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By

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Abstract

We consider the weighted maximum multiflow problem with respect to terminal weight μ. We show that if the dimension of the *tight span* associated with μ is at most 2, then there exists a 1/12-integral optimal multiflow in the μ-weighted maximum multiflow problem for every Eulerian supply graph. This result solves a weighted generalization of Karzanov's conjecture for classifying commodity graphs H with *finite fractionality*. Also we prove the existence of an integral or half-integral optimal multiflow for a larger class of Eulerian multiflow maximization problems including previously known classes, and give a strongly polynomial time algorithm to find it.

1 Introduction

Let G be an undirected graph with nonnegative edge capacity $c : EG \to \mathbf{R}_+$. Let $S \subseteq VG$ be a set of terminals. Let H be a simple undirected graph on S, called a *commodity graph*. A *multiflow* (*multicommodity flow*) f is a pair $(\mathcal{P}, \lambda)$ of a set $\mathcal{P}$ of paths connecting the ends of some edge of H and its nonnegative flow-value function $\lambda : \mathcal{P} \to \mathbf{R}_+$ satisfying capacity constraint $\sum_{P \in \mathcal{P}: e \in P} \lambda(P) \leq c(e)$ for $e \in EG$. The total flow-value $\|f\|$ is defined to be $\sum_{P \in \mathcal{P}} \lambda(P)$. The *maximum multiflow problem* with respect to $(G, c; H)$ is formulated as:

(1.1) Maximize $\|f\|$ over all multiflows f for $(G, c; H)$.

Suppose that H consists of one edge, i.e., $H = K_2$. Then problem (1.1) is the maximum flow problem. The max-flow min-cut theorem, due to Ford-Fulkerson [7], says that if c is integral, then there exists an integral maximum flow. Suppose that H consists of two vertex-disjoint edges, i.e., $H = K_2 + K_2$. Then problem (1.1) is the maximum 2-commodity flow problem. Hu [14] showed that there exists a *half-integral* maximum flow whenever c is integral. However, in 3-commodity flow problem, an analogous theorem does not hold. It is known that there is no positive integer k such that all integer-capacitated 3-commodity flow problems have a $1/k$-integral maximum flow. On the other hand, suppose that H is the complete graph K_n ($n = \#S$). Then Lovász [27] and Cherrkasky [3] independently showed that there exists a half-integer maximum flow whenever c is integral.

So these integrality or half-integrality phenomena crucially depend on the structure of commodity graphs. Motivated by these facts, Karzanov [17] defined the *fractionality* frac(H) of a commodity graph H by the least positive integer k with the property that there exists a $1/k$-integral maximum flow in problem (1.1) for every integer-capacitated graph (G, c) having H as a commodity graph. If such a positive integer k does not exist, then frac(H) is defined to be $+\infty$. The above-described examples show $\mathrm{frac}(K_2) = 1$, $\mathrm{frac}(K_2 + K_2) = \mathrm{frac}(K_n) = 2$, and $\mathrm{frac}(K_2 + K_2 + K_2) = +\infty$.

Karzanov raised the following fundamental problem:

(1.2) *Classify the commodity graphs having finite fractionality.*

It is rather hard to determine the exact value of the fractionality for more complicated commodity graphs. The linear programming dual to (1.1) gives a lower bound of the fractionality. The *dual fractionality* $\mathrm{frac}^*(H)$ is defined to be the least positive integer k with the property that there exists a $1/k$-integral optimum in the LP-dual to (1.1) for every capacitated graph (G, c) having H as a commodity graph. Then the standard TDI argument implies $\mathrm{frac}(H) \geq \mathrm{frac}^*(H)$ [17]. Therefore the finiteness of the dual fractionality is a necessary condition for the finiteness of the (primal) fractionality. Karzanov [17] gave a necessary and sufficient condition for the finiteness of the dual fractionality, and determined its possible values as follows. A commodity graph H has *property P* if it satisfies the following condition:

(P) For any triple A, B, C of pairwise intersecting maximal stable sets of H, we have $A \cap B = B \cap C = C \cap A$.

Theorem 1.1 ([17]). *For a commodity graph H, we have the following:*

(1) *If H has property P, then* $\mathrm{frac}^*(H) \in \{1, 2, 4\}$.

(2) *If H does not have property P, then* $\mathrm{frac}^*(H) = +\infty$ *and thus* $\mathrm{frac}(H) = +\infty$.

See also [28, Section 73.3b]. Karzanov conjectured that property P is also *sufficient* for the finiteness of primal fractionality, and, more strongly, that the possible values are also $1, 2, 4, +\infty$, as follows.

Conjecture 1.2 ([18]). Suppose that a commodity graph H has property P. Then the following holds:

(1) $\mathrm{frac}(H) < +\infty$.

(2) $\mathrm{frac}(H) \in \{1, 2, 4\}$.

Recently, Theorem 1.1 and Conjecture 1.2 were extended to a more general setting of *weighted* maximum multiflow problems. Instead of a commodity graph, we are given a nonnegative terminal weight $\mu : \binom{S}{2} \to \mathbf{R}_+$. Here a multiflow f is a pair $(\mathcal{P}, \lambda)$ of a set $\mathcal{P}$ of paths connecting distinct terminals in S and its nonnegative flow-value function $\lambda : \mathcal{P} \to \mathbf{R}_+$ satisfying the capacity constraint. The total flow-value $\|f\|_\mu$ is defined to be $\sum_{P \in \mathcal{P}} \mu(s_P, t_P)\lambda(P)$, where s_P and t_P denote the ends of P. The *μ-weighted maximum multiflow problem* (the *μ-problem* for short) is formulated as:

(1.3) Maximize $\|f\|_\mu$ over all multiflows f for $(G, c; S)$.

If μ is 0-1 valued, then the μ-problem (1.3) coincides with (1.1) for a commodity graph H_μ specified by $st \in EH \Leftrightarrow \mu(s, t) = 1$. Similarly, the *fractionality* frac(μ) of terminal weight μ is the least positive integer k with the property that the μ-problem (1.3) has a $1/k$-integral optimal multiflow for every integer-capacitated graph, and the *dual fractionality*

$\mathrm{frac}^*(\mu)$ is the least positive integer k with the property that the LP-dual to (1.3) has a $1/k$-integral optimal solution for every capacitated graph. Again $\mathrm{frac}(\mu) \geq \mathrm{frac}^*(\mu)$ holds for an *integral* weight $\mu : \binom{S}{2} \to \mathbf{Z}_+$.

Karzanov [20] extended Theorem 1.1 for integral metric-weights, and our previous paper [11] further extended it for general integral weights. For a terminal weight $\mu : \binom{S}{2} \to \mathbf{R}_+$, a polyhedron P_μ and its subset T_μ in $\mathbf{R}^S_+$ are defined by

$$\begin{aligned} P_\mu &= \{p \in \mathbf{R}^S_+ \mid p(s) + p(t) \geq \mu(s,t) \ (s,t \in S)\}, \\ T_\mu &= \text{the set of minimal elements of } P_\mu. \end{aligned}$$

This polyhedral set T_μ is called the *tight span* or the *injective envelope*, introduced independently by Isbell [15] and Dress [4]; their motivations were completely irrelevant to the multiflow theory. The dimension $\dim T_\mu$ is defined to be the largest dimension of faces of T_μ.

Theorem 1.3 ([20] for metrics and [11] for general). *For an integral weight* $\mu : \binom{S}{2} \to \mathbf{Z}_+$*, we have the following:*

(1) *If* $\dim T_\mu \leq 2$*, then* $\mathrm{frac}^*(\mu) \in \{1, 2, 4\}$.

(2) *If* $\dim T_\mu \geq 3$*, then* $\mathrm{frac}^*(\mu) = \mathrm{frac}(\mu) = +\infty$.

In particular, the property P of H is equivalent to the 2-dimensionality of the tight span of the corresponding 0-1 weight μ_H, as remarked in [11, Section 7]. So Conjecture 1.2 is naturally generalized into the following:

Conjecture 1.4. Suppose that a terminal weight μ satisfies $\dim T_\mu \leq 2$. Then the following holds:

(1) $\mathrm{frac}(\mu) < +\infty$.

(2) $\mathrm{frac}(\mu) \in \{1, 2, 4\}$.

The main result of this paper affirmatively solves the weaker statement (1) of this generalized conjecture and thus (1) of Conjecture 1.2.

Theorem 1.5. *For a terminal weight* μ *on* S*, if* $\dim T_\mu \leq 2$*, then the* μ*-problem* (1.3) *has a* $1/12$*-integral optimal multiflow for every Eulerian graph.*

This completes the classification of terminal weights and commodity graphs having finite fractionality.

Corollary 1.6. *A terminal weight* μ *has finite fractionality if and only if* $\dim T_\mu \leq 2$*, and a commodity graph* H *has finite fractionality if and only if* H *has property P.*

The possible values of the fractionality are $1, 2, 3, 4, 6, 8, 12, 24$, and $+\infty$. However we do not know any example of terminal weights having fractionality except $1, 2, 4, +\infty$.

Organization. The goal is the proof of 1/12-integrality theorem (Theorem 1.5). We use the duality framework using *folder complexes* (*F-complexes* for short), developed by the previous paper [13]. An F-complex is a CAT(0) polygonal complex obtained by gluing Euclidean right triangles, introduced and studied by Chepoi [2]. If $\dim T_\mu \leq 2$, then μ is *embedded into* some F-complex $\mathcal{K}$, and the maximum value of the μ-problem is equal to the minimum value of a *discrete location problem* on $\mathcal{K}$. In Section 2, we introduce

F-complexes and related concepts, and summarize basic properties for F-complexes and the multiflow duality.

Our proof is based on a fractional version of the splitting-off method combining the dual update, called *SPUP*, which is a framework for proving the existence of a $1/k$-integral optimal multiflow for a bounded integer k. This framework was originally devised by the previous paper [12] for a special case. In Section 3, we develop the SPUP framework for a general setting and give several key lemmas for it. Based on these arguments, we prove the main theorem in Section 4.

Our framework not only brings a unified understanding to previously known results but also provides a powerful algorithmic tool for proving the existence of an integral or half-integral optimal multiflow. We describe them in Sections 5, 6 and 7. In Section 5, we show that if weight μ is embedded into a sufficiently nice (*nonsingular*) F-complex, then Eulerian μ-problem always has an integral optimal multiflow. Furthermore we can sometimes *blow up a singular F-complex into a nonsingular one* to prove the integrality theorem. This idea provides a powerful method for proving the integrality theorem. We give an illustrative application to *multiterminal weighted 2-commodity flows*. In Section 6, we focus on maximum multiflow problem (1.1) for a commodity graph H with property P. We give a geometric interpretation of the *anticlique-bipartite* condition for H in terms of the nonsingularity of an F-complex associated with H (Theorem 6.2). As a corollary, we obtain Karzanov-Lomonosov integrality theorem [16, 23, 25]. Furthermore we prove a powerful *fractionality relation* $\mathrm{frac}(H) \leq 2\,\mathrm{frac}(\mu^s_H)$ for a *smoothed metric* μ^s_H associated with H (Theorem 6.4). This reduces the fractionality study of (1.1) to that of a relatively simpler metric-weighted maximum multiflow problem. By using this we prove the stronger conjecture (Conjecture 1.2 (2)) for a larger class of commodity graphs beyond previously known such classes [22, 26]; see Theorems 6.6 and 6.8. Our proof is constructive, and gives, in some cases, a strongly polynomial time algorithm to find a $1/k$-integral optimal multiflow for $k \leq 12$. The final Section 7 summarizes algorithmic consequences.

Notation. Let $\mathbf{R}$, $\mathbf{R}_+$, $\mathbf{Z}$, and $\mathbf{Z}_+$ denote the sets of reals, nonnegative reals, integers, and nonnegative integers, respectively. For a set V, let $\mathbf{R}^V$ and $\mathbf{R}^V_+$ denote the sets of functions from V to $\mathbf{R}$ and V to $\mathbf{R}_+$, respectively. For a subset S of V, let χ_S denote the characteristic vector of S defined as $\chi_S(x) = 1$ if $x \in S$ and $\chi_S(x) = 0$ otherwise. The characteristic vector $\chi_{\{x\}}$ for a singleton $\{x\}$ is simply denoted by χ_x.

By a graph we mean finite (capacitated) undirected graph possibly having multiple edges and loops. For a graph G, the vertex set and the edge set are denoted by VG and EG, respectively. An edge joining x and y is denoted by xy. In this paper there are two types of graphs G and Γ; G represents a supply graph for multiflows and Γ represents a space of potentials. To distinguish their roles, we particularly called a vertex of G a *node*. A node except terminals is called an *inner node*. By a path we mean a simple path. For node sets $A_1, A_2, \ldots, A_m$, a path P is said to be an $(A_1, A_2, \ldots, A_m)$*-path* if P connects A_1 and A_m passing through $A_1, A_2, \ldots, A_m$ in order. If some A_i is a singleton $\{x\}$, then we simply denote it by x. An (A, A)-path is particularly called an A*-path*.

We consider (1.3) only for *rational*-valued edge-capacity. Then we can always take an optimal multiflow $f = (\mathcal{P}, \lambda)$ for a *rational-valued* flow-value function λ. Therefore, by allowing $\mathcal{P}$ to be a *multiset*, we can represent $f = (\mathcal{P}, \lambda)$ by a pair of a multiset $\mathcal{P}$ of S-paths and a *uniform* flow-value function $\lambda = 1/\kappa$ for some positive integer κ. We shall adopt this expression, denoted by $f = (\mathcal{P}; \kappa)$. For an edge e, the subset of paths in $\mathcal{P}$ containing e is denoted by $\mathcal{P}^e$. Its total flow-value $|\mathcal{P}^e|/\kappa$ is denoted by f^e. For consecutive two edges e, e', the subset of paths passing e and e' is denoted by $\mathcal{P}^{e,e'}$, and its flow value is denoted by $f^{e,e'}$. If a path P in $\mathcal{P}^{xy}$ is an (s,x,y,t)-path, then terminal s is called the *x-end* of P (with respect to xy). The *total support* of multiflow f is defined to be $\sum_{e \in EG} f^e$.

A function d on $S \times S$ is called a *distance* if $d(s,t) = d(t,s) \geq d(s,s) = 0$ for $s,t \in S$. A distance d is call a *metric* if it satisfies the triangle inequalities $d(s,t)+d(t,u) \geq d(s,u)$ for $s,t,u \in S$. We shall regard a terminal weight μ as a distance. We often regard a metric d on vertex set VG of graph G as an edge-length $d : EG \to \mathbf{R}_+$ by $d(e) := d(x,y)$ for $e = xy$. For a path or cycle P, $d(P)$ denotes the sum of $d(e)$ along all edges e in P. For a metric d on S and two subsets $A, B \subseteq S$, the distance $d(A,B)$ between A and B is defined as

$$d(A,B) = \min\{d(s,t) \mid (s,t) \in A \times B\}.$$

We denote $d(A,\{p\})$ simply by $d(A,p)$. For a graph Γ with uniform edge-length δ, the shortest path metric on $V\Gamma$ is denoted by $d_{\Gamma,\delta}$.

A *piecewise Euclidean cell complex* $\mathcal{K}$ is a space formed by gluing together Euclidean convex polyhedra via isometries of their faces, together with the subdivision of $\mathcal{K}$ into cells; see [1, Chapter I.7] for a precise definition. A 1-dimensional cell of segment $[p,q]$ is also called an *edge*, denoted by pq. A 0-dimensional cell is called a *vertex*.

2 Multiflow combinatorial dualities

In this section we introduce a duality framework for μ-weighted maximum multiflow problems (1.3) by folder complexes. In the following paragraph we introduce the notation of an F-complex and related concepts, and then we describe a combinatorial duality relation for (1.3). Section 2.1 summarizes basic properties and operations of F-complexes. Section 2.2 summarizes basic properties of μ-problem and its dual, including optimality criteria, the locking property, reductions of degrees, and so on.

Folder complexes. We consider a finite 2-dimensional piecewise Euclidean cell complex $\mathcal{K}$ by obtained by gluing squares and isosceles right triangles along edges of the same isometry type. More precisely, for some positive real δ, each 2-dimensional cell (2-cell) is isometric to

$$\begin{array}{rl} \textit{square} & \{(x_1,x_2) \in \mathbf{R}^2 \mid 0 \leq x_1 \leq \delta,\ 0 \leq x_2 \leq \delta\} \text{ or} \\ \textit{triangle} & \{(x_1,x_2) \in \mathbf{R}^2 \mid 0 \leq x_1 \leq x_2 \leq \delta\} \end{array}$$

in the Euclidean plane. Also we suppose that each maximal 1-dimensional cell is isometric to segment $[0,\delta]$. A longer edge of a triangle is called a *hypotenuse*, and other edges are called *legs*; a maximal 1-cell is a leg. δ is called the *leg-length*. Therefore if two 2-cells share a common edge e, then e is either the hypotenuse of both of them or a leg of both of them. A *folder* of $\mathcal{K}$ is a square, or the union of all triangles sharing one common hypotenuse; see Figure 1 (a). Then $\mathcal{K}$ is called a *folder complex* (an *F-complex* for short) [2, Section 7] if

(0) it is simply-connected,

(1) the intersection of any two folders does not contain incident legs, and

(2) there are no three folders F_i $(i = 1,2,3)$ and three distinct legs e_i $(i = 1,2,3)$ sharing a common vertex such that e_i belongs to F_j exactly when $i \neq j$.

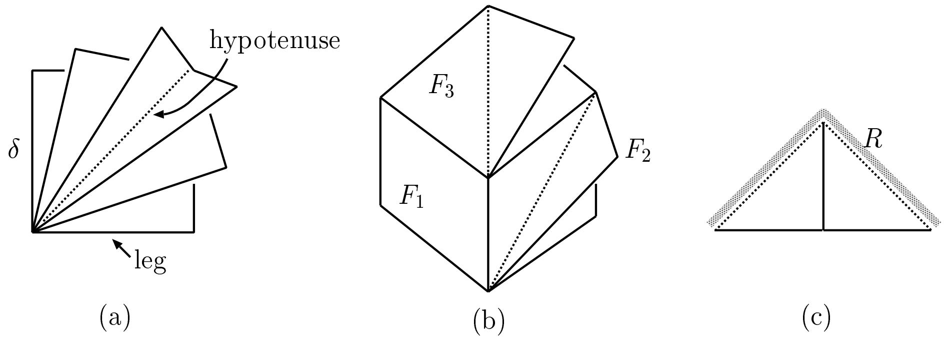


Figure 1: Basic concepts

See Figure 1 (b) for a violation of (2). This condition says that $\mathcal{K}$ is a CAT(0) space [2]; also see [1].

A subset R of $\mathcal{K}$ is called *normal* if it satisfies the following axiom:

(1) R is a connected subcomplex of $\mathcal{K}$ with the property that if R contains a leg e, then every cell containing e belongs to R.

(2) there are no two triangles σ and σ' sharing a leg and a right angle such that $(\sigma \cup \sigma') \cap R$ coincides with the union of the hypotenuses of σ and σ'.

See Figure 1 (c) for the violation of (2).

Although $\mathcal{K}$ has the l_2-length metric by definition, we are mainly interested in the l_1-length metric. Note that each 2-cell has a natural l_1-metric so that the coordinate axes are parallel to legs. Then the l_1-length of a path P in $\mathcal{K}$ is the sum, over all cells σ, of l_1-length of $\sigma^\circ \cap P$ measured by the l_1-metric on σ, where σ° denotes the relative interior of σ. The l_1*-length metric* $d_{\mathcal{K}}(p,q)$ between p and q is defined to be the infimum of the lengths of all paths connecting p, q in $\mathcal{K}$.

Let $\Gamma = \Gamma^{\mathcal{K}}$ be the (simple undirect) graph consisting of all legs of $\mathcal{K}$, called the *leg-graph*. Equivalently, Γ is the graph obtained by deleting all hypotenuses from the 1-skeleton graph of $\mathcal{K}$. The edge-length of Γ is given by δ uniformly. For two normal sets N and M, one can easily see

$$d_{\mathcal{K}}(N, M) = d_{\Gamma,\delta}(N \cap V\Gamma, M \cap V\Gamma). \tag{2.1}$$

An F-complex $\mathcal{K}$ is said to be *orientable* if its leg-graph Γ has an orientation the property that

(2.2) (i) in each square its diagonal edges have same direction (in the local coordinate), and

(ii) in each folder consisting of triangles, the hypotenuse joins a source and a sink.

See Figure 4 for a portion of an orientation.

F-complex realization and multiflow duality. For a distance μ on set S, an *F-complex realization* of μ is a pair $(\mathcal{K}; \{R_s\}_{s\in S})$ of an F-complex $\mathcal{K}$ and a family $\{R_s\}_{s\in S}$ of normal sets satisfying

$$\mu(s,t) = d_{\mathcal{K}}(R_s, R_t) \quad (s,t \in S).$$

$$
\mu = \begin{array}{|c|cccccc|}
\hline
 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 \\
\hline
s_1 & 1 & 2 & 3 & 3 & 5 & 2 \\
s_2 & & 1 & 4 & 4 & 4 & 1 \\
s_3 & & & 3 & 3 & 5 & 2 \\
s_4 & & & & 2 & 2 & 3 \\
s_5 & & & & & 2 & 3 \\
s_6 & & & & & & 1 \\
\hline
\end{array}
$$

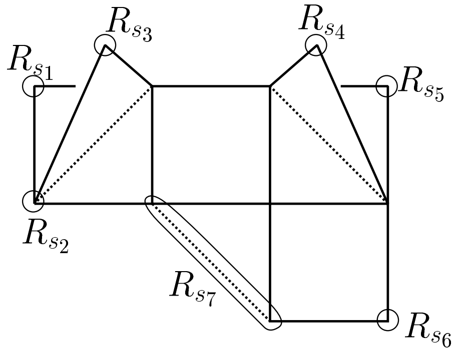


Figure 2: F-complex realization

Namely μ is realized by the distances among normal sets R_s. Figure 2 illustrates an example, where s_7 is embedded into a hypotenuse, and others are embedded into vertices.

Theorem 2.1 ([13, Theorem 4.5]). *For a rational distance μ on a finite set S, the following two conditions are equivalent:*

(1) $\dim T_\mu \leq 2$.

(2) *μ has an F-complex realization.*

It is known that a realization of μ is obtained by subdividing 2-dimensional polyhedral set T_μ into triangles and squares [11, 13].

Suppose that a distance μ on S has an F-complex realization $(\mathcal{K}; \{R_s\}_{s \in S})$. Consider the μ-problem (1.3) for $(G, c; S)$ and consider the following *discrete location problem* on the leg-graph $\mathit{\Gamma}$ of $\mathcal{K}$:

$$
\begin{array}{lll}
(2.3) \qquad & \text{Minimize} & \displaystyle\sum_{xy \in EG} c(xy) d_{\mathit{\Gamma},\delta}(\rho(x), \rho(y)) \\
 & \text{subject to} & \rho : VG \to V\mathit{\Gamma}, \\
 & & \rho(s) \in R_s \cap V\mathit{\Gamma} \quad (s \in S).
\end{array}
$$

Our previous paper established the following duality relation, extending a result in [19].

Theorem 2.2 ([13, Theorem 2.1]). *Suppose that $\mathcal{K}$ is orientable. Then the maximum value of the μ-problem* (1.3) *is equal to the minimum value of the discrete location problem* (2.3).

By multiplying a positive rational to μ, the leg-length δ can be taken to be 1. Also, as will be seen in Section 2.1.1, we can always replace a nonorientable realization by orientable one. So, in the sequel,

(1) *by an F-complex we mean an orientable one.*

(2) *the leg-length δ is supposed to be* 1.

2.1 Geometry of F-complex

Here we summarize basic facts on an F-complex $\mathcal{K}$. We use the same d for $d_{\mathit{\Gamma},\delta}$ and $d_{\mathcal{K}}$ (thanks to (2.1)). We note the following bipartite properties:

(2.4) (1) the leg-graph Γ is bipartite.

(2) for a normal set R and a leg pq, if $pq \not\subseteq R$, then $d(R,p) - d(R,q) \in \{\pm 1\}$.

See [13] for proof.

2.1.1 Subdivision

An F-complex $\mathcal{K}$ has a natural subdivision operation. For a positive integer m, subdivide each square into $m \times m$ squares of leg-length δ/m, each triangle into m triangles and $m(m-1)/2$ squares of leg-length δ/m, and each maximal edge into m edges of length δ/m; see [13, Figure 5]. The resulting complex is denoted by $\mathcal{K}^m$, called the *m-subdivision* of $\mathcal{K}$. One can easily see the following facts:

(2.5) (1) $\mathcal{K}^m$ is an F-complex.

(2) $\mathcal{K}^2$ is orientable.

See also Figure 14 for verifying (2). Therefore, if μ has a realization by a nonorientable F-complex $\mathcal{K}$, then we can replace $\mathcal{K}$ by orientable one $\mathcal{K}^2$.

2.1.2 Star-shaped F-complex

An F-complex $\mathcal{K}$ is said to be *star-shaped* if there is a vertex p, called a *center*, such that every maximal cell contains p and no triangle has p as its right angle. A star-shaped F-complex is obviously orientable; it is oriented so that the center p is a unique source, and each vertex nonincident to p is a sink.

For a star-shaped F-complex $\mathcal{K}$ with center p, its *boundary leg-graph* Π is the bipartite graph $\Gamma \setminus p$ obtained by deleting p from the leg-graph Γ. Then the axiom (1-2) of F-complex can be rephrased by the following:

(2.6) Π has girth at least 8.

The bipartition of Π consists of the set Q of vertices incident to p and the set L of vertices nonincident to p. For $u, v \in Q \cup L$, we write $u \sim v$ if $u = v$, u is incident to v, or u and v have a common neighbor. Particularly we write $u \sim_1 v$ if $u = v$ or u is incident to v. We also denote Π by $(Q, L; \sim)$.

Remark 2.3. A bipartite graph $\Pi = (Q, L; \sim)$ of girth at least 8 has an analogy of an *incidence geometry* of points Q and lines L. In particular, a bipartite graph with girth 8 and *diameter* 4 is known as a *generalized quadrangle*; see, e.g., [6, Chapter 5].

2.1.3 Local structure and shortest path

The following problem naturally arises in the local multiflow rearrangement in Section 2.2.3:

(2.7) Let p be a vertex in $\mathcal{K}$, and M, N normal sets. Suppose that we are given two shortest paths P and P' such that P connects p and M, and P' connects p and N. *Is the concatenation $P + P'$ at p shortest between M and N ?*

We answer this problem in term of the position of (M, N) relative to a *neighborhood* around p.

Let $\mathcal{K}_p$ denote the subcomplex of $\mathcal{K}$ consisting of cells containing p and their faces. Namely $\mathcal{K}_p$ is the *closed star* at p. Clearly $\mathcal{K}_p$ is an F-complex. Although $\mathcal{K}_p$ may not be star-shaped, $(\mathcal{K}^m)_p$ for $m \geq 2$ is always star-shaped. Let $\mathit{\Pi}_p = (Q_p, L_p; \sim)$ denote the boundary leg-graph of $(\mathcal{K}^m)_p$ for $m \geq 2$ (well-defined).

Suppose that $\mathcal{K}_p$ is star-shaped (by subdivision). We identify the boundary leg-graph of $\mathcal{K}_p$ with $\mathit{\Pi}_p = (Q_p, L_p; \sim)$. For a normal set R not containing p, the vertex $g \in Q_p \cup L_p$ in $\mathcal{K}_p$ with $d(g, R) = d(\mathcal{K}_p, R)$ is uniquely determined [13, Lemma 3.8]. We call this vertex g the *gate* of R in $\mathcal{K}_p$, denoted by gR. Then the solution of problem (2.7) depends only on gates of M and N. Namely $P + P'$ is shortest between M and N if and only if $g(P + P')$ is shortest between gM and gN, as follows.

Lemma 2.4 ([13, Lemmas 3.6 and 3.9])**.** *Let M and N be two normal sets.*

(1) *If $p \in M$ and $p \notin N$, then the following conditions are equivalent:*

  (a) *$d(M, N) = d(p, N)$.*

  (b) *$gN \notin M$.*

(2) *If $p \notin M$ and $p \notin N$, then the following conditions are equivalent:*

  (a) *$d(M, N) = d(M, p) + d(p, N)$.*

  (b) *$d(gM, gN) = d(gM, p) + d(p, gN)$.*

  (c) *There is no $q \in Q_p$ with $gM \sim_1 q \sim_1 gN$.*

From (2.10) (1), the uniqueness of the gate, and Lemma 2.4, we see

$$d(R, u) = d(R, gR) + d(gR, u) \quad (u \in \{p\} \cup Q_p \cup L_p). \tag{2.8}$$

This roughly means that a shortest path from R to u can enter $\mathcal{K}_p$ through the gate gR; our definition of the gate is compatible to that in [5]. The next lemma describes the gate of R when a shortest path from R to p enters $\mathcal{K}_p$ via $u \neq p$.

Lemma 2.5. *For a normal set R and a vertex $u \in Q_p \cup L_p$, suppose $d(R, p) = d(R, u) + d(u, p)$.*

(1) *If $u \in L_p$, then $gR = u$.*

(2) *If $u \in Q_p$, then $gR \sim_1 u$.*

Proof. By (2.8), we have $d(R, p) = d(R, gR) + d(gR, u) + d(u, p)$, and $d(gR, u) + d(u, p) = d(gR, p) \in \{1, 2\}$. Hence $d(u, p) = 2$ $(u \in L_p)$ implies $d(gR, u) = 0$, and $d(u, p) = 1$ $(u \in Q_p)$ implies $d(gR, u) = 0$ or 1. $\square$

2.1.4 Orbits and summands

We recall the notion of orbits [19, 20] with a slight modification by [11]. An *orbit* is an equivalence class of the transitive closure of the following relation on edge set $E\mathit{\Gamma}$: $e \simeq e'$ if e and e' are nonincident edges in some square, or belong to a common folder consisting of triangles. For a (disjoint) union U of several orbits, we can construct a new complex $\mathcal{K}^U$ from $\mathcal{K}$ by identifying the ends of each edge not in U. Again $\mathcal{K}^U$ consists of squares and right isosceles triangles. We call $\mathcal{K}^U$ a *summand* of $\mathcal{K}$. See Figure 3. We also denote the leg-graph of $\mathcal{K}^U$ by $\mathit{\Gamma}^U$. This identification naturally induces a map $(\cdot)^U : V\mathit{\Gamma} \to V\mathit{\Gamma}^U$ by defining p^U to be the contracted vertex. By extending linearly, we obtain a map $(\cdot)^U : \mathcal{K} \to \mathcal{K}^U$.

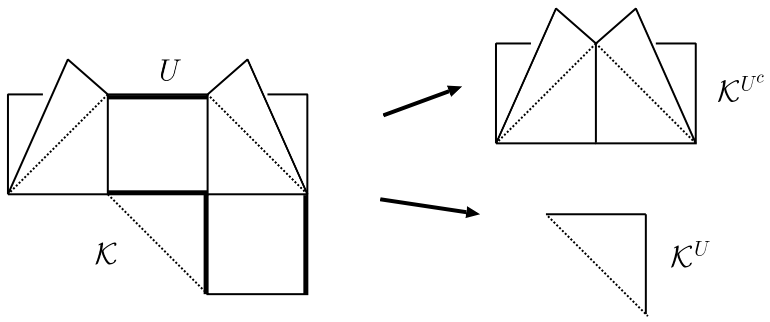


Figure 3: Summands

Proposition 2.6 ([13, Proposition 3.15]). *Let U be the union of any subset of the orbits and U^c its complement.*

(1) *$\mathcal{K}^U$ is an F-complex.*

(2) *For a normal set R in $\mathcal{K}$, R^U is also normal in $\mathcal{K}^U$.*

(3) *For normal sets M, N, we have $d_{\mathcal{K}}(M,N) = d_{\mathcal{K}^U}(M^U, N^U) + d_{\mathcal{K}^{U^c}}(M^{U^c}, N^{U^c})$.*

2.1.5 Frame and metric

In the case where μ is a metric, one can handle its F-complex realization in a purely graph-theoretical way. A graph Γ is called a *frame* if it is bipartite, has no isometric cycles of length at least 6, and is orientable in the sense that Γ can be oriented so that in every 4-cycle, diagonal edges have opposite directions in a cyclic orientation. Karzanov [19, 20] showed that for a rational metric μ, $\dim T_\mu \leq 2$ if and only if there are a frame Γ and a positive integer k such that μ is a submetric of $d_{\Gamma,1/k}$. For a frame Γ, we can naturally associate it with an (orientable) F-complex as follows. Every maximal complete bipartite subgraph of Γ is necessarily $K_{2,m}$ (for varying $m \geq 2$) by orientability. For each maximal complete subgraph $K_{2,m}$, fill m triangles as in Figure 1 (a) if $m \geq 3$, and fill a square if $m = 2$; see [19, Section 4] for a precise construction. Then the resulting complex is an orientable F-complex; conversely the leg-graph of an orientable F-complex is always a frame [2]. Therefore we obtain an F-complex realization with property that each R_s is a single vertex. So, if μ is a metric, then we are sufficient to retain a frame Γ and an isometric embedding $\phi : S \to V\Gamma$. Since the leg-graph loses the information of folders consisting of at most two triangles, it has a difficulty to treat nonmetric problems. This is a motivation for introducing a framework by F-complexes.

Let us rephrase several concepts for frames. A frame is said to be *star-shaped* if the corresponding F-complex is star-shaped. There is a one-to-one correspondence between star-shaped frames Γ and bipartite graphs $\Pi = (Q, L; \sim)$ of girth at least 8 and degree at least 2 in L. Indeed, we have already seen the construction of Π from Γ in Section 2.1.2. From Π, we can construct a star-shaped frame Γ by adding new vertex p and joining it to each vertex in Q. An *orbit* of a frame Γ is an equivalence class of the transitive closure of the relation: $e \simeq e'$ if there is a 4-cycle containing e, e' as nonincident edges. For the union U of several orbits, the *summand* Γ^U is the graph obtained from Γ by contracting all edges not in U, identifying parallel edges, and deleting loops. If U is a single orbit, then Γ^U is particularly called the *orbit graph* with respect to U.

2.2 Weighted maximum multiflow problem and its dual

The aim of this subsection is to describe basic properties of the μ-problem (1.3) and its dual (2.3). Let μ be a distance on S, and let $(\mathcal{K}; \{R_s\}_{s\in S})$ be its (orientable) F-complex realization of unit leg-length. Let (G, c) be a capacitated graph with terminal set S. The optimal value of (1.3) is denoted by $\mathrm{opt}(G, c)$. A map ρ feasible to the dual (2.3) is called a *potential*. For a potential ρ, a metric d^ρ on VG is defined by

$$d^\rho(x, y) := d(\rho(x), \rho(y)) \quad (x, y \in VG).$$

For a potential ρ, the objective value of (2.3) is denoted by $c \cdot d^\rho$.

2.2.1 Optimality criteria

Here we give optimality criteria to the μ-problem (1.3) and its dual (2.3). For a multiflow $f = (\mathcal{P}; \kappa)$ and a potential ρ, the duality gap $c \cdot d^\rho - \|f\|_\mu$ is given by

$$\sum_{e\in EG} d^\rho(e)(c(e) - f^e) + \sum_{P\in\mathcal{P}} (d^\rho(P) - \mu(s_P, t_P))/\kappa. \tag{2.9}$$

For a potential ρ, an S-path P in G is called *ρ-shortest* if $d^\rho(P) = d(R_{s_P}, R_{t_P}) = \mu(s_P, t_P)$. Hence the optimality criterion of primal-dual type is given as follows.

Lemma 2.7. *A multiflow f and a potential ρ are both optimal if and only if each path in f is ρ-shortest, and each edge e with $d^\rho(e) > 0$ is saturated by f, i.e., $f^e = c(e)$.*

This means that paths in f are embedded into $\mathcal{K}$ by ρ and embedded paths are shortest paths connecting terminal regions R_s. This view is very important in every place of this paper. For example, one can immediately see the following properties of an optimal multiflow f and an optimal potential ρ:

(2.10) (1) If f has a path passing through nodes x, y, z in order, then $d^\rho(x, z) = d^\rho(x, y) + d^\rho(y, z)$ holds.

(2) Subdivide an edge xy into xz and zy, and extend ρ by $\rho(z) := p$ for a vertex p satisfying $d^\rho(x, y) = d(\rho(x), p) + d(p, \rho(y))$. Then ρ is optimal to the new graph.

(3) If f has a path of end s passing through node y, then there is no point $p \in R_s$ with $p \neq \rho(s)$ and $d(R_s, \rho(y)) = d(R_s, p) + d(p, \rho(y))$.

Next we describe an optimal criterion involving potential only. We use an orientation of Γ. An orientation of Γ with property (2.2) is said to be *admissible*. An admissible orientation is obtained by orienting orbits independently. Such an orientation of an orbit is also said to be *admissible*. See Figure 4. Each orbit has exactly two admissible orientations; one is the reverse to the other one. Let ρ be a potential. Let $\overrightarrow{O}$ be an oriented orbit by an admissible orientation. A potential ρ' is called a *neighbor* of ρ with respect to $\overrightarrow{O}$ if for each $x \in VG$ with $\rho'(x) \neq \rho(x)$

(2.11) (i) there is an oriented edge $\overrightarrow{pq} \in \overrightarrow{O}$ such that $(\rho(x), \rho'(x)) = (p, q)$, or

(ii) there are two oriented edges $\overrightarrow{pq}, \overrightarrow{qr} \in \overrightarrow{O}$ belonging to a common folder such that $(\rho(x), \rho'(x)) = (p, r)$.

Namely ρ' is obtained by *moving ρ along direction $\overrightarrow{O}$*. The following theorem is a basis for the SPUP framework.

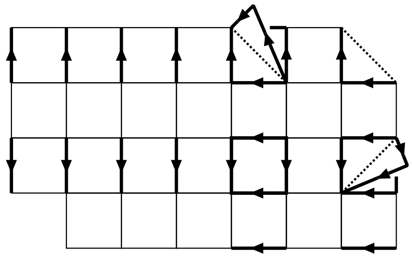


Figure 4: An orbit with an admissible orientation

Theorem 2.8 ([13, Theorem 4.1])**.** *If a potential ρ is not optimal, then there exists a neighbor ρ' of ρ having smaller objective value.*

2.2.2 Orbits and the locking property

In Section 2.1.4 we saw that orbits decompose an F-complex into several smaller F-complexes. Furthermore orbits also decomposes the corresponding μ-problem into several smaller problems. This gives a natural explanation to the *locking phenomenon*, that is, the existence of a multiflow simultaneously optimal to both μ- and μ'-problems for distinct terminal weights μ, μ'; see [19, Section 5] for metric cases.

Let U be the union of any subsets of the orbits in $\mathcal{K}$. Let μ^U be the distance on S defined by

$$\mu^U(s,t) = d_{\mathcal{K}^U}((R_s)^U, (R_t)^U) \quad (s, t \in S).$$

μ^U is also called a *summand* of μ with respect to U. By construction and Proposition 2.6, $(\mathcal{K}^U; \{(R_s)^U\}_{s\in S})$ is an F-complex realization of μ^U.

Proposition 2.9. *Let f be an optimal multiflow and ρ an optimal potential. For the union U of any subset of the orbits, we have the following:*

(1) *f is optimal to the μ^U-problem.*

(2) *ρ^U is optimal to the dual* (2.3) *with respect to realization $(\mathcal{K}^U; \{(R_s)^U\}_{s\in S})$.*

Proof. Let U^c be the complement of U. By Proposition 2.6, we have $\|f\|_\mu = \|f\|_{\mu^U} + \|f\|_{\mu^{U^c}}$ and $d^\rho = d^{\rho^U} + d^{\rho^{U^c}}$. Moreover, ρ^U and ρ^{U^c} are feasible to (2.3) for $(\mathcal{K}^U; \{(R_s)^U\}_{s\in S})$ and for $(\mathcal{K}^{U^c}; \{(R_s)^{U^c}\}_{s\in S})$, respectively. Thus we have $\|f\|_\mu = \|f\|_{\mu^U} + \|f\|_{\mu^{U^c}} \leq c \cdot d^{\rho^U} + c \cdot d^{\rho^{U^c}} = c \cdot d^\rho = \|f\|_\mu$. □

2.2.3 Local multiflow rearrangements

In the analysis of the splitting-off, the local multiflow rearrangement plays crucial roles. Suppose that we are given an optimal multiflow $f = (\mathcal{P}; \kappa)$. Let y be a node. We will face the following rearrangement problem:

Split some of paths in $\mathcal{P}$ at y, and reconnect them with *keeping optimality.*

Suppose further that we are given an optimal potential ρ. Let $p = \rho(y)$. Such a rearrangement can be carried out, according to the local structure of $\mathcal{K}$ at p. Recall the notions in Section 2.1.3. By subdivision, we assume that $\mathcal{K}_p$ is star-shaped, and we identify $\Pi_p = (Q_p, L_p; \sim)$ with the boundary leg-graph of $\mathcal{K}_p$.

Split some of paths in $\mathcal{P}$ at y. Then the resulting paths induce shortest paths between p and terminal regions R_s since f and ρ are optimal. As a consequence of Lemma 2.4,

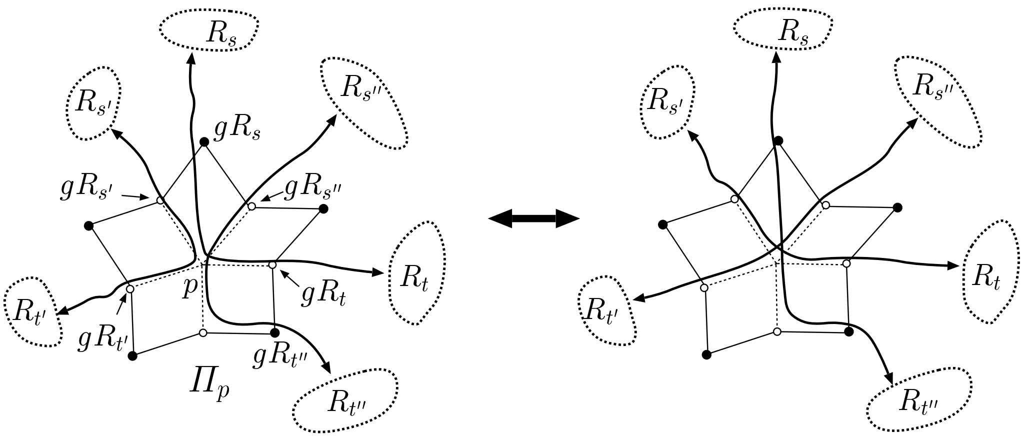


Figure 5: Local multiflow rearrangement to keep optimality

to keep optimality, it suffices to avoid to reconnect a (y, s)-path and a (y, t)-path of the following types:

(1) $p \in R_s$, $p \notin R_t$, and $gR_t \in R_s$.

(2) $p \notin R_s$, $p \notin R_t$, and $gR_s \sim_1 q \sim_1 gR_t$ for some $q \in Q_p$.

Motivated by this fact, for $u \in Q_p \cup L_p$, we define terminal subset $su \subseteq S$ by

$$su = \{s \in S \mid p \notin R_s, u = gR_s\}. \tag{2.12}$$

In the local multiflow rearrangements at $\rho^{-1}(p)$, we can regard su as a *single terminal.* Figure 5 illustrates an intuition of local multiflow rearrangement to keep optimality.

Exchange operation and homogeneity. We will often use the following simple flow rearrangement through an edge $e = xy$. Take two paths P_1 and P_2 from $\mathcal{P}^e$. The *exchange operation* of P_1 and P_2 at e is the following: For $i = 1, 2$, split P_i at x into two paths P_i^1 and P_i^2 so that P_i^2 contains y. Reconnect P_1^1 and P_2^2 at x, and reconnect P_2^1 ant P_1^2 at x. If the resulting multiflow has nonsimple paths (resp. cycles), then simplify (resp. delete) them. A subset $\mathcal{P}' \subseteq \mathcal{P}^e$ is said to be *homogeneous* if the exchange operation of every pair of paths in $\mathcal{P}'$ at e does not decrease the optimal value $\|f\|_\mu$. If $\mathcal{P}^e$ itself is homogeneous, then we also say "*f is homogeneous at e.*"

From the position $(\rho(x), \rho(y))$ in $\mathcal{K}_p$, we can check the homogeneity of $\mathcal{P}^e$. The next lemma simply rephrases Lemma 2.5.

Lemma 2.10. *Suppose $d(\rho(x), \rho(y)) = d(\rho(x), p) + d(p, \rho(y))$ and $\rho(x) = u \in Q_p \cup L_p$.*

(1) *If $u \in L_p$, then $\mathcal{P}^{xy}$ consists of paths with x-end belonging to su, and thus $\mathcal{P}^{xy}$ is homogeneous.*

(2) *If $u \in Q_p$, then $\mathcal{P}^{xy}$ consists of paths with x-end belonging to $\bigcup_{v \sim_1 u} sv$.*

By taking p as the center of a folder and considering the 2-subdivision, we obtain a useful lemma.

Lemma 2.11. *If $(\rho(x), \rho(y))$ is a nonadjacent pair of vertices in a folder, then $\mathcal{P}^{xy}$ is homogeneous.*

Anti-exchange operation. There is a reverse way of exchanging two paths P_1 and P_2 at edge $e = xy$. For each $i = 1, 2$, split P_i at x into two paths P_i^1 and P_i^2 so that P_i^2 contains y, as above. Reconnect P_1^1 and P_2^1 at x, and reconnect P_2^2 ant P_1^2 at x. Then the resulting multiflow contains nonsimple paths. So simplify them. This operation is called an *anti-exchange operation* at e. The anti-exchange operation decreases the total support of f. Therefore, if we take an optimal multiflow of the minimum total support, then the anti-exchange operation never keeps optimality. We often use this logic in Section 4.

2.2.4 Degree conditions and reductions

A capacitated graph (G, c) with terminal set S is said to be *inner Eulerian* if c is integral and each inner node has even degree. A terminal s is said to be *proper* (with respect to realization $(\mathcal{K}; \{R_s\}_{s \in S})$) if R_s contains no legs. In other words, $R_s \cap V\Gamma$ belongs to one color class of bipartite graph Γ. Other terminal is called *improper*. We consider the following degree condition (*terminal condition I*) for terminals:

(I) each improper terminal has even degree.

Lemma 2.12. *Suppose that (G, c) is an inner Eulerian graph with terminal condition I. For two potentials ρ, ρ', we have $c \cdot d^{\rho'} - c \cdot d^{\rho} \in 2\mathbf{Z}$.*

Proof. Let $T \subseteq S$ be the set of proper terminals. Since each node except proper terminal has even degree, c can be decomposed into the sum of the characteristic vectors of (nonsimple) cycles C_i and T-paths P_j. Hence we have $c \cdot d^{\rho'} - c \cdot d^{\rho} = \sum_i \{d^{\rho'}(C_i) - d^{\rho}(C_i)\} + \sum_j \{d^{\rho'}(P_j) - d^{\rho}(P_j)\}$. Since Γ is bipartite, both $d^{\rho'}(C_i)$ and $d^{\rho}(C_i)$ are even. For each proper terminal $s \in T$, $\rho(s)$ and $\rho'(s)$ belong to the same color class, and thus $d^{\rho'}(P_j) - d^{\rho}(P_j)$ is even. □

Next we describe a method to transform the problem so that each node has degree at most 4. Suppose that (G, c) is inner Eulerian. By multiplying edges, we can make each edge have unit capacity. For an inner node x of degree greater than four, we can transform (G, c) into (G', c') by changing the incidence at x as in [8, p. 50] (or [12, Figure 3]). Then the problem is unchanged.

Consider a terminal s. Let m be the degree of s. There are two ways of reducing the degree. The first way is reducing the degree to one. Add new terminals $s_1, s_2, \ldots, s_m$, connect s and each s_i by an edge with unit capacity, make s being an inner node, and define the distance μ' on $S \setminus s \cup \{s_1, s_2, \ldots, s_m\}$ as $\mu'(s_i, t) = \mu(s, t)$ for $t \in S \setminus s$ and other distances are same as μ. Then any multiflow for the resulting graph $(G', c'; S', \mu')$ can be transformed into a multiflow for $(G, c; S, \mu)$ having the same objective value. The reverse way is also possible. Moreover, an F-complex realization of (S', μ') is obtained by setting $R_{s_i} := R_s$ for each i. The second way is reducing the degree to two when m *is even.* Add new terminals $s_1, s_2, \ldots, s_{m/2}$, connect s and each s_i by two parallel edges (with unit capacity). The rest is the same as above.

In particular, if (G, c) is inner Eulerian with terminal condition I, then we can convert (G, c) so that it is inner Eulerian with terminal condition I, each inner node has degree four, each proper terminal has degree one, and each improper terminal has degree two.

In addition, if we are given an optimal potential ρ for (G, c), then we can extend ρ to an optimum for the new problem by setting $\rho(x') := \rho(x)$ for each new node x' replacing original node x.

2.2.5 Terminal modifications and creations

Suppose that we are given an optimal potential ρ. Sometimes we can simplify R_s, and can create new terminals to decompose the problem.

Terminal modification. Suppose that a terminal s is incident to exactly one node x with $\rho(x) \neq \rho(s) = p$. Suppose further that $\mathcal{K}_p$ is star-shaped. Replace R_s by

$$R'_s = \{q \in \mathcal{K}_p \cap R_s \mid d(q, g\rho(x)) = d(p, g\rho(x))\}.$$

Then R'_s is a single vertex or union of hypotenuses, and thus normal. Modify μ by setting $\mu(s,t) := d(R'_s, R_t)$ for $t \in S$. Note that s becomes proper in the new problem. By Lemma 2.4 (1), any optimal multiflow for the original problem fulfills the optimality criterion (Lemma 2.7) to the new problem with ρ. In particular ρ is also optimal to the new problem. To see the converse, take an arbitrary optimal multiflow f for the new problem. Suppose that f is not optimal to the original problem. Then there is an (s,t)-path in $\mathcal{P}^{sx}$ such that it is not ρ-shortest for original R_s. Lemma 2.4 (1) implies $p \notin R_t$, and $gR_t \in R_s$. However this implies $gR_t \in R'_s$ by definition of R'_s and Lemma 2.5; a contradiction to optimality in the new problem. Therefore f is also optimal to the original problem.

Terminal creation I. Suppose that there are an edge $e = xy$ and a nonadjacent pair (p, p') of vertices of a folder F such that $d(\rho(x), \rho(y)) = d(\rho(x), p) + d(p, p') + d(p', \rho(y))$. Then delete edge xy, add new terminals s_x, s_y, add new edges xs_y, ys_x with capacity $c(xs_y) = c(ys_x) = c(xy)$. Set $R_{s_x} := \{p\}$ and $R_{s_y} := \{p'\}$, and extend μ to $S \cup \{s_x, s_y\}$ by $\mu(s_x, t) := d(R_{s_x}, R_t)$ and $\mu(s_y, t) := d(R_{s_y}, R_t)$. Extend ρ by $(\rho(s_x), \rho(s_y)) := (p, p')$. Take an optimal multiflow $f = (\mathcal{P}; \kappa)$ for the original problem. For each path in $\mathcal{P}^{xy}$, delete xy to split it into two paths, add edge xs_y to one of the two paths having x, and add edge ys_x to the other path. Then we obtain a multiflow and a potential for the new problem. Both are optimal by Lemmas 2.4 and 2.7; consider $(\mathcal{K}^2)_{p^*}$ for the center p^* of folder F. Conversely, take an arbitrary optimal multiflow $f = (\mathcal{P}; \kappa)$ to the new problem. For any pair $(P', P'') \in \mathcal{P}^{xs_y} \times \mathcal{P}^{ys_x}$, delete edges xs_y, ys_x and join them by adding edge xy. The the resulting path P is ρ-shortest by Lemma 2.4. Apply it to all pairs in any matching between $\mathcal{P}^{xs_y}$ and $\mathcal{P}^{ys_x}$. Then we obtain an optimal multiflow for the original graph.

Terminal creation II. Let p be a vertex in $\mathcal{K}$ such that Π_p is an 8-cycle. Namely $\mathcal{K}$ is *flat* at p. So p is incident to four vertices q_1, q_2, r_1, r_2 by legs. We suppose that pq_i and pr_j belong to a common folder for $i, j \in \{1, 2\}$, and suppose that Π_p is an 8-cycle $(\bar{q}_1, l_{11}, \bar{r}_1, l_{21}, \bar{q}_2, l_{22}, \bar{r}_2, l_{12})$, where $\bar{q}_i$ and $\bar{r}_j$ denote the members of Q_p lying on legs pq_i and pr_j, respectively.

Suppose that there is an edge $e = xy \in EG$ with $(\rho(x), \rho(y)) = (q_1, q_2)$. In this case, we can apply the following terminal creation. Subdivide e into two edges xz and zy with capacity $c(xz) = c(zy) = c(xy)$, add new terminals s^*, t^*, and join them to z with capacity $c(s^*z) = c(t^*z) = c(xy)$. Set $R_{s^*} := \{r_1\}$ and $R_{t^*} := \{r_2\}$, and extend μ to $S \cup \{s^*, t^*\}$. Also extend ρ by $(\rho(s^*), \rho(z), \rho(t^*)) := (r_1, p, r_2)$. Take an optimal multiflow $f = (\mathcal{P}; \kappa)$ for the original problem. We can extend f for new graph by subdividing each path in $\mathcal{P}^e$ at z. Add paths $(s^*, s^*z, z, zt^*, t^*)$ of flow-value $c(xy)$ to $\mathcal{P}$. Then the resulting ρ and f are both optimal. Conversely, take an arbitrary optimal multiflow $f = (\mathcal{P}; \kappa)$. From this, we can construct an optimal multiflow for the original problem by the following way. Since four edges incident to z are all saturated, by a simple calculation we have $(f^{xz,zs^*}, f^{xz,zy}, f^{xz,zt^*}) = (f^{t^*z,zy}, f^{t^*z,zs^*}, f^{s^*z,zy})$. If $f^{xz,zs^*} = f^{t^*z,zy} = f^{xz,zt^*} = f^{s^*z,zy} = 0$, then the deletion of $\mathcal{P}^{s^*z,zt^*}$ gives an optimal multiflow for the original problem. Suppose $f^{xz,zs^*} = f^{t^*z,zy} > 0$. Split each path in $\mathcal{P}^{xz,zs^*}$ and in $\mathcal{P}^{t^*z,zy}$ at z. Reconnect paths from s^* and paths from t^*, and reconnect paths from x and paths from y. Then the resulting multiflow is also optimal. Indeed, $\mathcal{P}^{xz,zs^*}$ consists of $(s\bar{r}_1, z, s\bar{q}_1 \cup sl_{12})$-paths, and $\mathcal{P}^{xz,zs^*}$ consists of $(s\bar{r}_2, z, s\bar{q}_2 \cup sl_{21})$-paths; draw a picture such as Figure 5 for 8-cycle Π_p. So the rearrangement criterion keeps, and thus we can make f fulfill $f^{xz,zs^*} = f^{t^*z,zy} = f^{xz,zt^*} = f^{s^*z,zy} = 0$.

2.2.6 Irrational distances

An irrational distance μ may have no F-complex realizations even if $\dim T_\mu \leq 2$. In this case, we can perturb μ into rational one μ' with the problem unchanged. By Dress' dimension criterion [4, Theorem 9] (also see [10, 11]), the set $\mathcal{D}$ of distances μ' with $\dim T_{\mu'} \leq \dim T_\mu$ is the union of rational polyhedral cones. For a multiflow $f = (\mathcal{P}, \lambda)$, the *boundary* ∂f is the vector in $\mathbf{R}^{\binom{S}{2}}$ defined as $(\partial f)(s,t) = \sum\{\lambda(P) \mid (s,t)\text{-path } P \text{ in } \mathcal{P}\}$ for $s,t \in S$. Let $\mathcal{B}_{G,c;S} = \{\partial f \mid \text{multiflow } f\}$ be the set of all multiflow boundaries. Then $\mathcal{B}_{G,c;S}$ is a convex polytope, and the μ-problem can be regarded as a linear optimization over $\mathcal{B}_{G,c;S}$ with linear objective function μ. Suppose that c is rational. Then $\mathcal{B}_{G,c;S}$ is also rational. Consider the normal cone $\mathcal{N}$ of $\mathcal{B}_{G,c;S}$ containing μ as its relative interior $\mathcal{N}^\circ$. $\mathcal{N}$ is also rational. We can take a rational distance μ' from $\mathcal{N}^\circ \cap \mathcal{D}$. By $\mu' \in \mathcal{D}$, we have $\dim T_{\mu'} \leq 2$. By $\mu' \in \mathcal{N}^\circ$, every optimal multiflow for μ'-problem is also optimal to the μ-problem. Therefore, for proving Theorem 1.5, we may assume that μ is rational.

3 Splitting-off fractionally

Our proof of the main theorem is based on a fractional version of splitting-off operation. Recall the splitting-off operation; see, e.g., [8, 16, 27]. For two edges $e = xy$ and $e' = yz$ incident to y, a triple (e, y, e') is called a *fork*. When both $e = xy$ and $e' = yz$ have no multiple edges, fork (e, y, e') is simply denoted by a triple xyz. For a fork $\tau = (e, y, e')$, the *splitting-off* operation at τ is to decrease the capacity of edges e and e' by one, and add a new edge e^* joining y, z with unit capacity. A fork τ is said to *splittable* if the splitting-off operation at τ does not decrease the optimal value $\mathrm{opt}(G, c)$. Otherwise τ is said to be *unsplittable*.

Suppose that τ is splittable. Apply the splitting-off operation at τ. Then, (i) from an optimal multiflow for the new graph, we obtain an optimal multiflow for the original graph, (ii) if the original graph is Eulerian, then the new graph is also Eulerian, and (iii) the total sum of edge-capacities decreases. From (i-iii) we can apply the inductive argument to prove the existence of an integral optimal multiflow.

However we will face the situation that there is no splittable fork. Our previous paper [12] devised a framework to overcome this difficulty by a fractional splitting-off operation combining potential update, called *SPUP*. The aim of this section is to describe the SPUP framework. Throughout this section, we are given an instance $(G, c; S, \mu)$ of the μ-problem, and given an F-complex realization $(\mathcal{K}; \{R_s\}_{s \in S})$ of μ with unit leg-length. In graph operations, we delete isolated nodes, loops, and edges of zero capacity whenever appeared. Deceasing the capacity of an edge $e = xy$ by 2 is also regarded as splitting-off for a degenerate fork (e, y, e). A node y is said to be *splittable* if we can repeat splitting-off operations by splittable forks at y to make y isolated. Otherwise y is said to be *unsplittable*.

3.1 SPUP: Splitting-off with Potential UPdate

According to [12], we begin with introducing a fractional version of the splitting-off operation. For a fork $\tau = (e, y, e')$, consider the graph (G^τ, c) obtained from (G, c) by adding a new inner node y^τ and a new edge $e^\tau = yy^\tau$ and reconnecting e and e' to y^τ. The capacity of e^τ is defined by $c(e) + c(e')$. Then we can identify a multiflow for (G, c) with a multiflow for (G^τ, c). So this does not change the problem. We use this identification in the sequel. In particular, $\mathcal{P}^{e^\tau} = (\mathcal{P}^e \setminus \mathcal{P}^{e,e'}) \cup (\mathcal{P}^{e'} \setminus \mathcal{P}^{e,e'})$ for a multiflow $f = (\mathcal{P}; \kappa)$.

The *fractional splitting-off* operation at τ is to decrease $c(e^\tau)$ as much as possible keeping the optimal value invariant. The maximum possible value, denoted by $\alpha(\tau) = \alpha^{G,c}(\tau)$, is called the *splitting capacity*, i.e.,

$$\alpha(\tau) := \max\{0 \leq \alpha \leq c(e^\tau) \mid \mathrm{opt}(G^\tau, c - \alpha\chi_{e^\tau}) = \mathrm{opt}(G, c)\}.$$

Let $c^\tau = c - \alpha(\tau)\chi_{e^\tau}$. Clearly if (G^τ, c^τ) has a $1/k$-integral optimal multiflow, then so does (G, c). However a naive inductive argument does not work since (G^τ, c^τ) violates the Eulerian condition. So we need another principle to prove the existence of a $1/k$-integral optimal solution for a bounded integer k.

By the max-min relation (Theorem 2.2), $\alpha(\tau)$ can also be represented as

$$\alpha(\tau) = \min\left\{ \frac{c \cdot d^{\rho'} - \mathrm{opt}(G, c)}{d^{\rho'}(e^\tau)} \;\middle|\; \rho'\text{: potential with } d^{\rho'}(e^\tau) > 0 \right\}.$$

A potential ρ' attaining the minimum in RHS is called *critical*. Note that ρ' is optimal to the new graph (G^τ, c^τ).

Suppose that we are given an optimal potential ρ for (G, c). Then ρ can also be extended to an optimum for (G^τ, c) by setting $\rho(y^\tau) := \rho(y)$. By Theorem 2.8, we can take a critical potential that is a neighbor of ρ.

Proposition 3.1. *Let τ be a fork and ρ an optimal potential. Then we have the following.*

$$\alpha(\tau) = \min\left\{ \frac{c \cdot d^{\rho'} - c \cdot d^{\rho}}{d^{\rho'}(e^\tau)} \;\middle|\; \rho'\text{: neighbor of } \rho \text{ with } d^{\rho'}(e^\tau) > 0 \right\}. \tag{3.1}$$

In particular, if (G, c) is inner Eulerian with terminal condition I, then we have

$$\alpha(\tau) \in \left\{0, \frac{1}{2}, \frac{2}{3}, 1, \frac{4}{3}, \frac{3}{2}, 2, \ldots\right\} = \frac{1}{2}\mathbf{Z}_+ \cup \frac{2}{3}\mathbf{Z}_+.$$

The latter part follows from: (i) the numerator $c \cdot d^{\rho'} - c \cdot d^{\rho}$ is even by Lemma 2.12, and (ii) the denominator $d^{\rho'}(e^\tau)$ is one of $1, 2, 3, 4$ since both $\rho'(y)$ and $\rho'(y^\tau)$ belong to $\mathcal{K}_p$ for $p = \rho(y)$.

SPUP framework. Now we are ready to describe the SPUP framework. After splitting-off operations, we assume that there is no splittable fork. We keep graph (G, c) together with an optimal potential ρ, denoted by $(G, c; \rho)$. The *SPUP operation* for an unsplittable fork τ is to replace $(G, c; \rho)$ by $(G^\tau, c^\tau; \rho')$ for a critical neighbor ρ' of ρ with respect to τ. We also call it γ-SPUP if $\gamma = \alpha(\tau)$.

As will be seen in Section 3.3, there are many vertices p in $\mathcal{K}$ (or $\mathcal{K}^m$) with the following nice property:

(3.2) *Under an appropriate Eulerian condition, if inner node y has potential $\rho(y) = p$, then y has a splittable fork.*

Let V^s be a set of vertices with this property (3.2). Suppose further that Γ has an admissible orientation $\overrightarrow{\Gamma}$ so that the set of sinks coincides with V^s. This orientation is called *forward*. An orientation of an orbit induced by the forward orientation is also called *forward*. The opposite orientation is called *backward*. A neighbor of ρ with respect to an orbit of the forward (resp. backward) orientation is called a *forward neighbor* (resp. *backward neighbor*). An SPUP operation is said to be *forward* (resp. *backward*) if the corresponding critical neighbor is forward (resp. backward). Let $S_\rho = \{y \in VG \mid \rho(y) \in V^s\}$. To prove the existence of a $1/k$-integral optimal solution, we conduct SPUP operations *only for the forward direction* as follows:

(1) Take a fork τ at an inner node y in $VG \setminus S_\rho$.

(2) Take a critical neighbor ρ' of ρ with respect to τ.

(3) If ρ' is *forward*, then apply SPUP $(G, c; \rho) \leftarrow (G^\tau, c^\tau; \rho')$.

(4) Repeat (1-3) until all inner nodes belong to S_ρ.

In the procedure, potentials of nodes are moving toward sinks V^s, or equivalently, nodes are moving toward S_ρ. Once a node y falls into S_ρ, it never moves in the subsequent process. So, in the numerator $c \cdot d^{\rho'} - c \cdot d^\rho$ of (3.1), *the terms by edges joining S_ρ cancel out*. Therefore the even degree condition at S_ρ is unnecessary to keep the numerator even. Let us formalize this observation. $(G, c; \rho)$ is called *restricted Eulerian* (with respect to V^s) if each edge has an integral capacity and each node in $VG \setminus S_\rho$ except proper terminals has even degree. Then one can easily see the following.

Lemma 3.2. *Suppose that $(G, c; \rho)$ is restricted Eulerian. For a fork τ, if its critical neighbor ρ' of ρ is forward, then $c \cdot d^{\rho'} - c \cdot d^\rho$ is even, and thus $\alpha(\tau)$ is a half- or 2/3-integer.*

We try to repeat forward SPUP operations keeping $(G, kc; \rho)$ restricted Eulerian for a bounded integer k. After the procedure, suppose that all inner nodes belong to S_ρ. Multiply $2k$ to the edge-capacity to make (G, c) fulfill an Eulerian condition. Then we can apply the splitting-off to each inner node according to (3.2), and thus can apply the ordinary inductive argument to prove the existence of an integral optimal solution. By reversing operations we can conclude that the original graph has a $1/(2k)$-integral optimal solution. This is our proof scheme.

We end this subsection with listing basic properties of the fractional splitting-off. The proof is not difficult and completely same as in [12, Section 3.2].

Lemma 3.3. *Let $\tau = (e, y, e')$ be a fork, f an optimal multiflow, and ρ an optimal potential.*

(1) *τ is splittable if and only if $\alpha(\tau) \geq 2$.*

(2) *If τ is splittable, then ρ is optimal after the splitting-off at τ.*

(3) $\alpha(\tau) \geq c(e^\tau) - f^{e^\tau} \geq 2f^{e,e'}$.

(4) *If τ and τ' are two forks at distinct nodes, then $\alpha^{G^\tau, c^\tau}(\tau') \leq \alpha^{G,c}(\tau')$.*

(5) *If edge e satisfies $d^\rho(e) = 0$ and is saturated by every optimal multiflow, then there is a neighbor ρ' of ρ such that $d^{\rho'}(e) > 0$ and ρ' is optimal.*

Note that (5) is obtained by considering a critical neighbor for a fractional splitting-off at a degenerate fork (e, y, e). Here a fractional splitting-off at a degenerate fork (e, y, e) is to decrease capacity $c(e)$ as much as possible.

3.2 Perturbation methods

By perturbing an optimal potential to its critical neighbors, we can obtain valuable information for the local multiflow configurations. The aim of this subsection is to describe them. Let $f = (\mathcal{P}; \kappa)$ be an optimal multiflow and ρ an optimal potential. Let y be a node with $\rho(y) = p$, and $\tau = (e, y, e')$ a fork at y. Let ρ' be a critical neighbor with respect to τ.

Suppose that $\alpha(\tau) = c(e^\tau) - f^{e^\tau}$ (or $2f^{e,e'}$) holds (see Lemma 3.3 (3)). Then f can also be regarded as an optimal multiflow for (G^τ, c^τ). By optimality criterion (Lemma 2.7) for (f, ρ'), each path P in $\mathcal{P}^{e^\tau}$ fulfills

$$d(R_s, R_t) = d(R_s, \rho'(y)) + d(\rho'(y), \rho'(y^\tau)) + d(\rho'(y^\tau), R_t), \tag{3.3}$$

where P is supposed to be an (s, y, y^τ, t)-path. From the position $(\rho'(y), \rho'(y^\tau))$ in $\mathcal{K}_p$ with a help of Lemma 2.10, we can determine the gate of R_s or R_t in $\mathcal{K}_p$ and the homogeneity of $\mathcal{P}^{e^\tau}$. This is a lucky case.

We need to analyze $\mathcal{P}^{e^\tau}$ for general case $\alpha(\tau) \geq c(e^\tau) - f^{e^\tau} \geq 2f^{e,e'}$ with possibly strict inequality. Let $\mathcal{P}^{e^\tau;\rho'}$ be the set of paths P in $\mathcal{P}^{e^\tau}$ satisfying (3.3). Its flow-value is denoted by $f^{e^\tau;\rho'}$. We can estimate $f^{e^\tau;\rho'}$ by the following lemma, which is the basis for our perturbation arguments.

Lemma 3.4. *Under the notation above, suppose that $d(\rho'(y), \rho'(y^\tau)) = d(\rho'(y), p) + d(p, \rho'(y^\tau))$, and there is no improper terminal s with $\rho(s) = \rho(y) = p$. Then the following holds:*

(1) $d^{\rho'}(e^\tau) f^{e^\tau;\rho'} + (d^{\rho'}(e^\tau) - 2)(f^{e^\tau} - f^{e^\tau;\rho'}) \geq d^{\rho'}(e^\tau)(c(e^\tau) - \alpha(\tau))$.

(2) *If $d^{\rho'}(e^\tau) \geq 2$, then* $f^{e^\tau;\rho'} \geq c(e^\tau) + (d^{\rho'}(e^\tau) - 2) f^{e,e'} - \dfrac{d^{\rho'}(e^\tau)\alpha(\tau)}{2}$.

Proof. We use the formula (2.9) of the duality gap. By definition of α, we have

$$\mathrm{opt}(G, c) = \mathrm{opt}(G^\tau, c^\tau) = c^\tau \cdot d^{\rho'}.$$

Let f' be the multiflow for (G^τ, c^τ) obtained by deleting all paths in $\mathcal{P}^{e^\tau}$ from f. Then the duality gap between f' and ρ' in (G^τ, c^τ) is

$$c^\tau \cdot d^{\rho'} - \|f'\|_\mu = \sum_{P \in \mathcal{P}^{e^\tau}} \mu(s_P, t_P)/\kappa.$$

We next estimate the first term $\delta_1 := \sum_{e \in EG} d^{\rho'}(e)(c(e) - (f')^e)$ in (2.9), which means the unsaturation of edges. Since there is no path passing e^τ in (G^τ, c^τ), this contributes $d^{\rho'}(e^\tau)(c(e^\tau) - \alpha(\tau))$ for δ_1. The deletion of an (s_P, y, y^τ, t_P)-path P contributes at least $\{d(R_{s_P}, \rho'(y)) + d(\rho'(y^\tau), R_{t_P})\}/\kappa$ for the unsaturation of edges except e^τ. Therefore we have

$$\delta_1 \geq d^{\rho'}(e^\tau)(c(e^\tau) - \alpha(\tau)) + \sum_{P \in \mathcal{P}^{e^\tau}} \{d(R_{s_P}, \rho'(y)) + d(\rho'(y^\tau), R_{t_P})\}/\kappa.$$

Since the duality gap is at least δ_1, we have

$$\frac{1}{\kappa} \sum_{P \in \mathcal{P}^{e^\tau}} \left[d^{\rho'}(e^\tau) - \{d(R_{s_P}, \rho'(y)) + d(\rho'(y), \rho'(y^\tau)) + d(\rho'(y^\tau), R_{t_P}) - d(R_{s_P}, R_{t_P})\} \right]$$
$$\geq d^{\rho'}(e^\tau)(c(e^\tau) - \alpha(\tau)).$$

By the assumption and $d(R_{s_P}, R_{t_P}) = d(R_{s_P}, p) + d(p, R_{t_P})$ for $P \in \mathcal{P}$ passing through y, we have

$$
\begin{aligned}
(3.4) \qquad & d(R_{s_P}, \rho'(y)) + d(\rho'(y), \rho'(y^\tau)) + d(\rho'(y^\tau), R_{t_P}) - d(R_{s_P}, R_{t_P}) \\
& = \{d(R_{s_P}, \rho'(y)) + d(\rho'(y), p) - d(R_{s_P}, p)\} \\
& \qquad + \{d(p, \rho'(y^\tau)) + d(\rho'(y^\tau), R_{t_P}) - d(p, R_{t_P})\}.
\end{aligned}
$$

We show that (3.4) is a nonnegative *even* integer, and is at least 2 if $P \not\in \mathcal{P}^{e^\tau;\rho'}$. Suppose $d(\rho'(y), p) = 2$. Then $\rho'(y)$ and p belong to a common folder F and are nonadjacent. If F consists of triangles, then $\rho'(y)$ and p are joined by a hypotenuse. By the normality, if R_{s_P} meets a leg of F, then R_{s_P} contains both $p, \rho'(y)$. By (2.4) (2) and this fact, we have $d(R_{s_P}, \rho'(y)) - d(R_{s_P}, p) \in \{0, \pm 2\}$, and the first term is even; this argument does not use the assumption of improper terminals. Suppose $d(\rho'(y), p) = 1$; p and $\rho'(y)$ are joined by a leg e. Then R_{s_P} does not contain e, which implies that the first term is even by (2.4) (2). Indeed, suppose $e \subseteq R_{s_P}$. Then s_P is improper. By the assumption, $\rho(s_P) \neq p = \rho(y) \in R_{s_P}$. However this contradicts to (2.10) (3) with (f, ρ).

Consequently (3.4) is even, and we obtain the first inequality. The second follows from the first by substituting $f^{e^\tau} \leq c(e^\tau) - 2f^{e,e'}$. □

SPUP at inner node of degree four. Motivated by the reductions in Section 2.2.4. we describe some properties for SPUP at inner nodes of degree four. Let y be an inner node incident to four edges with unit capacity. Then we easily see:

Lemma 3.5. *If y has multiple edges, then y is splittable.*

So we are interested in the case where y has distinct four neighbors and is unsplittable. Suppose that y is incident to four nodes x_0, x_1, x_2, x_3 by edges $e_0 = x_0y$, $e_1 = x_1y$, $e_2 = x_2y$, $e_3 = x_3y$. By symmetry, for distinct $i, j, k \in \{1, 2, 3\}$, two forks (e_0, y, e_i) and (e_j, y, e_k) have the same property, e.g., $\alpha(e_0, y, e_i) = \alpha(e_j, y, e_k)$. Fork (e_0, y, e_i) is particularly denoted by τ_i, and $\alpha(\tau_i)$ is simply denoted by α_i.

Remark 3.6. In SPUP procedure, sometimes $\alpha(\tau) = 0$ occurs in a fork τ at an inner node y of degree four. In this case, we can replace ρ by an optimal neighbor ρ' with $\rho'(y) \neq \rho(y)$. Indeed, take a critical neighbor ρ' of ρ with respect to τ. Then $\rho'(y) \neq \rho(y)$ or $\rho'(y^\tau) \neq \rho(y)$. We may assume the former $\rho'(y) \neq \rho(y)$ by symmetry. Then y^τ has degree four and has three neighbors in (G^τ, c^τ), and thus is splittable by Lemma 3.5. Split y^τ off in (G^τ, c^τ), the resulting graph coincides with the original one (G, c).

Let f be an optimal multiflow, ρ an optimal potential, and ρ_i a critical neighbor of ρ with respect to τ_i for $i = 1, 2, 3$. Behaviors of ρ_1, ρ_2, and ρ_3 are interrelated, and often determine the local flow configuration at y. To explain and prove them, we use the following simplified notation:

$$
\begin{aligned}
(3.5) \qquad & \mathcal{P}_i = \mathcal{P}^{e_i}, \ \mathcal{P}_{ij} = \mathcal{P}^{e_i, e_j}, \\
& \mathcal{P}^{\tau_i} = \mathcal{P}^{e^{\tau_i}}, \ \mathcal{P}^{\tau_i}_j = \mathcal{P}^{\tau_i} \cap \mathcal{P}_j, \ \mathcal{P}^{\tau_i}_{jk} = \mathcal{P}^{\tau_i} \cap \mathcal{P}_{jk}, \\
& \mathcal{P}^{*\tau_i} = \mathcal{P}^{e^{\tau_i};\rho_i}, \ \mathcal{P}^{*\tau_i}_j = \mathcal{P}^{*\tau_i} \cap \mathcal{P}_j, \ \mathcal{P}^{*\tau_i}_{jk} = \mathcal{P}^{*\tau_i} \cap \mathcal{P}_{jk}.
\end{aligned}
$$

The corresponding flow-values are denoted by $f_i, f_{ij}, f^{\tau_i}, f^{\tau_i}_j, f^{\tau_i}_{jk}, f^{*\tau_i}, f^{*\tau_i}_j, f^{*\tau_i}_{jk}$, respectively. We note an obvious relation $\mathcal{P}^{\tau_i} = \mathcal{P}_{0j} \cup \mathcal{P}_{0k} \cup \mathcal{P}_{ij} \cup \mathcal{P}_{ik}$ for distinct $i, j, k \in \{1, 2, 3\}$. The following property will be used in many places.

Lemma 3.7. *Under the notation above, we have the following:*

(1) $\alpha_1 + \alpha_2 + \alpha_3 \geq 2$.

(2) $2\alpha_1 + \alpha_2 + \alpha_3 \geq 4$ *if there is an optimal multiflow f for (G^{τ_1}, c^{τ_1}) being homogeneous at e^{τ_1}.*

Proof. Take an optimal multiflow f for (G^{τ_1}, c^{τ_1}), and regard it as an optimum for (G, c). By Lemma 2.7, edge e^{τ_1} is saturated by f in (G^{τ_1}, c^{τ_1}), and thus $f^{\tau_1} = 2 - \alpha_1$. By Lemma 3.3 (3) and symmetry, $\alpha_2 + \alpha_3 \geq f_{02} + f_{03} + f_{12} + f_{13} = f^{\tau_1} = 2 - \alpha_1$. Thus we have (1). Suppose that f is homogeneous at e^{τ_1}. By symmetry and relabeling we may assume $f_2^{\tau_1} \geq f_0^{\tau_1} \geq 1 - \alpha_1/2 \geq f_1^{\tau_1} \geq f_3^{\tau_1}$; recall $f^{\tau_1} = f_0^{\tau_1} + f_1^{\tau_1} = f_3^{\tau_1} + f_4^{\tau_1}$. Since $f_2^{\tau_1} \geq f_0^{\tau_1} \geq f_1^{\tau_1}$, by exchange operations at e^{τ_1}, we can make f fulfill $f_{02} = f_0^{\tau_1}$, and we can also make f fulfill $f_{12} = f_1^{\tau_1}$. Thus $\alpha_1 + \alpha_2 \geq 2(f_0^{\tau_1} + f_1^{\tau_1}) = 2(2 - \alpha_1)$. □

A typical obstruction to proceed SPUP is an occurrence of *backward* SPUP. Let $p = \rho(y)$. In our framework, if all critical neighbors ρ_1, ρ_2, ρ_3 are backward, then the following case occurs:

(3.6) (i) there is a leg pq such that $\{\rho_i(y), \rho_i(y^{\tau_i})\} = \{p, q\}$ for $i = 1, 2, 3$, or

(ii) there are two consecutive legs qp and pq' not belonging to a common folder such that $\{\rho_i(y), \rho_i(y^{\tau_i})\} = \{q, p\}$ or $\{p, q'\}$ for $i = 1, 2, 3$.

The following lemma will be used to avoid such an obstruction.

Lemma 3.8. *Suppose that c is integral and there is no improper terminal s with $\rho(s) = \rho(y) = p$. Then* (3.6) *never occurs.*

Proof. Suppose that (3.6) (ii) occurs; (i) can be regarded as a special case of (ii) in our argument. By the assumption and Lemma 2.5, we have the following:

(3.7) There is no terminal s with $d(R_s, p) = d(R_s, q') + 1 = d(R_s, q) + 1$.

Indeed, suppose that such a terminal s exists. Consider the gate of R_s in $(\mathcal{K}^2)_p$. Then gR_s is the center point of a folder of $\mathcal{K}$ containing both qp and pq'; a contradiction.

By relabeling and symmetry we may assume

(3.8) $$(\rho_i(y^{\tau_i}), \rho_i(y)) = (q', p) \quad \text{or} \quad (p, q) \quad (i = 1, 2, 3).$$

Let $\bar{f}^{*\tau_i} = f^{\tau_i} - f^{*\tau_i}$. By Lemma 3.4 (1) for $c(e^{\tau_i}) = 2$ and $d^{\rho_i}(e^{\tau_i}) = 1$, we have

(3.9) $$f^{*\tau_i} - \bar{f}^{*\tau_i} \geq 2 - \alpha_i \quad (i = 1, 2, 3).$$

We claim

(3.10) $$\mathcal{P}_{ij}^{*\tau_i} \cap \mathcal{P}_{ij}^{*\tau_j} = \emptyset \quad (1 \leq i < j \leq 3).$$

Take $P \in \mathcal{P}_{ij}^{*\tau_i} \cap \mathcal{P}_{ij}^{*\tau_j}$. Suppose that P is an (s, x_i, y, x_j, t)-path. Since P can be regarded as an (s, y^{τ_i}, y, t)-path and as an (s, y, y^{τ_j}, t)-path, we have $d(R_s, \rho_i(y)) = 1 + d(R_s, \rho_i(y^{\tau_i}))$, $d(R_s, \rho_j(y^{\tau_j})) = 1 + d(R_s, \rho_j(y))$, $d(R_t, \rho_i(y^{\tau_i})) = 1 + d(R_t, \rho_i(y))$, and $d(R_t, \rho_j(y)) = 1 + d(R_t, \rho_j(y^{\tau_j}))$. By (3.7) and (3.8), any case yields a contradiction. For example, say $(\rho_i(y^{\tau_i}), \rho_i(y)) = (q', p)$. Then $(*)$ $d(R_s, p) = 1 + d(R_s, q')$. If $(\rho_j(y^{\tau_j}), \rho_j(y)) = (q', p)$, then $d(R_s, q') = 1 + d(R_s, p)$, that contradicts to $(*)$. If $(\rho_j(y^{\tau_j}), \rho_j(y)) = (p, q)$, then $d(R_s, p) = 1 + d(R_s, q)$, that contradicts to (3.7) with $(*)$.

By (3.10), we have $\bar{f}^{*\tau_i} \geq f_{ij}^{*\tau_j} + f_{ik}^{*\tau_k}$. Substitute it and $f^{*\tau_i} = f_0^{*\tau_i} + f_{ij}^{*\tau_i} + f_{ik}^{*\tau_i}$ to (3.9). Then we have

$$f_0^{*\tau_i} + f_{ij}^{*\tau_i} + f_{ik}^{*\tau_i} - f_{ij}^{*\tau_j} - f_{ik}^{*\tau_k} \geq 2 - \alpha_i \quad \text{(distinct } i, j, k \in \{1, 2, 3\}).$$

Summing up these three inequalities yields $f_0^{*\tau_1} + f_0^{*\tau_2} + f_0^{*\tau_3} \geq 6 - \alpha_1 - \alpha_2 - \alpha_3$. Since $f_0^{*\tau_i} \leq f_{0j} + f_{0k}$, we have $2f_0 = 2(f_{01} + f_{02} + f_{03}) \geq f_0^{*\tau_1} + f_0^{*\tau_2} + f_0^{*\tau_3}$. From $f_0 \leq c(e_0) = 1$, we have $\alpha_1 + \alpha_2 + \alpha_3 \geq 4$. However, since c is integral and $d^{\rho_i}(e^{\tau_i}) = 1$, we have $\alpha_i \in \{0, 1\}$ by Proposition 3.1. This is a contradiction. □

The next lemma describes a case where positions $(\rho_i(y), \rho_i(y^{\tau_i}))$ in $\mathcal{K}_p$ $(i = 1, 2, 3)$ completely determine the local flow configuration at y, which motivates the concepts of the sparsity in Section 3.3 and tri-fixed nodes in Section 4. We use the notation in Section 2.1.3. Suppose that $\mathcal{K}_p$ is star-shaped for $\rho(y) = p$, and we identify the boundary leg-graph of $\mathcal{K}_p$ with $\mathit{\Pi}_p = (Q_p, L_p; \sim)$.

Lemma 3.9. *Suppose that $\alpha_i = 1$ and $(\rho_i(y), \rho_i(y^{\tau_i}))$ is a nonadjacent pair of vertices of some folder for $i = 1, 2, 3$. Then there exists a triple of $l_1, l_2, l_3 \in L_p$ such that, by an appropriate relabeling of e_0, e_1, e_2, e_3,*

(1) $(\rho_i(y), \rho_i(y^{\tau_i})) = (p, l_i)$ *for $i = 1, 2, 3$, and*

(2) $\mathcal{P}_{ij}$ *consists of $(sl_i, x_i, y, x_j, sl_j)$-paths with flow-value $f_{ij} = 1/2$ for $1 \leq i < j \leq 3$ in every optimal multiflow $f = (\mathcal{P}; \kappa)$.*

Proof. Take an optimal multiflow $f = (\mathcal{P}; \kappa)$ for (G^{τ_1}, c^{τ_1}), and regard it as an optimum for (G, c). By Lemma 2.7 we have $f^{\tau_1} = 1$. Then f is homogeneous at e^{τ_1} by the assumption and Lemma 2.11. By relabeling and exchange operation at e^{τ_1} we may assume $f_{12} + f_{13} \geq f_{13} \geq 1/2 \geq f_{02} + f_{12} \geq f_{02}$ and $f_{03} = 0$. Then $f_{13} > 1/2$ is impossible by $2f_{13} \leq \alpha_2 = 1$ (Lemma 3.3 (3)). So $f_{13} = 1/2$. Then f can be regarded as an optimal multiflow for (G^{τ_2}, c^{τ_2}). Therefore $f^{\tau_2} = f_{01} + f_{23} + f_{12} = 1$, $f_{23} = 1/2$, and $\mathcal{P}^{\tau_2}$ is also homogeneous. Then $f_{01} > 0$ is impossible. Indeed we can exchange f at e^{τ_2} for two paths, one from $\mathcal{P}_{23}$ and one from $\mathcal{P}_{01}$, and exchange f at e^{τ_1} so that $f_{12} > f_{12}^{\tau_1} = 1/2 = \alpha_3/2 \geq f_{12}$; a contradiction. Therefore $f_{12} = f_{23} = f_{13} = 1/2$ and $f_0 = 0$. Then f is optimal to (G^{τ_i}, c^{τ_i}) for each $i = 1, 2, 3$. In particular, $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ are homogeneous.

Next consider the position $(\rho_1(y), \rho_1(y^{\tau_1}))$ in $\mathcal{K}_p$. There are three cases: (i) (p, l) for $l \in L_p$, (ii) (l, p) for $l \in L_p$, and (iii) (q, q') for $q, q' \in Q_p$ with $q \sim q'$. Both (ii) and (iii) are impossible. Indeed, suppose $\rho_1(y) = u \neq p$. Lemma 2.10 implies that $\mathcal{P}^{e^{\tau_1}} (= \mathcal{P}^{yx_2} \cup \mathcal{P}^{yx_3})$ consists of paths with y-end belonging to su if $u \in L_p$ and $\bigcup_{v \sim_1 u} sv$ if $u \in Q_p$. Since we can exchange f at e_2 and at e_3, $\mathcal{P}_{23}$ necessarily has an (su, x_2, y, x_3, su)-path if $u \in L_p$ and an (sv, x_2, y, x_3, sv')-path for $v \sim_1 u \sim_1 v'$ if $u \in Q_p$. Such a path is never ρ-shortest by Lemma 2.4 (2). A contradiction to optimality. By the same argument, we have $(\rho_2(y), \rho_2(y^{\tau_2})) = (p, l_2)$ and $(\rho_3(y), \rho_3(y^{\tau_3})) = (p, l_3)$ for $l_2, l_3 \in L_p$. Hence we have (1).

Take an *arbitrary* optimal multiflow $f = (\mathcal{P}; \kappa)$, and consider $\mathcal{P}^{*\tau_i}$. Since $d(\rho_i(y^{\tau_i}), p) = 2$, (3.4) in the proof of Lemma 3.4 is even. Therefore the inequalities in Lemma 3.4 hold in this case (without assumption of improper terminals). So $f^{*\tau_i} \geq 1$ and $\mathcal{P}^{*\tau_i}$ consists of paths with y^{τ_i}-ends in sl_i. By applying the same argument for $\mathcal{P}^{*\tau_i}$ instead of $\mathcal{P}^{\tau_i}$, we have (2). □

3.3 Sparse vertices

Here we introduce the concept of sparse vertices. A vertex p in $\mathcal{K}$ is said to be *sparse* if the following condition is fulfilled:

(3.11) For any two consecutive legs uv and vw in $\mathcal{K}_p$, if they belong to a common orbit, then they belong to a common folder.

See Section 5 for the origin of the name *sparse*. A sparse vertex has a desired property (3.2) for our SPUP framework.

Theorem 3.10. *Suppose that (G, c) is an inner Eulerian graph with terminal condition I. Let ρ be an optimal potential. Suppose that an inner node y has potential $\rho(y) = p$ that is sparse. Then y has a splittable fork if there is no terminal s such that*

(3.12) (1) *$\rho(s) = \rho(y) = p$,*

(2) *s has odd degree, and*

(3) *R_s has three hypotenuses meeting at p.*

Proof. By applying the operations in Section 2.2.4 at $\rho^{-1}(p)$, we may assume that $\rho^{-1}(p)$ consists of inner nodes of degree four, improper terminals of degree two, proper terminals of degree one without property (3), and proper terminals of degree two with property (3). If p is sparse in $\mathcal{K}$, then p is also sparse in $\mathcal{K}^2$. So we work on $\mathcal{K}^2$ by regarding ρ as $VG \to V\mathit{\Gamma}^2$.

It suffices to show that there exists a splittable inner node in $\rho^{-1}(p)$; if true, then all inner nodes are splittable (thanks to Lemma 3.3 (2)). Let y be an inner node with $\rho(y) = p$. Suppose that y is unsplittable, and y is incident to four nodes x_0, x_1, x_2, x_3. We use the notation in Section 3.2. Consider a critical neighbor ρ_i with respect to τ_i for $i = 1, 2, 3$. By the sparsity condition, $\rho_i(y)$ and $\rho_i(y^{\tau_i})$ belong to a common folder in $\mathcal{K}_p$. Therefore by degree condition $\alpha(\tau_i) > 0$ implies $\alpha(\tau_i) = 1$ and $d^{\rho_i}(e^{\tau_i}) = 2$; $(\rho_i(y), \rho_i(y^{\tau_i}))$ is a nonadjacent pair of vertices of a folder. We may assume $\alpha(\tau_1) = 1$ (by Lemma 3.7 (1)). Since any optimal multiflow for (G^{τ_1}, c^{τ_1}) is homogeneous at e^{τ_1} (Lemma 2.11), we have $\alpha(\tau_2) = \alpha(\tau_3) = 1$ (Lemma 3.7 (2)). So Lemma 3.9 is applicable. There is a triple $l_1, l_2, l_3 \in L_p$ with properties (1-2) in Lemma 3.9. Take an optimal multiflow f. Then $\mathcal{P}^{e_i, e_j}$ consists of (sl_i, y, sl_j)-paths with flow-value 1/2 for $1 \leq i < j \leq 3$. Edge $x_0 y$ is unsaturated, and thus $\rho(x_0) = \rho(y) = p$.

Consider the splitting property at x_0. Suppose first that x_0 is an inner node of degree four, incident to y, y_1, y_2, y_3. Edge $y_i x$ is denoted by $\tilde{e}_i$ for $i = 1, 2, 3$. Fork $(e_0, x_0, \tilde{e}_i)$ is denoted by $\tilde{\tau}_i$ for $i = 1, 2, 3$. If x_0 has a splittable fork, this is a desired node. Suppose not. Again, by Lemma 3.9, there is a triple $l'_1, l'_2, l'_3 \in L_p$ such that $\mathcal{P}^{\tilde{e}_i, \tilde{e}_j}$ consists of $(sl'_i, y_i, x_0, y_j, sl'_j)$-paths with flow-value 1/2 for $1 \leq i < j \leq 3$. Suppose $l_1 \not\sim l'_1$. Then we can rearrange f as in Figure 6 (a). The resulting multiflow f is also optimal, and $f^{e_2, e_3} > 1/2$, that contradicts to $\alpha(\tau_1) = 1$. Therefore $l_i \sim l'_j$ for any i, j. Then $\mathit{\Pi}_p$ contains the subdivision of $K_{3,3}$, and all edges incident to l_i, l'_j in $\mathit{\Pi}_p$ belong to a common orbit. Therefore p is not sparse; a contradiction.

Suppose that x_0 is a terminal (of degree one or two). Since edge e_0 is unsaturated, we have $\rho(x_0) = \rho(y)$ and $\rho_i(x_0) = \rho_i(y)$ $(i = 1, 2, 3)$. Hence we have $p, l_1, l_2, l_3 \in R_{x_0}$. Recall that we work on the 2-subdivision, and l_i is the center point of a folder in $\mathcal{K}$. So if x_0 is a (proper) terminal of degree one, then p and l_i are joined by a hypotenuse in R_x. This is a contradiction to (2-3). Therefore x_0 is necessarily a terminal of degree two, incident to y and z. If unique fork $\tau' = yx_0z$ is splittable, then split it off, and

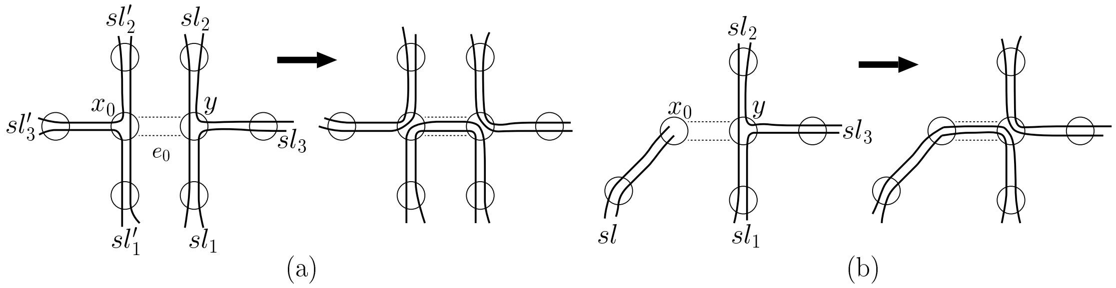


Figure 6: Flow rearrangements

consider node z, which is now incident to x_0 and also has potential $\rho(z) = p$. So we may assume that τ' is not splittable. By $f^{e_0} = 0$, we have $\alpha(\tau') = 1$ and $f^{e'} = 1$ (Lemma 3.3 (3)). Consider a critical neighbor ρ' of ρ with respect to τ', and consider the position $(\rho'(x_0), \rho'(x_0^{\tau'}))$. By sparsity, the possible positions are (i) (p, l) for $l \in L_p$, (ii) (l, p) for $l \in L_p$, and (iii) (q, q') for $q, q' \in Q_p$ with $q \sim q'$. Then both (ii) and (iii) are impossible. Suppose (ii). Then $p, l \in R_{x_0}$, and this contradicts to (2.10) (3) with (f, ρ'). Suppose (iii). Then $q \in R_{x_0}$, and hence leg pq belongs to R_{x_0}. Since q is the midpoint of a leg in $\mathcal{K}$, it is in the interior of R_{x_0} by normality. Again this contradicts to (2.10) (3). Since f can be regarded as an optimal multiflow for $(G^{\tau'}, c^{\tau'})$, $\mathcal{P}^{zx_0}$ consists of paths with z-end in sl. If $l \not\sim l_1$, then by a rearrangement as in Figure 6 (b), we have $f^{e_2,e_3} > 1/2$; a contradiction. So suppose $l \sim l_i$ for $i = 1, 2, 3$. Then p and l_i are joined by a hypotenuse since $p, l_1, l_2, l_3 \in R_{x_0}$ and $l \notin R_{x_0}$. Again all edges incident to l and l_i in $\mathit{\Pi}_p$ belong to the same orbit; $\mathcal{K}_p$ has a subcomplex around p^D in Figure 11. So p is never sparse. A contradiction. $\square$

The goal of the SPUP procedure is the following.

Lemma 3.11. *Suppose that (G, c) is an inner Eulerian graph with terminal condition I. If we are given an optimal potential ρ such that*

(1) *each inner node has sparse potential,*

(2) *there is no terminal s having sparse potential p with properties* (2-3) *in* (3.12)*, and*

(3) *for each terminal s having nonsparse potential, each edge incident to s is saturated by paths of end s in every optimal multiflow,*

then there exists an integral optimal multiflow.

A typical situation satisfying (3) is: (3') each terminal s having nonsparse potential is incident to only one node x with $\rho(x) \neq \rho(s)$.

Proof. After the degree reduction and splitting-off (Theorem 3.10) at nodes having sparse potential, we may assume that there is no inner node, and each terminal s has degree at most 2 whenever $\rho(s)$ is sparse. We may assume that such degree 2 terminals are unsplittable. We show that

(3.13) for each edge st with $\rho(s) \neq \rho(t)$, we have $d(\rho(s), \rho(t)) = d(R_s, R_t)$.

If true, then the set of one-edge paths $\{(s, st, t) \mid s, t \in VG, \rho(s) \neq \rho(t)\}$ with unit flow-value is an integral optimal multiflow by optimality criterion. Take an edge st with $\rho(s) \neq \rho(t)$. By (3), we may assume that s is an unsplittable terminal of degree two having sparse potential $\rho(s) = p$. Take an optimal multiflow $f = (\mathcal{P}; \kappa)$. Then $\mathcal{P}^{st}$

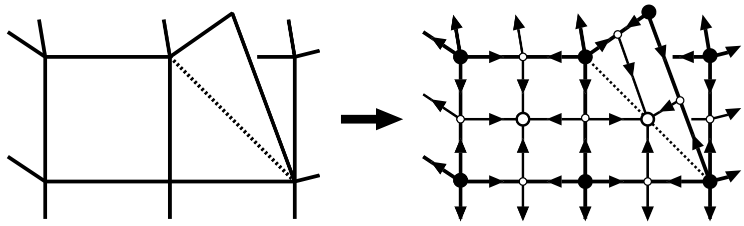


Figure 7: 2-subdivision with forward orientation

contains a path of end s and a path of end t. Otherwise s or t is splittable. Therefore $d(\rho(s), \rho(t)) = d(p, \rho(t)) = d(p, R_t)$. We may assume that $\mathcal{K}_p$ is star-shaped (by subdivision). By Lemma 2.4 (1), it suffices to show $gR_t \not\in R_s$. Consider the gate u of $\rho(t)$ in $\mathcal{K}_p$. Necessarily $u \sim_1 gR_t$ by Lemma 2.5. Since $\mathcal{P}^{st}$ contains a path of end s, we have $u \not\in R_s$. If $u \in L_p$, then $u = gR_t$ (Lemma 2.5), and thus $gR_t \not\in R_s$. So suppose $u \in Q_p$. Consider a critical neighbor ρ' with respect to a unique fork τ at s. By sparsity we have $(\rho'(s), \rho'(s^\tau)) = (p, l)$ for $l \in L_p$. Note that both (q, q') and (l, p) are impossible; see the proof of Theorem 3.10. Also $l \not\in R_s$. Necessarily $l \sim_1 u$. Indeed, consider an optimal multiflow f for (G^τ, c^τ), and regard it as an optimum for (G, c). Then $\mathcal{P}^{st}$ contains a path of end sl; apply Lemma 2.5. In particular pu and ul belong to a common orbit. By sparsity and normality, other folder containing pu is necessarily a square meeting R_s at only p. Therefore $gR_t \not\in R_s$. $\square$

Although there may be no sparse vertices in $\mathcal{K}$, there are many sparse vertices in the subdivision $\mathcal{K}^m$. A point $p \in \mathcal{K}$ is *rational* if p is a vertex of $\mathcal{K}^m$ for some m. Then we can extend the sparsity concept for any rational point in $\mathcal{K}$ by subdivision. If a leg e in $\mathcal{K}$ is subdivided into m edges $e_1, e_2, \ldots, e_m$ in $\mathcal{K}^m$, then they belong to distinct orbits in $\mathcal{K}^m$. From this, one can see the following:

Lemma 3.12. *Any rational point in the interior of a folder is sparse.*

4 Proof of bounded fractionality

This section is devoted to proving the main result of this paper:

Theorem 4.1. *Suppose that distance μ is realized by an F-complex $\mathcal{K}$. Then the μ-problem has a $1/12$-integral optimal multiflow for every inner Eulerian graph with terminal condition I in $\mathcal{K}$.*

Our argument basically follows the line of [12] with some care for algorithmic consequences in Section 7.

4.1 Setting up

We work on the 2-subdivision $\mathcal{K}^2$ instead of $\mathcal{K}$. The SPUP scheme is constructed as follows. Let V^s be the set of vertices lying on the center of a folder in $\mathcal{K}$. They are sparse by Lemma 3.12. The forward orientation is a unique orientation so that vertices in V^s are sinks and vertices in $\mathcal{K}$ are sources. See Figure 7. Let ρ be an optimal potential.

Partition VG into three sets:

$$\begin{aligned} S_\rho &= \{x \in VG \mid \rho(x) \in V^s\}, \\ M_\rho &= \{x \in VG \mid \rho(x) \text{ is the midpoint of a leg in } \mathcal{K}\}, \\ C_\rho &= \{x \in VG \mid \rho(x) \in V\Gamma\}. \end{aligned}$$

In the initial step, $C_\rho = VG$ and $S_\rho = M_\rho = \emptyset$ by taking an optimal potential for $\mathcal{K}$. Our goal is to repeat forward SPUP until there is no inner node in $M_\rho \cup C_\rho$. By reductions in Section 2.2.4, we may assume that each inner node has degree four, each proper terminal has degree one, and each improper terminal has degree two. Also we may assume that there is no splittable fork. For each $p \in V\Gamma$, we identify the boundary leg-graph of $\mathcal{K}_p$ and $\Pi_p = (Q_p, L_p; \sim)$. We remark $L_p \subseteq V^s$.

4.1.1 Eliminating improper terminals at C_ρ

To make use of Lemma 3.4 we need to eliminate improper terminals at C_ρ. Let $s \in C_\rho$ be an improper terminal incident to nodes x, z, and let $\tau = xsz$ be a unique fork at s. Take a critical neighbor ρ' at τ. Let $p = \rho(s)$. Recall (2.10) (3). By $\rho(s), \rho'(s) \in R_s$, it is impossible to satisfy both $\rho'(s) \neq p$ and $d(\rho'(s), \rho'(s^\tau)) = d(\rho'(s), p) + d(p, \rho'(s^\tau))$. Also $\rho'(s) = q \in Q_p$ is impossible since q must be in the interior of R_s. Then the position $(\rho'(s), \rho'(s^\tau))$ is given as (i) $\rho'(s) = p \neq \rho'(s^\tau)$, (ii) $(\rho'(s), \rho'(s^\tau)) \in L_p \times L_p$ with $\rho'(s) \sim \rho'(s^\tau)$, or (iii) $(\rho'(s), \rho'(s^\tau)) \in L_p \times Q_p$ with $\rho'(s) \sim_1 \rho'(s^\tau)$. In any case, $d^{\rho'}(e^\tau) \in \{1, 2\}$ and thus $\alpha(\tau) = \{0, 1\}$. Apply SPUP $(G, c; \rho) \leftarrow (G^\tau, c^\tau; \rho')$. If case (ii) or (iii) occurs, then the SPUP keeps $(G, c; \rho)$ restricted Eulerian, and s falls into S_ρ. Suppose case (i). Then s is incident to only one node s^τ with $\rho'(s) \neq \rho'(s^\tau)$. Apply the terminal modification in Section 2.2.5 to s. Then s becomes proper, and $(G, c; \rho)$ is restricted Eulerian. Apply this procedure for the remaining improper terminals in C_ρ. Now that C_ρ has no improper terminal. Also M_ρ has no terminal. Indeed, such a terminal s is necessarily an improper terminal of degree two since $\rho(s)$ is the midpoint of a leg in $\mathcal{K}$. By normality $\rho(s)$ is in the interior of R_s, and thus there is no optimal multiflow flowing into s by (2.10) (3). So the unique fork at s is splittable, and has already been split off.

4.1.2 Keeping $\alpha(\tau)$ half- or $2/3$-integral

In analysis on SPUP at C_ρ, we need a more precise condition to keep $\alpha(\tau)$ half- or 2/3-integral.

An inner node $y \in C_\rho$ is called *tri-fixed* if there is a triple of $l_1, l_2, l_3 \in L_p$ for $p = \rho(y)$ such that every optimal multiflow contains (sl_i, y, sl_j)-paths for $1 \leq i < j \leq 3$; recall the situation in Lemma 3.9.

For $p \in V\Gamma$, consider an edge $e = xy$ having potential $(\rho(x), \rho(y)) = (q, q') \in Q_p \times Q_p$ with $q \not\sim q'$. Such an edge e is called *unmixed* if there are $l \in L_p$ and an optimal multiflow f such that e is saturated by paths with end in sl. Otherwise e is called *mixed*.

$(G, c; \rho)$ is called *admissible* if it satisfies:

(1) each edge incident to $M_\rho \cup C_\rho$ has an integer capacity,

(2) for a set $\tilde{E}$ of mixed edges, each inner node in M_ρ has even degree in the graph obtained by deleting $\tilde{E}$ from G, and

(3) each inner node in C_ρ except tri-fixed nodes has even degree.

	$\{\rho'(y), \rho'(y^\tau)\}$	$d^{\rho'}(e^\tau)$	$\alpha(\tau)$, c admissible	$\alpha(\tau)$, $3c$ admissible	e^τ
(1a)	$\{q, l\}$, $q \sim l$	1	$0, 2$	$0, 2/3, 4/3, 2$	
(1b)	$\{p, q\}$	1	$0, 2$	$0, 2/3, 4/3, 2$	
(2a)	$\{p, l\}$	2	$0, 1, 2$	$0, 1/3, 2/3, 1, 4/3, 5/3, 2$	homog.
(2b)	$\{q, q'\}$, $q \sim q'$	2	$0, 1, 2$	$0, 1/3, 2/3, 1, 4/3, 5/3, 2$	homog.
(2c)	$\{q, q'\}$, $q \not\sim q'$	2	$0, 1, 2$	$0, 1/3, 2/3, 1, 4/3, 5/3, 2$	
(2d)	$\{l, l'\}$, $l \sim l'$	2	$0, 1, 2$	$0, 1/3, 2/3, 1, 4/3, 5/3, 2$	
(3)	$\{q, l\}$, $q \not\sim l$	3	$0, 2/3, 4/3, 2$	$2m/9$ $(0 \leq m \leq 9)$	homog.
(4)	$\{l, l'\}$, $l \not\sim l'$	4	$0, 1/2, 1, 3/2, 2$	$m/6$ $(0 \leq m \leq 12)$	homog.

Table 1: Classification of $\{\rho'(y), \rho'(y^\tau)\}$, where $q, q' \in Q_p$ and $l, l' \in L_p$

It will turn out that one can conduct SPUP keeping the admissibility without examining all optimal multiflows. Clearly, if $(G, c; \rho)$ is restricted Eulerian, then it is also admissible since $\tilde{E}$ can be taken to be empty.

Lemma 4.2. *Suppose that $(G, c; \rho)$ is admissible. Let τ be a fork at an inner node $y \in C_\rho$ of even degree, and let ρ' be a critical neighbor of ρ with respect to τ. Then $c \cdot d^{\rho'} - c \cdot d^\rho$ is even, and thus $\alpha(\tau)$ is half- or 2/3-integral.*

Proof. Note that ρ' is necessarily forward. As in the proof of [12, Lemma 3.12], the assertion immediately follows from:

(1) for a mixed edge e, we have $d^{\rho'}(e) = d^\rho(e) = 2$.

(2) for a tri-fixed node $x (\neq y)$, $\rho(x)$ and $\rho'(x)$ belong to the same color class of Γ^2.

Take a mixed edge $e = xy$. Suppose $(\rho(x), \rho(y)) \in Q_p \times Q_p$ with $\rho(x) \not\sim \rho(y)$ for $p \in V\Gamma$. Then $d^{\rho'}(e) \in \{2, 3, 4\}$. Suppose $d^{\rho'}(e) \in \{3, 4\}$. For one of ends x, y, say x, we have $\rho'(x) = l \in L_p$ with $\rho'(y) \not\sim l \sim_1 \rho(x)$, and e is saturated by paths of ends sl in every optimal multiflow f for (G^τ, c^τ). A contradiction. Thus we have (1).

For a tri-fixed node x $(\neq y)$, $\rho'(x)$ necessarily satisfies $d(l_i, l_j) = d(l_i, \rho'(x)) + d(\rho'(x), l_j)$ for $1 \leq i < j \leq 3$ by (2.10) (1) and (2.8). Since $l_i \not\sim l_j$, we have $d(l_i, l_j) = 4$, $d(\rho'(x), l_i) = 2$, and (2). $\square$

4.2 SPUP at C_ρ

In the initial step $(G, c; \rho)$ is trivially restricted Eulerian and admissible. For an arbitrary inner node $y \in C_\rho$ with $\rho(y) = p$ and a fork τ at y, consider a critical neighbor ρ' of ρ with respect to τ, which is always forward. Then $\{\rho'(y), \rho'(y^\tau)\}$ belongs to $\mathcal{K}_p$ and therefore $Q_p \cup L_p \cup \{p\}$. Table 1 summarizes the possible cases of $\{\rho'(y), \rho'(y^\tau)\}$ with $d^\rho(e^\tau)$ and $\alpha(\tau)$, where the last column indicates whether every optimal multiflow f for (G^τ, c^τ) is homogenous at e^τ, according to Lemmas 2.10 and 2.11.

We apply SPUP at a fork having maximum α at first three stages. Then by Lemma 3.3 (4) the maximum value of $\alpha(\tau)$ over forks τ at C_ρ decreases. When $\alpha(\tau)$ becomes close to 1, the estimation by Lemmas 3.4 and 3.7 becomes more effective.

4.2.1 3/2-SPUP

By searching all forks at C_ρ, take a fork τ at inner node $y \in C_\rho$ with $\alpha(\tau) = 3/2$. Let $p = \rho(y)$. Take a critical neighbor ρ' of ρ with respect to τ. Then $d^{\rho'}(e^\tau) = 4$, and thus $\{\rho'(y), \rho'(y^\tau)\}$ is of case (4) in Table 1. Apply 3/2-SPUP $(G, c; \rho) \leftarrow (G^\tau, c^\tau; \rho')$. Then

$(G, c; \rho)$ is admissible and $(G, 2c; \rho)$ is restricted Eulerian. Repeat this process until there is no fork τ at C_ρ with $\alpha(\tau) = 3/2$. After that, the possible values of α of forks at C_ρ are $0, 2/3, 1, 4/3$. Note that $1/2$ never occurs since $\alpha(\tau) = 1/2$ implies the existence of another fork τ' with $\alpha(\tau') = 3/2$ by Lemma 3.7 (2); see also the last column of Table 1.

4.2.2 4/3-SPUP and 7/6-SPUP

By searching all forks at C_ρ, take a fork τ at an inner node y in C_ρ with $\alpha(\tau) = 4/3$ (case (3) in Table 1). Apply 4/3-SPUP $(G, c; \rho) \leftarrow (G^\tau, c^\tau; \rho')$. Then

(4.1) $(G, 3c; \rho)$ is admissible and $(G, 6c; \rho)$ is restricted Eulerian.

From now on we keep this condition (4.1). In the next SPUP, $\alpha(\tau)$ belongs to $1/3(2\mathbf{Z}_+/3 \cup \mathbf{Z}_+/2)$; see the fifth column in Table 1. Note that $\alpha(\tau) > 4/3$ is impossible by Lemma 3.3 (4). By this fact together with Lemma 3.7 (2), $\alpha(\tau) \in \{1/6, 2/9, 4/9\}$ is also impossible. So the possible values of $\alpha(\tau)$ are $0, 1/3, 2/3, 5/6, 8/9, 1, 10/9, 7/6, 4/3$.

Apply SPUP for a fork τ at inner node of degree four in C_ρ with $\alpha(\tau) = 4/3$. If $\alpha(\tau) = 3/4$ in (1b, 2a) occurs, then one of $\rho'(y), \rho'(y^\tau)$ does not move, and C_ρ does not decrease. However the following hidden property holds:

(4.2) Suppose that $\{\rho'(y), \rho'(y^\tau)\}$ is of case (1b) or (2a) with $\alpha(\tau) = 4/3$. Then there exists an optimal potential ρ'' for (G^τ, c^τ) such that $\{\rho''(y), \rho''(y^\tau)\}$ is of case (3). In particular every optimal multiflow f for (G^τ, c^τ) is homogeneous at e^τ.

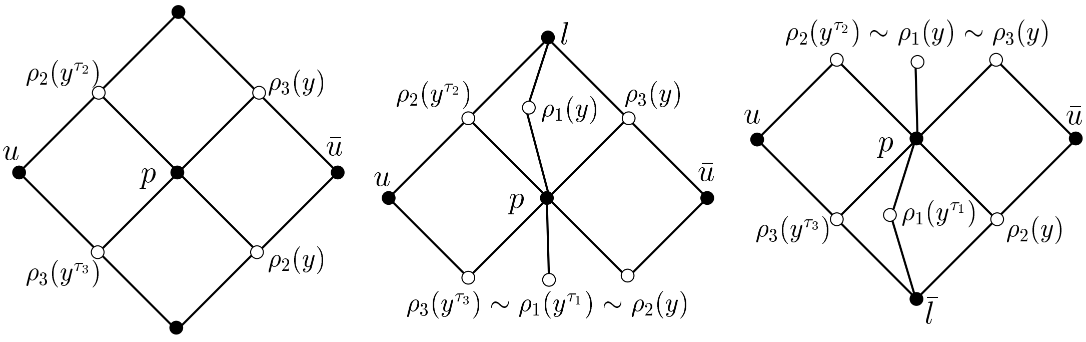
Proof. After 3/2-SPUP in the previous subsection, $\alpha(\tau) = 4/3$ holds by Lemma 3.3 (4). Therefore *we could choose this fork τ in the first 4/3-SPUP*. Consider a critical neighbor ρ'' with respect to τ at the first 4/3-SPUP. ρ'' is necessarily of case (3), and can be regarded as an optimal potential for the current graph by setting $\rho''(\tilde{y}^{\tilde{\tau}}) := \rho''(\tilde{y})$ for forks $\tilde{\tau}$ at $\tilde{y}$ processed after 3/2-SPUP. □

After the procedure, the possible values of α are $0, 1/3, 2/3, 5/6, 8/9, 1, 10/9, 7/6$. Next apply SPUP for any fork τ at inner node $y \in C_\rho$ of degree four with $\alpha(\tau) = 7/6$. In this case, its critical neighbor ρ' is of case (4). Thus 7/6-SPUP keeps (4.1). After the procedure, the possible values of α are $0, 1/3, 2/3, 8/9, 1, 10/9$; Lemma 3.7 (2) and $\alpha < 7/6$ exclude $5/6$.

4.2.3 1-SPUP

Take any inner node $y \in C_\rho$ of degree four, and take a critical neighbor ρ'. Let $p = \rho(y)$. The possible cases of $(\alpha(\tau), \rho')$ are $\alpha(\tau) = 1/3$ in (2c, 2d), $\alpha(\tau) = 2/3$ in (1a, 1b, 2c, 2d), $\alpha(\tau) = 8/9$ in (3), $\alpha(\tau) = 1$ in (2a, 2b, 2c, 2d, 4), and $\alpha(\tau) = 10/9$ in (3). Note that Lemma 3.7 (2) and $\alpha < 4/3$ exclude $\alpha(\tau) \in \{1/3, 2/3\}$ in (2a, 2b, 3, 4).

The obstruction to keep (4.1) is the occurrence of $\alpha(\tau) = 10/9$, or $\alpha(\tau) = 1$ in (2c) with e^τ unmixed in $(G^\tau, c^\tau; \rho')$. We can avoid such an SPUP by examining all three forks τ_1, τ_2, τ_3 at y and their critical neighbors ρ_1, ρ_2, ρ_3. Here we use the notation (3.5) in Section 3.2. The main claim here is the following:



(2-1) (2-2) (2-3)

Figure 8: Configurations of (ρ_1, ρ_2, ρ_3)

(4.3) Suppose that none of ρ_2 and ρ_3 is of case (2d).

(1) If $\alpha_1 = 10/9$, then ρ_2 or ρ_3 is of case (2c).

(2) If ρ_1 is of case (2c) with e^{τ_1} unmixed in $(G^{\tau_1}, c^{\tau_1}; \rho_1)$, then both ρ_2 and ρ_3 are of case (2c), and by a relabeling fixing $\{\{e_0, e_1\}, \{e_2, e_3\}\}$ one of the following holds:

(2-1) $\rho_3(y) \sim \rho_2(y) \sim \rho_3(y^{\tau_3}) \sim \rho_2(y^{\tau_2}) \sim \rho_3(y)$.

(2-2) $\rho_3(y) \sim \rho_2(y) \sim \rho_1(y^{\tau_1}) \sim \rho_3(y^{\tau_3}) \sim \rho_2(y^{\tau_2}) \sim \rho_1(y) \sim \rho_3(y)$ and $\rho_2(y^{\tau_2}) \sim \rho_3(y)$.

(2-3) $\rho_3(y) \sim \rho_2(y) \sim \rho_1(y^{\tau_1}) \sim \rho_3(y^{\tau_3}) \sim \rho_2(y^{\tau_2}) \sim \rho_1(y) \sim \rho_3(y)$ and $\rho_3(y^{\tau_3}) \sim \rho_2(y)$.

See Figure 8 for the configurations of (ρ_1, ρ_2, ρ_3) in (4.3) (2). The proof of this claim is complicated, and we give it in the end. Let us proceed, assuming (4.3). If some ρ_i is of case (2d) or (4), then apply SPUP for τ_i, which keeps (4.1). So suppose that neither (2d) nor (4) occurs. Suppose $\alpha_i = 10/9$. By (4.3), for $j \neq i$, ρ_j is of case (2c), and e^{τ_j} is guaranteed to be mixed in $(G^{\tau_j}, c^{\tau_j}; \rho_j)$. Apply 1-SPUP for τ_j with adding e^{τ_j} to $\tilde{E}$. If $\alpha_i = 8/9$, then $\alpha_j = \alpha_k = 10/9$ by Lemma 3.7 (2). However this is impossible by (4.3) (1).

So suppose further $\max(\alpha_1, \alpha_2, \alpha_3) \leq 1$. Suppose $\alpha_i = 1$ with (2c). If (ρ_i, ρ_j, ρ_k) violates the configuration (4.3) (2), then e^{τ_i} is guaranteed to be mixed in $(G^{\tau_i}, c^{\tau_i}; \rho_i)$, and apply 1-SPUP for τ_i with adding e^{τ_i} to $\tilde{E}$. Suppose that (ρ_i, ρ_j, ρ_k) fulfills the configuration (4.3) (2) by a relabeling fixing τ_i. Then at least one of ρ_j and ρ_k, say ρ_j, violates the configuration (4.3) (2), and apply 1-SPUP for τ_j with adding e^{τ_j} to $\tilde{E}$. Indeed, it is impossible that all ρ_1, ρ_2, ρ_3 fulfill (4.3) (2). To verify it, suppose that all ρ_i fulfill (4.3) (2). Then all $\rho_1(y), \rho_1(y^{\tau_1}), \rho_2(y), \rho_2(y^{\tau_2}), \rho_3(y), \rho_3(y^{\tau_3})$ are distinct. Suppose that ρ_1 fulfills (2-2) in Figure 8. Then by (4.3) (2) for ρ_2, we have $\rho_1(y) \sim \rho_3(y^{\tau_3})$ or $\rho(y^{\tau_1}) \sim \rho_3(y)$. The first case finds a 6-cycle in Π_p. So suppose the second case. Then necessarily $\rho_1(y^{\tau_1}) \sim \bar{u}$. By (4.3) (2) for ρ_3, we have $\rho_1(y) \sim \rho_2(y)$ or $\rho(y^{\tau_1}) \sim \rho_2(y^{\tau_2})$. Similarly we have $\rho_1(y^{\tau_1}) \sim u$. Then Π_p has a 6-cycle $(u, \rho_2(y^{\tau_2}), l, \rho_3(y), \bar{u}, \rho_1(y^{\tau_1}))$. A contradiction to (2.6). The case (2-3) is similar. If all ρ_1, ρ_2, ρ_3 fulfill (2-1), then again one can find a 6-cycle in Π_p.

Suppose that $\alpha_i = 1$ in case (2a) or (2b) occurs. Then necessarily $\alpha_j = \alpha_k = 1$ (by

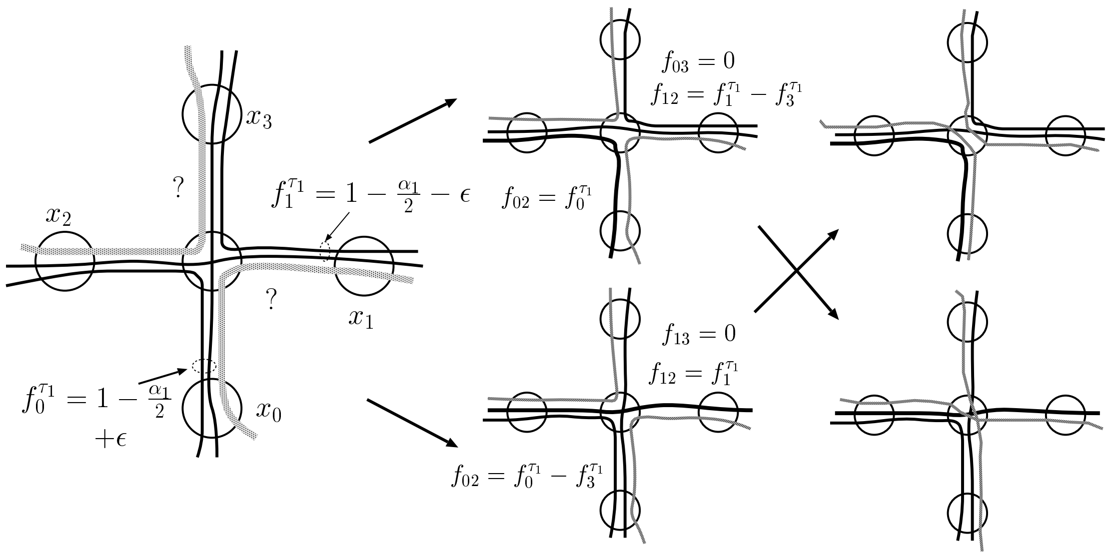


Figure 9: Flow configuration at y

Lemma 3.7 (2)). We may assume that both ρ_j and ρ_k are of case (2a) or (2b); other cases reduce to the above. By Lemma 3.9, all ρ_i are of case (2a), and y is tri-fixed. Apply SPUP for any τ_i. Then one of y and y^{τ_i}, say y, remains in C_ρ, and is a tri-fixed node of degree three, and thus (4.1) keeps. All three neighbors of y are saturated by every optimal multiflow. If one x of the neighbors is not in C_ρ, then replace ρ by its forward optimal neighbor ρ' with $\rho'(x) \neq \rho'(y)$ according to Lemma 3.3 (5). The remaining possible values of α_i are $1/3, 2/3$. By Lemma 3.7 (1), we have $\alpha_1 = \alpha_2 = \alpha_3 = 2/3$. Then every optimal multiflow f satisfies

$$f_{01} = f_{02} = f_{03} = f_{12} = f_{13} = f_{23} = 1/3. \tag{4.4}$$

Namely all edges incident to y are saturated by every optimal multiflow. According to Lemma 3.3 (5), if $\rho(y) = \rho(x_i)$ for a neighbor x_i of y, then replace ρ by an optimal forward neighbor ρ' with $\rho'(y) \neq \rho'(x_i)$. Apply it for all inner nodes in C_ρ of degree four.

The proof of (4.3). Suppose $\alpha_1 = 10/9$ in case (3) or $\alpha_1 = 1$ in case (2c) with e^{τ_1} unmixed in $(G^{\tau_1}, c^{\tau_1}; \rho_1)$. In both cases, we can take an optimal multiflow $f = (\mathcal{P}; \kappa)$ for (G^{τ_1}, c^{τ_1}) such that

(4.5) (i) $\mathcal{P}^{\tau_1}$ consists of paths with y-end in sl for some $l \in L_p$, or

(ii) $\mathcal{P}^{\tau_1}$ consists of paths with y^{τ_1}-end in $s\bar{l}$ for some $\bar{l} \in L_p$.

Take such an optimal multiflow f with minimum total support. Then f is homogeneous at e^{τ_1}, and $f^{\tau_1} = 2 - \alpha_1$. Since $f_0^{\tau_1} + f_1^{\tau_2} = f_2^{\tau_1} + f_3^{\tau_1} = f^{\tau_1}$, by relabeling (fixing τ_1) we may assume $f_2^{\tau_1} \geq f_0^{\tau_1} \geq 1 - \alpha_1/2 \geq f_1^{\tau_1} \geq f_3^{\tau_1}$. Let $f_0^{\tau_1} = 1 - \alpha_1/2 + \epsilon$ for $\epsilon \geq 0$. See the left of Figure 9.

We try to estimate $\mathcal{P}_{01}$ and $\mathcal{P}_{23}$. Consider τ_2 and τ_3, and their critical neighbors ρ_2 and ρ_3. Since $\mathcal{P}^{\tau_1}$ is homogeneous, we can make f fulfill $f_{02} = f_0^{\tau_1} = 1 - \alpha_1/2 + \epsilon \geq 4/9 + \epsilon$; see the upper central in Figure 9. So $\alpha_2 \in \{8/9, 1, 10/9\}$ (Lemma 3.3 (3)) and ρ_2 is of case (2a), (2b), (2c), (3), or (4). By Lemma 3.4 (2) we have

$$(4.6)\quad f^{*\tau_2} \geq 2 + (d^{\rho_2}(e^{\tau_2}) - 2)f_0^{\tau_1} - \frac{d^{\rho_2}(e^{\tau_2})\alpha_2}{2} = \begin{cases} 7/9 + \epsilon & \text{if } \alpha_1 = \alpha_2 = 10/9, \\ 5/6 + \epsilon & \text{if } \alpha_1 = 1, \alpha_2 = 10/9, \\ 1 & \text{if } \alpha_2 = 1, \\ 10/9 & \text{if } \alpha_1 = 10/9, \alpha_2 = 8/9. \end{cases}$$

Next consider τ_3. We can make f fulfill $f_{12} = f_1^{\tau_1} = 1 - \alpha_1/2 - \epsilon \geq 4/9 - \epsilon$; see the lower central in Figure 9. Then $\alpha_3 \in \{8/9, 1, 10/9\}$ also holds. Indeed, $\alpha_3 = 2/3$ implies $\alpha_1 = 10/9$, $f_0^{\tau_1} = 5/9$, and $f_1^{\tau_1} = 1/3$. So we can rearrange f so that $f_{02} = 5/9$. Thus $\alpha_2 = 10/9$, and f can be regarded as an optimal multiflow for (G^{τ_2}, c^{τ_2}). Then both $\mathcal{P}_{01}^{\tau_2}$ and $\mathcal{P}_{23}^{\tau_2}$ are nonempty, and we can rearrange f so that $f_{12} > 1/3$ ($\mathcal{P}^{\tau_2}$ is homogeneous); a contradiction to $\alpha_3 = 2/3$. Therefore ρ_3 is also of case (2a), (2b), (2c), (3) or (4). Again by Lemma 3.4 (2) we have

$$(4.7)\quad f^{*\tau_3} \geq 2 + (d^{\rho_3}(e^{\tau_3}) - 2)f_1^{\tau_1} - \frac{d^{\rho_3}(e^{\tau_3})\alpha_3}{2} = \begin{cases} 7/9 - \epsilon & \text{if } \alpha_1 = \alpha_3 = 10/9, \\ 5/6 - \epsilon & \text{if } \alpha_1 = 1, \alpha_3 = 10/9, \\ 1 & \text{if } \alpha_3 = 1, \\ 10/9 & \text{if } \alpha_1 = 10/9, \alpha_3 = 8/9. \end{cases}$$

Our analysis is based on (4.6), (4.7), and the following three inequalities:

$$(4.8)\qquad \begin{aligned} f_{01}^{*\tau_2} + f_{01}^{*\tau_3} &\leq \begin{cases} \alpha_1/2 - \epsilon & \text{if } \mathcal{P}_{01}^{*\tau_2} \cap \mathcal{P}_{01}^{*\tau_3} = \emptyset, \\ \alpha_1 - 2\epsilon & \text{otherwise,} \end{cases} \\ f_{23}^{*\tau_2} + f_{23}^{*\tau_3} &\leq \begin{cases} \alpha_1 - 1 + f_3^{\tau_1} & \text{if } \mathcal{P}_{23}^{*\tau_2} \cap \mathcal{P}_{23}^{*\tau_3} = \emptyset, \\ 2(\alpha_1 - 1 + f_3^{\tau_1}) & \text{otherwise,} \end{cases} \\ f_{01}^{*\tau_2} + f_{01}^{*\tau_3} + f_{23}^{*\tau_2} + f_{23}^{*\tau_3} &\geq f^{*\tau_2} + f^{*\tau_3} - 2 + \alpha_1 + 2f_3^{\tau_1}. \end{aligned}$$

The first and second follow from $f_{01}^{*\tau_i} \leq f_{01} \leq 1 - f_0^{\tau_1}$ and $f_{23}^{*\tau_i} \leq f_{23} \leq 1 - f_2^{\tau_1}$, respectively. The third follows from summing $f^{*\tau_2} \leq f_{01}^{*\tau_2} + (f_1^{\tau_1} - f_3^{\tau_1}) + f_{23}^{*\tau_2}$ and $f^{*\tau_3} \leq f_{01}^{*\tau_3} + (f_0^{\tau_1} - f_3^{\tau_1}) + f_{23}^{*\tau_3}$.

We begin with showing:

(4.9) $\quad \rho_2$ or ρ_3 is of case (2c).

Suppose not. In each case $\mathcal{P}^{*\tau_i}$ is homogeneous. We have

$$(4.10)\qquad \mathcal{P}^{*\tau_2} \cap \mathcal{P}_1^{\tau_1} \neq \emptyset.$$

Indeed, $\mathcal{P}^{*\tau_2} \cap \mathcal{P}_1^{\tau_1} = \emptyset$ implies that $f_{01}^{*\tau_2} + f_{23}^{*\tau_2} = f^{*\tau_2}$. By $\max(f_{01}^{*\tau_2}, f_{23}^{*\tau_2}) \leq 1 - f_0^{\tau_1}$, both $\mathcal{P}_{01}^{*\tau_2}$ and $\mathcal{P}_{23}^{*\tau_2}$ have flow-value at least $2/9 + 2\epsilon$ in each case. Then we can rearrange f so that $f_{12} \geq 4/9 - \epsilon + 2/9 + 2\epsilon > 5/9$; see the lower right in Figure 9. A contradiction to $\alpha_3 \leq 10/9$. Similarly,

$$(4.11)\qquad \mathcal{P}^{*\tau_3} \cap \mathcal{P}_0^{\tau_1} \neq \emptyset.$$

Otherwise, both $\mathcal{P}_{01}^{*\tau_3}$ and $\mathcal{P}_{23}^{*\tau_3}$ have flow-value at least $2/9$, and we can rearrange f so that $f_{02} \geq 4/9 + \epsilon + 2/9 > 5/9$; see the upper right in Figure 9.

The next claim is:

(4.12) $\quad$ the case (4.5) (i) holds, and thus $l \not\sim q = \rho_1(y^{\tau_1}) \in Q_p$.

Suppose that (4.5) (ii) holds. $\mathcal{P}_{01}^{*\tau_2}$ is nonempty; otherwise $7/9 \leq f^{*\tau_2} = f_2^{*\tau_2} \leq 1 - f_0^{\tau_1} \leq$

5/9. Since the exchange operation for $\mathcal{P}^{*\tau_2}$ at e_1 works, by (4.10) $\mathcal{P}^{*\tau_2}_{01}$ necessarily has a path with x_1-end in $s\bar{l}$. Consequently $\mathcal{P}_0$ necessarily has a path with y-end in $s\bar{l}$ and a path with x_0-end in $s\bar{l}$. Then the anti-exchange operation for the two paths at e_0 works; a contradiction to the minimality. Next we claim

$$\mathcal{P}^{*\tau_2}_{01} \cap \mathcal{P}^{*\tau_3}_{01} = \emptyset. \tag{4.13}$$

Suppose not. By (4.12) and Lemma 2.10, $\mathcal{P}^{\tau_1}$ consists of paths with y^{τ_1}-end belonging to $\bigcup_{u \sim_1 q} su$. Since both $\mathcal{P}^{*\tau_2} \cap \mathcal{P}^{\tau_1}_1$ and $\mathcal{P}^{*\tau_3} \cap \mathcal{P}^{\tau_1}_0$ are nonempty and homogeneous, by exchange operations at e_1 and at e_0 we can conclude that $\mathcal{P}^{*\tau_2}_{01} \cap \mathcal{P}^{*\tau_3}_{01}$ necessarily has an (su, x, y, x_1, sv)-path for $u \sim_1 q \sim_1 v$. However such a path is never ρ-shortest by Lemma 2.4; a contradiction to optimality.

By (4.8) and (4.13) we have

$$2(\alpha_1 - 1 + f^{\tau_1}_3) + \alpha_1/2 - \epsilon \geq f^{*\tau_2} + f^{*\tau_3} - 2 + \alpha_1 + 2f^{\tau_1}_3 \tag{4.14}$$

Then only the case $\alpha_1 = \alpha_2 = \alpha_3 = 10/9$ is possible; other cases yield LHS $<$ RHS. So suppose $\alpha_1 = \alpha_2 = \alpha_3 = 10/9$. In particular $(\rho_1(y), \rho_1(y^{\tau_1})) = (l, q)$ holds by (4.12). Consider $(\rho_2(y), \rho_2(y^{\tau_2}))$. Then $\rho_2(y^{\tau_2}) = l \in L_p$ or $\rho_2(y) = \bar{l}_1 \in L_p$ with $\bar{l}_1 \sim_1 q$; otherwise $\mathcal{P}^{*\tau_2} \cap \mathcal{P}^{\tau_1}_1 = \emptyset$. If the former case occurs, then $\mathcal{P}^{\tau_2}_{01}$ has a path connecting sl, x_0, y in order, and the anti-exchange operation at e_0 works; a contradiction. So the latter case holds. Similarly $\rho_3(y^{\tau_3}) = \bar{l}_2 \in L_p$ with $\bar{l}_2 \sim_1 q$. Then $\bar{l}_1 \neq \bar{l}_2$ necessarily holds. Otherwise the anti-exchange operation at e_0 works. Consequently $\mathcal{P}^{*\tau_2}_{23} \cap \mathcal{P}^{*\tau_3}_{23} = \emptyset$ also holds. By (4.8), the flow configuration at y is completely determined as $\epsilon = 0$, $f^{\tau_1}_3 = 0$, $f^{\tau_1}_0 = f^{\tau_1}_1 = f_{02} = f_{12} = f^{*\tau_3}_{02} = f^{*\tau_2}_{12} = 4/9$, and $f^{*\tau_2} = f^{*\tau_3} = 7/9$. Since $f^{*\tau_2}_{01} + f^{*\tau_3}_{01} + f^{*\tau_2}_{23} + f^{*\tau_3}_{23} = 6/9$ and $f_{01} \leq 5/9$, we may assume that both $\mathcal{P}^{*\tau_3}_{01}$ and $\mathcal{P}^{*\tau_3}_{23}$ are nonempty. By exchange operation at e^{τ_3} we can make f fulfill $f_{02} > 4/9$. Again, by applying Lemma 3.4 (2) to τ_2, we have $f^{*\tau_2} > 7/9$. A contradiction. Thus we have (4.9) and (4.3) (1).

Suppose that ρ_1 is of case (2c) with e^{τ_1} unmixed in $(G^{\tau_1}, c^{\tau_1}; \rho_1)$. Suppose that ρ_2 is not of case (2c). Then ρ_3 is necessarily of case (2c) by (4.9). Then $f^{*\tau_2} \geq 5/6 + \epsilon$, and (4.10) and (4.12) hold by the same argument. Then $\mathcal{P}^{*\tau_3} \cap \mathcal{P}^{\tau_1}_0$ or $\mathcal{P}^{*\tau_3}_{01} \cap \mathcal{P}^{*\tau_2}_{01}$ must be empty. Suppose not. Any path in $\mathcal{P}^{*\tau_3} \cap \mathcal{P}^{\tau_1}_0$ connects y, x_2, sl in order. This implies $\rho_3(y) \sim_1 l$ (by Lemma 2.10). Since the exchange at $\mathcal{P}^{*\tau_2}_1$ works, $\mathcal{P}^{*\tau_3}_{01} \cap \mathcal{P}^{*\tau_2}_{01}$ has a path with x_1-end belonging to sq or sl' for $l' \in L_p$ with $q \sim_1 l' \not\sim l$. Thus $\rho_3(y) = q$ or $\rho_3(y) \sim_1 l'$. This is a contradiction to $d(l, q) = 3$ or $d(l, l') = 4$. If $\mathcal{P}^{*\tau_3} \cap \mathcal{P}^{\tau_1}_0$ is empty, then by $f_{01} \leq 1/2 - \epsilon$ and $f_{23} < 1/2 - \epsilon$ (by $\mathcal{P}^{*\tau_2} \cap \mathcal{P}^{\tau_1}_1 \neq \emptyset$) it is impossible to fulfill $f^{*\tau_3}_{01} + f^{*\tau_3}_{23} = f^{*\tau_3} \geq 1$. If $\mathcal{P}^{*\tau_3}_{01} \cap \mathcal{P}^{*\tau_2}_{01}$ is empty, then we obtain a contradiction $5/2 \geq 25/9$ in (4.14).

Thus ρ_2 is also of case (2c), and necessarily $f^{\tau_1}_0 = f^{\tau_1}_1 = 1/2$. Then ρ_3 is also of case (2c); we can interchange roles of x_0 and x_1. Thus $\alpha_1 = \alpha_2 = \alpha_3 = 1$. Now f can be regarded as an optimal multiflow for (G^{τ_2}, c^{τ_2}) by exchanging f as $f^{\tau_1}_0 = f_{02} = 1$ and for (G^{τ_3}, c^{τ_3}) by exchanging f as $f^{\tau_1}_1 = f_{12} = 1$.

From this with help of Lemma 2.10 we immediately obtain the potential configuration in (4.3) (2). Indeed, take a path P from $\mathcal{P}_{01}$, and suppose that P is an $(su, x_0, y, x_1, s\bar{u})$-path for $u, \bar{u} \in Q_p \cup L_p$. Regard f as an optimum for (G^{τ_2}, c^{τ_2}) and for (G^{τ_3}, c^{τ_3}). Then P is regarded as an $(su, y^{\tau_2}, y, s\bar{u})$-path and as an $(su, y^{\tau_3}, y, s\bar{u})$-path. By Lemma 2.10, $\rho_2(y^{\tau_2}) \sim_1 u \sim_1 \rho_3(y^{\tau_3})$, and consequently $\rho_2(y^{\tau_2}) \sim \rho_3(y^{\tau_3})$. Also we have $\rho_2(y) \sim \rho_3(y)$. If $f_{23} \neq 0$, then similarly we have $\rho_2(y^{\tau_2}) \sim \rho_3(y)$ and $\rho_3(y^{\tau_3}) \sim \rho_2(y)$, and thus the configuration (2-1). Suppose $f_{23} = 0$. Then $f_{01} = f_{02} = f_{12} = 1/2$. Apply the same argument for a path in $\mathcal{P}_{02}$ and a path in $\mathcal{P}_{12}$. If (4.5) (i) holds, then we obtain the configuration (2-2), and if (4.5) (ii) holds, then we obtain (2-3). We are done.

4.3 SPUP on rings

Now that any inner node in C_ρ is one of the following three types:

(1) an end of e^τ produced in 4/3-SPUP (of cases (1b, 2a)).

(2) a tri-fixed node in produced 1-SPUP.

(3) a node each of whose fork τ satisfies $\alpha(\tau) = 2/3$.

Any terminal in C_ρ is incident to only one node, and is one of the following two types:

(0) s is incident to a terminal t with $\rho(s) = \rho(t)$.

(1) s is incident to an inner node y of type (1) with $\rho(s) = \rho(y)$.

(2) s is incident to a node x with $\rho(s) \neq \rho(x)$.

We can delete terminals of type 0 since $\mu(s,t) = 0$. By subdividing edges and extend ρ as in (2.10) (2), we may assume that

(4.15) there is no edge xy joining nodes $x, y \in C_\rho$ with $\rho(x) \neq \rho(y)$.

Note that all neighbors of an inner node of type (2-3) are in the outside of C_ρ. Set $(G,c) \leftarrow (G, 3c)$. Then $(G,c;\rho)$ is admissible, and $(G, 2c; \rho)$ is restricted Eulerian. Inner nodes in C_ρ of type (3) are splittable by (4.4). So split them off. Keep nodes in C_ρ of type (2). Let C^*_ρ consist of inner nodes in C_ρ of type (1) and terminals of type (1), and let D^*_ρ be the set of inner nodes joined to C^*_ρ by e^τ for a fork τ at the past 4/3-SPUP. Then we clearly have the following (thanks to (4.15)).

(4.16) Each node y in C^*_ρ is incident to at most two nodes not in D^*_ρ, and each terminal s in C^*_ρ is incident to at most one node not in D^*_ρ.

Therefore the subgraph induced by C^*_ρ consists of cycles, paths, and isolated nodes. We try to apply splitting-off and SPUP, keeping the condition (4.16). First we examine splitting-off at forks $\tau = xyz$ for $(x,y) \in D^*_\rho \times C^*_\rho$. If such a fork τ is splittable, then split it off (and simplify multiple edges appeared). By (4.15) this splitting-off keeps the condition (4.16). Repeat it. So suppose that there is no such a splittable fork.

Next consider SPUP at a fork $\tau = xyz$ for $(x,y,z) \in D^*_\rho \times C^*_\rho \times D^*_\rho$. Let $p = \rho(y)$. Suppose $\alpha(\tau) > 0$. Let $f = (\mathcal{P};\kappa)$ be an optimal multiflow for (G^τ, c^τ). Since both xy and yz are saturated, we have $f^{xy^\tau, y^\tau y} > 0$, $f^{zy^\tau, y^\tau y} > 0$, and $f^{xy^\tau, y^\tau z} = \alpha(\tau)/2 > 0$. By (4.2), for some $l \in L_p$, $\mathcal{P}^{xy}$ consists of paths with x-end in sl or consists of paths with y-end in sl. Suppose the latter case. Since $\mathcal{P}^{yz}$ is also homogeneous by (4.2), we can exchange f at yz so that $\mathcal{P}^{e^\tau}$ has a path with y^τ-end in sl. However $\mathcal{P}^{xy^\tau, y^\tau y}$ consists of paths with y-end in sl. By the anti-exchange operation at $\mathcal{P}^{e^\tau}$, we have $f^{e^\tau} < c(e^\tau) - \alpha(\tau)$. A contradiction. Therefore $\mathcal{P}^{xy}$ consists of paths with x-end in sl. Similarly $\mathcal{P}^{yz}$ consists of paths with z-end in sl' for $l' \in L_p$. Necessarily $l \not\sim l'$, i.e., $d(l,l') = 4$. Consider a critical neighbor ρ' with respect to τ. By $f^{xy^\tau, y^\tau y} > 0$, $f^{zy^\tau, y^\tau y} > 0$, and $f^{xy^\tau, y^\tau z} > 0$, we have $d(l, \rho'(y^\tau)) + d(\rho'(y^\tau), l') = d(l,l') = 4$, $d(l, \rho'(y^\tau)) + d(\rho'(y^\tau), \rho'(y)) = d(l, \rho'(y)) \leq 4$, and $d(l', \rho'(y^\tau)) + d(\rho'(y^\tau), \rho'(y)) = d(l', \rho'(y)) \leq 4$; recall (2.10) (1) and (2.8). Then $d^{\rho'}(e^\tau) \in \{3,4\}$ is impossible. $d^{\rho'}(e^\tau) = 1$ implies $\alpha(\tau) \geq 2$, i.e., τ is splittable. So $d^{\rho'}(e^\tau) = 2$. Then $d(l, \rho'(y^\tau)) = d(l', \rho'(y^\tau)) = 2$ necessarily holds. So (i) $\rho'(y^\tau) = \rho(y) = p$ or (ii) $\rho'(y^\tau) = \tilde{l} \in L_p$ with $l \sim \tilde{l} \sim l'$. Further we have $\rho'(y) = l'' \in L_p$ with $l \not\sim l'' \not\sim l'$. Therefore $\mathcal{P}^{e^\tau}$ consists of paths with y-end in sl''. So we can apply 1-SPUP $(G,c;\rho) \leftarrow (G^\tau, c^\tau; \rho')$ for both cases; if (i) occurs, then y^τ

is a tri-fixed node of odd degree. Apply 1-SPUP to all such forks, which keeps $(G, c; \rho)$ admissible and $(G, 2c; \rho)$ restricted Eulerian. Then we have:

(4.17) There is no fork $\tau = xyz$ with $(x, y, z) \in D^*_\rho \times C^*_\rho \times D^*_\rho$ and $\alpha(\tau) > 0$.

Also we have the following:

(4.18) There is no edge xs joining $x \in D^*_\rho$ and a terminal $s \in C^*_\rho$.

Suppose not. s has at most one neighbor z not in D^*_ρ. Since $\mathcal{P}^{xs}$ has no path of end s by (4.2), a fork xsz is splittable or there is a fork $\tau = xsz'$ for $(x, s, z') \in D^*_\rho \times C^*_\rho \times D^*_\rho$ with $\alpha(\tau) > 0$; a contradiction. One more immediate consequence of (4.17) is:

(4.19) For an edge xy with $(x, y) \in D^*_\rho \times C^*_\rho$, there are two neighbors $u, v \notin D^*_\rho$ of y such that $f^{yu} > 0$, $f^{yv} > 0$, and $f^{xy} = f^{yu} + f^{yv}$.

From this, we have:

(4.20) For an edge xy with $(x, y) \in D^*_\rho \times C^*_\rho$, we have $c(xy) = 1$, and, by even degree condition, y is connected to another node z' in D^*_ρ.

Otherwise y has a splittable fork to keep (4.16).

(4.21) For a fork $\tau = xyz$ with $(x, y) \in D^*_\rho \times C^*_\rho, z \notin D^*_\rho$, we have $\alpha(\tau) = 1$.

Suppose $\alpha(\tau) \neq 1$. Take a critical neighbor ρ' and an optimal multiflow f for (G^τ, c^τ). Then $d^{\rho'}(e^\tau) = 3$ or 4. In both cases, f is homogeneous at e^τ. By (4.20), y is connected to $z' \in D^*_\rho$. By (4.19), $f^{z'y, e^\tau} > 0$. Also $f^{zy, e^\tau} > 0$. We can exchange paths at e^τ so that $f^{zy, yz'} > 0$. This contradicts to (4.17). Therefore,

(4.22) for an edge xy with $(x, y) \in D^*_\rho \times C^*_\rho$, there are two neighbors $u, v \notin D^*_\rho$ of y such that $f^{xy,yu} = f^{xy,yv} = 1/2$ and $f^{xy} = 1 = c(xy)$ for any optimal multiflow f.

Furthermore, both yu and yv are necessarily saturated. According to Lemma 3.3 (5), replace ρ to its forward neighbor ρ' with $\rho'(y) \neq \rho'(u)$ or $\rho'(y) \neq \rho'(v)$. Apply this procedure for all (inner) nodes $y \in C^*_\rho$ joined to D^*_ρ.

Now the subgraph induced by C_ρ consists of isolated nodes and paths having no nodes connected to D^*_ρ. Set $(G, c) \leftarrow (G, 2c)$. Then $(G, c; \rho)$ is restricted Eulerian, and each inner node y in C_ρ is splittable since y is a tri-fixed node having three neighbors, had flow configuration in (4.22), or has at most two neighbors. Split them off. For terminals in C_ρ, the following holds:

(4.23) Any terminal s in C_ρ is incident to at most one node y. If y is an inner node, then $\rho(s) \neq \rho(y)$.

So we can delete edge joining terminals $s, t \in C^*_\rho$ with $\rho(s) = \rho(t)$ since $\mu(s, t) = 0$.

4.4 SPUP at M_ρ

Now that there is no inner node in C_ρ, and any terminal in C_ρ is incident to only one node y with $\rho(s) \notin C_\rho$. Make all inner nodes in M_ρ have degree four. Note that there is no terminal s in M_ρ; see the end of Section 4.1.1. Apply the splitting-off at nodes in M_ρ if possible. Suppose that there is no splittable fork at M_ρ. Take arbitrary an inner node $y \in M_\rho$ (of degree four), and take a fork τ at y. Let $p = \rho(y)$ be the midpoint of a leg qq' of $\mathcal{K}$. Then qp and pq' belong to distinct orbits, and belong to no common folder. Consider a critical neighbor ρ' of ρ with respect to τ. If ρ' is backward, then $\{\rho'(y), \rho'(y^\tau)\} = \{q, p\}$ or $\{p, q'\}$. By Lemma 3.8 there exists a fork τ at y such that its critical neighbor ρ' is forward. Then $\alpha(\tau) \in \{0, 1\}$. If $\alpha(\tau) = 1$, then both $\rho'(y)$ and $\rho'(y^\tau)$ belong to S_ρ, and thus forward 1-SPUP $(G, c; \rho) \leftarrow (G^\tau, c^\tau; \rho')$ keeps $(G, c; \rho)$ restricted Eulerian. If $\alpha(\tau) = 0$, then we can replace ρ by an optimal forward neighbor ρ' with $\rho'(y) \neq \rho(y)$ (Remark 3.6). Then M_ρ decreases. Repeat it until $M_\rho = \emptyset$. Note that any terminal s in C_ρ never moves into M_ρ since s is proper. In the process, each edge incident to a terminal s in C_ρ is always saturated by paths of end s in every optimal multiflow. Replace $(G, c; \rho)$ by $(G, 2c; \rho)$. Now that all inner nodes belong to S_ρ. Now that Lemma 3.11 is applicable; any terminal region R_s meet at most two hypotenuses at $p \in V^s$. The current problem has an integral optimal multiflow, and thus the original problem has a 1/12-integral optimal multiflow; we multiplied $3 \times 2 \times 2$ to the edge-capacity so far.

5 Integrality, sparsity, and blow-up

Theorem 3.10 provides a powerful method for proving the existence of an integral optimal multiflow for Eulerian μ-problems. An F-complex $\mathcal{K}$ is said to be *locally sparse* if each vertex is sparse. Motivated by (3.12) in Theorem 3.10, we consider the following terminal condition:

(II) each terminal s whose R_s is neither a path of hypotenuses nor a single vertex has even degree.

Then by Theorem 3.10 and Lemma 3.11 we have the following.

Theorem 5.1. *Suppose that μ has a realization by a locally sparse F-complex $\mathcal{K}$. Then the μ-problem has an integral optimal multiflow for every inner Eulerian graph with terminal condition II in $\mathcal{K}$.*

An F-complex $\mathcal{K}$ without triangles is locally sparse. For each orbit O, the summand $\mathcal{K}^O$ is one leg, and thus μ^O is a cut distance for some disjoint subsets $A, B \subseteq S$. By Proposition 2.9, any optimal multiflow for the μ-problem is a maximum (A, B)-flow. From this, one can derive (a slight extension of) the *multiflow locking theorem* due to Karzanov-Lomonosov [23]; see [19, Section 5]. A folder itself is locally sparse. The leg-graph is a complete bipartite graph $K_{2,r}$. So Theorem 5.1 includes Karzanov-Mannoussakis integrality theorem for $K_{2,r}$-metric weighted maximum multiflow problem [24].

The class of distances admitting a realization by a locally sparse F-complex seems be narrow. Interestingly, although $\mathcal{K}$ is not locally sparse, sometimes *we can represent $\mathcal{K}$ as a summand of locally sparse one*; see $\mathcal{K}^{U^c}$ in Figure 3. By combining Proposition 2.9 we can prove the integrality theorem for $\mathcal{K}$.

Theorem 5.2. *Suppose that μ has a realization by a summand of a locally sparse F-complex $\mathcal{K}^*$. Then the μ-problem has an integral optimal multiflow for every inner Eulerian graph with terminal condition II in $\mathcal{K}^*$.*

A sparse (resp. nonsparse) vertex is an analogue of a *nonsingular* (resp. *singular*) point in an algebraic variety. We call the process constructing an F-complex $\mathcal{K}^*$ having $\mathcal{K}$ as a summand a *blow-up*. Actually a blow-up subdivides the normal fan of the feasible polyhedron in the LP-dual (7.1), and therefore is compatible to the *blow-up* in the theory of toric varieties. We give an illustrative application of Theorem 5.2 below.

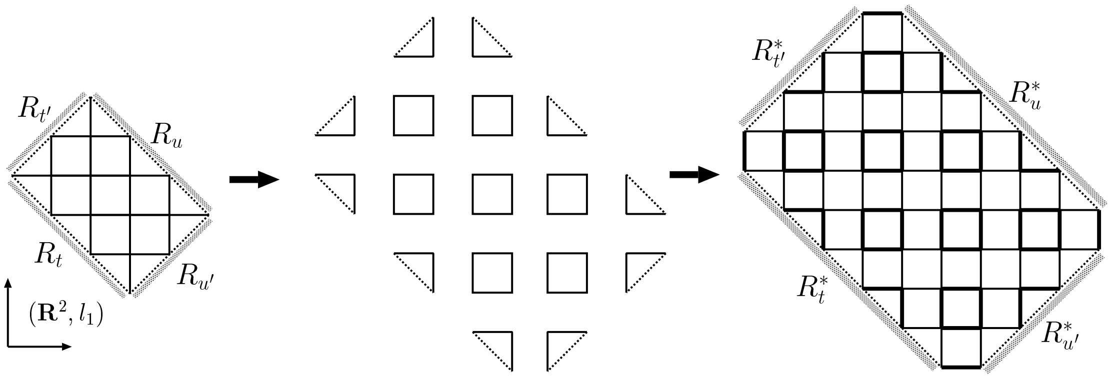


Figure 10: Blowing up 2-commodity F-complex

Multiterminal weighted 2-commodity flows. Suppose that terminal set S is partitioned into four sets $\{T, T', U, U'\}$. For relatively prime positive integers a and b, let μ be a distance on S defined as $(\mu(t,u), \mu(t',u')) = (a,b)$ for $(t,t',u,u') \in T \times T' \times U \times U'$, and other distances are zero. Then the corresponding μ-problem is a *weighted* version of the multiterminal 2-commodity flow maximization. An F-complex realization for μ is constructed as follows. Consider a rectangle in the l_1-plane $\mathbf{R}^2$ with edge parallel to $(1,1)$ or $(1,-1)$, and edge-length given by a and b. Subdivide this rectangle into squares and right triangles along lines parallel to coordinate axes as in Figure 10. Set the leg-length to be $1/2$. The resulting complex $\mathcal{K}$ is clearly an (orientable) F-complex. Set (R_t, R_u) and $(R_{t'}, R_{u'})$ to be diagonal pairs of edges of length b and a, respectively. Then we obtain an F-complex realization of μ. Although $\mathcal{K}$ is not locally sparse, we can blow up $\mathcal{K}$ into a locally sparse one as follows. Delete all legs from $\mathcal{K}$, and insert squares and triangles along deleted legs as in Figure 10. The resulting F-complex $\mathcal{K}^*$ is locally sparse, and has $\mathcal{K}$ as a summand. Each R_s is naturally extended to series of hypotenuses R_s^*. Thus we have:

Theorem 5.3. *The multiterminal weighted 2-commodity flow problem has an integral optimal flow for every inner Eulerian graph.*

Sparse frames and locally sparse frames. Recall the definition of a frame and related concepts in Section 2.1.5. A frame is called *locally sparse* if the corresponding F-complex $\mathcal{K}$ is locally sparse. The concept of sparsity was originally introduced by Karzanov [21] for frames. A frame Γ is called *sparse* if every orbit graph of Γ is K_2 or $K_{2,r}$ for $r \geq 3$. Clearly, if a frame Γ is sparse, then it is also locally sparse, and therefore the integrality theorem holds for a metric represented by as a submetric of a sparse frame.

The blow-up for a frame is intriguing from both multiflow- and graph-theoretical points of the view. However we do not know any characterization for a frame represented by a summand of a locally sparse frame, even for a star-shaped frame.

For star-shaped frames, the sparsity and the local sparsity are same, and can be easily characterized. Recall that a star-shaped frame Γ with center p is obtained by the bipartite graph $\Gamma \setminus p = (Q, L; \sim)$ of girth at least 8.

Lemma 5.4. *Let Γ be a star-shaped frame with the center p and let $\Gamma \setminus p = (Q, L; \sim)$. Then Γ is locally sparse if and only if there is no pair $l, l' \in L$ such that $l \sim l'$ and both l and l' have degree at least 3.*

6 0-1 problems

In this section, we investigate the maximum multiflow problem (1.1) in more detail. Let H be a commodity graph with the property P, and let μ_H be the corresponding 0-1 distance defined by $\mu_H(s,t) = 1 \Leftrightarrow st \in EH$. Although the 1/12-integrality theorem holds for the μ_H-problem by Theorem 1.5, there is an interesting hierarchy of classes admitting the integrality or half-integrality theorem. The aim of this section is to reveal it. In Section 6.1 we associate H with three F-complexes $\mathcal{K}_H$, $\mathcal{K}^s_H$, and $\mathcal{K}^e_H$ such that

(1) $\mathcal{K}_H$ realizes μ_H,

(2) $\mathcal{K}^s_H$ is star-shaped, and

(3) $\mathcal{K}^e_H$ has both $\mathcal{K}_H$ and $\mathcal{K}^s_H$ as summands.

From $\mathcal{K}^s_H$ we define a metric μ^s_H, and from $\mathcal{K}^e_H$ we define a distance μ^e_H having both μ_H and μ^s_H as summands. By Proposition 2.9, any optimal multiflow for μ^e_H-problem is also optimal to both μ_H- and μ^s_H-problems. In Section 6.2 we observe that $\mathcal{K}^e_H$ is locally sparse if and only if H is anticlique-bipartite. As a corollary of Theorem 5.2 we obtain Karzanov-Lomonosov integrality theorem [16, 23, 25]. In Section 6.3 we prove a powerful *fractionality relation* $\mathrm{frac}(H) \leq 2\,\mathrm{frac}(\mu^s_H)$. This reduces the fractionality study of the μ_H-problem to that of the μ^s_H-problem. By the sparsity and the blow-up of $\mathcal{K}^s_H$, we prove the half-integrality theorem (Conjecture 1.2 (2)) for a larger class of commodity graphs.

In this section we assume that commodity graph H has no isolated nodes, i.e., the corresponding distance matrix μ_H has no zero columns. H is said to be *reduced* if distance matrix μ_H has no same columns. A maximal stable set of H is simply called an *anticlique*.

6.1 F-complexes for a commodity graph with the property P

Let $\mathcal{A}$ be the set of all anticliques of H, and let $\mathcal{D}$ be the set of nonempty subsets $D \subseteq S$ represented by the intersection of two distinct anticliques. By property P, we have $D = \bigcap\{A \in \mathcal{A} \mid D \subseteq A\}$. Let $\mathcal{D}_0 \subseteq \mathcal{D}$ be the set of subsets belonging to exactly two anticliques. Let $\mathcal{A}_0 \subseteq \mathcal{A}$ be the set of anticliques A with property $A \cap A' = \emptyset$ for each $A' \in \mathcal{A} \setminus \{A\}$. Then $\mathcal{A}_0 \cup \mathcal{D}$ is a subpartition of S. Let $\mathit{\Pi}_H$ be the bipartite graph with bipartition $(\mathcal{D}, \mathcal{A})$ and edge set $\{DA \mid D \subseteq A\}$. $\mathit{\Pi}_H$ has girth at least 8. Indeed, a 6-cycle corresponds to an intersecting triple of anticliques with distinct intersections; it is impossible by property P.

The first F-complex $\mathcal{K}_H$. Let us construct the first F-complex $\mathcal{K}_H$. To construct an F-complex, we are sufficient to designate the set of vertices and the set of maximal cells with their vertices and edges. The set of vertices of $\mathcal{K}_H$ consists of

$$p^O, \quad p^D \ (D \in \mathcal{D}), \quad p^A, q^A \ (A \in \mathcal{A}).$$

The set of maximal cells consists of

triangles:	$p^D q^A p^A$	$((D,A) \in \mathcal{D} \times \mathcal{A} : D \subseteq A)$,
triangles:	$p^O q^A p^D$	$((D,A) \in \mathcal{D} \setminus \mathcal{D}_0 \times \mathcal{A} : D \subseteq A)$,
squares:	$p^O q^A p^D q^{A'}$	$(D \in \mathcal{D}_0 : D = A \cap A' \ (A, A' \in \mathcal{A}))$,
maximal 1-cells:	$p^O q^A,\ q^A p^A$	$(A \in \mathcal{A}_0)$.

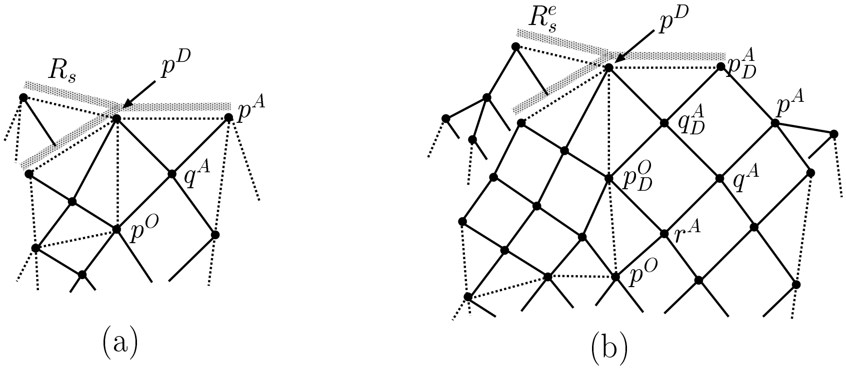


Figure 11: Constructions of (a) $\mathcal{K}_H$ and (b) $\mathcal{K}_H^e$

Here a triangle of vertices p, q, r with right angle q is denoted by pqr, and a square of four edges op, pq, qr, ro is denoted by $opqr$. The leg-length is defined to be 1/4. See Figure 11 (a) for a portion of $\mathcal{K}_H$. This is an F-complex. Indeed, $\mathcal{K}_H$ is contractible and thus simply-connected. It suffices to verify the axiom (1-2) of F-complex for the center p^O. This immediately follows from $\mathit{\Pi}_{p^O} = \mathit{\Pi}_H$, which has girth at least 8. Furthermore, we can orient $\mathcal{K}_H$ so that each p^D is a source, and p^O and each p^A are sinks. For $s \in S$, we define set $R_s \subseteq \mathcal{K}_H$ by

$$R_s = \begin{cases} \bigcup\{p^D p^A \mid A \in \mathcal{A} : D \subseteq A\} & \text{if } s \text{ belongs to } D \in \mathcal{D}, \\ p^A & \text{if } s \text{ belongs to a unique anticlique } A \in \mathcal{A}. \end{cases}$$

Then each R_s is clearly normal; R_s is a star of hypotenuses or a single vertex. One can easily verify $\mu_H(s,t) = d_{\mathcal{K}_H}(R_s, R_t)$. Indeed, $R_s \cap R_t \neq \emptyset \Leftrightarrow s$ and t belong to a common anticlique $\Leftrightarrow \mu_H(s,t) = 0$. Conversely, $R_s \cap R_t = \emptyset$ implies $d_{\mathcal{K}_H}(R_s, R_t) = 1$. Therefore $(\mathcal{K}_H; \{R_s\}_{s \in S})$ is an F-complex realization of μ_H. Note that the combinatorial duality relation from (2.3) coincides with that given in [17].

The second F-complex $\mathcal{K}_H^s$. The second F-complex $\mathcal{K}_H^s$ is the subcomplex of $\mathcal{K}_H$ consisting of cells having p^O and their faces, i.e., $\mathcal{K}_H^s = (\mathcal{K}_H)_{p^O}$. Let $\mathit{\Gamma}_H^s$ be the leg-graph of $\mathcal{K}_H^s$, which is a star-shaped frame. In particular $\mathit{\Gamma}_H^s \setminus p^O = \mathit{\Pi}_H$. For a terminal s, the vertex p_s is defined by

$$p_s = \begin{cases} p^D & \text{if } s \text{ belongs to } D \in \mathcal{D}, \\ q^A & \text{if } s \text{ belongs to a unique anticlique } A \in \mathcal{A}. \end{cases}$$

Then a metric μ_H^s is defined by

$$\mu_H^s(s,t) = d_{\mathcal{K}_H^s}(p_s, p_t) \quad (s, t \in S).$$

By construction, $(\mathcal{K}_H^s; \{p_s\}_{s \in S})$ is an F-complex realization of μ_H^s.

The third F-complex $\mathcal{K}_H^e$. Next we construct the third F-complex $\mathcal{K}_H^e$. The set of vertices of $\mathcal{K}_H^e$ consists of

$$\begin{array}{ll} p^O,\ p^D,\ p^O_D & (D \in \mathcal{D}), \\ r^A,\ q^A,\ p^A & (A \in \mathcal{A}), \\ q^A_D,\ p^A_D & ((D, A) \in \mathcal{D} \times \mathcal{A} : D \subseteq A). \end{array}$$

The set of maximal cells consists of

triangles:	$p^D q^A_D p^A_D$	$((D, A) \in \mathcal{D} \times \mathcal{A} : D \subseteq A)$,
squares:	$p^O_D r^A q^A q^A_D,\ \ q^A_D q^A p^A p^A_D$	$((D, A) \in \mathcal{D} \times \mathcal{A} : D \subseteq A)$,
triangles:	$p^O r^A p^O_D,\ \ p^O_D q^A_D p^D$	$((D, A) \in \mathcal{D} \setminus \mathcal{D}_0 \times \mathcal{A} : D \subseteq A)$,
squares:	$p^O_D q^A_D p^D q^{A'}_D,\ \ p^O r^A p^O_D r^{A'}$	$(D \in \mathcal{D}_0 : D = A \cap A'\ (A, A' \in \mathcal{A}))$,
maximal 1-cells:	$p^O r^A,\ \ r^A q^A,\ \ q^A p^A$	$(A \in \mathcal{A}_0)$.

Obviously $\mathcal{K}^e_H$ is an orientable F-complex; see Figure 11 (b). For a terminal $s \in S$, normal set R^e_s is defined by

$$R^e_s = \begin{cases} \bigcup\{p^D p^A_D \mid A \in \mathcal{A} : D \subseteq A\} & \text{if } s \text{ belongs to } D \in \mathcal{D}, \\ p^A & \text{if } s \text{ belongs to a unique anticlique } A \in \mathcal{A}. \end{cases}$$

Then distance μ^e_H is defined by

$$\mu^e_H(s, t) = d_{\mathcal{K}^e_H}(R^e_s, R^e_t) \quad (s, t \in S).$$

Any edge in $(\mathcal{K}^e_H)_{p^O}$ and any edge in $(\mathcal{K}^e_H)_{p^D}$ belong to distinct orbits. Let U be the union of orbits containing an edge in $(\mathcal{K}^e_H)_{p^O}$, and U^c the complement. Then $(\mathcal{K}^e_H)^U = \mathcal{K}^s_H$ and $(\mathcal{K}^e_H)^{U^c} = \mathcal{K}_H$. Also $(R^e_s)^U = \{p_s\}$ and $(R^e_s)^{U^c} = R_s$. Then Proposition 2.9 implies the following.

Theorem 6.1. *Both μ_H and μ^s_H are summands of μ^e_H, and thus any optimal multiflow for μ^e_H-problem is also optimal to both μ_H- and μ^s_H-problems.*

This locking property was proved by Lomonosov [25] for special commodity graphs.

Complement-triangle-free commodity graphs. A commodity graph H is called *complement-triangle-free* if the complement $\overline{H}$ has no triangle K_3 (girth at least 4). A complement-triangle-free commodity graph H has property P since every anticlique has cardinality at most 2. In addition, if H is reduced, then the construction of $\mathcal{K}^s_H$ is very simple; $\mathit{\Pi}_H (= \mathit{\Gamma}^s_H \setminus p^O)$ is the subdivision of the complement $\overline{H}$. Figure 12 illustrates three F-complexes $\mathcal{K}_H$, $\mathcal{K}^s_H$, and $\mathcal{K}^e_H$ for a reduced complement-triangle-free commodity graph H.

6.2 Anticlique-bipartite commodity graphs

A commodity graph H is called *anticlique-bipartite* if

(1) for every triple $A, B, C \in \mathcal{A}$, at least one of $A \cap B$, $B \cap C$, $C \cap A$ is empty, and

(2) the intersection graph of $\mathcal{A}$ is bipartite.

By (1) an anticlique-bipartite commodity graph has property P. In this case, by (1), $(\mathcal{K}^e_H)_{p^O} (= \mathcal{K}^s_H)$ has no triangle, and p^O is sparse in $\mathcal{K}^e_H$. Moreover each p^D is also sparse since $q^A_D p^A_D$ and $q^A_D p^O_D$ belong to distinct orbits by (1) and (2). One can verify it by tracing the orbit started from $q^A_D p^A_D$. By bipartiteness (2) this orbit never meets $q^A_D p^O_D$. Consequently q^A_D and q^A are also sparse.

Theorem 6.2. *H is anticlique-bipartite if and only if $\mathcal{K}^e_H$ is locally sparse.*

Each R_s is a path of hypotenuses or a single vertex. As a corollary of Theorem 5.1, we have the following fundamental result in the literature.

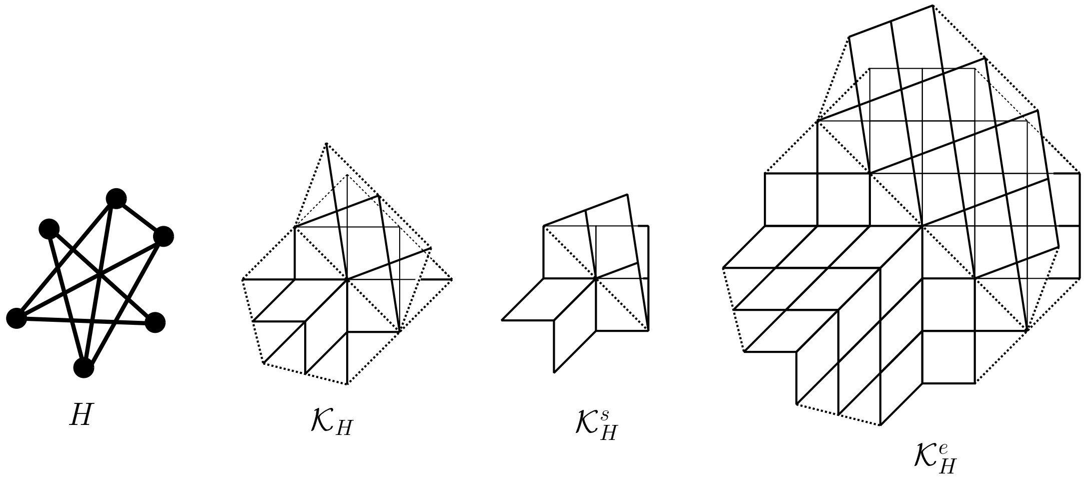


Figure 12: Three F-complexes

Theorem 6.3 ([16, 23, 25])**.** *If H is anticlique-bipartite, then the μ_H-problem has an integral optimal multiflow for every inner Eulerian graph.*

Frank, Karzanov, and Sebö [9] gave a polymatroidal proof of this result. Relation between their approach and our approach is not clear.

A commodity graph H is called *anticlique-nonbipartite* if it satisfies (1) and violates (2). Also in this case, p^O is sparse in $\mathcal{K}^e_H$. However there is $D \in \mathcal{D}$ such that p^D is not sparse. In fact, it is known that the integrality theorem fails for anticlique-nonbipartite commodity graphs. So one might interpret it as: *this failure of the integrality theorem comes from the singularity at p^D.*

6.3 Fractionality relation and its consequences

If H is not anticlique-bipartite, then $\mathcal{K}^e_H$ has a nonsparse vertex p^D that cannot not be blown up. Sometimes the center vertex p^O is sparse, or equivalently, star-shaped frame $\mathit{\Gamma}^s_H$ is sparse, and hence the μ^s_H-problem has an integral optimal multiflow. In this case, the μ_H-problem is guaranteed to have a half-integral optimal multiflow by the following *fractionality relation* between μ_H and μ^s_H. This means that the singularity at p^D is relatively *tame.*

Theorem 6.4. *Let H be a commodity graph with property P. Suppose that the μ^s_H-problem has a $1/k$-integral optimal multiflow for every inner Eulerian graph.*

(1) *If k is even, then the μ_H-problem has a $1/k$-integral optimal multiflow for every inner Eulerian graph.*

(2) *If k is odd, then the μ_H-problem has a $1/(2k)$-integral optimal multiflow for every inner Eulerian graph.*

In particular $\mathrm{frac}(H) \leq 2\,\mathrm{frac}(\mu^s_H)$ *holds.*

Before the proof, we describe several consequences. Consider $H = K_2 + K_r$, i.e., the vertex-disjoint union of one edge and complete graph K_r ($r \geq 3$), which is reduced complement-triangle-free. Then $\mathit{\Pi}_H$ is the subdivision of $K_{2,r}$, and μ^s_H is the graph metric of $K_{2,r}$. From Karzanov-Mannoussakis integrality theorem for $K_{2,r}$-metric weighted maximum multiflow problem [24], we obtain the following.

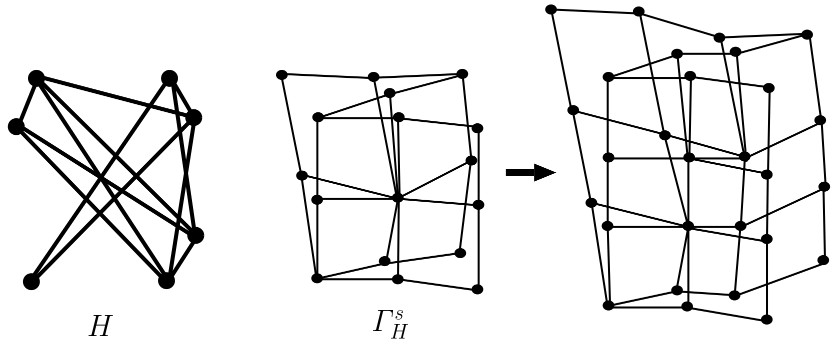


Figure 13: A sparsible commodity graph H, star-shaped frame Γ_H^s, and a blow-up

Theorem 6.5 ([21] for $r = 3$ and [26] for $r > 3$)**.** *If $H = K_2 + K_r$, then the μ_H-problem has a half-integral optimal multiflow for every inner Eulerian graph.*

We can extend it for a more general commodity graph. A commodity graph H with property P is called *sparse* if star-shaped frame Γ_H^s is (locally) sparse. By Theorems 5.1 and 6.4, we have the following.

Theorem 6.6. *If H is sparse, the μ_H-problem has a half-integral optimal multiflow for every inner Eulerian graph.*

The half-integrality theorem for anticlique-nonbipartite commodity graphs by Karzanov-Lomonosov [16, 23, 25] also follows from this theorem. A sparse commodity graph can be easily characterized according to Lemma 5.4.

Proposition 6.7. *A commodity graph H with property P is sparse if and only if H has no five anticliques A_1, A_2, B, C_1, C_2 with $\emptyset \neq A_1 \cap A_2 = A_2 \cap B = A_3 \cap B \neq C_1 \cap C_2 = C_2 \cap B = C_3 \cap B \neq \emptyset$.*

Again Theorem 5.2 widens the class admitting the half-integrality theorem. A commodity graph H with property P is called *sparsible* if Γ_H^s is a summand of locally sparse frame.

Theorem 6.8. *If H is sparsible, then the μ_H-problem has a half-integral optimal multiflow for every inner Eulerian graph.*

However we do not know any nice characterization of a sparsible commodity graph, or equivalently, a star-shaped frame represented by a summand of a locally sparse frame. We give one example together with a blow-up in Figure 13.

A commodity graph H with property P is called *integral* if μ_H^s-problem has an integral optimal multiflow for every inner Eulerian graph. Then the following inclusion holds:

$$\text{sparse} \subset \text{sparsible} \subseteq \text{integral} \subset \text{property P}.$$

The vertex-disjoint union of two triangles $H_{3,3} := K_3 + K_3$ is a typical nonintegral example (the fractionality is unknown). So $H_{3,3}$ is not sparsible. One can directly see it from $\Pi_{H_{3,3}}$, which is the subdivision of $K_{3,3}$. A natural question is:

$$\textit{sparsible} = \textit{integral ?}$$

For example, consider commodity graph $H_{3,3}^+ := H_{3,3}$ plus one edge. Its complement is $K_{3,3}^- := K_{3,3}$ minus one edge, and $\Pi_{H_{3,3}^+}$ is the subdivision of $K_{3,3}^-$. One can verify that $H_{3,3}^+$ is not sparsible. However we do not know whether $H_{3,3}^+$ is integral.

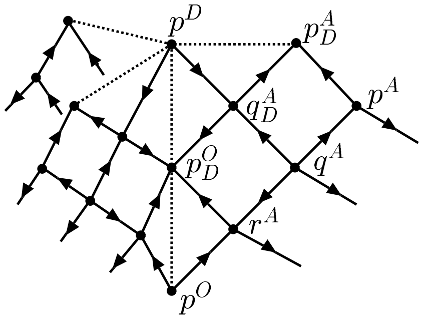


Figure 14: Forward orientation

Finally, we shall rephrase these results by using notion of the fractionality frac(H). The commodity graphs of fractionality 1 or 2 have already been classified as follows:

(1) $\mathrm{frac}(H) = 1$ if and only if H is a complete bipartite graph.

(2) $\mathrm{frac}(H) = 2$ if and only if H is $K_2 + K_3$ or anticlique-bipartite (not complete bipartite).

See [17, 22]. Other commodity graphs are known to have fractionality at least 4.

Corollary 6.9. *A sparsible commodity graph being neither anticlique-bipartite nor $K_2 + K_3$ has fractionality* 4.

Proof of Theorem 6.4. We begin with μ_H^{e}-problem and reduce it to μ_H^{s}-problem; recall the locking property (Theorem 6.1). Let (G, c) be an inner Eulerian graph with terminal set S. Since there is no improper terminal, the terminal condition I is fulfilled. Let us construct the SPUP scheme for $\mathcal{K}_H^{\mathrm{e}}$. One can easily see that p_D^A and p_D^O are sparse. So we let V^s consist of p_D^O and p_D^A for all $D \in \mathcal{D}$ and $(D, A) \in \mathcal{D} \times \mathcal{A}$ with $D \subseteq A$. The forward orientation is a unique orientation so that p^O is a source and each p^D is a source; see Figure 14. Let ρ be an optimal potential, and let $S_\rho = \{x \in VG \mid \rho(x) \in V^s\}$. Recall the restricted Eulerian condition. We keep it during the proof. By the degree reduction, we make (G, c) so that each inner node has degree four and each terminal has degree one. We may assume that there exists no splittable fork. Note that by sparsity there is no inner node in $\rho^{-1}(\{p^A, q^A, r^A\})$ for $A \in \mathcal{A}_0$.

First we try to repeat the forward 1-SPUP until $\rho^{-1}(p^D)$ is empty for all $D \in \mathcal{D}$. Take $D \in \mathcal{D}$ with $\rho^{-1}(p^D) \neq \emptyset$. We may assume that there is an edge xy with exactly one end y having potential $p^D = \rho(y)$. Consider the gate $g\rho(x)$ of $\rho(x)$ in $(\mathcal{K}_H^{\mathrm{e}})_{p^D}$. Then there are three cases below:

(0) $g\rho(x) = p_D^A$ for $A \in \mathcal{A}$ with $D \subseteq A$.

(1) $g\rho(x) = q_D^A$ for $A \in \mathcal{A}$ with $D \subseteq A$.

(2) $g\rho(x) = p_D^O$.

Consider case (1). Suppose $\rho(x) = q_D^A$. Then x is an inner node of degree four. Moreover $p^D q_D^A$ and $q_D^A q^A$ belong to distinct orbits, and belong to no common folder. So a critical backward neighbor ρ' for a fork τ at x satisfies $\{\rho'(y^\tau), \rho'(y)\} = \{q_D^A, p_D^A\}$ or $\{q_D^A, p_D^O\}$. Therefore Lemma 3.8 is applicable. Then there is a fork τ at x such that its critical neighbor ρ' is forward. If $\alpha(\tau) = 1$, then $\{\rho'(y^\tau), \rho'(y)\} = \{p_D^A, p_D^O\}$, and thus 1-SPUP at τ succeeds. If $\alpha(\tau) = 0$, then we can replace ρ by its optimal forward neighbor ρ' with

$\rho'(x) \neq q_D^A$ (Remark 3.6). Therefore we can decrease the edges of type (1). So suppose $\rho(x) \neq q_D^A$. By subdividing xy according to (2.10) (2), we may assume that $\rho(x) = q^A$. Here we use the terminal creation II in Section 2.2.5; q_D^A is flat. Subdivide xy into xz and zy. Add two new terminals s, t joined to z. Set $R_s = \{p_D^A\}$ and $R_t = \{p_D^O\}$. Extend potential ρ for the new problem by defining $(\rho(s), \rho(z), \rho(t)) = (p_D^A, q_D^A, p_D^O)$. Next consider splitting-off at fork $\tau = yzs$ or yzt. We may assume $\alpha(\tau) > 0$ for $\tau = yzs$ or yzt; by reversing the argument in Section 2.2.5 we can make any optimal multiflow f fulfill $f^{yz,zs} > 0$ or $f^{yz,zt} > 0$. Here we suppose $\tau = yzs$; the argument for yzt is similar. Then $\alpha(\tau) \in \{1, 2\}$ by the orbit structure around q_D^A. Suppose $\alpha(\tau) = 2$. Then split it off. Thus edges of type (1) decrease. Suppose $\alpha(\tau) = 1$. Take a critical neighbor ρ' of ρ with respect to τ. Then ρ' is forward. Otherwise we have $(\rho'(z^\tau), \rho'(z)) \in \{(p^D, q_D^A), (q_D^A, p^D), (q_D^A, q^A), (q^A, q_D^A)\}$. Consider an optimal multiflow f for (G^τ, c^τ). All four edges incident to z are saturated, and thus $f^{sz^\tau, z^\tau z} > 0$ and $f^{tz, zz^\tau} > 0$. Recall (2.10) (1). Since $(\rho'(s), \rho'(t)) = (p_D^A, p_D^O)$, it is impossible to satisfy $d^{\rho'}(s, z) = d^{\rho'}(s, z^\tau) + d^{\rho'}(z^\tau, z)$ and $d^{\rho'}(t, z^\tau) = d^{\rho'}(t, z) + d^{\rho'}(z, z^\tau)$. Thus ρ' is necessarily forward, and 1-SPUP at τ succeeds. Then we can decrease edges of type (1).

Suppose case (2). By subdividing xy, we may assume $\rho(x) = p_D^O$. If y is a terminal (of degree one), then replace $\rho(y)$ by p_D^A, that keeps optimality. So suppose that y is an inner node. Take a fork τ at y with $\alpha(\tau) > 0$, and consider a critical neighbor ρ' of ρ with respect to τ. Then ρ' is necessarily forward. So $\rho'(x) = p_D^O$. For every optimal multiflow f, we have $f^{xy^\tau, y^\tau y} > 0$ by $c(xy) = f^{xy}$. By (2.10) (1), we have $d^{\rho'}(x, y) = d^{\rho'}(x, y^\tau) + d^{\rho'}(y^\tau, y)$. Therefore the possible configurations are (i) $(\rho'(y^\tau), \rho'(y)) = (p_D^O, p^D)$ and (ii) $(\rho'(y^\tau), \rho'(y)) = (p_D^O, p_D^A)$ for some $A \in \mathcal{A}$ with $D \subseteq A$. By Lemma 3.9, we can take a fork τ at y with case (ii). Then $\alpha(\tau) = 1$, and both $\rho'(y^\tau)$ and $\rho'(y)$ fall into S_ρ. Thus 1-SPUP is successful, and update $(G, c; \rho) \leftarrow (G^\tau, c^\tau; \rho')$.

Repeat this procedure. Then there are no edges of types (1-2), and thus there is no flow connecting terminal $s \in \rho^{-1}(p^D)$ (Lemma 2.4 (1)). Delete all terminals in $\rho^{-1}(p^D)$ and edges connecting them, and replace $\rho(x)$ for all $x \in \rho^{-1}(p^D)$ by p_D^O. Then the resulting ρ is also optimal, $(G, c; \rho)$ is restricted Eulerian, and $\rho^{-1}(p^D)$ is empty. Apply this procedure for each $D \in \mathcal{D}$ with $\rho^{-1}(p^D) \neq \emptyset$. Then $\rho^{-1}(p^D) = \emptyset$ for each $D \in \mathcal{D}$.

Apply forward SPUP at each inner node y with $\rho(y) = r^A$, p^A, or q_D^A, according to Lemma 3.8. Then there is no such an inner node.

Consider an edge $e = xy$ connecting $x \in S_\rho$ and inner node $y \in VG \setminus S_\rho$. Then $\rho(x) = p_D^O$ or p_D^A, and $\rho(y) = p^O$ or $q^{A'}$. Then there are a folder F and a nonadjacent pair p, p' of its vertices such that $d(\rho(x), \rho(y)) = d(\rho(x), p) + d(p, p') + d(p', \rho(y))$. Here we apply terminal creation I at (p, p'). We can take such a pair (p, p') as (p_D^A, q^A), (p_D^O, q^A), or (p^O, p_D^O). Apply this procedure for all such edges. Then the supply graph G is separated into two disjoint graphs G_0 and G_1. Here G_1 consists of edges joined to an inner node having potential p^O or q^A for some $A \in \mathcal{A}$, and G_0 consists of other edges. So we may consider the problems for G_0 and for G_1 separately. All inner nodes of G_0 have sparse vertices, and each terminal s with $\rho(s) = p^A, p^O$, or q^A is incident to one node x with $\rho(x) \neq \rho(s)$. By Lemma 3.11, (G_0, c) has a half-integral optimal multiflow. Therefore if (G_1, c) has a $1/k$-integral optimal multiflow, then the original graph has a $1/k$-integral multiflow if k is even and a $1/(2k)$-integral multiflow if k is odd. Now terminal region R_s for $s \in S \cap VG_1$ is p_D^O, p_D^A, or p^A. Therefore we can delete all cells containing p^D from $\mathcal{K}_H^e$ for all $D \in \mathcal{D}$. The the resulting F-complex $(\mathcal{K}_H^e)'$ together with $\{R_s\}_{s \in S \cap VG_0}$ is also a realization of μ restricted to $S \cap VG_1$. Therefore we may consider the problem on $(\mathcal{K}_H^e)'$. Then $p^A q^A$ and $q^A r^A$ belong to distinct orbits in $(\mathcal{K}_H^e)'$. Now that q_D^A is also sparse. So add each q_D^A to V^s. Apply forward SPUP at each inner node having potential q^A or r^A according to Lemma 3.8, until there is no such an inner node.

Take an edge $e = xy$ whose exactly one end x is an inner node having potential p^O. Then potential of the other end x is p^O_D, q^A_D, p^A_D, or p^A. For the first three cases, again apply the terminal creation I for (p^O, p^O_D). If $\rho(x) = p^A$, then x is necessarily a terminal s with $R_s = \{p^A\}$, and replace R_s by $\{r^A\}$ and $\rho(s)$ by r^A; this does not change the problem. Again the graph G_1 is separated into two disjoint graphs G'_1 and G''_1. Here G'_1 consists of edge joined to an inner node having a potential p^O, and G''_1 consists of other edges. By the same argument as above, G''_1 has a half-integral optimal multiflow. For the problem for G'_1, again we can delete from $(\mathcal{K}^e_H)'$ all cells not belonging to $(\mathcal{K}^e_H)_{p^O}$. Then the problem for G'_1 is (essentially) the μ^s_H-problem for H, and has a $1/k$-integral optimal multiflow by the assumption. We are done.

7 Algorithmic consequences

Our proof is constructive. Each step searches all nodes and forks of required properties, and applies splitting-off or SPUP to decrease the number of nodes in question. Once the problem becomes trivial to have an integral optimum, we obtain a $1/k$-integral optimum for the original problem by reversing the operations. The aim of this section is to show that our proof indeed yields a (strongly) polynomial time algorithm under a certain assumption.

We begin to study the complexity of an F-complex realization for integral weight μ. The size of realization $\mathcal{K}$ (the number of maximal cells) is already not polynomially bounded by $\{\lceil \log_2(\mu(st) + 1) \rceil\}_{s,t \in S}$; see the 2-commodity F-complex in Figure 10. For 0-1 cases, we gave explicit constructions for $\mathcal{K}_H$, $\mathcal{K}^s_H$, and $\mathcal{K}^e_H$ in Section 6. Their sizes are $O(|S|^2)$. Indeed, $\mathcal{D} \cup \mathcal{A}_0$ is disjoint, and $\{A \setminus D \mid D \subseteq A \in \mathcal{A}\}$ is also disjoint (by property P). Thus $|\mathcal{D}| + |\mathcal{A}_0| = O(|S|)$ and each $D \in \mathcal{D}$ is incident to at most $|S|$ anticliques. However we do not know whether a general distance μ of $\dim T_\mu \leq 2$ has a realization of pseudo polynomial size. Also there is a possibility to handle (or solve) the dual problem (2.3) without an explicit realization. We leave these problems as future research topics. So our analysis here mainly assumes that *an F-complex realization $\mathcal{K}$ of μ is given and its size is fixed.*

Let us consider the complexity of computing an optimal potential, the splitting capacity, and a critical neighbor. The μ-problem (1.3) is a linear program, and its LP-dual is given by

$$
\begin{array}{lll}
(7.1) & \text{Minimize} & \displaystyle\sum_{e \in EG} c(e)d(e) \\
& \text{subject to} & d\text{: metric on } VG, \\
& & d(s,t) \geq \mu(s,t) \quad (s,t \in S).
\end{array}
$$

A feasible solution d is said to be *minimal* if there is no other feasible solution d' with $d' \leq d$. By adding dummy terminals we can make $\{R_s\}_{s \in S}$ include all singletons $\{p\}$ ($p \in V\Gamma$). Then, for a potential ρ, metric d^ρ is a minimal extreme solution, and conversely every minimal extreme solution d is uniquely represented by $d = d^\rho$ for a potential ρ [13, p. 21]. Since the constraint matrix of LP (7.1) consists of 0-1 entries, we can find a minimal extreme optimal solution d^* in strongly polynomial time by the method of Tardos [29] (that uses a polynomial time linear programming algorithm, the ellipsoid or the interior point method, as a subroutine). From d^*, an optimal potential ρ^* is given by $\rho^*(x) = p$ with $d^*(x,s) = 0$ and $R_s = \{p\}$. The splitting capacity α can be computed by the bisection method (without realization $\mathcal{K}$) in weakly polynomial time; solving (7.1) in $\lceil \log_2 c(e^\tau) \rceil$ times. There is another way. Let $h(\alpha) := \mathrm{opt}(G^\tau, c - \alpha\chi_{e^\tau}) - \mathrm{opt}(G, c)$. Then

$h(\alpha)$ is a monotone nonincreasing piecewise linear concave function. The gradient of h at α is given by $d^{\rho}(e^{\tau})$ for an optimal potential ρ of $(G^{\tau}, c - \alpha\chi_{e^{\tau}})$. So the possible values of the gradients are $0, 1, 2, \ldots, \operatorname{diam} \Gamma$, where $\operatorname{diam} \Gamma$ denotes the diameter of Γ. Therefore we can determine α by solving (7.1) at most $\operatorname{diam} \Gamma$ time. Next consider a critical neighbor. Suppose that we are given an optimal potential ρ. Then d^{ρ} is an optimal minimal extreme solution. By Theorem 2.8, every minimal extreme solution d' adjacent to d^{ρ} is necessarily represented by $d' = d^{\rho'}$ for a neighbor ρ' of ρ. Therefore it suffices to consider minimal extreme points adjacent to d^*. For the purpose, we *cut* d^* from the feasible region by the following way. Define capacity c^* (in the complete graph on VG) by $c^*(xy) = 1$ if $d^*(x, y) = 0$ and $c^*(xy) = 0$ otherwise. Then d^* is a *unique* minimal optimal solution with respect to c^*. Then $\operatorname{opt}(G, c^*) = \sum c^*(xy) d^*(x, y) = 0$. Add constraint $\sum c^*(xy) d(x, y) \leq 1/2$ to LP (7.1), and solve this modified LP for $(G^{\tau}, c - (\alpha(\tau) + \epsilon)\chi_{e^{\tau}})$ for small $\epsilon > 0$. The resulting minimal optimal solution $\tilde{d}$ lies on the edge between d^* and its adjacent minimal extreme point d'. From $d'(xy) - d^*(xy) \in \{0, 1, 2, 3, 4\}$, we can determine d' and a critical neighbor ρ'. An optimal neighbor in Lemma 3.3 (5) is also computed by the same manner. Thus we can find a critical neighbor in strongly polynomial time, provided the size of $\mathcal{K}$ is fixed. Consequently, a naive adaptation of our proof gives a pseudo polynomial time algorithm to find a $1/12$-integral optimal multiflow, provided the size of $\mathcal{K}$ is fixed.

To get a polynomial time algorithm, we need to reduce the sum of capacities before making the input supply graph have degree at most four. This is naturally achieved by conducting the splitting-off before the degree reduction. We work on the complete graph G. Let $n = |VG|$. Here, by the splitting-off at a fork $\tau = xyz$ with $x \neq z$ we mean updating c by $c(xy) := c(xy) - \beta(\tau)$, $c(yz) := c(yz) - \beta(\tau)$, and $c(xz) := c(xz) + \beta(\tau)$ for $\beta(\tau) := \min\{c(xy), c(yz), \lfloor \alpha(\tau)/2 \rfloor\}$. We also consider the splitting-off for degenerate fork xyx, which decreases edge-capacity $c(xy)$ by an *even* integer as much as possible. The maximum possible value is also determined by the same manner. By repeating splitting-off appropriately we can make (G, c) fulfill $\beta(\tau) = 0$ for every fork τ, in strongly polynomial time. This is not so obvious. We sketch to prove this fact based on [16, Section 4]. We examine the splitting-off for all forks in some ordering. Fix an arbitrary ordering of nodes. According to this node ordering, choose a node one-by-one, and apply splitting-off to all forks at the chosen node in any order. After the procedure (of $O(n^3)$ steps), $\beta(\tau) = 0$ holds for every fork. We prove it by induction. Suppose now that fork xyz is processed. Obviously each fork τ at y processed before xyz satisfies $\beta(\tau) = 0$. Suppose $\beta(x'y'z') \geq 1$ for a fork $x'y'z'$ at a previously processed node y'. By the same argument in [16, p. 98] we may assume that $x' = x$ and $y' = z$. Apply (again) the splitting-off at $x'y'z'$, and consider an optimal multiflow for the resulting graph. By reversing splitting-off operations at $x'y'z'$ and at xyz we can make this optimal multiflow have paths (with unsaturation) of flow-value at least 1 passing through x, y, y', z' in order, which implies $\beta(yy'z') \geq 1$ before xyz is processed. However, by induction, $\beta(yy'z') = 0$ holds at that time. This is a contradiction.

At this moment, if the existence of an integral optimal solution is guaranteed, then the set of all one-edge paths (s, st, t) with flow-value $c(st)$ is obviously optimal. Thus Theorem 5.2 implies the following.

Theorem 7.1. *Let μ be a distance realized by a summand of a locally sparse F-complex $\mathcal{K}^*$. Suppose that an F-complex realization of μ is given and its size is fixed. There exists a strongly polynomial time algorithm to find an integral optimal multiflow in the μ-problem for every inner Eulerian graph with terminal condition II in $\mathcal{K}^*$.*

If we adopt the bisection method for computing α, then this algorithm is weakly polynomial without realization.

Let us return back to the analysis. After the splitting-off procedure above,

(7.2) each inner node has degree $O(n^2)$.

Indeed, consider an optimal multiflow f. For each inner node y, the following obvious equation holds:

$$\sum_{x \in VG \setminus y} c(xy) = \sum_{x \in VG \setminus y} (c(xy) - f^{xy}) + 2 \sum_{x,z \in VG \setminus y : x \neq z} f^{xy,yz}.$$

Then $f^{xy,yz} \leq 1$; otherwise a fork xyz is splittable (Lemma 3.3 (1,3)). Also $c(xy) - f^{xy} \leq 2$; otherwise degenerate fork xyx is splittable. Thus the degree of y is at most $2(n-1) + 2\binom{n-1}{2} = O(n^2)$. A terminal may have a large degree. We can reduce it by the following way. Compute an optimal multiflow $f = (\mathcal{P}, \lambda)$ by solving LP; we use a compact representation for multiflows. Here we assume that $\mathcal{P}$ has no duplication. For each terminal pair (s,t), check the flow-value $\lambda(P)$ of the one-edge path $P = (s, st, t)$, and decrease the edge capacity $c(st)$ by the maximum even integer l_{st} not exceeding $\lambda(P)$. For any optimal multiflow of the resulting problem, we recover an optimum for the original problem by adding the path $P = (s, st, t)$ of flow-value l_{st}. Apply this procedure for all terminal pairs. Then

(7.3) each terminal s has degree $O(n^2)$.

Indeed, consider the optimal multiflow $\tilde{f} = (\tilde{\mathcal{P}}; \kappa)$ obtained by deleting each one-edge path (s, st, t) of flow-value l_{st} from the above-computed optimal multiflow. Then we have

$$\begin{aligned}
&\sum_{x \in VG \setminus s} c(sx) = \sum_{x \in VG \setminus s} (c(sx) - \tilde{f}^{sx}) + 2 \sum_{x,y \in VG \setminus s : x \neq y} \tilde{f}^{xs,sy} \\
&+ \sum_{x,y \in VG \setminus s : x \neq y} |\{P \in \tilde{\mathcal{P}}^{sx,xy} \mid P \text{ connects } s\}|/\kappa + \sum_{t \in S \setminus s} |\{P \in \tilde{\mathcal{P}}^{st} \mid P = (s, st, t)\}|/\kappa \\
&\leq 2(n-1) + 2\binom{n-1}{2} + (n-1)(n-2) + 2(|S|-1).
\end{aligned}$$

Apply the degree reduction in Section 2.2.4. Then we obtain a supply graph of degree at most four and $O(n^5)$ vertices, and we can directly apply the proof in Section 4.

Theorem 7.2. *Let μ be a distance with $\dim T_\mu \leq 2$. Suppose that an F-complex realization of μ is given and its size is fixed. Then there exists a strongly polynomial time algorithm to find a 1/12-integral optimal multiflow in the μ-problem for every inner Eulerian graph with terminal condition I.*

Corollary 7.3. *Let H be a commodity graph with property P. Then there exists a strongly polynomial time algorithm to find a 1/12-integral optimal multiflow in the μ_H-problem for every inner Eulerian graph.*

Our reduction in Theorem 6.4 can also be applied in the straightforward manner.

Theorem 7.4. *Suppose that a commodity graph H is sparse, sparsible, or integral. Then there exists a strongly polynomial time algorithm to find a half-integral optimal multiflow in the μ_H-problem for every inner Eulerian graph.*

There remain many algorithmic problems. Although we can easily check whether H is sparse by Proposition 6.7, we do not know how to check whether H is sparsible (or integral). More generally we do not know how to check whether an F-complex or a frame is a summand of a locally sparse one. Also it is a challenge to design a *combinatorial* polynomial time algorithm finding a $1/k$-integral optimal multiflow $k \leq 12$.

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