paths, one from $\mathcal{P}(e_2, e_3)$ and the other one from $\mathcal{P}(e, e_1) \neq \emptyset$, keeps the optimality and results in $f^{e_1,e_2} > 1/2$. A contradiction to $1 \geq \alpha_2 \geq 2f^{e_1,e_2}$. Therefore $f^{e^{\tau_i}} = f^{e_i} = 1$, $f^e = 0$, $f^{e_i,e_j} = 1/2$, and $\alpha_i = 1$. In particular, $\mathcal{P}(e_i)(=\mathcal{P}(e^{\tau_i}))$ is exchangeable at e_i.

Next consider the position $(\rho_1(y), \rho_1(y^{\tau_1}))$ in $\mathcal{K}_p$; (2a) or (2b). Then $d^{\rho_1}(e^{\tau_1}) = d_{\mathcal{K}_p}(\rho_1(y), p) + d_{\mathcal{K}_p}(p, \rho_1(y^{\tau_1}))$ holds. Since $\mathcal{P}(e_i)$ is exchangeable at e_i, any path in $\mathcal{P}(e_2, e_3)$ is exchangeable with any path in $\mathcal{P}(e^{\tau_1})$ at e_2 and at e_3. By Lemma 4.5 (2), if $\rho_1(y) \neq p$, then $\mathcal{P}(e_2, e_3)$ is a $([q]_1, x_2yx_3, [q]_1)$-set for some $q \in Q_p$; a contradiction to the local geodesic condition. Thus $\rho_1(y) = p$ must hold. Consequently $(\rho_1(y), \rho_1(y^{\tau_1})) = (p, l_1)$ for $l_1 \in L_p$. By the same argument, we have $(\rho_2(y), \rho_2(y^{\tau_2})) = (p, l_2)$ and $(\rho_3(y), \rho_3(y^{\tau_3})) = (p, l_3)$ for $l_2, l_3 \in L_p$. By Lemma 4.5 (1), $\mathcal{P}(e_i, e_j)$ is an $([l_i], x_iyx_j, [l_j])$-set; the vertices l_1, l_2, l_3 are distinct by the local geodesic condition. □

Proof of Theorem 4.3 (2). Suppose to the contrary that $q = q'$, or $q \neq q'$ and there is no common folder containing pq and pq'. By relabeling and symmetry, we may assume

$$(\rho_i(y^{\tau_i}), \rho_i(y)) = \begin{cases} (p, q) & \text{if } q = q', \\ (q', p) \ \text{ or } \ (p, q) & \text{if } q \neq q', \end{cases} \quad (i = 1, 2, 3). \tag{4.11}$$

They are in case (1b) in Figure 21. Let $\bar{f}^{e^{\tau_i}} := f^{e^{\tau_i}} - f^{e^{\tau_i}; \rho_i}$. By Lemma 4.6 (1) for $d^{\rho_i}(e^{\tau_i}) = 1$, we have

$$f^{e^{\tau_i}; \rho_i} - \bar{f}^{e^{\tau_i}} \geq 2 - \alpha_i \quad (i = 1, 2, 3). \tag{4.12}$$

Claim 4.7. $\mathcal{P}(e_i, e_j; \rho_i) \cap \mathcal{P}(e_i, e_j; \rho_j) = \emptyset$ for $1 \leq i < j \leq 3$.

Proof. Take P from $\mathcal{P}(e_i, e_j; \rho_i) \cap \mathcal{P}(e_i, e_j; \rho_j)$. Suppose (say) that $(\rho_i(y^{\tau_i}), \rho_i(y)) = (p, q)$. According to Table 1, P is a $([q]_1, yy^{\tau_i}, *)$-path, and hence is a $([q]_1, x_jyx_i, *)$-path. If $(\rho_j(y^{\tau_j}), \rho_j(y)) = (p, q)$, then P is a $(*, x_jyx_i, [q]_1)$-path, and hence P is a $([q]_1, x_jyx_i, [q]_1)$-path; a contradiction to the local geodesic condition. If $(\rho_j(y^{\tau_j}), \rho_j(y)) = (q', p)$, then P is a $([q']_1, x_jyx_i, *)$-path, and hence P is a $([q']_1 \cap [q]_1, x_jyx_i, *)$-path, implying $[q']_1 \cap [q]_1 \neq \emptyset$, and the existence of u in $\mathit{\Pi}_p$ with $q' \sim_1 u \sim_1 q$. This in turn implies the existence of a folder containing pq and pq'; a contradiction to the assumption. □

Hence $\bar{f}^{e^{\tau_i}} \geq f^{e_i, e_j; \rho_j} + f^{e_i, e_k; \rho_k}$. By substituting this and (4.10) to (4.12), we get

$$f^{e, e^{\tau_i}; \rho_i} + f^{e_i, e_j; \rho_i} + f^{e_i, e_k; \rho_i} - (f^{e_i, e_j; \rho_j} + f^{e_i, e_k; \rho_k}) \geq 2 - \alpha_i \quad (\text{for distinct } i, j, k \in \{1, 2, 3\}).$$

Addition of these three inequalities yields

$$f^{e, e^{\tau_1}; \rho_1} + f^{e, e^{\tau_2}; \rho_2} + f^{e, e^{\tau_3}; \rho_3} \geq 6 - \alpha_1 - \alpha_2 - \alpha_3.$$

Since $f^{e, e^{\tau_i}; \rho_i} \leq f^{e, e_j} + f^{e, e_k}$, we have $2f^e = 2(f^{e, e_1} + f^{e, e_2} + f^{e, e_3}) \geq f^{e, e^{\tau_1}; \rho_i} + f^{e, e^{\tau_2}; \rho_2} + f^{e, e^{\tau_3}; \rho_3}$. From $f^e \leq c(e) = 1$, we have $\alpha_1 + \alpha_2 + \alpha_3 \geq 4$. However this contradicts $\alpha_i \leq 1$ for $i = 1, 2, 3$. □

5 Proof of (A), (B), and (C) and algorithmic implication

In this section, we complete the proof of Theorem 1.5 by proving three claims (A), (B), and (C) in Section 3, which are given in Sections 5.1 and 5.2.

In a key step of the proof of claim (C), we will make use of the following lemma, called the *uncrossing lemma*. Recall the notions of the forward orientation of $\mathcal{K}^2$, partition S_ρ, M_ρ, C_ρ, and forward semi-neighbors. The proof of Lemma 5.1 is given in Section 5.3.

Lemma 5.1. *For two optimal potentials ρ, ρ', there exists a forward semi-neighbor ρ^* of ρ that is optimal with $C_{\rho^*} = \{x \in C_\rho \mid \rho(x) = \rho'(x)\}$.*

In Section 5.4, we show that our proof indeed gives a polynomial time algorithm to find a 1/12-integral optimal multiflow provided the size of a realization of μ is fixed.

Table 2: Classification of $\{\rho'(y), \rho'(y^\tau)\}$, where $q, q' \in Q_p$ and $l, l' \in L_p$

case	$\{\rho'(y), \rho'(y^\tau)\}$	$d^{\rho'}(e^\tau)$	α_τ, G admissible	α_τ, $3G$ admissible
(1a)	$\{q, l\}$, $q \sim l$	1	0	0, 2/3, 4/3
(1b)	$\{p, q\}$	1	0	0, 2/3, 4/3
(2a)*	$\{p, l\}$	2	0, 1	0, 1/3, 2/3, 1, 4/3, 5/3
(2b)*	$\{q, q'\}$, $q \sim q'$	2	0, 1	0, 1/3, 2/3, 1, 4/3, 5/3
(2c)	$\{q, q'\}$, $q \not\sim q'$	2	0, 1	0, 1/3, 2/3, 1, 4/3, 5/3
(2d)	$\{l, l'\}$, $l \sim l'$	2	0, 1	0, 1/3, 2/3, 1, 4/3, 5/3
(3)*	$\{q, l\}$, $q \not\sim l$	3	0, 2/3, 4/3	$2m/9$ $(0 \leq m \leq 8)$
(4)*	$\{l, l'\}$, $l \not\sim l'$	4	0, 1/2, 1, 3/2	$m/6$ $(0 \leq m \leq 11)$

($*$ means that every optimal multiflow derived from $G^{\tau,\alpha}$ is exchangeable at e^τ)

5.1 Proof of (A) and (B)

Claims (A) and (B) are easy consequences of Theorem 4.3.

(A). We may assume that y has degree four. Suppose (to the contrary) that all three forks at y are unsplittable. Consider critical neighbors ρ_1, ρ_2, ρ_3 of ρ for three forks τ_1, τ_2, τ_3. As was seen in the proof of Lemma 3.6, for $i = 1, 2, 3$, $\alpha_i \in \{0, 1\}$ and $\{\rho_i(y), \rho_i(y^{\tau_i})\}$ belongs to folder F or F' in Figure 13 (a). By Lemma 4.4 (1), $\alpha_j = 1$ for some j, and then $d^{\rho_j}(e^{\tau_j}) = 2$ (since the numerator of (3.2) is even). This means that $\rho_j(y)$ and $\rho_j(y^{\tau_j})$ are not adjacent by a leg ((2a) or (2b) in Table 1). Then any optimal multiflow derived from G^{τ_j, α_j} is exchangeable at e^{τ_j}. By Lemma 4.4 (2), we have $\alpha_i = 1$ for all $i \in \{1, 2, 3\}$. Hence $\rho_i(y)$ and $\rho_i(y^{\tau_i})$ are not adjacent by a leg, and belong to a common folder (F or F') for $i \in \{1, 2, 3\}$. So Theorem 4.3 (1) is applicable. However, the configuration of (i) (Figure 19) is impossible. Therefore y must have a splittable fork. □

(B). Suppose that $\rho(y)$ is the midpoint p of a leg qq' in $\mathcal{K}$, and all ρ_i are backward. Then $\{\rho_i(y), \rho_i(y^{\tau_i})\} = \{p, q\}$ or $\{p, q'\}$ for all i (see Figure 13 (b)). Since G has an integer capacity and $d^{\rho_i}(e^{\tau_i}) = 1$, we have $\alpha_i \in \{0, 1\}$; the numerator of (3.2) is integral. By Theorem 4.3 (2), there is a folder containing pq and pq'. However, such a folder does not exist. A contradiction. This means that at least one of ρ_i is forward. □

5.2 Proof of (C)

We will repeat SPUP at inner nodes in C_ρ, which is always forward (Lemma 3.6 (1)). Then the number of inner nodes in C_ρ is nonincreasing. To bound the denominator of created fractional edges, we introduce a sharper degree condition than the restricted Eulerian condition. $(G; \rho)$ is called *admissible* if the deletion of edges between S_ρ makes $(G; \rho)$ restricted Eulerian. Namely, we allow edges between S_ρ to have a fractional capacity. In view of the paragraph after Lemma 3.4, if $(G; \rho)$ is admissible and τ is a fork at C_ρ, then the numerator of formula (3.2) of α_τ is even. Thus, for a critical neighbor ρ' of ρ, the possible cases of $\{\rho'(y), \rho'(y^\tau)\}$ with $(d^{\rho'}(e^\tau), \alpha_\tau)$ are summarized as in Table 2; see also Figure 21. Here, for $p \in V(\mathcal{K})$, $(\mathcal{K}^2)_p$ is star-shaped, and the leg-graph of the boundary of $(\mathcal{K}^2)_p$ is identified with $\mathit{\Pi}_p$.

Our goal is to sweep out inner nodes from C_ρ. We will use the following fact for this purpose.

(5.1) For an edge $e = xy$ with $x, y \in C_\rho$ and $\rho(x) = \rho(y)$, if $c(e) = f^e$ for every optimal multiflow f, then there exists a forward neighbor ρ' of ρ such that $\rho'(x) \neq \rho'(y)$ and ρ' is optimal.

Proof. Decrease $c(e)$ by $\beta \geq 0$. The resulting graph is denoted by $G^{e,\beta}$. Obviously $\mathrm{opt}(\mu; G^{e,\beta}) \leq \mathrm{opt}(\mu; G)$. By the same argument as in the proof of Proposition 3.2, the maximum possible $\beta \geq 0$ with $\mathrm{opt}(\mu; G^{e,\beta}) = \mathrm{opt}(\mu; G)$ is the minimum of $\{d^{\rho'}(G) - d^{\rho}(G)\}/d^{\rho'}(e)$ over all neighbors ρ' of ρ with $d^{\rho'}(e) > 0$. By $c(e) = f^e$, this must be zero. Any neighbor ρ' attaining the maximum β is an optimal forward neighbor as required. □

In successive SPUP, the value of α_τ is monotone nonincreasing.

(5.2) For two forks τ and τ' on distinct nodes, we have $\alpha_{\tau'}(G^{\tau,\alpha_\tau}) \leq \alpha_{\tau'}(G)$.

Proof. For $\alpha_\tau := \alpha_\tau(G)$ and $\alpha' := \alpha_{\tau'}(G^{\tau,\alpha_\tau})$, $(G^{\tau,\alpha_\tau})^{\tau',\alpha'}$ is well-defined, and $\mathrm{opt}(\mu; G) = \mathrm{opt}(\mu; (G^{\tau,\alpha_\tau})^{\tau',\alpha'})$ by definition. Since $\mathrm{opt}(\mu; G) \geq \mathrm{opt}(\mu; G^{\tau',\alpha'}) \geq \mathrm{opt}(\mu; (G^{\tau,\alpha_\tau})^{\tau',\alpha'})$, we have $\mathrm{opt}(\mu; G^{\tau',\alpha'}) = \mathrm{opt}(\mu; G)$. This means $\alpha_{\tau'}(G) \geq \alpha'$. □

Let us start the proof of claim (C). In the initial step, G is properly-inner Eulerian. For any optimal potential ρ (for $\mathrm{DLP}(\mathcal{K}^2; \{R_s\}_{s\in S})$), $(G; \rho)$ is restricted Eulerian and admissible. By the degree-reductions (Section 3.3), we can make G so that each inner node in C_ρ has degree four, each proper terminal in C_ρ has degree one, and each improper terminal in C_ρ has degree two. We may assume that there is no splittable fork at C_ρ and all improper terminals are essential (see (3.4)). By edge-subdivisions, we can further assume:

(5.3) For every edge xy with $y \in C_\rho$, we have $\rho(x) \in V((\mathcal{K}^2)_{\rho(y)})$.

After the preprocessing (Section 5.2.1), at first three stages, we apply SPUP at a fork having maximum α_τ so that split nodes go out C_ρ (Sections 5.2.2 and 5.2.3). Then the number of inner nodes in C_ρ decreases, and also the maximum α_τ decreases by (5.2). When the maximum α_τ becomes close to 1, the estimate by Lemmas 4.4 and 4.6 becomes effective, and finally we can apply 1-SPUP to reach the goal (Section 5.2.4).

5.2.1 Preprocessing

We first modify $(G; \rho)$ so that $(G; \rho)$ is restricted Eulerian and each terminal in C_ρ is incident to a unique node (while keeping (3.4)). Take an improper terminal s in C_ρ of degree two, incident to two nodes x, y. For a fork $\tau := (xs, s, sy)$ we have $\alpha_\tau < 2$ (since s is essential). If $\alpha_\tau = 0$, then every optimal multiflow f has paths connecting s with the flow-value 2 (by Lemma 3.1), and hence we can apply the degree-1 reduction to s; the new terminals remain essential; see (3.3). So consider the case where $0 < \alpha_\tau < 2$. Take a critical neighbor ρ' of ρ. By $0 < \alpha_\tau < 2$, we have $d^{\rho'}(e^\tau) > 1$ (the numerator of (3.2) is even). Also, every optimal multiflow in G^{τ,α_τ} must have paths connecting s passing through e^τ. This means that $\rho'(s)$ must lie in the boundary of R_s, and $\rho'(s^\tau)$ is not in R_s with $d_{\mathcal{K}}(R_s, \rho'(s^\tau)) = d_{\mathcal{K}}(\rho'(s), \rho'(s^\tau))$. By Lemma 3.3, $\rho'(s)$ and $\rho(s)$ must lie in the same connected component of the boundary of R_s. So the possible positions of $\{\rho'(s), \rho'(s^\tau)\}$ are $\rho'(s) = p$ and $\rho'(s^\tau) \in L_p$ (case (2a)) or $(\rho'(s), \rho'(s^\tau)) \in L_p \times L_p$ with $\rho'(s) \sim \rho'(s^\tau)$ (case (2d)), where $p := \rho(s)$; see Figure 21. In both cases, we have $d^{\rho'}(e^\tau) = 2$ and hence $\alpha_\tau = 1$. Apply the corresponding 1-SPUP. Then s^τ falls into S_ρ, and hence $(G; \rho)$ keeps the restricted Eulerian condition. Furthermore s has degree one and is essential (by (3.3)). Repeat this process to improper terminals until each terminal in C_ρ has degree one and a unique neighbor.

5.2.2 3/2-SPUP

From here, we consider SPUP at inner nodes in C_ρ. By searching all forks at C_ρ, take a fork τ at inner node $y \in C_\rho$ with (maximum) $\alpha_\tau = 3/2$. Let $p := \rho(y)$. Take a critical neighbor ρ' of ρ with respect to τ. Then $d^{\rho'}(e^\tau) = 4$, and thus $\{\rho'(y), \rho'(y^\tau)\}$ is in case (4) in Table 2; both y and y^τ fall into $S_{\rho'}$. Apply 3/2-SPUP: $(G;\rho) \leftarrow (G^{\tau,\alpha_\tau};\rho')$.

Then $(G;\rho)$ is admissible and $(2G;\rho)$ is restricted Eulerian. Repeat this process until there is no fork τ at C_ρ with $\alpha_\tau = 3/2$. After that, the possible values of α_τ of forks τ at C_ρ are 0, 2/3, 1, 4/3. Here $\alpha_\tau = 1/2$ (case (4)) never occurs since this implies the existence of another fork τ' with $\alpha_{\tau'} = 3/2$ by Lemma 4.4 (2) and the exchangeability in case (4).

5.2.3 4/3-SPUP and 7/6-SPUP

By searching all forks at C_ρ, take a fork τ at an inner node y in C_ρ with (maximum) $\alpha_\tau = 4/3$. Then a critical neighbor ρ' is in case (3) in Table 2. Apply 4/3-SPUP: $(G;\rho) \leftarrow (G^{\tau,\alpha_\tau};\rho')$. Then one of y and y^τ falls into S_ρ, and the other falls into M_ρ and has degree 8/3. Therefore

(5.4) $(3G;\rho)$ is admissible and $(6G;\rho)$ is restricted Eulerian.

From now on, we keep this condition (5.4). In the next SPUP, α_τ belongs to $1/3(2\mathbf{Z}_+/3 \cup \mathbf{Z}_+/2)$; see the fifth column in Table 2. Note that $\alpha_\tau > 4/3$ is impossible by (5.2). By this fact together with Lemma 4.4 (2), $\alpha_\tau \in \{1/6, 2/9, 4/9\}$ is also impossible. So the possible values of α_τ are 0, 1/3, 2/3, 5/6, 8/9, 1, 10/9, 7/6, 4/3.

Apply SPUP for a fork τ at an inner node in C_ρ with $\alpha_\tau = 4/3$. Here, if $\alpha_\tau = 4/3$ in (1b, 2a) occurs, then $\rho'(y)$ or $\rho'(y^\tau)$ does not move, and the number of inner nodes in C_ρ does not decrease. However, by the uncrossing lemma (Lemma 5.1), we can take a forward critical semi-neighbor ρ^* of ρ with $y, y^\tau \notin C_{\rho^*}$ as follows.

Let $(\tilde{G};\tilde{\rho})$ be the graph with the optimal potential at just after the final 3/2-SPUP. By (5.2), necessarily $\alpha_\tau(\tilde{G}) = 4/3$ holds. This means that *we could have chosen this fork τ in the first 4/3-SPUP*. Consider a critical neighbor ρ'' of $\tilde{\rho}$ with respect to τ in $\tilde{G}$. ρ'' is necessarily in case (3), and can be regarded as an optimal potential for the current graph G^{τ,α_τ} by $\rho''(\tilde{y}^{\tilde{\tau}}) := \rho''(\tilde{y})$ for processed forks $\tilde{\tau}$ at $\tilde{y}$. By the uncrossing lemma for (ρ', ρ''), there is another optimal forward semi-neighbor ρ^* of ρ' with $C_{\rho^*} = \{y \in C_{\rho'} \mid \rho'(y) = \rho''(y)\}$. Both $\rho''(y)$ and $\rho''(y^\tau)$ are in $V((\mathcal{K}^2)_p) \setminus \{p\}$. Hence $y, y^\tau \notin C_{\rho^*}$. Let $(G;\rho) \leftarrow (G^{\tau,\alpha_\tau};\rho^*)$; the number of inner nodes in C_ρ strictly decreases. In this way, repeat 4/3-SPUP. After the procedure, the possible values of α_τ are 0, 1/3, 2/3, 5/6, 8/9, 1, 10/9, 7/6.

Next apply SPUP for a fork τ at inner node $y \in C_\rho$ with $\alpha_\tau = 7/6$. In this case, its critical neighbor ρ' is in case (4). Thus 7/6-SPUP keeps (5.4), and the number of inner nodes in C_ρ decreases. After the procedure, the possible values of α_τ are $0, 1/3, 2/3, 8/9, 1, 10/9$; note that $\alpha_\tau < 7/6$ excludes 5/6 (case (4)) by Lemma 4.4 (2).

5.2.4 1-SPUP

Take any inner node $y \in C_\rho$, a fork τ, and a critical neighbor ρ' of ρ with respect to τ. Let $p := \rho(y)$. The possible cases of (α_τ, ρ') are $\alpha_\tau = 1/3$ in (2c, 2d), $\alpha_\tau = 2/3$ in (1a, 1b, 2c, 2d), $\alpha_\tau = 8/9$ in (3), $\alpha_\tau = 1$ in (2a, 2b, 2c, 2d, 4), and $\alpha_\tau = 10/9$ in (3). Note that $\alpha < 4/3$ excludes $\alpha_\tau \in \{1/3, 2/3\}$ in (2a, 2b, 3, 4) by Lemma 4.4 (2).

The main obstructions to keeping (5.4) are the occurrences of $\alpha_\tau = 10/9$ in (3) and $\alpha_\tau = 1$ in (2c). Sometimes we can proceed SPUP when the latter case occurs. Suppose $\alpha_\tau = 1$ in (2c) with $(\rho'(y), \rho'(y^\tau)) = (q, q')$. Then, for every optimal multiflow $f = (\mathcal{P}; \kappa)$ in G derived from G^{τ,α_τ}, $\mathcal{P}(e^\tau)$ is a $([q]_1, yy^\tau, [q']_1)$-set (see Lemma 4.5 (1) with Table 1). Here consider the following condition:

(5.5) There is no optimal multiflow $f = (\mathcal{P}; \kappa)$ derived from G^{τ,α_τ} such that

(i) $\mathcal{P}(e^\tau)$ is an $([l], yy^\tau, [q']_1)$-set for some $l \in L_p$ with $l \sim_1 q$, or

(ii) $\mathcal{P}(e^\tau)$ is a $([q]_1, yy^\tau, [l'])$-set for some $l' \in L_p$ with $l' \sim_1 q'$.

Suppose that this condition is met. Apply SPUP: $(G; \rho) \leftarrow (G^{\tau,\alpha_\tau}; \rho')$. Continue SPUP at C_ρ. Then $d^\rho(e^\tau)$ keeps 2, and hence α_τ remains in $1/3(2\mathbf{Z}_+/3 \cup \mathbf{Z}_+/2)$ (see the paragraph after Lemma 3.4).

Indeed, suppose that one of $\rho(y)$ and $\rho(y^\tau)$, say $\rho(y)$, moves at some SPUP. Then $\rho(y) = l \in L_p$ with $l \sim_1 q$ (since SPUP is forward). Consider an optimal multiflow f in the current graph. Then any (s, yy^τ, t)-path in f induces by ρ a path passing through $R_s \to l \to \rho(y^\tau) \to R_t$. As in Lemma 4.5, the type of (s,t) is determined by the position of $(l, \rho(y^\tau))$. Also $\rho(y^\tau)$ is q' or $l' \in L_p$ with $l' \sim_1 q'$. If $\rho(y^\tau) = q'$ or $l' \in L_p$ with $l' \not\sim l$, then any (s, yy^τ, t)-path is an $([l], yy^\tau, [q']_1)$-path (see Table 1). By contracting edges $e^{\tau'}$ for processed τ' (after τ), we get an optimal multiflow f in G^{τ,α_τ} so that any (s, yy^τ, t)-path is an $([l], yy^\tau, [q']_1)$-path. Then the optimal multiflow in G derived from f violates (5.5); a contradiction. Hence $\rho(y^\tau) = l'$ with $l \sim l'$; in particular both y and y^τ fall into S_ρ. In this way, we can continue SPUP without an increase in the denominator of α_τ.

1-SPUP with (2c) is called *mixed* if it satisfies (5.5), and called *unmixed* otherwise. We can avoid 10/9-SPUP and unmixed 1-SPUP by examining all three forks τ_1, τ_2, τ_3 at y and their critical neighbors ρ_1, ρ_2, ρ_3. The main technical statement here is the following.

Proposition 5.2. *Suppose that ρ_j is in case neither* (2d) *nor* (4) *for $j = 2, 3$.*

1. *If $\alpha_1 = 10/9$, then, for $j = 2$ or 3, ρ_j is in case* (2c) *with $\alpha_j = 1$.*
2. *If ρ_1 is in case* (2c) *with $\alpha_1 = 1$ such that the corresponding 1-SPUP is unmixed, then both ρ_2 and ρ_3 are in case* (2c) *with $\alpha_2 = \alpha_3 = 1$, and by a relabeling fixing $\{\{e, e_1\}, \{e_2, e_3\}\}$, one of the following holds:*

   (2-0) *$\rho_3(y) \sim \rho_2(y) \sim \rho_3(y^{\tau_3}) \sim \rho_2(y^{\tau_2}) \sim \rho_3(y)$.*

   (2-1) *$\rho_3(y) \sim \rho_2(y) \sim \rho_1(y^{\tau_1}) \sim \rho_3(y^{\tau_3}) \sim \rho_2(y^{\tau_2}) \sim \rho_1(y) \sim \rho_3(y)$ and $\rho_2(y^{\tau_2}) \sim \rho_3(y)$.*

   (2-2) *$\rho_3(y) \sim \rho_2(y) \sim \rho_1(y^{\tau_1}) \sim \rho_3(y^{\tau_3}) \sim \rho_2(y^{\tau_2}) \sim \rho_1(y) \sim \rho_3(y)$ and $\rho_3(y^{\tau_3}) \sim \rho_2(y)$.*

See Figure 22 for the positions of $\{\rho_i(y), \rho_i(y^{\tau_i})\}$ in (2). The proof of Proposition 5.2 is rather technical. Before the proof, let us proceed, assuming Proposition 5.2. Take an inner node $y \in C_\rho$ having a fork τ with maximum $\alpha_\tau \leq 10/9$. Consider three critical neighbors ρ_i for τ_i $(i = 1, 2, 3)$. If some ρ_i is in case (2d) or (4), then both y and y^{τ_i} fall into S_{ρ_i}, and apply 1-SPUP for (τ_i, ρ_i), which keeps (5.4). So suppose that neither (2d) nor (4) occurs. Suppose $\alpha_i = 10/9$. By Proposition 5.2 (1), for $j \neq i$, ρ_j is in case

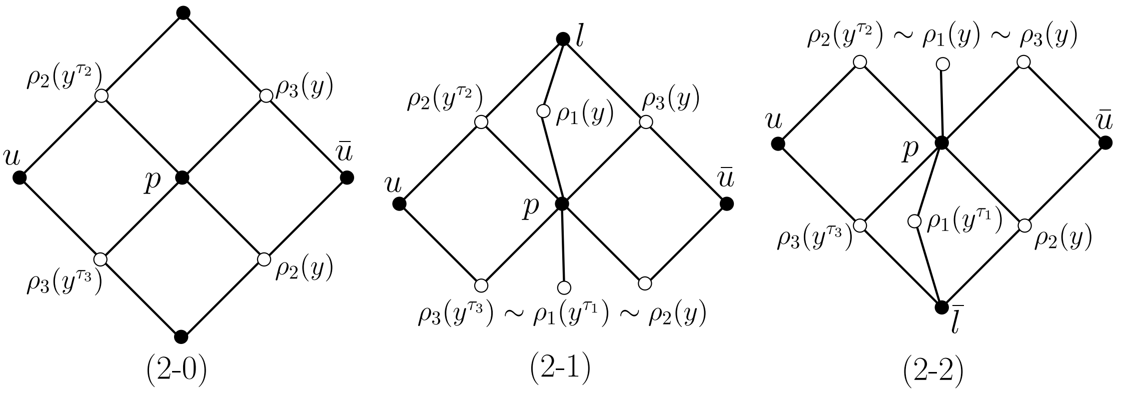


Figure 22: Positions of $\{\rho_i(y), \rho_i(y^{\tau_i})\}$

(2c), and, by Proposition 5.2 (2) for ρ_j, the corresponding 1-SPUP is guaranteed to be mixed. Let $(G;\rho) \leftarrow (G^{\tau_j,\alpha_j};\rho_j)$.

Suppose $\max(\alpha_1,\alpha_2,\alpha_3) \leq 1$; then $\alpha_i = 8/9$ is impossible by Lemma 4.4 (2). Suppose $\alpha_i = 1$ with (2c). If (ρ_i,ρ_j,ρ_k) violates the configuration of Proposition 5.2 (2), then $(G;\rho) \leftarrow (G^{\tau_i,\alpha_i};\rho_i)$ is guaranteed to be mixed 1-SPUP.

Here, it is impossible that all ρ_i satisfy Proposition 5.2 (2). To verify this fact, suppose (to contrary) that all ρ_i satisfy (2). Then all $\rho_i(y), \rho_i(y^{\tau_i})$ $(i = 1,2,3)$ are distinct. To derive a contradiction, we utilize the girth condition (2.5) for Π_p. Suppose (first) that ρ_1 satisfies (2-1) as in Figure 22. By (2) for ρ_2, we have $\rho_1(y) \sim \rho_3(y^{\tau_3})$ or $\rho_1(y^{\tau_1}) \sim \rho_3(y)$. The first case finds a 6-cycle (using $\rho_3(y^{\tau_3}), u, \rho_2(y^{\tau_2}), l, \rho_1(y)$) in Π_p; a contradiction to (2.5). Consider the second case. Then $\rho_1(y^{\tau_1})$ must be incident to $\bar{u}$; otherwise, by $\rho_1(y^{\tau_1}) \sim \rho_2(y)$, we find a 6-cycle (using $\rho_1(y^{\tau_1}), \rho_3(y), \bar{u}, \rho_2(y)$). Again, by (2) for ρ_3, we have $\rho_1(y) \sim \rho_2(y)$ or $\rho(y^{\tau_1}) \sim \rho_2(y^{\tau_2})$. Similarly we have $\rho_1(y^{\tau_1}) \sim u$. Then Π_p has a 6-cycle $(u, \rho_2(y^{\tau_2}), l, \rho_3(y), \bar{u}, \rho_1(y^{\tau_1}))$. The case (2-2) is similar. Also, if all ρ_i satisfy (2-0), we can find a 6-cycle, as above, more easily.

Apply such mixed 1-SPUP as far as possible. Suppose that $\alpha_i = 1$ with (2a) or (2b) occurs. Then necessarily $\alpha_j = \alpha_k = 1$ (by Lemma 4.4 (2)). Then both ρ_j and ρ_k are also in case (2a) or (2b). By Theorem 4.3 (1), all ρ_i are necessarily in case (2a). So every multiflow configuration around y is given as in Figure 19 (after relabeling):

$$f^{e_1,e_2} = f^{e_2,e_3} = f^{e_1,e_3} = 1/2, \quad f^e = 0. \tag{5.6}$$

In particular, $f^{e_1} = f^{e_2} = f^{e_3} = 1$. If $\rho(y) = \rho(x')$ for $x' \in \{x_1,x_2,x_3\}$, then replace ρ by an optimal forward neighbor ρ' with $\rho'(y) \neq \rho'(x')$, according to (5.1). Here $\rho(x) = \rho(y)$ by $f^e = 0$ and the saturation condition. If x is a terminal, then x has degree one and has no flow. If x is an inner node, then x has the same configuration (5.6) as y.

The remaining case is $\alpha_i = 1/3$ or $2/3$. By Lemma 4.4 (1), we have $\alpha_1 = \alpha_2 = \alpha_3 = 2/3$. By Lemma 3.1, every optimal multiflow f satisfies

$$f^{e,e_1} = f^{e,e_2} = f^{e,e_3} = f^{e_1,e_2} = f^{e_1,e_3} = f^{e_2,e_3} = 1/3. \tag{5.7}$$

Also $f^e = f^{e_1} = f^{e_2} = f^{e_3} = 1$. If $\rho(y) = \rho(x')$ for $x' \in \{x,x_1,x_2,x_3\}$, then replace ρ by an optimal forward neighbor ρ' with $\rho'(y) \neq \rho'(x')$ as above. By (5.3), each x' above belongs to $M_\rho \cup S_\rho$. By (5.6) and (5.7), we can split off all inner nodes in C_ρ in $6G$. Split them off. Then $(6G;\rho)$ is restricted Eulerian, there is no inner node in C_ρ, and each terminal in C_ρ has a unique neighbor (Section 5.2.1). Then the proof of claim (C) is done. □

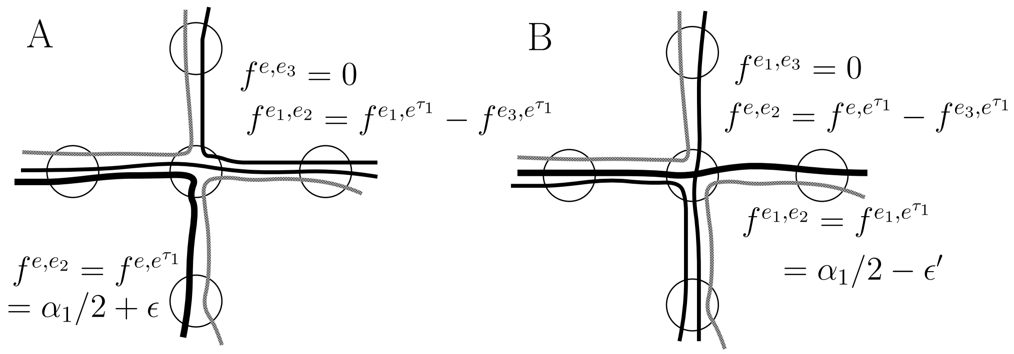


Figure 23: A-configuration and B-configuration

Proof of Proposition 5.2. Consider the case where $\alpha_1 = 10/9$ in case (3) or $\alpha_1 = 1$ in case (2c) such that the corresponding 1-SPUP for (τ_1, ρ_1) is unmixed. In either cases, we can take an optimal multiflow $f = (\mathcal{P}; \kappa)$ in G derived from G^{τ_1,α_1} such that

(5.8) (i) $\mathcal{P}(e^{\tau_1})$ is an $([l], yy^{\tau_1}, [q']_1)$-set for some $(l, q') \in L_p \times Q_p$, or

(ii) $\mathcal{P}(e^{\tau_1})$ is a $([q]_1, yy^{\tau_1}, [l'])$-set for some $(q, l') \in Q_p \times L_p$.

(Necessarily $l \not\sim q'$ for (i), $q \not\sim l'$ for (ii) by the local geodesic condition). Take such an optimal multiflow f of fractionality κ with minimum total support $\sum_{e \in E} f^e$. Then every exchange operation keeping the optimality and (5.8) does not decrease the support; note that any optimality-keeping (anti-)exchange operation at edges except e^{τ_2}, e^{τ_3} keeps the property that f is derived from G^{τ_1,α_1}.

0. By (4.4), we may assume $f^{e_2,e^{\tau_1}} \geq f^{e,e^{\tau_1}} \geq f^{e^{\tau_1}}/2 \geq f^{e_1,e^{\tau_1}} \geq f^{e_3,e^{\tau_1}}$ (after a relabeling fixing τ_1). Now $f^{e^{\tau_1}} \geq 2 - \alpha_1$ (Lemma 3.1). Let $f^{e,e^{\tau_1}} = 1 - \alpha_1/2 + \epsilon$ and $f^{e_1,e^{\tau_1}} = 1 - \alpha_1/2 - \epsilon'$ for $\epsilon \geq \epsilon' \geq 0$. Since $\mathcal{P}(e^{\tau_1})$ is exchangeable, by the exchange operations (as in Figure 17) at e^{τ_1}, we can make f satisfy $f^{e,e_2} = f^{e^{\tau_1},e} = 1 - \alpha_1/2 + \epsilon \geq 4/9 + \epsilon$, and also make f satisfy $f^{e_1,e_2} = f^{e_1,e^{\tau_1}} = 1 - \alpha_1/2 - \epsilon' \geq 4/9 - \epsilon'$. The former is called *A-configuration*, and the latter is called *B-configuration*. See Figure 23.

By Lemma 4.4 (2), for $j = 2$ and 3, $\alpha_j \in \{8/9, 1, 10/9\}$ and ρ_j is in case (2a), (2b), (2c), (3), or (4). Also $\epsilon \leq 1/9$ (otherwise $\alpha_2 \geq 2f^{e,e_2} > 10/9$), By applying Lemma 4.6 (2) to τ_2 in A-configuration and to τ_3 in B-configuration, we get

(5.9)

$$f^{e^{\tau_2};\rho_2} \geq 2 + (d^{\rho_2}(e^{\tau_2}) - 2) f^{e,e^{\tau_1}} - \frac{d^{\rho_2}(e^{\tau_2})\alpha_2}{2} = \begin{cases} 7/9 + \epsilon & \text{if } \alpha_1 = \alpha_2 = 10/9, \\ 5/6 + \epsilon & \text{if } \alpha_1 = 1, \alpha_2 = 10/9, \\ 1 & \text{if } \alpha_2 = 1, \\ 10/9 & \text{if } \alpha_1 = 10/9, \alpha_2 = 8/9, \end{cases}$$

$$f^{e^{\tau_3};\rho_3} \geq 2 + (d^{\rho_3}(e^{\tau_3}) - 2) f^{e_1,e^{\tau_1}} - \frac{d^{\rho_3}(e^{\tau_3})\alpha_3}{2} = \begin{cases} 7/9 - \epsilon' & \text{if } \alpha_1 = \alpha_3 = 10/9, \\ 5/6 - \epsilon' & \text{if } \alpha_1 = 1, \alpha_3 = 10/9, \\ 1 & \text{if } \alpha_3 = 1, \\ 10/9 & \text{if } \alpha_1 = 10/9, \alpha_3 = 8/9. \end{cases}$$

Furthermore $f^{e^{\tau_2};\rho_2} + f^{e^{\tau_3};\rho_3}$ has the following upper bounds.

Claim 5.3. If $\mathcal{P}(e, e_1; \rho_2) \cap \mathcal{P}(e, e_1; \rho_3) = \emptyset$, then

$$f^{e^{\tau_2};\rho_2} + f^{e^{\tau_3};\rho_3} \leq \frac{3}{2}\alpha_1,$$

and, if, in addition, $\mathcal{P}(e_2, e_3; \rho_2) \cap \mathcal{P}(e_2, e_3; \rho_3) = \emptyset$, then

$$f^{e^{\tau_2};\rho_2} + f^{e^{\tau_3};\rho_3} \le 1 + \frac{\alpha_1}{2} - \epsilon - f^{e_3, e^{\tau_1}}.$$

Proof. This follows from substituting

$$f^{e,e_1;\rho_2} + f^{e,e_1;\rho_3} \le \begin{cases} \alpha_1/2 - \epsilon & \text{if } \mathcal{P}(e, e_1; \rho_2) \cap \mathcal{P}(e, e_1; \rho_3) = \emptyset, \\ \alpha_1 - 2\epsilon & \text{otherwise,} \end{cases}$$
$$f^{e_2,e_3;\rho_2} + f^{e_2,e_3;\rho_3} \le \begin{cases} 1 - f^{e^{\tau_1}} + f^{e_3,e^{\tau_1}} & \text{if } \mathcal{P}(e_2, e_3; \rho_2) \cap \mathcal{P}(e_2, e_3; \rho_2) = \emptyset, \\ 2(1 - f^{e^{\tau_1}} + f^{e_3,e^{\tau_1}}) & \text{otherwise,} \end{cases}$$

into $f^{e,e_1;\rho_2} + f^{e,e_1;\rho_3} + f^{e_2,e_3;\rho_2} + f^{e_2,e_3;\rho_3} \ge f^{e^{\tau_2};\rho_2} + f^{e^{\tau_3};\rho_3} - f^{e^{\tau_1}} + 2f^{e_3,e^{\tau_1}}$. Then use $f^{e^{\tau_1}} = 2 - \alpha_1 + \epsilon - \epsilon'$ for $\epsilon \ge \epsilon'$.

The first and second inequalities follow from $f^{e,e_1;\rho_i} \le f^{e,e_1} \le 1 - f^{e,e^{\tau_1}} = \alpha_1/2 - \epsilon$ and $f^{e_2,e_3;\rho_i} \le f^{e_2,e_3} \le 1 - f^{e_2,e^{\tau_1}} = 1 - f^{e^{\tau_1}} + f^{e_3,e^{\tau_1}}$, respectively. The third follows from adding $f^{e^{\tau_2};\rho_2} \le f^{e,e_1;\rho_2} + (f^{e_1,e^{\tau_1}} - f^{e_3,e^{\tau_1}}) + f^{e_2,e_3;\rho_2}$ and $f^{e^{\tau_3};\rho_3} \le f^{e,e_1;\rho_3} + (f^{e,e^{\tau_1}} - f^{e_3,e^{\tau_1}}) + f^{e_2,e_3;\rho_3}$, and using $f^{e_1,e^{\tau_1}} + f^{e,e^{\tau_1}} = f^{e^{\tau_1}}$. □

1. We first show the following, which includes Proposition 5.2 (1):

(5.10) For $j = 2$ or 3, ρ_j is in case (2c) with $\alpha_j = 1$.

To prove this, suppose not. For all cases (2a, 2b, 3, 4) and $j = 2, 3$, $\mathcal{P}(e^{\tau_j}; \rho_j)$ is exchangeable at e^{τ_j}.

(5.11) In A-configuration, $\mathcal{P}(e_1, e_2; \rho_2) \ne \emptyset$.

Otherwise, $\mathcal{P}(e_1, e_2; \rho_2) = \emptyset$ would imply $f^{e,e_1;\rho_2} + f^{e_2,e_3;\rho_2} = f^{e^{\tau_2};\rho_2} \ge 7/9 + \epsilon$. Since $\max\{f^{e,e_1;\rho_2}, f^{e_2,e_3;\rho_2}\} \le 1 - f^{e,e^{\tau_1}} = 1 - \alpha_1/2 - \epsilon \le 5/9 - \epsilon$, both $\mathcal{P}(e, e_1; \rho_2)$ and $\mathcal{P}(e_2, e_3; \rho_2)$ have the flow-value at least $2/9 + 2\epsilon$ (in each case). In B-configuration, by the exchange operation (without simplification) between $\mathcal{P}(e, e_1; \rho_2)$ and $\mathcal{P}(e_2, e_3; \rho_2)$ at e^{τ_2} we can make f satisfy $f^{e_1,e_2} \ge 4/9 - \epsilon' + (2/9 + 2\epsilon) > 5/9$ (while keeping optimality). Since $f^{e^{\tau_3}} \le 2 - 2f^{e_1,e_2}$, we have $f^{e^{\tau_3}} < 8/9$ (this holds after the simplification). Then $\alpha_3 \ge 2 - f^{e^{\tau_3}} > 10/9$; a contradiction to $\alpha_3 \le 10/9$.

Similarly,

(5.12) In B-configuration, $\mathcal{P}(e, e_2; \rho_3) \ne \emptyset$.

Otherwise, both $\mathcal{P}(e, e_1; \rho_3)$ and $\mathcal{P}(e_2, e_3; \rho_3)$ would have the flow-value at least $2/9 + (\epsilon - \epsilon')$, and in A-configuration we can make f satisfy $f^{e,e_2} \ge 4/9 + \epsilon + 2/9 + \epsilon - \epsilon' > 5/9$, implying $\alpha_2 > 10/9$; a contradiction, as above. Then we have the following:

(5.13) $\mathcal{P}(e, e_1; \rho_2) \cap \mathcal{P}(e, e_1; \rho_3)$ is empty.

Indeed, take P from $\mathcal{P}(e, e_1; \rho_2) \cap \mathcal{P}(e, e_1; \rho_3)$ if exists. Then, by (5.11) and the fact that $\mathcal{P}(e^{\tau_2}; \rho_2)$ is exchangeable, P is exchangeable with a path in $\mathcal{P}(e^{\tau_1})$ at e_1. Also, by (5.12) and the fact that $\mathcal{P}(e^{\tau_3}; \rho_3)$ is exchangeable, P is exchangeable with a path in $\mathcal{P}(e^{\tau_1})$ at e. By Lemma 4.5 (2), P is a $([q]_1, xyx_1, [q]_1)$-path for some $q \in Q$; a contradiction to the local geodesic condition.

Therefore the first inequality in Claim 5.3 holds. Then the case $\alpha_1 = \alpha_2 = \alpha_3 = 10/9$ (case (3)) is the only possibility; the other cases yield LHS < RHS. In particular, $(\rho_1(y), \rho_1(y^{\tau_1})) = (l, q')$ or (q, l') holds. Suppose that the latter case holds. Here $\mathcal{P}(e_1, e_2)$

is an $([l'], x_1yx_2, *)$-set, and $\mathcal{P}(e, e_2)$ is an $([l'], xyx_2, *)$-set (by Table 1). $\mathcal{P}(e, e_1; \rho_2)$ is also nonempty; otherwise $7/9 \leq f^{e^{\tau_2};\rho_2} = f^{e_2, e^{\tau_2};\rho_2} \leq 1 - f^{e, e^{\tau_1}} \leq 5/9$. Then a path $P \in \mathcal{P}(e, e_1; \rho_2)$ is exchangeable with a path $P' \in \mathcal{P}(e_1, e_2; \rho_2)$ ($\neq \emptyset$ by (5.11)) at e, since $\mathcal{P}(e^{\tau_2}; \rho_2)$ is exchangeable at e^{τ_2}. Therefore P is an $([l'], x_1yx, *)$-path. Then f includes an $([l'], yx, *)$-path and a $(*, yx, [l'])$-path at e. Then the anti-exchange operation for these two paths at e decreases the total support, while keeping optimality. A contradiction to the minimality of the support. Therefore $(\rho_1(y), \rho_1(y^{\tau_1})) = (l, q')$ holds.

Consider $(\rho_2(y), \rho_2(y^{\tau_2}))$, which is also in case (3): $\rho_2(y^{\tau_2}) = \bar{l}_1' \in L_p$ or $\rho_2(y) = \bar{l}_1 \in L_p$. Take a path P from $\mathcal{P}(e_1, e_2; \rho_2)$ (by (5.11)) in A-configuration, which is a $(*, x_1yx_2, [\bar{l}_1'])$-path or an $([\bar{l}_1], x_1yx_2, *)$-path. Since $P \in \mathcal{P}(e^{\tau_1})$, P is also a $([q']_1, x_1yx_2, [l])$-path. Therefore $\rho_2(y^{\tau_2}) = \bar{l}_1' = l$ or $\rho_2(y) = \bar{l}_1$ with $\bar{l}_1 \sim_1 q'$. Suppose that the former case occurs. Then $\mathcal{P}(e, e_1; \rho_2)$ is a $([l], xyx_1, *)$-set. Also $\mathcal{P}(e, e_2)$ is a $(*, xyx_2, [l])$-set. Then the anti-exchange operation at $e = xy$ works, as above; a contradiction to the minimality of the support. Hence the latter case ($\rho_2(y) = \bar{l}_1$) holds. Similarly $\rho_3(y^{\tau_3}) = \bar{l}_2 \in L_p$ with $\bar{l}_2 \sim_1 q'$. Then $\bar{l}_1 \neq \bar{l}_2$ necessarily holds; this means $[\bar{l}_1] \cap [\bar{l}_2] = \emptyset$. Otherwise the anti-exchange operation at e, which has both an $([\bar{l}_2], xy, *)$-path and a $(*, xy, [\bar{l}_1])$-path, works. In particular, $\mathcal{P}(e_2, e_3; \rho_2) \cap \mathcal{P}(e_2, e_3; \rho_3)$ has no path; otherwise such a path is an $([\bar{l}_1] \cap [\bar{l}_2], x_2yx_3, *)$-path.

Hence the second inequality in Claim 5.3 also holds. Then this completely determines the multiflow configuration at y as $\epsilon = \epsilon' = 0$, $f^{e_3, e^{\tau_1}} = 0$, $f^{e, e^{\tau_1}} = f^{e_1, e^{\tau_1}} = f^{e, e_2} = f^{e_1, e_2} = f^{e, e_2; \rho_3} = f^{e_1, e_2; \rho_2} = 4/9$, and $f^{e^{\tau_2};\rho_2} = f^{e^{\tau_3};\rho_3} = 7/9$. In particular, both equalities hold in (5.9). Since $f^{e, e_1; \rho_2} + f^{e, e_1; \rho_3} + f^{e_2, e_3; \rho_2} + f^{e_2, e_3; \rho_3} = 6/9$ and $f^{e, e_1} \leq 5/9$, we may assume that both $\mathcal{P}(e, e_1; \rho_3)$ and $\mathcal{P}(e_2, e_3; \rho_3)$ are nonempty. By the exchange operation at e^{τ_3} for two paths, one from $\mathcal{P}(e, e_1; \rho_3)$ and another from $\mathcal{P}(e_2, e_3; \rho_3)$, we can make f satisfy $f^{e, e_2} = f^{e_1, e^{\tau_1}} > 4/9$, while keeping the optimality and $f^{e^{\tau_2};\rho_2} = 7/9$. This means that the inequality in Lemma 4.6 (2) fails; a contradiction. Thus we have (5.10), and hence Proposition 5.2 (1).

2. Next we show: if the condition of Proposition 5.2 (2) holds, i.e., $\alpha_1 = 1$ with unmixed (2c), then ρ_2 is in case (2c) with $\alpha_2 = 1$. If this is true, then necessarily $f^{e, e^{\tau_1}} = f^{e_1, e^{\tau_1}} = 1/2$ ($\epsilon = \epsilon' = 0$), and ρ_3 is also in case (2c) with $\alpha_3 = 1$ since we can interchange the roles of x and x_1.

Suppose (indirectly) that ρ_2 is not in case (2c). Then $f^{e^{\tau_2};\rho_2} \geq 5/6 + \epsilon$, (5.11) holds by the same argument, and (5.13) does not hold. By (5.10), ρ_3 is necessarily in case (2c) with $\alpha_3 = 1$ and $(\rho_3(y), \rho_3(y^{\tau_3})) = (\bar{q}, \bar{q}')$. Consider f in B-configuration. Then $f^{e^{\tau_3}} = f^{e, e_1} + (f^{e, e_2} + f^{e_2, e_3}) \leq (1 - f^{e, e^{\tau_1}}) + (1 - f^{e_1, e_2}) = 2 - f^{e^{\tau_1}} \leq \alpha_1 = 1$, and $1 = 2 - \alpha_3 \leq f^{e^{\tau_3}} \leq 1$. Therefore $f^{e^{\tau_3}} = 1$, and f is an optimum for G^{τ_3, α_3}. We may assume $f^{e, e_2} > 0$. Otherwise $f^{e, e^{\tau_1}} = f^{e_1, e^{\tau_1}} = 1$; we can change the role of x and x_1. Take a path P in $\mathcal{P}(e_1, e_2; \rho_2) \neq \emptyset$. Since $\mathcal{P}(e^{\tau_1})$ is exchangeable at e^{τ_1}, P is exchangeable with a path in $\mathcal{P}(e, e_2) \neq \emptyset$. Since $\mathcal{P}(e^{\tau_2}; \rho_2)$ is exchangeable at e^{τ_2}, P is exchangeable with a path in $\mathcal{P}(e, e_1; \rho_2) \subseteq \mathcal{P}(e, e_1)$. By Lemma 4.5 (2), P is a $([\bar{q}], y, [\bar{q}])$-path; a contradiction to the local geodesic condition.

3. Finally we show that $\{\rho_i(y), \rho_i(y^{\tau_i})\}$ $(i = 1, 2, 3)$ satisfy (2-0), (2-1), or (2-2). Now $f^{e, e^{\tau_1}} = f^{e, e^{\tau_2}} = 1/2$, and f is an optimum for G^{τ_2, α_2} in A-configuration ($f^{e^{\tau_2}} = 1$), and is an optimum for G^{τ_3, α_3} in B-configuration ($f^{e^{\tau_3}} = 1$). In particular, $f^{e, e_1} = 1/2$. Take a path P from $\mathcal{P}(e, e_1)$, and suppose that P is a $([u], xyx_1, [\bar{u}])$-path. Here P is a $([u], y^{\tau_2}y, [\bar{u}])$-path and a $([u], y^{\tau_3}y, [\bar{u}])$-path. By Lemma 4.5 (1), $\rho_2(y^{\tau_2}) \sim_1 u \sim_1 \rho_3(y^{\tau_3})$ and $\rho_2(y) \sim_1 \bar{u} \sim_1 \rho_3(y)$. If $f^{e_2, e_3} > 0$, then we can apply the same argument for $\mathcal{P}(e_2, e_3)$ and we get $\rho_2(y^{\tau_2}) \sim \rho_3(y)$ and $\rho_2(y) \sim \rho_3(y^{\tau_3})$, i.e., (2-0) holds. Suppose that $f^{e_2,e_3} = 0$; necessarily $f^{e,e_2} = f^{e,e_1} = f^{e_1,e_2} = 1/2$. Suppose that (5.8) (i) holds. Take a path P' from $\mathcal{P}(e_1, e_2)$, which is a $([q']_1, x_1yx_2, [l])$-path for $q' = \rho_1(y^{\tau_1})$ and $l \sim_1 \rho_1(y)$. Here P' is an $([l], y^{\tau_2}y, [q']_1)$-path. By Lemma 4.5 (1), we get $\rho_2(y^{\tau_2}) \sim_1 l$ and $q' \sim \rho_2(y)$. By the same argument for $\mathcal{P}(e, e_2)$, we get $\rho_3(y) \sim_1 l$ and $q' \sim \rho_3(y^{\tau_3})$. Hence (2-1) holds; see Figure 22 (2-1). Similarly, for the case of (5.8) (ii), (2-2) holds; see Figure 22 (2-2). We are done. □

5.3 Proof of the uncrossing lemma

Here we prove Lemma 5.1; the proof technique is due to [11]. We use the relation between DLP and LP-dual, which is revealed in [12] and is summarized by Section 5.3.1. Our argument is algorithmic, and will be used in the next Section 5.4. For an F-complex $\mathcal{K}$ with unit leg-length $\delta = 1$, let $\operatorname{diam} \mathcal{K}$ denote the diameter of $\mathcal{K}$.

5.3.1 Relation between LP-dual and DLP

Consider the following continuous relaxation of DLP:

$$\mathrm{CLP}(\mathcal{K}; \{R_s\}_{s \in S}): \qquad \text{Minimize} \quad \sum_{xy \in E} c(xy) d_{\mathcal{K}}(\rho(x), \rho(y))$$
$$\text{subject to} \quad \rho : V \to \mathcal{K}, \quad \rho(s) \in R_s \ (s \in S).$$

We also call a feasible map ρ in CLP a *potential*. For a potential ρ to CLP, metric d^ρ is feasible to LP-dual (2.1) with the same objective value. Conversely, for any metric d feasible to LP-dual (2.1), we can greedily construct a potential ρ in CLP with $d^\rho \leq d$ as follows.

Let $V = \{x_1, x_2, \ldots, x_n\} (\supseteq S)$. For $k = 1, 2, \ldots, n$, define $\rho(x_k)$ to be an arbitrary point in

$$\bigcap_{s \in S} B(R_s, d(s, x_k)) \cap \bigcap_{i=1}^{k-1} B(\rho(x_i), d(x_i, x_k)) \quad (k = 1, 2, \ldots, n), \tag{5.14}$$

where $B(R, r)$ is the set of points p with $d_{\mathcal{K}}(R, p) \leq r$. This construction is well-defined, since (5.14) is nonempty for every k [12]. Then, by construction, ρ is a potential in CLP with $d^\rho \leq d$ (since $\rho(x_i) \in B(\rho(x_i), d(x_i, x_k))$). Hence, if d is optimal to LP-dual, then ρ is optimal to CLP. Therefore, from an optimal metric d, we can construct an optimal potential ρ in CLP in polynomial time.

Next we round a potential in CLP to a potential in DLP. Fix an admissible orientation $\overrightarrow{\mathcal{K}}$ of $\mathcal{K}$. This orientation determines an orientation of the local coordinate of every cell.

A leg uv oriented as $\overrightarrow{uv}$ is identified with a segment in $\mathbf{R}$ with ends $u = 0, v = 1$. A triangle σ with oriented legs $\overrightarrow{uv}, \overrightarrow{vw}$ and hypotenuse $\overrightarrow{uw}$ is identified with a triangle in $\mathbf{R}^2$ with vertices $(u, v, w) = ((0, 0), (1, 0), (1, 1))$. For simplicity, we regard a square-folder F as a $K_{2,2}$-folder with the hypotenuse joining the sink and the source in F.

For $a \in [0, 1)$, we can define a rounding map $\phi^a : \mathcal{K} \to V(\mathcal{K})$ as follows. For a point $p \in \mathcal{K}$, we can take a cell σ containing p. In the case where σ is a triangle with vertices u, v, w oriented as above, p is locally represented as a point $(x, y) \in \mathbf{R}^2$ with $0 \leq y \leq x \leq 1$. Define ϕ^a by

$$\phi^a(p) := \begin{cases} u & \text{if } y \leq x \leq a, \\ v & \text{if } y \leq a < x, \\ w & \text{if } a < x \leq y. \end{cases}$$

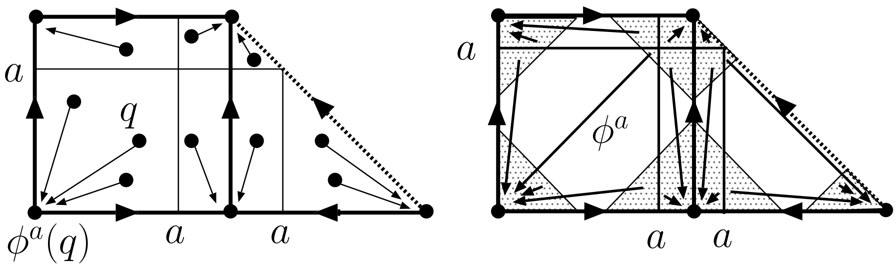


Figure 24: Rounding map ϕ^a

In the case where σ is a leg $\overrightarrow{uv}$, p is locally represented as a point x with $0 \leq x \leq 1$. Define $\phi^a(p) := u$ if $x \leq a$ and $\phi^a(p) := v$ if $a < x$. See Figure 24. This map ϕ^a is well-defined. This rounding is due to [18], and it is known in [12, 18] that

(5.15) If ρ is optimal to CLP, so is $\phi^a \circ \rho$.

Consequently an optimal potential can be obtained from any optimal metric in polynomial time.

5.3.2 Constructing a semi-neighbor from two potentials

We can consider an analogue of a convex combination of two distinct potentials ρ and ρ'. Here we assume the following condition:

(5.16) For every $p \in V(\mathcal{K})$, there is a terminal $s \in S$ with $R_s = \{p\}$.

This is achieved by adding at most $|V(\mathcal{K})|$ dummy terminals. Take $\sigma \in [0, 1]$. Let $d_\sigma = (1-\sigma)d^\rho + \sigma d^{\rho'}$. Then d_σ is feasible to LP-dual (2.1). According to (5.14), we can construct a potential ρ_σ in CLP with $d^{\rho_\sigma} \leq d$. In particular, if both ρ and ρ' are optimal to CLP, then ρ_σ is also optimal to CLP.

(5.17) Under assumption (5.16), we have $d_{\mathcal{K}}(\rho(x), \rho_\sigma(x)) = \sigma d_{\mathcal{K}}(\rho(x), \rho'(x))$ and $d_{\mathcal{K}}(\rho_\sigma(x), \rho'(x)) = (1-\sigma) d_{\mathcal{K}}(\rho(x), \rho'(x))$ for $x \in V$.

Indeed, for each $x \in V$ there are terminals s, t with $R_s = \{\rho(x)\}$ and $R_t = \{\rho'(x)\}$; so $\rho(x) = \rho(s) = \rho'(s)$ and $\rho'(x) = \rho(t) = \rho'(t)$. Thus, by construction (5.14), we have

$$\begin{aligned}\rho_\sigma(x) &\in B(R_s, d_\sigma(s,x)) \cap B(R_t, d_\sigma(t,x)) \\ &= B(\rho(x), \sigma d_{\mathcal{K}}(\rho(x), \rho'(x))) \cap B(\rho'(x), (1-\sigma) d_{\mathcal{K}}(\rho(x), \rho'(x))).\end{aligned}$$

Hence $d_{\mathcal{K}}(\rho(x), \rho_\sigma(x)) \leq \sigma d_{\mathcal{K}}(\rho(x), \rho'(x))$ and $d_{\mathcal{K}}(\rho_\sigma(x), \rho'(x)) \leq (1-\sigma) d_{\mathcal{K}}(\rho(x), \rho'(x))$. By the triangle inequality $d_{\mathcal{K}}(\rho(x), \rho'(x)) \leq d_{\mathcal{K}}(\rho(x), \rho_\sigma(x)) + d_{\mathcal{K}}(\rho_\sigma(x), \rho'(x))$, we obtain the equalities in (5.17).

Let ρ be a potential in DLP. If we are given a potential ρ' in CLP *close to* ρ, we can construct a semi-neighbor of ρ from ρ'. A semi-neighbor is called *forward* if it is a semi-neighbor with respect to $\overrightarrow{\mathcal{K}}$ and is called *backward* if it is a semi-neighbor with respect to the opposite orientation of $\overrightarrow{\mathcal{K}}$.

(5.18) For a potential ρ to DLP and a potential ρ' to CLP, if $d_{\mathcal{K}}(\rho(x), \rho'(x)) < 1/2$ for $x \in V$, then $\phi^a \circ \rho'$ is a forward (resp., backward) semi-neighbor of ρ for $a \in [0, 1/2)$ (resp., $a \in [1/2, 1)$).

This property can be easily seen from the right of Figure 24, where shaded regions depict disjoint balls around vertices in $\mathcal{K}$ with radius less than $1/2$; if $p = \rho(x)$, then $\rho'(x)$ is contained by the ball around p, and is rounded to $\phi^a \circ \rho'(x)$ along the arrows. The uncrossing lemma is now immediate.

Proof of the uncrossing lemma. Take positive $\sigma \leq 1/(2 \operatorname{diam} \mathcal{K}^2 + 1)$. Consider $d_\sigma := (1-\sigma)d^\rho + \sigma d^{\rho'}$. Then d_σ is optimal. Next, according to (5.14), we construct a potential ρ_σ to CLP with $d^{\rho_\sigma} \leq d_\sigma$, which is also optimal. By (5.17), $d_{\mathcal{K}^2}(\rho(x), \rho_\sigma(x)) < 1/2$, and $\rho(x) \neq \rho'(x)$ implies $\rho(x) \neq \rho_\sigma(x)$. In the forward orientation, round ρ_σ to $\rho^* := \phi^a \circ \rho_\sigma$. Here take $a = 0$. By (5.18), ρ^* is a forward semi-neighbor and is a desired one. □

5.4 Algorithmic implication

The proof of Theorem 1.5, we have shown above, is constructive. Each step searches all forks for one having required properties, and applies SPUP or splitting-off to decrease the number of nodes in question. Once the problem becomes trivial to have an integral optimum, we obtain a $1/k$-integral optimum for the original problem by reversing the operations.

Here we verify that our proof indeed yields a (strongly) polynomial time algorithm under the assumption that the size (the number of cells) of a realization is fixed.

Theorem 5.4. *Suppose that a realization of μ is given and its size is fixed. Then there exists a strongly polynomial time algorithm to find a $1/12$-integral optimal multiflow in μ-MFP for every property-inner Eulerian graph.*

The size of a realization is not polynomially bounded by the bit size of μ in general; see the 2-commodity F-complex in Section 6. In the case of 0-1 weight, there is a realization of $O(|S|^2)$ size; see (7.2) in Section 7.1.

In the case where the edge capacity is not so large, our proof gives a strongly polynomial time algorithm, assuming the oracles of finding an optimal potential, the splitting capacity α_τ, a critical neighbor, and a forward semi-neighbor in the uncrossing lemma. We note that our proof goes on without any explicit multiflow calculation; the mixed 1-SPUP (in Section 5.2.4) can be done without checking all optimal multiflows by Proposition 5.2. Also a critical neighbor can be relaxed to a critical semi-neighbor.

These computations can be done (in a combinatorial way) if we get an optimal metric in LP-dual (2.1). Since LP (2.1) is given by a $\{-1, 0, 1\}$ coefficient matrix of polynomial size, we can evaluate the optimal value and find an optimal metric in strongly polynomial time by the method of Tardos [28]. In Section 5.3 we mentioned polynomial time constructions of an optimal potential from an optimal metric and of a forward semi-neighbor in the uncrossing lemma. Hence, in the rest of this subsection, we explain how to compute α_τ and a critical semi-neighbor, and how to reduce edge-capacities.

5.4.1 Computing α_τ and a critical semi-neighbor

The computation of α_τ is a fractional programming. Let ρ be an optimal potential to G and let τ be a fork. Let $h(\alpha) := \operatorname{opt}(\mu; G) - \operatorname{opt}(\mu; G^{\tau,\alpha})$. Then $\alpha_\tau = \max\{\alpha \mid h(\alpha) = 0\}$. The gradient of h at α is given by $d^\rho(e^\tau)$ for some optimal potential ρ of $G^{\tau,\alpha}$. So the possible values of the gradients are $0, 1, 2, \ldots, \operatorname{diam} \mathcal{K}$. Here h is a monotone nondecreasing piecewise linear convex function. Hence, by the discrete Newton method, we can determine α_τ by solving (2.1) at most $\operatorname{diam} \mathcal{K}$ time.

Next suppose $\alpha_\tau \leq 2$, and consider a critical semi-neighbor. A semi-neighbor ρ' of ρ is critical with respect to τ if and only if it satisfies $d^{\rho'}(e^\tau) > 0$ and it is optimal to G^{τ,α_τ}. We can construct a critical neighbor from any feasible metric d such that it satisfies $d(e^\tau) > 0$ and it is optimal for G^{τ,α_τ}. Such a metric, also called *critical*, can be naturally obtained at the computation of α_τ above.

Consider $d_\sigma = (1-\sigma)d^\rho + \sigma d$ for positive $\sigma(\leq 1/(2\,\mathrm{diam}\,\mathcal{K}+1))$; obviously d_σ is also critical. Next take a potential $\tilde{\rho}$ to CLP with $d^{\tilde{\rho}} \leq d_\sigma$, according to (5.14). Again $\tilde{\rho}$ is optimal to G^{τ,α_τ}. Also $d^{\tilde{\rho}}(e^\tau) > 0$. Indeed, if $d^{\tilde{\rho}}(e^\tau) = 0$, then for small positive $\epsilon > 0$ we have $d^{\tilde{\rho}}(G^{\tau,\alpha_\tau}) = d^{\tilde{\rho}}(G^{\tau,\alpha_\tau+\epsilon}) \leq d_\sigma(G^{\tau,\alpha_\tau+\epsilon}) < d_\sigma(G^{\tau,\alpha_\tau}) = d^{\tilde{\rho}}(G^{\tau,\alpha_\tau})$. A contradiction. Thus $d^{\tilde{\rho}}$ is also critical. Fix an admissible orientation. Take a and round $\tilde{\rho}$ to $\phi^a \circ \tilde{\rho}$, which is a semi-neighbor of ρ. Since $\tilde{\rho}(y) \neq \tilde{\rho}(y^\tau)$ we can choose a so that $\phi^a \circ \tilde{\rho}(y) \neq \phi^a \circ \tilde{\rho}(y^\tau)$. Then $\phi^a \circ \tilde{\rho}$ is a critical semi-neighbor of ρ as required. This construction can be done in strongly polynomial time.

5.4.2 Reducing edge-capacities

Finally, we explain a preprocess to reduce the edge-capacities. This can be done in splitting-off. We may assume that $G = (V, E)$ is a complete graph. Let $n = |V|$. We use a capacitated version of the splitting-off. For a fork $\tau = (xy, y, yz)$ and a nonnegative integer $\beta \leq \min\{c(xy), c(yz)\}$, decrease $c(xy)$ and $c(yz)$ by β and increase $c(xz)$ by β. The splitting-off operation is to decrease the maximum possible value β_τ keeping the optimal value. We also consider the degenerate fork (xy, y, yx). In this case the splitting-off operation is to decrease $c(xy)$ by the maximum possible even integer β_τ keeping the optimal value. We can recover an optimal multiflow in the original graph from any optimal multiflow in the graph obtained by a splitting-off. Again β_τ is also computed in the same manner as in the previous section.

By repeating the splitting-off $O(n^3)$ times, we can make (G, c) so that $\beta_\tau = 0$ for every fork τ. Indeed, take a node x, and apply the splitting-off for all forks at x in an arbitrary order. Then $\beta_\tau = 0$ for every fork τ at x. If we apply the splitting-off to a fork at another node x', then this does not increase the degree of x, and also does not produce a new splittable fork at x. Apply this procedure to all nodes. Then $\beta_\tau = 0$ for all forks in G. At this moment,

(5.19) each inner node y has $O(n^2)$ degree.

Indeed, consider an optimal multiflow f. Then (5.19) follows from:

$$\sum_{x \in V \setminus \{y\}} c(xy) = \sum_{x \in V \setminus \{y\}} (c(xy) - f^{xy}) + 2 \sum_{x,z \in V \setminus \{y\}} f^{xy,yz}.$$

Then $f^{xy,yz} \leq 1$; otherwise the fork (xy, y, yz) is splittable (Lemma 3.1). Also $c(xy) - f^{xy} \leq 2$; otherwise the degenerate fork (xy, y, yx) is splittable. Thus the degree of y is at most $2(n-1) + 2\binom{n-1}{2} = O(n^2)$.

Terminals may have a large degree. Next compute an optimal multiflow $f = (\mathcal{P}, \lambda)$ by solving LP; we can use a compact representation for multiflows. For each pair (s, t) of terminals, check the flow-value $\lambda(P)$ of the path P of a single edge st, and decrease the edge capacity $c(st)$ by the maximum even integer l_{st} not exceeding $\lambda(P)$. Again we can recover an optimum in the original problem from any optimum in the new problem by adding the path of a single edge st of flow-value l_{st}. Then

(5.20) each terminal s has $O(n^2)$ degree.

Indeed we have

$$\begin{aligned}\sum_{x\in V\setminus\{s\}} c(sx) &= \sum_{x\in V\setminus\{s\}} (c(sx)-f^{sx}) + \sum_{x,y\in V\setminus\{s\}} 2f^{xs,sy} + \sum_{x,y\in V\setminus\{s\}} f_0^{sx,xy} + \sum_{t\in S\setminus\{s\}} f_0^{st}, \\ &\leq 2(n-1) + 2\binom{n-1}{2} + (n-1)(n-2) + 2(|S|-1),\end{aligned}$$

where $f_0^{sx,xy}$ denotes the total amount of $(s, sxy, *)$-flow, and f_0^{st} denotes the total amount of (s, st, t)-flow.

At this moment, if the existence of an integral optimal solution is guaranteed, then the degree of every inner node is zero, and the multiflow of one-edge paths is optimal.

By edge multiplication, make each edge have unit capacity, and apply the degree reduction in Section 3.3. Then we obtain a graph G^* with degree at most four, $O(n^5)$ vertices, and unit capacity. Consequently, our proof of Theorem 1.5 to G^* finds a 1/12-integral optimal multiflow for G^* in strongly polynomial time. By reversing the process above, we get a 1/12-integral optimal multiflow for the original graph G in strongly polynomial time.

6 Sparsity and integrality

In this section, we give a powerful geometric criterion of the splittabiliy/integrality. We introduce the concept of a *sparse* vertex in an orientable F-complex, and show that *if an inner node y is mapped to a sparse vertex by some optimal potential ρ, then y has a splittable fork* (under Eulerian condition). This generalizes claim (A), and enables us to prove the integrality theorem for a large class of μ-MFP.

6.1 Sparsity

A vertex p in an orientable F-complex $\mathcal{K}$ is said to be *sparse* if, for every oriented orbit $\overrightarrow{O}$, every pair of vertices q, q' with $p \succeq_{\overrightarrow{O}} q$ and $p \succeq_{\overrightarrow{O}} q'$ belongs to a common folder in $\mathcal{K}_p$. This concept generalizes and localizes the one due to Karzanov [20, Definition 1.3], who introduced the sparseness concept for a different purpose. The main result in this section connects the geometric notion of the sparseness and the splittability/integrality in μ-MFP.

Let G be a graph with terminal set S, and let μ be a terminal weight having a realization $(\mathcal{K}; \{R_s\}_{s\in S})$. We consider μ-MFP and DLP$(\mathcal{K}; \{R_s\}_{s\in S})$. A terminal s is said to be *strong* if R_s is a path of hypotenuses or a single vertex. Note that any strong terminal is proper. G is said to be *strongly-inner Eulerian* (with respect to $(\mathcal{K}; \{R_s\}_{s\in S})$) if each node other than strong terminals has an even degree.

Theorem 6.1. *Suppose that G is strongly-inner Eulerian. If there exists an optimal potential ρ such that $\rho(x)$ is sparse for every node x, then there exists an integral optimal multiflow.*

The large part of this theorem follows from the following splittability criterion. Recall that G is said to be properly-inner Eulerian (with respect to $(\mathcal{K}; \{R_s\}_{s\in S})$) if each node other than proper terminals has an even degree (see Section 3.2 for the definition of proper terminals).

Theorem 6.2. *Suppose that G is properly-inner Eulerian. For an optimal potential ρ, an inner node y has a splittable fork if*

(i) *$\rho(y)$ is sparse, and*

(ii) *there exists no odd-degree terminal s such that $\rho(s) = \rho(y)$ and R_s has three hypotenuses incident to $\rho(y)$.*

The proofs of Theorems 6.1 and 6.2 are given in Section 6.3. We first describe consequences of Theorem 6.1.

6.2 Locally sparse F-complex and blow-up

The integrality of μ-MFP is closely related to the embeddability of μ into a nice F-complex. An orientable F-complex $\mathcal{K}$ is said to be *locally sparse* if each vertex is sparse. An immediate, but powerful, consequence of Theorem 6.1 is the following.

Theorem 6.3. *Suppose that μ has a realization $(\mathcal{K}; \{R_s\}_{s \in S})$ with a locally sparse F-complex $\mathcal{K}$. Then μ-MFP has an integral optimal multiflow for every strongly-inner Eulerian graph with respect to $(\mathcal{K}; \{R_s\}_{s \in S})$.*

The local sparsity is an easily checkable property. The following F-complexes are all locally sparse:

- a folder itself.
- a subdivision of a locally sparse F-complex.
- an F-complex without $K_{2,*}$-folders.
- a star-shaped F-complex without a pair of $K_{2,*}$-folders having a common leg.
- an F-complex each of whose summands is a single leg or a single folder.

By Theorem 6.3, for any weight μ realized by these F-complexes, μ-MFP admits an integral optimal multiflow for every Eulerian graph. For example, take μ as the graph metric $d_{K_{2,r}}$ of $K_{2,r}$. Then μ is realized by a single folder. Hence we obtain the integrality theorem due to Karzanov-Manoussakis [23]: *There exists an integral optimal multiflow in $d_{K_{2,r}}$-MFP for every inner Eulerian graph.* Consider an F-complex $\mathcal{K}$ without $K_{2,*}$-folders, i.e., $\mathcal{K}$ is a cubical complex. The corresponding integrality theorem is nothing but the *multiflow locking theorem* due to Karzanov-Lomonosov [22]; see for [19, Section 5] for the detail of this relation. Theorem 6.3 includes many other integrality instances. For example, consider μ in Figure 9. Then the F-complex in the right is locally sparse, and hence the integrality theorem holds for this weight.

Interestingly, even if $\mathcal{K}$ is not locally sparse, sometimes *we can represent $\mathcal{K}$ as a summand of a locally sparse one*; see $\mathcal{K}/U$ in Figure 8. By combining Theorem 6.1 with the locking property (Proposition 2.6 in Section 2.2.3), we can prove the integrality theorem for such μ that is a summand of another weight μ^* having a locally sparse realization.

Theorem 6.4. *Suppose that μ is a summand of μ^* having a realization $(\mathcal{K}^*; \{R^*_s\}_{s \in S})$ with a locally sparse F-complex $\mathcal{K}^*$. Then μ-MFP has an integral optimal multiflow for every strongly-inner Eulerian graph with respect to $(\mathcal{K}^*; \{R^*_s\}_{s \in S})$.*

A sparse (resp., nonsparse) vertex is an analogue of a *nonsingular* (resp., *singular*) point in an algebraic variety. We call the process of constructing an F-complex $\mathcal{K}^*$ having $\mathcal{K}$ as a summand a *blow-up*.

An illustrative application of Theorem 6.4 is shown.

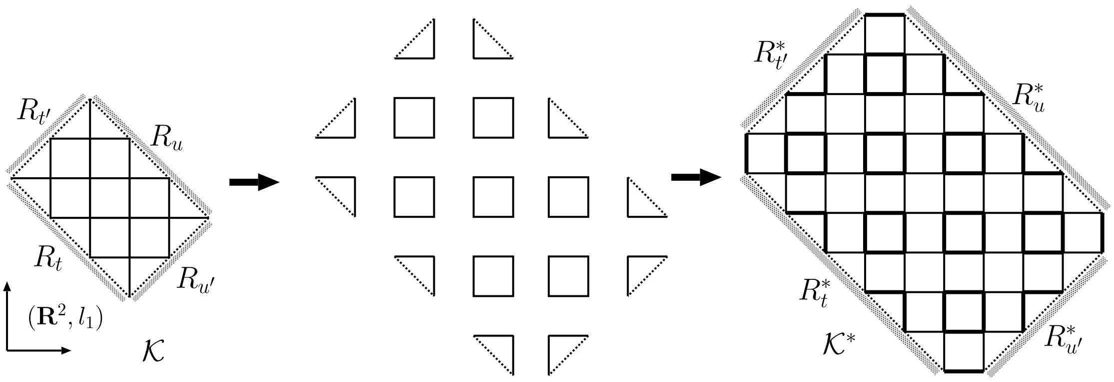


Figure 25: Blowing up 2-commodity F-complex

Multiterminal weighted 2-commodity flows. Suppose that S is partitioned into four sets $\{T, T', U, U'\}$. For relatively prime positive integers a and b, let μ be the weight on S such that $\mu(t, u) = a$ for $(t, u) \in T \times U$, $\mu(t', u') = b$ for $(t', u') \in T' \times U'$, and μ vanishes for other pairs. Then the corresponding μ-MFP is a *weighted* version of the multiterminal 2-commodity flow maximization problem; note that the standard super-terminal technique does not work to reduce this problem to a single-terminal problem.

Theorem 6.5. *The multiterminal weighted 2-commodity flow problem has an integral optimal flow for every inner Eulerian graph.*

We prove this from Theorem 6.4. We first construct a realization of μ. Consider a rectangle in the l_1-plane $\mathbf{R}^2$ such that the edge-directions are $(1, 1)$ and $(1, -1)$, and the edge-lengths are a and b; see Figure 25. Subdivide this rectangle into squares and right isosceles triangles along lines parallel to coordinate axes as in the left of Figure 25. Set the leg-length to be 1/2. The resulting complex $\mathcal{K}$ is clearly an (orientable) F-complex. Let (R_t, R_u) and $(R_{t'}, R_{u'})$ be opposite pairs of edges of length b and a, respectively. Then we obtain a realization of μ. Although $\mathcal{K}$ is not locally sparse (with all edges belonging to a common orbit), we can blow up $\mathcal{K}$ to a locally sparse F-complex as follows. Delete all legs from $\mathcal{K}$, and insert squares and triangles along deleted legs as in the middle of Figure 25. The inserted edges form two orbits, different from the orbit to which the original edges belong. From this, one can see that the resulting F-complex $\mathcal{K}^*$ is locally sparse, and has $\mathcal{K}$ as a summand. Each R_s is naturally extended to series of hypotenuses R^*_s; each terminal is strong. Thus by Theorem 6.4 we get Theorem 6.5.

6.3 Proof

We first prove Theorem 6.2 and then Theorem 6.1. Theorem 6.2 is a consequence of Theorem 4.3 (1).

Proof of Theorem 6.2. Let $p := \rho(y)$ and $X_p := \rho^{-1}(p)$. By applying the degree reductions (Section 3.3) at X_p, we may assume that each inner node in X_p has degree four, each proper terminal having no three hypotenuses at p has degree one, and the other terminals have degree two. We may assume that all improper terminals are essential. We consider $\mathcal{K}^2$; the sparsity of p is kept. We regard ρ as $V \to V(\mathcal{K}^2)$.

It suffices to show that some inner node in X_p has a splittable fork; then so does each inner node in X_p. Let y be an inner node in X_p, incident to four edges $e = xy$, $e_i = x_i y$ $(i = 1, 2, 3)$. Suppose to the contrary that all three forks $\tau_i = (xy, y, yx_i)$ are unsplittable. Consider a critical neighbor ρ_i with respect to τ_i for $i = 1, 2, 3$. We have

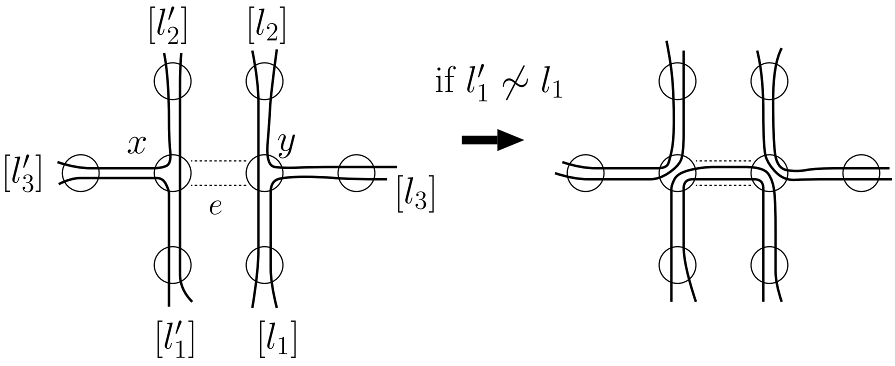


Figure 26: Flow rearrangement

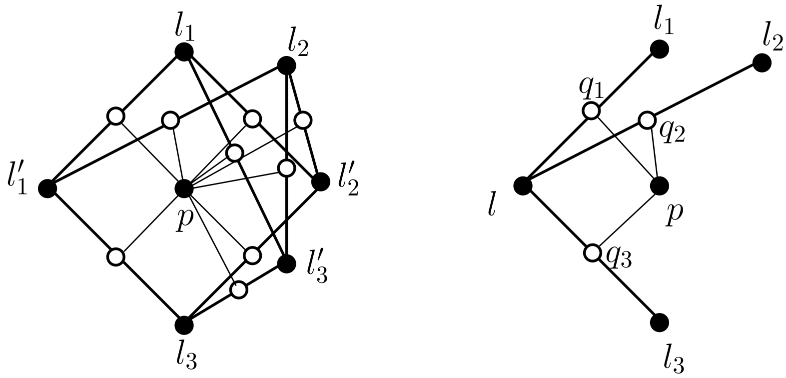


Figure 27: Forbidden folder structures around p

$p \succeq_{\overrightarrow{O_i}} \rho_i(y)$ and $p \succeq_{\overrightarrow{O_i}} \rho_i(y^{\tau_i})$ for some oriented orbit $\overrightarrow{O_i}$. By the sparsity condition, $\rho_i(y)$ and $\rho_i(y^{\tau_i})$ belong to a common folder in $\mathcal{K}_p$, i.e., $d^{\rho_i}(e^{\tau_i}) \in \{1, 2\}$. By the properly-inner Eulerian condition and Lemma 3.4, the numerator of formula (3.2) of α_τ is even. Therefore $\alpha_i > 0$ implies $\alpha_i = 1$ and $d^{\rho_i}(e^{\tau_i}) = 2$, i.e., $\rho_i(y)$ and $\rho_i(y^{\tau_i})$ are not adjacent by a leg; (2a, 2b) in Figure 21. Thus Theorem 4.3 (1) is applicable. There is a triple $l_1, l_2, l_3 \in L_p$ with properties (i) and (ii) (in Theorem 4.3 (1)). Take an optimal multiflow f. Then $\mathcal{P}(e_i, e_j)$ is an $([l_i], x_i y x_j, [l_j])$-set with $f^{e_i,e_j} = 1/2$ for $1 \leq i < j \leq 3$. Edge xy has no flow, and thus $\rho(x) = \rho(y) = p$ (by the saturation condition in Lemma 2.4).

Consider next the splittability property at $x \in X_p$, which is an inner node or a terminal. Suppose first that x is an inner node incident to y, y_1, y_2, y_3. The edge $y_i x$ is denoted by $\tilde{e}_i$ for $i = 1, 2, 3$. The fork $(e, x, \tilde{e}_i)$ is denoted by $\tilde{\tau}_i$ for $i = 1, 2, 3$. If x has a splittable fork, this is a desired node. Suppose not. Again, by Theorem 4.3 (1), there is a triple $l_1', l_2', l_3' \in L_p$ such that $\mathcal{P}(\tilde{e}_i, \tilde{e}_j)$ is an $([l_i'], y_i x y_j, [l_j'])$-set with $f^{\tilde{e}_i,\tilde{e}_j} = 1/2$ for $1 \leq i < j \leq 3$. See the left of Figure 26. Suppose $l_1 \not\sim l_1'$. Then we can rearrange f as in Figure 26. By the local geodesic condition, the resulting multiflow f is also optimal, and $f^{e_2,e_3} > 1/2$, which contradicts $\alpha_1 = 1$. Therefore $l_i \not\sim l_j$, $l_i' \not\sim l_j'$, and $l_i \sim l_j'$ for any i, j. Then $\mathit{\Pi}_p$ contains the subdivision of $K_{3,3}$, and all edges incident to l_i, l_j' in $\mathit{\Pi}_p$ belong to a common orbit. See the left of Figure 27. Therefore p cannot be sparse; a contradiction.

Suppose that x is a terminal (of degree one or two). Since $f^{e^{\tau_i}} = 1$, f is optimal for G^{τ_i,α_i}, and $f^e = 0$, we have $\rho(x) = \rho(y)$ and $\rho_i(x) = \rho_i(y)$ $(i = 1, 2, 3)$. Hence we have $p, l_1, l_2, l_3 \in R_x$. Here l_i is the midpoint of a folder in $\mathcal{K}$. By the normality of R_x in $\mathcal{K}$, for $i = 1, 2, 3$, R_x has a hypotenuse pl_i or a square-folder including p and l_i. By the properly-inner-Eulerian condition and the condition (ii), x must have degree two, incident to y and z. Since s is essential, $\tau' := (zx, x, xy)$ is unsplittable. By $f^e = 0$, we have $\alpha_{\tau'} = 1$ and $f^{zx} = 1$ (Lemma 3.1). In particular, $f^{e^{\tau'}} = 1$, and f is also optimal for $G^{\tau',\alpha_{\tau'}}$. Consider a critical neighbor ρ' of ρ with respect to τ', and consider the position $(\rho'(x), \rho'(x^{\tau'}))$. Necessarily $\rho'(x)$ is on the boundary of R_x, and $\rho'(x^{\tau'})$ is not in R_x. By the sparsity, $\rho'(x)$ and $\rho'(x^{\tau'})$ belong to a common folder. So $(\rho'(x), \rho'(x^{\tau'})) = (p, l)$ for some $l \in L_p$ (case (2a) in Figure 21) is the only possibility. Since f is also optimal for $G^{\tau',\alpha_{\tau'}}$, $\mathcal{P}(zx)(= \mathcal{P}(e^{\tau'}))$ is an $([l], zx, *)$-set (Table 1). If $l \not\sim l_1$, then by a rearrangement similar to Figure 26, we have $f^{e_2,e_3} > 1/2$; a contradiction to $1 = \alpha_1 \geq 2f^{e_2,e_3}$. So suppose $l \sim l_i$ for $i = 1, 2, 3$, i.e., l and l_i have a common neighbor q_i as in the right of Figure 27. Necessarily q_ip and q_il belong to a common orbit O_i (otherwise $(\rho'(y), \rho'(y^{\tau_i})) = (p, l_i)$ does not occur). Since there is a $K_{2,*}$-folder including p, l, q_1, q_2, q_3, we have $O_1 = O_2 = O_3$. So p cannot be sparse. A contradiction. □

Proof of Theorem 6.1. By the degree reduction (Section 3.3), we can assume that each inner node has degree four, each strong terminal has degree one, and the other terminals have degree two. All inner nodes are splittable by Theorem 6.2. So we may assume that there is no inner node. We may assume that each terminal of degree two is unsplittable; otherwise we can split it off. We show that the multiflow consisting of one-edge paths is optimal. By the optimality criterion, it suffices to show that $d_{\mathcal{K}}(\rho(s), \rho(t)) = d_{\mathcal{K}}(R_s, R_t)$ for each edge st. Again we consider $\mathcal{K}^2$ and regard ρ as $V \to V(\mathcal{K}^2)$.

Take an edge st with $\rho(s) \neq \rho(t)$. Let $p := \rho(s)$. Take an optimal multiflow $f = (\mathcal{P}; \kappa)$; necessarily $f^{st} = 1$. Since $\mathcal{P}(st)$ contains a $(*, st, t)$-path (otherwise t is splittable), we have $d_{\mathcal{K}}(\rho(s), R_t) = d_{\mathcal{K}}(\rho(s), \rho(t))$. Consider the gate g of R_t at p. If $g \notin R_s$, then, by Lemma 4.1, we have $d_{\mathcal{K}}(R_s, R_t) = d_{\mathcal{K}}(\rho(s), R_t) = d_{\mathcal{K}}(\rho(s), \rho(t))$ as required.

Suppose (to the contrary) that $g \in R_s$, i.e., $d_{\mathcal{K}}(\rho(s), R_t) > d_{\mathcal{K}}(R_s, R_t)$. Take $u \in Q_p$ with $d_{\mathcal{K}}(p, \rho(t)) = d_{\mathcal{K}_p}(p, u) + d_{\mathcal{K}}(u, \rho(t))$. Then $d_{\mathcal{K}}(p, R_t) = d_{\mathcal{K}_p}(p, u) + d_{\mathcal{K}}(u, R_t)$. By Lemma 4.2, we have $u \sim_1 g$. Since $\mathcal{P}(st)$ also contains an $(s, st, *)$-path (otherwise s is splittable), we have $d_{\mathcal{K}}(\rho(s), \rho(t)) = d_{\mathcal{K}}(R_s, \rho(t)) = d_{\mathcal{K}}(R_s, u) + d_{\mathcal{K}}(u, \rho(t))$. In particular, $d_{\mathcal{K}}(R_s, u) = d_{\mathcal{K}_p}(p, u)(> 0)$, and $u \notin R_s$. So $p, g \in R_s \not\ni u$, and g is incident to u. In particular, g belongs to L_p and is the midpoint of a folder in $\mathcal{K}$; recall that we are working on $\mathcal{K}^2$. By the normality, p, g, and u form a triangle σ in some $K_{2,*}$-folder F with hypotenuse $pg = \sigma \cap R_s$. Consequently, F belongs to a common orbit O. See Figure 28.

The terminal s must have degree two and a unique (unsplittable) fork τ with $\alpha_\tau > 0$; otherwise f has a path connecting s and t in st, implying $d_{\mathcal{K}}(\rho(s), \rho(t)) = d_{\mathcal{K}}(R_s, R_t)$. Consider a critical neighbor ρ' of ρ with respect to τ. Since the numerator of formula (3.2) of α_τ is even (Lemma 3.4), we have $d^{\rho'}(e^\tau) \geq 2$. Moreover $\rho'(s)$ belongs to the boundary of R_s, and $\rho'(s^\tau)$ is not in R_s. By the sparsity, $\rho'(s)$ and $\rho'(s^\tau)$ belong to a common folder. Therefore $(\rho'(s), \rho'(s^\tau)) = (p, l)$ for $l \in L_p$ (case (2a) in Figure 21) and $\alpha_\tau = 1$. Necessarily $g \neq l(\notin R_s)$. Consider an optimal multiflow $f = (\mathcal{P}; \kappa)$ for G^{τ,α_τ} and take a path $P \in \mathcal{P}(e^\tau, s^\tau t)(\neq \emptyset)$, which connects s and some terminal t' through e^τ. By the geodesic condition for (f, ρ'), $d_{\mathcal{K}}(p, R_{t'}) = d_{\mathcal{K}_p}(p, l) + d_{\mathcal{K}}(l, R_{t'})$ holds, and hence the gate of $R_{t'}$ is l by Lemma 4.2. Since (f, ρ) is also optimal for $G(= G^{\tau,0})$, $d_{\mathcal{K}}(p, R_{t'}) = d_{\mathcal{K}_p}(p, u) + d_{\mathcal{K}}(u, R_{t'})$ holds. By Lemma 4.2, u and l are adjacent, and hence pu and ul belong to a common folder $F'(\neq F)$ and a common orbit O'. Consequently F and F' belong to common orbit $O = O'$. This is impossible by the sparsity. □

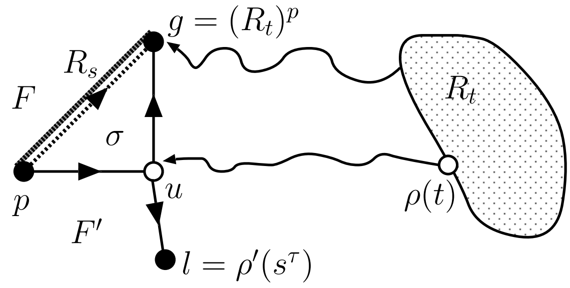


Figure 28: p, q, u, l in the proof of Theorem 6.1

7 0-1 problems

Here we focus on μ_H-MFP for a commodity graph H with property P, where μ_H is the 0-1 weight corresponding to H by the relation: $\mu_H(s,t) = 1 \Leftrightarrow st \in E(H)$. Application of our results in the previous sections reveals an interesting hierarchy of problem classes admitting integrality or half-integrality theorems. This gives a unified understanding to previously known results, as well as to new half-integrality results.

In Section 7.1 we introduce three F-complexes $\mathcal{K}_H$, $\mathcal{K}_H^s$, and $\mathcal{K}_H^e$ with the properties that $\mathcal{K}_H$ realizes μ_H, $\mathcal{K}_H^s$ is star-shaped, and $\mathcal{K}_H^e$ has both $\mathcal{K}_H$ and $\mathcal{K}_H^s$ as summands. From $\mathcal{K}_H^s$ and $\mathcal{K}_H^e$, we define weights μ_H^s and μ_H^e such that μ_H^s is a metric and μ_H^e has both μ_H^s and μ_H^e as summands. Recall Sections 2.1.5 and 2.2.3 for summands.

In Section 7.2, we show that the local sparsity of $\mathcal{K}_H^e$ is equivalent to the *anticlique-bipartite* condition on H. This fact and Theorem 6.4 immediately imply the classical Karzanov-Lomonosov integrality theorem [22].

In Section 7.3, the *fractionality relation* $\mathrm{frac}(H) \leq 2\,\mathrm{frac}(\mu_H^s)$ is stated. This relation reduces the fractionality study for μ_H-MFP to that for μ_H^s-MFP. Since $\mathcal{K}_H^s$ is star-shaped, μ_H^s-MFP has a much simpler structure than μ_H-MFP has. Applying the result to $\mathcal{K}_H^s$ in the previous section, we prove the half-integrality theorem for a large class of commodity graphs including the previously known. In Section 7.4, we prove, algorithmically, the fractionality relation.

In this section we assume that commodity graph H has no isolated nodes. A maximal stable set of H is called an *anticlique*. In constructions of F-complexes, a square-folder with legs $pp', p'q', q'q, qp$ is denoted by $pp'q'q$, and a triangle with hypotenuse pp' and legs pq and qp' is denoted by pqp'.

7.1 F-complexes for a commodity graph with the property P

Let $\mathcal{A}$ be the set of anticliques of H, and $\mathcal{D}$ be the set of nonempty subsets $D \subseteq S$ represented as the intersection of (at least) two distinct anticliques. By property P, we have $D = \bigcap\{A \in \mathcal{A} \mid D \subseteq A\}$ for any $D \in \mathcal{D}$. Let $\mathcal{A}_0 \subseteq \mathcal{A}$ be the set of anticliques A with $A' \cap A = \emptyset$ for every $A' \in \mathcal{A} \setminus \{A\}$.

Let $\mathit{\Pi}_H$ be the bipartite graph with bipartition $\{\mathcal{D}, \mathcal{A}\}$ and edge set $\{DA \mid D \subseteq A\}$. By property P, we easily see:

(7.1) $\qquad \mathit{\Pi}_H$ has girth at least 8.

Indeed, a 6-cycle corresponds to an intersecting triple of anticliques with distinct intersections.

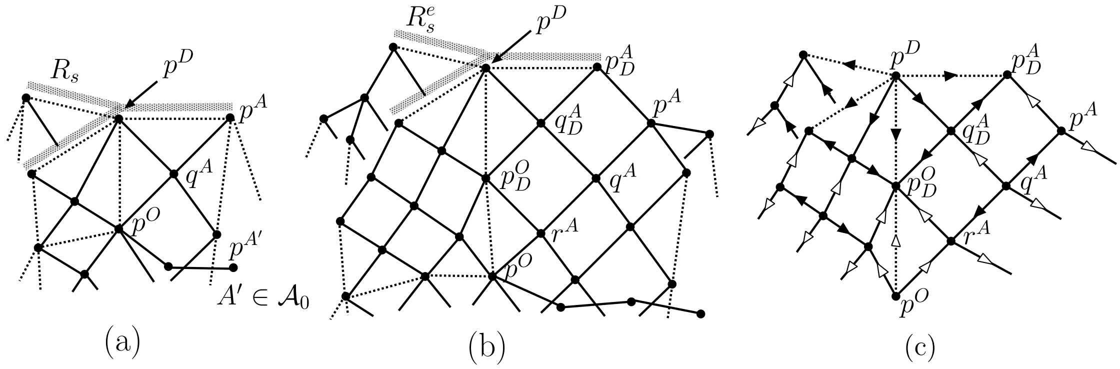


Figure 29: (a) $\mathcal{K}_H$, (b) $\mathcal{K}^e_H$, and (c) the orientation of $\mathcal{K}^e_H$

The first F-complex $\mathcal{K}_H$. The first F-complex $\mathcal{K}_H$ is constructed as follows. The vertices of $\mathcal{K}_H$ are p^O, p^D $(D \in \mathcal{D})$, $p^A, q^A (A \in \mathcal{A})$. For $D \in \mathcal{D}$, consider $K_{2,*}$-folder F_D consisting of triangles $p^D q^A p^O$ over all anticliques A including D. If F_D is a $K_{2,2}$-folder, then replace F_D by a square-folder (on the same vertices). Such a $K_{2,2}$-folder corresponds to a member of $\mathcal{D}$ which is the intersection of exactly two anticliques. Next, for each anticlique A including D, attach triangles $p^D q^A p^A$ to F_D. Let K_D be the resulting complex. Glue K_D over all $D \in \mathcal{D}$. Finally, for each $A \in \mathcal{A}_0$, attach series of two legs $p^O q^A, q^A p^A$ to p^O. Let $\mathcal{K}_H$ be the resulting complex. The leg-length is defined to be $1/4$. See Figure 29 (a).

$\mathcal{K}_H$ is an F-complex. Indeed, it is contractible, and hence simply-connected. It suffices to verify the flag condition at p^O. Observe $\mathit{\Pi}_{p^O} = \mathit{\Pi}_H$. Thus $\mathit{\Pi}_{p^O}$ has girth at least 8. Furthermore, $\mathcal{K}_H$ is orientable; we can orient $\mathcal{K}_H$ so that p^O and p^A are sources.

To realize μ_H, normal sets R_s $(s \in S)$ are defined as follows. By property P, each $s \in S$ belongs to either a unique $D \in \mathcal{D}$ or a unique $A \in \mathcal{A}$. In the former case, define R_s as the union of hypotenuses $p^D p^A$ over all anticliques A including D. In the latter case, define R_s as the single vertex p^A. Then each R_s is clearly normal; R_s is a star of hypotenuses or a single vertex. Then $\mu_H(s,t) = d_{\mathcal{K}_H}(R_s, R_t)$. Indeed, $R_s \cap R_t \neq \emptyset \Leftrightarrow s$ and t belong to a common anticlique $\Leftrightarrow \mu_H(s,t) = 0$. Conversely, $R_s \cap R_t = \emptyset$ implies $d_{\mathcal{K}_H}(R_s, R_t) = 1$. Therefore $(\mathcal{K}_H; \{R_s\}_{s \in S})$ is a realization of μ_H. The corresponding combinatorial duality relation (Theorem 2.3) coincides with that given in [16].

The second F-complex $\mathcal{K}^s_H$. The second F-complex $\mathcal{K}^s_H$ is the neighborhood of p^O in $\mathcal{K}_H$, i.e., $\mathcal{K}^s_H := (\mathcal{K}_H)_{p^O}$. For each terminal s, let $p_s := p^D$ if s belongs to a unique $D \in \mathcal{D}$, and let $p_s := q^A$ if s belongs to a unique $A \in \mathcal{A}$. Define μ^s_H by $\mu^s_H(s,t) := d_{\mathcal{K}^s_H}(p_s, p_t)$ for $s, t \in S$. Then $(\mathcal{K}^s_H; \{p_s\}_{s \in S})$ is a realization of μ^s_H.

The third F-complex $\mathcal{K}^e_H$. In the construction of K_D in $\mathcal{K}_H$ above, relabel (p^A, q^A, p^O) by (p^A_D, q^A_D, p^O_D). For each anticlique A including D, attach squares $p^A_D q^A_D q^A p^A$ and $p^O_D q^A_D q^A r^A$ to K_D. Also attach to K_D the $K_{2,*}$-folder, denoted by F'_D, consisting of triangles $p^O r^A p^O_D$ over all anticliques A including D. Replace F'_D by a square-folder if F'_D is a $K_{2,2}$-folder. Glue K_D over all $D \in \mathcal{D}$. For each $A \in \mathcal{A}_0$, attach series of three legs $p^A q^A, q^A r^A, r^A p^O$ to p^O. Let $\mathcal{K}^e_H$ be the resulting complex. See Figure 29 (b). Clearly $\mathcal{K}^e_H$ is also an orientable F-complex; see Figure 29 (c) for an admissible orientation.

Let R^e_s be the union of hypotenuses $p^D p^A_D$ if s belongs to a unique $D \in \mathcal{D}$ and let R^e_s be the vertex p^A if s belongs to a unique $A \in \mathcal{A}$. Then define μ^e_H on S as $\mu^e_H(s,t) := d_{\mathcal{K}^e_H}(R^e_s, R^e_t)$ for $s, t \in S$. Again $(\mathcal{K}^e_H; \{R^e_s\}_{s \in S})$ is a realization of μ^e_H.

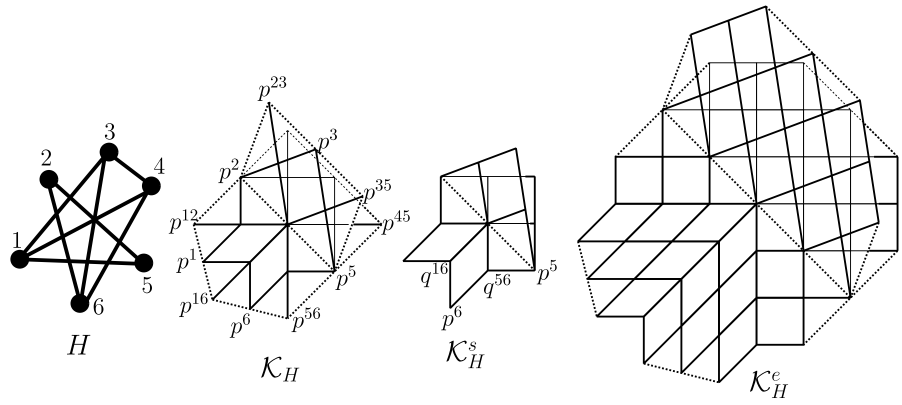


Figure 30: Three F-complexes

Example. We consider *complement-triangle-free commodity graphs* as a class of commodity graphs having a simpler construction. A commodity graph H is called *complement-triangle-free* if the complement $\overline{H}$ has no triangle K_3. Such a commodity graph has property P since every anticlique has cardinality at most 2. In this case, the construction of $\mathcal{K}^s_H$ is quite simple; $\mathit{\Pi}_H$ is the subdivision of $\overline{H}$. Figure 30 illustrates three F-complexes $\mathcal{K}_H$, $\mathcal{K}^s_H$, and $\mathcal{K}^e_H$ for a complement-triangle-free commodity graph H.

Summand relation between $\mathcal{K}_H$, $\mathcal{K}^s_H$ and $\mathcal{K}^e_H$. These F-complexes and the corresponding weights are in a relation of summands (Sections 2.1.5 and 2.2.3). Observe that $(\mathcal{K}^e_H)_{p^O}$ and $(\mathcal{K}^e_H)_{p^D}$ belong to distinct orbits; see Figure 29, where the black and the white arrows indicate different orbits. Let U be the union of the orbits meeting $(\mathcal{K}^e_H)_{p^O}$. Then $(\mathcal{K}^e_H)/U = \mathcal{K}^s_H$ and $(\mathcal{K}^e_H)\backslash U = \mathcal{K}_H$. Also $(R^e_s)/U = \{p_s\}$ and $(R^e_s)\backslash U = R_s$. Thus Proposition 2.6 implies the following.

Theorem 7.1. *Both μ_H and μ^s_H are summands of μ^e_H, and thus any optimal multiflow to μ^e_H-MFP is also optimal to both μ_H-MFP and μ^s_H-MFP.*

This locking property is known in Lomonosov [24] for special commodity graphs.

Algorithmic implication: Proof of Theorem 1.7. The sizes of these F-complexes are bounded:

(7.2) $\mathcal{K}_H$, $\mathcal{K}^s_H$ and $\mathcal{K}^e_H$ have $O(|S|^2)$ cells.

Indeed, $\mathcal{A}_0 \cup \mathcal{D}$ is a subpartition of S. This implies $|\mathcal{A}_0| + |\mathcal{D}| = O(|S|)$. Also $\{A \setminus D \mid D \subseteq A \in \mathcal{A}\}$ for $D \in \mathcal{D}$ is a disjoint family. This implies that there exist at most $O(|S|)$ anticliques containing $D \in \mathcal{D}$. Thus the number of cells in K_D is $O(|S|)$, and consequently we have (7.2).

By the general theorem, Theorem 5.4, there exists a strongly polynomial time to find a 1/12-integral optimal multiflow in every inner Eulerian graph for μ_H-, μ^e_H-, and μ^s_H-MFP. Note that there is no improper terminal in this case. This implies Theorem 1.7 in Introduction.

7.2 Local sparsity of $\mathcal{K}^e_H$ and anticlique-bipartite commodity graphs

We show that the local sparsity of $\mathcal{K}^e_H$ is equivalent to the classical anticlique-bipartite condition [22]. We first note that the following sparse/nonsparse properties of $\mathcal{K}^e_H$:

(7.3) (0) r^A, q^A, and r^A are sparse if $A \in \mathcal{A}_0$.

(1) p^O_D and p^A_D are sparse.

(2) p^D is not sparse if D is the intersection of at least three anticliques.

(0) is obvious. (1) can be seen from Figure 29 (c), where the black and the white arrows indicate distinct orbits. (2) follows from the fact that all edges incident to p^D belong to a common orbit (by $K_{2,*}$-folders around p^D).

A commodity graph H is said to be *loose* if it satisfies:

(7.4) for every triple $A, B, C \in \mathcal{A}$, at least one of $A \cap B$, $B \cap C$, $C \cap A$ is empty.

This condition, due to [22], is stronger than property P. So a loose commodity graph has property P. This condition is equivalent to that each $D \in \mathcal{D}$ is the intersection of two anticliques. Geometrically, this condition says that there is no $K_{2,m}$-folder for $m \geq 2$ in the three F-complexes. In particular, $(\mathcal{K}^e_H)_{p^O} (= \mathcal{K}^s_H)$ consists of square-folders, each of which meets two distinct orbits. Hence p^O is sparse in $\mathcal{K}^e_H$, and consequently $\mathcal{K}^s_H$ is locally sparse.

A loose commodity graph H is called *anticlique-bipartite* if the intersection graph of $\mathcal{A}$ is bipartite, and otherwise it is called *anticlique-nonbipartite*. The complement $\overline{C_n}$ of n-cycle C_n ($n \geq 4$) is loose, and $\overline{C_n}$ is anticlique-bipartite if and only if n is even. Figure 31 illustrates $\mathcal{K}^e_{\overline{C_n}}$ for the case $n = 5, 6$.

Each p^D may be sparse or nonsparse. Trace the orbit starting from $p^A q^A$, as in Figure 31. If H is anticlique-bipartite, then this orbit never meets $q^A r^A$ (by the bipartiteness), and hence p^D, q^A_D, and q^A are sparse; all vertices are sparse. On the other hand, if H is anticlique-nonbipartite, then for some D the orbit returns to $q^A r^A$, and hence p^D, q^A_D, and q^A are not sparse.

Theorem 7.2. *H is anticlique-bipartite if and only if $\mathcal{K}^e_H$ is locally sparse.*

By virtue of this characterization, we can derive, as a corollary of Theorem 6.4, the following fundamental result; see also [8]. Here each R_s is a path of hypotenuses or a single vertex; each terminal is strong in the sense of the previous section.

Theorem 7.3 ([15, 22, 24])**.** *If H is anticlique-bipartite, then μ_H-MFP has an integral optimal multiflow for every inner Eulerian graph.*

It is known that the integrality theorem fails for anticlique-nonbipartite commodity graphs. Nevertheless Karzanov-Lomonosov [22] proved that the half-integrality theorem still holds.

Theorem 7.4 ([15, 22, 24])**.** *If H is anticlique-nonbipartite, then μ_H-MFP has a half-integral optimal multiflow for every inner Eulerian graph.*

We will prove this theorem as an immediate consequence of the fractionality relation between μ^s_H and μ_H in the next section.

7.3 Fractionality relation and its consequences

The *fractionality relation*, which is the main result in this section, says that $1/k$-integrality of μ^s_H-MFP guarantees $1/(2k)$-integrality of μ_H-MFP.

Theorem 7.5. *Let H be a commodity graph with property P. Suppose that μ^s_H-MFP has a $1/k$-integral optimal multiflow for every inner Eulerian graph.*

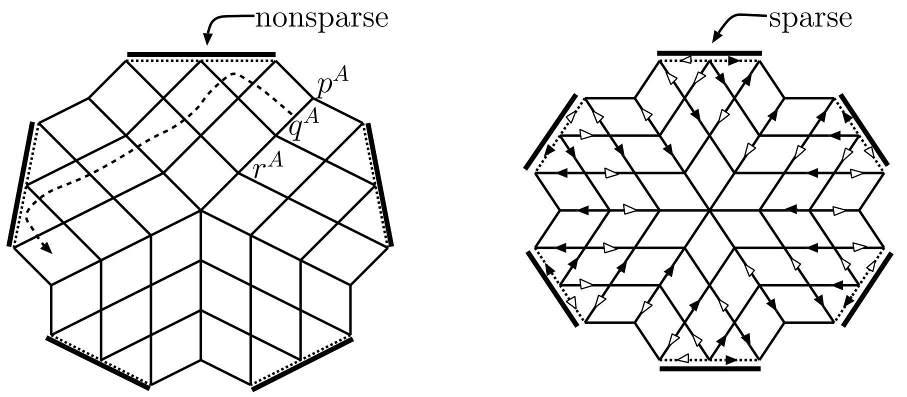


Figure 31: $\mathcal{K}^e_H$ for $H = \overline{C_5}$ and $\overline{C_6}$

(1) *If k is even, then μ_H-MFP has a $1/k$-integral optimal multiflow for every inner Eulerian graph.*

(2) *If k is odd, then μ_H-MFP has a $1/(2k)$-integral optimal multiflow for every inner Eulerian graph.*

In particular, $\mathrm{frac}(H) \leq 2\,\mathrm{frac}(\mu^s_H)$ *holds.*

The proof is given in Section 7.4. Here we describe consequences. Theorem 7.4 immediately follows from this theorem and the integrality of μ^s_H-MFP. The integrality of μ^s_H-MFP follows from the multiflow locking theorem [22] or, in our framework, by Theorem 6.4 and the local sparsity of $\mathcal{K}^s_H$, which consists of square-folders.

Consider $H = K_2 + K_r$, i.e., the vertex-disjoint union of a single edge and complete graph K_r ($r \geq 3$), which is complement-triangle-free. Then Π_H is the subdivision of $K_{2,r}$ and, μ^s_H is the graph metric $d_{K_{2,r}}$ of $K_{2,r}$. Thus μ^s_H admits an integral optimal multiflow by Karzanov-Manoussakis integrality theorem [23] or, in our framework, by Theorem 6.4 and the local sparsity of $\mathcal{K}^s_H$, which is the subdivision of a single folder. Hence Theorem 7.5 implies the following.

Theorem 7.6 ([20] for $r = 3$ and [25] for $r > 3$)**.** *If $H = K_2 + K_r$, then μ_H-MFP has a half-integral optimal multiflow for every inner Eulerian graph.*

A commodity graph H with property P is called *sparse* if $\mathcal{K}^s_H$ is locally sparse. By Theorems 6.3 and 7.5, we have the following, which includes the two theorems above.

Theorem 7.7. *If H is sparse, then μ_H-MFP has a half-integral optimal multiflow for every inner Eulerian graph.*

The commodity graph H in Figure 30 is sparse, and hence the half-integrality result holds for this H. A sparse commodity graph can be easily characterized by the following observation: $\mathcal{K}^s_H$ is locally sparse if and only if it has no adjacent pair of $K_{2,*}$-folders. This characterization can be rephrased in terms of $\mathcal{A}$ as follows.

Proposition 7.8. *A commodity graph H with property P is sparse if and only if H has no five anticliques A_1, A_2, B, C_1, C_2 with $\emptyset \neq A_1 \cap A_2 = A_2 \cap B = A_3 \cap B \neq C_1 \cap C_2 = C_2 \cap B = C_3 \cap B \neq \emptyset$.*

Again Theorem 6.4 enlarges the class of commodity graphs admitting the half-integrality property. A commodity graph H with property P is called *sparsible* if $\mathcal{K}^s_H$ is a summand of a locally sparse F-complex.

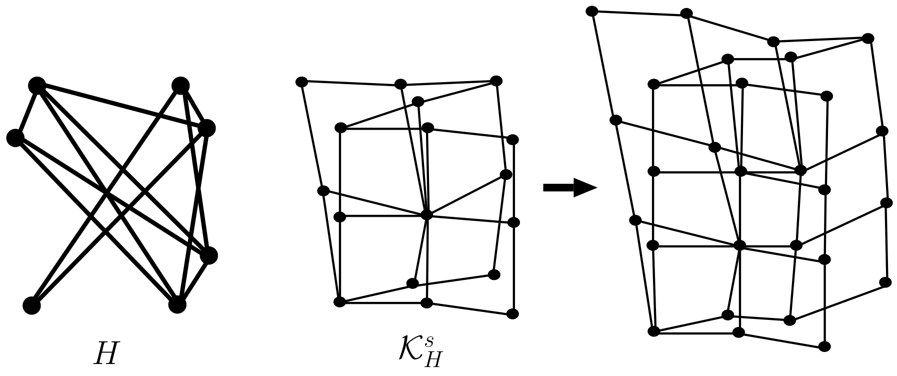


Figure 32: A sparsible commodity graph H, $\mathcal{K}_H^s$, and a blow-up

Theorem 7.9. *If H is sparsible, then μ_H-MFP has a half-integral optimal multiflow for every inner Eulerian graph.*

We give an example of sparsible commodity graph H together with a blow-up of $\mathcal{K}_H^s$ in Figure 32; the half-integrality theorem holds for this commodity graph. However, we do not know any nice characterization of a sparsible commodity graph.

A commodity graph H with property P is called *weakly-integral* if μ_H^s-MFP has an integral optimal multiflow for every inner Eulerian graph. Obviously, by Theorem 7.5, the half-integral theorem holds for weakly-integral commodity graphs. Thus we have the following hierarchy:

$$\text{loose} \subset \text{sparse} \subset \text{sparsible} \subseteq \text{weakly-integral} \subset \text{property P}.$$

The vertex-disjoint union of two triangles $H_{3,3} := K_3 + K_3$ is a typical nonintegral example. This implies that $H_{3,3}$ is not sparsible. One can directly see the nonsparsibility from $\Pi_{H_{3,3}}$, which is the subdivision of $K_{3,3}$ (Figure 27). We do not know whether sparsible = weakly-integral holds or not.

Let us rephrase these results by using the notion of the fractionality $\mathrm{frac}(H)$. The commodity graphs of fractionality 1 or 2 have already been classified by Karzanov [16, 21] as follows:

(1) $\mathrm{frac}(H) = 1$ if and only if H is a complete bipartite graph.

(2) $\mathrm{frac}(H) = 2$ if and only if H is $K_2 + K_3$ or anticlique-bipartite (not complete bipartite).

Other commodity graphs have fractionality at least 4. Combining this classification with our results, we get the following.

Corollary 7.10. *A sparse/sparsible/weakly-integral commodity graph that is neither anticlique-bipartite nor $K_2 + K_3$ has fractionality* 4.

Our proof of Theorem 7.5 is based on SPUP framework, and constructs algorithmically a half-integral optimum in μ_H^e-MFP from an integral optimum in μ_H^s-MFP. An integral optimum of μ_H^s-MFP is obtained by splitting-off if its existence is guaranteed (Section 5.4.2). As a by-product we obtain the following.

Theorem 7.11. *Suppose that H is sparse/sparsible/weakly-integral. Then there exists a strongly polynomial time algorithm to find a half-integral optimal multiflow in μ_H-MFP for every inner Eulerian graph.*

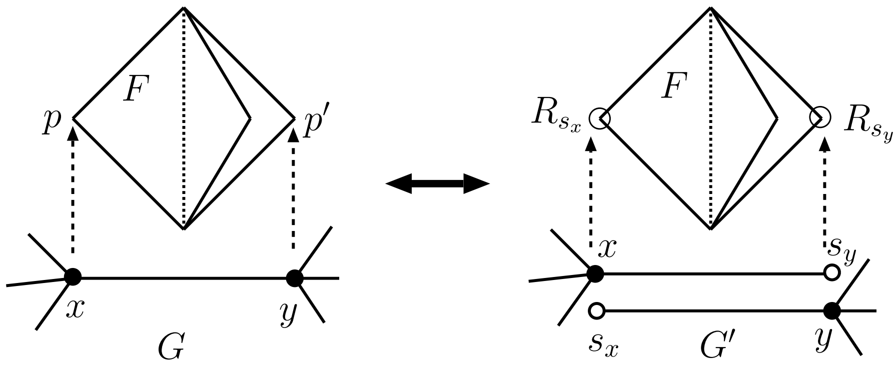


Figure 33: Terminal creation I

7.4 Proof of the fractionality relation

Here we give an algorithmic proof of the fractionality relation (Theorem 7.5) according to the SPUP framework (Section 3).

7.4.1 Preliminary: terminal creations

As a preliminary, we introduce terminal creation techniques under an optimal potential ρ. This technique works for general DLP($\mathcal{K}$; $\{R_s\}_{s \in S}$). We assume that G has unit capacity.

Terminal creation I. Suppose that there are an edge $e = xy$ and a folder F such that $\rho(x)$ and $\rho(y)$ are distinct vertices in F, nonadjacent by a leg; see Figure 33. In this case, we can make the following change on μ, S, G, ρ.

Delete edge xy, add new terminals s_x, s_y, and add new edges xs_y, ys_x. Set $R_{s_x} := \{p\}$ and $R_{s_y} := \{p'\}$, and extend μ to $S \cup \{s_x, s_y\}$ by $\mu(s_x, t) := d(R_{s_x}, R_t)$ and $\mu(s_y, t) := d(R_{s_y}, R_t)$. Extend ρ by $(\rho(s_x), \rho(s_y)) := (p, p')$.

Take an optimal multiflow $f = (\mathcal{P}; \kappa)$ for the original problem. For each path in $\mathcal{P}(xy)$, delete xy to split it into two paths, add edge xs_y to one of the two paths having x, and add edge ys_x to the other path. Then we obtain a multiflow and a potential for the new problem. Both are optimal. Indeed, the saturation condition holds, and the new paths are all geodesic by Lemmas 4.1 and 4.2; consider $(\mathcal{K}^2)_{p^*}$ for the center p^* of folder F.

Conversely, take an arbitrary optimal multiflow $f = (\mathcal{P}; \kappa)$ to the new problem. Take a pair $(P', P'') \in \mathcal{P}(xs_y) \times \mathcal{P}(ys_x)$, and concatenate P' and P'' by deleting edges xs_y and ys_x and by adding edge xy to get a path in $\mathcal{P}(xy)$. Repeating this concatenation, we obtain a multiflow f' in the original graph of fractionality κ. Again all the new paths satisfy the geodesic condition by Lemmas 4.1 and 4.2. Hence f' is optimal.

Terminal creation II. Let p be a vertex in $\mathcal{K}$ such that p is incident to four vertices q_1, q_2, r_1, r_2 by legs and $\mathit{\Pi}_p$ is an 8-cycle $(q_1, l_{11}, r_1, l_{21}, q_2, l_{22}, r_2, l_{12})$, as in Figure 34. Suppose that there is an edge $e = xy \in E$ with $(\rho(x), \rho(y)) = (q_1, q_2)$. In this case, we can make the following change on μ, S, G, ρ.

Subdivide e into two edges xz and zy, add new terminals s and t, and join them to z. Set $R_s := \{r_1\}$ and $R_t := \{r_2\}$, and extend μ to $S \cup \{s, t\}$. Also extend ρ by $(\rho(s), \rho(z), \rho(t)) := (r_1, p, r_2)$.

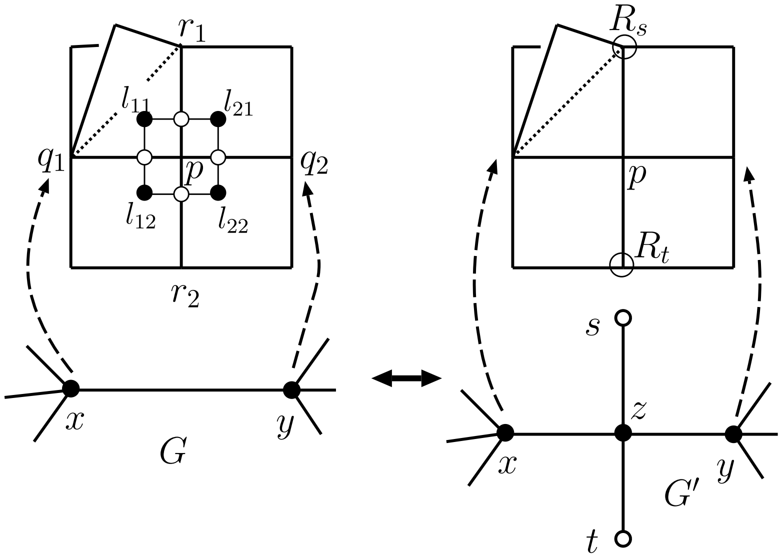


Figure 34: Terminal creation II

Take an optimal multiflow $f = (\mathcal{P};\kappa)$ for the original problem. We can extend f for the new graph by subdividing each path in $\mathcal{P}(e)$ at z and adding (s,t)-paths of two edges sz and zt so that $f^{sz,zt} = 1$. Then the resulting ρ and f are both optimal.

Conversely, take an arbitrary optimal multiflow $f = (\mathcal{P};\kappa)$ in the new problem. We can construct an optimal multiflow for the original problem by the following way. Since four edges incident to z are all saturated, we have $(f^{xz,zs}, f^{xz,zy}, f^{xz,zt}) = (f^{tz,zy}, f^{tz,zs}, f^{sz,zy})$. If $f^{xz,zs} = f^{tz,zy} = f^{xz,zt} = f^{sz,zy} = 0$, then the deletion of $\mathcal{P}(sz,zt)$ gives an optimal multiflow (of fractionality κ) in the original problem. Suppose that $f^{xz,zs} = f^{tz,zy} > 0$. Then $\mathcal{P}(xz,zs)$ is a $([q_1]\cup[l_{12}], xzs, [r_1])$-set, and $\mathcal{P}(xz,zs)$ is an $([r_2], xzs, [q_2]\cup[l_{21}])$-set. Reconnect paths from s and paths from t, and reconnect paths from x and paths from y. Then the local geodesic condition (Section 4.1) is kept, and thus we can make f satisfy $f^{xz,zs} = f^{tz,zy} = f^{xz,zt} = f^{sz,zy} = 0$ (while keeping the optimality), and we get an optimal multiflow of fractionality κ in the original graph.

7.4.2 Proof of Theorems 7.5 and 7.11

We reduce μ^{e}_{H}-MFP and DLP$(\mathcal{K}^{e}_{H}; \{R^{e}_{s}\}_{s\in S})$ to μ^{s}_{H}-MFP and DLP$(\mathcal{K}^{s}_{H}; \{p_s\}_{s\in S})$ (thanks to Theorem 7.1). Let G be an inner Eulerian graph with terminal set S. There is no improper terminal. We may assume that G has unit capacity, also in the algorithmic sense explained in Section 5.4.2.

0. Let us construct the SPUP scheme for $\mathcal{K}^{e}_{H}$, as in Section 3.4. The forward orientation is a unique orientation such that p^O, p^D, and q^A are sources; see Figure 29 (c). Then p^A_D and p^O_D are sinks, and are sparse by (7.3) (1). For an optimal potential ρ, partition V into three subsets S_ρ, M_ρ, and C_ρ:

$$
\begin{aligned}
S_\rho &:= \{x \in V \mid \rho(x) = p^A_D, \text{or } p^O_D\},\\
M_\rho &:= \{x \in V \mid \rho(x) = r^A, q^A_D, \text{or } p^A\},\\
C_\rho &:= \{x \in V \mid \rho(x) = p^D, q^A, \text{or } p^O\}.
\end{aligned}
$$

Recall the restricted Eulerian condition; each inner node not in S_ρ has an even degree. Then all properties for S_ρ, M_ρ, C_ρ in Section 3.4 hold. For example, by Theorem 6.2, claim (A) holds. Also claim (B) holds since r^D, q^A_D, and p^A have the same local orbit structure as the midpoint of $\mathcal{K}^2$ (Theorem 4.3 (2) is applicable).

Let $C^D_\rho := \rho^{-1}(p^D)$, $C^A_\rho := \rho^{-1}(q^A)$, and $C^O_\rho := \rho^{-1}(p^O)$. Then C_ρ is the union of C^O_ρ, C^A_ρ, and C^D_ρ over all D, A. Since the folder structures around p^D and q^A are rather special, we do not need claim (C) to make both C^D_ρ and C^A_ρ empty while keeping the restricted Eulerian condition, which we will show below.

By the degree reduction (Section 3.3), we modify G so that each inner node has degree four and each terminal has degree one. We may assume that there exists no splittable fork. Then there is no inner node x with $\rho(x) = p^A, q^A$ or r^A for $A \in \mathcal{A}_0$ by the sparsity (7.3) (2) and Theorem 6.2.

1. First we repeat applying the forward 1-SPUP until C^D_ρ is empty for all $D \in \mathcal{D}$. Take $D \in \mathcal{D}$ with $C^D_\rho \neq \emptyset$. We may assume that there is an edge xy with $y \in C^D_\rho$ and $x \notin C^D_\rho$. Consider the gate of $\rho(x)$ at $(\mathcal{K}^e_H)_{p^D}$, which is (i) p^A_D, (ii) q^A_D, or (iii) p^O_D.

First consider case (ii). Suppose that $\rho(x) = q^A_D$, i.e., $x \in M_\rho$. According to claim (B) there is a fork τ at x such that its critical neighbor ρ' is forward. If $\alpha_\tau = 1$, then $\{\rho'(y^\tau), \rho'(y)\} = \{p^A_D, p^O_D\}$, and thus 1-SPUP for (τ, ρ') succeeds. If $\alpha_\tau = 0$, then we can replace ρ by its optimal forward neighbor ρ' with $\rho'(x) = p^O_D$ or p^A_D (by (4.3)). Therefore we can decrease the number of edges in case (ii). So suppose $\rho(x) \neq q^A_D$. By the edge-subdivision (Section 2.2.1), we may assume that $\rho(x) = q^A$. Here we use the terminal creation II. Subdivide xy into xz and zy. Add two new terminals s and t joined to z. Set $R_s := \{p^A_D\}$ and $R_t := \{p^O_D\}$. Extend potential ρ for the new problem by $(\rho(s), \rho(z), \rho(t)) := (p^A_D, q^A_D, p^O_D)$. For a fork $\tau := (yz, z, zs)$, consider α_τ. If $\alpha_\tau = 2$, then split off τ. Consider the case of $\alpha_\tau < 2$. Take a critical neighbor ρ' of ρ with respect τ. We show that ρ' is forward. Consider an optimal multiflow f for G^{τ,α_τ}. Since $f^{sz} = f^{zt} = 1$ (by the saturation condition), $\mathcal{P}(e^\tau)$ contains paths P_s and P_t such that P_s connects s and P_t connects t. If ρ' is backward, then $\{\rho'(y), \rho'(y^\tau)\} = \{p^D, q^A_D\}$ or $\{q^A_D, q^A\}$. Since $(\rho'(s), \rho'(t)) = (p^A_D, p^O_D)$, for $u \in \{p^D, q^A\}$, $\rho'(P_s)$ passes through $q^A_D \to u \to p^A_D$ and $\rho'(P'_t)$ passes through $q^A_D \to u \to p^O_D$. Then one of $\rho'(P_s)$ and $\rho'(P'_t)$ is not geodesic. A contradiction to the optimality. Thus ρ' is necessarily forward. As in claim (B), if $\alpha_\tau = 1$, then $\{\rho'(z), \rho(z^\tau)\} = \{p^A_D, p^O_D\}$, and apply 1-SPUP for (τ, ρ'). If $\alpha_\tau = 0$, then we can replace ρ by a forward neighbor ρ' with $\rho'(z) \in \{p^A_D, p^O_D\}$. In this way, we can decrease the number of edges in case (ii).

Consider case (iii). We may assume $\rho(x) = p^O_D$. Otherwise, subdivide xy and extend ρ by defining the potential of the new node as p^O_D; see (4.1). If y is a terminal (of degree one), then replace $\rho(y)$ by p^A_D; this keeps the feasibility and the optimality. So suppose that y is an inner node. Take a fork τ at y with $0 < \alpha_\tau < 2$ (by Lemma 4.4 (1)), and consider a critical neighbor ρ' of ρ with respect to τ. Then ρ' is necessarily forward. So $\rho'(x) = p^O_D$. Consider an optimal multiflow $f = (\mathcal{P}; \kappa)$ for G^{τ,α_τ} and take a path P from $\mathcal{P}(xy^\tau, y^\tau y) (\neq \emptyset)$. Then $\rho'(P)$ passes through $p^O_D \to \rho'(y^\tau) \to \rho'(y)$, which must be geodesic. Also $d^{\rho'}(e^\tau) > 1$ is necessary (otherwise $\alpha_\tau = 2$). Thus the possible configurations of ρ' are (2a) $(\rho'(y^\tau), \rho'(y)) = (p^O_D, p^D)$ and (2d) $(\rho'(y^\tau), \rho'(y)) = (p^O_D, p^A_D)$. If all three forks at y have a critical neighbor in case (2a), this contradicts Theorem 4.3 (1). Thus there exists a fork τ having a critical neighbor in case (2d). Then $\alpha_\tau = 1$, and both $\rho'(y^\tau)$ and $\rho'(y)$ fall into S_ρ. Apply 1-SPUP. In this way, we can decrease the number of edges in (iii).

So suppose that all edges entering C^D_ρ from the outside are in case (i). Then there is no flow connecting a terminal s in C^D_ρ. Indeed, take an arbitrary edge xy with $y \in C^D_\rho \not\ni x$. By edge-subdivision with (4.1), we may assume that $\rho(x) = p^A_D$ for some

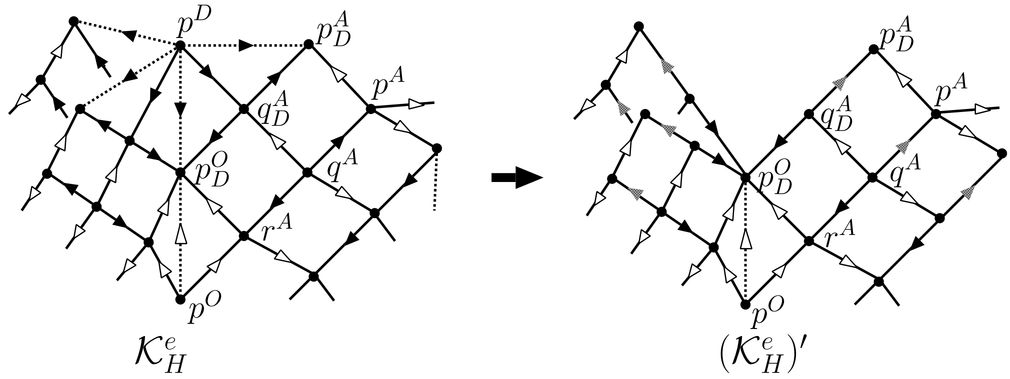


Figure 35: Construction of $(\mathcal{K}^e_H)'$

$A \in \mathcal{A}$. Since $p^A_D \in R_s$ for each terminal s in C^D_ρ, the flow on xy cannot connect s (by the geodesic condition), and goes out C^D_ρ through another edge $x'y'$ with $y' \in C^D_\rho \not\ni x'$ and $\rho(x) = p^{A'}_D$.

Delete all terminals in C^D_ρ and edges connecting them; at this moment, inner nodes in C^D_ρ may have an odd degree. Next replace $\rho(x)$ by p^O_D for all $x \in C^D_\rho$. This change keeps the saturation condition and the geodesic condition (for any optimal multiflow in the original graph). Thus the resulting ρ is also optimal, $(G;\rho)$ is restricted Eulerian, and C^D_ρ is empty. Apply this procedure until $C^D_\rho = \emptyset$ for all $D \in \mathcal{D}$. Next, according to claim (B), apply the forward SPUP to make M_ρ empty.

2. Second, by using terminal creation I, we decompose the current primal-dual pair of MFP and DLP into two primal-dual pairs; one admitting a half-integral optimal multiflow, and the other realized by a subcomplex $(\mathcal{K}^e_H)'$ of $\mathcal{K}^e_H$; see Figure 35.

Consider an edge $e = xy$ connecting $y \in C_\rho$ and an inner node $x \notin C_\rho$. Then $\rho(y) = p^O$ or q^A for some $A \in \mathcal{A}$. Consider the gate g of $\rho(y)$ at $(\mathcal{K}^e_H)_{\rho(y)}$, and, by edge-subdivision with (4.1), we may assume that $\rho(x) = g$. Since M_ρ is empty, $(\rho(x), \rho(y))$ is a nonadjacent (by leg) pair of some folder. Hence we can apply the terminal creation I at $(\rho(x), \rho(y))$. Apply this procedure for all such edges. Then G is separated into two disjoint graphs G_0 and G_1, with terminal sets S_0 and S_1, respectively. Here G_1 consists of edges joining an inner node in C^O_ρ or in C^A_ρ for some $A \in \mathcal{A}$. G_0 consists of the other edges. Recall that each terminal has degree one. So we can consider the multiflow problem for G_0 and for G_1 separately. All inner nodes of G_0 belong to S_ρ. Each (new) terminal s in C_ρ is incident to a single node x with $\rho(x) \neq \rho(s)$. Consider $2G_0$ and apply the degree-1 reduction (Section 3.3) to terminals; this does not produce inner nodes in C_ρ. All inner nodes are splittable by claim (A), and $2G_0$ has an integral optimal multiflow. Hence G_0 has a half-integral optimal multiflow. Therefore if G_1 has a $1/k$-integral optimal multiflow, then the original graph has a $1/k$-integral optimal multiflow if k is even and a $1/(2k)$-integral optimal multiflow if k is odd.

Here terminal region R_s for $s \in S_1$ is p^O_D, p^A_D, or p^A. Therefore, as in Figure 35, we can delete all cells containing p^D from $\mathcal{K}^e_H$ for all $D \in \mathcal{D}$. Then the resulting F-complex $(\mathcal{K}^e_H)'$ together with $\{R_s\}_{s \in S_1}$ is still a realization of μ restricted to S_1. Therefore we may consider DLP on $(\mathcal{K}^e_H)'$. In $(\mathcal{K}^e_H)'$, legs p^Aq^A and q^Ar^A belong to distinct orbits, and hence q^A_D is now sparse. So include $\rho^{-1}(q^A_D)$ in S_ρ; claim (A) holds. Moreover q^A has the same local orbit structure as that of the midpoint of legs in $\mathcal{K}^2$. We can apply claim (B) to sweep out inner nodes from C^A_ρ to S_ρ, while keeping restricted Eulerian condition (for new S_ρ).

3. Finally, we decompose the current MFP for G_1 into two MFPs, one admitting a half-integral optimal multiflow, and the other being μ_H^s-MFP. This completes the reduction from μ_H^e-MFP to μ_H^s-MFP.

Take an edge $e = xy$ with $x \notin C_\rho^O$ and $y \in C_\rho^O$. Then $\rho(x)$ is p_D^O, q_D^A, p_D^A, or p^A for some A, D. For the first three cases, we can apply the terminal creation I as above. If $\rho(x) = p^A$, then x is necessarily a terminal s with $R_s = \{p^A\}$. Replace R_s by $\{r^A\}$ and $\rho(s)$ by r^A, and, accordingly modify μ; this does not change the problem. Again the graph G_1 is separated into two disjoint graphs G_1' and G_1''. Here G_1' consists of edges joining an inner node having a potential p^O, and G_1'' consists of the other edges. By the same argument as above, G_1'' has a half-integral optimal multiflow. As above, we can consider MFP/DLP for G_1' by deleting all cells except $(\mathcal{K}_H^e)_{p^O}$ from $(\mathcal{K}_H^e)'$. Then the MFP for G_1' is nothing but μ_H^s-MFP, and has a $1/k$-integral optimal multiflow by the assumption. Thus we obtain an optimal multiflow in the original graph, which is $1/k$-integral if k is even and $1/2k$-integral if k is odd. The proof of Theorem 7.5 is completed. □

It is worth noting that this reduction can be done in strongly polynomial time. If μ_H^s-MFP has an integral optimal multiflow f (for G_1'), then f is obtained in strongly polynomial time (see Section 5.4). Thus the half-integral optimal multiflow in the original problem can also be found in strongly polynomial time. This implies Theorem 7.11.

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References

[1] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer-Verlag, Berlin, 1999.

[2] V. Chepoi, Graphs of some CAT(0) complexes, *Advances in Applied Mathematics* **24** (2000), 125–179.

[3] B. V. Cherkasski, A solution of a problem of multicommodity flows in a network, *Ekonomika i Matematicheskie Metody* **13** (1977), 143–151 (in Russian).

[4] A. W. M. Dress, Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: a note on combinatorial properties of metric spaces, *Advances in Mathematics* **53** (1984), 321–402.

[5] A. W. M. Dress and R. Scharlau, Gated sets in metric spaces, *Aequationes Mathematicae* **34** (1987), 112–120.

[6] L. R. Ford, Jr. and D. R. Fulkerson, *Flow in Networks*, Princeton University Press, Princeton, 1962.

[7] A. Frank, Packing paths, circuits, and cuts —– a survey, in: *Paths, Flows, and VLSI-Layout* (B. Korte, L. Lovász, H. J. Prömel, A. Schrijver, eds.), Springer, Berlin, 1990, pp. 47–100.

[8] A. Frank, A. V. Karzanov, and A. Sebö, On integer multiflow maximization, *SIAM Journal on Discrete Mathematics* **10** (1997), 158–170.

[9] H. Hirai, Characterization of the distance between subtrees of a tree by the associated tight span, *Annals of Combinatorics* **10** (2006), 111–128.

[10] H. Hirai, Tight spans of distances and the dual fractionality of undirected multiflow problems, *Journal of Combinatorial Theory, Series B* **99** (2009), 843–868.

[11] H. Hirai, Bounded fractionality of the multiflow feasibility problem for demand graph $K_3 + K_3$ and related maximization problems, RIMS-preprint 1645, (2008).

[12] H. Hirai, Folder complexes and multiflow combinatorial dualities, *SIAM Journal on Discrete Mathematics*, **25** (2011), 1119–1143.

[13] T. C. Hu, Multi-commodity network flows, *Operations Research* **11** (1963), 344–360.

[14] J. R. Isbell, Six theorems about injective metric spaces, *Commentarii Mathematici Helvetici* **39** (1964), 65–76.

[15] A. V. Karzanov, On a class of maximum multicommodity flow problems with integer optimal solutions, in: *Selected Topics in Discrete Mathematics* (A. K. Kelmans, ed.), American Mathematical Society Translations Series 2, Volume 158, American Mathematical Society, Providence, 1994, pp. 81–99.

[16] A. V. Karzanov, Polyhedra related to undirected multicommodity flows, *Linear Algebra and its Applications* **114/115** (1989), 293–328.

[17] A. V. Karzanov, Undirected multiflow problems and related topics – some recent developments and results, in: *Proceedings of the International Congress of Mathematician, Volume II*, Kyoto, Japan 1991, 1561–1571.

[18] A. V. Karzanov, Minimum 0-extensions of graph metrics, *European Journal of Combinatorics* **19** (1998), 71–101.

[19] A. V. Karzanov, Metrics with finite sets of primitive extensions, *Annals of Combinatorics* **2** (1998), 211–241.

[20] A. V. Karzanov, On one maximum multiflow problem and related metrics, *Discrete Mathematics* **192** (1998) 187–204.

[21] A. V. Karzanov, A combinatorial algorithm for the minimum $(2, r)$-metric problem and some generalizations, *Combinatorica* **18** (1998) 549–568.

[22] A. V. Karzanov and M. V. Lomonosov, Systems of flows in undirected networks, in: *Mathematical Programming. Problems of Social and Economical Systems. Operations Research Models. Work collection. Issue 1* (O.I. Larichev, ed.), Institute for System Studies, Moscow, 1978, 59–66 (in Russian).

[23] A. V. Karzanov and Y. G. Manoussakis, Minimum $(2, r)$-metrics and integer multiflows. *European Journal of Combinatorics*, **17** (1996), 223–232.

[24] M. V. Lomonosov, Combinatorial approaches to multiflow problems, *Discrete Applied Mathematics* **11** (1985), 93 pp.

[25] M. V. Lomonosov, On return path packing, *European Journal of Combinatorics* **25** (2004), 35–53.

[26] L. Lovász, On some connectivity properties of Eulerian graphs, *Acta Mathematica Academiae Scientiarum Hungaricae* **28** (1976), 129–138.

[27] A. Schrijver, *Combinatorial Optimization—Polyhedra and Efficiency*, Springer-Verlag, Berlin, 2003.

[28] É. Tardos, A strongly polynomial algorithm to solve combinatorial linear programs, *Operations Research* **34** (1986), 250–256.