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By

Hiroshi HIRAI

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京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

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Hiroshi HIRAI
Research Institute for Mathematical Sciences,
Kyoto University, Kyoto 606-8502, Japan
hirai@kurims.kyoto-u.ac.jp

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Abstract

This note addresses the undirected multiflow (multicommodity flow) theory. A multiflow in a network with terminal set T can be regarded as a single commodity $(A, T \setminus A)$-flow for any nonempty proper subset $A \subseteq T$ by ignoring flows not connecting A and $T \setminus A$. A set system $\mathcal{A}$ on T is said to be *lockable* if for every network having T as terminal set there exists a multiflow being simultaneously a maximum $(A, T \setminus A)$-flow for every $A \in \mathcal{A}$. The *multiflow locking theorem*, due to Karzanov and Lomonosov, says that $\mathcal{A}$ is lockable if and only if it is *3-cross-free*.

A multiflow can also be regarded as a single commodity (A, B)-flow for every partial cut (A, B) of terminals, where a *partial cut* is a pair of disjoint subsets (not necessarily a bipartition). Based on this observation, we study the locking property for partial cuts, and prove an analogous characterization for a lockable family of partial cuts.

1 Introduction

By a network (G, T, c) we mean a tuple of an undirected graph $G = (VG, EG)$, a specified node subset $T \subseteq VG$, and a nonnegative edge-capacity $c : EG \to \mathbf{R}_+$. A node in T is called a *terminal*, and a node in $VG \setminus T$ is called an *inner node*. A *multiflow* (*multicommodity flow*) is a pair $f = (\mathcal{P}, \lambda)$ of paths $\mathcal{P}$ connecting distinct terminals and a nonnegative flow-value function $\lambda : \mathcal{P} \to \mathbf{R}_+$ satisfying the capacity constraint $\sum_{P \in \mathcal{P}: e \in P} \lambda(P) \leq c(e)$ for each edge $e \in EG$. A multiflow is said to be *integral* if its flow-value function is integer-valued. For a nonempty proper subset $A \subseteq T$, any multiflow f can be regarded as a *single commodity* $(A, T \setminus A)$*-flow* by ignoring paths not connecting A and $T \setminus A$. One of interesting phenomena in multiflows is: for a special set system $\mathcal{A} \subseteq 2^T \setminus \{\emptyset, T\}$ there always exists a multiflow f being *simultaneously a maximum* $(A, T \setminus A)$*-flow for all* $A \in \mathcal{A}$. For example, take $\mathcal{A}$ as the set of all singletons $\{\{s\} \mid s \in T\}$. Then Lovász [16] and Cherkassky [2] independently showed that there exists a multiflow being simultaneously a maximum $(s, T \setminus s)$-flow for all $s \in T$. Moreover, if the capacity c is integer-valued, then there exists a *half-integral* multiflow of this property.

For a multiflow f, we say "f *locks* $\mathcal{A}$" if f is a maximum $(A, T \setminus A)$-flow for all $A \in \mathcal{A}$. Then $\mathcal{A}$ is said to be *lockable* if for every network (G, T, c) there exists a multiflow locking $\mathcal{A}$. The *multiflow locking theorem*, due to Karzanov and Lomonosov [14], gives a complete characterization of such a lockable set system. Two subsets $A, B \subseteq T$ are said to be *crossing* if none of $A \setminus B$, $B \setminus A$, $A \cap B$, and $T \setminus (A \cup B)$ is empty. A network (G, T, c) is said to be *inner Eulerian* if c is integer-valued and every inner node has even degree.

Theorem 1.1 ([14]). *Let $\mathcal{A} \subseteq 2^T \setminus \{\emptyset, T\}$ be a set system on T. Then $\mathcal{A}$ is lockable if and only if $\mathcal{A}$ has no pairwise crossing triple. In addition, if lockable, then there exists an integral multiflow locking $\mathcal{A}$ in every inner Eulerian network (G, T, c)*

See [7, 15] for proofs, and also see [12] and [18, 73.3c] for further information. The aim of this note is to give an extension of this result. A pair (A, B) of two disjoint nonempty subsets is called a *partial cut*; we do not distinguish (A, B) and (B, A). For a partial cut (A, B) on T, any multiflow f can also be regarded as a single commodity (A, B)-flow (by ignoring paths not connecting A and B). So we can extend the locking concept for partial cuts. Let $\mathcal{A}$ be a set of partial cuts on T. For a multiflow f, we say " *f locks $\mathcal{A}$*" if f is a maximum (A, B)-flow for all $(A, B) \in \mathcal{A}$. Then $\mathcal{A}$ is said to be *lockable* if there exists a multiflow locking $\mathcal{A}$ in every network (G, T, c).

Our main result is an analogous characterization of a lockable system of partial cuts. Two partial cuts (A, B) and (C, D) are said to be *laminar* if one of the following four cases holds: (i) $A \subseteq C, B \supseteq D$, (ii) $A \subseteq D, B \supseteq C$, (iii) $A \supseteq C, B \subseteq D$, (iv) $A \supseteq D, B \subseteq C$. Otherwise (A, B) and (C, D) are said to be *crossing*, and, in addition, said to be *regularly crossing* if $A \cup B = C \cup D$ and *irregularly crossing* if $A \cup B \neq C \cup D$. A terminal s is said to be *proper* if $s \in A \cup B$ for all $(A, B) \in \mathcal{A}$, and *improper* otherwise. A network (G, T, c) is said to be *properly inner Eulerian* (with respect to $\mathcal{A}$) if it is inner Eulerian and each improper terminal has even degree.

Theorem 1.2. *Let $\mathcal{A}$ be a set of partial cuts on T. Then $\mathcal{A}$ is lockable if and only if $\mathcal{A}$ has no pairwise crossing triple and no irregularly crossing pair. In addition, if lockable, then there exists an integral multiflow locking $\mathcal{A}$ in every properly inner Eulerian network (G, T, c)*

This theorem includes the previous one for a special case where $A \cup B = T' \subseteq T$ for each $(A, B) \in \mathcal{A}$. Although one can prove this theorem from a general framework developed in [10, 11], we here prove it by a basic technique involving splitting-off and submodularity of cuts, similar to that in [7, 16].

2 Proof

Let (G, T, c) be a network, possibly having multiple edges an loops. An edge e joining nodes x and y is denoted as xy. We begin to prove the only-if part.

Only-if part. Let $\mathcal{A}$ be a set of partial cut on T. Suppose first that $\mathcal{A}$ has an irregularly crossing pair (A, B), (C, D). We may assume $A \setminus (C \cup D) \neq \emptyset$. Take $s \in A \setminus (C \cup D)$. Suppose the case where both $B \cap C$ and $B \cap D$ are nonempty. Take $t \in B \cap C$ and $u \in B \cap D$. Consider the network on T consisting of only two edges st, tu with unit capacity. Then there is no multiflow locking $\{(A, B), (C, D)\}$. Indeed, to lock (C, D), we need to push unit (t, u)-flow passing through t, s, u in order. However if push, then it is impossible to add flows to lock (A, B). Next suppose the case where $B \setminus (C \cup D)$ is nonempty. We may assume that $A \cap C$ or $A \cap D$ is empty; otherwise it reduces the case above. Take $s' \in B \setminus (C \cup D)$, $t \in C$, and $t' \in D$. Consider the network consisting of four edges $st, ts', s't', t's$ with unit capacity. Again there is no multiflow locking $\{(A, B), (C, D)\}$. Indeed, we may assume $t' \notin A$. To lock (A, B), we need to push unit (s, s')-flow passing through s, t', s' in order. If push, then it is impossible to lock (C, D). So we may assume $B \subseteq C$. Since (A, B) and (C, D) are not laminar, $D \setminus A$ is nonempty. Take $t \in B \subseteq C$ and $u \in D \setminus A$. Consider the network consisting of two edges st, su of unit capacity. Again there is no multiflow locking $\{(A, B), (C, D)\}$.

So we may assume that $\mathcal{A}$ has no irregularly crossing pair and has pairwise crossing triple (A_i, B_i) $(i = 1, 2, 3)$. Then $A_1 \cup B_1 = A_2 \cup B_2 = A_3 \cup B_3$. In this case, we can directly use a proof [15, p. 44] of the only-if part of the ordinary locking theorem (Theorem 1.1).

If part. We need some notions. For a partial cut (X, Y), the capacity $c(X, Y)$ is the sum of capacity of edges e joining X and Y. A *cut* is a bipartition (X, Y) of node set VG. For a partial cut (A, B) on T, a cut (X, Y) is called an (A, B)-cut if $A \subseteq X$ and $B \subseteq Y$. The minimum capacity of (A, B)-cuts is denoted by $\kappa_{A,B}$. The *cut distance* $\delta_{A,B} : T \times T \to \mathbf{R}_+$ is defined by

$$\delta_{A,B}(s,t) = \begin{cases} 1 & \text{if } (s,t) \in A \times B \text{ or } (t,s) \in A \times B, \\ 0 & \text{otherwise,} \end{cases} \qquad (s, t \in T).$$

Let $\mathcal{A}$ be a set of partial cuts on T. For a nonnegative weight $\alpha : \mathcal{A} \to \mathbf{R}_+$, let $\mu^\alpha_{\mathcal{A}} = \sum_{(A,B)\in\mathcal{A}} \alpha(A, B)\delta_{A,B}$. Consider the following maximum multiflow problem:

$$\text{(2.1)} \qquad \text{Max.} \sum_{P\in\mathcal{P}} \mu^\alpha_{\mathcal{A}}(s_P, t_P)\lambda(P) \ \text{ s.t. } \ f = (\mathcal{P}, \lambda) : \text{multiflow in } (G, T, c).$$

Here s_P, t_P denote the ends of P. For a multiflow f, let $\mathrm{val}(f)$ denote the objective value of (2.1). Then we have

$$\text{(2.2)} \qquad \mathrm{val}(f) \le \sum_{(A,B)\in\mathcal{A}} \alpha(A, B)\kappa_{A,B}.$$

Indeed, this follows from

$$\sum_{P\in\mathcal{P}} \delta_{A,B}(s_P, t_P)\lambda(P) = (\text{the total flow-value of } (A, B)\text{-flows in } f) \le \kappa_{A,B}.$$

By the max-flow min-cut theorem [5], a multiflow f locks $\mathcal{A}$ if and only if f attains (2.2) with equality. Therefore the if-part of Theorem 1.2 is rephrased as follows:

Theorem 2.1. *Suppose that $\mathcal{A}$ has no pairwise crossing triple and no irregularly crossing pair. Then, for every nonnegative weight $\alpha : \mathcal{A} \to \mathbf{R}_+$ and every properly inner Eulerian network (G, T, c), there exists an integral optimal multiflow f with* $\mathrm{val}(f) = \sum_{(A,B)\in\mathcal{A}} \alpha(A, B)\kappa_{A,B}$ *in* (2.1).

Suppose that (G, T, c) and $\mathcal{A}$ fulfill the condition of the theorem. Our proof is based on the splitting-off method. For a simplification of the proof, we use a standard technique to make the input network have a small degree, as in [6, p. 51]. First, by multiplying edges, make each edge have unit capacity. Suppose that there is an inner node y of degree at least 6, incident to edges $e_i = x_i y$ $(i = 0, 1, \ldots, m+1)$ (some nodes x_i, x_j may coincide). Change the incidence around y by the following way. Subdivide edge $e_i = x_i y$ into two edges $x_i z_i, z_i y$ for $i = 1, 2, \ldots, m$. Replace e_0 by $x_0 z_1$, and replace e_{m+1} by $x_{m+1} z_m$. Add new edge $z_i z_{i+1}$ for $i = 1, 2, \ldots, m-1$. Then any integer multiflow in the new network can be transformed into an integer multiflow in the original network having the same $\mathrm{val}(\cdot)$. The converse transformation is also possible. In particular, the min-cut value $\kappa_{A,B}$ is invariant. By repeating this process, make y have degree four. Suppose that there is an improper terminal s of (even) degree $m \ge 4$. Add $m/2$ new terminals $s_1, s_2, \ldots, s_{m/2}$, and join s and each s_i by two parallel edges (of unit capacity). Make s being an inner node, i.e., $T \leftarrow T \setminus \{s\}$. For each partial cut $(A, B) \in \mathcal{A}$ with $s \in A$ (resp.

$s \in B$), replace A by $A \cup \{s_1, s_2, \ldots, s_{m/2}\} \setminus \{s\}$ (resp. B by $B \cup \{s_1, s_2, \ldots, s_{m/2}\} \setminus \{s\}$). Then the problem is unchanged. By this reduction, we may assume that each edge has unit capacity, each inner node has degree four, and each improper terminal has degree two.

Recall the splitting-off operation. A pair $\{xy, yz\}$ of consecutive edges incident to a common node y is called a *fork*. The *splitting-off* operation at $\{xy, yz\}$ is to delete edges xy, yz and add a new edge joining y and z of unit capacity (if $y \neq z$). A fork is said to be *splittable* if the splitting-off operation does not decrease the min-cut value $\kappa_{A,B}$ for all $(A, B) \in \mathcal{A}$. The splitting-off decreases the total number of edges. From an integral multiflow in the new network, we obtain an integral multiflow in the original network. Therefore, if we find a splittable fork, then by induction on the number of edges we can prove the existence of an integral multiflow attaining (2.2) in equality.

By the degree condition, each node $x \notin A \cup B$ has even degree for $(A, B) \in \mathcal{A}$. Therefore, for two (A, B)-cuts (X, Y) and (X', Y'), the difference $c(X, Y) - c(X', Y')$ is an even integer. Moreover the splitting-off at $\{xy, yz\}$ decreases the cut-capacity of (X, Y) if and only if $x \in X, y, z \in Y$ or $x \in Y, y, z \in X$. If decreases, then it decreases by 2. Therefore, if $\{xy, yz\}$ is unsplittable, then there exist a partial cut $(A, B) \in \mathcal{A}$ and a minimum (A, B)-cut (X, Y) such that $x \in X, y, z \in Y$ or $x \in Y, y, z \in X$. We call this cut (X, Y) a *critical* (A, B)*-cut* with respect to $\{xy, yz\}$.

Take an inner node y of degree four incident to four edges $e_i = yx_i$ for $i = 0, 1, 2, 3$. We show that at least one of three forks $\{x_0y, yx_i\}$ $(i = 1, 2, 3)$ is splittable. Suppose (to the contrary) that all three forks are unsplittable. Then there is a critical (A_i, B_i)-cut (X_i, Y_i) with respect to $\{x_0y, yx_i\}$ for $i = 1, 2, 3$. We claim:

$(*1)$ (A_i, B_i) and (A_j, B_j) are crossing if $i \neq j$.

This immediately leads a contradiction to the hypothesis of $\mathcal{A}$. Suppose that (A_1, B_1) and (A_2, B_2) are laminar. We may assume $x_0, x_i \in X_i$ and $y \in Y_i$ for $i = 1, 2$. Then $x_2, x_3 \in Y_1$ and $x_1, x_3 \in Y_2$ necessarily hold. Indeed, if $x_2 \in X_1$, then $c(X_1, Y_1) > c(X_1 \cup \{y\}, Y_1 \setminus \{y\})$; this is a contradiction to the assumption that (X_1, Y_1) is a minimum (A_1, B_1)-cut. It suffices to consider two cases: (i) $A_1 \subseteq A_2, B_1 \supseteq B_2$ and (ii) $A_1 \subseteq B_2, B_1 \supseteq A_2$. Suppose (i). Recall the submodular inequality of cuts:

$$c(X_1, Y_1) + c(X_2, Y_2) \geq c(X_1 \cap X_2, Y_1 \cup Y_2) + c(X_1 \cup X_2, Y_1 \cap Y_2). \tag{2.3}$$

Then $(X_1 \cap X_2, Y_1 \cup Y_2)$ is a minimum (A_1, B_1)-cut and $(X_1 \cup X_2, Y_1 \cap Y_2)$ is a minimum (A_2, B_2)-cut. However $c(X_1 \cup X_2 \cup \{y\}, Y_1 \cap Y_2 \setminus \{y\}) < c(X_1 \cup X_2, Y_1 \cap Y_2)$. A contradiction to the minimality. So suppose (ii). Recall one more submodular inequality:

$$c(X_1, Y_1) + c(X_2, Y_2) \geq c(X_1 \cap Y_2, Y_1 \cup X_2) + c(X_1 \cup Y_2, Y_1 \cap X_2). \tag{2.4}$$

Then $(X_1 \cap Y_2, Y_1 \cup X_2)$ is a minimum (A_1, B_1)-cut, and $(X_1 \cup Y_2, Y_1 \cap X_2)$ is a minimum (A_2, B_2)-cut. Thus the equality in (2.4) holds. However the equality in (2.4) holds if and only if there is no edge connecting $X_1 \cap X_2$ and $Y_1 \cap Y_2$. This contradicts to $x_0 \in X_1 \cap X_2$, $y \in Y_1 \cap Y_2$, and $e_0 = x_0y \in EG$. Therefore every inner node has splittable fork.

So it suffices to consider the case where there is no inner node, i.e., $VG = T$. Suppose further that a (unique) fork at every improper terminal is unsplittable; if splittable, then split it off and apply induction. We claim:

$(*2)$ for each $(A, B) \in \mathcal{A}$ we have $c(A, B) = \kappa_{A,B}$.

If true, then the set of all one-edge paths of unit flow-value is obviously an integral multiflow attaining (2.2) with equality. So suppose $c(A, B) < \kappa_{A,B}$. By Menger's theorem, there is a path $(s_0, s_1, \ldots, s_m)$ such that $s_0 \in A$, $s_m \in B$, and $s_j \notin A \cup B$ for

$j = 1, 2, \ldots, m-1$ ($m \geq 2$). For $j = 1, 2, \ldots, m-1$, terminal s_j is improper and is incident only to s_{j-1} and s_{j+1}. Consider the splitting-off at a fork $\{s_0s_1, s_1s_2\}$. By assumption, it is unsplittable. Take its critical (A', B')-cut (X, Y) with $s_1 \in X$ and $s_0, s_2 \in Y$. Then necessarily $s_1 \in A'$; otherwise $(X \setminus \{s_1\}, Y \cup \{s_1\})$ is an (A', B')-cut having smaller cut capacity. Thus $A' \cup B' \neq A \cup B$, and consequently $(A', B'), (A, B)$ are laminar. By laminarity we have $s_m \in B \subseteq A' \subseteq X$ (and $A \supseteq B'$). Then $s_2 \in B$ ($m = 2$) is impossible. So $m > 2$. However we have $c(X, Y) > c(X \cup \{s_2, s_3, \ldots, s_{m-1}\}, Y \setminus \{s_2, s_3, \ldots, s_{m-1}\})$; a contradiction to the minimality of (X, Y).

3 Concluding remarks

The size of lockable family. Karzanov and Lomonosov [14] asked: *how large is the size of a lockable family ?* A set system without crossing triple is said to be *3-cross-free*. Pevzner [17] showed that the cardinality of any 3-cross-free family on n-set is $O(n)$; see [4] for a shorter proof. Recently, Dress, Koolen and Moulton [3] proved the tight upper bound $8n - 20$ ($n \geq 3$).

A similar question arises: *how large is the size of a lockable family of partial cuts ?* We show that the size of a lockable family of partial cuts on n-set is *not* linearly bounded, but is bounded by $O(n^2)$. We give an example of $O(n^2)$ size. Let $n = 2k$ be a positive even integer. Consider a tree Γ consisting of edges $x_0x_1^i, x_1^ix_2^i, \ldots, x_{k-1}^ix_k^i$ for $i = 1, 2, \ldots, k$. Namely Γ is obtained by subdividing a star of k leaves. Let $[n] = \{1, 2, \ldots, n\}$ be an n-element set. We associate each element j in $[n]$ with a subtree F_j in Γ as follows. For $j = 1, 2, \ldots, k$, let F_j be the subtree induced by nodes x with $d_\Gamma(x_0, x) \leq j - 1$, where d_Γ is the shortest path metric on node set of Γ. For $j = k+1, k+2, \ldots, 2k$, let F_j be the subtree consisting of one node x_k^{j-k}. For each edge e in Γ, the deletion of e divides Γ into two connected components Γ', Γ''. Let A_e be the set of indices i for which F_i belongs to Γ', and let B_e be the set of indices i for which F_i belongs to Γ''. Note that if F_j contains e, then neither A_e nor B_e contains j. Now we obtain a partial cut family $\mathcal{A} = \{(A_e, B_e) \mid e \text{ is an edge of } \Gamma\}$ of cardinality $(n/2)^2 = O(n^2)$. One can easily verify that $\mathcal{A}$ is laminar, i.e., each pair is laminar. In fact, it is known that every laminar partial cut family is obtained by a family of subtrees on a tree in this way [9].

To see the upper bound $O(n^2)$, we use a result from the *split decomposition theory* [1, 8]; a partial cut is called a *partial split* in [8, Section 4]. Then one can easily see that a set of partial cuts without crossing triple and irregularly crossing pair is *weakly compatible* in the sence of [8, Section 4]. A general result [8, Theorem 4.13] says that for a weakly compatible family of partial cuts $\mathcal{A}$ on an n-set T, the corresponding set of cut distances $\{\delta_{A,B} \mid (A, B) \in \mathcal{A}\}$ is linearly independent in the vector space of symmetric functions with zero diagonals $\{d \in \mathbf{R}^{T \times T} \mid d(s, t) = d(t, s), d(s, s) = 0 \ (s, t \in T)\}$. Therefore $|\mathcal{A}| \leq n(n-1)/2 = O(n^2)$.

Fractionality. Theorem 2.1 says the existence of an integral optimal multiflow in a class of μ-weighted maximum multiflow problems. For a weight $\mu : T \times T \to \mathbf{R}_+$, the *fractionality* of μ is defined to be the least positive integer k with property that the μ-weighted maximum multiflow problem has a $1/k$-integral optimal multiflow for every integer-capacitated network (G, T, c). If such an integer does not exist, then the fractionality is defined to be infinity. Recently, [11] proved a complete characterization of weights having finite fractionality, and showed that if the fractionality is finite, then it is a divisor of 24 (the conjectured tight upper bound is 4). So it is still interesting to identify a class of weights having small fractionality. From the point of the view, Theorem 2.1 provides a new class of weights having fractionality 2.

A distance among subtrees in a tree, considered in [9], is a natural example of such a weight; it is exactly a nonnegative sum of cut distances for a laminar partial cut family. By the *extended split decomposition* in [8, Section 4], we can determine, in strongly polynomial time, whether a given weight μ is decomposed into a nonnegative sum of cut distances for a lockable family of partial cuts, and we can also obtain an explicit decomposition if decomposable.

A polynomial time algorithm. The splitting-off proof provides a strongly polynomial time algorithm to find an integral multiflow locking $\mathcal{A}$ in a properly inner Eulerian network. We sketch it. Once the existence of an integral solution is established, it is unnecessary to reduce the degrees as in the proof of Theorem 2.1. So we may assume that the input network (G, T, c) is complete and has no multiple edges and loops. Let $n = |VG|$. We use a capacitated version of splitting-off. The maximum capacity of the splitting-off can be computed, in strongly polynomial time, by a minimum (A, B)-cut algorithm for all $(A, B) \in \mathcal{A}$. By applying splitting-off for all $n^2(n+1)/2$ forks (in some ordering), we can make the network have no splittable fork. This is nontrivial; see [11, Section 7] and [13, Section 4] for details. Then the multiflow consisting of all one-edge paths is obviously an integral solution. By reversing the splitting-off operations, we obtain an integral multiflow locking $\mathcal{A}$ in the original network (in an edge-node form if necessarily). So the whole complexity is $O(n^3|\mathcal{A}|\varphi(n))$, where $\varphi(n)$ denotes the complexity of a maximum flow algorithm for an n-node network. An augmenting path approach, as in [2, 12, 15], would yield a more faster algorithm, which is left to readers.

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