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On certain arithmetic functions $\tilde{M}(s; z_1, z_2)$ associated with global fields: Analytic properties.

Dedicated to Professor Mikio Sato

By

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Introduction

0.1 – In [2], we started our study of the complex analytic function $\tilde{M}(s; z_1, z_2)$ (denoted there as $\tilde{M}_s(z_1, z_2)$) in three variables s, z_1, z_2 ($\Re(s) > 1/2$), in connection with the value distribution of $\{d \log L(s, \chi)/ds\}_\chi$. Here, χ runs over a suitable family of abelian characters of a global field K and $L(s, \chi)$ denotes the associated L-function. The connection is that for each fixed s with $\Re(s) = \sigma > 1/2$, the inverse Fourier transform $M_\sigma(w)$ of $\tilde{M}_\sigma(z) = \tilde{M}(\sigma, z, \bar{z})$ is the density function for the distribution of $\{d \log L(s, \chi)/ds\}_\chi$ on the complex w-plane (generally conjectural, proved in various cases [2, 4, 6]). In the joint work with K. Matsumoto [5, 6] (cf. also a survey [7]), we continued this study treating also the corresponding M-and $\tilde{M}$-functions related to the value distribution of $\{\log L(s, \chi)\}_\chi$. We use the same symbols M,$\tilde{M}$ etc., and distinguish the former $d \log$-case as Case 1, the latter log case as Case 2. They are different systems of functions having various properties in common. Each also depends on the pair (K, P_∞), where K is a global field (either an algebraic number field or an algebraic function field of one variable over a finite field) and P_∞ is a given finite set of prime divisors of K including all archimedean primes in the number field case. When $(K, P_\infty) = (\mathbf{Q}, (\infty))$,

$$\tilde{M}(s; z_1, z_2) = \sum_{n=1}^{\infty} \lambda_{z_1}(n) \lambda_{z_2}(n) n^{-2s} \qquad (\Re(s) > 1/2), \tag{0.1.1}$$

where each $\lambda_z(n)$ ($z \in \mathbf{C}$; $n = 1, 2, \cdots$) is a polynomial of z determined by

$$\sum_{n=1}^{\infty} \lambda_z(n) n^{-s} := \begin{cases} \exp(\frac{iz}{2} \frac{d}{ds} \log \zeta(s)) & \text{(Case 1)} \\ \exp(\frac{iz}{2} \log \zeta(s)) & \text{(Case 2)} \end{cases}$$

($\Re(s) > 1$, $i = \sqrt{-1}$). For example, $\tilde{M}(s, -2i, -2ix) = \zeta(2s)^x$ ($x \in \mathbf{C}$) in Case 2. It seems to the author that these functions are interesting in themselves.

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0.2 – We shall pursue further analytic properties of $\tilde{M}(s; z_1, z_2)$ and $M_\sigma(w)$. In the present article, we shall first study the variance μ_σ and the "Plancherel volume"

$$\nu_\sigma = \int M_\sigma(w)^2 |dw| = \int |\tilde{M}_\sigma(z)|^2 |dz|; \tag{0.2.1}$$

especially the limit behaviours $\lim_{\sigma\to 1/2}$ and $\lim_{\sigma\to+\infty}$ of the "natural invariant" $\mu_\sigma\nu_\sigma$, and of the variance-normalized measure $\mu_\sigma M_\sigma(\mu_\sigma^{1/2} w)$ and its Fourier transform $\tilde{M}_\sigma(\mu_\sigma^{-1/2} z)$ (§1,§2). One of the key points is the limit behaviour of the complex analytic version

$$\tilde{M}(s; \mu(s)^{-1/2} z_1, \mu(s)^{-1/2} z_2) \tag{0.2.2}$$

of $\tilde{M}_\sigma(\mu_\sigma^{-1/2} z)$, which is partly related to the second main subject of this article, namely the analytic continuation. We shall prove (§3) that $\tilde{M}(s; z_1, z_2)$ extends analytically to $\mathcal{D} \times \mathbf{C^2}$, where

$$\mathcal{D} = \{\Re(s) > 0\} \setminus \{\rho/2n\,;\, n \in \mathbf{N},\, \zeta(\rho) = 0 \text{ } or \text{ } \infty\}, \tag{0.2.3}$$

$\zeta(s) = \zeta_{K,P_\infty}(s)$ being the zeta function of K without P_∞ factors. In fact, $\tilde{M}(s; z_1, z_2)$ is univalent on $\mathcal{D} \times \mathbf{C^2}$ in Case 1, but multivalent in Case 2 (univalent on $\mathcal{D}^{urab} \times \mathbf{C^2}$, $\mathcal{D}^{urab}$ being the maximal unramified abelian cover of $\mathcal{D}$). This property is closely related to the infinite product expansion which, in Case 2, looks like

$$\tilde{M}(s; z_1, z_2) = \prod_{n=1}^{\infty} \zeta(2ns)^{R_n(z_1,z_2)}, \tag{0.2.4}$$

where each $R_n(z_1, z_2)$ is a polynomial of degree $\deg_{z_i} \le n$ $(i = 1, 2)$. This means that for any $N \in \mathbf{N}$, (i) the quotient of $\tilde{M}(s; z_1, z_2)$ by the partial product over $n \le N$ on the right hand side extends to a holomorphic function on $\Re(s) > 1/(2N + 2)$, and (ii) *on some subdomain* of $\{\Re(s) > 1/2\} \times \mathbf{C^2}$, the remaining product converges absolutely to a non-vanishing holomorphic function which gives that quotient. The case $N = 1$ will be used to show that (0.2.2) converges to $\exp(-z_1 z_2/4)$ as $s \to 1/2$. This, together with an upper bound for $|\tilde{M}_\sigma(z)|^2$ near $\sigma = 1/2$, valid for all $z \in \mathbf{C}$ studied in §4, leads to our limit formulas for $\mu_\sigma\nu_\sigma$ and $\mu_\sigma M_\sigma(\mu_\sigma^{1/2} w)$.

0.3 – In §1.1, we first discuss general continuous density functions $M(x)|dx|$ on $\mathbf{R}^d$ $(d = 1, 2, \cdots)$ with center 0, in particular, the rigorous lower bound for the quantity $\mu^{d/2}\nu$ (Theorem 1), where μ is the variance and ν is the Plancherel volume. For $d = 2$, this gives $\mu\nu \ge 8/9$. Then in §1.2, we briefly review (from [6]§4) the definition and the basic properties of our functions $\tilde{M}(s; z_1, z_2)$ and $M_\sigma(w)$.

In §2, we study the limits as $\sigma \to 1/2, +\infty$ of $\mu_\sigma \nu_\sigma$ and $\mu_\sigma M_\sigma(\mu_\sigma^{1/2} w)$ (Theorems 2,3). Some of the key lemmas used will be proved later (§3,§4). This logically inverted ordering of sections is due to the introductory nature of §2 and the "heaviness" of §3,§4.

In §3, we shall prove the analytic continuation of $\tilde{M}(s; z_1, z_2)$ (Theorem 5).

In §4, we shall study the rapid decay property of $|\tilde{M}_\sigma(z)|^2$, especially when σ is arbitrarily close to 1/2 and $|z|$ not being bounded.

0.4 – Now we mention something about the zero divisor of $\tilde{M}(s; z_1, z_2)$ on which no information is shown in the product formula (0.2.4). First, as is already shown in the previous articles (reviewed in §1.2), $\tilde{M}(s; z_1, z_2)$ has an Euler product decomposition

$$\tilde{M} = \tilde{M}(s; z_1, z_2) = \prod_{\mathfrak{p} \notin P_\infty} \tilde{M}_{\mathfrak{p}}(s; z_1, z_2) \qquad (\Re(s) > 1/2), \tag{0.4.1}$$

where each local factor $\tilde{M}_{\mathfrak{p}} = \tilde{M}_{\mathfrak{p}}(s; z_1, z_2)$ is holomorphic on $\{\Re(s) > 0\} \times \mathbf{C}^2$. In Case 2, $\tilde{M}_{\mathfrak{p}}$ can be expressed by the Gauss hypergeometric function $F(a, b; c; x)$, as

$$\tilde{M}_{\mathfrak{p}}(s; z_1, z_2) = F(iz_1/2, iz_2/2; 1;\ N(\mathfrak{p})^{-2s}). \tag{0.4.2}$$

Each $\tilde{M}_{\mathfrak{p}}$ has a non-trivial zero divisor $\mathcal{Z}_{\mathfrak{p}}$, $\{\mathcal{Z}_{\mathfrak{p}}\}_{\mathfrak{p}}$ is locally finite, and the intersection with $\mathcal{D} \times \mathbf{C}^2$ of $\sum_{\mathfrak{p}} \mathcal{Z}_{\mathfrak{p}}$ gives the zero divisor of $\tilde{M}$.

The local zero divisor $\mathcal{Z}_{\mathfrak{p}}$ seems worth studying fully[1]. But let us touch here the main property of its restriction to the hyperplane $z_1 + z_2 = 0$, say, in Case 2. Put $t = N(\mathfrak{p})^{-s}$, $x = iz_1$, and consider the "locally normalized" function

$$f_t(x) = F(x/(2\arcsin(t)), -x/(2\arcsin(t)); 1;\ t^2). \tag{0.4.3}$$

Then $f_0(x) = J_0(x)$, the Bessel function of order 0. If $\pm\{\gamma_\nu\}_{\nu=1}^\infty$ with $0 < \gamma_1 < \gamma_2 < \cdots$ denote all the zeros of $J_0(x)$, then there exists $0 < t_0 < 1$ such that each γ_ν extends uniquely to a zero $\gamma_\nu(t)$ of $f_t(x)$ for all $|t| < t_0$ (*real* if t is so), and we have

$$f_t(x) = \prod_{\nu=1}^\infty \left(1 - \frac{x^2}{\gamma_\nu(t)^2}\right). \tag{0.4.4}$$

This gives rise to another infinite product decomposition

$$\tilde{M}(s; z, -z) = \prod_{\mathfrak{p} \notin P_\infty} \prod_{\nu=1}^\infty \left(1 + \left(\frac{\arcsin(N(\mathfrak{p})^{-s})}{\gamma_\nu(N(\mathfrak{p})^{-s})}\right)^2 z^2\right) = \prod_{\mu=1}^\infty (1 + \theta_\mu^2 z^2) \tag{0.4.5}$$

for $\Re(s) > 1/2$, where $\{\theta_\mu\}$ is a reordering according to the absolute values. For $s = \sigma \in \mathbf{R}$, θ_μ^2 are all positive real, as long as $N(\mathfrak{p})^\sigma$ is sufficiently large. The comparison of two decompositions (0.2.4) and (0.4.5) would be a future subject of study.

[1] Left to future articles; cf. [3] for some partial results for Case 1.

1 Preliminaries

1.1 – The Plancherel volume. Let $\mathbf{R}^d = \{x = (x_1, ..., x_d);\ x_i \in \mathbf{R}\,(1 \leq i \leq d)\}$ be the d-dimensional Euclidean space ($d = 1, 2, \cdots$), and $|dx| = (dx_1...dx_d)/(2\pi)^{d/2}$ be the self-dual Haar measure with respect to the self-dual pairing $e^{i\langle x,x'\rangle}$ of $\mathbf{R}^d$, where $\langle x, x'\rangle = \sum_{i=1}^d x_i x_i'$. Write, as usual, $|x| = \langle x, x\rangle^{1/2}$. Consider any density measure $M(x)|dx|$ ($M(x)$: a measurable function) on $\mathbf{R}^d$ with center 0, for which the standard formulas in Fourier analysis hold; namely,

$$M(x) \geq 0, \qquad \int M(x)|dx| = 1; \tag{1.1.1}$$

$$\int M(x)x_i|dx| = 0 \qquad (1 \leq i \leq d); \tag{1.1.2}$$

$$\tilde{M}(y) := \int M(x)e^{i\langle x,y\rangle}|dx|, \qquad M(x) = \int \tilde{M}(y)e^{-i\langle x,y\rangle}|dy|; \tag{1.1.3}$$

$$\nu := \nu_M = \int M(x)^2|dx| = \int |\tilde{M}(y)|^2|dy| \qquad \text{(Plancherel formula)}. \tag{1.1.4}$$

We shall compare the two invariants

$$\mu := \mu_M = \int M(x)|x|^2|dx| \qquad \text{(the variance)} \tag{1.1.5}$$

and the above ν_M which will be called the *Plancherel volume* of $M(x)$ (or of $M(x)|dx|$). Note that ν_M can also be expressed as

$$\nu_M = M(x) * M(-x)\ |_{x=0} \tag{1.1.6}$$

($*$: the convolution product with respect to $|dx|$). Thus, ν_M may be regarded as the density at the origin of the differences of two points in the measure space $(\mathbf{R}^d, M(x)|dx|)$.

In general, the two invariants, the average μ of the square of the distance from the center and the density ν at the origin of $x - x'$ ($x, x' \in \mathbf{R}^d$), both with respect to the given density measure $M(x)|dx|$, are unrelated invariants. But the product

$$\mu^{d/2}\nu \tag{1.1.7}$$

seems to be an interesting basic invariant. Note that this is invariant by the scalar transform

$$M(x) \longmapsto c^d M(cx) \tag{1.1.8}$$

for any $c > 0$; in fact, μ (resp. ν) is multiplied by c^{-2} (resp. c^d).

If we denote by $M^\star(x) = \mu^{d/2}M(\mu^{1/2}x)$ the scalar transform (1.1.8) for $c = \mu^{1/2}$, then $M^\star(x)$ has the Fourier dual $\tilde{M}(\mu^{-1/2}y)$, the variance $= 1$, and the Plancherel volume $\mu^{d/2}\nu$. This scalar transform $M(x) \to M^\star(x)$ will be called *the variance-normalization.*

Let us pay attention to the following 3 special cases and the theorem to come thereafter.

Example 1 If $M(x)|dx|$ is Gaussian, i.e., $M(x) = ce^{-a|x|^2}$ $(a,\, c > 0)$, then

$$\mu^{d/2}\nu = \left(\frac{d}{2}\right)^{d/2}. \tag{1.1.9}$$

In particular, the 2 dimensional Gaussian distribution satisfies $\mu\nu = 1$.

Indeed, we have $c = (2a)^{d/2}$ by (1.1.1), and $\mu = d/(2a)$, $\nu = a^{d/2}$.

Example 2 If $M(x) = c$ $(|x| \leq R)$ and $= 0$ $(|x| > R)$, where $c,\, R > 0$, then

$$\mu^{d/2}\nu = \left(\frac{2d}{d+2}\right)^{d/2}\Gamma\left(\frac{d}{2}+1\right). \tag{1.1.10}$$

In particular, when $d = 2$, we again have $\mu\nu = 1$.

Indeed, $c = 2^{d/2}\Gamma(\frac{d}{2}+1)R^{-d}$, $\mu = \frac{d}{d+2}R^2$, $\nu = 2^{d/2}\Gamma(\frac{d}{2}+1)R^{-d}$.

Thus, when $d = 2$, $\mu\nu = 1$ holds in these two special cases.

Example 3 Define the function $f_d^*(r)$ of $r \geq 0$ by

$$f_d^*(r) = \begin{cases} \frac{d(d+2)}{2}\gamma_d \cdot (1-r^2) \cdots 0 \leq r \leq 1, \\ 0 \qquad\qquad\qquad\quad\; \cdots r \geq 1, \end{cases} \tag{1.1.11}$$

where

$$\gamma_d = (2\pi)^{\frac{d}{2}}/\mathrm{Vol}(\mathrm{S}_{d-1}) = 2^{\frac{d}{2}-1}\Gamma(d/2), \tag{1.1.12}$$

$\mathrm{Vol}(\mathrm{S}_{d-1})$ being the Euclidean volume of the $(d-1)$-dimensional unit sphere. And for any fixed $c > 0$, consider the function $M(x) = c^d \cdot f_d^*(c|x|)$ on $\mathbf{R}^d$. Then $M(x)$ also satisfies (1.1.1)(1.1.2) and we have

$$\mu^{d/2}\nu = \left(\frac{2d}{d+4}\right)^{d/2}\frac{4\Gamma(\frac{d+4}{2})}{d+4}. \tag{1.1.13}$$

Indeed, $\mu = c^{-2}\mu_d^*$ and $\nu = c^d\nu_d^*$, where

$$\mu_d^* = \frac{d}{d+4}, \qquad \nu_d^* = \frac{2d(d+2)}{d+4}\gamma_d. \tag{1.1.14}$$

Now, intuitively, μ and ν cannot be too small at the same time and hence there must be some inequality showing this. The following elementary but seemingly basic inequality was obtained in passing. Since I could not find this in the past literatures (including e.g. [1]), I take this opportunity to present it with a full proof (a sketch was given in [3]).

Theorem 1 *For each $d \geq 1$ and each measurable function $M(x)$ on $\mathbf{R}^d$ satisfying (1.1.1)(1.1.2), we have, for $\mu = \mu_M$ and $\nu = \nu_M$* [1]*:*

$$\mu^{d/2}\nu \geq \left(\frac{2d}{d+4}\right)^{d/2} \frac{4\Gamma(\frac{d+4}{2})}{d+4}. \tag{1.1.15}$$

Moreover, the equality holds if and only if $M(x)$ is the function given in Example 3.

The minimum-giving Example 3 was found by using small deformations, which led to a simple differential equation of order 1. And once found, the proof is simple (and somewhat miraculous).

Proof Let $M(x)$ be as at the beginning of this subsection, with their invariants μ, ν. We shall prove

$$\mu^{d/2}\nu \geq (\mu_d^*)^{d/2}\nu_d^*, \tag{1.1.16}$$

where μ_d^*, ν_d^* are as defined by (1.1.14). We may assume that $M(x)$ is rotation invariant, because averaging over $|x| = r$ does not change μ, while ν either decreases or remains the same. Therefore, $M(x) = f(|x|)$ with some non-negative real valued function $f(r)$ of $r \geq 0$, and

$$\frac{1}{\gamma_d}\int_0^\infty f(r)r^{d-1}dr = 1, \quad \frac{1}{\gamma_d}\int_0^\infty f(r)r^{d+1}dr = \mu, \quad \frac{1}{\gamma_d}\int_0^\infty f(r)^2r^{d-1}dr = \nu. \tag{1.1.17}$$

By a suitable scalar transform (1.1.8) we may assume that μ is any given positive real number, and so we assume $\mu = \mu_d^*$. We then have

$$\frac{1}{\gamma_d}\int_0^1 f(r)(1-r^2)r^{d-1}dr \geq \frac{1}{\gamma_d}\int_0^\infty f(r)(1-r^2)r^{d-1}dr = 1 - \mu_d^* = \frac{4}{d+4}, \tag{1.1.18}$$

[1] Here we just need the first definition of ν in (1.1.4) involving only $M(x)$.

because the corresponding integral over $(1,\infty)$ is obviously non-positive. Now the Schwarz inequality gives

$$(1.1.19)\qquad \left(\int_0^1 f_d^*(r)^2 r^{d-1}dr\right)\left(\int_0^1 f(r)^2 r^{d-1}dr\right) \geq \left(\int_0^1 f_d^*(r) f(r) r^{d-1}dr\right)^2.$$

Here, the first integral on the LHS is nothing but $\gamma_d \nu_d^*$, while the RHS is

$$(1.1.20)\qquad \left(\frac{d(d+2)}{2}\right)^2 \gamma_d^2 \left(\int_0^1 f(r)(1-r^2) r^{d-1}dr\right)^2 \geq \left(\frac{d(d+2)}{2}\right)^2 \gamma_d^4 \left(\frac{4}{d+4}\right)^2,$$

by (1.1.11),(1.1.18). Therefore, (1.1.19) gives

$$(1.1.21)\qquad \frac{1}{\gamma_d}\int_0^1 f(r)^2 r^{d-1}dr \geq \gamma_d^2 (\nu_d^*)^{-1}\left(\frac{2d(d+2)}{d+4}\right)^2 = \nu_d^*$$

by (1.1.14), and hence the desired inequality $\nu \geq \nu_d^*$. The last statement of Theorem 1 is clear from the above proof. $\square$.

In particular, for $d=1,2$,

Corollary 1.1.22 *We have*

$$(1.1.23)\qquad \mu^{1/2}\nu \;\geq\; (18\pi/125)^{1/2} \qquad (d=1),$$

$$(1.1.24)\qquad \mu\nu \;\geq\; 8/9 \qquad (d=2).$$

On the other hand, there is no upper bound for $\mu^{d/2}\nu$; indeed, if the support of $M(x)$ is concentrated to the sphere with center 0 and radius r, then μ is close to r^2 while ν can be as large as possible.

1.2 – The function $\tilde{M}(s;z_1,z_2)$. We shall review, mainly from [6]§4, the definition and some main properties of the function $\tilde{M}(s;z_1,z_2)$ and its local factors $\tilde{M}_{\mathfrak{p}}(s;z_1,z_2)$. Let K be any global field, i.e., either an algebraic number field of finite degree, or an algebraic function field of one variable over a finite field. Let $\mathfrak{p}$ be any non-archimedean prime of K. Define $\lambda_z(\mathfrak{p}^n)$ ($z\in\mathbf{C}$, $n\geq 0$) to be the coefficient of the power series

$$(1.2.1)\qquad \sum_{n=0}^{\infty}\lambda_z(\mathfrak{p}^n)N(\mathfrak{p})^{-ns} = \begin{cases}\exp\left(\frac{iz}{2}\frac{d}{ds}\log((1-N(\mathfrak{p})^{-s})^{-1})\right) & \text{(Case 1)}\\ \exp\left(\frac{iz}{2}\log((1-N(\mathfrak{p})^{-s})^{-1})\right) & \text{(Case 2)}\end{cases}$$

of $N(\mathfrak{p})^{-s}$. It is a polynomial

$$(1.2.2)\qquad \lambda_z(\mathfrak{p}^n) = \begin{cases}F_n(-\frac{iz}{2}\log N(\mathfrak{p})) & \text{(Case 1)},\\ F_n(\frac{iz}{2}) & \text{(Case 2)},\end{cases}$$

with

$$(1.2.3) \qquad F_n(x) = \begin{cases} \sum_{k=1}^{n} \frac{1}{k!} \binom{n-1}{k-1} x^k & \text{(Case 1)}, \\ \sum_{k=1}^{n} \frac{1}{k!} \delta_k(n) x^k = \frac{1}{n!} x(x+1)...(x+n-1) & \text{(Case 2)}, \end{cases}$$

$(n \geq 1)$, $F_0(x) = 1$, where

$$(1.2.4) \qquad \delta_k(n) = \sum_{\substack{n=n_1+...+n_k \\ n_1,...,n_k \geq 1}} \frac{1}{n_1...n_k} \leq \sum_{\substack{n=n_1+...+n_k \\ n_1,...,n_k \geq 1}} 1 = \binom{n-1}{k-1}.$$

Now the local $\mathfrak{p}$-factor $\tilde{M}_{\mathfrak{p}}(s; z_1, z_2)$ of $\tilde{M}(s; z_1, z_2)$ is a holomorphic function of (s, z_1, z_2) on $\{\Re(s) > 0\} \times \mathbf{C}^2$ defined by the power series of $N(\mathfrak{p})^{-2s}$ given by

$$(1.2.5) \qquad \tilde{M}_{\mathfrak{p}}(s; z_1, z_2) = \sum_{n=0}^{\infty} \lambda_{z_1}(\mathfrak{p}^n) \lambda_{z_2}(\mathfrak{p}^n) N(\mathfrak{p})^{-2ns}.$$

For a given finite set P_∞ of prime divisors of K including all the archimedean primes in the number field case, the global function $\tilde{M}(s; z_1, z_2)$, which is a holomorphic function of (s, z_1, z_2) on $\{\Re(s) > 1/2\} \times \mathbf{C}^2$, is defined by the Euler product

$$(1.2.6) \qquad \tilde{M}(s; z_1, z_2) = \prod_{\mathfrak{p} \notin P_\infty} \tilde{M}_{\mathfrak{p}}(s; z_1, z_2)$$

which is absolutely convergent on $\Re(s) > 1/2$ in the following sense. For any given $\sigma_0 > 1/2$, $R > 0$, let $|z_1|, |z_2| \leq R$ and $\Re(s) \geq \sigma_0$. Then for all but finitely many primes $\mathfrak{p}$, we have $|\tilde{M}_{\mathfrak{p}}(s; z_1, z_2) - 1| < 1$, and the sum of $\log \tilde{M}_{\mathfrak{p}}(s; z_1, z_2)$ (the principal branch) over these $\mathfrak{p}$ converges absolutely and uniformly. It has a Dirichlet series expansion

$$(1.2.7) \qquad \tilde{M}(s; z_1, z_2) = \sum_{D: integral} \lambda_{z_1}(D) \lambda_{z_2}(D) N(D)^{-2s} \qquad (\Re(s) > 1/2),$$

where D runs over the *integral divisors*; i.e., divisors of K of the form $D = \prod_{\mathfrak{p} \notin P_\infty} \mathfrak{p}^{n_{\mathfrak{p}}}$ ($n_{\mathfrak{p}} \geq 0$, $n_{\mathfrak{p}} = 0$ for almost all $\mathfrak{p}$), and $\lambda_z(D) = \prod_{\mathfrak{p} \notin P_\infty} \lambda_z(\mathfrak{p}^{n_{\mathfrak{p}}})$.

(**Other expressions**) The local function $\tilde{M}_{\mathfrak{p}}(s; z_1, z_2)$ has an integral expression

$$(1.2.8) \qquad \tilde{M}_{\mathfrak{p}}(s; z_1, z_2) = \int_{\mathbf{C}^1} \exp\left(\frac{i}{2}(z_1 g_{s,\mathfrak{p}}(t^{-1}) + z_2 g_{s,\mathfrak{p}}(t))\right) d^\times t,$$

where $g_{s,\mathfrak{p}}(t)$ is a continuous function on $\mathbf{C}^1 = \{t \in \mathbf{C}; |t| = 1\}$ defined by

$$(1.2.9) \qquad g_{s,\mathfrak{p}}(t) = \begin{cases} \frac{-(\log N(\mathfrak{p})) N(\mathfrak{p})^{-s} t}{1 - N(\mathfrak{p})^{-s} t} & \text{(Case 1)}, \\ -\log(1 - N(\mathfrak{p})^{-s} t) & \text{(Case 2)}, \end{cases}$$

(the principal branch of the logarithm), and $d^\times t$ is the normalized Haar measure of $\mathbf{C}^1$. It also has the following power series expansion in z_1, z_2;

$$\tilde{M}_{\mathfrak{p}}(s; z_1, z_2) = 1 + \sum_{a,b\geq 1} (\pm i/2)^{a+b} \mu_{\mathfrak{p}}^{(a,b)}(s) \frac{z_1^a z_2^b}{a!b!}, \tag{1.2.10}$$

where the sign is *minus* (resp. *plus*) for Case 1(resp. Case 2), and

$$\mu_{\mathfrak{p}}^{(a,b)}(s) = \begin{cases} (\log N(\mathfrak{p}))^{a+b} \sum_{n\geq \mathrm{Max}(a,b)} \binom{n-1}{a-1}\binom{n-1}{b-1} N(\mathfrak{p})^{-2ns} & (\text{Case } 1), \\ \sum_{n\geq \mathrm{Max}(a,b)} \delta_a(n)\delta_b(n) N(\mathfrak{p})^{-2ns} & (\text{Case } 2). \end{cases} \tag{1.2.11}$$

In particular,

$$\mu_{\mathfrak{p}}^{(1,1)}(s) = \begin{cases} (\log N(\mathfrak{p}))^2/(N(\mathfrak{p})^{2s} - 1) & (\text{Case } 1), \\ \sum_{n\geq 1} n^{-2} N(\mathfrak{p})^{-2ns} & (\text{Case } 2). \end{cases} \tag{1.2.12}$$

The global function $\tilde{M}(s; z_1, z_2)$, for each s with $\Re(s) > 1/2$, has an everywhere absolutely convergent power series expansion in z_1, z_2;

$$\tilde{M}(s; z_1, z_2) = 1 + \sum_{a,b\geq 1} (\pm i/2)^{a+b} \mu^{(a,b)}(s) \frac{z_1^a z_2^b}{a!b!}, \tag{1.2.13}$$

with the same choice of the sign as above. Here, each $\mu^{(a,b)}(s)$ denotes the following Dirichlet series which is absolutely convergent on $\Re(s) > 1/2$;

$$\mu^{(a,b)}(s) = \sum_{D: integral} \Lambda_a(D)\Lambda_b(D)N(D)^{-2s}, \tag{1.2.14}$$

where $\Lambda_k(D)$ (≥ 0) for each integral divisor D is defined by

$$\Lambda_k(D) = \sum_{D=D_1...D_k} \Lambda_1(D_1)...\Lambda_1(D_k), \tag{1.2.15}$$

where

$$\Lambda_1(D) = \begin{cases} \log N(\mathfrak{p}) & (\text{Case } 1), \\ 1/n & (\text{Case } 2), \end{cases} \tag{1.2.16}$$

if $D = \mathfrak{p}^n$ with some $\mathfrak{p} \notin P_\infty$ and $n \geq 1$, and $\Lambda_1(D) = 0$ otherwise. By comparing the coefficients of $z_1 z_2$ for $\tilde{M}_{\mathfrak{p}}(s; z_1, z_2)$ and $\tilde{M}(s; z_1, z_2)$ in the formula (1.2.6), we obtain the Euler sum expansion (only for $(a, b) = (1, 1)$):

$$\mu(s) := \mu^{(1,1)}(s) = \sum_{\mathfrak{p}\notin P_\infty} \mu_{\mathfrak{p}}^{(1,1)}(s) \qquad (\Re(s) > 1/2). \tag{1.2.17}$$

Finally, let $M_\sigma(w)$ ($\sigma > 1/2$, $w \in \mathbf{C}$) denote the "M-function" defined and studied in [2](Case 1) and [5](Case 2). (In the latter, it is denoted as $\mathcal{M}_\sigma(w)$.) Then its Fourier dual is $\tilde{M}_\sigma(z) := \tilde{M}(\sigma; z, \bar{z})$. In fact, if ψ_{z_1,z_2} ($z_1, z_2 \in \mathbf{C}$) denotes the quasi-character $\mathbf{C} \to \mathbf{C}^\times$ defined by

$$\psi_{z_1,z_2}(w) = \exp\left(\frac{i}{2}(z_1\overline{w} + z_2 w)\right), \tag{1.2.18}$$

and if we put $\psi_z = \psi_{z,\bar{z}}$ (which is a character $\mathbf{C} \to \mathbf{C}^1$), then we have

$$\tilde{M}(\sigma; z_1, z_2) = \int_{\mathbf{C}} M_\sigma(w)\psi_{z_1,z_2}(w)|dw|, \tag{1.2.19}$$

$$M_\sigma(w) = \int_{\mathbf{C}} \tilde{M}_\sigma(z)\psi_{-w}(z)|dz|, \tag{1.2.20}$$

where $|dw| = dxdy/2\pi$ for $w = x + yi$. Both $M_\sigma(w)$ and $\tilde{M}_\sigma(z)$ are continuous functions on $\mathbf{C}$ belonging to L^1; hence the Plancherel formula holds. Recall also ([6] §4.2) that the center of gravity of $M_\sigma(w)|dw|$ is 0, and that $\mu(\sigma) = \mu^{(1,1)}(\sigma)$ ($\sigma > 1/2$) is equal to the variance

$$\mu_\sigma := \mu(\sigma) = \int M_\sigma(w)|w|^2|dw|. \tag{1.2.21}$$

It is easy to see (cf. §3 below) that $\lim_{\sigma\to 1/2}\mu_\sigma = +\infty$ and $\lim_{\sigma\to+\infty}\mu_\sigma = 0$ (Cases 1,2).

Now let ν_σ denote the Plancherel volume of $M_\sigma(w)$. In connection with Examples 1,2,3 (§1.1), where $\mu\nu = 1,\ 1,\ 8/9$ (the minimal possible value) respectively for $d = 2$, we are interested in studying the product $\mu_\sigma\nu_\sigma$. First, some numerical evidences suggest that $\mu_\sigma\nu_\sigma$ is often quite close to 1. For example, when $K = \mathbf{Q}$ (resp. $\mathbf{Q}(\sqrt{-1})$) and P_∞ consists of the unique archimedean prime of K, then $1 - \mu_1\nu_1 = 0.017...$ (resp. $0.018...$). In §2, we shall study the limit behaviors of the variance-normalized function $\mu_\sigma M_\sigma(\mu_\sigma^{1/2} w)$ and that of $\mu_\sigma\nu_\sigma$ as $\sigma \to 1/2$ and $\sigma \to \infty$ for general cases of (K, P_∞). Here, we just add, without proof (cf. [3] for a sketch of proof) the following

Example 4 Let $K = \mathbf{F}_q(x)$ be the rational function field over a finite field $\mathbf{F}_q$ and $P_\infty = \{\mathfrak{p}_\infty\}$ the unique prime at which $x = \infty$. Write $\mu_\sigma^{(q)}$ (resp. $\nu_\sigma^{(q)}$) for the variance (resp. the Plancherel volume) of $M_\sigma(w)|dw|$. Then for *any fixed* $\sigma > 1/2$, at least in Case 1, we have

$$\lim_{q\to\infty}(\mu_\sigma^{(q)}\nu_\sigma^{(q)}) = 1. \tag{1.2.22}$$

Is $\mu_\sigma\nu_\sigma$ related to some invariant with a different origin? Is there a complex analytic version of ν_σ?

2 Limits at $\sigma = 1/2$ and $\sigma = +\infty$.

Let μ_σ (resp. ν_σ) denote the variance (resp. the Plancherel volume; cf.§1.1) of the measure $M_\sigma(w)|dw|$ ($\sigma > 1/2$). We shall study the limits, first at $\sigma = 1/2$, then (briefly) those at $\sigma = +\infty$, of the invariant $\mu_\sigma\nu_\sigma$ and of the variance-normalized function $\mu_\sigma M_\sigma(\mu_\sigma^{1/2} w)$. In this section, we shall state the main results, Theorem 2 for $\sigma \to 1/2$ and Theorem 3 for $\sigma \to +\infty$, and reduce their proofs to Lemmas A,B (for Theorem 2) and to Lemmas A',B' (for Theorem 3). The Lemmas A,A' are for the limits of

$$\tilde{M}(s; \frac{z_1}{\mu(s)^{1/2}}, \frac{z_2}{\mu(s)^{1/2}})$$

as $s \to 1/2, +\infty$ respectively, where $\mu(s) = \mu^{(1,1)}(s)$ is the complex analytic version of μ_σ. Lemmas B, B' are on the rapid decay property of the normalized Fourier dual $\tilde{M}_\sigma(z/\mu_\sigma^{1/2})$. The proofs of these lemmas will be postponed to later sections (except Lemma A'). Because of its introductory nature, we have set this section right after §1, in spite of its logical dependence on later sections.

2.1 – The main results for $\sigma \to 1/2$.

Theorem 2 *(i) As $\sigma \to 1/2$,*

$$\text{(2.1.1)} \qquad \mu_\sigma \sim \begin{cases} (2\sigma-1)^{-2} & \text{(Case 1)}, \\ \log \frac{1}{2\sigma - 1} & \text{(Case 2)}, \end{cases}$$

where $\sim$ means that the ratio of two sides tends to 1.
(ii)

$$\text{(2.1.2)} \qquad \lim_{\sigma \to 1/2} (\mu_\sigma \nu_\sigma) = 1 \qquad \text{(Cases 1, 2).}$$

(iii)[1]

$$\text{(2.1.3)} \qquad \lim_{\sigma \to 1/2} (\mu_\sigma M_\sigma(\mu_\sigma^{1/2} w))) = 2e^{-|w|^2} \qquad (w \in \mathbf{C}) \text{ (Cases 1, 2).}$$

These answer "the $\lim_{\sigma\to 1/2}$-version" of the questions raised in [2] Remark 3.11.17.

[1] The author is grateful to S. Takanobu for helpful discussions which lead to this generalized form of the result.

2.2 – The proof of Theorem 2(i). This follows directly from (1.2.12) and (1.2.17). But we also note that (with the notations of §3.3) the easiest case of Theorem 4 asserts that the difference $\mu(s) - \phi^{(2\kappa)}(2s)$ extends to a holomorphic function on $\Re(s) > 1/4$. Hence

$$\lim_{\substack{s\to 1/2\\ |\mathrm{Arg}(2s-1)|<\pi}} \frac{\mu(s)}{\phi^{(2\kappa)}(2s)} = 1; \tag{2.2.1}$$

hence

$$\lim_{s\to 1/2} (2s-1)^2\mu(s) = 1 \qquad \text{(Case 1)}, \tag{2.2.2}$$

$$\lim_{\substack{s\to 1/2\\ |\mathrm{Arg}(2s-1)|<\pi}} \frac{\mu(s)}{\log\frac{1}{2s-1}} = 1 \qquad \text{(Case 2)}, \tag{2.2.3}$$

as desired.

For any s with $|2s-1| \ll 1$ and $|\mathrm{Arg}(2s-1)| < \pi$, we define $\mu(s)^{1/2}$ to be the square root taking positive value when s is real and $> 1/2$.

2.3 – The Key Lemmas A,B. The first key lemma is Corollary 3.4.8 (§3.4) of Theorem 5 to be proved in the next section.

Lemma A *We have*

$$\lim_{\substack{s\to 1/2\\ |\mathrm{Arg}(2s-1)|<\pi}} \tilde{M}\left(s; \frac{z_1}{\mu(s)^{1/2}}, \frac{z_2}{\mu(s)^{1/2}}\right) = \exp\left(-\frac{z_1 z_2}{4}\right), \tag{2.3.1}$$

and the convergence is uniform on $|z_1|, |z_2| \le R$ *for any given* $R > 0$.

The second key lemma is related to a rapid decay property of the function $\tilde{M}_\sigma(z) := \tilde{M}(\sigma; z, \bar{z})$ of $z \in \mathbf{C}$, to be proved in §4.6.

Lemma B *Fix any* ϵ *with* $0 < \epsilon < 1$. *If* $(2\sigma-1)^{-1} \gg_\epsilon 1$, *then the inequality*

$$|\tilde{M}_\sigma(z)|^2 \le \exp\left(-\frac{1-\epsilon}{2}\mu_\sigma |z|^{2(1-\epsilon)}\right) \tag{2.3.2}$$

holds for all $z \in \mathbf{C}$.

2.4 – Proof of Theorem 2(ii)(iii) assuming Lemmas A,B.

[**Proof of (ii)**] Note first that

$$\mu_\sigma \nu_\sigma = \int |\tilde{M}_\sigma(\mu_\sigma^{-1/2} z)|^2 |dz|. \tag{2.4.1}$$

For each fixed z, the integrand tends to $\exp(-|z|^2/2)$ by Lemma A. In order to apply Lebesgue's convergence theorem to the effect that $\lim_{\sigma\to 1/2}$ operation commutes with the integration, we only need to show that the integrand is uniformly bounded near $\sigma = 1/2$ by an integrable function of z. But this follows directly from Lemma B. In fact, Lemma B for $\epsilon = 1/2$ gives $|\tilde{M}_\sigma(z)|^2 \leq \exp(-\mu_\sigma |z|/4)$. Since $\mu_\sigma > 1$ if σ is sufficiently close to $1/2$, we have for such σ

$$|\tilde{M}_\sigma(\mu_\sigma^{-1/2} z)|^2 \leq \exp(-\mu_\sigma^{1/2}|z|/4) \leq \exp(-|z|/4), \tag{2.4.2}$$

which is integrable. Therefore,
(2.4.3)

$$\lim_{\sigma\to 1/2}(\mu_\sigma \nu_\sigma) = \int \lim_{\sigma\to 1/2} |\tilde{M}_\sigma(\mu_\sigma^{-1/2} z)|^2 |dz| = \int \exp(-|z|^2/2)|dz| = \int_0^\infty e^{-r^2/2} r dr = 1,$$

as desired.

[**Proof of (iii)**] The Fourier inversion formula (1.2.20) gives

$$\mu_\sigma M_\sigma(\mu_\sigma^{1/2} w) = \int \tilde{M}_\sigma(\mu_\sigma^{-1/2} z)\psi_{-w}(z)|dz|. \tag{2.4.4}$$

By Lemma A and (2.4.2), we can also apply Lebesgue's convergence theorem and hence obtain

$$\lim_{\sigma\to 1/2}(\mu_\sigma M_\sigma(\mu_\sigma^{1/2} w)) = \int \exp(-|z|^2/4)\psi_{-w}(z)|dz| = 2e^{-|w|^2}, \tag{2.4.5}$$

as desired. □

2.5 – The main results for $\sigma \to +\infty$.

The following numerical invariants of the pair (K, P_∞),

$$\alpha := \mathrm{Min}_{\mathfrak{p}\notin P_\infty} N(\mathfrak{p}), \qquad m := |\{\mathfrak{p} \notin P_\infty;\ N(\mathfrak{p}) = \alpha\}|$$

($|\;|$: the cardinality), and the Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n} \tag{2.5.1}$$

are involved. Clearly, $\alpha \geq 2$ and $m \geq 1$. The main results corresponding to Theorem 2 are the following:

Theorem 3 *(i) As $\sigma \to +\infty$,*

$$\mu_\sigma \sim \begin{cases} (\log\alpha)^2 m\alpha^{-2\sigma} & (\text{Case}\,1), \\ m\alpha^{-2\sigma} & (\text{Case}\,2). \end{cases} \tag{2.5.2}$$

(ii) In each of Cases 1,2,

$$\lim_{\sigma\to+\infty}(\mu_\sigma\nu_\sigma) = m\int_0^\infty J_0(x)^{2m}xdx \qquad \begin{cases} = \infty & (m \le 2), \\ < \infty & (m \ge 3). \end{cases} \tag{2.5.3}$$

(iii) In each of Cases 1,2, (at least) if $m \ge 5$, we have

$$\lim_{\sigma\to+\infty}(\mu_\sigma M_\sigma(\mu_\sigma^{1/2}w))) = \int_0^\infty J_0(|w|x)J_0(x/\sqrt{m})^m xdx. \tag{2.5.4}$$

Moreover, the support of this function is compact, being contained in $\{w \in \mathbf{C};\ |w| \le \sqrt{m}\}$.

2.6 – Proof of Theorem 3(i).

We shall show a slightly stronger result;

$$\lim_{\sigma=\Re(s)\to+\infty}(\alpha^{2s}\mu(s)) = \begin{cases} (\log\alpha)^2 m & (\text{Case 1}), \\ m & (\text{Case 2}), \end{cases} \tag{2.6.1}$$

the convergence being uniform in $\Im(s)$. First, by (1.2.12) and (1.2.17) we have

$$\alpha^{2s}\mu(s) = \alpha^{2s}\sum_{\mathfrak{p}\notin P_\infty}\mu_{\mathfrak{p}}^{(1,1)}(s) = \sum_{\substack{\mathfrak{p}\notin P_\infty \\ n\ge 1}} a(\mathfrak{p}^n)(\alpha/N(\mathfrak{p})^n)^{2s}, \tag{2.6.2}$$

where $a(\mathfrak{p}^n) = (\log N(\mathfrak{p}))^2$ (resp. $1/n^2$) for Case 1 (resp. Case 2). Now decompose the sum into three parts; the first sum over those $(\mathfrak{p}, n)$ satisfying $N(\mathfrak{p}) = \alpha$ and $n = 1$ gives the RHS of (2.6.1); the second, over $N(\mathfrak{p}) > \alpha$ is $\ll (\alpha/\alpha')^{2\sigma-2}$, where α' denotes the second smallest norm outside P_∞; the rest is over $N(\mathfrak{p}) = \alpha$, $n \ge 2$, which is $\ll \alpha^{-2\sigma}$. Since the latter two partial sums tend to 0 uniformly w.r.t. $\Im(s)$, this proves (2.6.1).

In particular, $\mu(s) \ne 0$ for $\Re(s)$ sufficiently large. We shall denote by $\mu(s)^{1/2}$ its unique square root that takes positive values when $s = \sigma > 1$.

2.7 – The Key Lemmas A',B'.

The counterparts of Lemmas A,B for the case $\lim_{\sigma\to+\infty}$ are the following.

Lemma A' *We have*

$$\lim_{\sigma=\Re(s)\to+\infty}\tilde{M}\left(s;\ \frac{z_1}{\mu(s)^{1/2}}, \frac{z_2}{\mu(s)^{1/2}}\right) = J_0\left(\sqrt{\frac{z_1z_2}{m}}\right)^m, \tag{2.7.1}$$

and the convergence is uniform on $|z_1|, |z_2| \leq R$ for any given $R > 0$ and w.r.t. $\Im(s)$.

The proof will be sketched in §2.9.

Lemma B' *There exists a constant $C > 0$ depending only on (K, P_∞) such that*

$$|\tilde{M}_\sigma(\mu_\sigma^{-1/2} z)| \leq C|z|^{-m/2} \tag{2.7.2}$$

for all $\sigma \geq 1$ and all $z \in \mathbf{C}$.

The proof of this key lemma will be postponed until §4.1.

2.8 – Proof of Theorem 3(ii)(iii) assuming Lemmas A',B'.

[**Proof of (ii)**] The limit formula (2.5.3) for $m \geq 3$ can be obtained from Lemmas A',B' exactly in the same manner as in the proof of Theorem 2 (ii). For $m \leq 2$, the divergences can be checked easily.

[**Proof of (iii)**] When $m \geq 5$, there is again no problem. (The term $J_0(|w|x)$ appears as the average of $\psi_{-w}(z)$ over the circle $|z| = x$.) It is likely that the same equality holds also for smaller m. But it should be noted that the limit function of w need not be continuous. Especially when $m = 1$, the limit is not even a function; a hyperfunction with support on the unit circle $|w| = 1$. This is because at the limit $\sigma \to \infty$, only the contribution of the unique prime $\mathfrak{p}$ with $N(\mathfrak{p}) = \alpha$ remains.

As for the statement on the support, we can see this in two ways. Firstly, by construction, the support of $M_\sigma(w)$ for $\sigma > 1$ is contained in $|w| \leq \rho_\sigma$, where

$$\rho_\sigma = \begin{cases} -\frac{d}{d\sigma} \log \zeta_{K,P_\infty}(\sigma) \sim m\alpha^{-\sigma} \log \alpha & \text{(Case 1)}, \\ \log \zeta_{K,P_\infty}(\sigma) \sim m\alpha^{-\sigma} & \text{(Case 2)}; \end{cases} \tag{2.8.1}$$

hence $\lim_{\sigma \to +\infty}(\rho_\sigma / \mu_\sigma^{1/2}) = \sqrt{m}$. Secondly, on the RHS of (2.5.4), one can also see this by a result of Nicholson (cf.[9]§13.46), which asserts that if $\Re(\nu) > -1$, $a_1, \cdots, a_m > 0$, $b > a_1 + ... + a_m$, then

$$\int_0^\infty x^{\nu(1-m)+1} J_\nu(bx) \prod_{i=1}^m J_\nu(a_i x) dx = 0 \tag{2.8.2}$$

(our m corresponds to $m-1$ in [9]). Apply this for $\nu = 0$, $a_1 = ... = a_m = 1/\sqrt{m}$, $b = |w|$, to see that the RHS of (2.5.4) vanishes for $|w| \geq \sqrt{m}$.

Remark 2.8.3 *As for the value of the RHS of (2.5.3), i.e.,*

$$a(m) := m \int_0^\infty J_0(x)^{2m} x dx, \tag{2.8.4}$$

we have $a(3) = 1.01\cdots$, $a(4) = 0.951\cdots$, $a(5) = 0.953\cdots$, etc., and one can prove that $\lim_{m\to\infty} a(m) = 1$. *Numerical evidences suggest* $\lim_{m\to\infty} m(1 - a(m)) = 1/4$.

2.9 – Sketch of the proof of Lemma A'. The power series expansion (1.2.13) of $\tilde{M}(s; z_1, z_2)$ gives

$$\tilde{M}\left(s; \frac{z_1}{\mu(s)^{1/2}}, \frac{z_2}{\mu(s)^{1/2}}\right) = 1 + \sum_{a,b\geq 1} (\pm i/2)^{a+b} \frac{\mu^{(a,b)}(s)}{\mu(s)^{\frac{a+b}{2}}} \frac{z_1^a z_2^b}{a!b!}. \tag{2.9.1}$$

Here, as in (1.2.13), the sign of $i/2$ is *minus* (Case 1), *plus* (Case 2). On the other hand, the expansion (2.5.1) for $J_0(x)$ gives

$$J_0\left(\left(\frac{z_1 z_2}{m}\right)^{1/2}\right)^m = 1 + \sum_{a,b\geq 1} (-i/2)^{a+b} \tilde{\mu}^{(a,b)} \frac{z_1^a z_2^b}{a!b!}, \tag{2.9.2}$$

where

$$\tilde{\mu}^{(a,b)} = \begin{cases} 0 & (a \neq b), \\ m^{-k} \sum_{k=k_1+\ldots+k_m} \binom{k}{k_1, ..., k_m}^2 & (a = b = k \geq 1). \end{cases} \tag{2.9.3}$$

So, it is enough to prove that there exist constants $\sigma_0 > 1$ and $C > 0$, each depending only on (K, P_∞), such that

$$\frac{\mu^{(a,b)}(s)}{\mu(s)^{\frac{a+b}{2}}} - \tilde{\mu}^{(a,b)} \ll \frac{C^{a+b}}{\sigma - \sigma_0 - 1} \qquad (\sigma = \Re(s) > \sigma_0 + 1) \tag{2.9.4}$$

holds. Note that the LHS of (2.9.4) is 0 when $a = b = 1$.

To prove (2.9.4), note first that (1.2.14) gives

(2.9.5)

$$\alpha^{(a+b)s} \mu^{(a,b)}(s) = \sum_{D:integral} \Lambda_a(D)\Lambda_b(D)(\alpha^{a+b}/N(D)^2)^s = \sum_{D:integral}{}' \Lambda_a(D)\Lambda_b(D)(\alpha^{a+b}/N(D)^2)^s,$$

where $\sum'$ denotes the sum over non-vanishing terms.

Proposition 2.9.6 *Let $\{\mathfrak{p}_1, ..., \mathfrak{p}_m\}$ be all the distinct primes $\notin P_\infty$ with norm α. Let $k \geq 1$ and D be any integral divisor. If $\Lambda_k(D) \neq 0$, then $N(D) \geq \alpha^k$ and the equality holds if and only if D has the form $D = \prod_{i=1}^m \mathfrak{p}_i^{k_i}$ with $\sum k_i = k$. Moreover, in this case,*

$$\Lambda_k(D) = \binom{k}{k_1, ..., k_m} (\log \alpha)^{\kappa k}, \tag{2.9.7}$$

where $\kappa = 1(\text{Case1}), \kappa = 0(\text{Case2})$.

This is almost obvious. By this Proposition, we may rewrite (2.9.5) as $I^{(a,b)} + II^{(a,b)}(s)$, where

$$\begin{aligned} I^{(a,b)} &= \sum_{N(D)^2 = \alpha^{a+b}} {}' \Lambda_a(D)\Lambda_b(D) \\ &= \begin{cases} 0 & (a \neq b) \\ \left(\sum_{k=k_1+\ldots+k_m} \binom{k}{k_1, \ldots, k_m}^2 \right) (\log \alpha)^{\kappa(a+b)} & (a = b = k), \end{cases} \\ II^{(a,b)}(s) &= \sum_{N(D)^2 > \alpha^{a+b}} {}' \Lambda_a(D)\Lambda_b(D)(\alpha^{a+b}/N(D)^2)^s. \end{aligned}$$

In particular, $I^{(1,1)} = m(\log \alpha)^{2\kappa}$; hence

$$\tilde{\mu}^{(a,b)} = \frac{I^{(a,b)}}{(I^{(1,1)})^{\frac{a+b}{2}}}. \tag{2.9.8}$$

Therefore,

$$\frac{\mu^{(a,b)}(s)}{\mu(s)^{\frac{a+b}{2}}} - \tilde{\mu}^{(a,b)} = \frac{I^{(a,b)} + II^{(a,b)}(s)}{(I^{(1,1)} + II^{(1,1)}(s))^{\frac{a+b}{2}}} - \frac{I^{(a,b)}}{(I^{(1,1)})^{\frac{a+b}{2}}}. \tag{2.9.9}$$

In order to estimate the quantity $II^{(a,b)}(s)$, we need the following

Proposition 2.9.10 *There exists $\sigma_0 > 1$ depending only on (K, P_∞) such that*

$$\Lambda_k(D) < N(D)^{\sigma_0} \tag{2.9.11}$$

holds for any D and any $k \geq 1$.

The point is that the present bound is independent of k.

Proof Since $\lim_{\sigma \to +\infty}(\zeta'_{K,P_\infty}/\zeta_{K,P_\infty})(\sigma) = 0$, we have

$$0 < -\frac{\zeta'_{K,P_\infty}}{\zeta_{K,P_\infty}}(\sigma_0) = \sum_D \frac{\Lambda_1(D)}{N(D)^{\sigma_0}} < 1$$

for sufficiently large $\sigma_0 > 1$. But then its k-th power is also < 1; hence

$$\sum_D \frac{\Lambda_k(D)}{N(D)^{\sigma_0}} < 1. \tag{2.9.12}$$

Since each summand is non-negative, this implies $\Lambda_k(D) < N(D)^{\sigma_0}$ for each D, as desired.

By using Prop 2.9.10, we can easily derive

$$(2.9.13) \qquad |II^{(a,b)}(s)| \ll \frac{(\alpha^{\sigma_0+1})^{a+b}}{\sigma-\sigma_0-1} \qquad (\sigma = \Re(s) > \sigma_0+1),$$

and by combining these we obtain (2.9.4) directly. □

3 Analytic continuations

3.1 – Local formal power series. In connection with the local factors of $\tilde{M}(s; z_1, z_2)$, we consider, in each of Cases 1,2, the following power series $F = F(x_1, x_2; t)$ in 3 variables

$$(3.1.1) \qquad F(x_1, x_2; t) = \sum_{n=0}^{\infty} F_n(x_1)F_n(x_2)t^n = 1 + \sum_{n=1}^{\infty} F_n(x_1)F_n(x_2)t^n,$$

where each $F_n(x)$ is a polynomial of x of degree n defined by (1.2.3), or equivalently, by the generating functions

$$(3.1.2) \qquad \exp\left(\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} F_n(x)t^n \qquad \text{(Case 1)},$$

$$(3.1.3) \qquad \exp(-x\log(1-t)) = (1-t)^{-x} = \sum_{n=0}^{\infty} F_n(x)t^n \qquad \text{(Case 2)}.$$

Note that each monomial $x_1^a x_2^b t^n$ appearing in $F-1$ satisfies $1 \le a, b \le n$, and has a positive rational coefficient. Define also the formal power series $\log F$, by $\sum_{k=1}^{\infty}(-1)^{k-1}(F-1)^k/k$, and express it as a power series of x_1, x_2, t as

$$(3.1.4) \qquad \log F(x_1, x_2; t) = \sum_{a,b,n\ge 1} \beta_n^{(a,b)} \frac{x_1^a x_2^b}{a!b!} t^n \qquad (\beta_n^{(a,b)} \in \mathbf{Q})$$

$$= \sum_{n\ge 1} B_n(x_1, x_2)t^n = \sum_{a,b\ge 1} B^{(a,b)}(t)\frac{x_1^a x_2^b}{a!b!}.$$

Note that $\beta_n^{(a,b)} = 0$ if $n < \mathrm{Max}(a,b)$; hence

$$(3.1.5) \qquad B_n(x_1, x_2) = \sum_{1\le a,b\le n} \beta_n^{(a,b)} \frac{x_1^a x_2^b}{a!b!},$$

$$(3.1.6) \qquad B^{(a,b)}(t) = \sum_{n\ge \mathrm{Max}(a,b)} \beta_n^{(a,b)} t^n.$$

For example,

$$B^{(1,1)}(t) = \begin{cases} t(1-t)^{-1} & \text{(Case 1)}, \\ \sum_{n=1}^{\infty} t^n/n^2 & \text{(Case 2)}; \end{cases} \tag{3.1.7}$$

hence $\beta_n^{(1,1)} = 1$ (Case 1), $= 1/n^2$ (Case 2).

In connection with the local factors of higher logarithmic derivatives of the zeta function, we also consider the power series

$$\ell(t) = \ell_0(t) = -\log(1-t), \tag{3.1.8}$$

and for each $k \geq 0$,

$$\ell_k(t) = \left(t\frac{d}{dt}\right)^k \ell(t) = \sum_{n=1}^{\infty} n^{k-1} t^n = t + \cdots. \tag{3.1.9}$$

They have the generating function

$$\ell(te^u) = \sum_{k=0}^{\infty} \frac{\ell_k(t)}{k!} u^k. \tag{3.1.10}$$

Put

$$\kappa = \begin{cases} 1 & \text{(Case 1)}, \\ 0 & \text{(Case 2)}. \end{cases}$$

For each fixed $a, b \geq 1$, $\{\ell_{\kappa(a+b)}(t^n)\}_{n=1,2,\ldots}$ forms a $\mathbf{Q}$-linear topological basis of the power series algebra $\mathbf{Q}[[\mathbf{t}]]$ equipped with the t-adic topology. Hence there exists a unique system $\{\gamma_n^{(a,b)}\}_{n,a,b\geq 1}$ of rational numbers such that

$$B^{(a,b)}(t) = \sum_{n\geq 1} \gamma_n^{(a,b)} \ell_{\kappa(a+b)}(t^n) \tag{3.1.11}$$

holds for any $a, b \geq 1$. It is clear from the definition that $\gamma_n^{(a,b)} = 0$ if $n < \mathrm{Max}(a,b)$, and that

$$\beta_m^{(a,b)} = \sum_{n|m} \gamma_n^{(a,b)} (m/n)^{\kappa(a+b)-1} \tag{3.1.12}$$

$(m = 1, 2, \cdots)$; hence the Möbius inversion formula gives

$$\gamma_n^{(a,b)} = \sum_{d|n} \mu(n/d)(n/d)^{\kappa(a+b)-1} \beta_d^{(a,b)}. \tag{3.1.13}$$

For example, $\gamma_1^{(1,1)} = 1$, and for $n > 1$, $\gamma_n^{(1,1)} = \prod_{\ell|n}(1-\ell)$ (Case 1), and n^{-2}-times this quantity in Case 2, where ℓ runs over all prime factors of n. By (3.1.4) and (3.1.11), we have the formal equality

$$\log F(x_1, x_2; t) = \sum_{\substack{n,a,b\geq 1 \\ n\geq \mathrm{Max}(a,b)}} \gamma_n^{(a,b)} \ell_{\kappa(a+b)}(t^n) \frac{x_1^a x_2^b}{a!b!}. \tag{3.1.14}$$

3.2 – Local analytic functions. We start with the following

Proposition 3.2.1 *(i) $F(x_1, x_2; t)$ defines a holomorphic function of $x_1, x_2, t \in \mathbf{C}$ on $|t| < 1$. (ii) Let $R > 0$, $0 < r < 1$; $|x_1|, |x_2| \leq R, |t| \leq r$, and suppose that one of r, R is fixed and the other is sufficiently small. Then $|F(x_1, x_2; t) - 1| < 1$; hence the principal logarithm $\log F(x_1, x_2; t)$ is holomorphic on this domain.*

Proof Note first that in each of Cases 1,2, the equality (3.1.2) resp. (3.1.3) is valid also as a formula for analytic functions of x, t on $|t| < 1$. Recall also that the coefficients of $F_n(x)$ are non-negative. Thus, for any $N \geq 1$,

$$\begin{aligned} \sum_{n=1}^{N} |F_n(x_1)F_n(x_2)t^n| &\leq \sum_{n=1}^{N} F_n(R)^2 r^n \leq \left(\sum_{n=1}^{N} F_n(R) r^{n/2}\right)^2 \\ &< \left(\sum_{n=1}^{\infty} F_n(R) r^{n/2}\right)^2 = \begin{cases} \left(\exp(\frac{Rr^{1/2}}{1-r^{1/2}}) - 1\right)^2 & \text{(Case 1)}, \\ \left((1-r^{1/2})^{-R} - 1\right)^2 & \text{(Case 2)}. \end{cases} \end{aligned} \tag{3.2.2}$$

The rest is obvious.

Corollary 3.2.3 *(i) For each $a, b \geq 1$, the series (3.1.6) converges absolutely on $|t| < 1$ and hence defines a holomorphic function $B^{(a,b)}(t)$ on $|t| < 1$. (ii) The assumptions being as in that of (ii) of Proposition 3.2.1, the three series in (3.1.4) are absolutely convergent, and the three equalities there are valid as those for analytic functions.*

Now let $\mathfrak{p}$ be any non-archimedean prime divisor of the base field K, and put

$$\lambda_{\mathfrak{p}} = (-\log N(\mathfrak{p}))^{\kappa} = \begin{cases} -\log N(\mathfrak{p}) & \text{(Case 1)}, \\ 1 & \text{(Case 2)}. \end{cases} \tag{3.2.4}$$

Then it follows directly from the definitions (§1.2) that

$$\tilde{M}_{\mathfrak{p}}(s; z_1, z_2) = F((i\lambda_{\mathfrak{p}}/2)z_1, (i\lambda_{\mathfrak{p}}/2)z_2; N(\mathfrak{p})^{-2s}) \tag{3.2.5}$$

$(s, z_1, z_2 \in \mathbf{C}, \Re(s) > 0)$. For each $a, b \geq 1$, define the holomorphic function $\mathbf{B}_{\mathfrak{p}}^{(a,b)}(s)$ on $\Re(s) > 0$ by

$$\mathbf{B}_{\mathfrak{p}}^{(a,b)}(s) = \lambda_{\mathfrak{p}}^{a+b} B^{(a,b)}(N(\mathfrak{p})^{-2s}). \tag{3.2.6}$$

In the special case $a = b = 1$, we have by (1.2.12) and (3.1.7),

$$\mathbf{B}_{\mathfrak{p}}^{(1,1)}(s) = \mu_{\mathfrak{p}}^{(1,1)}(s) = \begin{cases} (\log N(\mathfrak{p}))^2 (N(\mathfrak{p})^{2s} - 1)^{-1} & \text{(Case 1)}, \\ \sum_{n=1}^{\infty} \frac{1}{n^2} N(\mathfrak{p})^{-2ns} & \text{(Case 2)}. \end{cases} \tag{3.2.7}$$

Corollary 3.2.8 *Let $R > 0$, $\alpha \geq 2, \sigma_0 > 0$, and $|z_1|, |z_2| \leq R$, $N(\mathfrak{p}) \geq \alpha$, $\Re(s) \geq \sigma_0$. Suppose that two of R, α, σ_0 are fixed and the remaining one, if R, is sufficiently small while if α or σ_0, is sufficiently large. Then $|\tilde{M}_{\mathfrak{p}}(s; z_1, z_2) - 1| < 1$, and*

$$\log \tilde{M}_{\mathfrak{p}}(s; z_1, z_2) = \sum_{a,b \geq 1} \mathbf{B}_{\mathfrak{p}}^{(a,b)}(s) (i/2)^{a+b} \frac{z_1^a z_2^b}{a! b!}. \tag{3.2.9}$$

Proof In Case 2, this is obvious by (3.2.5) and Cor 3.2.3(ii). In Case 1, the difference between $|z_\nu|$ and $|x_\nu|$ $(\nu = 1, 2)$ involves $\log N(\mathfrak{p})$. But since $(\log N(\mathfrak{p})) N(\mathfrak{p})^{-\sigma_0}$ is bounded and it tends to 0 when one of α, σ_0 tends to ∞, the same proof works. □

Now put

$$\phi_{\mathfrak{p}}(s) = \ell(N(\mathfrak{p})^{-s}) = -\log(1 - N(\mathfrak{p})^{-s}) \qquad (\Re(s) > 0), \tag{3.2.10}$$

and for $k = 0, 1, 2, ...$,

$$\phi_{\mathfrak{p}}^{(k)}(s) = \frac{d^k \phi_{\mathfrak{p}}}{ds^k}(s) = (-\log N(\mathfrak{p}))^k \ell_k(N(\mathfrak{p})^{-s}) \quad (\Re(s) > 0). \tag{3.2.11}$$

In particular, for $k = \kappa(a+b)$ and $n = 1, 2, ...$,

$$\phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns) = \lambda_{\mathfrak{p}}^{a+b} \ell_{\kappa(a+b)}(N(\mathfrak{p})^{-2ns}). \tag{3.2.12}$$

The formal equalities (3.1.11)(3.1.14) suggest that the corresponding analytic equalities

$$\mathbf{B}_{\mathfrak{p}}^{(a,b)}(s) = \sum_{n \geq \mathrm{Max}(a,b)} \gamma_n^{(a,b)} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns) \tag{3.2.13}$$

$$\log \tilde{M}_{\mathfrak{p}}(s; z_1, z_2) = \sum_{a,b,n \geq 1} \gamma_n^{(a,b)} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns) (i/2)^{a+b} \frac{z_1^a z_2^b}{a! b!} \tag{3.2.14}$$

would hold on some suitable domain where $\tilde{M}_{\mathfrak{p}}(s; z_1, z_2)$ does not vanish. Note that the coefficients $\gamma_n^{(a,b)}$ are independent of $\mathfrak{p}$, so that under some further conditions the globalization would be possible. Our aim is to establish these results (Theorems 4,5).

3.3 – The global analytic functions of s. First, we define the functions $\mathbf{B}^{(a,b)}(s)$ $(a, b \geq 1)$ of s.

Proposition 3.3.1 *Let $a, b \geq 1$. Then*

$$\mathbf{B}^{(a,b)}(s) := \sum_{\mathfrak{p} \notin P_\infty} \mathbf{B}_{\mathfrak{p}}^{(a,b)}(s) \tag{3.3.2}$$

converges absolutely and uniformly on $\sigma = \Re(s) \geq \frac{1+\epsilon}{2\mathrm{Max}(a,b)}$ for any $\epsilon > 0$, thereby defining a holomorphic function on $\sigma > \frac{1}{2\mathrm{Max}(a,b)}$

Proof Since $N(\mathfrak{p})^{-2\sigma} \leq 2^{-1/\mathrm{Max}(a,b)}$, and since $B^{(a,b)}(t)/t^{\mathrm{Max}(a,b)}$ is holomorphic on $|t| < 1$ and hence is bounded on $|t| \leq 2^{-1/\mathrm{Max}(a,b)}$, we have by (3.2.6),

$$|\mathbf{B}_{\mathfrak{p}}^{(a,b)}(s)| \ll_{a,b} (\log N(\mathfrak{p}))^{a+b} N(\mathfrak{p})^{-2\sigma \mathrm{Max}(a,b)} \leq (\log N(\mathfrak{p}))^{a+b} N(\mathfrak{p})^{-1-\epsilon},$$

whose sum over $\mathfrak{p} \notin P_\infty$ converges. □

In the special case $a = b = 1$, we have, by (1.2.17) and (3.2.7),

$$\mathbf{B}^{(1,1)}(s) = \mu^{(1,1)}(s). \tag{3.3.3}$$

Now we shall define the functions $\phi^{(k)}(s)$. Let $\zeta(s) = \zeta_{K,P_\infty}(s)$ be the zeta function of K without P_∞ factors, defined by the Euler product expansion

$$\prod_{\mathfrak{p} \notin P_\infty} (1 - N(\mathfrak{p})^{-s})^{-1} \qquad (\Re(s) > 1) \tag{3.3.4}$$

and by analytic continuation to the whole complex plane. Let

$$\phi(s) = \log \zeta(s), \tag{3.3.5}$$

where the branch of the logarithm is the one that tends to 0 as $\Re(s)$ tends to $+\infty$. It is holomorphic on $\Re(s) > 1$ and is a multivalued analytic function on $\mathbf{C}$ where $\zeta(s) \neq \infty, 0$. For each $k \geq 0$, $\phi^{(k)}(s)$ will denote its k-th derivative with respect to s. Thus, $\phi^{(0)}(s) = \log \zeta(s)$, and for each $k \geq 1$,

$$\phi^{(k)}(s) = \frac{d^{k-1}}{ds^{k-1}}(\zeta'(s)/\zeta(s)) \tag{3.3.6}$$

is a meromorphic function on $\mathbf{C}$. By these definitions we have, for each $k \geq 0$,

$$\phi^{(k)}(s) = \sum_{\mathfrak{p} \notin P_\infty} \phi_{\mathfrak{p}}^{(k)}(s) \qquad (\Re(s) > 1); \tag{3.3.7}$$

hence for each $n \geq 1$,

$$\phi^{(k)}(2ns) = \sum_{\mathfrak{p} \notin P_\infty} \phi^{(k)}_{\mathfrak{p}}(2ns) \qquad (\Re(s) > 1/2n). \tag{3.3.8}$$

In particular, if $n \geq \mathrm{Max}(a, b)$, then $\phi^{(\kappa(a+b))}(2ns)$ is holomorphic on $\Re(s) > 1/(2\mathrm{Max}(a, b))$.

Theorem 4 *Let $a, b \geq 1$. Then the equality*

$$\mathbf{B}^{(a,b)}(s) = \sum_{n \geq \mathrm{Max}(a,b)} \gamma_n^{(a,b)} \phi^{(\kappa(a+b))}(2ns) \tag{3.3.9}$$

holds in the following sense. (i) For any $N \geq \mathrm{Max}(a, b) - 1$ and $\epsilon > 0$, the sum over $n \geq N + 1$ on the RHS converges absolutely and uniformly on $\sigma = \Re(s) \geq \frac{1+\epsilon}{2(N+1)}$, and (ii) the equality (3.3.9) holds on $\sigma > 1/(2\mathrm{Max}(a, b))$.

In other words, the holomorphic function

$$\mathbf{B}^{(a,b)}(s) - \sum_{n \leq N} \gamma_n^{(a,b)} \phi^{(\kappa(a+b))}(2ns) \tag{3.3.10}$$

on $\sigma > 1/(2\mathrm{Max}(a, b))$ extends to a holomorphic function

$$\sum_{n \geq N+1} \gamma_n^{(a,b)} \phi^{(\kappa(a+b))}(2ns) \tag{3.3.11}$$

on $\sigma > 1/(2(N+1))$. In particular, $\mu^{(1,1)}(s) - \phi^{(2\kappa)}(2s)$ extends to a holomorphic function on $\sigma > 1/4$.

The proof will be given in §3.7 after the preliminaries §3.5-3.6.

3.4 – The analytic continuation of $\tilde{M}(s; z_1, z_2)$.

Theorem 5 *(i) For any $N \geq 0$, the holomorphic function*

$$\tilde{M}(s; z_1, z_2) \exp\left(- \sum_{1 \leq a,b \leq n \leq N} \gamma_n^{(a,b)} \phi^{(\kappa(a+b))}(2ns)(i/2)^{a+b} \frac{z_1^a z_2^b}{a!b!} \right) \tag{3.4.1}$$

of (s, z_1, z_2) on $\Re(s) > 1/2$ extends to that on $\Re(s) > 1/(2(N+1))$. In particular ($N = 1$),

$$\tilde{M}(s; z_1, z_2) \exp(\frac{1}{4} \phi^{(2\kappa)}(2s) z_1 z_2) \tag{3.4.2}$$

extends to a holomorphic function of s, z_1, z_2 on the domain defined by $\sigma > 1/4$.
(ii) Let $\sigma_0 > 1/2, R > 0$, and $\Re(s) \geq \sigma_0$, $|z_1|, |z_2| \leq R$. Suppose that either σ_0 is fixed and R is sufficiently small, or R is fixed and σ_0 is sufficiently large. Then the two series

$$\sum_{a,b\geq 1} \mathbf{B}^{(a,b)}(s)(i/2)^{a+b}\frac{z_1^a z_2^b}{a!b!} \tag{3.4.3}$$

$$\sum_{\substack{a,b,n \\ n\geq \mathrm{Max}(a,b)}} \gamma_n^{(a,b)}\phi^{(\kappa(a+b))}(2ns)(i/2)^{a+b}\frac{z_1^a z_2^b}{a!b!} \tag{3.4.4}$$

both converge absolutely and uniformly to $\log \tilde{M}(s; z_1, z_2)$. In Case 2, this means that $\tilde{M}(s; z_1, z_2)$ has an absolutely convergent infinite product expansion

$$\tilde{M}(s; z_1, z_2) = \prod_{n=1}^{\infty} \zeta(2ns)^{R_n(z_1,z_2)} \qquad (\Re(s) \geq \sigma_0,\ |z_1|, |z_2| \leq R) \tag{3.4.5}$$

(σ_0, R as above), where

$$R_n(z_1, z_2) = \sum_{a,b=1}^{n} \gamma_n^{(a,b)}(i/2)^{a+b}\frac{z_1^a z_2^b}{a!b!}. \tag{3.4.6}$$

The proof will be given in §3.8 after the preliminary subsections.

For example, let K, P_∞ be as in Example 4 (§1.2). Then $\zeta(s) = \zeta_{K,P_\infty}(s) = (1-q^{1-s})^{-1}$; hence

$$\tilde{M}(s; z_1, z_2) = \prod_{n=1}^{\infty}(1 - q^{1-2ns})^{-R_n(z_1,z_2)}. \tag{3.4.7}$$

Corollary 3.4.8 (Lemma A §2.3) *We have*

$$\lim_{\substack{s\to 1/2 \\ |\mathrm{Arg}(2s-1)|<\pi}} \tilde{M}\left(s;\ \frac{z_1}{\mu(s)^{1/2}}, \frac{z_2}{\mu(s)^{1/2}}\right) = \exp\left(-\frac{z_1 z_2}{4}\right), \tag{3.4.9}$$

and the convergence is uniform on $|z_1|, |z_2| \leq R$ for any given $R > 0$.

Proof The above theorem shows in particular that

$$f(s; z_1, z_2) := \tilde{M}(s; z_1, z_2)\exp\left(\frac{\phi^{(2\kappa)}(2s)}{4}z_1 z_2\right) \tag{3.4.10}$$

extends to a holomorphic function of (s, z_1, z_2) on $\Re(s) > 1/4$. Clearly, $f(s, 0, 0) = 1$, $f(s, z_1, z_2)$ is continuous at $(1/2, 0, 0)$, and $\lim_{\sigma \to 1/2} \mu(s)^{-1/2} = 0$. Therefore,
(3.4.11)

$$f(s;\, z_1/\mu(s)^{1/2}, z_2/\mu(s)^{1/2}) = \tilde{M}\left(s;\, z_1/\mu(s)^{1/2}, z_2/\mu(s)^{1/2}\right) \exp\left(\frac{\phi^{(2\kappa)}(2s)}{4\mu(s)} z_1 z_2\right)$$

tends uniformly to 1 as $s \to 1/2$ (on $|z_1|, |z_2| \leq R$). But by (2.2.1), the exponential factor tends uniformly to $\exp(z_1 z_2/4)$. These together prove the Corollary.

Now let

$$(3.4.12) \qquad \mathcal{D} = \{s \in \mathbf{C};\, \Re(s) > 0,\, \zeta(2ns) \neq 0, \infty\, (n = 1, 2, ...)\},$$

where $\zeta(s) = \zeta_{K,P_\infty}(s)$. (In the number field case, the condition $\zeta(2ns) \neq \infty$ is of course equivalent to $s \neq 1/(2n)$.) Then Theorem 5 gives directly:

Corollary 3.4.13 $\tilde{M}(s; z_1, z_2)$ *extends to a single-valued (Case 1) or multi-valued (Case 2) analytic function of* (s, z_1, z_2) *on* $\mathcal{D} \times \mathbf{C}^2$.

As regards Case 2, if s_0 is a point with $\Re(s_0) > 0$, $s_0 \notin \mathcal{D}$, and s encircles s_0 in a small neighborhood in the positive direction (z_1, z_2 being fixed), then $\tilde{M}(s; z_1, z_2)$ is multiplied by

$$(3.4.14) \qquad \exp(2\pi i \sum_{\nu=1}^{r} k_\nu R_{n_\nu}(z_1, z_2)).$$

Here, $(n_\nu)_{\nu=1}^r$ are the distinct positive integers such that $\zeta(2n_\nu s_0) = 0$ or ∞, and k_ν is the order of $\zeta(s)$ at $s = 2n_\nu s_0$. Thus, $\tilde{M}(s; z_1, z_2)$ can be regarded as a univalent analytic function on $\mathcal{D}^{urab} \times \mathbf{C}^2$, where $\mathcal{D}^{urab}$ denotes the maximal unramified *abelian* covering of $\mathcal{D}$. Moreover, although $\tilde{M}(s; z_1, z_2)$ is multi-valued, its *divisor* on $\mathcal{D} \times \mathbf{C}^2$ is well-defined. Note also that for $y_1, y_2 \in \mathbf{R}$, $|\tilde{M}(s, iy_1, iy_2)|$ is a univalent function on $\mathcal{D} \times \mathbf{R}^2$ (because $R_n(iy_1, iy_2) \in \mathbf{R}$).

Now each local factor $\tilde{M}_\mathfrak{p}(s; z_1, z_2)$ is a holomorphic function on $\{\Re(s) > 0\} \times \mathbf{C}^2$, having a non-trivial zero divisor. It is clear from the Euler product expansion

$$(3.4.15) \qquad \tilde{M}(s; z_1, z_2) = \prod_{\mathfrak{p} \notin P_\infty} \tilde{M}_\mathfrak{p}(s; z_1, z_2) \qquad (\Re(s) > 1/2)$$

(cf. §1.2) that the zero divisor of $\tilde{M}(s; z_1, z_2)$ on $\{\Re(s) > 1/2\} \times \mathbf{C}^2$ is simply the sum of zero divisors of local factors. But moreover, we have

Corollary 3.4.16 *The zero divisor of* $\tilde{M}(s; z_1, z_2)$ *on* $\mathcal{D} \times \mathbf{C}^2$ *is the sum of zero divisors of* $\tilde{M}_\mathfrak{p}(s; z_1, z_2)$ *(restricted to* $\mathcal{D} \times \mathbf{C}^2$*) for all* $\mathfrak{p} \notin P_\infty$.

This will be proved in the course of the proof of Theorem 5 (i)(in §3.8).

3.5 – Preliminaries for the proofs of Theorems 4,5; Some estimates.

A main point in their proofs is the exchangeability of the order of (various) summations, over $\mathfrak{p}$, n, (a,b). To justify this we need the absolute convergence of various sums over all $\mathfrak{p}$, n, (a,b), and for this, some estimations of each summand will be needed. In this §3.5, we shall give some estimates of $|B^{(a,b)}(t)|$, $|\beta_n^{(a,b)}|$, $|\gamma_n^{(a,b)}|$.

Proposition 3.5.1 *Let* $|t|<1$. *Then*

$$|B^{(a,b)}(t)| \leq (2\mathrm{Min}(a,b))^{a+b}|t|^{\mathrm{Max}(a,b)}(1-|t|^{1/2})^{-2(a+b)}. \tag{3.5.2}$$

Proposition 3.5.3 [1] *We have*

$$\begin{array}{llr} (i) & |\beta_n^{(a,b)}| < (4en)^{a+b-1} & (\text{Cases } 1,\ 2), \\ (ii) & \displaystyle\sum_{a,b,n}\left(\frac{\beta_n^{(a,b)}}{3^{a+b}a!b!}\right)^2 < \frac{1}{2} & (\text{Case } 2). \end{array}$$

Proposition 3.5.4 *We have*

$$\begin{array}{llr} (i) & |\gamma_n^{(a,b)}| < (4en)^{a+b} & (\text{Case } 1), \\ (ii) & |\gamma_n^{(a,b)}| < 3^{a+b}a!b! & (\text{Case } 2). \end{array}$$

First, some preparatory materials for these proofs. First, by (1.2.3), the coefficient of $x_1^a x_2^b t^n$ in $F(x_1,x_2;t)=\sum_{n\geq 0}F_n(x_1)F_n(x_2)t^n$ for $n\geq \mathrm{Max}(a,b)$ is given by $\binom{n-1}{a-1}\binom{n-1}{b-1}/a!b!$ in Case 1 and by $\delta_a(n)\delta_b(n)/a!b!$ in Case 2, and is $=0$ otherwise; hence $F(x_1,x_2;t)$ may be rewritten as

$$F(x_1,x_2;t) = 1+\sum_{a,b\geq 1} f^{(a,b)}(t)\frac{x_1^a x_2^b}{a!b!}, \tag{3.5.5}$$

where

$$f^{(a,b)}(t) = \begin{cases} \sum_{n\geq \mathrm{Max}(a,b)} \binom{n-1}{a-1}\binom{n-1}{b-1} t^n & (\text{Case } 1), \\ \sum_{n\geq \mathrm{Max}(a,b)} \delta_n(a)\delta_n(b) t^n & (\text{Case } 2). \end{cases} \tag{3.5.6}$$

[1] Since so many positive absolute constants appear, instead of denoting them C_1, C_2, etc., we shall simply give an explicit choice for each (e.g., $4e$ in (i) below). Later arguments will not depend on these specific choices.

Therefore,

$$\log F(x_1,x_2;t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\sum_{a,b\geq 1} f^{(a,b)}(t) \frac{x_1^a x_2^b}{a!b!} \right)^k ; \tag{3.5.7}$$

hence the coefficient $B^{(a,b)}(t)$ of $\frac{x_1^a x_2^b}{a!b!}$ in (3.5.7) is given by

(3.5.8)

$$B^{(a,b)}(t) = \sum_{k=1}^{\mathrm{Min}(a,b)} \frac{(-1)^{k-1}}{k} \sum_{\substack{a=a_1+\ldots+a_k \\ a_1,\ldots,a_k\geq 1}} \sum_{\substack{b=b_1+\ldots+b_k \\ b_1,\ldots,b_k\geq 1}} \left(\begin{array}{c} a \\ a_1,\ldots,a_k \end{array} \right) \left(\begin{array}{c} b \\ b_1,\ldots,b_k \end{array} \right) \prod_{\nu=1}^{k} f^{(a_\nu,b_\nu)}(t).$$

(A priori, the outer sum is over all $k \geq 1$, but the inner sum is 0 unless $k \leq \mathrm{Min}(a,b)$.)

For any formal power series f and g with *non-negative* real coefficients, $f \leq_{cf} g$ will denote the *coefficientwise inequality* $\leq$. Note that this inequality is preserved by additions and multiplications. By (3.1.6)(3.5.8), we have

(3.5.9)

$$\sum_{n\geq \mathrm{Max}(a,b)} |\beta_n^{(a,b)}| t^n \leq_{cf} \sum_{k=1}^{\mathrm{Min}(a,b)} \frac{1}{k} \sum_{\substack{a=a_1+\ldots+a_k \\ a_1,\ldots,a_k\geq 1}} \sum_{\substack{b=b_1+\ldots+b_k \\ b_1,\ldots,b_k\geq 1}} \left(\begin{array}{c} a \\ a_1,\ldots,a_k \end{array} \right) \left(\begin{array}{c} b \\ b_1,\ldots,b_k \end{array} \right) \prod_{\nu=1}^{k} f^{(a_\nu,b_\nu)}(t).$$

We shall need the following two $\leq_{cf}$ inequalities;

$$f^{(a,b)}(u^2) \leq_{cf} \begin{cases} (u(1-u)^{-1})^{a+b}, \\ \binom{\mathrm{Max}(a,b)-1}{\mathrm{Min}(a,b)-1} (u(1-u)^{-1})^{2\mathrm{Max}(a,b)}. \end{cases} \tag{3.5.10}$$

To verify these we may assume $a \geq b$. By (1.2.4)(3.5.6),

(3.5.11)

$$f^{(a,b)}(u^2) \leq_{cf} \sum_{n\geq a} \binom{n-1}{a-1} \binom{n-1}{b-1} u^{2n} \leq_{cf} \left(\sum_{n\geq a} \binom{n-1}{a-1} u^n \right) \left(\sum_{n\geq a} \binom{n-1}{b-1} u^n \right).$$

But

$$\sum_{n\geq a} \binom{n-1}{a-1} u^n = (u(1-u)^{-1})^a, \tag{3.5.12}$$

and hence

$$\sum_{n\geq a} \binom{n-1}{b-1} u^n \leq_{cf} \begin{cases} (u(1-u)^{-1})^b \\ \binom{a-1}{b-1} (u(1-u)^{-1})^a. \end{cases} \tag{3.5.13}$$

(The first is obvious by (3.5.12) and $b \leq a$; the second is by (3.5.12) and $\binom{n-1}{b-1} \leq \binom{n-b}{a-b}\binom{n-1}{b-1} = \binom{a-1}{b-1}\binom{n-1}{a-1}$). Therefore, (3.5.10) follows directly from (3.5.11)∼(3.5.13).

[**Proof of Prop 3.5.3(i)**]. By (3.5.9) and the first inequality of (3.5.10), we obtain (3.5.14)

$$\sum_{n \geq \mathrm{Max}(a,b)} |\beta_n^{(a,b)}| u^{2n} \leq_{cf} \sum_{k=1}^{\mathrm{Min}(a,b)} k^{a+b-1}(u(1-u)^{-1})^{a+b} \leq_{cf} \mathrm{Min}(a,b)^{a+b}(u(1-u)^{-1})^{a+b}.$$

Therefore, by (3.5.12),

$$|\beta_n^{(a,b)}| \leq \mathrm{Min}(a,b)^{a+b}\binom{2n-1}{a+b-1} \leq (a+b-1)^{a+b}\frac{(2n-1)^{a+b-1}}{(a+b-1)!}.$$

By using $n! > e^{-n}n^n$ and $a+b-1 \leq 2^{a+b-1}$, we obtain

$$|\beta_n^{(a,b)}| < (a+b-1)e^{a+b-1}(2n-1)^{a+b-1} < (4en)^{a+b-1},$$

as desired.

[**Proof of Prop 3.5.1**] We use the second inequality of (3.5.10), and proceed similarly. The only difference is that we finally turn to "real inequalities" by using $|t| < 1$ and $a+b \geq \sum_\nu \mathrm{Max}(a_\nu, b_\nu) \geq \mathrm{Max}(a,b)$.

[**Proof of Prop 3.5.3(ii)**] This is more delicate. In Case 2, by (3.1.1) and (1.2.3), our $F(x_1, x_2; t)$ is nothing but the Gauss hypergeometric series

$$F(a,b;c;t) = 1 + \frac{a.b}{1.c}t + \frac{a(a+1)b(b+1)}{1.2.c(c+1)}t^2 + \cdots, \tag{3.5.15}$$

for $a = x_1, b = x_2, c = 1$;

$$F(x_1, x_2; t) = F(x_1, x_2; 1; t). \tag{3.5.16}$$

When $\Re(c) > 0$ and $\Re(c-a-b) > 0$, the series (3.5.15) converges also for $t = 1$, and the Gauss formula

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{3.5.17}$$

holds. In particular, $F(1/3, 1/3; 1) = \Gamma(1/3)\Gamma(2/3)^{-2} = 1.461\cdots < 3/2$. Therefore, when $|x_1|, |x_2| \leq 1/3$, $|t| < 1$, we have

$$|F(x_1, x_2; t) - 1| \leq \sum_{n \geq 1} F_n(1/3)^2 = F(1/3, 1/3; 1) - 1 < 1/2. \tag{3.5.18}$$

Note now the following. For any $\alpha_1, \alpha_2, ... \in \mathbf{C}$, if we define the formal power series

$$\sum_{n\geq 1} \beta_n t^n = \log(1 + \sum_{n\geq 1} \alpha_n t^n) \tag{3.5.19}$$

then for any $\{a_n\}_{n\geq 1}$ with $|\alpha_n| \leq a_n$, the coefficientwise inequality

$$\sum_{n\geq 1} |\beta_n| t^n \leq_{cf} -\log(1 - \sum_{n\geq 1} a_n t^n) \tag{3.5.20}$$

holds. Apply this for $\alpha_n = F_n(x_1)F_n(x_2)$ (for $|x_1|, |x_2| \leq 1/3$), $a_n = F_n(1/3)^2$ and $\beta_n = B_n(x_1, x_2)$, to obtain

$$\sum_{n\geq 1} |B_n(x_1, x_2)| t^n \leq_{cf} -\log(1 - \sum_{n\geq 1} F_n(1/3)^2 t^n). \tag{3.5.21}$$

This of course carries over to an actual inequality for any t with $0 \leq t < 1$. Therefore, by letting $t \to 1$ and using (3.5.18) (and Abel's theorem), we obtain

$$\sum_{n\geq 1} |B_n(x_1, x_2)| \leq \log 2 \qquad (|x_1|, |x_2| \leq 1/3); \tag{3.5.22}$$

hence

$$\sum_{n\geq 1} |B_n(x_1, x_2)|^2 \leq (\log 2)^2 < 1/2 \qquad (|x_1|, |x_2| \leq 1/3). \tag{3.5.23}$$

Now by (3.1.5) and the orthogonality relation we obtain

$$\int_{|x_1|=|x_2|=1/3} |B_n(x_1, x_2)|^2 d^\times x_1 d^\times x_2 = \sum_{a,b=1}^{n} \left(\frac{\beta_n^{(a,b)}}{a!b!} (\frac{1}{3})^{a+b} \right)^2 ; \tag{3.5.24}$$

where $d^\times x_\nu$ ($\nu = 1, 2$) denotes the normalized Haar measure of the circle $|x_\nu| = 1/3$ (note that $\beta_n^{(a,b)}$ are rational and hence real). Hence

$$\sum_{n,a,b} \left(\frac{\beta_n^{(a,b)}}{a!b!} (\frac{1}{3})^{a+b} \right)^2 = \int_{|x_1|=|x_2|=1/3} \sum_{n=1}^{\infty} |B_n(x_1, x_2)|^2 d^\times x_1 d^\times x_2 < 1/2, \tag{3.5.25}$$

as desired. This settles the proof of Prop 3.5.3(ii).

[**Proof of Prop 3.5.4**] (Case 1) By (3.1.13), we have

$$\gamma_n^{(a,b)} = n^{a+b-1} \sum_{d|n} \mu(n/d) d^{1-a-b} \beta_d^{(a,b)},$$

and by Prop 3.5.3(i), we have $d^{1-a-b}|\beta_d^{(a,b)}| < (4e)^{a+b-1}$; hence

$$|\gamma_n^{(a,b)}| < (4en)^{a+b-1} \sum_{d|n} 1 \leq (4e)^{a+b-1} n^{a+b}.$$

(Case 2) In this case,

$$\gamma_n^{(a,b)} = \frac{1}{n} \sum_{d|n} \mu(n/d) d \beta_d^{(a,b)};$$

hence

$$|\gamma_n^{(a,b)}| \leq \frac{1}{n} \sum_{d|n} d |\beta_n^{(a,b)}| \leq \frac{1}{n} \left((\sum_{d|n} d^2)(\sum_{d|n} |\beta_d^{(a,b)}|^2) \right)^{1/2}. \tag{3.5.26}$$

By Prop 3.5.3(ii) we have

$$\sum_{d|n} |\beta_d^{(a,b)}|^2 < (3^{a+b} a!b!)^2/2 \tag{3.5.27}$$

for each $a, b \geq 1$, and on the other hand, $n^{-2} \sum_{d|n} d^2 < \sum_{m \geq 1} m^{-2} = \pi^2/6$; hence

$$|\gamma_n^{(a,b)}| < \frac{\pi}{\sqrt{12}} 3^{a+b} a!b! < 3^{a+b} a!b!, \tag{3.5.28}$$

as desired. $\square$

Remark 3.5.29 *From (3.5.16)(3.5.17) and the power series expansion*

$$\log \Gamma(1-x) = \gamma x + \sum_{n=2}^{\infty} \frac{\zeta_{\mathbf{Q}}(n)}{n} x^n \qquad (|x| < 1) \tag{3.5.30}$$

(γ: the Euler constant, $\zeta_{\mathbf{Q}}(s)$: the Riemann zeta function), we obtain, in Case 2 for $|x_1|, |x_2| < 1/2$,

$$\begin{aligned} \log F(x_1, x_2; 1) &= \log \Gamma(1 - x_1 - x_2) - \log \Gamma(1 - x_1) - \log \Gamma(1 - x_2) \\ &= \sum_{n=2}^{\infty} \frac{\zeta_{\mathbf{Q}}(n)}{n} ((x_1 + x_2)^n - x_1^n - x_2^n) = \sum_{a,b \geq 1} (a+b-1)! \zeta_{\mathbf{Q}}(a+b) \frac{x_1^a x_2^b}{a!b!}, \end{aligned} \tag{3.5.31}$$

and hence also

$$\mathbf{B}^{(a,b)}(1) = (a+b-1)! \zeta_{\mathbf{Q}}(a+b) \qquad (a, b \geq 1)(\text{Case 2}). \tag{3.5.32}$$

3.6 – Further estimations and convergences.

Proposition 3.6.1 *Fix $a, b \geq 1$, $0 < r < 1$ and let $|t| \leq r$. Then the series*

$$\sum_{n \geq \mathrm{Max}(a,b)} \gamma_n^{(a,b)} \ell_{\kappa(a+b)}(t^n) \tag{3.6.2}$$

is absolutely and uniformly convergent, and has $B^{(a,b)}(t)$ as its limit. Moreover,

$$\left| B^{(a,b)}(t) - \sum_{n=\mathrm{Max}(a,b)}^{N} \gamma_n^{(a,b)} \ell_{\kappa(a+b)}(t^n) \right| \ll_{a,b,r} (N+1)^{\kappa(a+b)+1} |t|^{N+1} \tag{3.6.3}$$

holds for any $N \geq \mathrm{Max}(a,b) - 1$.

The second inequality will be needed for globalization.

Proof Since (a,b) is fixed, $|\gamma_n^{(a,b)}| \ll n^{\kappa(a+b)}$ by Prop 3.5.4. And clearly, $|\ell_k(t)| \ll_{k,r} |t|$ for $|t| \leq r$. Therefore,

$$|\gamma_n^{(a,b)} \ell_{\kappa(a+b)}(t^n)| \ll_{a,b,r} n^{\kappa(a+b)} r^n. \tag{3.6.4}$$

Therefore, (3.6.2) is absolutely and uniformly convergent. Now, by definitions,

$$\mathrm{Coeff}\left(B^{(a,b)}(t) - \sum_{n=\mathrm{Max}(a,b)}^{N} \gamma_n^{(a,b)} \ell_{\kappa(a+b)}(t^n),\ t^m \right) = \beta_m^{(a,b)} - \sum_{\substack{n \leq N \\ n|m}} \gamma_n^{(a,b)} (m/n)^{\kappa(a+b)-1} \tag{3.6.5}$$

holds for the coefficient of t^m. This is $= 0$ when $m \leq N$, and is $\ll_{a,b} m^{\kappa(a+b)+1}$ for $m \geq N+1$, by Prop 3.5.3, Prop 3.5.4. Therefore, the LHS of (3.6.3) is

$$\ll_{a,b} \sum_{m \geq N+1} m^{\kappa(a+b)+1} |t|^m \ll_{a,b,r} (N+1)^{\kappa(a+b)+1} |t|^{N+1},$$

as desired.

By (3.2.6) and (3.2.12), this gives:

Corollary 3.6.6 *The holomorphic function $\mathbf{B}_{\mathfrak{p}}^{(a,b)}(s)$ on $\Re(s) > 0$ can be expressed as an absolutely (and uniformly on $\Re(s) \geq \epsilon > 0$) convergent series*

$$\mathbf{B}_{\mathfrak{p}}^{(a,b)}(s) = \sum_{n \geq \mathrm{Max}(a,b)} \gamma_n^{(a,b)} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns). \tag{3.6.7}$$

Proposition 3.6.8 *Let $|x_1|, |x_2| \le R$, $|t| \le r < 1$, and consider the triple series*

$$\sum_{n,a,b\ge 1} \gamma_n^{(a,b)} \frac{x_1^a x_2^b}{a!b!} \ell_{\kappa(a+b)}(t^n). \tag{3.6.9}$$

If one of R, r is fixed and the other is sufficiently small, then (3.6.9) converges absolutely and uniformly to $\log F(x_1, x_2; t)$. *Moreover, for each $N \ge 0$ and $0 < c < 1$, we have*

$$\sum_{\substack{a,b,n \\ n\ge N+1}} |\gamma_n^{(a,b)} \frac{x_1^a x_2^b}{a!b!} \ell_{\kappa(a+b)}(t^n)| \ll_c (N+1)^2 (re^{8eR})^{N+1} \qquad if\ re^{8eR} \le c. \tag{3.6.10}$$

Proof We shall first prove (3.6.10).

(Case 1) By Prop 3.5.4 and (3.1.10)(which holds analytically for $|t|, |te^u| < 1$), the LHS of (3.6.10) is

$$\begin{aligned}\le \sum_{n\ge N+1} \sum_{k\ge 2} \sum_{\substack{a+b=k \\ 1\le a,b\le n}} \frac{(4enR)^k}{a!b!} \ell_k(r^n) &\le \sum_{n\ge N+1} \left(\sum_{k\ge 0} \frac{1}{k!} (8enR)^k \ell_k(r^n) \right) \\ = \sum_{n\ge N+1} \ell((re^{8eR})^n) &\ll_c \sum_{n\ge N+1} (re^{8eR})^n \ll_c (re^{8eR})^{N+1},\end{aligned}$$

provided that $re^{8eR} \le c < 1$.

(Case 2) Since $\ell(r^n) = -\log(1-r^n) \le r^n(1-r^n)^{-1} \le r^n(1-r)^{-1}$, Prop 3.5.4 (ii) gives

$$\sum_{a,b=1}^{n} |\gamma_n^{(a,b)} \frac{x_1^a x_2^b}{a!b!} \ell(t^n)| \le \sum_{a,b=1}^{n} (3R)^{a+b} r^n (1-r)^{-1} \ll_r \begin{cases} (3R)^2 n^2 r^n & (3R < 1), \\ (3R)^{2n} n^2 r^n & (3R \ge 1). \end{cases} \tag{3.6.11}$$

But since $(aR)^2/2 < e^{aR}$ $(a > 0)$ and hence $(3R)^2 \le e^{3\sqrt{2}R} < e^{8eR}$, this gives $(3R)^2 < e^{8eRn}$ and also $(3R)^{2n} < e^{8eRn}$ $(n \ge 1)$; hence the LHS of (3.6.10) in this case is $\ll_c (N+1)^2 (re^{8eR})^{N+1}$ if $re^{8eR} \le c$, as desired. This settles the proof of (3.6.10) for both cases.

By (3.6.10), the series (3.6.9) converges absolutely and uniformly, as long as $re^{8eR} \le c$. Therefore, we may change the order of summation. Since we already know by Prop 3.6.1 that

$$\sum_{n\ge 1} \gamma_n^{(a,b)} \ell_{\kappa(a+b)}(t^n) = B^{(a,b)}(t) \qquad (|t| < 1), \tag{3.6.12}$$

and by Cor 3.2.3 that

$$\sum_{a,b\ge 1} B^{(a,b)}(t) \frac{x_1^a x_2^b}{a!b!} = \log F(x_1, x_2; t), \tag{3.6.13}$$

when one of r, R is fixed and the other is sufficiently small, we conclude that (3.6.9) tends to $\log F(x_1, x_2; t)$ for such r, R. $\square$

Corollary 3.6.14 *Let $R, \sigma_0 > 0$, $\sigma \geq \sigma_0, |z_1|, |z_2| \leq R$. Suppose that either σ_0 is fixed and R is sufficiently small, or R is fixed and σ_0 is sufficiently large. Then for any non-archimedean prime $\mathfrak{p}$, $\log \tilde{M}_{\mathfrak{p}}(s; z_1, z_2)$ (cf. Cor 3.2.8) can be expressed as an absolutely convergent series*

$$\log \tilde{M}_{\mathfrak{p}}(s; z_1, z_2) = \sum_{\substack{n,a,b \geq 1 \\ n \geq \mathrm{Max}(a,b)}} \gamma_n^{(a,b)} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns)(i/2)^{a+b} \frac{z_1^a z_2^b}{a!b!}. \tag{3.6.15}$$

Proof Again, in Case 2, this follows immediately from the above Proposition. In Case 1, we may take $r = N(\mathfrak{p})^{-2\sigma_0}$, but R is replaced by $(\log N(\mathfrak{p}))R/2$; hence re^{8eR} will be replaced by $N(\mathfrak{p})^{4eR-2\sigma_0}$. The exponent $4eR - 2\sigma_0$ is $\leq -\sigma_0$ if and only if $4eR \leq \sigma_0$; which is satisfied under our assumptions on σ_0 and R. Hence this case is also settled. $\square$

3.7 – Proof of Theorem 4.

Write $\sigma = \Re(s)$. Fix $N \geq \mathrm{Max}(a, b) - 1$ and $\epsilon > 0$. We shall prove first that the double sum

$$\sum_{\substack{n \geq N+1 \\ \mathfrak{p} \notin P_\infty}} |\gamma_n^{(a,b)} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns)| \tag{3.7.1}$$

is finite and bounded on $\sigma \geq (1+\epsilon)/(2(N+1))$. By (3.2.12), $\phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns) = \lambda_{\mathfrak{p}}^{a+b} \ell_{\kappa(a+b)}(N(\mathfrak{p})^{-2ns})$. But we have $|\gamma_n^{(a,b)}| \ll_{a,b} n^{\kappa(a+b)} \leq n^{a+b}$ (by Prop 3.5.4), $|\lambda_{\mathfrak{p}}| \leq \log N(\mathfrak{p})$, $\ell_{\kappa(a+b)}(t) \ll_{a,b,r} |t|$ for $|t| \leq r < 1$, and $N(\mathfrak{p})^{-2\sigma n} \leq 2^{-2\sigma(N+1)} \leq 2^{-1-\epsilon} < 1/2$; hence (3.7.1) is

$$\ll_{a,b} \sum_{n \geq N+1} n^{a+b} \Big(\sum_{\mathfrak{p} \notin P_\infty} (\log N(\mathfrak{p}))^{a+b} N(\mathfrak{p})^{-2n\sigma} \Big). \tag{3.7.2}$$

Put $\alpha = \mathrm{Min}_{\mathfrak{p} \notin P_\infty} N(\mathfrak{p})$. Then since $2n\sigma \geq 1+\epsilon$ and $\alpha N(\mathfrak{p})^{-1} \leq 1$, we have $(\alpha N(\mathfrak{p})^{-1})^{2n\sigma} \leq (\alpha N(\mathfrak{p})^{-1})^{1+\epsilon}$; hence

$$\alpha^{2n\sigma} \Big(\sum_{\mathfrak{p} \notin P_\infty} (\log N(\mathfrak{p}))^{a+b} N(\mathfrak{p})^{-2n\sigma} \Big) \leq \alpha^{1+\epsilon} \Big(\sum_{\mathfrak{p} \notin P_\infty} (\log N(\mathfrak{p}))^{a+b} N(\mathfrak{p})^{-1-\epsilon} \Big) \ll_{a,b,\epsilon} 1; \tag{3.7.3}$$

hence (3.7.2) is

$$\ll_{a,b,\epsilon} \sum_{n \geq N+1} n^{a+b} \alpha^{-2n\sigma} \leq \sum_{n \geq N+1} n^{a+b} (\alpha^{-(1+\epsilon)/(N+1)})^n \ll_{a,b,\epsilon,N} 1, \tag{3.7.4}$$

as desired.

Since the sum

$$\phi^{(k)}(2ns) = \sum_{\mathfrak{p}\notin P_\infty} \phi_{\mathfrak{p}}^{(k)}(2ns) \tag{3.7.5}$$

is absolutely convergent, because $2n\sigma \geq 2(N+1)\sigma \geq 1+\epsilon$, the convergence of (3.7.1) implies that the global sum

$$\sum_{n\geq N+1} \gamma_n^{(a,b)} \phi^{(\kappa(a+b))}(2ns) \tag{3.7.6}$$

is also absolutely and uniformly convergent on $\sigma \geq (1+\epsilon)/(2(N+1))$; whence (i).

To prove (ii), let $\sigma > 1/(2\mathrm{Max}(a,b))$. By Prop 3.3.1 and Cor 3.6.6,

$$\mathbf{B}^{(a,b)}(s) = \sum_{\mathfrak{p}\notin P_\infty} \mathbf{B}_{\mathfrak{p}}^{(a,b)}(s) = \sum_{\mathfrak{p}\notin P_\infty} \Big(\sum_{n\geq \mathrm{Max}(a,b)} \gamma_n^{(a,b)} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns)\Big), \tag{3.7.7}$$

which, by the absolute convergence (3.7.1)(for $N=0$) can be reordered as

$$\sum_{n\geq \mathrm{Max}(a,b)} \gamma_n^{(a,b)} \Big(\sum_{\mathfrak{p}\notin P_\infty} (\phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns)\Big) = \sum_{n\geq \mathrm{Max}(a,b)} \gamma_n^{(a,b)} \phi^{(\kappa(a+b))}(2ns) \tag{3.7.8}$$

(cf. (3.3.8)), as desired. This completes the proof of Theorem 4.

3.8 – Proof of Theorem 5.

Proof of (i) Fix $N \geq 0$. We may also fix any $R \geq 1$ and assume $|z_1|, |z_2| \leq R$. Depending on R we may remove a finite set of prime components $\mathfrak{p}$ from both $\tilde{M}(s; z_1, z_2)$ and $\phi^{(\kappa(a+b))}(2ns)$, so that the following conditions are satisfied for any remaining primes $\mathfrak{p}$; (i)$N(\mathfrak{p})^{-1/(2(N+1))} \leq 1/2$, and more strongly, $4eR(\log N(\mathfrak{p}))N(\mathfrak{p})^{-1/(2(N+1))} \leq 1/2$; (ii) $\alpha = \mathrm{Min}(N(\mathfrak{p}))$ is so large that the assumption of Cor 3.2.8 for $\sigma_0 = 1/(2(N+1))$ (and for the above given R) is satisfied. Thus, we have $|\tilde{M}_{\mathfrak{p}}(s; z_1, z_2) - 1| < 1$, and

$$\log \tilde{M}_{\mathfrak{p}}(s; z_1, z_2) = \sum_{a,b\geq 1} \mathbf{B}_{\mathfrak{p}}^{(a,b)}(s)(i/2)^{a+b}\frac{z_1^a z_2^b}{a!b!} \qquad (\textit{absolutely convergent}) \tag{3.8.1}$$

holds for all the remaining primes. Write

$$\log \tilde{M}_{\mathfrak{p}}(s; z_1, z_2) - \sum_{1\leq a,b\leq n\leq N} \gamma_n^{(a,b)} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns)(i/2)^{a+b}\frac{z_1^a z_2^b}{a!b!} = I_{\mathfrak{p}} + II_{\mathfrak{p}}, \tag{3.8.2}$$

with

$$I_{\mathfrak{p}} = \sum_{a,b=1}^{N} \left(\mathbf{B}_{\mathfrak{p}}^{(a,b)}(s) - \sum_{n=\mathrm{Max}(a,b)}^{N} \gamma_n^{(a,b)} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns) \right) (i/2)^{a+b}\frac{z_1^a z_2^b}{a!b!}, \tag{3.8.3}$$

$$II_{\mathfrak{p}} = \sum_{\mathrm{Max}(a,b)\geq N+1} \mathbf{B}_{\mathfrak{p}}^{(a,b)}(s)(i/2)^{a+b}\frac{z_1^a z_2^b}{a!b!}. \tag{3.8.4}$$

Note that $I_{\mathfrak{p}}$ is a finite sum. Let $I_{\mathfrak{p}}^{\star}$ (resp. $II_{\mathfrak{p}}^{\star}$) denote the sums (3.8.3)(resp.(3.8.4)) where each outer summand is replaced by its absolute value.

First, when $\Re(s) > 1/2$, we have

$$\sum_{\mathfrak{p}} \log \tilde{M}_{\mathfrak{p}}(s; z_1, z_2) = \log \tilde{M}(s; z_1, z_2) \qquad (\textit{absolutely convergent}), \tag{3.8.5}$$

by the argument of [6] §4, applied to the present situation, and also

$$\sum_{\mathfrak{p}} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns) = \phi^{(\kappa(a+b))}(2ns) \qquad (\textit{absolutely convergent}). \tag{3.8.6}$$

Hence the sum over $\mathfrak{p}$ of the LHS of (3.8.2) for $\Re(s) > 1/2$ converges to

$$\log \tilde{M}(s; z_1, z_2) - \sum_{\substack{a,b,n \\ 1 \le a,b \le n \le N}} \gamma_n^{(a,b)} \phi^{(\kappa(a+b))}(2ns)(i/2)^{a+b} \frac{z_1^a z_2^b}{a!b!}. \tag{3.8.7}$$

So, in order to prove Theorem 5 (i)and Corollary 3.4.16, it suffices to show that (3.8.7) extends to a holomorphic function on $\sigma > 1/(2(N+1))$, and for this it suffices to prove that $\sum_{\mathfrak{p}} I_{\mathfrak{p}}^{\star}$ and $\sum_{\mathfrak{p}} II_{\mathfrak{p}}^{\star}$ are finite and uniformly bounded on $\sigma \ge (1+\epsilon)/(2(N+1))$.

As for $I_{\mathfrak{p}}^{\star}$, by Prop 3.6.1,

$$\begin{aligned} I_{\mathfrak{p}}^{\star} &\le \sum_{a,b=1}^{N} \left| \mathbf{B}_{\mathfrak{p}}^{(a,b)}(s) - \sum_{\mathrm{Max}(a,b) \le n \le N} \gamma_n^{(a,b)} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns) \right| \frac{(R/2)^{a+b}}{a!b!} \\ &\ll_{N,\epsilon} \sum_{a,b=1}^{N} (N+1)^{\kappa(a+b)+1} (\log N(\mathfrak{p}))^{a+b} N(\mathfrak{p})^{-2\sigma(N+1)} \frac{(R/2)^{a+b}}{a!b!} \\ &\qquad \ll_{N,\epsilon,R} (\log N(\mathfrak{p}))^{2N} N(\mathfrak{p})^{-1-\epsilon}; \end{aligned}$$

Hence $\sum_{\mathfrak{p}} I_{\mathfrak{p}}^{\star} \ll \sum_{\mathfrak{p}} (\log N(\mathfrak{p}))^{2N} N(\mathfrak{p})^{-1-\epsilon} \ll 1$.

As for $II_{\mathfrak{p}}^{\star}$, we first estimate this by using Prop 3.5.1, which, together with (3.2.6) gives

$$|\mathbf{B}_{\mathfrak{p}}^{(a,b)}(s)| \le (\log N(\mathfrak{p}))^{a+b} (2\mathrm{Min}(a,b))^{a+b} N(\mathfrak{p})^{-2\sigma \mathrm{Max}(a,b)} (1 - N(\mathfrak{p})^{-\sigma})^{-2(a+b)}. \tag{3.8.8}$$

But since $N(\mathfrak{p})^{-\sigma} < N(\mathfrak{p})^{-1/(2(N+1))} \le 1/2$ (by the assumption (i) above) and $\mathrm{Min}(a,b)^{a+b} \le a^a b^b \le e^{a+b} a!b!$, we obtain

$$\frac{1}{a!b!} |\mathbf{B}_{\mathfrak{p}}^{(a,b)}(s)| \le (8e \log N(\mathfrak{p}))^{a+b} N(\mathfrak{p})^{-2\sigma \mathrm{Max}(a,b)}. \tag{3.8.9}$$

Since $a+b \leq 2\mathrm{Max}(a,b)$ and $R \geq 1$, we obtain by reordering the sum using $\nu = \mathrm{Max}(a,b)$,
(3.8.10)

$$II_{\mathfrak{p}}^{\star} \leq \sum_{\mathrm{Max}(a,b) \geq N+1} (4eR \log N(\mathfrak{p}))^{2\mathrm{Max}(a,b)} N(\mathfrak{p})^{-2\sigma \mathrm{Max}(a,b)} \leq 2 \sum_{\nu \geq N+1} \nu \left(\frac{4eR \log N(\mathfrak{p})}{N(\mathfrak{p})^{\sigma}} \right)^{2\nu}.$$

By the assumption (i) for $N(\mathfrak{p})$, we have $4eR(\log N(\mathfrak{p}))N(\mathfrak{p})^{-\sigma} \leq 1/2$; hence

$$II_{\mathfrak{p}}^{\star} \ll_N \left(\frac{4eR \log N(\mathfrak{p})}{N(\mathfrak{p})^{\sigma}} \right)^{2(N+1)} \leq \frac{(4eR \log N(\mathfrak{p}))^{2(N+1)}}{N(\mathfrak{p})^{1+\epsilon}}, \tag{3.8.11}$$

because $2(N+1)\sigma \geq 1+\epsilon$. Therefore, $\sum_{\mathfrak{p}} II_{\mathfrak{p}}^{\star} \ll_{N,R,\epsilon} 1$.

This settles the proof of (i) and Corollary 3.4.16.

Proof of (ii) First, we shall prove the statement related to (3.4.3). By Cor 3.2.8, $|\tilde{M}_{\mathfrak{p}}(s; z_1, z_2) - 1| < 1$ and

$$\log \tilde{M}_{\mathfrak{p}}(s; z_1, z_2) = \sum_{a,b \geq 1} \mathbf{B}_{\mathfrak{p}}^{(a,b)}(s)(i/2)^{a+b} \frac{z_1^a z_2^b}{a!b!}. \tag{3.8.12}$$

Moreover, by the finiteness of $\sum_{\mathfrak{p}} II_{\mathfrak{p}}^{\star}$ for $N=0$ shown above, the double sum

$$\sum_{\mathfrak{p}} \sum_{a,b} \mathbf{B}_{\mathfrak{p}}^{(a,b)}(s)(i/2)^{a+b} \frac{z_1^a z_2^b}{a!b!} \tag{3.8.13}$$

is absolutely convergent. Therefore, we may interchange the summation order , and since $\sigma_0 > 1/2$, we have

$$\begin{aligned} \text{(3.8.14)}\ \log \tilde{M}(s; z_1, z_2) &= \sum_{\mathfrak{p}} \log \tilde{M}_{\mathfrak{p}}(s; z_1, z_2) = \sum_{\mathfrak{p}} \sum_{a,b} \mathbf{B}_{\mathfrak{p}}^{(a,b)}(s)(i/2)^{a+b} \frac{z_1^a z_2^b}{a!b!} \\ &= \sum_{a,b} (\sum_{\mathfrak{p}} \mathbf{B}_{\mathfrak{p}}^{(a,b)}(s))(i/2)^{a+b} \frac{z_1^a z_2^b}{a!b!} = \sum_{a,b} \mathbf{B}^{(a,b)}(s)(i/2)^{a+b} \frac{z_1^a z_2^b}{a!b!}, \end{aligned}$$

as desired.

As for (3.4.4), by Cor 3.6.14,

$$\log \tilde{M}_{\mathfrak{p}}(s; z_1, z_2) = \sum_{n,a,b} \gamma_n^{(a,b)} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns)(i/2)^{a+b} \frac{z_1^a z_2^b}{a!b!} \tag{3.8.15}$$

holds for all $\mathfrak{p}$. Put

$$III_{\mathfrak{p}} = \sum_{n,a,b} \left| \gamma_n^{(a,b)} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns)(i/2)^{a+b} \frac{z_1^a z_2^b}{a!b!} \right|. \tag{3.8.16}$$

We shall show, by using Prop 3.6.8 that $\sum_{\mathfrak{p}} III_{\mathfrak{p}} < \infty$. In Case 1, $r = N(\mathfrak{p})^{-2\sigma_0}$, and R should be replaced by $(R/2)\log N(\mathfrak{p})$. Hence re^{8eR} in Prop 3.6.8 is $N(\mathfrak{p})^{-2\sigma_0+4eR}$. Hence if either $\sigma_0 \gg_R 1$ or $R \ll_{\sigma_0} 1$, then $-2\sigma_0 + 4eR < -1-\epsilon$; hence by (3.6.10) for $N = 0$, $III_{\mathfrak{p}} \ll N(\mathfrak{p})^{-1-\epsilon}$; hence $\sum_{\mathfrak{p}} III_{\mathfrak{p}} < \infty$. In Case 2, the same conclusion follows more directly. Therefore, we may interchange $\sum_{\mathfrak{p}}$ with $\sum_{n,a,b}$, and use (3.3.8) to conclude the convergence of (3.4.4) to $\log \tilde{M}(s; z_1, z_2)$. This settles the proof of (ii), and hence completes that of Theorem 5.

4 Rapid decay of $|\tilde{M}_\sigma(z)|$

The main purpose of §4 is to give some reasonably strong estimations of $|\tilde{M}_\sigma(z)|^2$, for $\tilde{M}_\sigma(z) = \tilde{M}(\sigma; z, \bar{z})$ ($\sigma > 1/2, z \in \mathbf{C}$). The main results are Theorem 6 (§4.3) and Theorem 7C (§4.6). The proofs of Lemmas B, B' of §2 will also be supplied (cf. §4.6 resp. §4.1).

4.1 – Local estimations; large $|z|$. For any non-archimedean prime $\mathfrak{p}$ of K and a positive real number $\sigma > 0$, write as before

$$(4.1.1)\qquad \mu_{\sigma,\mathfrak{p}} := \mu_{\mathfrak{p}}^{(1,1)}(\sigma) = \begin{cases} (\log N(\mathfrak{p}))^2/(N(\mathfrak{p})^{2\sigma}-1) & \text{(Case 1)}, \\ \sum_{n\geq 1} n^{-2} N(\mathfrak{p})^{-2n\sigma} & \text{(Case2)}, \end{cases}$$

cf, (1.2.12), and put

$$(4.1.2)\qquad \tilde{M}_{\sigma,\mathfrak{p}}(z) = \tilde{M}_{\mathfrak{p}}(\sigma; z, \bar{z}) = \int_{\mathbf{C}^1} \exp(i\Re(z g_{\sigma,\mathfrak{p}}(t^{-1})))d^\times t$$

cf. (1.2.8). Note that

$$(4.1.3)\qquad |\tilde{M}_{\sigma,\mathfrak{p}}(z)| \leq 1.$$

A basic universal estimate of $|\tilde{M}_{\sigma,\mathfrak{p}}(z)|$ is the following:

Lemma C *Fix any $\sigma_0 > 0$. Then*

$$(4.1.4)\qquad |\tilde{M}_{\sigma,\mathfrak{p}}(z)|^2 \ll_{\sigma_0} (\mu_{\sigma,\mathfrak{p}}^{1/2}|z|)^{-1} \qquad (\sigma \geq \sigma_0),$$

where $\ll_{\sigma_0}$ depends only on σ_0.

Proof Roughly speaking, this follows from the integral expression (4.1.2) and classical analysis: if $f(\theta)$ ($\theta \in \mathbf{R}/2\pi$) is a real-valued C^2-function such that $f'(\theta)$, f"(θ) are "sufficiently close" to trigonometric functions $\sin\theta$, $\cos\theta$ respectively, then

$$\int_0^{2\pi} e^{i|z|f(\theta)} d\theta \ll |z|^{-1/2}.$$

But to save space, we shall simply reduce its proof of each Case to an established result.

In Case 1 this is proved in [2] §3.3 (By (3.3.12), $|\tilde{M}_{\sigma,\mathfrak{p}}(z)|^2 = |H_{\sigma,\mathfrak{p}}(z)|^2 \ll_{\sigma_0} (r_{\sigma,\mathfrak{p}}|z|)^{-1}$, but $r_{\sigma,\mathfrak{p}}^{-1} = (N(\mathfrak{p})^{\sigma} - N(\mathfrak{p})^{-\sigma})/\log N(\mathfrak{p}) < \mu_{\sigma,\mathfrak{p}}^{-1/2}$.) In Case 2, this follows directly from [8] §7 Theorem 13, for $F(z) = -\log(1-z)$ in which case we can take $\rho_0 = 1$ (cf. the first paragraph of [8] §10). This asserts that for any $\rho_1 < 1$,

$$\text{(4.1.5)} \qquad \frac{1}{2\pi}\int_0^{2\pi} \exp\{-i\Re(\bar{z}\log(1-re^{i\theta}))\}d\theta \ll_{\rho_1} r^{-1/2}|z|^{-1/2} \qquad (0 < r \leq \rho_1).$$

Since the LHS of (4.1.5) for $r = N(\mathfrak{p})^{-\sigma}$ gives $\tilde{M}_{\sigma,\mathfrak{p}}(z)$, and since $\mu_{\sigma,\mathfrak{p}}N(\mathfrak{p})^{2\sigma} \ll_{\sigma_0} 1$ $(\sigma \geq \sigma_0)$, (4.1.5) gives $\tilde{M}_{\sigma,\mathfrak{p}}(z) \ll_{\sigma_0} \mu_{\sigma,\mathfrak{p}}^{-1/4}|z|^{-1/2}$, and hence the desired result. □

Corollary 4.1.6 (Lemma B' of §2.7) *There exists a constant $C > 0$ depending only on (K, P_∞) such that*

$$\text{(4.1.7)} \qquad |\tilde{M}_\sigma(\mu_\sigma^{-1/2}z)| \leq C|z|^{-m/2}$$

holds for all $\sigma \geq 1$ and all $z \in \mathbf{C}$, where m is as in §2.5.

Proof By (4.1.1) we have $\mu_{\sigma,\mathfrak{p}} \gg N(\mathfrak{p})^{-2\sigma}$; hence

$$\text{(4.1.8)} \qquad \mu_{\sigma,\mathfrak{p}}^{-1/2} \ll N(\mathfrak{p})^{\sigma},$$

and Theorem 3 (i) (§2) gives $\alpha^{2\sigma}\mu_\sigma \ll_K 1$ for $\sigma \geq 1$; hence by Lemma C,

$$\text{(4.1.9)} \qquad |\tilde{M}_\sigma(\mu_\sigma^{-1/2}z)|^2 \leq \prod_{\substack{\mathfrak{p}\notin P_\infty \\ N(\mathfrak{p})=\alpha}} |\tilde{M}_{\sigma,\mathfrak{p}}(\mu_\sigma^{-1/2}z)|^2 \ll \prod_{\substack{\mathfrak{p}\notin P_\infty \\ N(\mathfrak{p})=\alpha}} \left(\mu_{\sigma,\mathfrak{p}}^{1/2}|\mu_\sigma^{-1/2}z|\right)^{-1}$$
$$\ll \left(\alpha^{\sigma}|\mu_\sigma^{-1/2}z|^{-1}\right)^m = (\alpha^{2\sigma}\mu_\sigma)^{m/2}|z|^{-m} \ll |z|^{-m},$$

as desired.

4.2 – Local estimations; small $|z|$. Since we always have (4.1.3), the bound (4.1.4) is effective only when $\mu_{\sigma,\mathfrak{p}}^{1/2}|z| \gg 1$. If we fix both $z \in \mathbf{C}$ and $\sigma > 0$, then $\mu_{\sigma,\mathfrak{p}}^{1/2}z$ tends to 0 as $N(\mathfrak{p}) \to \infty$. For small $\mu_{\sigma,\mathfrak{p}}^{1/2}|z|$, the following estimate will be useful.

Lemma D *There exists an absolute constant $q_0 > 1$ such that*

$$\text{(4.2.1)} \qquad |\tilde{M}_{\sigma,\mathfrak{p}}(z)|^2 \leq \exp(-\frac{\mu_{\sigma,\mathfrak{p}}}{2}|z|^2)$$

holds whenever $N(\mathfrak{p})^{\sigma} \geq q_0$ and $\mu_{\sigma,\mathfrak{p}}^{1/2}|z| \leq 2$.

Proof Put $t = N(\mathfrak{p})^{-\sigma}$. Then (4.1.1) gives a power series expansion of $\mu_{\sigma,\mathfrak{p}}$ in t^2 starting with $\lambda_{\mathfrak{p}}^2 t^2$, where $\lambda_{\mathfrak{p}} = \log N(\mathfrak{p})$ (resp. $= 1$) for Case 1 (resp. Case 2). Note that this power series is nowhere vanishing on $|t| < 1$ (in Case 1 this is obvious; in Case 2 note only that $\sum_{n\geq 2} n^{-2} = \pi^2/6 - 1 < 1$). Thus, $\mu_{\sigma,\mathfrak{p}}^{1/2} = \lambda_{\mathfrak{p}} t + \cdots$ extends to a holomorphic and nowhere vanishing function of t on $|t| < 1$. We shall first show that for any $\vartheta \in \mathbf{R}/2\pi$,

$$\left|\tilde{M}_{\sigma,\mathfrak{p}}\left(\frac{re^{i\vartheta}}{\mu_{\sigma,\mathfrak{p}}^{1/2}}\right)\right|^2 - J_0(r)^2 \tag{4.2.1}$$

extends to a holomorphic function of (r,t) on $|t| < 1$ *whose Taylor series at* $(0,0)$ *is divisible by* t^2r^4. By (1.2.10),

$$\tilde{M}_{\sigma,\mathfrak{p}}\left(\frac{re^{i\vartheta}}{\mu_{\sigma,\mathfrak{p}}^{1/2}}\right) = 1 + \sum_{a,b\geq 1} (\pm i/2)^{a+b} \frac{\mu_{\sigma,\mathfrak{p}}^{(a,b)}}{(\mu_{\sigma,\mathfrak{p}}^{1/2})^{a+b}} \frac{r^{a+b}}{a!b!} \cos((a-b)\vartheta), \tag{4.2.2}$$

where $\mu_{\sigma,\mathfrak{p}}^{(a,b)} = \mu_{\mathfrak{p}}^{(a,b)}(\sigma)$. Note that no $\sin((a-b)\vartheta)$ term appears, due to cancellations caused by adding (a,b) and (b,a) terms. On the other hand, by (1.2.11) we see easily that

$$\frac{\mu_{\sigma,\mathfrak{p}}^{(a,b)}}{(\mu_{\sigma,\mathfrak{p}}^{1/2})^{a+b}} \equiv \begin{cases} 0 \mod t^{|b-a|} & (a \neq b), \\ 1 \mod t^2 & (a = b). \end{cases} \tag{4.2.3}$$

Note also that this quotient is a power series of t depending only on Cases and (a,b) (the $(a+b)$-th power of $\log N(\mathfrak{p})$ appearing in Case 1 in the numerator and the denominator cancels with each other). Therefore, the real (resp. imaginary) part $f_1(r,t)$ (resp. $f_2(r,t)$) of (4.2.2) (for $r > 0$, $\vartheta \in \mathbf{R}/2\pi$) are

$$\begin{aligned} f_1(r,t) &= 1 + \sum_{a\geq 1} (-1)^a \frac{\mu_{\sigma,\mathfrak{p}}^{(a,a)}}{\mu_{\sigma,\mathfrak{p}}^{a}} \frac{(r/2)^{2a}}{a!^2} \\ &+ 2 \sum_{\substack{b>a\geq 1 \\ b\equiv a \mod 2}} (-1/4)^{(a+b)/2} \frac{\mu_{\sigma,\mathfrak{p}}^{(a,b)}}{(\mu_{\sigma,\mathfrak{p}}^{1/2})^{a+b}} \frac{r^{a+b}}{a!b!} \cos((a-b)\vartheta) \\ &\equiv J_0(r) \mod t^2r^4, \end{aligned} \tag{4.2.4}$$

$$\begin{aligned} f_2(r,t) &= \pm \sum_{\substack{b>a\geq 1 \\ b\equiv a+1 \mod 2}} (-1/4)^{(a+b-1)/2} \frac{\mu_{\sigma,\mathfrak{p}}^{(a,b)}}{(\mu_{\sigma,\mathfrak{p}}^{1/2})^{a+b}} \frac{r^{a+b}}{a!b!} \cos((a-b)\vartheta) \\ &\equiv 0 \mod tr^3. \end{aligned} \tag{4.2.5}$$

Hence $f_1^2 + f_2^2 - J_0(r)^2 \equiv 0 \mod t^2r^4$, as desired. Therefore, the quotient

$$(f_1^2 + f_2^2 - J_0(r)^2)/t^2r^4 \tag{4.2.6}$$

is bounded on $|t| \leq 1/\sqrt{2}$ and $|r| \leq 2$ (say), independent of the continuous parameter $\vartheta \in \mathbf{R}/2\pi$. Call c_1 an upper bound, so that

$$(4.2.7) \qquad |\tilde{M}_{\sigma,\mathfrak{p}}(\mu_{\sigma,\mathfrak{p}}^{-1/2}w)|^2 - J_0(|w|)^2 \leq c_1 N(\mathfrak{p})^{-2\sigma}|w|^4 \qquad (N(\mathfrak{p})^\sigma \geq \sqrt{2}, |w| \leq 2).$$

Now we shall verify another inequality

$$(4.2.8) \qquad \exp(-|w|^2/2) - J_0(|w|)^2 \geq c_2|w|^4 \qquad (|w| \leq 2),$$

where c_2 is another positive absolute constant. These two combined will give Lemma D; indeed, if $q_0 \geq \sqrt{2}$ and $q_0^2 > c_1/c_2$, then $c_1 N(\mathfrak{p})^{-2\sigma} \leq c_1 q_0^{-2} < c_2$; hence by (4.2.7)(4.2.8),

$$(4.2.9) \qquad |\tilde{M}_{\sigma,\mathfrak{p}}(\mu_{\sigma,\mathfrak{p}}^{-1/2}w)|^2 \leq J_0(|w|)^2 + c_2|w|^4 \leq \exp(-|w|^2/2).$$

[**Verification of (4.2.8)**] First, the power series expansion at $r = 0$ gives

$$(4.2.10) \qquad \exp(-r^2/2) - J_0(r)^2 \equiv (1 - \frac{r^2}{2} + \frac{r^4}{8}) - (1 - \frac{r^2}{4} + \frac{r^4}{64})^2 \equiv \frac{r^4}{32} \mod r^6;$$

hence

$$(4.2.11) \qquad \frac{1}{r^4}(\exp(-r^2/2) - J_0(r)^2) > 0 \qquad (0 \leq r \leq r_0)$$

with some $r_0 > 0$. That we may take $r_0 = 2.72$ can be checked by computor. That we may take $r_0 = 2$ can also be shown as follows. Put $f(r) = \exp(r^2/4)J_0(r)$. Then $f(0) = 1$, and

$$f'(r) = (\frac{r}{2}J_0(r) - J_1(r))\exp(r^2/4) = -\frac{r}{2}J_2(r)\exp(r^2/4).$$

But $J_0(r) > 0$ for $r < 2.4$, and $J_2(r) > 0$ for $r < 5.1$; hence for $0 < r < 2.4$, we have $f(r) > 0$ and $f'(r) < 0$; hence $f(r) < f(0) = 1$; hence $f(r)^2 < 1$, i.e., $J_0(r)^2 < \exp(-r^2/2)$ on this region. Therefore, (4.2.11) takes a positive minimal value c_2 on $0 \leq r \leq 2$. This settles the proof of (4.2.8) and hence that of Lemma D.

4.3 – Global estimations; large $|z|$ and general $\sigma > 1/2$. Here and in the following, all primes $\mathfrak{p}$ considered are those *outside* P_∞; in particular, $\sum_{\mathfrak{p}\notin P_\infty}$ will be abbreviated as $\sum_{\mathfrak{p}}$. As an easy consequence of Lemma C and the prime number theorems, we obtain

Theorem 6 *For any fixed $\sigma_1 > 1/2$, $\delta > 0$, $a > 0$, there exists $R = R_{\sigma_1,\delta,a} > 0$ such that*

$$(4.3.1) \qquad |\tilde{M}_\sigma(z)|^2 < \exp\left(-a|z|^{\frac{1}{\sigma+\delta}}\right) \qquad (1/2 < \sigma \leq \sigma_1,\ |z| \geq R).$$

Remark 4.3.2 *This exponent $1/(\sigma+\delta)(<2)$ of $|z|$ cannot be replaced by 2. This is because for each fixed $\sigma > 1/2$ the Fourier dual $M_\sigma(w)$ satisfies $M_\sigma(w) \ll e^{-\lambda|w|^2}$ for any $\lambda > 0$ cf. [2]§5.2. By Hardy's theorem* [1]*, this implies that $\tilde{M}_\sigma(z) \ll e^{-c|z|^2}$ does not hold for any $c > 0$.*

Proof We may assume $|z| > 1$. For each $y > 1$, write

$$P_y = \{\mathfrak{p}; N(\mathfrak{p}) \le y\}, \qquad \tilde{M}_{\sigma,P_y}(z) = \prod_{\mathfrak{p}\in P_y} \tilde{M}_{\sigma,\mathfrak{p}}(z), \tag{4.3.3}$$

so that $|\tilde{M}_\sigma(z)| \le |\tilde{M}_{\sigma,P_y}(z)|$. By Lemma C and (4.1.8), $|\tilde{M}_{\sigma,\mathfrak{p}}(z)|^2 \le CN(\mathfrak{p})^\sigma |z|^{-1}$ holds with some $C > 1$; hence

$$|\tilde{M}_{\sigma,P_y}(z)|^2 \le C^{|P_y|}(\prod_{\mathfrak{p}\in P_y} N(\mathfrak{p}))^\sigma |z|^{-|P_y|}. \tag{4.3.4}$$

Choose

$$y = |z|^{1/(\sigma+\delta/2)}. \tag{4.3.5}$$

Since $\sigma \le \sigma_1$, $|z| \gg 1$ implies $y \gg 1$. We shall give a proof in the number field case; the function field case can be treated with minor modifications. For any $\epsilon > 0$, we have

$$(1-\epsilon)y/\log y \le |P_y| \le (1+\epsilon)y/\log y, \tag{4.3.6}$$

$$\sum_{\mathfrak{p}\in P_y} \log N(\mathfrak{p}) \le (1+\epsilon)y \tag{4.3.7}$$

for $y \gg_\epsilon 1$. Hence by (4.3.4) (including $\log 0 = -\infty$ in the inequality)

$$\begin{aligned} \text{(4.3.8)} \log(|\tilde{M}_{\sigma,P_y}(z)|^2) &\le y\left((1+\epsilon)\log C/\log y + (1+\epsilon)\sigma - (1-\epsilon)\log|z|/\log y\right) \\ &= |z|^{1/(\sigma+\delta/2)}(I + II), \end{aligned}$$

with

$$\begin{aligned} \text{(4.3.9)} \qquad I &\le (1+\epsilon)(\sigma_1+\delta/2)(\log C)/\log|z|, \\ \text{(4.3.10)} \qquad II &= (1+\epsilon)\sigma - (1-\epsilon)(\sigma+\delta/2) \le -\delta/2 + \epsilon(2\sigma_1+\delta/2). \end{aligned}$$

But $I < \delta/8$ for $|z| \gg 1$, and if we take such ϵ that satisfies $\epsilon(2\sigma_1+\delta/2) = \delta/8$, then $I + II < -\delta/4$ holds. Therefore,

$$\log(|\tilde{M}_{\sigma,P_y}(z)|^2) < -\frac{\delta}{4}|z|^{1/(\sigma+\delta/2)} \le -a|z|^{1/(\sigma+\delta)} \tag{4.3.11}$$

for $|z| \gg_{a,\delta,\sigma_1} 1$, as desired. □

[1] Recall that in the 1 dimensional case, it asserts that $f(x) \ll e^{-a|x|^2/2}$, $f\hat{}(\xi) \ll e^{-b|\xi|^2/2}$ $(a, b > 0)$ with $ab > 1$ implies $f \equiv 0$. Apply this to $f(x) = (1/\sqrt{2\pi})\int_{-\infty}^{\infty} M_\sigma(x+yi)dy$, $f\hat{}(\xi) = \tilde{M}_\sigma(\xi)$.

4.4 – Small $|z|$, large $(2\sigma-1)^{-1}$. An easy consequence of Lemma D is:

Theorem 7A *Fix any ϵ, R, with $0<\epsilon<1$, $R>0$. If $|z|\le R$ and $(2\sigma-1)^{-1}\gg_{\epsilon,R}1$, then*

$$|\tilde{M}_\sigma(z)|^2 \le \exp\{-\frac{1-\epsilon}{2}\mu_\sigma|z|^2\}. \tag{4.4.1}$$

Proof We may remove a finite set $P=P^{(R)}$ of primes $\mathfrak{p}$ and assume that $\mu_{1/2,\mathfrak{p}}^{1/2}R\le 2$ holds for $\mathfrak{p}\notin P$; hence that $\mu_{\sigma,\mathfrak{p}}^{1/2}|z|\le 2$ holds whenever $\mathfrak{p}\notin P$, $|z|\le R$, $\sigma>1/2$. We may also assume $N(\mathfrak{p})^{1/2}\ge q_0$ for the constant q_0 in Lemma D. Thus, Lemma D gives

$$|\tilde{M}_\sigma(z)|^2 \le \exp\{-\frac{|z|^2}{2}\sum_{\mathfrak{p}\notin P}\mu_{\sigma,\mathfrak{p}}\} \le \exp\{\frac{|z|^2}{2}(\sum_{\mathfrak{p}\in P}\mu_{1/2,\mathfrak{p}}-\mu_\sigma)\}. \tag{4.4.2}$$

Since $\lim_{\sigma\to+\infty}\mu_\sigma=\infty$, this gives

$$|\tilde{M}_\sigma(z)|^2 \le \exp\{-\frac{1-\epsilon}{2}\mu_\sigma|z|^2\}, \tag{4.4.3}$$

for $|z|\le R$ and $(2\sigma-1)^{-1}\gg_{\epsilon,R}1$. □

4.5 – Large $|z|$, large $(2\sigma-1)^{-1}$.

Theorem 7B *Fix any ϵ with $0<\epsilon<1$. If $|z|\gg_\epsilon 1$ and $(2\sigma-1)^{-1}\gg_\epsilon 1$, then*

$$|\tilde{M}_\sigma(z)|^2 \le \exp\{-\frac{\mu_\sigma}{2}|z|^{2(1-\epsilon)}\}. \tag{4.5.1}$$

The proof requires some global estimations, Lemma E below in §4.7.

4.6 – Large $(2\sigma-1)^{-1}$, all $|z|$. Now, Theorems 7A, 7B combined give immediately:

Theorem 7C (Lemma B,§2.3) *Fix any ϵ with $0<\epsilon<1$. If $(2\sigma-1)^{-1}\gg_\epsilon 1$, then the inequality*

$$|\tilde{M}_\sigma(z)|^2 \le \exp\{-\frac{1-\epsilon}{2}\mu_\sigma|z|^{2(1-\epsilon)}\} \tag{4.6.1}$$

holds for all $z\in\mathbf{C}$.

In fact, Theorem 7B shows that (4.5.1) and hence also (4.6.1) holds for $|z|\ge R_\epsilon$ with some R_ϵ. Now take $R=R_\epsilon$ in Theorem 7A and let $(2\sigma-1)^{-1}\gg_{\epsilon,R_\epsilon}1$. Then (4.4.1) and hence also (4.6.1) holds for $|z|\le R_\epsilon$, too. Thus, Theorem 7C is reduced to Theorem 7B.

4.7 – Key Lemma E. The key points for the proof of Theorem 7B are Lemma D (local) and the following global estimation *from below* of the error term for sums over primes.

Lemma E *Fix any* ϵ *with* $0 < \epsilon < 1/2$. *If* $(2\sigma - 1)^{-1} \gg_\epsilon 1$ *and* $T \gg_\epsilon 1$, *then*

$$\sum_{N(\mathfrak{p}) \geq T} \mu_{\sigma,\mathfrak{p}} > \begin{cases} (1-\epsilon)\mu_\sigma T^{1-2\sigma} & (\text{Case 1}), \\ (1-\epsilon)\mu_\sigma \frac{T^{1-2\sigma}}{\log T} & (\text{Case 2}). \end{cases}$$

Proof We shall give a proof for the number field case. The function field case can be treated with minor modifications.

(Case 1) By (4.1.1), we have $\mu_{\sigma,\mathfrak{p}} > (\log N(\mathfrak{p}))^2/N(\mathfrak{p})^{2\sigma}$. As usual, set

$$\pi(T) = \sum_{N(\mathfrak{p}) \leq T} 1 \sim T/\log T, \ \ \psi(T) = \sum_{N(\mathfrak{p}) \leq T} \log N(\mathfrak{p}) \sim T, \tag{4.7.1}$$

and also set

$$\psi_2(T) = \sum_{N(\mathfrak{p}) \leq T} (\log N(\mathfrak{p}))^2 \sim T \log T. \tag{4.7.2}$$

The last estimation follows from the first two by using only the trivial inequalities $\psi_2(T) \leq (\log T)\psi(T)$ and $\psi_2(T)/\pi(T) \geq (\psi(T)/\pi(T))^2$ (the Schwartz inequality). By partial summation and by (4.7.2), we easily obtain, for $T \gg_\epsilon 1$,

$$\sum_{N(\mathfrak{p}) \geq T} \mu_{\sigma,\mathfrak{p}} > -(1+\epsilon)T^{1-2\sigma}\log T + (1-\epsilon)\int_T^\infty \frac{\log t}{t^{2\sigma}} dt \tag{4.7.3}$$

for any $\sigma > 1/2$. But since the last integral can be explicitly given by

$$\left(\frac{1}{(2\sigma-1)^2} + \frac{\log T}{2\sigma - 1} \right) T^{1-2\sigma}, \tag{4.7.4}$$

we obtain

$$\begin{aligned} \sum_{N(\mathfrak{p}) \geq T} \mu_{\sigma,\mathfrak{p}} \ &> \ (1-\epsilon)\left(\frac{T^{1-2\sigma}}{(2\sigma-1)^2} + \left(\frac{1}{2\sigma-1} - \frac{1+\epsilon}{1-\epsilon} \right) T^{1-2\sigma}\log T \right) \\ &> \ (1-\epsilon)\frac{T^{1-2\sigma}}{(2\sigma-1)^2} > (1-2\epsilon)\mu_\sigma T^{1-2\sigma} \end{aligned} \tag{4.7.5}$$

for σ sufficiently close to 1/2, by §2.1 Theorem 2 (i). This settles Case 1.

(Case 2) In this case, where $\mu_{\sigma,\mathfrak{p}} > N(\mathfrak{p})^{-2\sigma}$, we first obtain easily

$$\sum_{N(\mathfrak{p})\geq T} \mu_{\sigma,\mathfrak{p}} > -(1+\epsilon)\frac{T^{1-2\sigma}}{\log T} + (1-\epsilon)\int_T^\infty \frac{dt}{t^{2\sigma}\log t}. \tag{4.7.6}$$

But a more delicate treatment of the integral

$$\int_T^\infty \frac{dt}{t^{2\sigma}\log t} = \int_{(2\sigma-1)\log T}^\infty e^{-u}u^{-1}du \tag{4.7.7}$$

is required.

Sublemma 4.7.8 *We have, for any $b > 0$,*

$$\int_b^\infty e^{-u}u^{-1}du = -\gamma + \log(1/b) + \int_0^b \frac{1-e^{-t}}{t}dt \tag{4.7.9}$$

$$\geq \begin{cases} c_0(\log(1/b)+1) & (0 < b \leq 2), \\ (b+1)^{-1}e^{-b-1} & (\mathit{all}\, b > 0), \end{cases} \tag{4.7.10}$$

where γ is the Euler constant $\gamma = 0.5772\cdots$, and c_0 is an absolute positive constant.

Proof As for the first equality, the derivative d/db of the two sides are equal, and the formula for $b = 1$ can be found, e.g., in [10] §12.2 Ex 4. When $0 < b \leq 2$, so that $\log(1/b) + 1 > 1/4$, the quotient

$$\left(\int_b^\infty e^{-u}u^{-1}du\right)/(\log(1/b)+1) \tag{4.7.11}$$

is a continuous positive-valued function, which, by the equality (4.7.9) tends to 1 as $b \to 0$. Therefore, (4.7.11) attains a positive minimal value $c_0 > 0$ on $0 < b \leq 2$. The second inequality is obvious, because

$$\int_b^\infty e^{-u}u^{-1}du > \int_b^{b+1} e^{-u}u^{-1}du > e^{-b-1}(b+1)^{-1}.$$

□

Corollary 4.7.12

$$e^{y/x}\int_{y/x}^\infty e^{-u}u^{-1}du > (1+\log x)/y \qquad (x, y \gg 1). \tag{4.7.13}$$

Proof Put $b = y/x$. Call LHS (resp. RHS) the left (resp. right) hand side of (4.7.13). First, let $0 < b \leq 2$. Then $1 + \log(1/b) > 1/4$, and by Sublemma 4.7.8,

$$LHS > e^b c_0(\log(1/b) + 1) > c_0(\log(1/b) + 1).$$

If y is so large that $1/y < c_0/2$ and $(\log y)/y < c_0/8$, then

$$RHS = (1 + \log(1/b) + \log y)/y < \frac{c_0}{2}(1 + \log(1/b)) + \frac{c_0}{8}.$$

But since $1/4 < 1 + \log(1/b)$, this is $< LHS$.

Now let $b \geq 2$. Then $(b+1)^{-1} \geq (2/3)b^{-1}$; hence by Sublemma 4.7.8,

$$LHS > e^b(b+1)^{-1}e^{-b-1} \geq 2/(3eb).$$

On the other hand, if x is so large that $(1 + \log x)/x < 2/(3e)$, then

$$RHS = (1 + \log x)/(bx) < 2/(3eb) < LHS.$$

$\square$

Now by (4.7.6) (4.7.7) and Cor 4.7.12 applied to $x = (2\sigma - 1)^{-1}$ and $y = \log T$ (hence $e^{y/x} = T^{2\sigma-1}$), we obtain

$$\text{(4.7.14)} \qquad \sum_{N(\mathfrak{p}) \geq T} \mu_{\sigma,\mathfrak{p}} > -(1+\epsilon)\frac{T^{1-2\sigma}}{\log T} + (1-\epsilon)\frac{T^{1-2\sigma}}{\log T}\left(1 + \log\frac{1}{2\sigma - 1}\right)$$

$$= \frac{T^{1-2\sigma}}{\log T}\left((1-\epsilon)\log\frac{1}{2\sigma-1} - 2\epsilon\right) > \frac{T^{1-2\sigma}}{\log T}\left((1-2\epsilon)\log\frac{1}{2\sigma-1}\right),$$

since we may assume $\log(1/(2\sigma - 1)) > 2$. Since $\mu_\sigma - \log(1/(2\sigma - 1))$ is bounded near $\sigma = 1/2$ (say, by Theorem 4 §3.3), this is

$$> \frac{T^{1-2\sigma}}{\log T}\left((1-3\epsilon)\mu_\sigma\right).$$

This settles the proof of Lemma E.

4.8 – Proof of Theorem 7B.

Let $z \in \mathbf{C}$ with $|z| > 1$ and put

$$T = T_z = \begin{cases} (2|z|\log|z|)^2 & \text{(Case 1)}, \\ |z|^2 & \text{(Case 2)}. \end{cases}$$

We claim that there exists a constant $C > 0$ depending only on (K, P_∞) such that *if $|z| \geq C$ and $N(\mathfrak{p}) \geq T_z$, then the assumptions of Lemma D are satisfied for any $\sigma > 1/2$ and hence*

$$\text{(4.8.1)} \qquad |\tilde{M}_{\sigma,\mathfrak{p}}(z)|^2 \leq \exp\{-\frac{\mu_{\sigma,\mathfrak{p}}}{2}|z|^2\}.$$

Proof (Case 1) First, note that if $x \geq 2y\log y$ and $y \gg 1$ then $(x-1)/\log x > y$. Now let $N(\mathfrak{p}) \geq T_z$. Then $N(\mathfrak{p})^{1/2} \geq 2|z|\log|z|$; hence $(N(\mathfrak{p})^{1/2}-1)/\log(N(\mathfrak{p})^{1/2}) > |z|$ for $|z| \gg 1$; hence

$$\mu_{\sigma,\mathfrak{p}}^{1/2}|z| = \frac{\log N(\mathfrak{p})}{(N(\mathfrak{p})^{2\sigma}-1)^{1/2}}|z| < \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{1/2}-1}|z| < 2, \tag{4.8.2}$$

as desired.

(Case 2) In this case, $N(\mathfrak{p}) \geq T_z = |z|^2$ implies

$$\mu_{\sigma,\mathfrak{p}}^{1/2}|z| \leq \left(N(\mathfrak{p})\sum_{n\geq 1}\frac{1}{n^2 N(\mathfrak{p})^{2n\sigma}}\right)^{1/2} \leq \left(\sum_{n\geq 1}\frac{1}{N(\mathfrak{p})^{n-1}}\right)^{1/2} \leq \sqrt{2} < 2, \tag{4.8.3}$$

as desired. □

Now we can finish the proof of Theorem 7B.

(Case 1) Let $0 < \epsilon < 1$, and $|z| \gg_\epsilon 1$, $(2\sigma-1)^{-1} \gg_\epsilon 1$. Then by the above claim and Lemmas D, E, we have, for $T = T_z$ as above,

$$\prod_{N(\mathfrak{p})\geq T}|\tilde{M}_{\sigma,\mathfrak{p}}(z)|^2 \leq \prod_{N(\mathfrak{p})\geq T}\exp\{-\frac{\mu_{\sigma,\mathfrak{p}}}{2}|z|^2\} \leq \exp\{-\frac{1-\epsilon}{2}\mu_\sigma T^{1-2\sigma}|z|^2\}. \tag{4.8.4}$$

But if $2\sigma - 1 < \epsilon/2$ and $|z| \gg_\epsilon 1$, then $T^{1-2\sigma} > T^{-\epsilon/2} = (2|z|\log|z|)^{-\epsilon} > (1-\epsilon)^{-1}|z|^{-2\epsilon}$; hence

$$|\tilde{M}_\sigma(z)|^2 \leq \prod_{N(\mathfrak{p})\geq T}|\tilde{M}_{\sigma,\mathfrak{p}}(z)|^2 \leq \exp\{-\frac{\mu_\sigma}{2}|z|^{2(1-\epsilon)}\}, \tag{4.8.5}$$

as desired.

(Case 2) In this case, $T = |z|^2$, and we obtain, similarly,

$$\prod_{N(\mathfrak{p})\geq T}|\tilde{M}_{\sigma,\mathfrak{p}}(z)|^2 \leq \exp\{-\frac{1-\epsilon}{2}\mu_\sigma\frac{T^{1-2\sigma}}{\log T}|z|^2\}. \tag{4.8.6}$$

But

$$T^{1-2\sigma}/\log T = \frac{1}{2}|z|^{2(1-2\sigma)}/\log|z| > \frac{1}{2}|z|^{-\epsilon}/\log|z| > (1-\epsilon)^{-1}|z|^{-2\epsilon}$$

for $2\sigma - 1 < \epsilon/2$ and $|z| \gg_\epsilon 1$; hence (4.8.6) is

$$\leq \exp\{-\frac{\mu_\sigma}{2}|z|^{2(1-\epsilon)}\},$$

also in this case. This completes the proof of Theorem 7B.

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