RIMS-1688

Holonomic $\mathcal{D}$-module with Betti structure

By

Takuro MOCHIZUKI

January 2010

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

Holonomic $\mathcal{D}$-module with Betti structure

Takuro Mochizuki

Abstract

This is an attempt to define a notion of Betti structure with nice functorial property for algebraic holonomic D-modules which are not necessarily regular singular.
Keywords: holonomic D-module, Betti structure, Stokes structure
MSC: 14F10, 32C38

1 Introduction

In this paper, we would like to introduce a notion of Betti structure for holonomic $\mathcal{D}$-modules in a naive way, motivated by a question in [9]. For regular holonomic $\mathcal{D}$-modules, it is clearly defined in terms of the Riemann-Hilbert correspondence. Namely, a Betti structure of a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is defined to be a $\mathbb{Q}$-perverse sheaf $\mathcal{F}$ with an isomorphism $\alpha : \mathcal{F} \otimes \mathbb{C} \simeq \mathrm{DR}_X \mathcal{M}$. It has a nice functorial property for standard functors such as pull back, push-forward, dual etc.. The non-regular version of the Riemann-Hilbert correspondence has not yet been established as far as the author knows, except for the case that the dimension of the support is one dimensional. Although it would be a natural and attractive to expect a correspondence between holonomic $\mathcal{D}$-modules and perverse sheaves equipped with "Stokes structure" in some sense, it seems to require some more complicated machinery for a precise formulation. Instead, we make an attempt to define just "Betti structure" of holonomic $\mathcal{D}$-modules with functorial property (at least in the algebraic case), by using only the classical machinery of holonomic $\mathcal{D}$-modules and perverse sheaves. It still requires a non-trivial task, and we hope that it would be useful for further study toward Riemann-Hilbert correspondence.

1.1 Betti structure

1.1.1 Pre-Betti structure

To define a Betti structure of a holonomic $\mathcal{D}_X$-module $\mathcal{M}$, it is a most naive idea to consider a pair $(\mathcal{F}, \alpha)$ as above, which is called a pre-Betti structure of $\mathcal{M}$ in this paper. We should say that pre-Betti structure is too naive for the following reasons:

- It is not so intimately related with Stokes structure.
- Although pre-Betti structures have nice functoriality with respect to dual and push-forward, they are not functorial with respect to pull back, nearby cycle and vanishing cycle functors. Recall that the de Rham functor is not compatible with the latter class of functors.

We would like to introduce a condition for a pre-Betti structure to be a "Betti structure" with an inductive way on the dimension of the support. In the zero dimensional case, we do not need any additional condition.

In the following, a $\mathbb{Q}$-structure of a $\mathbb{C}$-perverse sheaf $\mathcal{F}_{\mathbb{C}}$ is a $\mathbb{Q}$-perverse sheaf $\mathcal{F}_{\mathbb{Q}}$ with an isomorphism $\mathcal{F}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathcal{F}_{\mathbb{C}}$.

1.1.2 One dimensional case

Before explaining the condition for Betti structure in the one dimensional case, let us recall "Riemann-Hilbert correspondence" for holonomic $\mathcal{D}$-module *on curves*, which are not necessarily regular singular. For simplicity, we consider holonomic $\mathcal{D}$-modules on $X = \Delta = \{|z| < 1\}$ which may have a singularity at the origin $D = \{O\}$.

Meromorphic flat bundles Let V be a meromorphic flat bundle on (X, D). Let $\pi : \widetilde{X}(D) \longrightarrow X$ be the real blow up along D. Let $\mathcal{L}$ be the local system on $\widetilde{X}(D)$ associated to the flat bundle $V_{|X-D}$. Let P be any point of $\pi^{-1}(D)$. According to the classical asymptotic analysis, we have the Stokes filtration $\mathcal{F}^P$ of the stalk $\mathcal{L}_P$ given by the growth order of flat sections. The meromorphic flat bundle V can be reconstructed from the flat bundle $V_{|X-D}$ and the system of filtrations $\{\mathcal{F}^P \mid P \in \pi^{-1}(D)\}$, which is a Riemann-Hilbert correspondence for meromorphic flat bundles on a curve.

Let $V^\vee$ be the dual of V as a meromorphic flat bundle, and let $V_! := \mathbb{D}_X V^\vee$ be the dual of $V^\vee$ as a $\mathcal{D}_X$-module. Let us recall that the de Rham complexes $\mathrm{DR}_X(V)$ and $\mathrm{DR}_X(V_!)$ can be described in terms of Stokes filtrations. Let $\mathcal{L}^{\leq D}$ and $\mathcal{L}^{<D}$ be the constructible subsheaves of $\mathcal{L}$ such that $\mathcal{L}^{\leq D}_P = \mathcal{F}^P_{\leq 0}(\mathcal{L}_P)$ and $\mathcal{L}^{<D}_P = \mathcal{F}^P_{<0}(\mathcal{L}_P)$. Then, we have natural isomorphisms:

$$\mathrm{DR}(V) \simeq R\pi_* \mathcal{L}^{\leq D}, \qquad \mathrm{DR}(V_!) \simeq R\pi_* \mathcal{L}^{<D}. \tag{1}$$

Gluing Let us very briefly recall a key construction due to A. Beilinson [3] on the gluing of holonomic $\mathcal{D}$-modules, which we will review in Subsection 2.2 in more details. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module such that $V := \mathcal{M}(*D)$ is a meromorphic flat bundle on (X, D). We have the natural morphisms $V_! \xrightarrow{a_0} \mathcal{M} \xrightarrow{b_0} V$. According to [3], we have the $\mathcal{D}$-modules $\Xi_z(V)$ and $\psi_z(V)$ associated to V, with morphisms

$$\psi_z(V) \xrightarrow{a_1} \Xi_z(V) \xrightarrow{b_1} \psi_z(V), \qquad V_! \xrightarrow{a_2} \Xi_z(V) \xrightarrow{b_2} V. \tag{2}$$

It can be shown that $b_0 \circ a_0 = b_2 \circ a_2$. We also have $b_2 \circ a_1 = 0$ and $b_1 \circ a_2 = 0$. We obtain the $\mathcal{D}$-module $\phi_z(\mathcal{M})$ as the cohomology of the natural complex:

$$V_! \longrightarrow \Xi_z(V) \oplus \mathcal{M} \longrightarrow V \tag{3}$$

We have the naturally induced morphisms $\psi_z(V) \xrightarrow{\mathrm{can}} \phi_z(\mathcal{M}) \xrightarrow{\mathrm{var}} \psi_z(V)$. Then, $\mathcal{M}$ is reconstructed as the cohomology of the complex:

$$\psi_z(V) \longrightarrow \Xi_z(V) \oplus \phi_z(\mathcal{M}) \longrightarrow \psi_z(V) \tag{4}$$

Recall that $\Xi_z(V)$, $\psi_z(V)$, and $\phi_z(\mathcal{M})$ are called the maximal extension, the nearby cycle sheaf, and the vanishing cycle sheaf of $\mathcal{M}$.

Good $\mathbb{Q}$-structure of a meromorphic flat bundle Let V be a meromorphic flat bundle on (X, D), and let $\mathcal{L}$ denote the associated local system on $\widetilde{X}(D)$ with the Stokes structure. We say that V has a good $\mathbb{Q}$-structure, if $\mathcal{L}$ has a $\mathbb{Q}$-structure such that the Stokes filtrations $\mathcal{F}^P$ are defined over $\mathbb{Q}$. By the isomorphisms (1), we obtain the pre-Betti structures of V and $V_!$. Moreover, it is easy to observe that $\psi(V)$ and $\Xi(V)$ are also naturally equipped with pre-Betti structures such that the morphisms a_i and b_i $(i = 1, 2)$ are compatible with pre-Betti structures.

Betti structure of a holonomic $\mathcal{D}$-module Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module on (X, D) such that $V := \mathcal{M}(*D)$ is a meromorphic flat bundle. Let $(\mathcal{F}, \alpha)$ be a pre-Betti structure of $\mathcal{M}$. It is called a Betti structure, if the following holds:

- The induced $\mathbb{Q}$-structure on $\mathrm{DR}(V_{|X-D})$ induces a good $\mathbb{Q}$-structure of V. As remarked above, we have the induced pre-Betti structures on V and $V_!$.
- The natural morphisms a_0 and b_0 are compatible with the pre-Betti structures.

Note that we obtain a pre-Betti structure on $\phi(\mathcal{M})$ from the expression as the cohomology of the complex (3), and the morphisms var and can are compatible with the pre-Betti structures. The pre-Betti structure of $\mathcal{M}$ can be reconstructed from the pre-Betti structure of $\phi(\mathcal{M})$ and the good $\mathbb{Q}$-structure of V.

1.1.3 Higher dimensional case

We would like to generalize it in the higher dimensional case in a naive way.

Good meromorphic flat bundle and good $\mathbb{Q}$-structure Let X be a complex manifold with a simple normal crossing hypersurface D. Let (V,∇) be a good meromorphic flat bundle in the sense that it is equipped with a good lattice as in [32]. (See also [34], [35] and [33].) Asymptotic analysis for meromorphic flat bundles on curves can be naturally generalized for good meromorphic flat bundles (see [25], [34], [32]). Let $\pi:\widetilde{X}(D)\longrightarrow X$ be the real blow up along D, which means in this paper the fiber product of the real blow up at each irreducible component taken over X. Let $\mathcal{L}$ be the local system on $\widetilde{X}(D)$ associated to $V_{|X-D}$. For any point $P\in\pi^{-1}(D)$, we have the Stokes filtration $\mathcal{F}^P$ of the stalk $\mathcal{L}_P$. We can reconstruct V from $V_{|X-D}$ and the system of filtrations $\bigl\{\mathcal{F}^P\bigm| P\in\pi^{-1}(D)\bigr\}$. We obtain the constructible subsheaf $\mathcal{L}^{\leq D}$ of $\mathcal{L}$ which consists of flat sections with the growth of polynomial order, i.e., $\mathcal{L}^{\leq D}_P=\mathcal{F}^P_{\leq 0}(\mathcal{L}_P)$. Let $\mathcal{L}^{<D}$ be the constructible subsheaf of $\mathcal{L}$, which consists of flat sections with exponential decay along D. (It is also described in terms of Stokes filtrations. See Subsection 5.1.2.) We have natural generalization of the isomorphisms (1). For a holomorphic function g on X such that $g^{-1}(0)=D$, we obtain $\mathcal{D}_X$-modules $V_!$, $\psi_g(V)$ and $\Xi_g(V)$ with morphisms as in (2).

As in the one dimensional case, we say that V has a good $\mathbb{Q}$-structure, if $\mathcal{L}$ has a $\mathbb{Q}$-structure such that the Stokes filtrations are defined over $\mathbb{Q}$. Then, the $\mathcal{D}_X$-modules V, $V_!$, $\Xi_g(V)$ and $\psi_g(V)$ are naturally equipped with pre-Betti structures, and the natural morphisms are compatible with pre-Betti structures.

Remark 1.1 *We have resolution of turning points for any algebraic meromorphic flat bundles* [31], [32]*. Namely, let (V,∇) be an algebraic meromorphic flat bundle on (X,D), which is not necessarily good. Then, there exists a projective birational morphism $\varphi:(X',D')\longrightarrow(X,D)$ such that $\varphi^*(V,\nabla)$ has no turning points. In* [20]*, Kedlaya showed the existence of a resolution of turning points for meromorphic flat bundles on complex surfaces.* ∎

Cell and induced pre-Betti structure Let P be a point of X. For any closed analytic subset W of X, let $\dim_P W$ denote the dimension of W at P. Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module on X with $\dim_P\operatorname{Supp}\mathcal{M}\leq n$. An n-dimensional good cell of $\mathcal{M}$ at P is a tuple (Z,U,φ,V) as follows:

(Cell 1) $\varphi:Z\longrightarrow X$ is a morphism of complex manifolds such that $P\in\varphi(Z)$ and $\dim Z=n$. There exists a neighbourhood X_P of P in X such that $\varphi:Z\longrightarrow X_P$ is projective. We permit that Z may be non-connected or empty.

(Cell 2) $U\subset Z$ is the complement of a normal crossing hypersurface D_Z. The restriction $\varphi_{|U}$ is an immersion. Moreover, there exists a hypersurface H of X_P such that $\varphi^{-1}(H)=D_Z$.

(Cell 3) V is a good meromorphic flat bundle on (Z,D_Z). For a hypersurface H as in (Cell 2), we have $\mathcal{M}(*H)=\varphi_\dagger V$ and $\mathcal{M}(!H)=\varphi_\dagger V_!$. The restriction of V to some connected components may be 0. We obtain the natural morphisms $\varphi_\dagger V_!\longrightarrow\mathcal{M}\longrightarrow\varphi_\dagger V$.

(The conditions are stated in a slightly different way from that in Subsection 7.1.1.) A holomorphic function g on X is called a cell function for $\mathcal{C}$, if $U=\operatorname{Supp}\mathcal{M}\setminus g^{-1}(0)$. We set $g_Z:=\varphi^{-1}(g)$. We have natural isomorphisms $\varphi_\dagger\Xi_{g_Z}(V)\simeq\Xi_g\varphi_\dagger(V)$ and $\varphi_\dagger\psi_{g_Z}(V)\simeq\psi_g\varphi_\dagger(V)$. The $\mathcal{D}_X$-module $\phi_g(\mathcal{M})$ is obtained as the cohomology of the complex:

$$\varphi_\dagger V_!\longrightarrow\Xi_g\varphi_\dagger(V)\oplus\mathcal{M}\longrightarrow\varphi_\dagger V \tag{5}$$

We also have a description of $\mathcal{M}$ around P as the cohomology of the complex:

$$\psi_g(\varphi_\dagger V)\longrightarrow\Xi_g(\varphi_\dagger V)\oplus\phi_g(\mathcal{M})\longrightarrow\psi_g(\varphi_\dagger V).$$

Let $\mathcal{F}$ be a pre-Betti structure of $\mathcal{M}$. Let $\mathcal{C}=(Z,U,\varphi,V)$ be a good n-cell of $\mathcal{M}$ at P. We say that $\mathcal{F}$ and $\mathcal{C}$ are compatible, if the following holds:

- The induced $\mathbb{Q}$-structure of $V_{|U}$ is good, i.e., compatible with the Stokes filtrations. It implies that $\varphi_\dagger V$, $\varphi_\dagger V_!$, $\Xi_g\varphi_\dagger V$ and $\psi_g\varphi_\dagger V$ are equipped with the induced pre-Betti structures.
- The morphisms $\varphi_\dagger V_!\longrightarrow\mathcal{M}\longrightarrow\varphi_\dagger V$ are compatible with pre-Betti structures.

Such a cell $\mathcal{C}$ is called a $\mathbb{Q}$-cell of $\mathcal{M}$ at P. Since $\phi_g(\mathcal{M})$ is the cohomology of the complex (5), we have the induced pre-Betti structure on $\phi_g(\mathcal{M})$.

Inductive definition of Betti structure Let us define the notion of Betti structure of $\mathcal{M}$ at P, inductively on the dimension of $\operatorname{Supp}\mathcal{M}$. In the case $\dim_P \operatorname{Supp}\mathcal{M} = 0$, a Betti structure is defined to be a pre-Betti structure. Let us consider the case $\dim_P \operatorname{Supp}\mathcal{M} \leq n$. We say that a pre-Betti structure of $\mathcal{M}$ is a Betti structure at P, if there exists an n-dimensional $\mathbb{Q}$-cell $\mathcal{C} = (Z, \varphi, U, V)$ at P with the following property:

- $\dim_P\Big(\big(\operatorname{Supp}\mathcal{M} \cap X_P\big) \setminus \varphi(Z)\Big) < n$ for some neighbourhood X_P of P in X.
- For any cell function g for $\mathcal{C}$, the induced pre-Betti structure of $\phi_g(\mathcal{M})$ is a Betti structure at P.

A holonomic $\mathcal{D}$-module with Betti structure is called a $\mathbb{Q}$-holonomic $\mathcal{D}$-module. The category of $\mathbb{Q}$-holonomic $\mathcal{D}$-modules is abelian.

Remark 1.2 *The above definition is slightly different from that given in Subsection* 7.2. ∎

1.2 Main purpose

It is our main purpose to show the functoriality of Betti structures.

Theorem 1.3 *The category of $\mathbb{Q}$-holonomic $\mathcal{D}$-modules is equipped with the standard functors such as dual, push-forward, pull-back, tensor product, and inner homomorphism, compatible with those for the category of holonomic $\mathcal{D}$-modules with respect to the forgetful functor.*

It is not so trivial to show that obvious examples are $\mathbb{Q}$-holonomic $\mathcal{D}$-modules.

Theorem 1.4 *Let X be a complex projective manifold with a simple normal crossing divisor D. Let V be a good meromorphic flat bundle on (X, D) with a good $\mathbb{Q}$-structure. Then, the associated pre-Betti structure of V is a Betti structure.*

1.3 Acknowledgement

I am grateful to C. Sabbah, who kindly sent an earlier version of his monograph [35], by which I was inspired. I thank H. Esnault, who attracted my attention to Betti structure of holonomic $\mathcal{D}$-modules. This study much owes to the foundational works on holonomic $\mathcal{D}$-modules and perverse sheaves due to many people, among all A. Beilinson, J. Bernstein, P. Deligne, M. Kashiwara and M. Saito. I thank P. Schapira for a valuable discussion. Special thanks goes to K. Vilonen. I am grateful to A. Ishii and Y. Tsuchimoto for their constant encouragement. I am grateful to M. Hien, G. Morando, M-H. Saito, C. Simpson and Sz. Szabo for stimulating discussions.

2 Preliminary

2.1 Notation and words

2.1.1 Dual, push-forward and de Rham functor

We prepare some notation. See very useful text books [13] and [18] for more details and precise on $\mathcal{D}$-modules. Let X be a complex manifold with $\dim X = d_X$. Let $\mathcal{D}_X$ denote the sheaf of holomorphic differential operators on X. In this paper, $\mathcal{D}_X$-module means left $\mathcal{D}_X$-module. Let $\operatorname{Hol}(X)$ be the category of holonomic $\mathcal{D}_X$-modules, and let $D^b_{\mathrm{hol}}(\mathcal{D}_X)$ be the derived category of cohomologically holonomic $\mathcal{D}_X$-complexes. Let Ω^j_X denote the sheaf of holomorphic j-forms. The invertible sheaf $\Omega^{d_X}_X$ is denoted by Ω_X. The dual functor on the derived category of $\mathcal{D}_X$-modules is denoted by $\mathbb{D}_X$, i.e., $\mathbb{D}_X\mathcal{M}^\bullet := R\mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{M}^\bullet, \mathcal{D}_X \otimes \Omega_X^{\otimes -1}\big)[d_X]$. Recall that $\mathbb{D}_X\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, if $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module. For $\mathcal{D}_X$-modules $\mathcal{M}_i$ $(i = 1, 2)$, the tensor product $\mathcal{M}_1 \otimes_{\mathcal{O}_X} \mathcal{M}_2$ is naturally a $\mathcal{D}_X$-module. For a tangent vector field v, we have $v(m_1 \otimes m_2) = (vm_1) \otimes m_2 + m_1 \otimes (vm_2)$. The $\mathcal{D}_X$-module is denoted by $\mathcal{M}_1 \otimes^D \mathcal{M}_2$. It is also denoted by $\mathcal{M}_1 \otimes \mathcal{M}_2$, if there is no risk of confusion.

Lemma 2.1 *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Let V be a $\mathcal{D}_X$-module, coherent and locally free as an $\mathcal{O}_X$-module. Its dual is denoted by $V^\vee$. Then, we have a natural isomorphism*

$$\mathbb{D}_X\big(\mathcal{M} \otimes^D V\big) \simeq (\mathbb{D}_X \mathcal{M}) \otimes^D V^\vee$$

Proof We recall Remark 3.4 in [18]. For a left $\mathcal{D}_X$-module $\mathcal{N}$, we have the left $\mathcal{D}_X$-action on $\mathcal{D}_X \otimes^D \mathcal{N}$. It is also equipped with a right $\mathcal{D}_X$-action given by the multiplication $(f \otimes m) \cdot g = fg \otimes m$ for $g \in \mathcal{D}_X$. The two-sided $(\mathcal{D}_X, \mathcal{D}_X)$-module is denoted by $\mathcal{N}_1$. Similarly, we have a left action of $\mathcal{D}_X$ on $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{N}$ (the $\mathcal{O}_X$-module structure of $\mathcal{D}_X$ is given by a right multiplication) given by the multiplication $g \cdot (f \otimes m) = gf \otimes m$ for $g \in \mathcal{D}_X$, and a right $\mathcal{D}_X$-action given by $(f \otimes m) \cdot v = fv \otimes m - f \otimes vm$ for a tangent vector v. The two-sided $(\mathcal{D}_X, \mathcal{D}_X)$-module is denoted by $\mathcal{N}_2$. We have a naturally defined $\mathcal{O}_X$-morphism $\mathcal{N} \longrightarrow \mathcal{N}_1$ given by $m \longmapsto 1 \otimes m$. It is naturally extended to a morphism of left $\mathcal{D}_X$-modules $\mathcal{N}_2 \longrightarrow \mathcal{N}_1$. Actually, it is an isomorphism and compatible with the right $\mathcal{D}_X$-action, as remarked in [18].

We have two left $\mathcal{D}_X$-actions on $\mathcal{D}_X \otimes \Omega_X^{\otimes -1}$. The first one is the natural one, and the second one is induced by the right $\mathcal{D}_X$-action. They induce two $\mathcal{O}_X$-actions. Let $(\mathcal{D}_X \otimes \Omega_X^{\otimes -1}) \otimes^i_{\mathcal{O}_X} \mathcal{N}$ denote the tensor product with respect to the i-th one. Each is equipped with two left $\mathcal{D}_X$-actions. From the consideration in the previous paragraph, we obtain a natural isomorphism $\iota : \mathcal{N} \otimes^1_{\mathcal{O}_X} (\mathcal{D}_X \otimes \Omega_X^{\otimes -1}) \longrightarrow \mathcal{N} \otimes^2_{\mathcal{O}_X} (\mathcal{D}_X \otimes \Omega_X^{\otimes -1})$, compatible with the $\mathcal{D}_X$-actions.

Let us return to Lemma 2.1. We have the following natural isomorphisms of $\mathcal{D}_X$-modules:

$$\begin{aligned} \mathbb{D}_X(\mathcal{M} \otimes^D V) = R\mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{M} \otimes^D V, \mathcal{D}_X \otimes \Omega_X^{\otimes -1}\big) &\simeq R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, V^\vee \otimes^1_{\mathcal{O}_X} \big(\mathcal{D}_X \otimes \Omega_X^{\otimes -1}\big)\Big) \\ &\simeq R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, V^\vee \otimes^2_{\mathcal{O}_X} \big(\mathcal{D}_X \otimes \Omega_X^{\otimes -1}\big)\Big) = (\mathbb{D}_X \mathcal{M}) \otimes^D V^\vee \end{aligned} \tag{6}$$

Here, the first one is obtained by using Godment type injective resolution, and the second one is induced by ι above. ∎

For any field R, let R_X denote the sheaf on X associated to the constant presheaf valued in R. Let $D^b_c(R_X)$ denote the derived category of cohomologically constructible R_X-complexes, and let $\mathrm{Per}(X, R)$ denote the category of R-perverse sheaves. Let $\omega_{X,R}$ denote the dualizing complex of R_X-modules. It will be denoted by ω_X, if there is no risk of confusion. The dual functor on the derived category of R_X-modules is also denoted by $\mathbb{D}_X$, i.e., for a R_X-complex $\mathcal{F}^\bullet$, let $\mathbb{D}_X \mathcal{F}^\bullet := R\mathcal{H}om_{R_X}\big(\mathcal{F}^\bullet, \omega_{X,R}\big)$.

The de Rham functor is denoted by DR_X, i.e., $\mathrm{DR}_X \mathcal{M} := \Omega_X \otimes^L_{\mathcal{D}_X} \mathcal{M} = \Omega^\bullet_X \otimes_{\mathcal{O}_X} \mathcal{M}[d_X]$. According to [15], it gives a functor of triangulated categories $\mathrm{DR}_X : D^b_{\mathrm{hol}}(\mathcal{D}_X) \longrightarrow D^b_c(\mathbb{C}_X)$ compatible with the t-structures, where the t-structure of $D^b_{\mathrm{hol}}(\mathcal{D}_X)$ is the natural one, and the t-structure of $D^b_c(\mathbb{C}_X)$ is given by the middle perversity. In particular, it induces an exact functor $\mathrm{DR}_X : \mathrm{Hol}(X) \longrightarrow \mathrm{Per}(X, \mathbb{C})$. We can identify $\omega_X = \mathrm{DR}_X \mathcal{O}_X[d_X]$. It is easy to observe that $\mathrm{DR}_X \mathcal{M} = 0$ implies $\mathcal{M} = 0$ for $\mathcal{M} \in \mathrm{Hol}(X)$. Hence, $\mathrm{DR}_X : \mathrm{Hol}(X) \longrightarrow \mathrm{Per}(X, \mathbb{C})$ is faithful, although it is not full in general.

Let $F : X \longrightarrow Y$ be a morphism of complex manifolds. The push-forward for $\mathbb{C}_X$-complexes in the derived category is denoted by RF_*. (It is also denoted by F_*, if there is no risk of confusion.) Its i-th perverse cohomology is denoted by $F^i_\dagger$. Put

$$\mathcal{D}_{X\to Y} := \mathcal{O}_X \otimes_{F^{-1}\mathcal{O}_Y} F^{-1}\mathcal{D}_Y, \qquad \mathcal{D}_{Y\leftarrow X} := \Omega_X \otimes_{F^{-1}\mathcal{O}_Y} F^{-1}\big(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}\big).$$

The push-forward for $\mathcal{D}_X$-complexes is denoted by $F_\dagger$, i.e., $F_\dagger \mathcal{M} = RF_*\big(\mathcal{D}_{Y\leftarrow X} \otimes^L_{\mathcal{D}_X} \mathcal{M}\big)$. Its i-th cohomology is denoted by $F^i_\dagger$.

Recall that these functors are compatible on the derived category of cohomologically holonomic $\mathcal{D}$-modules. Let $F : X \longrightarrow Y$ be a proper morphism of complex manifolds. We have natural transformations

$$\mathrm{DR}_Y \circ F_\dagger \simeq RF_* \circ \mathrm{DR}_X, \quad \mathbb{D}_X \circ \mathrm{DR}_X \simeq \mathrm{DR}_X \circ \mathbb{D}_X, \quad \mathbb{D}_Y \circ F_\dagger \simeq F_\dagger \circ \mathbb{D}_X.$$

We have the following diagram, which is commutative as shown in [39].

$$\begin{array}{ccccc} RF_* \mathbb{D}_X \mathrm{DR}_X & \xrightarrow{\simeq} & RF_* \mathrm{DR}_X \mathbb{D}_X & \xrightarrow{\simeq} & \mathrm{DR}_Y F_\dagger \mathbb{D}_X \\ \simeq\big\downarrow & & & & \simeq\big\downarrow \\ \mathbb{D}_Y RF_* \mathrm{DR}_X & \xrightarrow{\simeq} & \mathbb{D}_Y \mathrm{DR}_Y F_\dagger & \xrightarrow{\simeq} & \mathrm{DR}_Y \mathbb{D}_Y F_\dagger \end{array} \tag{7}$$

2.1.2 Hypersurface

For a hypersurface $D \subset X$, let $\mathcal{O}_X(*D)$ denote the sheaf of meromorphic functions whose poles are contained in D. For $\mathcal{M} \in \mathrm{Hol}(X)$, we have $\mathcal{M}(*D), \mathcal{M}(!D) \in \mathrm{Hol}(X)$ given as follows:

$$\mathcal{M}(*D) := \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*D), \quad \mathcal{M}(!D) := \mathbb{D}_X\Big(\big(\mathbb{D}_X\mathcal{M}\big)(*D)\Big).$$

We have naturally defined morphisms:

$$\mathcal{M}(!D) \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}(*D)$$

If D is given as the zero set of a holomorphic function f, they are denoted by $\mathcal{M}(*f)$ and $\mathcal{M}(!f)$, respectively. They are also denoted by $j_*j^*\mathcal{M}$ and $j_!j^*\mathcal{M}$, where $j : X - D \longrightarrow X$. Note that $j_\star j^*$ ($\star = *, !$) are exact functors on $\mathrm{Hol}(X)$.

We put $\mathcal{D}_{X(*D)} := \mathcal{D}_X \otimes \mathcal{O}_X(*D)$. A $\mathcal{D}_{X(*D)}$-module $\mathcal{M}$ is called holonomic, if it is holonomic as a $\mathcal{D}_X$-module. Let $\mathrm{Hol}\big(X(*D)\big)$ be the category of holonomic $\mathcal{D}_{X(*D)}$-modules, which is a full subcategory of $\mathrm{Hol}(X)$. The dual functor on $\mathrm{Hol}\big(X(*D)\big)$ is denoted by $\mathbb{D}_{X(*D)}$, i.e., $\mathbb{D}_{X(*D)}(\mathcal{M}) = \mathbb{D}_X(\mathcal{M})(*D)$.

2.1.3 Pre-K-holonomic $\mathcal{D}$-modules

Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Let K be a subfield of $\mathbb{C}$. A pre-K-Betti structure of $\mathcal{M}$ is defined to be a K-perverse sheaf $\mathcal{F}$ with an isomorphism $\lambda : \mathcal{F} \otimes_K \mathbb{C} \simeq \mathrm{DR}_X \mathcal{M}$. Such a tuple $(\mathcal{M}, \mathcal{F}, \lambda)$ is called a pre-K-holonomic $\mathcal{D}_X$-module. We will often omit to denote λ. A morphism of K-holonomic $\mathcal{D}_X$-modules $(\mathcal{M}_1, \mathcal{F}_1) \longrightarrow (\mathcal{M}_2, \mathcal{F}_2)$ is defined to be a pair of a morphism of $\mathcal{D}_X$-modules $\mathcal{M}_1 \longrightarrow \mathcal{M}_2$ and a morphism of perverse sheaves $\mathcal{F}_1 \longrightarrow \mathcal{F}_2$ such that the following induced diagram is commutative:

$$\begin{array}{ccc}
\mathcal{F}_1 \otimes_K \mathbb{C} & \xrightarrow{\simeq} & \mathrm{DR}_X(\mathcal{M}_1) \\
\downarrow & & \downarrow \\
\mathcal{F}_2 \otimes_K \mathbb{C} & \xrightarrow{\simeq} & \mathrm{DR}_X(\mathcal{M}_2)
\end{array}$$

The following lemma is clear.

Lemma 2.2 *The category of pre-K-holonomic $\mathcal{D}_X$-modules is abelian.* ∎

Let $\mathcal{F}$ be a pre-K-Betti structure of $\mathcal{M}$. We have induced pre-K-Betti structures $\mathbb{D}\mathcal{F}$ and $F_\dagger^i\mathcal{F}$ of $\mathbb{D}\mathcal{M}$ and $F_\dagger^i\mathcal{M}$, where $F : X \longrightarrow Y$ be a proper morphism. We put $\mathbb{D}(\mathcal{M}, \mathcal{F}) := \big(\mathbb{D}\mathcal{M}, \mathbb{D}\mathcal{F}\big)$ and $F_\dagger^i(\mathcal{M}, \mathcal{F}) := \big(F_\dagger^i\mathcal{M}, F_\dagger^i\mathcal{F}\big)$.

Lemma 2.3 *The isomorphism $\mathbb{D}F_\dagger\mathcal{M} \simeq F_\dagger\mathbb{D}\mathcal{M}$ is compatible with the induced pre-K-Betti structures.*

Proof Because (7) is commutative, we have the commutativity of the following naturally induced diagram:

$$\begin{array}{ccccc}
\mathrm{DR}\,\mathbb{D}F_\dagger\mathcal{M} & \xrightarrow{\simeq} & \mathbb{D}F_\dagger\,\mathrm{DR}\,\mathcal{M} & \xrightarrow{\simeq} & \mathbb{D}F_\dagger\mathcal{F} \otimes \mathbb{C} \\
\simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\
\mathrm{DR}\,F_\dagger\mathbb{D}\mathcal{M} & \xrightarrow{\simeq} & F_\dagger\mathbb{D}\,\mathrm{DR}\,\mathcal{M} & \xrightarrow{\simeq} & F_\dagger\mathbb{D}\mathcal{F} \otimes \mathbb{C}
\end{array}$$

It means the claim of the lemma. ∎

2.1.4 Formal completion

Let Y be a real analytic manifold. Let $\mathcal{C}_Y^{\infty}$ denote the sheaf of C^{∞}-functions on Y. For a real analytic subset Z, let $\mathcal{C}_Y^{\infty<Z}$ denote the subsheaf of $\mathcal{C}_Y^{\infty}$ which consists of the sections f such that the Taylor expansion of f at each point $P\in Z$ are 0. We set $\mathcal{C}^{\infty}_{\widehat{Z}}:=\mathcal{C}_Y^{\infty}/\mathcal{C}_Y^{\infty<Z}$. We have other descriptions. (i) It is the sheaf of Whitney functions of class C^{∞} on Z, i.e., sections of ∞-jets along Z satisfying the conditions in Theorem I.2.2 of [26]. (ii) Let $\mathcal{I}_{Z,\infty}$ be the ideal sheaf of $\mathcal{C}_Y^{\infty}$ corresponding to Z. Then, $\mathcal{C}^{\infty}_{\widehat{Z}}$ is also isomorphic to $\varprojlim \mathcal{C}_Y^{\infty}/\mathcal{I}_{Z,\infty}^m$. (See the proof of Theorem I.4.1 of [26].) For any $\mathcal{C}_Y^{\infty}$-module $\mathcal{F}$, let $\mathcal{F}_{|\widehat{Z}}$ denote $\mathcal{F}\otimes_{\mathcal{C}_Y^{\infty}}\mathcal{C}^{\infty}_{\widehat{Z}}$. Let Z_i $(i=1,2)$ be real analytic subsets in Y. According to Corollary IV.4.4 of [26], the natural sequence $0\longrightarrow \mathcal{C}^{\infty}_{\widehat{Z_1\cup Z_2}}\longrightarrow \mathcal{C}^{\infty}_{\widehat{Z}_1}\oplus\mathcal{C}^{\infty}_{\widehat{Z}_2}\longrightarrow \mathcal{C}^{\infty}_{\widehat{Z_1\cap Z_2}}\longrightarrow 0$ is exact.

Let Z_i $(i\in\Lambda)$ be real analytic subsets of Y. For any subset $I\subset\Lambda$, we put $Z_I:=\bigcap_{i\in I}Z_i$. We put $Z(I):=\bigcup_{i\in I}Z_i$. We fix a total order on Λ. For $J\subset K\subset\Lambda$, we have the restriction $r_{J,K}:\mathcal{C}^{\infty}_{\widehat{Z}_J}\longrightarrow\mathcal{C}^{\infty}_{\widehat{Z}_K}$. If $K=J\sqcup\{i\}$, we put $\kappa(J,K):=\{k\in J\,|\,k<i\}$ and $d_{J,K}:=(-1)^{\kappa(J,K)}r_{J,K}$. We set $\mathcal{K}^m\bigl(\mathcal{C}^{\infty}_{\widehat{Z}(I)}\bigr):=\bigoplus_{|J|=m+1,\,J\subset I}\mathcal{C}^{\infty}_{\widehat{Z}_J}$. The above morphisms $d_{J,K}$ induce $d_m:\mathcal{K}^m\bigl(\mathcal{C}^{\infty}_{\widehat{Z}(I)}\bigr)\longrightarrow\mathcal{K}^{m+1}\bigl(\mathcal{C}^{\infty}_{\widehat{Z}(I)}\bigr)$. Thus, we obtain the complex $\mathcal{K}^{\bullet}\bigl(\mathcal{C}^{\infty}_{\widehat{Z}(I)}\bigr)$. By using the exactness in the previous paragraph, it can be shown that the natural inclusion $\mathcal{C}^{\infty}_{\widehat{Z}(I)}\longrightarrow\mathcal{K}^0(\mathcal{C}^{\infty}_{\widehat{Z}(I)})$ induces a quasi-isomorphism $\mathcal{C}^{\infty}_{\widehat{Z}(I)}\simeq\mathcal{K}^{\bullet}\bigl(\mathcal{C}^{\infty}_{\widehat{Z}(I)}\bigr)$. (See [34], for example.)

Let X be a complex manifold. For a complex analytic subset Z, we set $\mathcal{O}_{\widehat{Z}}:=\varprojlim\mathcal{O}_X/\mathcal{I}_Z^m$, where $\mathcal{I}_Z$ denote the ideal sheaf of Z. We set $\Omega^{\bullet,\bullet}_{\widehat{Z}}:=\Omega^{\bullet,\bullet}_{X|\widehat{Z}}$ which is equipped with the differential operators ∂ and $\overline{\partial}$. If Z is smooth, it is easy to see that the natural inclusion $\mathcal{O}_{\widehat{Z}}\longrightarrow\Omega^{0,\bullet}_{\widehat{Z}}$ is a quasi-isomorphism.

Let D be a simple normal crossing hypersurface with the irreducible decomposition $D=\bigcup_{i\in\Lambda}D_i$. By the above procedures, we obtain the complexes $\mathcal{K}^{\bullet}\bigl(\mathcal{O}_{\widehat{D}(I)}\bigr)$. It is known that the natural inclusion $\mathcal{O}_{\widehat{D}(I)}\longrightarrow\mathcal{K}^0(\mathcal{O}_{\widehat{D}(I)})$ induces a quasi-isomorphism $\mathcal{O}_{\widehat{D}(I)}\simeq\mathcal{K}^{\bullet}\bigl(\mathcal{O}_{\widehat{D}(I)}\bigr)$. (See [10] and [34].) We also have $\Omega^{0,\bullet}_{\widehat{D}(I)}\simeq\mathcal{K}^{\bullet}\bigl(\Omega^{0,\bullet}_{\widehat{D}(I)}\bigr)$. Then, we obtain $\mathcal{O}_{\widehat{D}(I)}\simeq\Omega^{0,\bullet}_{\widehat{D}(I)}$.

2.2 Beilinson's construction of some functors

Let us recall Beilinson's beautiful construction of nearby cycle functor, vanishing cycle functor and maximal functor, which is crucial in this paper. See [3] for more details and precise.

2.2.1 Preliminary

Let k be a field of characteristic 0. Let $A:=k(\!(s)\!)$ and $A^i:=s^ik[\![s]\!]$. The multiplication of s induces a nilpotent map N_A of $A^{i,j}:=A^i\big/A^j$. Let $\mathfrak{I}:=A\otimes\mathcal{O}_{G_m}$ be a meromorphic flat bundle on $G_m:=\operatorname{Spec}k[t,t^{-1}]$ of infinite rank, equipped with a connection given by

$$\nabla\alpha=\alpha\cdot\left(s\frac{dt}{t}\right),\quad \alpha\in A.$$

We have the flat subbundle $\mathfrak{I}^i:=A^i\otimes\mathcal{O}_{G_m}$. We formally set $\mathfrak{I}^{-\infty}=\mathfrak{I}$. We set $\mathfrak{I}^{a,b}:=\mathfrak{I}^a\big/\mathfrak{I}^b$ for $a\leq b$, and formally $\mathfrak{I}^{a,\infty}:=\mathfrak{I}^a$. We have a natural morphism $\mathfrak{I}^{a,b}\longrightarrow\mathfrak{I}^{c,d}$ for $a\geq c$ and $b\geq d$. We have a natural isomorphism $\mathfrak{I}^{a,a+1}\simeq\mathfrak{I}^{0,1}=\mathcal{O}_{G_m}$ given by $s^a\longleftrightarrow 1$.

This construction makes sense also in the analytic situation, in which multi-valued flat sections are formally given by $\alpha\cdot\exp\bigl(-s\log t\bigr)$ for $\alpha\in A$.

2.2.2 Nearby cycle functor and maximal functor

Let X be a complex manifold with a hypersurface D. Let Y be a hypersurface of X. Let $j:X-Y\longrightarrow X$ denote the inclusion. Functors j_*j^* and $j_!j^*$ for holonomic $\mathcal{D}_{X(*D)}$-modules $\mathcal{M}$ are given as follows:

$$j_*j^*\mathcal{M}:=\mathcal{M}(*Y),\qquad j_!j^*\mathcal{M}:=\mathbb{D}_X\bigl(j_*j^*\mathbb{D}_X\mathcal{M}\bigr)(*D)=\bigl(\mathcal{M}(!Y)\bigr)(*D).$$

We have a naturally defined morphism $j_! j^* \mathcal{M} \longrightarrow j_* j^* \mathcal{M}$.

Let f be a meromorphic function on (X, D), i.e., the pole of f is contained in D. We set $\mathfrak{I}_f^{a,b} := f^* \mathfrak{I}^{a,b}(*D)$, which is a meromorphic flat bundle on $\big(X, f^{-1}(0) \cup D\big)$. Let $j : X - f^{-1}(0) \longrightarrow X$. For a holonomic $\mathcal{D}_{X(*D)}$-module $\mathcal{M}$, we obtain the following holonomic $\mathcal{D}_{X(*D)}$-module:

$$\mathcal{M}_f^{a,b} := \mathcal{M} \otimes \mathfrak{I}_f^{a,b} = j_* j^* \big(\mathcal{M} \otimes \mathfrak{I}_f^{a,b}\big)$$

Put $\Pi_{f!}^{a,b} \mathcal{M} := j_! j^* \mathcal{M}_f^{a,b}$ and $\Pi_{f*}^{a,b} \mathcal{M} := j_* j^* \mathcal{M}_f^{a,b}$. In the case $b = \infty$, they are denoted by $\Pi_{f!}^a \mathcal{M}$ and $\Pi_{f*}^a \mathcal{M}$. Beilinson defined the functors $\psi_f^{(a)}$ and $\Xi_f^{(a)}$ as follows:

$$\psi_f^{(a)} \mathcal{M} := \Pi_{f*}^a \mathcal{M} / \Pi_{f!}^a \mathcal{M}, \quad \Xi_f^{(a)} \mathcal{M} := \Pi_{f*}^a \mathcal{M} / \Pi_{f!}^{a+1} \mathcal{M}.$$

In the case $a = 0$, they are denoted by $\psi_f \mathcal{M}$ and $\Xi_f \mathcal{M}$, respectively. The multiplication of s naturally induces isomorphisms $\psi_f^{(a)} \mathcal{M} \simeq \psi_f^{(a+1)} \mathcal{M}$ and $\Xi_f^{(a)} \mathcal{M} \simeq \Xi_f^{(a+1)} \mathcal{M}$. They will be implicitly identified. We have the exact sequences of holonomic $\mathcal{D}_{X(*D)}$-modules:

$$0 \longrightarrow \Pi_{f!}^{a,a+1} \mathcal{M} \xrightarrow{c_1^{(a)}} \Xi_f^{(a)} \mathcal{M} \xrightarrow{c_2^{(a)}} \psi_f^{(a)} \mathcal{M} \longrightarrow 0$$

$$0 \longrightarrow \psi_f^{(a+1)} \mathcal{M} \xrightarrow{d_1^{(a)}} \Xi_f^{(a)} \mathcal{M} \xrightarrow{d_2^{(a)}} \Pi_{f*}^{a,a+1} \mathcal{M} \longrightarrow 0$$

The multiplication of s and the endomorphism $c_2^{(a)} \circ d_1^{(a)}$ induce an endomorphism $N^{(a+1)}$ of $\psi_f^{(a+1)} \mathcal{M}$.

Recall the important observation $\underset{\leftrightarrow}{\lim} \Pi_{f!}^{a,b} \mathcal{M} \simeq \underset{\leftrightarrow}{\lim} \Pi_{f*}^{a,b} \mathcal{M} =: \Pi_f \mathcal{M}$ due to Beilinson. See [3] for $\underset{\leftrightarrow}{\lim}$. In particular, it implies that $N^{(a+1)}$ is nilpotent. We also obtain the following morphism of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Pi_{f!}^a \mathcal{M} & \longrightarrow & \Pi_f \mathcal{M} & \longrightarrow & \Pi_{f!}^{-\infty,a} \mathcal{M} & \longrightarrow & 0 \\
 & & \downarrow & & =\downarrow & & \downarrow & & \\
0 & \longrightarrow & \Pi_{f*}^b \mathcal{M} & \longrightarrow & \Pi_f \mathcal{M} & \longrightarrow & \Pi_{f*}^{-\infty,b} \mathcal{M} & \longrightarrow & 0
\end{array}$$

Hence, we have a natural isomorphism $\mathrm{Cok}\Big(\Pi_{f!}^a \mathcal{M} \longrightarrow \Pi_{f*}^b \mathcal{M}\Big) \simeq \mathrm{Ker}\Big(\Pi_{f!}^{-\infty,a} \mathcal{M} \longrightarrow \Pi_{f*}^{-\infty,b}\Big)$. In particular, we have the following identifications:

$$\psi_f^{(a)} \mathcal{M} \simeq \mathrm{Ker}\big(\Pi_{f!}^{-\infty,a} \mathcal{M} \longrightarrow \Pi_{f*}^{-\infty,a} \mathcal{M}\big), \quad \Xi_f^{(a)} \mathcal{M} \simeq \mathrm{Ker}\big(\Pi_{f!}^{-\infty,a+1} \mathcal{M} \longrightarrow \Pi_{f*}^{-\infty,a} \mathcal{M}\big). \tag{8}$$

Remark 2.4 *When we distinguish that we work on the category of $\mathcal{D}_{X(*D)}$-modules, we will use the symbols $\psi_f^{(a)}(\mathcal{M}, *D)$, $\Xi_f^{(a)}(\mathcal{M}, *D)$, etc..* ∎

2.2.3 Vanishing cycle functor and gluing

Let f be as above. Let $\mathcal{M}_X$ be a holonomic $\mathcal{D}_{X(*D)}$-module such that $\mathcal{M}_X(*f) = \mathcal{M}$. We have the natural identifications $\Pi_{f\star}^{a,b} \mathcal{M}_X = \Pi_{f\star}^{a,b} \mathcal{M}$ for $\star = *, !$ and the naturally defined morphisms:

$$\Pi_{f!}^{a,a+1} \mathcal{M} \xrightarrow{c_{1,X}^{(a)}} \mathcal{M}_X \xrightarrow{d_{2,X}^{(a)}} \Pi_{f*}^{a,a+1} \mathcal{M}$$

Beilinson defined the vanishing cycle functor $\phi_f^{(a)} \mathcal{M}_X$ as the H^1-cohomology of the following sequence of holonomic $\mathcal{D}_{X(*D)}$-modules:

$$\Pi_{f!}^{a,a+1} \mathcal{M} \xrightarrow{c_1^{(a)} \oplus c_{1,X}^{(a)}} \Xi_f^{(a)} \mathcal{M} \oplus \mathcal{M}_X \xrightarrow{d_2^{(a)} - d_{1,X}^{(a)}} \Pi_{f*}^{a,a+1} \mathcal{M}$$

The morphisms $d_1^{(a)}$ and $c_2^{(a)}$ induce can and var:

$$\psi_f^{(a+1)}\mathcal{M} \xrightarrow{\text{can}} \phi_f^{(a)}\mathcal{M} \xrightarrow{\text{var}} \psi_f^{(a)}\mathcal{M}$$

By construction, we have $\text{var}\circ\text{can} = c_2^{(a)} \circ d_1^{(a)}$.

Conversely, let $\mathcal{M}_Y$ be a holonomic $\mathcal{D}_{X(*D)}$-module whose support is contained in $Y = f^{-1}(0)$, with morphisms such as

$$\psi_f^{(1)}\mathcal{M} \xrightarrow{u} \mathcal{M}_Y \xrightarrow{v} \psi_f^{(0)}\mathcal{M}, \qquad v\circ u = c_2^{(0)} \circ d_1^{(0)}.$$

Then, we obtain a holonomic $\mathcal{D}_{X(*D)}$-module $\text{Glue}(\mathcal{M}_Y, u, v)$ as the cohomology of the complex:

$$\psi_f^{(1)}\mathcal{M} \xrightarrow{d_1^{(0)}\oplus u} \Xi_f(\mathcal{M}) \oplus \mathcal{M}_Y \xrightarrow{c_2^{(0)}-v} \psi_f^{(0)}\mathcal{M}$$

Beilinson made an excellent observation that the above two operations are mutually inverse. See [3] for more details.

2.2.4 Comparison with ordinary definitions

Let $\widetilde{\psi}_{f,-1}$ and $\widetilde{\phi}_f$ be ordinary nearby cycle functor and vanishing cycle functor defined in terms of V-filtrations [17], i.e., $\widetilde{\psi}_{f,-1}(\mathcal{M}) = \text{Gr}^V_{-1}\mathcal{M}$ and $\widetilde{\phi}_f(\mathcal{M}_X) := \text{Gr}^V_0\mathcal{M}_X$. For simplicity, $\widetilde{\psi}_{f,-1}$ is denoted by $\widetilde{\psi}_f$ in the following.

Lemma 2.5 *We have natural isomorphisms $\psi_f \simeq \widetilde{\psi}_f$, and $\phi_f \simeq \widetilde{\phi}_f$.*

Proof Recall that $\widetilde{\phi}_f(\mathcal{M}_X)$ and $\widetilde{\phi}_f(\mathcal{M}_X)$ are naturally equipped with the nilpotent endomorphisms N, which is the nilpotent part of the multiplication of $-\partial_t t$. We have natural identifications:

$$\widetilde{\phi}_f\big(\Pi^{a,b}_{f!}\mathcal{M}\big) \simeq \widetilde{\phi}_f\big(\Pi^{a,b}_{f*}\mathcal{M}\big) \simeq \widetilde{\psi}_f\mathcal{M}\otimes A^{a,b}$$

The natural nilpotent endomorphisms are given by $N\otimes\text{id} - \text{id}\otimes(s\bullet)$, which is denoted by $N-s$. Here, $s\bullet$ denotes the multiplication of s on $A^{a,b}$. In the following, we argue on any compact subset of X.

Let us look at the natural morphism $G^{a,b} : \Pi^{a,b}_{f!}\mathcal{M} \longrightarrow \Pi^{a,b}_{f*}\mathcal{M}$. The supports of the kernel and the cokernel are contained in $f^{-1}(0)$. The morphism $\widetilde{\phi}_f(G^{a,b}) : \widetilde{\phi}_f\big(\Pi^{a,b}_{f!}\mathcal{M}\big) \longrightarrow \widetilde{\phi}_f\big(\Pi^{a,b}_{f*}\mathcal{M}\big)$ is naturally identified with $N-s : \widetilde{\psi}_f\mathcal{M}\otimes A^{a,b} \longrightarrow \widetilde{\psi}_f\mathcal{M}\otimes A^{a,b}$. Hence, if b is sufficiently larger than a, $\text{Cok}(G^{a,b})$ is isomorphic to $\widetilde{\psi}_f\mathcal{M}\otimes A^{a,a+1}$, independently of b. Therefore, we obtain $\psi_f^{(a)}\mathcal{M} \simeq \widetilde{\psi}_f\mathcal{M}\otimes A^{a,a+1}$. In particular, we naturally have $\psi_f^{(0)}\mathcal{M} = \widetilde{\psi}_f\mathcal{M}$.

It follows that $\text{Cok}\Big(\Pi^{a+1,M}_{f!}\mathcal{M} \longrightarrow \Pi^{a,M}_{f*}\mathcal{M}\Big)$ are independent of any sufficiently large M, which should be isomorphic to $\Xi_f^{(a)}\mathcal{M}$. We obtain $\widetilde{\phi}_f\big(\Xi_f^{(a)}\mathcal{M}\big) \simeq \text{Cok}\Big(N-s : \psi_f\mathcal{M}\otimes A^{a+1,M} \longrightarrow \psi_f\mathcal{M}\otimes A^{a,M}\Big)$ for any sufficiently large M. Because $\phi_f^{(0)}(\mathcal{M}_X)$ is naturally isomorphic to the cohomology of the complex

$$\widetilde{\phi}_f\big(\Pi^{0,1}_{f!}\mathcal{M}\big) \longrightarrow \widetilde{\phi}_f\big(\Xi_f^{(0)}\mathcal{M}\big) \oplus \widetilde{\phi}_f\big(\mathcal{M}_X\big) \longrightarrow \widetilde{\phi}_f\big(\Pi^{0,1}_{f*}\mathcal{M}\big),$$

it is easy to obtain $\phi_f^{(0)}(\mathcal{M}) \simeq \widetilde{\phi}_f(\mathcal{M})$ by a direct calculation. ∎

As was observed in the proof, on any compact subset of X, we have the following identifications for any sufficiently large M:

$$\psi_f^{(a)}\mathcal{M} = \text{Cok}\Big(\Pi^{a,a+M}_{f!}\mathcal{M} \longrightarrow \Pi^{a,a+M}_{f*}\mathcal{M}\Big), \quad \Xi_f^{(a)}\mathcal{M} = \text{Cok}\Big(\Pi^{a+1,a+M}_{f*}\mathcal{M} \longrightarrow \Pi^{a,a+M}_{f*}\mathcal{M}\Big) \tag{9}$$

Similarly, on any compact subset of X, we have the following identifications for any sufficiently large M:

$$\psi_f^{(a)}\mathcal{M} = \text{Ker}\Big(\Pi^{a-M,a}_{f!}\mathcal{M} \longrightarrow \Pi^{a-M,a}_{f*}\mathcal{M}\Big), \quad \Xi_f^{(a)}\mathcal{M} = \text{Ker}\Big(\Pi^{a-M,a+1}_{f!}\mathcal{M} \longrightarrow \Pi^{a-M,a}_{f*}\mathcal{M}\Big) \tag{10}$$

2.2.5 Compatibility with dual

In [3], the pairing $A \times A \longrightarrow k = A^{-1}/A^0$ is given by $\big\langle f(s),\, g(s)\big\rangle = \mathrm{Res}_{s=0}\big(f(s)\,g(-s)\,ds\big)$. It induces pairings $A^{a,b} \otimes A^{-b,-a} \longrightarrow A^{-1}/A^0$. Then, we obtain flat pairings $\mathfrak{I} \otimes \mathfrak{I} \longrightarrow \mathfrak{I}^{-1,0}$ and $\mathfrak{I}^{a,b} \otimes \mathfrak{I}^{-b,-a} \longrightarrow \mathfrak{I}^{-1,0}$. We can identify $\mathfrak{I}^{a,b}$ with the dual of $\mathfrak{I}^{-b,-a}$ by the pairing.

Let $\mathbb{D}$ denote the dual functor on the category of holonomic $\mathcal{D}_{X(*D)}$-modules. By using the $\mathcal{D}_{X(*D)}$-version of Lemma 2.1, we obtain identifications:

$$\mathbb{D}\Big(\Pi^{a,b}_{f*}\mathcal{M}\Big) \simeq \Pi^{-b,-a}_{f!}\Big(\mathbb{D}(\mathcal{M})\Big), \qquad \mathbb{D}\Big(\Pi^{a,b}_{f!}\mathcal{M}\Big) \simeq \Pi^{-b,-a}_{f*}\Big(\mathbb{D}(\mathcal{M})\Big)$$

By (9) and (10), we obtain the following identifications:

$$\mathbb{D}\psi_f(\mathcal{M}) \simeq \psi_f(\mathbb{D}\mathcal{M}) \qquad \mathbb{D}\Xi_f(\mathcal{M}) \simeq \Xi_f(\mathbb{D}\mathcal{M}) \qquad \mathbb{D}\phi_f(\mathcal{M}) \simeq \phi_f(\mathbb{D}\mathcal{M})$$

2.2.6 Compatibility with push-forward

Let $F : X \longrightarrow Y$ be a proper morphism. Assume that $D = F^{-1}(D_Y)$, for simplicity. Let g be a holomorphic function on Y. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_{X(*D)}$-module. We set $\widetilde{g} := F^*g$. Let $j_Y : Y - g^{-1}(0) \longrightarrow Y$ and $j_X : X - \widetilde{g}^{-1}(0) \longrightarrow X$. We have natural isomorphisms $F^i_{\dagger}\big(\mathcal{M} \otimes \mathfrak{I}^{a,b}_{\widetilde{g}}\big) \simeq F^i_{\dagger}(\mathcal{M}) \otimes \mathfrak{I}^{a,b}_g$ of $\mathcal{D}_{Y(*D_Y)}$-modules. By a general theory, we have $(j_{Y\star}j_Y^*)F^i_{\dagger} = F^i_{\dagger} \circ (j_{X\star}j_X^*)$ for $\star = *, !$. Hence, it is easy to obtain the following identification:

$$F^i_{\dagger}\psi_{\widetilde{g}}\mathcal{M} = \psi_g F^i_{\dagger}\mathcal{M} \quad F^i_{\dagger}\Xi_g\mathcal{M} = \Xi_g F_{\dagger}\mathcal{M} \quad F^i_{\dagger}\phi_g\mathcal{M} = \phi_g F^i_{\dagger}\mathcal{M}$$

2.2.7 Choice of a function

Let f and h be meromorphic functions on (X, D). Assume that h is nowhere vanishing. We have natural isomorphisms of $\mathcal{O}_X$-modules $\mathfrak{I}^{a,b}_f \simeq \mathfrak{I}^{a,b}_{hf} \simeq A^{a,b} \otimes \mathcal{O}_{X(*D)}(*f)$. For their flat connections ∇_f and ∇_{hf} and for $\alpha \in A^{a,b}$, we have the formulas:

$$\nabla_f\alpha = \alpha \cdot s\frac{df}{f} \qquad \nabla_{hf}\alpha = \alpha \cdot s\left(\frac{df}{f} + \frac{dh}{h}\right)$$

We have the flat isomorphism $\Phi : \mathfrak{I}^{a,b}_f \simeq \mathfrak{I}^{a,b}_{hf}$ given by $\Phi(\alpha) = \exp\big(-s\log h\big)\,\alpha$. It induces isomorphisms:

$$\Xi^{(a)}_f \simeq \Xi^{(a)}_{hf}, \quad \psi^{(a)}_f \simeq \psi^{(a)}_{hf}, \quad \phi^{(a)}_f \simeq \phi^{(a)}_{hf}. \tag{11}$$

They depend on a choice of the branch of $\log h$.

2.2.8 $\mathbb{Q}$-structure of $\mathfrak{I}$

In the analytic case, the $\mathbb{Q}$-structure of $A^{a,b}$ is given as follows:

$$\mathbb{C} \cdot s^j \supset \mathbb{Q} \cdot (2\pi\sqrt{-1})^j s^j$$

It gives a $\mathbb{Q}$-structure of the fiber of $\mathfrak{I}^{a,b}$ over $1 \in \mathbb{C}^*$. We would like to extend it to a flat $\mathbb{Q}$-structure of the flat bundle $\mathfrak{I}_{|\mathbb{C}^*}$. Let $u := 2\pi\sqrt{-1}\,s$. The connection of $\mathfrak{I}^{a,b}$ is expressed as

$$\nabla(u^a, \ldots, u^{b-1}) = (u^a, \ldots, u^{b-1}) \cdot N\frac{1}{2\pi\sqrt{-1}}\frac{dt}{t}$$

Here, N denotes the constant matrix such that $N_{i,i+1} = 1$ and $N_{i,j} = 0$ otherwise. Since the monodromy is expressed by $\exp(-N)$, the $\mathbb{Q}$-structure is well defined. More generally, for any subfield $K \subset \mathbb{C}$, we obtain a K-structure of $\mathfrak{I}^{a,b}$ in this way.

Note that the pairing $\langle\cdot,\cdot\rangle$ is not defined over $\mathbb{Q}$. We have the following formula:

$$\big\langle f(u),\, g(u)\big\rangle = \operatorname*{Res}_{u=0}\Big(f(u)\,g(-u)\,du\Big)\frac{1}{2\pi\sqrt{-1}}$$

Namely, the pairing $\langle\cdot,\cdot\rangle$ is valued in the Tate twist $\mathbb{Q}(-1) = (2\pi\sqrt{-1})^{-1}\mathbb{Q}$.

2.2.9 Comparison with the functors for perverse sheaves

Let $\mathrm{Loc}(\mathfrak{I})_{\mathbb{Q}}$ denote the $\mathbb{Q}$-local system associated to $\mathfrak{I}$. The fiber over 1 is $\mathbb{Q}(\!(u)\!)$, and the monodromy along the loop with the clockwise direction is given by the multiplication of $\exp(u)$. Recall another expression of this local system as in [3].

Let $A_{\mathcal{P}} := \mathbb{Q}(\!(v)\!)$. We set $t := v + 1$. The pairing $A_{\mathcal{P}} \times A_{\mathcal{P}} \longrightarrow \mathbb{Q}(-1)$ is given as follows:

$$\big\langle f(t), g(t)\big\rangle = \operatorname*{Res}_{t=1}\Big(f(t)\,g(t^{-1})\frac{dt}{t}\Big)\frac{1}{2\pi\sqrt{-1}}$$

We have a $\mathbb{Q}$-local system $\mathfrak{I}_{\mathcal{P}}$ on $\mathbb{C}^*$ such that the fiber over 1 is $A_{\mathcal{P}}$, and the monodromy along the loop with the clockwise direction is given by the multiplication of $t = 1 + v$. Let us compare $\mathfrak{I}_{\mathcal{P}}$ and $\mathrm{Loc}(\mathfrak{I})_{\mathbb{Q}}$. We take an algebra homomorphism $\Phi : \mathbb{Q}(\!(u)\!) \longrightarrow \mathbb{Q}(\!(v)\!)$ determined by $\Phi\big(\exp(u)\big) = 1 + v$. We identify the fibers of $\mathrm{Loc}(\mathfrak{I})_{\mathbb{Q}}$ and $\mathfrak{I}_{\mathcal{P}}$ by Φ. Because it is compatible with the monodromies, it induces the identification $\mathrm{Loc}(\mathfrak{I})_{\mathbb{Q}} \simeq \mathfrak{I}_{\mathcal{P}}$. Note that $\Phi\big(f(-u)\big) = \Phi(f)(t^{-1})$ and $\Phi(du) = dt/t$. Hence the pairing is preserved.

Remark 2.6 *Recall that the functors ψ, Ξ and ϕ for perverse sheaves are given in terms of $\mathfrak{I}_{\mathcal{P}}$, according to* [3]. *Hence, the above comparison gives the compatibility of the de Rham functor* DR *with ϕ, ψ and Ξ in the regular singular case.* ∎

2.3 A resolution

This subsection is a preparation for the proof of Theorem 8.1.

2.3.1 Commutativity of push-forward in the non-characteristic case

Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module on a complex algebraic manifold X. We have natural isomorphisms

$$\mathcal{M}(*D) \simeq \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*D), \qquad \mathbb{D}_X\Big((\mathbb{D}_X\mathcal{M})(*D)\Big) \simeq \mathcal{M} \otimes^L \mathcal{O}_X(!D).$$

If a hypersurface $D \subset X$ is non-characteristic to $\mathcal{M}$, we obtain $\mathcal{M}(!D) \simeq \mathcal{M} \otimes \mathcal{O}_X(!D)$.

Lemma 2.7 *Let D_i $(i = 1, 2)$ be hypersurfaces of X. If D_i $(i = 1, 2)$ and $D_1 \cap D_2$ are non-characteristic to $\mathcal{M}$, we have a natural isomorphism:*

$$\big(\mathcal{M}(*D_1)\big)(!D_2) \simeq \big(\mathcal{M}(!D_2)\big)(*D_1) \tag{12}$$

Proof Note that $\mathrm{Ch}\big(\mathcal{M}(*D_1)\big) = \mathrm{Ch}\big(\mathcal{M}\big) \cup \mathrm{Ch}\big(i_{1*}i_1^*\mathcal{M}\big)$, where $i_1 : D_1 \longrightarrow X$. We have a stratification $\mathrm{Supp}\,\mathcal{M} = \coprod Z_i$ such that $\mathrm{Ch}(\mathcal{M}) = \coprod T^*_{Z_i}X$. We obtain a stratification $\mathrm{Supp}\,\mathcal{M} = \coprod (Z_i \setminus D_1) \sqcup \coprod (Z_i \cap D_1)$, for which we have the following:

$$\mathrm{Ch}\big(\mathcal{M}(*D_1)\big) = \coprod T^*_{Z_i\setminus D_1}X \sqcup \coprod T^*_{Z_i\cap D_1}X$$

Hence, D_2 is non-characteristic to $\mathcal{M} \otimes \mathcal{O}(*D_1)$. Similarly, we can show that D_1 is non-characteristic to $\mathcal{M} \otimes \mathcal{O}(!D_2)$. Then, the both sides of (12) are naturally isomorphic to $\mathcal{M} \otimes \mathcal{O}(!D_2) \otimes \mathcal{O}(*D_1)$. ∎

2.3.2 Transversality

Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module on a complex algebraic manifold X. There exists a stratification $\mathrm{Supp}(\mathcal{M}) = \coprod_{i\in\Lambda} Z_i$ such that (i) each Z_i is a smooth locally closed analytic subset of X, (ii) $\mathrm{Ch}(\mathcal{M}) = \coprod_{i\in\Lambda} T^*_{Z_i}X$.

Lemma 2.8 *An analytic subset $W \subset X$ is non-characteristic to $\mathcal{M}$, if and only if W and Z_i are transversal for any $i \in \Lambda$.*

Proof For $P \in W \cap Z_i$, we have subspaces $(T^*_{Z_i}X)_{|P}$ and $(T^*_W X)_{|P}$ of $(T^*X)_{|P}$. Then, W and Z_i are transversal at P if and only if $(T^*_W X)_{|P} \cap (T^*_{Z_0}X)_{|P} = \{0\}$. Then, the claim of the lemma is clear. ∎

2.3.3 Non-characteristic tuple of hyperplane subbundles

Let $\mathcal{E}$ be a locally free sheaf on a complex algebraic manifold Y. We put $X := \mathbb{P}(\mathcal{E})$ with the projection $G : X \longrightarrow Y$. The zero set of a section of $\mathcal{O}_{\mathbb{P}(\mathcal{E})/Y}(1)$ is called a hyperplane subbundle of X.

Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Let $\boldsymbol{H} := (H_1, \dots, H_N)$ be a tuple of hyperplane subbundles of X. We say that $\boldsymbol{H}$ is non-characteristic to $\mathcal{M}$, if $H_I := \bigcap_{i\in I} H_i$ are non-characteristic to $\mathcal{M}$ for any $I \subset \{1, \dots, N\}$. We can show the following lemma by a standard argument of genericity.

Lemma 2.9 *Let $\boldsymbol{H} = (H_1, \dots, H_N)$ be non-characteristic to $\mathcal{M}$. We can take a hyperplane subbundle H_{N+1} such that $(H_1, \dots, H_N, H_{N+1})$ is also non-characteristic to $\mathcal{M}$.* ∎

Recall the following general lemma.

Lemma 2.10 *Let (H_1, H_2) be a tuple of hyperplane bundles of X, which is non-characteristic to $\mathcal{M}$. Then, $G^i_\dagger\big(\mathcal{M}(*H_1!H_2)\big) = 0$ for any $i \neq 0$.*

Proof Let $\mathcal{M}_i$ $(i = 1, 2)$ be holonomic $\mathcal{D}_X$-modules, to which H_i is non-characteristic. It is easy to show that $G^i_\dagger\mathcal{M}_1(*H_1) = 0$ for any $i > 0$. By using the duality, we obtain that $G^i_\dagger\big(\mathcal{M}_2(!H_2)\big) = 0$ for any $i < 0$. Then, the claim follows from Lemma 2.7. ∎

2.3.4 A resolution

Let X, Y and $\mathcal{M}$ be as in Subsection 2.3.3. Let $\boldsymbol{H} = (H_i)$ be a tuple of hyperplane subbundles of X, non-characteristic to $\mathcal{M}$. Let $\underline{i} := \{1, \dots, i\}$, and let $\iota_{H_{\underline{i}}}$ denote the inclusion $H_{\underline{i}} \subset X$. We put $\mathcal{N}_0 := \mathcal{M}(*H_1)$. We also put $\mathcal{C}_i := \iota_{H_{\underline{i}}\dagger}\iota^*_{H_{\underline{i}}}\mathcal{M}$, and $\mathcal{N}_i := \mathcal{C}_i(*H_{i+1})$. We have the natural exact sequences:

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{N}_0 \longrightarrow \mathcal{C}_1 \longrightarrow 0, \qquad 0 \longrightarrow \mathcal{C}_i \longrightarrow \mathcal{N}_i \longrightarrow \mathcal{C}_{i+1} \longrightarrow 0$$

Hence, we obtain the following exact sequence:

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{N}_0 \longrightarrow \mathcal{N}_1 \longrightarrow \cdots \longrightarrow \mathcal{N}_n \longrightarrow \cdots$$

Let $\boldsymbol{H}' = (H'_j)$ be a tuple of hyperplane subbundles of X such that $\boldsymbol{H} \sqcup \boldsymbol{H}'$ is non-characteristic to $\mathcal{M}$. We set $\mathcal{Q}_{i,0} := \mathcal{N}_i(!H'_1)$. We also put $\mathcal{K}_{i,-j} := \iota_{H'_{\underline{j}}\dagger}\iota^*_{H'_{\underline{j}}}\mathcal{N}_i$ and $\mathcal{Q}_{i,-j} := \mathcal{K}_{i,-j}(!H_{j+1})$. We have the natural exact sequences:

$$0 \longrightarrow \mathcal{K}_{i,-1} \longrightarrow \mathcal{Q}_{i,0} \longrightarrow \mathcal{N}_i \longrightarrow 0, \qquad 0 \longrightarrow \mathcal{K}_{i,-j-1} \longrightarrow \mathcal{Q}_{i,-j} \longrightarrow \mathcal{K}_{i,-j} \longrightarrow 0$$

Hence, we obtain the following exact sequences:

$$0 \longleftarrow \mathcal{N}_i \longleftarrow \mathcal{Q}_{i,0} \longleftarrow \mathcal{Q}_{i,-1} \longleftarrow \mathcal{Q}_{i,-2} \longleftarrow \cdots$$

By construction, we have the naturally defined morphisms $\mathcal{Q}_{i,-j} \longrightarrow \mathcal{Q}_{i+1,-j}$ and the commutative diagrams:

$$\begin{array}{ccc}
\mathcal{Q}_{i,-j} & \longrightarrow & \mathcal{Q}_{i+1,-j} \\
\downarrow & & \downarrow \\
\mathcal{Q}_{i,-j+1} & \longrightarrow & \mathcal{Q}_{i+1,-j+1}
\end{array}$$

Let $\mathrm{Tot}\big(\mathcal{Q}_{\bullet,\bullet}\big)$ denote the total complex of the double complex $\mathcal{Q}_{\bullet,\bullet}$. We have natural quasi-isomorphisms $\mathrm{Tot}\big(\mathcal{Q}_{\bullet,\bullet}\big) \xrightarrow{\simeq} \mathcal{N}^{\cdot} \xleftarrow{\simeq} \mathcal{M}$.

3 Good holonomic $\mathcal{D}$-modules and their de Rham complexes

3.1 Good holonomic $\mathcal{D}$-modules

3.1.1 $\mathcal{I}$-good meromorphic flat bundle

We put $X := \Delta^n$, $D_i := \{z_i = 0\}$ and $D := \bigcup_{i=1}^{\ell} D_i$. For $I \subset \underline{\ell}$, we set $D(I) := \bigcup_{i\in I} D_i$ and $D_I := \bigcap_{i\in I} D_i$. We put $\partial D_I := D_I \cap D(I^c)$, where $I^c := \underline{\ell} - I$. Let $M(X, D)$ be the set of meromorphic functions on X whose poles are contained in D. Let $H(X)$ be the set of holomorphic functions on X. Let $\mathcal{I}\subset M(X,D)/H(X)$ be a good set of irregular values. For $I\subset \underline{\ell}$, let $\mathcal{I}'(I)$ be the set of the elements $\mathfrak{a}\in\mathcal{I}$ which are regular along z_i ($i\in I$), and we put $\mathcal{I}(I):=\bigl\{\mathfrak{a}_{|D_I}\,\big|\,\mathfrak{a}\in\mathcal{I}'(I)\bigr\}$.

Let $X^{(m)}:=\Delta^n=\bigl\{(z_1^{1/m},\ldots,z_\ell^{1/m},z_{\ell+1},\ldots,z_n)\bigr\}$, $D_i^{(m)}:=\bigl\{z_i^{(m)}=0\bigr\}$ and $D^{(m)}=\bigcup_{i=1}^{\ell}D_i^{(m)}$, i.e., $X^{(m)}\longrightarrow X$ is a ramified covering along D. We have the induced ramified covering $D_I^{(m)}:=\bigcap_{i\in I}D_i^{(m)}\longrightarrow D_I$. Let $\mathcal{I}\subset M(X^{(m)},D^{(m)})/H(X^{(m)})$ be a good set of irregular values. Let $I\subset\underline{\ell}$. A meromorphic flat bundle $\mathcal{E}$ on $(D_I,\partial D_I)$ is called $\mathcal{I}$-good, if it is the descent of an unramifiedly good meromorphic flat bundle $\mathcal{E}^{(m)}$ on $(D_I^{(m)},\partial D_I^{(m)})$ whose set of irregular values is contained in $\mathcal{I}(I)$.

In this subsection, we use the following notation for simplicity of the description.

Notation 3.1 *The vanishing cycle functor ϕ_{z_i} is denoted by ϕ_i. We use the symbols ψ_i, Ξ_i and $\Pi^{a,b}_{i\star}$ in similar meanings. For a holonomic $\mathcal{D}_X$-module $\mathcal{M}$, we set $\mathcal{M}(*i):=\mathcal{M}(*D_i)$ and $\mathcal{M}(!i):=\mathcal{M}(!D_i)$. If we are given a subset $I\subset\underline{\ell}$, we put $\mathcal{M}(!I):=\mathcal{M}\bigl(!D(I)\bigr)$ and $\mathcal{M}(*I):=\mathcal{M}\bigl(*D(I)\bigr)$.* ∎

Lemma 3.2 *Let $\mathcal{E}$ be an $\mathcal{I}$-good meromorphic flat bundle on (X,D). If $i\neq j$, the natural morphism $\phi_i(\mathcal{E})\longrightarrow\phi_i(\mathcal{E})(*j)$ is an isomorphism.*

Proof It follows from a direct computation of the Kashiwara-Malgrange filtration along z_i. We give only an indication. We use an order on $\mathbb{C}$ given by the lexicographic order on $\mathbb{R}\times\mathbb{R}$ and the identification $\mathbb{C}\simeq\mathbb{R}^2$ via $\alpha\longleftrightarrow(\operatorname{Re}\alpha,\operatorname{Im}\alpha)$. For $\boldsymbol{\alpha}=(\alpha_k\,|\,k\in\underline{\ell})$, we can take a good lattice $E_{\boldsymbol{\alpha}}$ of $\mathcal{E}$ such that any eigen values β of $\operatorname{Res}_i(\nabla)$ satisfy $-\alpha_i<\beta\leq-\alpha_i-1$. Let ${}^{i}V_0\mathcal{D}$ denote the sheaf of subalgebras of $\mathcal{D}$ generated by $\mathcal{O}_X$, ∂_k ($k\neq i$) and $z_i\partial_i$. Put $D(i^c):=\bigcup_{j\neq i,j\leq\ell}D_j$. For $\alpha\in\mathbb{C}$, take an $\boldsymbol{\alpha}$ whose i-th component is α, and let ${}^{i}V_\alpha(\mathcal{E})$ be the ${}^{i}V_0\mathcal{D}$-submodule of $\mathcal{E}$ generated by ${}^{i}_{\alpha}\mathcal{E}:={}_{\boldsymbol{\alpha}}E\bigl(*D(i^c)\bigr)$. We can check that ${}^{i}V_{-\alpha-1}(\mathcal{E})$ is generated by ${}_{\boldsymbol{\alpha}}E$, where the i-th component of $\boldsymbol{\alpha}$ is α, and the other components of $\boldsymbol{\alpha}$ are larger than 1. Hence, we can deduce that ${}^{i}V_\alpha(\mathcal{E})$ are ${}^{i}V_0\mathcal{D}_X$-coherent. We can also check that the induced action of $-\partial_iz_i-\alpha$ on ${}^{i}V_\alpha/{}^{i}V_{<\alpha}$ is nilpotent. Hence, ${}^{i}V(\mathcal{E})$ is the Kashiwara-Malgrange filtration of $\mathcal{E}$ along z_i. Then, the claim of the lemma is clear. ∎

Lemma 3.3 *If $i\neq j$, the natural morphism $\mathcal{E}(!i)\longrightarrow\mathcal{E}(!i)(*j)$ is an isomorphism.*

Proof Let N denote the nilpotent part of the action of $-\partial_iz_i$ on $\phi_i(\mathcal{E})$. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker}N & \longrightarrow & \mathcal{E}(!i) & \longrightarrow & \mathcal{E} & \longrightarrow & \operatorname{Cok}N & \longrightarrow & 0\\
 & & {\scriptstyle a}\big\downarrow & & {\scriptstyle b}\big\downarrow & & {\scriptstyle =}\big\downarrow & & {\scriptstyle c}\big\downarrow & & \\
0 & \longrightarrow & \operatorname{Ker}N(*j) & \longrightarrow & \mathcal{E}(!i)(*j) & \longrightarrow & \mathcal{E} & \longrightarrow & \operatorname{Cok}N(*j) & \longrightarrow & 0
\end{array}$$

By Lemma 3.8, we obtain that a and c are isomorphisms. Hence, b is also an isomorphism. ∎

3.1.2 $\mathcal{I}$-good holonomic $\mathcal{D}$-modules

We continue to use the notation in Subsection 3.1.1.

Definition 3.4 *A holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is called $\mathcal{I}$-good on (X,D), if the following holds:*

- *$\mathcal{M}(*D)$ is a good meromorphic flat bundle whose good set of irregular values is $\mathcal{I}$.*
- *For an ordered tuple $I=(i_1,\ldots,i_m)$ where $1\leq i_j\leq\ell$, we set $\phi_I=\phi_{i_1}\circ\cdots\circ\phi_{i_m}$. Then, $\phi_I(\mathcal{M})\bigl(*I^c\bigr)$ is the push-forward of a good meromorphic flat bundle on $(D_I,\partial D_I)$ whose set of irregular values is $\mathcal{I}(I)$.* ∎

The full subcategory of $\mathcal{I}$-good holonomic $\mathcal{D}$-modules is abelian, and it is closed under extensions. If V is a good meromorphic flat bundle, it is a good holonomic $\mathcal{D}_X$-module in the above sense. When we do not have to distinguish $\mathcal{I}$, we will omit to denote it. We will implicitly use the following obvious lemma.

Lemma 3.5 *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Assume (i) $\mathcal{M}(*D)$ is an $\mathcal{I}$-good meromorphic flat bundle, (ii) $\phi_i(\mathcal{M})$ are $\mathcal{I}$-good for any $i=1,\ldots,\ell$. Then, $\mathcal{M}$ is $\mathcal{I}$-good.* ∎

Lemma 3.6 *Let $\mathcal{M}$ be a good holonomic $\mathcal{D}$-module on (X, D). Then, $\mathbb{D}_X\mathcal{M}$ is also good.*

Proof We use an induction on the dimension of the support of $\mathcal{M}$. It is easy to check that $\mathbb{D}_X\mathcal{M}(*D)$ is a good meromorphic flat bundle. By the hypothesis of the induction, $\phi_i(\mathbb{D}_X\mathcal{M}) \simeq \mathbb{D}_X\phi_i(\mathcal{M})$ are also good. Hence, we obtain that $\mathcal{M}$ is good. ∎

For a good holonomic $\mathcal{D}$-module $\mathcal{M}$, let $\rho(\mathcal{M}) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$ denote the pair of $\dim \operatorname{Supp} \mathcal{M}$ and the number of the irreducible components of $\operatorname{Supp} \mathcal{M}$ with the maximal dimension. We use the lexicographic order on $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$. For a good holonomic $\mathcal{D}$-module $\mathcal{M}$, there exists $J \subset \underline{\ell}$ with $\dim \operatorname{Supp} \mathcal{M} = n - |J|$ such that $\mathcal{M}\bigl(*J^c\bigr) \neq 0$. The kernel $\mathcal{N}_1$ and the cokernel $\mathcal{N}_2$ of the natural morphism $\mathcal{M} \longrightarrow \mathcal{M}\bigl(*J^c\bigr)$ satisfy $\rho\bigl(\mathcal{N}_i\bigr) < \rho(\mathcal{M})$ $(i = 1, 2)$.

Lemma 3.7 *Let $\mathcal{M}$ be good on (X, D). Then, $\psi_i(\mathcal{M})$ are also good for any $i = 1, \ldots, \ell$.*

Proof We use an induction on $\rho(\mathcal{M})$. Let J and $\mathcal{N}_j$ $(j = 1, 2)$ be as above. By the assumption of the induction, $\psi_i(\mathcal{N}_j)$ $(j = 1, 2)$ are good. It is easy to show that $\psi_i\bigl(\mathcal{M}(*J^c)\bigr)$ is good by using the lattice as in the proof of Lemma 3.2. Hence, we obtain that $\psi_i(\mathcal{M})$ is also good. ∎

3.1.3 Commutativity of the functor along the coordinate functions

Let $\mathcal{M}$ be good on (X, D).

Lemma 3.8 *For $i \neq j$, we have natural isomorphisms $\phi_i\bigl(\mathcal{M}(*j)\bigr) \simeq \phi_i(\mathcal{M})(*j)$ and $\phi_i\bigl(\mathcal{M}(!j)\bigr) \simeq \phi_i(\mathcal{M})(!j)$.*

Proof The second isomorphism is obtained as the dual of the first one. Let us consider the first isomorphism. We have the following naturally defined morphisms:

$$\phi_i\bigl(\mathcal{M}(*j)\bigr) \xrightarrow{a} \phi_i\bigl(\mathcal{M}(*j)\bigr)(*j) \xleftarrow{b} \phi_i\bigl(\mathcal{M}\bigr)(*j)$$

Because the restriction of b to $X - D_j$ is an isomorphism, it is easy to show that b is an isomorphism. Let us show that a is an isomorphism by using an induction on $\rho(\mathcal{M})$. As in the proof of Lemma 3.7, the issue can be reduced to the case that $\mathcal{M}$ is a good meromorphic flat bundle, which is given in Lemma 3.2. ∎

Lemma 3.9 *$\mathcal{M}(*j)$ and $\mathcal{M}(!j)$ are also good.*

Proof Because $\phi_j\bigl(\mathcal{M}(*j)\bigr) \simeq \psi_j(\mathcal{M})$, we obtain that $\mathcal{M}(*j)$ is good from Lemmas 3.5, 3.7 and 3.8. By using Lemma 3.6, we obtain that $\mathcal{M}(!j)$ is also good. ∎

We have the following corollary of Lemma 3.9.

Corollary 3.10 *Let f be a meromorphic function on (X, D) whose zero and pole are contained in D. Take $D^{(1)} \subset D$ such that the pole of f is contained in D. The holonomic $\mathcal{D}_X$-module $\Pi^{a,b}_{f\star}(\mathcal{M}, *D^{(1)})$ is good on (X, D). Hence, $\psi_f(\mathcal{M}, *D^{(1)})$, $\Xi_f(\mathcal{M}, *D^{(1)})$ and $\phi_f(\mathcal{M}, *D^{(1)})$ are also good on (X, D).* ∎

We have the following naturally defined morphisms:

$$\mathcal{M}(*i)(!j) \xrightarrow{a} \mathcal{M}(*i)(!j)(*i) \xleftarrow{b} \mathcal{M}(!j)(*i)$$

It is easy to show that b is an isomorphism.

Lemma 3.11 *a is also an isomorphism, by which we can identify $\mathcal{M}(*i)(!j)$ and $\mathcal{M}(!j)(*i)$.*

Proof By using an induction on $\rho(\mathcal{M})$, we can reduce the issue to the case that $\mathcal{M}$ is a good meromorphic flat bundle, which is given in Lemma 3.3. ∎

In the following, we will not distinguish $\mathcal{M}(*i)(!j)$ and $\mathcal{M}(!j)(*i)$ for $i \neq j$, which will be denoted by $\mathcal{M}(*i!j)$. For $I \sqcup J \subset \underline{\ell}$, we have the natural identification $\mathcal{M}(!I*J) \simeq \mathcal{M}(*J!I)$, which will be used implicitly.

Lemma 3.12 *We have the commutativity $\Xi_i \circ \Xi_j = \Xi_j \circ \Xi_i$, $\psi_i \circ \psi_j = \psi_j \circ \psi_i$ and $\phi_i \circ \phi_j = \phi_j \circ \phi_i$. Moreover, the functors Ξ_i, ψ_j and ϕ_k are mutually commutative, where i, j and k are mutually distinct. In the following, we will not care about the order of these functors for good holonomic $\mathcal{D}$-modules on (X, D).*

Proof We obtain the natural identification $\Pi^{a,b}_{i\star} \circ \Pi^{c,d}_{j\star'} = \Pi^{c,d}_{j\star'} \circ \Pi^{a,b}_{i\star}$ from Lemma 3.11. Then, the claim of the lemma is clear. ∎

3.1.4 Globalization

Let X be a complex manifold with a normal crossing hypersurface D.

Definition 3.13 *A holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is called good on (X, D), if the following holds:*

- *Let P be any point of D. Let $(U, z_1, \ldots, z_n)$ be a coordinate neighbourhood around P such that $D \cap U = \bigcup_{i=1}^{\ell} \{z_i = 0\}$. Then, $\mathcal{M}_{|U}$ is good in the sense of Definition 3.4.* ∎

We obtain the following from the results in Subsections 3.1.2–3.1.3.

Lemma 3.14 *Let $\mathcal{M}$ be good on (X, D).*

- *The dual $\mathbb{D}_X \mathcal{M}$ is also good on (X, D).*
- *Let $D^{(1)} \subset D$ be a union of some irreducible components. Then, $\mathcal{M}(*D^{(1)})$ and $\mathcal{M}(!D^{(1)})$ are also good on (X, D).*
- *Let $D^{(i)} \subset D$ $(i = 1, 2)$ be unions of some irreducible components such that $\dim D^{(1)} \cap D^{(2)} < \dim X - 1$. Then, we have a natural isomorphism $\mathcal{M}(*D^{(1)})(!D^{(2)}) \simeq \mathcal{M}(!D^{(2)})(*D^{(1)})$.*
- *Let f be a meromorphic function on (X, D) whose zero and pole are contained in D. Take $D^{(1)} \subset D$ such that the pole of f is contained in D. Then, $\psi_f(\mathcal{M}, *D^{(1)})$, $\Xi_f(\mathcal{M}, *D^{(1)})$ and $\phi_f(\mathcal{M}, *D^{(1)})$ are also good on (X, D).* ∎

3.2 De Rham complexes

3.2.1 De Rham complex with infinite decay

For a complex manifold X, let $\Omega^{p,q}_X$ denote the sheaf of C^∞-(p, q)-forms on X. For any analytic subset $Z \subset X$, we set $\Omega^{p,q}_{\widehat{Z}} := \Omega^{p,q}_X \otimes_{\mathcal{C}^\infty_X} \mathcal{C}^\infty_{\widehat{Z}}$. If we are given a normal crossing hypersurface $D \subset X$, we set $\Omega^{p,q}_{\widehat{Z}}(*D) := \Omega^{p,q}_{\widehat{Z}} \otimes_{\mathcal{O}_X} \mathcal{O}_{\widehat{Z}}(*D)$. We say that $D_1 \cup D_2 = D$ is a decomposition of D, if $D_i \subset X$ $(i = 1, 2)$ are hypersurfaces such that $\operatorname{codim}_X(D_1 \cap D_2) > 1$. In that situation, we say that D_2 is the complement of D_1 in D. When we are given a normal crossing hypersurface $D \subset X$ with a decomposition $D = D_1 \cup D_2$, let $\Omega^{p,q}_X(*D_2)^{<D_1}$ denote the kernel of $\Omega^{p,q}_X(*D_2) \longrightarrow \Omega^{p,q}_{\widehat{D}_1}(*D_2)$.

Let D_0 be a normal crossing hypersurface of X with a decomposition $D_0 = D_1 \cup D_2$. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. We define $\mathrm{DR}^{<D_1 \leq D_2}_X \mathcal{M}$ in the derived category $D^b(\mathbb{C}_X)$ as follows:

$$\mathrm{DR}^{<D_1 \leq D_2}_X \mathcal{M} := \Omega^{\dim X, \bullet\, <D_1}_X(*D_2) \otimes^L_{\mathcal{D}_X} \mathcal{M} \simeq \Omega^{\bullet,\bullet\, <D_1}_X(*D_2) \otimes_{\mathcal{O}_X} \mathcal{M}[\dim X]$$

It is easy to observe that the natural morphism $\mathrm{DR}^{<D_1 \leq D_2}_X \mathcal{M} \longrightarrow \mathrm{DR}^{<D_1 \leq D_2}_X \big(\mathcal{M}(*D_0)\big)$ is an isomorphism in $D^b(\mathbb{C}_X)$. We also have the following natural isomorphisms:

$$\begin{aligned}\mathrm{DR}^{<D_1}_X \big(\mathbb{D}_X \mathcal{M}(*D_0)\big) &\simeq \Omega^{\dim X, \bullet}_X(*D_2)^{<D_1} \otimes^L_{\mathcal{D}_X} \mathbb{D}_X \mathcal{M}(*D_0) \\ &\simeq R\mathcal{H}om_{\mathcal{D}_X(*D_0)}\big(\mathcal{M}, \Omega^{0,\bullet}_X(*D_2)^{<D_1}\big) \simeq R\mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{M}, \Omega^{0,\bullet}_X(*D_2)^{<D_1}\big) \qquad (13)\end{aligned}$$

3.2.2 The identification in the case of good holonomic $\mathcal{D}$-modules

Let X and D_i $(i=0,1,2)$ be as above. Let D be a normal crossing hypersurface such that $D_0\subset D$. Let $\mathcal{M}$ be a good holonomic $\mathcal{D}$-module on (X,D).

Proposition 3.15 *If $\mathcal{M}(*D_1)=\mathcal{M}$, the natural morphism $\mathrm{DR}_X^{<D_1\leq D_2}\mathbb{D}_X\mathcal{M}\longrightarrow \mathrm{DR}_X^{\leq D_2}\mathbb{D}_X\mathcal{M}$ is a quasi-isomorphism.*

Proof We have only to consider the case $X=\Delta^n$ and $D=\bigcup_{i=1}^{\ell}\{z_i=0\}$. We have $I_a\subset\underline{\ell}$ $(a=1,2)$ such that $D_a=\bigcup_{i\in I_a}\{z_i=0\}$. By using an induction on $\rho(\mathcal{M})$ (Subsection 3.1.2), we can reduce the issue to the case that $\mathcal{M}$ is the push-forward of a good meromorphic flat bundle on D_J for some $J\subset\underline{\ell}\setminus I_1$ as in the proof of Lemma 3.7. Moreover, we have only to consider the case $\mathcal{M}$ is a good meromorphic flat bundle V on (X,D).

Note that the induced morphism $\partial_i:\widetilde{\psi}_{z_i,-1}(\mathbb{D}_XV)\longrightarrow\widetilde{\phi}_{z_i}(\mathbb{D}_XV)$ is an isomorphism, which can be checked by using the lattices in the proof of Lemma 3.2. Hence, we have the following vanishing for any $I\subset I_1$:

$$R\mathcal{H}om_{\mathcal{D}_X}\bigl(V,\mathcal{O}_{\widehat{D}_I}(*D_2)\bigr)=R\mathcal{H}om_{\mathcal{D}_X}\bigl(\mathcal{O}_X,\mathbb{D}_XV\otimes\mathcal{O}_{\widehat{D}_I}(*D_2)\bigr)=0$$

Here, $D_I:=\bigcap_{i\in I}\{z_i=0\}$. (Note $D_{\{i\}}\neq D_i$ for $i=1,2$ in general.) Then, we obtain the vanishing $R\mathcal{H}om_{\mathcal{D}_X}\bigl(V,\mathcal{O}_{\widehat{D}_1}(*D_2)\bigr)=0$ by using the standard resolution of $\mathcal{O}_{\widehat{D}_1}$ in terms of $\mathcal{O}_{\widehat{D}_I}$ $(I\subset I_1)$. (See Subsection 2.1.4.) Because the cone of $\Omega_X^{0,\bullet<D_1}(*D_2)\longrightarrow\Omega_X^{0,\bullet}(*D_2)$ is quasi isomorphic to $\Omega^{0,\bullet}_{\widehat{D}_1}(*D_2)\simeq\mathcal{O}_{\widehat{D}_1}(*D_2)$, we obtain the claim of the lemma. ∎

Let $\mathcal{M}$ be a good holonomic $\mathcal{D}$-module on (X,D). Let $D_1\subset D$. Applying Proposition 3.15 to $\mathbb{D}_X\mathcal{M}(*D_1)$, we obtain an isomorphism $\mathrm{DR}^{<D_1}\mathcal{M}\simeq\mathrm{DR}\,\mathcal{M}(!D_1)$. Note $\mathcal{M}(!D_1)\simeq\mathbb{D}_X\bigl(\mathbb{D}_X\mathcal{M}(*D_1)\bigr)$. In particular, we obtain the following corollary.

Corollary 3.16 *Let $D=D_1\cup D_2$ be a decomposition. Let V be a good meromorphic flat bundle on (X,D). We have a natural isomorphism $\mathrm{DR}_X^{<D_1}(V)\simeq\mathrm{DR}_X\bigl(V(!D)(*D_2)\bigr)\simeq\mathrm{DR}_X\bigl(V(!D_1)\bigr)$.* ∎

Lemma 3.17 *Let D_1 and D_1' be hypersurfaces of X such that (i) $D_1,D_1'\subset D$, (ii) $\dim(D_1\cap D_1')<\dim X-1$. We have the following commutative diagram:*

$$\begin{array}{ccc}
\mathrm{DR}_X\mathcal{M}\bigl(!D_1!D_1'\bigr) & \longrightarrow & \mathrm{DR}_X\mathcal{M}(!D_1)\\
\uparrow & & \uparrow\\
\mathrm{DR}_X^{<D_1}\mathcal{M}(!D_1') & \longrightarrow & \mathrm{DR}_X^{<D_1}\mathcal{M}\\
\uparrow & & \uparrow{=}\\
\mathrm{DR}_X^{<D_1\cup D_1'}\mathcal{M} & \longrightarrow & \mathrm{DR}_X^{<D_1}\mathcal{M}
\end{array}$$

Proof It follows from the commutativity of the following:

$$\begin{array}{ccc}
R\mathcal{H}om_{\mathcal{D}_X}\bigl(\mathbb{D}_X\mathcal{M}(*D_1''),\Omega_X^{0,\bullet}\bigr) & \longrightarrow & R\mathcal{H}om_{\mathcal{D}_X}\bigl(\mathbb{D}_X\mathcal{M}(*D_1),\Omega_X^{0,\bullet}\bigr)\\
\uparrow & & \uparrow\\
R\mathcal{H}om_{\mathcal{D}_X}\bigl(\mathbb{D}_X\mathcal{M}(*D_1''),\Omega_X^{0,\bullet<D_1}\bigr) & \longrightarrow & R\mathcal{H}om_{\mathcal{D}_X}\bigl(\mathbb{D}_X\mathcal{M}(*D_1),\Omega_X^{0,\bullet<D_1}\bigr)\\
\uparrow & & \uparrow{=}\\
R\mathcal{H}om_{\mathcal{D}_X}\bigl(\mathbb{D}_X\mathcal{M}(*D_1''),\Omega_X^{0,\bullet<D_1''}\bigr) & \longrightarrow & R\mathcal{H}om_{\mathcal{D}_X}\bigl(\mathbb{D}_X\mathcal{M}(*D_1),\Omega_X^{0,\bullet<D_1}\bigr)
\end{array}\tag{14}$$

Here, we put $D_1'':=D_1\cup D_1'$. ∎

3.2.3 Duality

We continue to use the notation in Subsection 3.2.2. For simplicity, we assume $D = D_0$. Let V be a good meromorphic flat bundle on (X, D). Let us construct a morphism $\mathrm{DR}_X^{<D_1 \leq D_2}(\mathbb{D}_X V) \longrightarrow \mathbb{D}_X \mathrm{DR}_X^{<D_2 \leq D_1} V$. Let Θ_X denote the sheaf of holomorphic tangent vectors on X. We set $\Theta_X^\bullet := \bigwedge^\bullet \Theta_X$. Because V and $\Omega_X^{0,\bullet<D_1}(*D_2)$ are $\mathcal{D}_X(*D)$-modules, we have a natural isomorphism:

$$R\mathcal{H}om_{\mathcal{D}_X}\bigl(V, \Omega^{0,\bullet\,<D_1}(*D_2)\bigr) \simeq \mathcal{H}om_{\mathcal{D}_X}\Bigl(\mathcal{D}_X \otimes \Theta_X^\bullet \otimes V,\, \mathcal{D}_X \otimes \Theta_X^\bullet \otimes \Omega^{0,\bullet\,<D_1}(*D_2)\Bigr). \tag{15}$$

By considering $\Omega_X^{\dim X,\bullet\,<D_2} \otimes_{\mathcal{D}_X(*D)}$, we have the following morphism:

$$\begin{aligned} \mathcal{H}om_{\mathcal{D}_X}\Bigl(\mathcal{D}_X \otimes \Theta_X^\bullet \otimes V,\, \mathcal{D}_X \otimes \Theta_X^\bullet \otimes \Omega^{0,\bullet\,<D_1}(*D_2)\Bigr) &\longrightarrow \mathcal{H}om_{\mathbb{C}_X}\Bigl(\Omega_X^{\bullet,\bullet\,<D_2} \otimes V,\, \Omega_X^{\bullet,\bullet\,<D}\Bigr) \\ &\longrightarrow R\mathcal{H}om_{\mathbb{C}_X}\Bigl(\mathrm{DR}_X^{<D_2 \leq D_1}(V),\, \Omega_X^{\bullet,\bullet<D}[\dim X]\Bigr) \end{aligned} \tag{16}$$

By using the inclusion $\Omega_X^{\bullet,\bullet\,<D} \subset \Omega_X^{\bullet,\bullet}$, we obtain the following morphism:

$$\begin{array}{ccc} \mathrm{DR}_X^{<D_1 \leq D_2}(\mathbb{D}_X V) & \longrightarrow & \mathbb{D}_X \mathrm{DR}_X^{<D_2 \leq D_1} V \\ \| & & \| \\ R\mathcal{H}om_{\mathcal{D}_X}\bigl(V, \Omega_X^{0,\bullet}(*D_2)^{<D_1}\bigr) & \longrightarrow & R\mathcal{H}om_{\mathbb{C}_X}\bigl(\mathrm{DR}_X^{<D_2 \leq D_1} V,\, \Omega_X^{\bullet,\bullet}[\dim X]\bigr) \end{array} \tag{17}$$

Note $\mathbb{D}_X V(*D) = V^\vee$.

Theorem 3.18 *The following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{DR}^{<D_1 \leq D_2}(V^\vee) & \xrightarrow{G_1} & \mathbb{D}_X \mathrm{DR}^{<D_2 \leq D_1}(V) \\ \simeq\big\downarrow & & \simeq\big\uparrow \\ \mathrm{DR}\, V^\vee(!D_1) & \xrightarrow[\simeq]{G_2} & \mathbb{D}_X \mathrm{DR}\bigl(V(!D_2)\bigr) \end{array} \tag{18}$$

The vertical isomorphisms are given in Proposition 3.15*, and G_2 is induced by the natural isomorphism of $\mathcal{D}$-modules $V^\vee(!D_1) \simeq \mathbb{D}_X\bigl(V(!D_2)\bigr)$. (See Subsection* 3.1.3*.) In particular, G_1 is also an isomorphism.*

Proof We have only to check the commutativity locally. Recall that we have used the identifications $V^\vee(!D_1) \simeq (\mathbb{D}_X V)(*D_2)$ and $V(!D_2) \simeq (\mathbb{D}_X V^\vee)(*D_1)$ in the construction of the vertical arrows. Since an isomorphism $\mathbb{D}_X\bigl(V(!D_2)\bigr) \longrightarrow (\mathbb{D}_X V)(*D_2)$ is uniquely determined by its restriction to $X - D$, we can regard that G_2^{-1} is induced by "$\mathcal{O}_X(*D_2)\otimes_{\mathcal{O}_X}$" as follows:

$$R\mathcal{H}om_{\mathcal{D}_X}\bigl(\mathbb{D}_X V^\vee(*D_1),\, \mathcal{D}_X \otimes \Omega_X^{\otimes -1}\bigr) \longrightarrow R\mathcal{H}om_{\mathcal{D}_X}\bigl(V, \mathcal{D}_X \otimes \Omega_X^{\otimes -1}(*D_2)\bigr) \tag{19}$$

Applying the de Rham functor to (19), we obtain the upper horizontal arrow in the following diagram:

$$\begin{array}{ccc} R\mathcal{H}om_{\mathcal{D}_X}\bigl(\mathbb{D}_X V^\vee(*D_1),\, \Omega_X^{0,\bullet}\bigr) & \xrightarrow[\simeq]{G_3} & R\mathcal{H}om_{\mathcal{D}_X}\bigl(V, \Omega_X^{0,\bullet}(*D_2)\bigr) \\ \simeq\big\uparrow b_0 & & \simeq\big\uparrow b_1 \\ R\mathcal{H}om_{\mathcal{D}_X}\bigl(\mathbb{D}_X V^\vee(*D_1),\, \Omega_X^{0,\bullet\,<D_1}\bigr) & \xrightarrow[\simeq]{a_0} & R\mathcal{H}om_{\mathcal{D}_X}\bigl(V, \Omega_X^{0,\bullet\,<D_1}(*D_2)\bigr) \end{array} \tag{20}$$

Up to shift of the degree, b_1 is the left vertical arrow in (18), and $G_3 = G_2^{-1}$. We have the following commutative diagram:

$$\begin{array}{ccc} R\mathcal{H}om_{\mathcal{D}_X}\Bigl(\mathbb{D}_X V^\vee(*D_1),\, \Omega_X^{0,\bullet\,<D_1}\Bigr) & \xrightarrow[\simeq]{b_0} & R\mathcal{H}om_{\mathcal{D}_X}\Bigl(\mathbb{D}_X V^\vee(*D_1),\, \Omega_X^{0,\bullet}\Bigr) \\ a_1\big\downarrow & & \simeq\big\downarrow \\ R\mathcal{H}om_{\mathbb{C}_X}\Bigl(\mathrm{DR}_X^{\leq D_1}(\mathbb{D}_X V^\vee),\, \mathrm{DR}_X\, \Omega_X^{0,\bullet\,<D_1}\Bigr) & \longrightarrow & R\mathcal{H}om_{\mathbb{C}_X}\Bigl(\mathrm{DR}_X^{\leq D_1}(\mathbb{D}_X V^\vee),\, \mathrm{DR}_X\, \Omega_X^{0,\bullet}\Bigr) \\ a_2\big\downarrow\simeq & & \simeq\big\downarrow \\ R\mathcal{H}om_{\mathbb{C}_X}\Bigl(\mathrm{DR}_X^{<D_2 \leq D_1}(\mathbb{D}_X V^\vee),\, \mathrm{DR}_X\, \Omega_X^{0,\bullet\,<D_1}\Bigr) & \xrightarrow{a_3} & R\mathcal{H}om_{\mathbb{C}_X}\Bigl(\mathrm{DR}_X^{<D_2 \leq D_1}(\mathbb{D}_X V^\vee),\, \mathrm{DR}_X\, \Omega_X^{0,\bullet}\Bigr) \end{array} \tag{21}$$

Let us consider the following diagram:

$$\begin{array}{ccc} R\mathcal{H}om_{\mathcal{D}_X}\big(\mathbb{D}_X V^\vee(*D_1),\, \Omega_X^{0,\bullet\,<D_1}\big) & \underset{\simeq}{\xrightarrow{a_0}} & R\mathcal{H}om_{\mathcal{D}_X}\big(V, \Omega_X^{0,\bullet\,<D_1}(*D_2)\big) \\ {\scriptstyle a_1}\big\downarrow & & {\scriptstyle a_4}\big\downarrow \\ R\mathcal{H}om_{\mathbb{C}_X}\big(\mathrm{DR}_X^{\leq D_1}(\mathbb{D}_X V^\vee),\, \mathrm{DR}_X\,\Omega_X^{0,\bullet\,<D_1}\big) & \underset{\simeq}{\xrightarrow{a_2}} & R\mathcal{H}om_{\mathbb{C}_X}\big(\mathrm{DR}_X^{<D_2\leq D_1}(\mathbb{D}_X V^\vee),\, \mathrm{DR}_X\,\Omega_X^{0,\bullet\,<D_1}\big) \end{array} \tag{22}$$

The morphism a_4 is given by (15), (16) and the inclusion $\Omega_X^{\bullet,\bullet\,<D} \longrightarrow \Omega_X^{\bullet,\bullet\,<D_1}$.

Lemma 3.19 *The diagram* (22) *is locally commutative, i.e., for any point $P \in X$, there exists a neighbourhood of U such that* (22) *considered on U is commutative.*

Proof We set $\mathcal{D}_X^\circ := \mathcal{D}_X \otimes \Omega_X^{\otimes -1}$, equipped with the $\mathcal{D}_X$-action induced by the right $\mathcal{D}_X$-action on $\mathcal{D}_X$. Because $\Omega_X^{0,\bullet\,<D_1}$ is an $\mathcal{O}_X(*D_1)$-module, we have

$$R\mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{D}_X^\circ(*D_1),\, \Omega_X^{0,\bullet\,<D_1}\big) \simeq \mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{D}_X^\circ(*D_1),\, \Omega_X^{0,\bullet\,<D_1}\big).$$

Similarly, we have $R\mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{D}_X^\circ(*D),\, \Omega_X^{0,\bullet\,<D_1}(*D_2)\big) \simeq \mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{D}_X^\circ(*D),\, \Omega_X^{0,\bullet\,<D_1}(*D_2)\big)$. The following naturally defined diagram is commutative:

$$\begin{array}{ccc} \mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{D}_X^\circ(*D_1),\, \mathcal{D}_X \otimes \Theta_X^\bullet \otimes \Omega_X^{0,\bullet\,<D_1}\Big) & \longrightarrow & \mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{D}_X^\circ(*D),\, \mathcal{D}_X \otimes \Theta_X^\bullet \otimes \Omega_X^{0,\bullet\,<D_1}(*D_2)\Big) \\ \big\downarrow & & \big\downarrow \\ \mathcal{H}om_{\mathbb{C}_X}\Big(\Omega_X^{0,\bullet}(*D_1),\, \mathrm{DR}_X\,\Omega_X^{0,\bullet\,<D_1}\Big) & \longrightarrow & \mathcal{H}om_{\mathbb{C}_X}\Big(\Omega_X^{0,\bullet\,<D_2}(*D_1),\, \mathrm{DR}_X\,\Omega_X^{0,\bullet\,<D_1}\Big) \end{array}$$

Then, we can check the commutativity of (22) by taking a free resolution of $V^\vee$. ∎

By construction, $a_3 \circ a_4$ is the equal to G_1 in (18). Then, the claim of Theorem 3.18 follows from the commutativity of the diagrams (20), (21) and (22). ∎

3.2.4 Functoriality

Let X be a complex manifold, and let D be a normal crossing hypersurface with a decomposition $D = D_1 \cup D_2$. Let D_3 be a hypersurface of X. Let $\varphi : X' \longrightarrow X$ be a proper birational morphism such that (i) $D' = \varphi^{-1}\big(D \cup D_3\big)$ is normal crossing, (ii) $X' \setminus D' \simeq X \setminus \big(D \cup D_3\big)$. We put $D_1' := \varphi^{-1}(D_1)$. We take D_2' such that $D' = D_1' \cup D_2'$ is a decomposition.

Let V be a meromorphic flat bundle on (X, D), and we set $V' := \varphi^* V \otimes \mathcal{O}_{X'}(*D')$. We have a natural isomorphism $\big(V(*D_3)\big)(!D_1) \simeq \varphi_\dagger\big(V'(!D_1')\big)$, which induces a morphism of $\mathcal{D}_X$-modules $V(!D_1) \longrightarrow \varphi_\dagger\big(V'(!D_1')\big)$. We have a naturally induced morphism $\varphi^{-1}\big(\Omega_X^{\bullet,\bullet\,<D_1}(*D_2) \otimes V\big) \longrightarrow \Omega_{X'}^{\bullet,\bullet\,<D_1'}(*D_2') \otimes V'$, from which we obtain the following:

$$\mathrm{DR}_X^{<D_1\leq D_2}(V) \longrightarrow R\varphi_*\,\mathrm{DR}_{X'}^{<D_1'\leq D_2'}(V') \tag{23}$$

By considering the dual with $V^\vee$ (see Theorem 3.18), we obtain the morphism

$$R\varphi_*\,\mathrm{DR}_{X'}^{<D_2'\leq D_1'}(V') \longrightarrow \mathrm{DR}_X^{<D_2\leq D_1}(V) \tag{24}$$

Theorem 3.20 *We have the following commutative diagram:*

$$\begin{array}{ccc} \mathrm{DR}_X^{<D_1\leq D_2}\, V & \longrightarrow & R\varphi_*\,\mathrm{DR}_{X'}^{<D_1'\leq D_2'}\, V' \\ \simeq\big\downarrow & & \simeq\big\downarrow \\ \mathrm{DR}_X\, V(!D_1) & \longrightarrow & R\varphi_*\,\mathrm{DR}_{X'}\, V'(!D_1') \end{array} \tag{25}$$

Here, the vertical isomorphisms are given in Proposition 3.15*, the upper horizontal arrow is* (23)*, and the lower horizontal arrow is induced by the morphism of $\mathcal{D}_X$-modules $V(!D_1) \longrightarrow \varphi_\dagger\big(V'(!D_1')\big)$.*

Similarly, we have the following commutative diagram:

$$\begin{array}{ccc} R\varphi_* \operatorname{DR}_{X'}^{<D_2'\le D_1'} V' & \longrightarrow & \operatorname{DR}_X^{<D_2\le D_1} V \\ \simeq\big\downarrow & & \simeq\big\downarrow \\ R\varphi_* \operatorname{DR}_{X'} V'(!D_2') & \longrightarrow & \operatorname{DR}_X V(!D_2) \end{array} \tag{26}$$

Here, the vertical isomorphisms are given in Proposition 3.15*, the upper horizontal arrow is* (24)*, and the lower horizontal arrow is induced by the natural morphism of $\mathcal{D}_X$-modules $\varphi_\dagger\big(V'(!D_2')\big) \longrightarrow V(!D_2)$.*

Proof We set $\mathcal{D}_X^\circ := \mathcal{D}_X \otimes \Omega_X^{\otimes -1}$. Put $d_X := \dim X$. Let us consider the commutativity of (26). By construction and the duality in Theorem 3.18, the morphism (24) is expressed as follows:

$$\begin{aligned} R\varphi_* R\mathcal{H}om_{\mathbb{C}_{X'}}\Big(\operatorname{DR}_{X'}^{<D_1'}(V'^\vee),\, \mathfrak{Db}_{X'}^{\bullet;\bullet}\Big)[d_X] &\longrightarrow R\mathcal{H}om_{\mathbb{C}_X}\Big(R\varphi_*\big(\operatorname{DR}_{X'}^{<D_1'} V'^\vee\big),\, \varphi_*\mathfrak{Db}_{X'}^{\bullet;\bullet}\Big)[d_X] \\ \longrightarrow R\mathcal{H}om_{\mathbb{C}_X}\Big(\operatorname{DR}_X^{<D_1} V^\vee,\, \varphi_*\mathfrak{Db}_{X'}^{\bullet;\bullet}\Big)[d_X] &\longrightarrow R\mathcal{H}om_{\mathbb{C}_X}\Big(\operatorname{DR}_X^{<D_1} V^\vee,\, \mathfrak{Db}_X^{\bullet;\bullet}\Big)[d_X] \end{aligned} \tag{27}$$

The morphism $\operatorname{DR}_X \varphi_\dagger\big(V'(!D_2')\big) \longrightarrow \operatorname{DR}_X V(!D_2)$ is represented as follows:

$$\begin{aligned} R\varphi_* R\mathcal{H}om_{\mathcal{D}_{X'}}\big(V'^\vee, \mathcal{O}_{X'}(*D_1')\big) &\longrightarrow R\mathcal{H}om_{\mathcal{D}_X}\big(\varphi_\dagger V'^\vee,\, \varphi_\dagger\mathcal{O}_{X'}(*D_1')\big) \\ \longrightarrow R\mathcal{H}om_{\mathcal{D}_X}\big(V^\vee, \varphi_\dagger\mathcal{O}_{X'}(*D_1')\big) &\longrightarrow R\mathcal{H}om_{\mathcal{D}_X}\big(V^\vee, \mathcal{O}_X(*D_1)\big) \end{aligned} \tag{28}$$

We have the following commutative diagram:

$$\begin{array}{ccc} R\varphi_* R\mathcal{H}om_{\mathcal{D}_{X'}}\big(V'^\vee, \mathcal{O}_{X'}(*D_1')\big) & \longrightarrow & R\varphi_* R\mathcal{H}om_{\mathbb{C}_{X'}}\big(\operatorname{DR}_{X'}^{<D_1'} V'^\vee,\, \mathfrak{Db}_{X'}^{\bullet;\bullet}\big)[d_X] \\ \big\downarrow & & \big\downarrow \\ R\mathcal{H}om_{\mathcal{D}_X}\big(\varphi_\dagger V'^\vee,\, \varphi_\dagger\mathcal{O}_{X'}(*D_1')\big) & \longrightarrow & R\mathcal{H}om_{\mathbb{C}_X}\Big(R\varphi_*\big(\operatorname{DR}_{X'}^{<D_1'} V'^\vee\big),\, \varphi_*\mathfrak{Db}_{X'}^{\bullet;\bullet}\Big)[d_X] \\ \big\downarrow & & \big\downarrow \\ R\mathcal{H}om_{\mathcal{D}_X}\big(V^\vee,\, \varphi_\dagger\mathcal{O}_{X'}(*D_1')\big) & \longrightarrow & R\mathcal{H}om_{\mathbb{C}_X}\Big(\operatorname{DR}_X^{<D_1} V^\vee,\, \varphi_*\mathfrak{Db}_{X'}^{\bullet;\bullet}\Big)[d_X] \end{array} \tag{29}$$

Lemma 3.21 *The following diagram is commutative:*

$$\begin{array}{ccc} R\mathcal{H}om_{\mathcal{D}_X}\big(V^\vee, \varphi_\dagger\mathcal{O}_{X'}(*D')\big) & \longrightarrow & R\mathcal{H}om_{\mathbb{C}_X}\big(\operatorname{DR}_X^{<D_1} V^\vee,\, \varphi_*\mathfrak{Db}_{X'}^{\bullet;\bullet}\big)[d_X] \\ \big\downarrow & & \big\downarrow \\ R\mathcal{H}om_{\mathcal{D}_X}\big(V^\vee, \mathcal{O}_X(*D_1)\big) & \longrightarrow & R\mathcal{H}om_{\mathbb{C}_X}\big(\operatorname{DR}_X^{<D_1} V^\vee,\, \mathfrak{Db}_X^{\bullet;\bullet}\big)[d_X] \end{array} \tag{30}$$

Proof Note that we have the commutativity of the following diagram:

$$\begin{array}{ccc} \varphi_*\big(\mathfrak{Db}_{X'}^{\bullet;\bullet}(*D_1')\big) \otimes \Omega_X^{\dim X, \bullet\, <D_1} & \longrightarrow & \varphi_*\big(\mathfrak{Db}_{X'}^{\bullet;\bullet}\big) \\ \big\downarrow & & \big\downarrow \\ \mathfrak{Db}_X^{\bullet;\bullet}(*D_1) \otimes \Omega_X^{\dim X, \bullet\, <D_1} & \longrightarrow & \mathfrak{Db}_X^{\bullet;\bullet} \end{array} \tag{31}$$

The vertical arrows are induced by the trace maps, and the horizontal arrows are multiplications. Hence, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{D}_X,\, \varphi_*\big(\mathfrak{Db}_{X'}^{\bullet;\bullet}(*D_1')\big) \otimes \mathcal{D}_X^\circ\Big) & \longrightarrow & \mathcal{H}om_{\mathbb{C}_X}\Big(\Omega_X^{\dim X, \bullet\, <D_1},\, \varphi_*\mathfrak{Db}_{X'}^{\bullet;\bullet}\Big) \\ \big\downarrow & & \big\downarrow \\ \mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{D}_X,\, \mathfrak{Db}_X^{\bullet;\bullet}(*D_1) \otimes \mathcal{D}_X^\circ\big) & \longrightarrow & \mathcal{H}om_{\mathbb{C}_X}\big(\Omega_X^{\dim X, \bullet\, <D_1},\, \mathfrak{Db}_X^{\bullet;\bullet}\big) \end{array} \tag{32}$$

The vertical arrows are induced by the trace maps, and the horizontal arrows are induced by the tensor product $\Omega_X^{\dim X,\bullet<D_1}\otimes_{\mathcal{D}_X}$ and (31). By considering a free resolution of V, we obtain the commutativity of (30) from (32). ∎

We obtain the commutativity of (26) from (29) and (30). Let us consider the commutativity of (25). In general, we have the following commutative diagram for $\mathcal{N}\longrightarrow\varphi_\dagger\mathcal{N}'$, where $\mathcal{N}$ (resp. $\mathcal{N}'$) is a coherent $\mathcal{D}_X$-module (resp. $\mathcal{D}_{X'}$-module):

$$\begin{array}{ccccccc}
R\varphi_*\,\mathrm{DR}\,\mathbb{D}\mathcal{N}' & \simeq & \mathrm{DR}\,\varphi_\dagger\mathbb{D}\mathcal{N}' & \simeq & \mathrm{DR}\,\mathbb{D}\varphi_\dagger\mathcal{N}' & \longrightarrow & \mathrm{DR}\,\mathbb{D}\mathcal{N}\\
\downarrow & & & & \downarrow & & \downarrow\\
R\varphi_*\mathbb{D}\,\mathrm{DR}\,\mathcal{N}' & \simeq & \mathbb{D}R\varphi_*\,\mathrm{DR}\,\mathcal{N}' & \simeq & \mathbb{D}\,\mathrm{DR}\,\varphi_\dagger\mathcal{N}' & \longrightarrow & \mathbb{D}\,\mathrm{DR}\,\mathcal{N}
\end{array}$$

The vertical arrows are also isomorphisms. Applying this commutativity to $V^\vee(!D_1)\longrightarrow\varphi_\dagger V'^\vee(!D_1')$, we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
R\varphi_*\,\mathrm{DR}\,V'(!D_3') & \longrightarrow & \mathrm{DR}\,\varphi_\dagger V'(!D_3') & \longrightarrow & \mathrm{DR}\,\mathbb{D}\varphi_\dagger V'^\vee(!D_1') & \longrightarrow & \mathrm{DR}\,V(!D_2)\\
\big\downarrow & & & & \big\downarrow & & \big\downarrow\\
R\varphi_*\mathbb{D}\,\mathrm{DR}\,V'^\vee(!D_1') & \longrightarrow & \mathbb{D}R\varphi_*\,\mathrm{DR}\,V'^\vee(!D_1') & \longrightarrow & \mathbb{D}\,\mathrm{DR}\,\varphi_\dagger V'^\vee(!D_1') & \longrightarrow & \mathbb{D}\,\mathrm{DR}\,V^\vee(!D_1)
\end{array}$$

It implies the commutativity of the following:

$$\begin{array}{ccc}
\mathbb{D}\,\mathrm{DR}\big(V(!D_2)\big) & \longrightarrow & \mathbb{D}R\varphi_*\,\mathrm{DR}\big(V'(!D_3')\big)\\
\simeq\big\downarrow & & \simeq\big\downarrow\\
\mathrm{DR}\big(V^\vee(!D_1)\big) & \longrightarrow & R\varphi_*\,\mathrm{DR}\big(V'^\vee(!D_1')\big)
\end{array}$$

Then, (25) is obtained as the dual of (26) with $V^\vee$, and hence it is commutative. Thus, the proof of Theorem 3.20 is finished. ∎

4 Some sheaves on a real blow up

4.1 C^∞-functions holomorphic functions

4.1.1 Preliminary

Let X be an n-dimensional complex manifold with a simply normal crossing hypersurface D with the irreducible decomposition $\bigcup_{i\in\Lambda}D_i$. In this paper, the real blow up $\pi:\widetilde{X}(D)\longrightarrow X$ means the fiber product of $\widetilde{X}(D_i)$ over X. For any subset $I\subset\Lambda$, we set $D_I:=\bigcap_{i\in I}D_i$ and $D(I):=\bigcup_{i\in I}D_i$. Formally, $D_\emptyset:=X$. For $J\subset I^c:=\Lambda\setminus I$, we put $D_I(J):=D_I\cap D(J)$. In particular, $\partial D_I:=D_I(I^c)$. Let D° be a (possibly empty) hypersurface of X such that (i) $D\cup D^\circ$ is simply normal crossing, (ii) $\dim D\cap D^\circ<n-1$. For $J\subset\Lambda$, we set $D(\overline{J}):=D(J)\cup D^\circ$. For $I\sqcup J\subset\Lambda$, we put $D_I(\overline{J}):=D_I\cap D(\overline{J})$.

Recall that a holomorphic function on $\widetilde{X}(D)$ is defined to be a C^∞-function f on $\widetilde{X}(D)$ such that $f_{|X-D}$ is holomorphic. Let $\mathcal{O}_{\widetilde{X}(D)}$ denote the sheaf of holomorphic functions on X. Let $\Omega^{0,q}_{\widetilde{X}(D)}$ denote the sheaf of C^∞-logarithmic $(0,q)$-forms on $\widetilde{X}(D)$, i.e., a section of $\Omega^{0,q}_{\widetilde{X}(D)}$ is locally described as a linear combination of

$$f\cdot d\overline{z}_{i_1}/\overline{z}_{i_1}\cdots d\overline{z}_{i_m}/\overline{z}_{i_m}\cdot d\overline{z}_{j_1}\cdots d\overline{z}_{j_k}\quad(1\le i_1,\ldots,i_m\le\ell,\ \ell+1\le j_1,\ldots,j_k\le n,\ f\in\mathcal{C}^\infty_{\widetilde{X}(D)})$$

in terms of a local coordinate $(z_1,\ldots,z_n)$ such that D is locally described as $\bigcup_i^\ell\{z_i=0\}$. We have the naturally defined operator $\overline{\partial}:\Omega^{0,q}_{\widetilde{X}(D)}\longrightarrow\Omega^{0,q+1}_{\widetilde{X}(D)}$. The complex $\Omega^{0,\bullet}_{\widetilde{X}(D)}$ is called the Dolbeault complex of $\widetilde{X}(D)$. We put $\Omega^{0,\bullet}_{\widehat{Z}}:=\Omega^{0,\bullet}_{\widetilde{X}(D)|\widehat{Z}}$ for any real analytic subset $Z\subset\widetilde{X}(D)$.

Let Z be $\pi^{-1}\big(D_I(J)\big)$ for some $I\sqcup J\subset\Lambda$. Let $\mathcal{I}_Z\subset\mathcal{O}_{\widetilde{X}(D)}$ be the ideal sheaf of Z, and put $\mathcal{O}_{\widehat{Z}}:=\varprojlim\mathcal{O}_X/\mathcal{I}_Z^m$. For a given $\mathcal{O}_{\widetilde{X}(D)}$-module $\mathcal{F}$, we set $\mathcal{F}_{|\widehat{Z}}:=\mathcal{F}\otimes_{\mathcal{O}_{\widetilde{X}(D)}}\mathcal{O}_{\widehat{Z}}$. According to a generalized Borel-Ritt theorem due to Majima and Sabbah ([25], [34]), the natural morphism $\mathcal{O}_{\widehat{\pi^{-1}(D_I)}} \longrightarrow \mathcal{O}_{\widehat{\pi^{-1}(D_I(J))}}$ is surjective. The kernel is denoted by $\mathcal{O}^{<D(J)}_{\widehat{\pi^{-1}(D_I)}}$.

For a given C^∞-manifold Y and a real analytic subset $W \subset X$, let $\mathcal{C}^{\infty\,\leq W}_{\widehat{\pi^{-1}(D_I)\times Y}}$ denote the sheaf $\mathcal{C}^{\infty\,<\pi^{-1}(W)\times Y}_{\widehat{\pi^{-1}(D_I)\times Y}}$ on $\widetilde{X}(D)\times Y$, for simplicity of the description. We also put $\Omega^{0,\bullet<W}_{\widehat{\pi^{-1}(D_I)\times Y}} := \Omega^{0,\bullet}_{\widetilde{X}(D)} \otimes_{\mathcal{C}^\infty_{\widetilde{X}(D)}} \mathcal{C}^{\infty<W}_{\widehat{\pi^{-1}(D_I)\times Y}}$ on $\widetilde{X}(D)\times Y$.

Let q_I denote the projection $\pi^{-1}(D_I) \longrightarrow \widetilde{D}_I(\partial D_I)$. We have $\mathcal{O}^{<D(J)}_{\widehat{\pi^{-1}(D_I)}} = q_I^{-1}\mathcal{O}^{<D_I(J)}_{\widetilde{D}_I(\partial D_I)}[\![z_i\,|\,i\in I]\!]$. By a natural diffeomorphism $\pi^{-1}(D_I) \simeq \widetilde{D}_I(\partial D_I)\times (S^1)^{|I|}$, we can identify $\mathcal{C}^{\infty\,<D(\overline{J})}_{\widehat{\pi^{-1}(D_I)}} = \mathcal{C}^{\infty\,<D_I(\overline{J})}_{\widetilde{D}_I(\partial D_I)\times(S^1)^{|I|}}[\![z_i\,|\,i\in I]\!]$.

Put $\mathcal{T}(m,I,J) := \bigl\{K\subset J\,\big|\,I\subset K,\,|K| = |I|+m+1\bigr\}$ for $m\geq 0$. We set $\mathcal{K}^m\Bigl(\mathcal{O}_{\widehat{\pi^{-1}(D_I(J))}}\Bigr) := \bigoplus_{K\in\mathcal{T}(m,I,J)}\mathcal{O}_{\widehat{\pi^{-1}(D_K)}}$. We obtain the complex $\mathcal{K}^\bullet\Bigl(\mathcal{O}_{\widehat{\pi^{-1}(D_I(J))}}\Bigr)$ as in Subsection 2.1.4. Similarly, we obtain complex $\mathcal{K}^\bullet\bigl(\Omega^{0,\bullet\,<D^\circ}_{\widehat{\pi^{-1}(D_I(J))\times Y}}\bigr)$.

Lemma 4.1 ([34]) *Let $\mathcal{B}$ be $\mathcal{O}_{\widehat{\pi^{-1}(D_I(J))}}$ or $\Omega^{0,\bullet\,<D^\circ}_{\widehat{\pi^{-1}(D_I(J))\times Y}}$. The natural inclusion $\mathcal{B}\longrightarrow\mathcal{K}^0(\mathcal{B})$ induces a quasi-isomorphism $\mathcal{B}\longrightarrow\mathcal{K}^\bullet(\mathcal{B})$.* ∎

4.1.2 Dolbeault resolution

In this subsection, we do not consider D°.

Proposition 4.2 ([25], [34]) *$\Omega^{0,\bullet}_{\widehat{\pi^{-1}(D_I(J))}}$ and $\Omega^{0,\bullet\,<D(J)}_{\widehat{\pi^{-1}(D_I)}}$ are resolutions of $\mathcal{O}_{\widehat{\pi^{-1}(D_I(J))}}$ and $\mathcal{O}^{<D(J)}_{\widehat{\pi^{-1}(D_I)}}$ respectively, where $J\subset I^c$.*

Proof We give only an outline. In each case, it is easy to compute the 0-th cohomology of the Dolbeault complexes. We have only to show the vanishing of the higher cohomology. We may assume $X=\Delta^n$, $D_i = \{z_i=0\}$ and $D=\bigcup_{i=1}^{\ell}D_i$. First, let us look at $\Omega^{0,\bullet}_{\widetilde{X}(D)}$. For $1\leq j\leq n$, let $\mathcal{P}^0_{\leq j}$ be the sheaf of C^∞-functions on $\widetilde{X}(D)$ which are $\overline{\partial}_i$-holomorphic for $i>j$. We set $X_j := \Delta^j = \{(z_1,\ldots,z_j)\}$ and $D_{j,\ell} := \bigcup_{i\leq\min\{j,\ell\}}\{z_i=0\}$. Let $q_{\leq j}$ be the projection $\widetilde{X}(D)\longrightarrow\widetilde{X}_j(D_{j,\ell})$. Let $\mathcal{P}^1_{\leq j}$ be the sheaf of C^∞-sections of $q_{\leq j}^{-1}\Omega^{0,1}_{\widetilde{X}_j(D_{j,\ell})}$, which are $\overline{\partial}_i$-holomorphic for $i>j$. We set $\mathcal{P}^\bullet_{\leq j} := \bigwedge^\bullet\mathcal{P}^1_{\leq j}$ over $\mathcal{P}^0_{\leq j}$. We have the naturally defined operator $\overline{\partial}:\mathcal{P}^\bullet_{\leq j}\longrightarrow\mathcal{P}^{\bullet+1}_{\leq j}$.

Because $\mathcal{P}^\bullet_{\leq 0}=\mathcal{O}_{\widetilde{X}(D)}$ and $\mathcal{P}^\bullet_{\leq n}=\Omega^{0,\bullet}_{\widetilde{X}(D)}$, we have only to show that the natural inclusions $\mathcal{P}^\bullet_{\leq j}\longrightarrow\mathcal{P}^\bullet_{\leq j+1}$ are quasi-isomorphisms for the vanishing of the higher cohomology of $\Omega^{0,\bullet}_{\widetilde{X}(D)}$. Let $\mathcal{Q}^0_{\leq j}=\mathcal{P}^0_{\leq j+1}$. Let $\mathcal{Q}^1_{\leq j}$ be the sheaf of $q_{\leq j}^{-1}\Omega^{0,1}_{\widetilde{X}_j(D_{j,\ell})}$ which are $\overline{\partial}_i$-holomorphic for $i>j+1$. We take the exterior product $\mathcal{Q}^\bullet_{\leq j}=\bigwedge^\bullet\mathcal{Q}^1_{\leq j}$ over $\mathcal{Q}^0_{\leq j}$. We have the naturally defined operator $\overline{\partial}_{j+1}:\mathcal{Q}^\bullet_{\leq j}\longrightarrow\mathcal{Q}^\bullet_{\leq j}\wedge d\overline{z}_{j+1}/\overline{z}_{j+1}$. We clearly have $\operatorname{Ker}\overline{\partial}_{j+1}=\mathcal{P}^\bullet_{\leq j}$. Let us show $\operatorname{Cok}\overline{\partial}_{j+1}=0$. In the case $j\geq\ell$, it can be shown by the argument for the standard Dolbeault's lemma. Let us consider the case $j<\ell$.

Lemma 4.3 *The cokernel of the morphism $\overline{\partial}_{j+1}:\mathcal{Q}^\bullet_{\leq j|\widehat{\pi^{-1}(D_{j+1})}}\longrightarrow\mathcal{Q}^\bullet_{\leq j|\widehat{\pi^{-1}(D_{j+1})}}\wedge d\overline{z}_{j+1}/\overline{z}_{j+1}$ is 0.*

Proof We use the polar coordinate $z_{j+1}=r_{j+1}\,e^{\sqrt{-1}\theta_{j+1}}$. The action of $\overline{\partial}_{j+1}$ is expressed as follows:

$$\overline{\partial}_{j+1}\Bigl(\sum_n f_n(\theta_{j+1})\,z_{j+1}^n\Bigr)=\sum_n\Bigl(\frac{\sqrt{-1}}{2}\partial_{\theta_{j+1}}\Bigr)f_n(\theta_{j+1})\,z_{j+1}^n\cdot d\overline{z}_{j+1}/\overline{z}_{j+1}$$

Then, it is easy to show the claim of Lemma 4.3. ∎

Put $D' := \bigcup_{i=1, i\neq j+1}^{\ell}\{z_i = 0\}$, and let us consider the real blow up $\pi' : \widetilde{X}(D') \longrightarrow X$. We have a naturally induced morphism $q'_{\leq j} : \widetilde{X}(D') \longrightarrow X_j(D_{j,\ell})$. Let $\mathcal{S}^1_{\leq j,X}$ be the sheaf of sections of $(q'_{\leq j})^{-1}\Omega^{0,1}_{X_j(D_{j,\ell})}$ on $\widetilde{X}(D')$, which are $\overline{\partial}_i$-holomorphic for $i > j+1$. Let $\mathcal{S}^0_{\leq j,X}$ be the sheaf of C^∞-functions on $\widetilde{X}(D')$, which are $\overline{\partial}_i$-holomorphic for $i > j+1$. We set $\mathcal{S}^\bullet_{\leq j} := \bigwedge^\bullet \mathcal{S}^1_{\leq j}$. It is easy to show the vanishing of the cokernel of $\overline{\partial}_{j+1} : \mathcal{S}^\bullet_{\leq j} \longrightarrow \mathcal{S}^\bullet_{\leq j} \wedge d\overline{z}_{j+1}$ by using the argument for standard Dolbeault's lemma.

Let $P \in \pi^{-1}(D)$. Let U be a small neighbourhood around P, which will be shrinked in the following argument. According to Lemma 4.3, for any section φ of $\mathcal{Q}^\bullet_{\leq j} \wedge d\overline{z}_{j+1}/\overline{z}_{j+1}$ on U, we can take a local section ψ of $\mathcal{Q}^\bullet_{\leq j}$ such that

$$\big(\varphi - \overline{\partial}_j\psi\big)_{|\widehat{\pi^{-1}(D_j)}\cap U} = 0.$$

We put $\lambda := \varphi - \overline{\partial}_j\psi$. We take a cut function ρ around P, i.e., ρ is constantly 1 around P and constantly 0 near the boundary of U. We can regard $\rho\,\lambda$ as a section of $\mathcal{S}^\bullet_{\leq j} \wedge d\overline{z}_{j+1}$. Then, we can find a section κ of $\mathcal{S}^\bullet_{\leq j}$ around $\pi_j(P)$ such that $\overline{\partial}_{j+1}\kappa = \rho\lambda$, where π_j denotes the natural projection $\widetilde{X}(D) \longrightarrow \widetilde{X}(D')$. We obtain $\varphi = \overline{\partial}_j(\psi + \kappa)$ around P. Thus, we obtain the vanishing of the cokernel of $\overline{\partial}_{j+1} : \mathcal{Q}^\bullet_{\leq j} \longrightarrow \mathcal{Q}^\bullet_{\leq j} \wedge d\overline{z}_{j+1}/\overline{z}_{j+1}$, and hence the vanishing of the higher cohomology of $\Omega^{0,\bullet}_{\widetilde{X}(D)}$.

Because $\pi^{-1}(D_I) = \widetilde{D}_I(\partial D_I) \times (S^1)^{|I|}$, we can reduce the vanishing of the higher cohomology of $\Omega^{0,\bullet}_{\widehat{\pi^{-1}(D_I)}}$ to the vanishing of $\Omega^{0,\bullet}_{\widetilde{D}_I(\partial D_I)}$ by a formal calculation as in Lemma 4.3. By using the resolution in Lemma 4.1, we obtain the vanishing of the higher cohomology of $\Omega^{0,\bullet}_{\widehat{\pi^{-1}(D(I))}}$. We have the following diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}^{<D(I)}_{\widetilde{X}(D)} & \longrightarrow & \mathcal{O}_{\widetilde{X}(D)} & \longrightarrow & \mathcal{O}_{\widehat{\pi^{-1}(D(I))}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^{0,\bullet\,<D(I)}_{\widetilde{X}(D)} & \longrightarrow & \Omega^{0,\bullet}_{\widetilde{X}(D)} & \longrightarrow & \Omega^{0,\bullet}_{\widehat{\pi^{-1}(D(I))}} & \longrightarrow & 0
\end{array}$$

Then, we obtain the vanishing of the higher cohomology of $\Omega^{0,\bullet\,<D(I)}_{\widetilde{X}(D)}$. By a formal calculation as in Lemma 4.3, we obtain the vanishing of the higher cohomology of $\Omega^{0,\bullet}_{\widehat{\pi^{-1}(D_I(J))}}$ and $\Omega^{0,\bullet\,<D(J)}_{\widehat{\pi^{-1}(D_I)}}$. ∎

4.1.3 Flatness

In this subsection D° is not necessarily empty.

Proposition 4.4 *Let $I \sqcup J \subset \Lambda$. The sheaves $\mathcal{C}^{\infty\,<D(\overline{J})}_{\widehat{\pi^{-1}(D_I)}}$, $\mathcal{C}^{\infty\,<D^\circ}_{\widehat{\pi^{-1}(D_I(J))}}$, $\mathcal{O}^{<D(J)}_{\widehat{\pi^{-1}(D_I)}}$ and $\mathcal{O}_{\widehat{\pi^{-1}(D_I(J))}}$ are flat over $\pi^{-1}\mathcal{O}_X$.*

Proof Let us recall a general result. For a real analytic manifold Y, let $\mathcal{O}^{\mathbb{R}}_Y$ denote the sheaf of real analytic functions on Y. If Y is the product of a complex manifold Y_1 and a real analytic manifold Y_2, let $\mathcal{O}^{Y_1-\mathrm{hol}}_Y$ denote the sheaf of real analytic functions which are holomorphic in the Y_1-direction. The extension $\mathcal{O}^{Y_1-\mathrm{hol}}_Y \subset \mathcal{O}^{\mathbb{R}}_Y$ is faithfully flat.

Lemma 4.5 *Let $W_1 \subset W_2 \subset Y$ be real analytic subsets. Then, $\mathcal{C}^{\infty\,<W_i}_Y$ and $\mathcal{C}^{\infty\,<W_1}_Y/\mathcal{C}^{\infty\,<W_2}_Y$ are flat over $\mathcal{O}^{\mathbb{R}}_Y$.*

Proof The sheaf $\mathcal{C}^\infty_Y$ is faithfully flat over $\mathcal{O}^{\mathbb{R}}_Y$ (Corollary 1.12 of [26]). Theorem VI.1.2 of [26] implies $\mathfrak{a}\,\mathcal{C}^{\infty\,<W_1}_Y \cap \mathcal{C}^{\infty\,<W_2}_Y = \mathfrak{a}\,\mathcal{C}^{\infty\,<W_2}_Y$ for any real analytic subsets $W_1 \subset W_2 \subset Y$ and for any ideal sheaf $\mathfrak{a}$ of $\mathcal{O}^{\mathbb{R}}_Y$. By using the argument in the proof of Proposition III.4.7 in [26], we can show the following:

- Let A be a ring. Let M be an A-flat module. Let N be an A-submodule of M. If $\mathfrak{a}M \cap N = \mathfrak{a}N$ for any ideal $\mathfrak{a}$ of A, then N and M/N are also A-flat.

We immediately obtain the claim of Lemma 4.5 from these results. ∎

Let Z_0 be a complex manifold with a normal crossing hypersurface D_0. Let Z_1 be a real analytic manifold. We put $Z := Z_0 \times Z_1$ and $D := D_0 \times Z_1$. Let G denote the composite of the maps $Z \longrightarrow Z_0 \longrightarrow Z_0 \times \mathbb{C}^n$, where the latter is induced by the inclusion $\{(0,\ldots,0)\} \subset \mathbb{R}^n$. Let $(t_1,\ldots,t_n)$ be the coordinate of $\mathbb{C}^n$.

Lemma 4.6 *$\mathcal{C}_Z^{\infty\,<D}[\![t_1,\ldots,t_n]\!]$ is flat over $G^{-1}\mathcal{O}_{Z_0\times\mathbb{C}^n}$.*

Proof Let ι_1 denote the inclusion $Z \longrightarrow Z_2 := Z \times \mathbb{R}^n$. We put $D_2 := D \times \mathbb{R}^n$. We regard that $(t_1,\ldots,t_n)$ is a real coordinate of $\mathbb{R}^n \subset \mathbb{C}^n$. We have the natural identification $\mathcal{C}_Z^{\infty\,<D}[\![t_1,\ldots,t_n]\!] = \mathcal{C}_{Z_2}^{\infty\,<D_2}\big/\mathcal{C}_{Z_2}^{\infty\,<D_2\cup Z}$. According to Lemma 4.5, it is flat over $\iota_1^{-1}\mathcal{O}_{Z_2}^{\mathbb{R}}$. Let G_1 be the composite of $Z \longrightarrow Z_0 \longrightarrow Z_0 \times \mathbb{R}^n$. We have a natural isomorphism $G_1^{-1}\mathcal{O}_{Z_0\times\mathbb{R}^n}^{Z_0-\mathrm{hol}} \simeq G^{-1}\mathcal{O}_{Z_0\times\mathbb{C}^n}$. Since the extension $G_1^{-1}\mathcal{O}_{Z_0\times\mathbb{R}^n}^{Z_0-\mathrm{hol}} \subset \mathcal{O}_{Z_2}^{\mathbb{R}}$ is faithfully flat, we obtain the claim of Lemma 4.6. ∎

Let us return to the proof of Proposition 4.4. We may assume that $X = \Delta^n$, $D_i = \{z_i = 0\}$, $D = \bigcup_{i=1}^{\ell} D_i$ and $D^\circ = \bigcup_{i=\ell+1}^{m} D_i$. For $I \subset \underline{\ell}$, let $\pi_I : \widetilde{X}(D(I)) \longrightarrow X$ be the real blow up. We have the natural identification $\pi_I^{-1}(D_I) = D_I \times (S^1)^{|I|}$ and $\pi_I^{-1}\big(D_I(\overline{I}^c)\big) = D_I(\overline{I}^c) \times (S^1)^{|I|}$. From Lemma 4.6, we obtain that $\mathcal{C}^{\infty\,<D(\overline{I}^c)}_{\widehat{\pi_I^{-1}(D_I)}} = \mathcal{C}^{\infty\,<D_I(\overline{I}^c)}_{\widehat{\pi_I^{-1}(D_I)}}[\![z_i \,|\, i \in I]\!]$ is flat over $\pi_I^{-1}\mathcal{O}_X$.

Lemma 4.7 *$\mathcal{C}^{\infty\,<D(\overline{I}^c)}_{\widehat{\pi^{-1}(D_I)}}$ is flat over $\pi^{-1}\mathcal{O}_X$. (Note that $\pi : \widetilde{X}(D) \longrightarrow X$.)*

Proof The claim is clear outside of $\pi^{-1}(\partial D_I)$. Let P be any point of ∂D_I. Let $\mathfrak{a}$ be a finitely generated ideal of $\mathcal{O}_{X,P}$. We take a free resolution $\mathcal{Q}_\bullet$ of $\mathfrak{a}$, i.e., $\cdots \to \mathcal{Q}_1 \to \mathcal{Q}_0 \longrightarrow \mathfrak{a}$. We obtain a $\pi^{-1}\mathcal{O}_X$-free resolution $\pi^{-1}\mathcal{Q}_\bullet$ of $\pi^{-1}\mathfrak{a}$. We set $\mathcal{Q}_{-1} := \mathfrak{a}$ for simplicity of the description. We have only to show that $\pi^{-1}\mathcal{Q}_\bullet \otimes \mathcal{C}^{\infty\,<D(\overline{I}^c)}_{\widehat{\pi^{-1}(D_I)}}$ is exact. Let $\rho : \widetilde{X}(D) \longrightarrow \widetilde{X}(D(I))$ be the naturally induced map. Note

$$\rho_*\big(\pi^{-1}\mathcal{Q}_\bullet \otimes \mathcal{C}^{\infty\,<D(\overline{I}^c)}_{\widehat{\pi^{-1}(D_I)}}\big) = \pi_I^{-1}(\mathcal{Q}_\bullet) \otimes \rho_*\big(\mathcal{C}^{\infty\,<D(\overline{I}^c)}_{\widehat{\pi^{-1}(D_I)}}\big) = \pi_I^{-1}\big(\mathcal{Q}_\bullet\big) \otimes \mathcal{C}^{\infty\,<D(\overline{I}^c)}_{\widehat{\pi_I^{-1}(D_I)}}.$$

Let $Q \in \pi^{-1}(P)$. Take a cycle φ of $\pi^{-1}\mathcal{Q}_i \otimes \mathcal{C}^{\infty\,<D(\overline{I}^c)}_{\widehat{\pi^{-1}(D_I)}}$ at Q. By using a cut function around Q, we can regard it as a global cycle of $\pi^{-1}\mathcal{Q}_i \otimes \mathcal{C}^{\infty\,<D(\overline{I}^c)}_{\widehat{\pi^{-1}(D_I)}}$ whose support is a small neighbourhood of Q. Then, it can be regarded as a cycle of $\pi_I^{-1}(\mathcal{Q}_i) \otimes \mathcal{C}^{\infty\,<D(\overline{I}^c)}_{\widehat{\pi_I^{-1}(D_I)}}$ around $\rho(Q)$. Because $\mathcal{C}^{\infty\,<D(\overline{I}^c)}_{\widehat{\pi_I^{-1}(D_I)}}$ is flat over $\pi_I^{-1}\mathcal{O}_X$, we obtain that φ is a boundary in the complex $\pi_I^{-1}(\mathcal{Q}_\bullet) \otimes \mathcal{C}^{\infty\,<D(\overline{I}^c)}_{\widehat{\pi_I^{-1}(D_I)}}$. Then, it is easy to deduce that φ is a boundary in the complex $\pi^{-1}(\mathcal{Q}_\bullet) \otimes \mathcal{C}^{\infty\,<D(\overline{I}^c)}_{\widehat{\pi^{-1}(D_I)}}$. Thus, the proof of Lemma 4.7 is finished. ∎

Let us show that $\mathcal{C}^{\infty\,<D(\overline{J})}_{\widehat{\pi^{-1}(D_I)}}$ is flat over $\pi^{-1}\mathcal{O}_X$, where $I \sqcup J \subset \underline{\ell}$. We put

$$\mathcal{S}(I,J,m) := \big\{K \subset \underline{\ell} - J \,\big|\, I \subset K,\ |K| = m\big\}.$$

Put $\mathcal{G}_{I,\ell+1} := \mathcal{C}^{\infty\,<D(\overline{J})}_{\widehat{\pi^{-1}(D_I)}}$, and descending inductively

$$\mathcal{G}_{I,m} := \mathrm{Ker}\Big(\mathcal{G}_{I,m+1} \longrightarrow \bigoplus_{K\in\mathcal{S}(I,J,m)} \mathcal{C}^{\infty\,<D(\overline{K}^c)}_{\widehat{\pi^{-1}(D_K)}}\Big).$$

We have $\mathcal{G}_{I,|I|+1} = \mathcal{C}^{\infty\,<D(\overline{I}^c)}_{\widehat{\pi^{-1}(D_I)}}$, which is flat over $\pi^{-1}\mathcal{O}_X$. By an induction, we obtain that $\mathcal{G}_{I,m}$ are flat over $\pi^{-1}\mathcal{O}_X$. Hence, we obtain that $\mathcal{C}^{\infty\,<D(\overline{J})}_{\widehat{\pi^{-1}(D_I)}}$ is flat over $\pi^{-1}\mathcal{O}_X$. By using the resolution of $\mathcal{C}^{\infty\,<D^\circ}_{\widehat{\pi^{-1}(D_I(J))}}$ in Lemma 4.1, we obtain that $\mathcal{C}^{\infty\,<D^\circ}_{\widehat{\pi^{-1}(D_I(J))}}$ is flat over $\pi^{-1}\mathcal{O}_X$. As a result, we obtain that $\Omega^{0,\bullet\,<D(\overline{J})}_{\widehat{\pi^{-1}(D_I)}}$ and $\Omega^{0,\bullet\,<D^\circ}_{\widehat{\pi^{-1}(D_I(J))}}$ are

flat over $\pi^{-1}\mathcal{O}_X$, where $J \subset I^c$. In particular, $\Omega^{0,\bullet\,<D(J)}_{\widehat{\pi^{-1}(D_I)}}$ and $\Omega^{0,\bullet}_{\widehat{\pi^{-1}(D_I(J))}}$ are flat over $\pi^{-1}\mathcal{O}_X$. Then, we obtain the $\pi^{-1}\mathcal{O}_X$-flatness of $\mathcal{O}^{<D(J)}_{\widehat{\pi^{-1}(D_I)}}$ and $\mathcal{O}_{\widehat{\pi^{-1}(D_I(J))}}$ by using Proposition 4.2. Thus, the proof of Proposition 4.4 is finished. ∎

4.2 Functions of Nilsson type

4.2.1 Preliminary

We set $\mathrm{Nil}(z) := \bigoplus_{\alpha\in\mathbb{C}} z^\alpha\mathbb{C}[\log z]$. For $(\alpha,k)\in\mathbb{C}\times\mathbb{Z}_{\geq 0}$, we put $\varphi_{\alpha,k}(z) := z^\alpha(\log z)^k \in \mathrm{Nil}(z)$. Let T be a finite subset $T\subset\mathbb{C}$ such that the induced map $T\longrightarrow\mathbb{C}/\mathbb{Z}$ is injective. For simplicity, we assume $0\in T$. Let N be a non-negative integer. We set

$$\mathrm{Nil}_{T,N}(z) := \Big\{\sum a_{\alpha,j,k}\,\varphi_{\alpha+j,k}(z)\,\Big|\, a_{\alpha,j,k}\in\mathbb{C},\ j\geq -N,\ k\leq N\Big\}\subset \mathrm{Nil}(z).$$

Note that $\mathrm{Nil}_{T,N}(z)$ is a finitely generated free $\mathbb{C}[z]$-module. For $T\subset T'$ and $N\leq N'$, we have a natural inclusion $\mathrm{Nil}_{T,N}(z)\subset\mathrm{Nil}_{T',N'}(z)$. We have $\mathrm{Nil}(z) = \varinjlim \mathrm{Nil}_{T,N}(z)$.

Let $\widetilde{\mathbb{C}}_z(0)$ be the real blow up of $\mathbb{C}_z$ along 0. Let ι be the inclusion $\iota:\mathbb{C}_z^*\longrightarrow\widetilde{\mathbb{C}}_z(0)$. We regard $\mathrm{Nil}(z)$ and $\mathrm{Nil}_{T,N}(z)$ as subsheaves of $\iota_*\mathcal{O}_{\mathbb{C}^*}$ on $\widetilde{\mathbb{C}}(0)$.

We put $\mathrm{Nil}(z_1,\ldots,z_\ell) := \mathrm{Nil}(z_1)\otimes_{\mathbb{C}}\cdots\otimes_{\mathbb{C}}\mathrm{Nil}(z_\ell)$ and $\mathrm{Nil}_{T,N}(z_1,\ldots,z_\ell) := \mathrm{Nil}_{T,N}(z_1)\otimes_{\mathbb{C}}\cdots\otimes_{\mathbb{C}}\mathrm{Nil}_{T,N}(z_\ell)$. We naturally regard $\mathrm{Nil}(z_1,\ldots,z_\ell)$ as a subsheaf of $\iota_*\mathcal{O}_{\mathbb{C}^n-D}$ on the real blow up $\widetilde{\mathbb{C}}(D)$, where $D=\bigcup_{i=1}^\ell\{z_i=0\}$ and $\iota:\mathbb{C}^n-D\longrightarrow\widetilde{\mathbb{C}}^n(D)$. For $(\boldsymbol{\alpha},\boldsymbol{k})\in\mathbb{C}^\ell\times\mathbb{Z}^\ell_{\geq 0}$, we put $\varphi_{\boldsymbol{\alpha},\boldsymbol{k}}(z_1,\ldots,z_n) := \prod_{i=1}^\ell\varphi_{\alpha_i,k_i}(z_i)$, which are regarded as multi-valued flat sections of $\mathrm{Nil}(z_1,\ldots,z_\ell)$.

4.2.2 Sheaves of functions of Nilsson type

Let X be a complex manifold with a simply normal crossing hypersurface D. Let $D = D^{(1)}\cup D^{(2)}$ be a decomposition. Let D° be a hypersurface of X such that (i) $D\cup D^\circ$ is simply normal crossing, (ii) $\dim D\cap D^\circ < n-1$. We put $D^{(3)} := D^{(1)}\cup D^\circ$. We would like to introduce sheaves $\mathcal{A}^{<D^{(1)}\leq D^{(2)}}_{\widetilde{X}(D)}$ and $\mathcal{C}^{\infty<D^{(3)}\leq D^{(2)}}_{\widetilde{X}(D)}$ on $\widetilde{X}(D)$. First, let us consider the case $X=\Delta^n$, $D=\bigcup_{i=1}^\ell\{z_i=0\}$ and $D^\circ=\bigcup_{i=\ell+1}^m\{z_i=0\}$. Let $\underline{\ell}=I_1\sqcup I_2$ be determined by $D^{(j)}=\bigcup_{i\in I_j}\{z_i=0\}$ for $j=1,2$. Let $\widetilde{j}$ denote the inclusion $X-D\longrightarrow\widetilde{X}(D)$. Let $\mathcal{A}^{<D^{(1)}\leq D^{(2)}}_{\widetilde{X}(D)}$ be the image of the naturally defined morphisms:

$$\mathcal{O}^{<D^{(1)}}_{\widetilde{X}(D)}\otimes\mathrm{Nil}(z_i\,|\,i\in I_2)\longrightarrow\widetilde{j}_*\mathcal{O}_{X-D}.$$

Similarly, let $\mathcal{C}^{\infty\,<D^{(3)}\leq D^{(2)}}_{\widetilde{X}(D)}$ be the image of the naturally defined morphisms:

$$\mathcal{C}^{\infty\,<D^{(3)}}_{\widetilde{X}(D)}\otimes\mathrm{Nil}(z_i\,|\,i\in I_2)\longrightarrow\widetilde{j}_*\mathcal{C}^{\infty\,<D^\circ}_{X-D}.$$

We can observe that they are independent of the choice of a coordinate $(z_1,\ldots,z_n)$. Hence, we obtain globally defined sheaves $\mathcal{A}^{<D^{(1)}\leq D^{(2)}}_{\widetilde{X}(D)}$ and $\mathcal{C}^{\infty<D^{(3)}\leq D^{(2)}}_{\widetilde{X}(D)}$ on $\widetilde{X}(D)$. They are also denoted by $\mathcal{A}^{\mathrm{nil}\,<D^{(1)}}_{\widetilde{X}(D)}$ and $\mathcal{C}^{\infty\ \mathrm{nil}\,<D^{(3)}}_{\widetilde{X}(D)}$. We put $\Omega^{0,\bullet\,<D^{(3)}\leq D^{(2)}}_{\widetilde{X}(D)} := \Omega^{0,\bullet}_{\widetilde{X}(D)}\otimes_{\mathcal{C}^\infty_{\widetilde{X}(D)}}\mathcal{C}^{\infty<D^{(3)}\leq D^{(2)}}_{\widetilde{X}(D)}$. We will show the following theorem in Subsection 4.2.7. (Actually, more refined claims will be proved.)

Theorem 4.8

- $\Omega^{0,\bullet\,<D^{(1)}\leq D^{(2)}}_{\widetilde{X}(D)}$ *naturally gives a c-soft resolution of* $\mathcal{A}^{<D^{(1)}\leq D^{(2)}}_{\widetilde{X}(D)}$. *(The case $D^\circ=\emptyset$.)*
- *The sheaves* $\mathcal{A}^{<D^{(1)}\leq D^{(2)}}_{\widetilde{X}(D)}$ *and* $\Omega^{0,\bullet\,<D^{(3)}\leq D^{(2)}}_{\widetilde{X}(D)}$ *are flat over* $\pi^{-1}\mathcal{O}_X$.

Let $D^{(i)} = \bigcup_{j\in\Lambda_i} D_j^{(i)}$ $(i=1,2)$ be the irreducible decomposition. Fix $k\in\Lambda_1\sqcup\Lambda_2$. We put

$$E^{(i)} := \bigcup_{j\in\Lambda_i\setminus\{k\}} D_j^{(i)} \quad (i=1,2).$$

We put $E := E^{(1)}\cup E^{(2)}$ and $E^{(3)} := D^{(3)}$. We have the naturally defined projection $\rho : \widetilde{X}(D) \longrightarrow \widetilde{X}(E)$. We will prove the following theorem in Subsection 4.2.8.

Theorem 4.9 *If $k\in\Lambda_1$, the following naturally defined morphism is an isomorphism:*

$$\Omega^{0,\bullet\,<E^{(3)}\leq E^{(2)}}_{\widetilde{X}(E)} \longrightarrow \rho_*\Omega^{0,\bullet\,<D^{(3)}\leq D^{(2)}}_{\widetilde{X}(D)}$$

If $k\in\Lambda_2$, the following naturally defined morphism is a quasi-isomorphism:

$$\Omega^{0,\bullet\,<E^{(3)}\leq E^{(2)}}_{\widetilde{X}(E)}(*D_k^{(2)}) \longrightarrow \rho_*\Omega^{0,\bullet\,<D^{(3)}\leq D^{(2)}}_{\widetilde{X}(D)}$$

Corollary 4.10 *The natural morphism*

$$\Omega^{0,\bullet\,<D^{(1)}}_{X}(*D^{(2)}) \longrightarrow \pi_*\Omega^{0,\bullet\,<D^{(1)}\leq D^{(2)}}_{\widetilde{X}(D)}$$

is a quasi-isomorphism. In particular, $R\pi_*\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)} \simeq \mathcal{O}_X(*D)$.

For the proof of the theorems, we may assume $X=\Delta^n$ and $D=\bigcup_{i=1}^{\ell}\{z_i=0\}$ and $D^\circ=\bigcup_{i=\ell+1}^{m}\{z_i=0\}$, where $1\leq\ell\leq m\leq n$. We set $D_i := \{z_i=0\}$ for $i=1,\ldots,m$. We use the notation in Subsection 4.1. For a subset $J\subset\underline{\ell}$, we set $\overline{J} := J\sqcup\big(\underline{m}\setminus\underline{\ell}\big)$.

4.2.3 Sheaves of holomorphic functions of Nilsson type

For any real analytic subset $Z\subset\widetilde{X}(D)$, we implicitly regard $\mathcal{O}_{\widehat{Z}}$ as the sheaf on $\widetilde{X}(D)$ in a natural way. For any $I\sqcup J\subset\underline{\ell}$, let $\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)}}$ denote the image of the following naturally defined morphism:

$$\mathcal{O}^{<D(J)}_{\widehat{\pi^{-1}(D_I)}}\otimes_{\mathbb{C}[z_1,\ldots,z_\ell]}\mathrm{Nil}(z_1,\ldots,z_\ell) \longrightarrow \mathcal{O}^{<D(J)}_{\pi^{-1}\widehat{(D_I\setminus\partial D_I)}}\otimes_{\mathbb{C}[z_i\,|\,i\in I]}\mathrm{Nil}(z_i\,|\,i\in I)$$

In the case $I=\emptyset$, it is $\mathcal{A}^{\mathrm{nil}<D(J)}_{\widetilde{X}(D)}$. For $I\sqcup J\subset\underline{\ell}$, let $\mathcal{A}^{\mathrm{nil}}_{\pi^{-1}\widehat{(D_I(J))}}$ denote the image of the following naturally defined morphism:

$$\mathcal{O}_{\pi^{-1}\widehat{(D_I(J))}}\otimes_{\mathbb{C}[z_1,\ldots,z_\ell]}\mathrm{Nil}(z_1,\ldots,z_\ell) \longrightarrow \bigoplus_{j\in J}\mathcal{O}_{\pi^{-1}\widehat{(D_{Ij}\setminus\partial D_{Ij})}}\otimes_{\mathbb{C}[z_i\,|\,i\in Ij]}\mathrm{Nil}(z_i\,|\,i\in Ij)$$

Here, $Ij := I\sqcup\{j\}$. In particular, $\mathcal{A}^{\mathrm{nil}}_{\pi^{-1}\widehat{(D(J))}}$ is the image of the following morphism:

$$\mathcal{O}_{\pi^{-1}\widehat{(D(J))}}\otimes_{\mathbb{C}[z_1,\ldots,z_\ell]}\mathrm{Nil}(z_1,\ldots,z_\ell) \longrightarrow \bigoplus_{j\in J}\mathcal{O}_{\pi^{-1}\widehat{(D_j\setminus\partial D_j)}}\otimes_{\mathbb{C}[z_j]}\mathrm{Nil}\big(z_j\big)$$

Let $\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)},T,N}$ and $\mathcal{A}^{\mathrm{nil}}_{\pi^{-1}\widehat{(D_I(J))},T,N}$ be the sheaves obtained from $\mathrm{Nil}_{T,N}(z_1,\ldots,z_\ell)$ instead of $\mathrm{Nil}(z_1,\ldots,z_\ell)$. For $T\subset T'$ and $N\leq N'$, we have natural inclusions $\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)},T,N}\subset\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)},T',N'}$ and $\mathcal{A}^{\mathrm{nil}}_{\pi^{-1}\widehat{(D_I(J))},T,N}\subset\mathcal{A}^{\mathrm{nil}}_{\pi^{-1}\widehat{(D_I(J))},T',N'}$. We have the following natural isomorphisms:

$$\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)}}\simeq\varinjlim\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)},T,N}\qquad\mathcal{A}^{\mathrm{nil}}_{\pi^{-1}\widehat{(D_I(J))}}\simeq\varinjlim\mathcal{A}^{\mathrm{nil}}_{\pi^{-1}\widehat{(D_I(J))},T,N}\tag{33}$$

Let $q_I:\pi^{-1}(D_I)\longrightarrow\widetilde{D}_I(\partial D_I)$ denote the projection. Let $\pi_I:\widetilde{D}_I(\partial D_I)\longrightarrow D_I$ be the real blow up. Then, we have

$$\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)},T,N}=q_I^{-1}\mathcal{A}^{\mathrm{nil}\,<D_I(J)}_{\widetilde{D}_I(\partial D_I),T,N}[\![z_i\,|\,i\in I]\!]\otimes_{\mathbb{C}[z_i|i\in I]}\mathrm{Nil}_{T,N}(z_i\,|\,i\in I)\tag{34}$$

$$\mathcal{A}^{\mathrm{nil}}_{\pi^{-1}\widehat{(D_I(J))},T,N}=q_I^{-1}\mathcal{A}^{\mathrm{nil}}_{\pi_I^{-1}\widehat{(D_I(J))},T,N}[\![z_i\,|\,i\in I]\!]\otimes_{\mathbb{C}[z_i|i\in I]}\mathrm{Nil}_{T,N}(z_i\,|\,i\in I)\tag{35}$$

4.2.4 Specialization

Let us construct a morphism $\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I)}} \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}}$ for any $I \sqcup J \subset \underline{\ell}$. First, let us construct $\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)} \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D)}}$ in the case $D = D_1$. Let Φ denote the natural morphism $\Phi : \mathcal{O}_{\widetilde{X}(D)} \otimes \mathrm{Nil}(z_1) \longrightarrow \widetilde{j}_*\mathcal{O}_{X-D}$, where $\widetilde{j} : X - D \longrightarrow \widetilde{X}(D)$.

Lemma 4.11 *Assume that $D = D_1$. Let $\mathcal{S} \subset \mathbb{C}$ be a finite subset such that the induced map $\mathcal{S} \longrightarrow \mathbb{C}/\mathbb{Z}$ is injective. Assume that we are given $f = \sum_{\alpha\in\mathcal{S}} \sum_{j=0}^{M} f_{\alpha,j} \otimes \varphi_{\alpha,j}(z_1) \in \mathcal{O}_{\widetilde{X}(D)} \otimes \mathrm{Nil}(z_1)$ such that $\Phi(f) \in \mathcal{O}^{<D}_{\widetilde{X}(D)}$. Then, we have $f_{\alpha,j} \in \mathcal{O}^{<D}_{\widetilde{X}(D)}$. In particular, we have the well defined map $\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)} \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D)}}$ in the case $D = \{z_1 = 0\}$.*

Proof Let us consider the growth order of $f_{\alpha,j}\, z_1^{\alpha} (\log z_1)^j$. For the polar coordinate $z_1 = re^{\sqrt{-1}\theta}$, we have $z_1^{\alpha} = \exp\bigl(\beta \log r - \gamma\theta + \sqrt{-1}(\gamma \log r + \beta\theta)\bigr)$, where $\beta = \mathrm{Re}\,\alpha$ and $\gamma = \mathrm{Im}\,\alpha$. Let V be the set of $(\alpha, j) \in \mathcal{S} \times \mathbb{Z}_{\geq 0}$ such that $f_{\alpha,j}$ is not contained in $\mathcal{O}^{<D}_{\widetilde{X}(D)}$. We will derive a contradiction by assuming $V \neq \emptyset$. For each $(\alpha, j) \in V$, there exists a unique integer $m(\alpha, j)$ such that (i) $h_{\alpha,j} := z_1^{-m(\alpha,j)} f_{\alpha,j} \in \mathcal{O}_{\widetilde{X}(D)}$, (ii) $h_{\alpha,j|\pi^{-1}(D)}$ is not constantly 0. We set

$$\kappa := \max_{(\alpha,j)\in V} \bigl\{\mathrm{Re}\,\alpha + m(\alpha, j)\bigr\}, \quad S := \bigl\{(\alpha, j) \in V \bigm| \mathrm{Re}\,\alpha + m(\alpha, j) = \kappa\bigr\}$$

For $(\alpha_1, j_1), (\alpha_2, j_2) \in S$, we have $\mathrm{Re}\,\alpha_1 = \mathrm{Re}\,\alpha_2$ and $m(\alpha_1, j_1) = m(\alpha_2, j_2)$. We also have $\mathrm{Im}\,\alpha_1 \neq \mathrm{Im}\,\alpha_2$ if $\alpha_1 \neq \alpha_2$. We obtain the following estimate for some $\epsilon > 0$:

$$\begin{aligned}\sum_{(\alpha,j)\in V} h_{\alpha,j|\pi^{-1}(D)}\, z_1^{\alpha+m(\alpha,j)} (\log z_1)^j &= r^{\kappa}\Bigl(\sum_{(\alpha,j)\in V} h_{\alpha,j|\pi^{-1}(D)}\, e^{-\mathrm{Im}\,\alpha\theta + \sqrt{-1}(\mathrm{Im}\,\alpha \log r + \mathrm{Re}\,\alpha\theta)} (\log z_1)^j\Bigr) \\ &= O(r^{\kappa+\epsilon}) \qquad (36)\end{aligned}$$

Let us deduce that $h_{\alpha,j|\pi^{-1}(D)}$ are constantly 0 from (36). Assume the contrary. Let $Q \in \pi^{-1}(D)$ at which $h_{\alpha,j}(Q) \neq 0$ for one of $(\alpha, j) \in V$. We may assume $\theta(Q) = 0$. We obtain the following from (36):

$$\sum_{(\alpha,j)\in V} h_{\alpha,j}(Q) \cdot e^{\sqrt{-1}\,\mathrm{Im}\,\alpha \log r} (\log r)^j = O(r^{\epsilon}) \qquad (37)$$

But, for any $\delta > 0$, we can take $0 < r < \delta$ such that the amplitudes of the complex numbers

$$(-1)^j h_{\alpha,j}(Q)\, e^{\sqrt{-1}\,\mathrm{Im}\,\alpha \log r} \quad (\alpha, j) \in V$$

are sufficiently close, which contradicts with (37). Hence, $h_{\alpha,j}$ $(\alpha, j) \in V$ are constantly 0. Thus, we obtain Lemma 4.11. ∎

Let us return to the general case. We take $\mathcal{S} \subset \mathbb{C}$ such that the induced map $\mathcal{S} \longrightarrow \mathbb{C}/\mathbb{Z}$ is bijective. Let $q_i : (\mathcal{S} \times \mathbb{Z})^{\ell} \longrightarrow \mathcal{S} \times \mathbb{Z}$ be the projection onto the i-th component, and $\pi_i : (\mathcal{S} \times \mathbb{Z})^{\ell} \longrightarrow (\mathcal{S} \times \mathbb{Z})^{\ell-1}$ be the projection forgetting the i-th component. For a given

$$\sum_{(\boldsymbol{\alpha},\boldsymbol{k})\in\mathcal{S}^{\ell}\times\mathbb{Z}^{\ell}_{\geq 0}} A_{\boldsymbol{\alpha},\boldsymbol{k}} \otimes \varphi_{\boldsymbol{\alpha},\boldsymbol{k}} \in \mathcal{O}_{\widetilde{X}(D)} \otimes \mathrm{Nil}(z_1, \ldots, z_{\ell}),$$

we set ${}^iF_{\beta,j} := \sum_{q_i(\boldsymbol{\alpha},\boldsymbol{k})=(\beta,j)} A_{\boldsymbol{\alpha},\boldsymbol{k}} \cdot \varphi_{\pi_i(\boldsymbol{\alpha},\boldsymbol{k})}(z_j \,|\, j \neq i)$. Put $i^c := \underline{\ell} - \{i\}$. If $\sum A_{\boldsymbol{\alpha},\boldsymbol{k}} \cdot \varphi_{\boldsymbol{\alpha},\boldsymbol{k}} \in \mathcal{O}^{<D_i}_{\widetilde{X}(D)\setminus\pi^{-1}(D(i^c))}$, we obtain ${}^iF_{\beta,j|\pi^{-1}\widehat{(D_i\setminus\partial D_i)}} = 0$ by applying Lemma 4.11 to $\sum {}^iF_{\beta,j} \cdot \varphi_{\beta,j}(z_i)$. It implies that the morphism

$$\mathcal{O}_{\widetilde{X}(D)} \otimes \mathrm{Nil}(z_1, \ldots, z_{\ell}) \longrightarrow \mathcal{O}_{\widehat{\pi^{-1}(D_i)}} \otimes \mathrm{Nil}(z_1, \ldots, z_{\ell}) \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_i)}}$$

factors through $\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)}$. Hence, we have a well defined morphism $\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)} \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_i)}}$. By construction, it is surjective. We also obtain that the following morphism factors through $\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)}$:

$$\mathcal{O}_{\widetilde{X}(D)} \otimes \mathrm{Nil}(z_1, \ldots, z_{\ell}) \longrightarrow \mathcal{O}_{\widehat{\pi^{-1}(D(I))}} \otimes \mathrm{Nil}(z_1, \ldots, z_{\ell}) \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D(I))}} \subset \bigoplus_{i\in I} \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_i)}}$$

Hence, we obtain the well defined map $\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)} \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\pi^{-1}\widehat{(D(I))}}$. We also obtain $\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D),T,N} \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\pi^{-1}\widehat{(D(I))},T,N}$. They are surjective by construction. By using (33), (34) and (35), we also obtain the surjective morphisms $\mathcal{A}^{\mathrm{nil}}_{\pi^{-1}\widehat{(D_I)}} \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\pi^{-1}\widehat{(D_I(J))}}$ and $\mathcal{A}^{\mathrm{nil}}_{\pi^{-1}\widehat{(D_I)},T,N} \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\pi^{-1}\widehat{(D_I(J))},T,N}$.

Lemma 4.12 *We have the following:*

$$\mathcal{A}^{\mathrm{nil}<D(J)}_{\pi^{-1}\widehat{(D_I)}} = \mathrm{Ker}\Big(\mathcal{A}^{\mathrm{nil}}_{\pi^{-1}\widehat{(D_I)}} \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\pi^{-1}\widehat{(D_I(J))}}\Big)$$

$$\mathcal{A}^{\mathrm{nil}<D(J)}_{\pi^{-1}\widehat{(D_I)},T,N} = \mathrm{Ker}\Big(\mathcal{A}^{\mathrm{nil}}_{\pi^{-1}\widehat{(D_I)},T,N} \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\pi^{-1}\widehat{(D_I(J))},T,N}\Big)$$

Proof The implication $\subset$ is clear. Let us show the converse. First, we consider the case $I=\emptyset$. Let $f=\sum A_{\boldsymbol{\alpha},\boldsymbol{k}}\,\varphi_{\boldsymbol{\alpha},\boldsymbol{k}}$ be a section of $\mathrm{Ker}\Big(\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)} \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\pi^{-1}\widehat{(D(J))}}\Big)$. Let us show the following equality on $\pi^{-1}(\widehat{D_K-\partial D_K})$ for any subset $K\subset \underline{\ell}$ such that $K\cap J\neq\emptyset$:

$$\sum_{q_K(\boldsymbol{\alpha},\boldsymbol{k})=(\boldsymbol{\beta},\boldsymbol{j})} A_{\boldsymbol{\alpha},\boldsymbol{k}|\pi^{-1}\widehat{(D_K)}} \prod_{i\notin K}\varphi_{\alpha_i,k_i}(z_i)=0 \tag{38}$$

We use an induction on $|K|$. In the case $|K|=1$, it follows from the assumption. Let $K=K'\sqcup\{j\}$. Assume that we have already known (38) for K'. By using Lemma 4.11, we obtain the claim for K. In particular, we obtain $A_{\boldsymbol{\alpha},\boldsymbol{k}|\pi^{-1}\widehat{(D_{\underline{\ell}})}}=0$.

Note that the expression of f is not unique. We would like to replace $A_{\boldsymbol{\alpha},\boldsymbol{k}}$ such that the following holds:

P(m): $A_{\boldsymbol{\alpha},\boldsymbol{k}|\pi^{-1}\widehat{(D_K)}}=0$ if $|K|\geq m$ and $K\cap J\neq\emptyset$.

We use a descending induction on m. In the case $m=\ell$, it holds as was already shown. Assume that $P(m+1)$ holds. Take $K\subset\underline{\ell}$ such that $|K|=m$ and $K\cap J\neq\emptyset$. We have

$$A_{\boldsymbol{\alpha},\boldsymbol{k}|\pi^{-1}\widehat{(D_K)}}\prod_{i\notin K}\varphi_{\alpha_i,k_i}(z_i)\in\mathcal{O}^{<D(K^c)}_{\pi^{-1}\widehat{(D_K)}}.$$

By a generalized Borel-Ritt theorem due to Majima and Sabbah, we can take $G_{\boldsymbol{\alpha},\boldsymbol{k}}\in\mathcal{O}^{<D(K^c)}_{\widetilde{X}(D)}$ satisfying $G_{\boldsymbol{\alpha},\boldsymbol{k}|\pi^{-1}\widehat{(D_K)}}=A_{\boldsymbol{\alpha},\boldsymbol{k}|\pi^{-1}\widehat{(D_K)}}\prod_{i\notin K}\varphi_{\alpha_i,k_i}(z_i)$. By (38), the following holds:

$$\sum_{q_K(\boldsymbol{\alpha},\boldsymbol{k})=(\boldsymbol{\beta},\boldsymbol{j})} G_{\boldsymbol{\alpha},\boldsymbol{k}|\pi^{-1}\widehat{(D_K)}}=0$$

We have the following equality:

$$f=\sum_{\boldsymbol{\alpha},\boldsymbol{k}}\Big(A_{\boldsymbol{\alpha},\boldsymbol{k}}-\frac{G_{\boldsymbol{\alpha},\boldsymbol{k}}}{\prod_{i\notin K}\varphi_{\alpha_i,k_i}(z_i)}\Big)\cdot\varphi_{\boldsymbol{\alpha},\boldsymbol{k}}(z_1,\ldots,z_\ell)+\sum_{\boldsymbol{\beta},\boldsymbol{j}}\Big(\sum_{q_K(\boldsymbol{\alpha},\boldsymbol{k})=(\boldsymbol{\beta},\boldsymbol{j})}G_{\boldsymbol{\alpha},\boldsymbol{k}}\Big)\cdot\varphi_{\boldsymbol{\beta},\boldsymbol{j}}(z_i|i\in K)$$

Note that $\sum_{q_K(\boldsymbol{\alpha},\boldsymbol{k})=(\boldsymbol{\beta},\boldsymbol{j})}G_{\boldsymbol{\alpha},\boldsymbol{k}}$ is 0 on $\pi^{-1}\widehat{(D_K)}\cup\pi^{-1}\widehat{(D(K^c))}$. In particular, it is 0 on $\bigcup_{|K_1|=m}\pi^{-1}\widehat{(D_{K_1})}$. By construction, $A_{\boldsymbol{\alpha},\boldsymbol{k}}-G_{\boldsymbol{\alpha},\boldsymbol{k}}\prod_{i\notin K}\varphi_{\alpha_i,k_i}(z_i)^{-1}$ vanishes on $\pi^{-1}\widehat{(D_K)}$. Moreover, if $A_{\boldsymbol{\alpha},\boldsymbol{k}|\pi^{-1}\widehat{(D_L)}}=0$ for some $|L|=m$ with $L\cap J\neq\emptyset$, $A_{\boldsymbol{\alpha},\boldsymbol{k}}-G_{\boldsymbol{\alpha},\boldsymbol{k}}\prod_{i\notin K}\varphi_{\alpha_i,k_i}(z_i)^{-1}$ also vanishes on $\pi^{-1}\widehat{(D_L)}$. Hence, by applying the above procedure to each K satisfying $|K|=m$ and $K\cap J\neq\emptyset$, we can arrive at $P(m)$. The status $P(0)$ means $f=\sum A_{\boldsymbol{\alpha},\boldsymbol{k}}\,\varphi_{\boldsymbol{\alpha},\boldsymbol{k}}$ with $A_{\boldsymbol{\alpha},\boldsymbol{k}}\in\mathcal{O}^{<D(J)}_{\widetilde{X}(D)}$, which implies that $f\in\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widetilde{X}(D)}$. Thus, we are done in the case $I=\emptyset$. We can reduce the general case to the case $I=\emptyset$ by using (33), (34) and (35). ∎

4.2.5 A resolution

Put $\mathcal{T}(m,I,J):=\bigl\{K\subset J\bigm| I\subset K,\, |K|=|I|+m+1\bigr\}$ for $m\geq 0$. We set

$$\mathcal{K}^m\Bigl(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}}\Bigr):=\bigoplus_{K\in\mathcal{T}(m,I,J)}\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_K)}}.$$

We obtain the complex $\mathcal{K}^{\bullet}\Bigl(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}}\Bigr)$ as in Subsection 2.1.4.

Lemma 4.13 *The* 0*-th cohomology of* $\mathcal{K}^{\bullet}\Bigl(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}}\Bigr)$ *is* $\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}}$*, and the higher cohomology sheaves are* 0*. Similarly, The* 0*-th cohomology of* $\mathcal{K}^{\bullet}\Bigl(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))},T,N}\Bigr)$ *is* $\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))},T,N}$*, and the higher cohomology sheaves are* 0*.*

Proof We have only to consider the issue for $\mathcal{K}^{\bullet}\Bigl(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))},T,N}\Bigr)$. First, let us consider the case $I=\emptyset$. We use an induction on $|J|$ and the dimension of X. The cases $|J|=1$ or $\dim X=1$ are clear. Let $J=J_0\sqcup\{j\}$. Assume that the claim holds for J_0. We set $\mathcal{L}^m_{T,N}:=\bigoplus_{|K|=m+1,j\in K\subset J}\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_K)},T,N}$. We have the exact sequence:

$$0\longrightarrow\mathcal{L}^{\bullet}_{T,N}\longrightarrow\mathcal{K}^{\bullet}\Bigl(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D(J))},T,N}\Bigr)\longrightarrow\mathcal{K}^{\bullet}\Bigl(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D(J_0))},T,N}\Bigr)\longrightarrow 0$$

Let $q_j:\pi^{-1}(D_j)\longrightarrow\widetilde{D}_j(\partial D_j)$ and $\pi_j:\widetilde{D}_j(\partial D_j)\longrightarrow D_j$ be the projections. We have a natural isomorphism:

$$\mathcal{L}^{\bullet}_{T,N}\simeq q_j^{-1}\mathcal{K}^{\bullet}\Bigl(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi_j^{-1}(D_j\cap D(J_0))},T,N}\Bigr)[\![z_j]\!]\otimes_{\mathbb{C}[z_j]}\mathrm{Nil}_{T,N}(z_j)$$

By the hypothesis of the induction, we obtain the vanishing of the higher cohomology sheaves of $\mathcal{L}^{\bullet}_{T,N}$ and $\mathcal{K}^{\bullet}\Bigl(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D(J_0))},T,N}\Bigr)$. Hence, we obtain the vanishing of the higher cohomology of $\mathcal{K}^{\bullet}\Bigl(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D(J))},T,N}\Bigr)$. The calculation of the 0-th cohomology is easy. The general case can be easily reduced to the case $I=\emptyset$ by (33), (34) and (35). ∎

4.2.6 The C^{∞}-version

Let Y be a C^{∞}-manifold. For $I\sqcup J\subset\underline{\ell}$, let $\mathcal{C}^{\infty\ \mathrm{nil}<D(\overline{J})}_{\widehat{\pi^{-1}(D_I)}\times Y}$ denote the image of the following morphism:

$$\mathcal{C}^{\infty\,<D(\overline{J})}_{\widehat{\pi^{-1}(D_I)}\times Y}\otimes_{\mathbb{C}[z_i|i\in J^c]}\mathrm{Nil}(z_i\,|\,i\in J^c)\longrightarrow\mathcal{C}^{\infty\,<D^{\circ}}_{\widehat{\pi^{-1}(D_I\setminus\partial D_I)}\times Y}\otimes_{\mathbb{C}[z_i|i\in I]}\mathrm{Nil}(z_i\,|\,i\in I)$$

Let $\mathcal{C}^{\infty\ \mathrm{nil}<D^{\circ}}_{\widehat{\pi^{-1}(D_I(J))}\times Y}$ be the image of the following morphism:

$$\mathcal{C}^{\infty\,<D^{\circ}}_{\widehat{\pi^{-1}(D_I(J))}\times Y}\otimes_{\mathbb{C}[z_1,\ldots,z_\ell]}\mathrm{Nil}(z_1,\ldots,z_\ell)\longrightarrow\bigoplus_{j\in J}\mathcal{C}^{\infty\,<D^{\circ}}_{\widehat{\pi^{-1}(D_{Ij}-\partial D_{Ij})}\times Y}\otimes_{\mathbb{C}[z_i|i\in Ij]}\mathrm{Nil}\bigl(z_i\bigm| i\in Ij\bigr)$$

In particular, $\mathcal{C}^{\infty\ \mathrm{nil}<D^{\circ}}_{\widehat{\pi^{-1}(D(J))}\times Y}$ is the image of the following morphism:

$$\mathcal{C}^{\infty\,<D^{\circ}}_{\widehat{\pi^{-1}(D(J))}\times Y}\otimes_{\mathbb{C}[z_1,\ldots,z_\ell]}\mathrm{Nil}(z_1,\ldots,z_\ell)\longrightarrow\bigoplus_{j\in J}\mathcal{C}^{\infty\,<D^{\circ}}_{\widehat{\pi^{-1}(D_j-\partial D_j)}\times Y}\otimes_{\mathbb{C}[z_j]}\mathrm{Nil}\bigl(z_j\bigr)$$

Similarly, let $\mathcal{C}^{\infty\ \mathrm{nil}<D(\overline{J})}_{\widehat{\pi^{-1}(D_I)}\times Y,T,N}$ and $\mathcal{C}^{\infty\ \mathrm{nil}<D^{\circ}}_{\widehat{\pi^{-1}(D_I(J))}\times Y,T,N}$ denote the sheaves obtained from $\mathrm{Nil}_{T,N}(z_1,\ldots,z_\ell)$ instead of $\mathrm{Nil}(z_1,\ldots,z_\ell)$. We have

$$\mathcal{C}^{\infty\ \mathrm{nil}<D(\overline{J})}_{\widehat{\pi^{-1}(D_I)}\times Y,T,N}=\mathcal{C}^{\infty\ \mathrm{nil}<D_I(\overline{J})}_{\widetilde{D}_I(\partial D_I)\times(S^1)^{|I|}\times Y,T,N}[\![z_i|i\in I]\!]\otimes_{\mathbb{C}[z_i|i\in I]}\mathrm{Nil}_{T,N}(z_i|i\in I)\tag{39}$$

$$\mathcal{C}^{\infty\ \mathrm{nil}\,<D^\circ}_{\widehat{\pi^{-1}(D_I(J))}\times Y,T,N}=\mathcal{C}^{\infty\ \mathrm{nil}\,<D^\circ\cap D_I}_{\widehat{\pi_I^{-1}(D_I(J))\times(S^1)^{|I|}}\times Y,T,N}[\![z_i|i\in I]\!]\otimes_{\mathbb{C}[z_i|i\in I]}\mathrm{Nil}_{T,N}(z_i|i\in I) \tag{40}$$

By the argument in Subsection 4.2.4, we obtain the well defined surjective morphisms:

$$\mathcal{C}^{\infty\ \mathrm{nil}\,<D^\circ}_{\widehat{\pi^{-1}(D_I)}\times Y}\longrightarrow\mathcal{C}^{\infty\ \mathrm{nil}\,<D^\circ}_{\widehat{\pi^{-1}(D_I(J))}\times Y},\qquad \mathcal{C}^{\infty\ \mathrm{nil}\,<D^\circ}_{\widehat{\pi^{-1}(D_I)}\times Y,T,N}\longrightarrow\mathcal{C}^{\infty\ \mathrm{nil}\,<D^\circ}_{\widehat{\pi^{-1}(D_I(J))}\times Y,T,N} \tag{41}$$

By the argument in the proof of Lemma 4.12, we can show that the kernels of the morphisms in (41) are $\mathcal{C}^{\infty\ \mathrm{nil}\,<D(\overline{J})}_{\widehat{\pi^{-1}(D_I)}\times Y}$ and $\mathcal{C}^{\infty\ \mathrm{nil}\,<D(\overline{J})}_{\widehat{\pi^{-1}(D_I)}\times Y,T,N}$, respectively.

We set $\mathcal{K}^m\Bigl(\mathcal{C}^{\infty\ \mathrm{nil}\,<D^\circ}_{\widehat{\pi^{-1}(D_I(J))}\times Y}\Bigr):=\bigoplus_{K\in\mathcal{T}(m,I,J)}\mathcal{C}^{\infty\ \mathrm{nil}\,<D^\circ}_{\widehat{\pi^{-1}(D_K)}\times Y}$. We obtain the complex $\mathcal{K}^\bullet\Bigl(\mathcal{C}^{\infty\ \mathrm{nil}\,<D^\circ}_{\widehat{\pi^{-1}(D_I(J))}\times Y}\Bigr)$. It is easy to see that the 0-th cohomology is $\mathcal{C}^{\infty\ \mathrm{nil}\,<D^\circ}_{\widehat{\pi^{-1}(D_I(J))}\times Y}$. By using an argument in the proof of Lemma 4.13, we can show the vanishing of the higher cohomology. Similar claims hold for $\mathcal{K}^\bullet\Bigl(\mathcal{C}^{\infty\ \mathrm{nil}\,<D^\circ}_{\widehat{\pi^{-1}(D_I(J))}\times Y,T,N}\Bigr)$.

4.2.7 Proof of Theorem 4.8

In this subsection, we do not consider D°. We put $\Omega^{0,\bullet\ \mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)}}:=\Omega^{0,\bullet}_{\widetilde{X}(D)}\otimes_{\mathcal{C}^\infty_{\widetilde{X}(D)}}\mathcal{C}^{\infty\,\mathrm{nil}<D(J)}_{\widehat{\pi^{-1}(D_I)}}$ and $\Omega^{0,\bullet\ \mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}}:=\Omega^{0,\bullet}_{\widetilde{X}(D)}\otimes_{\mathcal{C}^\infty_{\widetilde{X}(D)}}\mathcal{C}^{\infty\ \mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}}$. We use the symbols $\Omega^{0,\bullet\ \mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)},T,N}$ and $\Omega^{0,\bullet\ \mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))},T,N}$ in similar meanings. The following proposition implies the first claim of Theorem 4.8.

Proposition 4.14 $\Omega^{0,\bullet\ \mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)}}$ *and* $\Omega^{0,\bullet\ \mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}}$ *give c-soft resolutions of* $\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)}}$ *and* $\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}}$*, respectively. Similarly,* $\Omega^{0,\bullet\ \mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)},T,N}$ *and* $\Omega^{0,\bullet\ \mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))},T,N}$ *give c-soft resolutions of* $\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)},T,N}$ *and* $\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))},T,N}$*, respectively.*

Proof We use an induction on $\dim X$. In the case $\dim X=0$, the claim is trivial. Let us show the claim for $\widehat{\pi^{-1}(D_I)}$. For $I\neq\emptyset$, let $q_I:\pi^{-1}(D_I)\longrightarrow\widetilde{D}_I(\partial D_I)$ denote the naturally induced morphism. We put $\widehat{\mathrm{Nil}}_{T,N}(I):=\mathbb{C}[\![z_i|i\in I]\!]\otimes_{\mathbb{C}[z_i|i\in I]}\mathrm{Nil}_{T,N}(z_i|i\in I)$. By using the hypothesis of the induction and a formal calculation as in Lemma 4.3, we can show that the following morphisms are quasi-isomorphisms:

$$q_I^{-1}\mathcal{A}^{\mathrm{nil}\,<D_I(J)}_{\widetilde{D}_I(\partial D_I),T,N}\otimes\widehat{\mathrm{Nil}}_{T,N}(I)\longrightarrow q_I^{-1}\Omega^{0,\bullet\ \mathrm{nil}\,<D_I(J)}_{\widetilde{D}_I(\partial D_I),T,N}\otimes\widehat{\mathrm{Nil}}_{T,N}(I)\longrightarrow\Omega^{0,\bullet\ \mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)},T,N}$$

It implies the claim for $\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)},T,N}$. We obtain the claim for $\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)}}$ from (33). For any subset $I\subset\underline{\ell}$ (I can be $\emptyset$), by using the resolutions $\mathcal{K}^\bullet\bigl(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}}\bigr)$ and $\mathcal{K}^\bullet\bigl(\Omega^{0,\bullet\ \mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}}\bigr)$, we can reduce the claim for $\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}}$ to the claims for $\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_K)}}$ ($I\subsetneq K$). The claim for $\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))},T,N}$ can be obtained in a similar way. By using the exact sequences

$$0\longrightarrow\Omega^{0,\bullet\,<D}_{\widetilde{X}(D)}\longrightarrow\Omega^{0,\bullet\ \mathrm{nil}}_{\widetilde{X}(D)}\longrightarrow\Omega^{0,\bullet\ \mathrm{nil}}_{\widehat{\pi^{-1}(D)}}\longrightarrow 0,\qquad 0\longrightarrow\mathcal{O}^{\leq D}_{\widetilde{X}(D)}\longrightarrow\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)}\longrightarrow\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D)}}\longrightarrow 0, \tag{42}$$

we obtain the claim for $\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)}$. By using the exact sequences

$$0\longrightarrow\Omega^{0,\bullet\,<D(J)}_{\widetilde{X}(D)}\longrightarrow\Omega^{0,\bullet\ \mathrm{nil}}_{\widetilde{X}(D)}\longrightarrow\Omega^{0,\bullet\ \mathrm{nil}}_{\widehat{\pi^{-1}(D(J))}}\longrightarrow 0,\qquad 0\longrightarrow\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widetilde{X}(D)}\longrightarrow\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)}\longrightarrow\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D(J))}}\longrightarrow 0,$$

we obtain the claim for $\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widetilde{X}(D)}$. The claims for $\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D),T,N}$ and $\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widetilde{X}(D),T,N}$ can be obtained similarly. ∎

The following proposition implies the second claim of Theorem 4.8.

Proposition 4.15 *The sheaves* $\mathcal{C}^{\infty\ \mathrm{nil}\,<D(\overline{J})}_{\widehat{\pi^{-1}(D_I)}}$, $\mathcal{C}^{\infty\ \mathrm{nil}\,<D^\circ}_{\widehat{\pi^{-1}(D_I(J))}}$, $\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)}}$ *and* $\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}}$ *are flat over* $\pi^{-1}\mathcal{O}_X$*. The sheaves* $\mathcal{C}^{\infty\ \mathrm{nil}\,<D(\overline{J})}_{\widehat{\pi^{-1}(D_I)},T,N}$, $\mathcal{C}^{\infty\ \mathrm{nil}\,<D^\circ}_{\widehat{\pi^{-1}(D_I(J))},T,N}$, $\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)},T,N}$ *and* $\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))},T,N}$ *are also flat over* $\pi^{-1}\mathcal{O}_X$*.*

Proof We have $\mathcal{C}^{\infty\,\mathrm{nil}\,<D(I^c)}_{\widehat{\pi^{-1}(D_I)}}=\mathcal{C}^{\infty<D(I^c)}_{\widehat{\pi^{-1}(D_I)}}\otimes_{\mathbb{C}[z_i|i\in I]}\mathrm{Nil}(z_i|\,i\in I)$, which is flat over $\pi^{-1}\mathcal{O}_X$, according to Lemma 4.7. Then, we can show Proposition 4.15 by the arguments in the last part of the proof of Proposition 4.4. ∎

4.2.8 Proof of Theorem 4.9

The first claim of Theorem 4.9 is obvious. We give a preliminary for the second claim. Put $X' := \mathbb{C}\times X$, $X'_0 := \{0\}\times X$ and $D' := (\mathbb{C}\times D)\cup(\{0\}\times X)$. Let $J\subset \underline{\ell}$. Put $D'(\overline{J}) := \mathbb{C}\times D(\overline{J})$. Let $\pi_0 : \widetilde{X}'(D') \longrightarrow X'$ and $\pi_1 : \mathbb{C}\times\widetilde{X}(D)\longrightarrow \mathbb{C}\times X$ be the real blow up. We have a natural diffeomorphism $\pi_0^{-1}(X'_0)\simeq S^1\times\widetilde{X}(D)$. Let $\rho_0 : \widetilde{X}'(D')\longrightarrow \mathbb{C}\times\widetilde{X}(D)$ be the naturally induced map. We use the coordinate $z = r\,e^{\sqrt{-1}\theta}$ of $\mathbb{C}$. We have a natural inclusion:

$$\mathcal{C}^{\infty\ \mathrm{nil}<D'(\overline{J})}_{\widehat{\pi_1^{-1}(X'_0)}}(*X'_0)\longrightarrow \rho_{0*}\Bigl(\mathcal{C}^{\infty\ \mathrm{nil}<D'(\overline{J})}_{\widehat{\pi_0^{-1}(X'_0)}}\Bigr) \tag{43}$$

The differential operator $\overline{z}\overline{\partial}_z$ induces the endomorphisms of $\mathcal{C}^{\infty\ \mathrm{nil}<D'(\overline{J})}_{\widehat{\pi_1^{-1}(X'_0)}}(*X'_0)$ and $\rho_{0*}\Bigl(\mathcal{C}^{\infty\ \mathrm{nil}<D'(\overline{J})}_{\widehat{\pi_0^{-1}(X'_0)}}\Bigr)$, which are denoted by F_1 and F_2, respectively.

Lemma 4.16 *The cokernel of F_i $(i=1,2)$ are 0, and (43) induces the isomorphism $\operatorname{Ker}F_1\simeq\operatorname{Ker}F_2$.*

Proof It is easy to obtain the vanishing of $\operatorname{Cok}F_1$ by a formal calculation. Let us show the other claims. We take $\mathcal{S}\subset\mathbb{C}$ such that (i) the induced map $\mathcal{S}\longrightarrow\mathbb{C}/\mathbb{Z}$ is bijective, (ii) $0\in\mathcal{S}$. Corresponding to the decomposition $\mathrm{Nil}(z) = \bigoplus_{\alpha\in\mathcal{S}} z^{\alpha}\mathbb{C}[z,z^{-1}]\,[\log z]$, we have the decomposition $\mathcal{C}^{\infty\ \mathrm{nil}<D'(\overline{J})}_{\widehat{\pi_0^{-1}(X'_0)}} = \bigoplus_{\alpha\in\mathcal{S}}\mathcal{C}^{\infty\ \mathrm{nil}<D'(\overline{J})}_{\widehat{\pi_0^{-1}(X'_0)},\alpha}$. Let $U\subset\widetilde{X}(D)$ be an open subset. Let f be a section of $\mathcal{C}^{\infty\ \mathrm{nil}<D'(\overline{J})}_{\widehat{\pi_0^{-1}(X'_0)},\alpha}$ on $S^1\times U\subset\pi_0^{-1}(X'_0)$ expressed as follows:

$$f=\sum_{\boldsymbol{\beta},\boldsymbol{k}}\sum_{n,j} f_{\boldsymbol{\beta},\boldsymbol{k},n,j}\,\varphi_{\boldsymbol{\beta},\boldsymbol{k}}\,e^{-\sqrt{-1}\theta\alpha}z^{\alpha+n}\bigl(\log|z|^2\bigr)^j\qquad\bigl(f_{\boldsymbol{\beta},\boldsymbol{k},n,j}\in\mathcal{C}^{\infty\,<D(\overline{J})}_{S^1\times\widetilde{X}(D)}\bigr)$$

We have the following equality:

$$\begin{aligned}\overline{z}\overline{\partial}_z f = &\sum_{\boldsymbol{\beta},\boldsymbol{k}}\sum_{n,j}\Bigl(\frac{\sqrt{-1}}{2}\partial_\theta+\frac{\alpha}{2}\Bigr)f_{\boldsymbol{\beta},\boldsymbol{k},n,j}\,\varphi_{\boldsymbol{\beta},\boldsymbol{k}}\,e^{-\sqrt{-1}\theta\alpha}z^{\alpha+n}(\log|z|^2)^j\\ &+\sum_{\boldsymbol{\beta},\boldsymbol{k}}\sum_{n,j} f_{\boldsymbol{\beta},\boldsymbol{k},n,j}\varphi_{\boldsymbol{\beta},\boldsymbol{k}}\,e^{-\sqrt{-1}\theta\alpha}z^{\alpha+n}j(\log|z|^2)^{j-1}\end{aligned}\tag{44}$$

For any section g of $\mathcal{C}^{\infty\,<D(\overline{J})}_{S^1\times\widetilde{X}(D)}$ on $S^1\times U$, we can solve the equation

$$\partial_\theta G-\sqrt{-1}\alpha\,G = g\qquad(\alpha\neq 0)$$

in $\mathcal{C}^{\infty\ \mathrm{nil}<D(\overline{J})}_{S^1\times\widetilde{X}(D)}$. We remark $\int_0^{2\pi}e^{-\sqrt{-1}\alpha\theta}g(\theta)\,d\theta=0$. Then, it is easy to obtain $\operatorname{Cok}(\overline{z}\partial_{\overline{z}})=0$ and $\operatorname{Ker}(\overline{z}\partial_{\overline{z}})=0$ in the part $\alpha\neq 0$ by using (44). Let us consider the part $\alpha=0$. We use the filtration with respect to the order of $\log|z|^2$. If we take Gr with respect to this filtration, the second term in (44) with $\alpha=0$ disappears. We obtain $\mathcal{H}^0\,\mathrm{Gr}_j=\mathcal{H}^1\,\mathrm{Gr}_j$ for each j, and they are represented by constants with respect to θ. Then, the second term in (44) induces $\mathcal{H}^0\,\mathrm{Gr}_j\simeq\mathcal{H}^1\,\mathrm{Gr}_{j-1}$ for $j\geq 1$. Hence, we obtain the vanishing of the cokernel of $\overline{z}\partial_{\overline{z}}$, and the kernel is $\mathcal{H}^0\,\mathrm{Gr}_0$. Then, the claim of Lemma 4.16 are clear. ∎

We have the following morphism of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^{0,\bullet\,<D'(\overline{J})\cup X'_0}_{\mathbb{C}\times\widetilde{X}(D)} & \longrightarrow & \Omega^{0,\bullet<D'(\overline{J})}_{\mathbb{C}\times\widetilde{X}(D)}(*X'_0) & \longrightarrow & \Omega^{0,\bullet\,<D'(\overline{J})}_{\widehat{\pi_1^{-1}(X'_0)}} & \longrightarrow & 0\\
 & & {\scriptstyle =}\big\downarrow & & \big\downarrow & & {\scriptstyle\simeq}\big\downarrow & & \\
0 & \longrightarrow & \rho_{0*}\Omega^{0,\bullet\,<D'(\overline{J})\cup X'_0}_{\widetilde{X}'(D')} & \longrightarrow & \rho_{0*}\Omega^{0,\bullet\,<D'(\overline{J})}_{\widetilde{X}(D')} & \longrightarrow & \rho_{0*}\Omega^{0,\bullet\,<D'(\overline{J})}_{\widehat{\pi_0^{-1}(X'_0)}} & \longrightarrow & 0
\end{array}$$

The left vertical arrow is an isomorphism. According to Lemma 4.16, the right vertical arrow is a quasi-isomorphism. Thus, the central vertical arrow is also a quasi-isomorphism, which is the second claim of Theorem 4.9. ∎

5 Some complexes associated to meromorphic flat bundles

5.1 De Rham complex on the real blow up

5.1.1 De Rham complex and a description by dual

Let X be a complex manifold and D be a normal crossing hypersurface with a decomposition $D = D_1 \cup D_2$. (Note that D_i are not necessarily irreducible. See Subsection 3.2.1.) Let $\pi : \widetilde{X}(D) \longrightarrow X$ be the real blow up. We set $\mathcal{A}^{<D_1 \leq D_2}_{\widetilde{X}(D)} := \mathcal{A}^{\mathrm{nil}<D_1}_{\widetilde{X}(D)}$ and $\Omega^{0,\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)} := \Omega^{\mathrm{nil}\,<D_1}_{\widetilde{X}(D)}$. We put

$$\Omega^{\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)} := \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\Omega^{\bullet}_X, \qquad \Omega^{\bullet,\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)} := \Omega^{0,\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\Omega^{\bullet}_X.$$

For a holonomic $\mathcal{D}$-module $\mathcal{M}$ on X, we define $\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(\mathcal{M}) := \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\,\mathrm{DR}_X(\mathcal{M})$, i.e.,

$$\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}\bigl(\mathcal{M}\bigr) := \Omega^{\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M}[d_X] \simeq \Omega^{\bullet,\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M}[d_X].$$

Note that $\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}\bigl(\mathcal{M}\bigr) \simeq \mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}\bigl(\mathcal{M}(*D)\bigr)$ because $\Omega^{\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)}(*D) = \Omega^{\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)}$.

By Theorem 4.8, we have an isomorphism $R\pi_*\,\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}\bigl(\mathcal{M}\bigr) \simeq \mathrm{DR}^{<D_1\leq D_2}_X\,\mathcal{M}$ induced as follows:

$$R\pi_*\Bigl(\Omega^{\bullet,\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M}\Bigr) \simeq R\pi_*\Omega^{\bullet,\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)} \otimes_{\mathcal{O}_X} \mathcal{M} \simeq \Omega^{\bullet,\bullet\,<D_1}_X(*D_2) \otimes_{\mathcal{O}_X} \mathcal{M} \tag{45}$$

Lemma 5.1 *We have a natural isomorphism* $R\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}\bigl(\pi^{-1}\mathcal{M}, \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)}\bigr) \simeq \mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}\bigl(\mathbb{D}\mathcal{M}\bigr)$.

Proof We have the following isomorphisms:

$$\begin{aligned}
&R\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}\bigl(\pi^{-1}\mathcal{M},\, \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)}\bigr) \simeq R\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}\bigl(\pi^{-1}\mathcal{M},\, \pi^{-1}\mathcal{D}_X\bigr) \otimes^L_{\pi^{-1}\mathcal{D}_X} \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)} \\
&= \pi^{-1}\Bigl(\Omega_X \otimes_{\mathcal{O}_X} \mathbb{D}\mathcal{M}\Bigr) \otimes^L_{\pi^{-1}\mathcal{D}_X} \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)}[-d_X] \simeq \Bigl(\pi^{-1}\Omega_X \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)}\Bigr) \otimes^L_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathbb{D}\mathcal{M}[-d_X]
\end{aligned} \tag{46}$$

Because $\mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)}$ is flat over $\pi^{-1}\mathcal{O}_X$ (Theorem 4.8), $\pi^{-1}\mathcal{D}_X \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)}$ is flat over $\pi^{-1}\mathcal{D}_X$. Therefore,

$$\mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)} \simeq \pi^{-1}\Bigl(\mathcal{D}_X \otimes_{\mathcal{O}_X} \Theta^{\bullet}_X\Bigr) \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)}$$

is a $\pi^{-1}\mathcal{D}_X$-flat resolution. Hence, (46) is quasi-isomorphic to the following:

$$\Bigl(\pi^{-1}\Bigl(\Omega^{\bullet}_X \otimes \mathcal{D}_X\Bigr) \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)}\Bigr) \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathbb{D}\mathcal{M} \simeq \Omega^{\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathbb{D}\mathcal{M}$$

Thus, we obtain the desired isomorphism. ∎

According to Lemma 5.1, we have a natural isomorphism

$$\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}\bigl(\mathcal{M}\bigr) \simeq R\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}\bigl(\pi^{-1}\mathbb{D}\mathcal{M},\, \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)}\bigr) \simeq R\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}\bigl(\pi^{-1}\mathbb{D}\mathcal{M}(*D),\, \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)}\bigr) \tag{47}$$

We will implicitly identify them in the following argument.

5.1.2 A combinatorial description in the case of good meromorphic flat bundles

Let X be a complex manifold with a normal crossing hypersurface D. Let $\pi : \widetilde{X}(D) \longrightarrow X$ be the real blow up. Let V be a good meromorphic flat bundle on (X, D). We have the local system on $X - D$ associated to $V_{|X-D}$. Its prolongment over $\widetilde{X}(D)$ is denoted by $\mathcal{L}$. For any $P \in \pi^{-1}(D)$, we have the Stokes filtration $\mathcal{F}^P$ of the stalk $\mathcal{L}_P$ indexed by the set of irregular values $\mathrm{Irr}(V, P) \subset \mathcal{O}_X(*D)_P\big/\mathcal{O}_{X,P}$ with the order $\leq_P$. The system of filtrations $\bigl\{\mathcal{F}^P \bigm| P \in \pi^{-1}(D)\bigr\}$ satisfies some compatibility condition. See [32] or [33] for more details.

Let $D = D_1 \cup D_2$ be a decomposition. Let us describe $\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(V)$ in terms of the Stokes filtrations. For $P \in \widetilde{X}(D)$, let $\mathcal{L}^{<D_1\leq D_2}_P$ be the union of the subspaces $\mathcal{F}^P_{\mathfrak{a}}(\mathcal{L}_P) \subset \mathcal{L}_P$ such that (i) $\mathfrak{a} \leq_P 0$, (ii) the poles of $\mathfrak{a}$ contains the germ of D_1 at P.

Lemma 5.2 *The family* $\{\mathcal{L}_P^{<D_1\leq D_2}\}$ *gives a constructible sheaf* $\mathcal{L}^{<D_1\leq D_2}$ *on* $\widetilde{X}(D)$.

Proof We have only to consider the case $X=\Delta^n$ and $D=\bigcup_{i=1}^{\ell}\{z_i=0\}$. By a decomposition around P as in Section 7.7.4 of [32], we have only to consider the case $V=\mathcal{O}_X(*D)$ with the flat connection $\nabla e=e\,d\mathfrak{a}$, where $\mathfrak{a}=\prod_{i=1}^m z_i^{-m_i}$ $(m_i>0)$. We have a decomposition $\underline{\ell}=I_1\sqcup I_2$ such that $D_j=\bigcup_{i\in I_j}\{z_i=0\}$. For $P\in\widetilde{X}(D)$, we set $I_j(P):=\{i\in I_j\,|\,z_i(\pi(P))=0\}$. We set $F_{\mathfrak{a}}:=-|\mathfrak{a}|^{-1}\operatorname{Re}\mathfrak{a}$. We put $R_0:=\bigcup_{i=1}^m\{z_i=0\}$ and $R_1:=\bigcup_{i=m+1}^{\ell}\{z_i=0\}\setminus R_0$.

- For $P\in X-D$, we have $\mathcal{L}_P^{<D_1\leq D_2}\neq 0$.
- For $P\in\pi^{-1}(R_1)$, we have $\mathcal{L}_P^{<D_1\leq D_2}\neq 0$ if and only if $I_1(P)=\emptyset$.
- For $P\in\pi^{-1}(R_0)$, we have $\mathcal{L}_P^{<D_1\leq D_2}\neq 0$ if and only if (i) $F_{\mathfrak{a}}(P)<0$, (ii) $I_1(P)\subset\underline{m}$.

Then, the claim of the lemma is clear. ∎

We recall the following proposition. (See [25] and [34]. See also [12].)

Proposition 5.3 *The natural inclusion* $\mathcal{L}^{<D_1\leq D_2}\longrightarrow\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(V)$ *is a quasi-isomorphism.*

Proof We give a preparation from elementary analysis on multi-sectors. We set $Y:=\Delta_z\times\Delta^n_{\boldsymbol{w}}$ and $D_Y=\{z=0\}\cup\bigcup_{i=1}^{\ell}\{w_i=0\}$. Let $\pi:\widetilde{Y}(D_Y)\longrightarrow Y$ be the real blow up. For $m>0$ and $\boldsymbol{m}=(m_1,\ldots,m_k)\in\mathbb{Z}^k_{>0}$ $(0\leq k\leq\ell)$, we put $\mathfrak{a}=z^{-m}\prod_{i=1}^k w_i^{-m_i}$. We put $F_{\mathfrak{a}}=-|\mathfrak{a}^{-1}|\operatorname{Re}(\mathfrak{a})$, which naturally gives a C^∞-function on $\widetilde{Y}(D_Y)$. Take a point $P\in\pi^{-1}(O)\subset\widetilde{Y}(D_Y)$. Let $S=S_z\times S_{\boldsymbol{w}}$ be a small multi-sector in $Y-D_Y$ such that P is contained in the interior part of the closure of $\overline{S}$ in $\widetilde{Y}(D_Y)$.

- If $F_{\mathfrak{a}}(P)<0$ (resp. $F_{\mathfrak{a}}(P)>0$), we assume that $F_{\mathfrak{a}}<0$ (resp. $F_{\mathfrak{a}}>0$) on $\overline{S}$.
- If $F_{\mathfrak{a}}(P)=0$, we assume that $F_{\mathfrak{a}}$ is monotonous with respect to θ, where $z=re^{\sqrt{-1}\theta}$ is the polar coordinate. Let θ_i $(i=1,2)$ be the arguments of the edges of S_z, i.e., $S_z=\{(r,\theta)\,|\,\theta_1\leq\theta\leq\theta_2,\ 0<r\leq r_0\}$. Let θ_+ be one of θ_i such that $F_{\mathfrak{a}}>0$ on $\{re^{\sqrt{-1}\theta_+}\}\times\overline{S}_{\boldsymbol{w}}$.

Let f be a holomorphic function on S of polynomial order with respect to z and $\boldsymbol{w}$. We set

$$\Phi(f)(z,\boldsymbol{w}):=\int_{\gamma(z,\boldsymbol{w})}\exp\bigl(-\mathfrak{a}(z,\boldsymbol{w})+\mathfrak{a}(\zeta,\boldsymbol{w})\bigr)\,f(\zeta,\boldsymbol{w})\,d\zeta. \tag{48}$$

Here, $\gamma(z,\boldsymbol{w})$ is a path contained in $S_z\times\{\boldsymbol{w}\}$ taken as follows.

Case $F_{\mathfrak{a}}(P)<0$: We fix a point $z_0\in S_z$, and $\gamma(z,\boldsymbol{w})$ is a path from z_0 to z.

Case $F_{\mathfrak{a}}(P)>0$: Let $\gamma(z,\boldsymbol{w})$ be the segment from 0 to z.

Case $F_{\mathfrak{a}}(P)=0$: Let θ_- be as above. For the polar coordinate $z=re^{\sqrt{-1}\theta}$, let $\gamma(z,\boldsymbol{w})$ be the union of the ray $\{\rho e^{\sqrt{-1}\theta_+}\,|\,0\leq\rho\leq r\}$ and the arc connecting $r\,e^{\sqrt{-1}\theta_+}$ and z.

Lemma 5.4 *For each $N>0$, there exists a constant C_N such that $\bigl|\Phi(f)(z,\boldsymbol{w})\bigr|\leq C_N\cdot C\,|z|^N\prod_{i=1}^{\ell}|w_i|^{N_i}$, if $|f(z,\boldsymbol{w})|\leq C\,|z|^N\prod_{i=1}^{\ell}|w_i|^{N_i}$.*

Proof We give only an outline. Let us consider the case $F_{\mathfrak{a}}(P)<0$. Let $z_0=r_0e^{\sqrt{-1}\theta_0}$ and $z=re^{\sqrt{-1}\theta}$. We may assume that the path γ is the union of (i) the arc γ_1 connecting z_0 and $z_1=r_0e^{\sqrt{-1}\theta}$, (ii) the segment γ_2 connecting z_1 and z_0. The segment γ_2 is divided into $\gamma_{2,1}=\gamma_1\cap\{|\zeta|>3|z|/2\}$ and $\gamma_{2,2}=\gamma_1\cap\{|\zeta|\leq 3|z|/2\}$. The contributions of γ_1 and $\gamma_{2,1}$ are dominated by $\bigl|\exp\bigl(-\mathfrak{a}(z,\boldsymbol{w})\bigr)\bigr|\prod_{i=k+1}^{\ell}|w_i|^{N_i}$. The function $\operatorname{Re}\mathfrak{a}$ is monotone on $\gamma_{2,2}$. We also have $|f(\zeta,\boldsymbol{w})|\leq C'\,|z^N|\prod_{i=1}^{\ell}|w_i|^{N_i}$ on $\gamma_{2,2}$. Hence, the contribution of $\gamma_{2,2}$ is dominated by $|z|^N \prod_{i=1}^{\ell} |w_i|^{N_i}$. Let us consider the case $F_{\mathfrak{a}}(P) \geq 0$. On γ, we have $\bigl|f(\zeta,\boldsymbol{w})\bigr| \leq C' \, |z^N| \prod_{i=1}^{\ell} |w_i|^{N_i}$, and $\mathrm{Re}(\mathfrak{a})$ is monotone. Hence, it is easy to obtain the desired estimate. ▮

Let us return to the proof of Proposition 5.3. We have only to consider the case $X = \Delta^n$ and $D = \bigcup_{i=1}^{\ell}\{z_i = 0\}$. Let $P \in \pi^{-1}(0,\dots,0)$. By using the local decomposition around P as in Section 7.7.4 of [32], we can reduce the issue to the case $V = \bigoplus_{i=1}^{M} \mathcal{O}_X(*D)\, e_i$ with the flat connection

$$\nabla \boldsymbol{e} = \boldsymbol{e}\Bigl(d\mathfrak{a} + \sum_{i=1}^{\ell}\bigl(\alpha_i\, I_M + N_i\bigr)\frac{dz_i}{z_i}\Bigr),$$

where I_M denotes the identity matrix, N_i $(i = 1,\dots,\ell)$ are mutually commuting nilpotent matrices, α_i are complex numbers, and we put $\boldsymbol{e} := (e_1,\dots,e_n)$ and $\mathfrak{a} := \prod_{i=1}^{m} z_i^{-m_i}$. Then, it is easy to show that $\mathcal{L}^{<D_1 \leq D_2}$ is naturally isomorphic to the 0-th cohomology of $\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(V)$. Hence, we have only to show the vanishing of the higher cohomology of $\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(V)$. We have only to consider the case $\operatorname{rank} V = 1$, and we put $v = e_1$.

First, let us consider the case $D_1 = D$. For a subset $J \subset \{1,\dots,n\}$, we set $dz_J = dz_{j_1}\wedge\cdots\wedge dz_{j_k}$. For a section ω of $\Omega^{\bullet\,<D}_{\widetilde{X}(D)}$, we have the unique decomposition $\omega = \sum \omega_J\, dz_J$, where $\omega_J \in \mathcal{A}^{<D}_{\widetilde{X}(D)}$. Let S_i $(i = 1,\dots,\ell)$ be a small sector in $\Delta^*_{z_i}$, and let U be a small neighbourhood of $(0,\dots,0)$ in $\prod_{i=\ell+1}^{n}\Delta_{z_i}$, such that the closure $\overline{S}$ of $S := \prod S_i \times U$ in $\widetilde{X}(D)$ is a neighbourhood of P. In the following, S will be shrinked without mention. It is easy to observe that we have only to consider the case $\alpha_i = 0$ $(i = 1,\dots,\ell)$.

Take $h = 1,\dots,n$. Assume $\nabla(\omega\, v) = 0$ for some section ω of $\Omega^{\bullet\,<D}_{\widetilde{X}(D)}$ on S such that $\omega_J = 0$ unless $J \subset \{1,\dots,h\}$. We have $d\bigl(\exp(\mathfrak{a})\omega\bigr) = 0$. For the expression $\exp(\mathfrak{a})\,\omega = \sum_{h\notin J} f_J\, dz_h\, dz_J + \sum_{h\notin J} f_J\, dz_J$, we set $\tau(\boldsymbol{z}) = \sum_{h\notin J}\exp(-\mathfrak{a})\Bigl(\int_{\gamma(\boldsymbol{z})} f_J\, dz_h\Bigr) dz_J$, where $\gamma(\boldsymbol{z})$ is a path taken as follows:

- If $h \leq m$, the condition is similar to that for the path in (48).
- If $m < h$, γ is a path connecting $(z_1,\dots,z_{h-1},0,z_{h+1},\dots,z_n)$ and $(z_1,\dots,z_n)$.

By using Lemma 5.4, we obtain that $\tau \in \Omega^{\bullet\,<D}_{\widetilde{X}(D)} \otimes V$. By a formal computation, we can show that $\omega\, v - \nabla(\tau\, v)$ does not contain dz_j for $j \geq h$. Hence, we can show the vanishing of the higher cohomology of $\Omega^{\bullet\,<D}_{\widetilde{X}(D)} \otimes V$ by an induction.

We have the decomposition $I_1 \sqcup I_2 = \underline{\ell}$ such that $D_j = \bigcup_{i\in I_j}\{z_i = 0\}$. Let us consider $\Omega^{\bullet\,<D(J^c)\leq D(J)}_{\widehat{\pi^{-1}(D_J)}} \otimes V$ for any subset $J \subset I_2$, where $J^c := \underline{\ell} \setminus J$. If $\underline{m} \cap J \neq \emptyset$, it is easy to show that $\Omega^{\bullet\,<D(J^c)\leq D(J)}_{\widehat{\pi^{-1}(D_J)}} \otimes V$ is acyclic by a formal computation. Assume $\underline{m} \cap J = \emptyset$. Let $V_J = \mathcal{O}_{D_J}(*\partial D_J)\, v_J$ be equipped with the flat connection $\nabla v_J = v_J \cdot d\mathfrak{a}_{|D_J}$ on D_J. Let q_J be the projection $\pi^{-1}(D_J) \longrightarrow \widetilde{D}_J(\partial D_J)$. Then, it is easy to obtain a natural quasi-isomorphism $q_J^{-1}\bigl(\Omega^{\bullet\,<\partial D_J}_{\widetilde{D}_J(\partial D_J)} \otimes V_J\bigr) \simeq \Omega^{\bullet\,<D(J^c)\leq D(J)}_{\widehat{\pi^{-1}(D_J)}} \otimes V$ by a formal computation. Hence, we obtain the vanishing of the higher cohomology of $\Omega^{\bullet\,<D(J^c)\leq D(J)}_{\widehat{\pi^{-1}(D_J)}} \otimes V$.

We put $h := |I_2|$. Let $\mathcal{G}_h^{\bullet}$ denote the kernel of the surjection $\Omega^{\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)} \otimes V \longrightarrow \Omega^{\bullet\,<D_1\leq D_2}_{\widehat{\pi^{-1}(D_{I_2})}} \otimes V$. Inductively, let $\mathcal{G}_k^{\bullet}$ be the kernel of the following surjection:

$$\mathcal{G}_k^{\bullet} \longrightarrow \bigoplus_{\substack{J\subset I_2\\ |J|=k}} \Omega^{\bullet\,<D(J^c)\leq D(J)}_{\widehat{\pi^{-1}(D_J)}} \otimes V$$

Because $\mathcal{G}_1^{\bullet} = \Omega^{\leq D}_{\widetilde{X}(D)} \otimes V$, we obtain the vanishing of the higher cohomology by an induction on k. Thus, the proof of Proposition 5.3 is finished. ▮

5.2 Duality

5.2.1 Duality morphism

Let X, D and $\mathcal{M}$ be as in Subsection 5.1.1. We have the following naturally defined morphism:

$$\begin{array}{ccc} \mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(\mathbb{D}\mathcal{M}) & \longrightarrow & \mathbb{D}\,\mathrm{DR}^{<D_2\leq D_1}_{\widetilde{X}(D)}(\mathcal{M}) \\ \| & & \| \\ R\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}\bigl(\pi^{-1}\mathcal{M},\Omega^{0,\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)}\bigr) & \longrightarrow & R\mathcal{H}om_{\mathbb{C}_{\widetilde{X}(D)}}\bigl(\Omega^{\bullet,\bullet\,<D_2\leq D_1}_{\widetilde{X}(D)}\otimes\pi^{-1}\mathcal{M},\ \Omega^{\bullet,\bullet\,<D}_{\widetilde{X}(D)}\bigr) \end{array} \tag{49}$$

Proposition 5.5 *The following diagram is commutative:*

$$\begin{array}{ccc} R\pi_*\,\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(\mathbb{D}\mathcal{M}) & \longrightarrow & R\pi_*\mathbb{D}\,\mathrm{DR}^{<D_2\leq D_1}_{\widetilde{X}(D)}(\mathcal{M}) \\ \simeq\big\downarrow & & \simeq\big\downarrow \\ \mathrm{DR}^{<D_1\leq D_2}_{X}(\mathbb{D}\mathcal{M}) & \longrightarrow & \mathbb{D}\,\mathrm{DR}^{<D_2\leq D_1}_{X}(\mathcal{M}) \end{array} \tag{50}$$

Here, the upper horizontal arrow is induced by (49)*, the lower horizontal arrow is given as in* (17)*, the left vertical arrow is given as in* (45)*, and the right vertical arrow is given by* $R\pi_*\mathbb{D}\,\mathrm{DR}^{<D_2\leq D_1}_{\widetilde{X}(D)}\mathcal{M}\simeq\mathbb{D}R\pi_*\,\mathrm{DR}^{<D_2\leq D_1}_{\widetilde{X}(D)}\mathcal{M}\simeq\mathbb{D}\,\mathrm{DR}^{<D_2\leq D_1}_{X}(\mathcal{M})$*.*

Proof By Lemma 5.1, we have a morphism $R\pi_*\,\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(\mathbb{D}\mathcal{M})\longrightarrow\mathrm{DR}^{<D_1\leq D_2}_{X}(\mathbb{D}\mathcal{M})$ given as follows:

$$\begin{aligned} R\pi_*R\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}\bigl(\pi^{-1}\mathcal{M},\ \Omega^{0,\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)}\bigr) &\simeq R\mathcal{H}om_{\mathcal{D}_X}\bigl(\mathcal{M},\ R\pi_*\Omega^{0,\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)}\bigr) \\ &\simeq R\mathcal{H}om_{\mathcal{D}_X}\bigl(\mathcal{M},\ \Omega^{0,\bullet}_X(*D_2)^{<D_1}\bigr) \end{aligned} \tag{51}$$

It is easy to check that it is equal to the morphism obtained as in (45). The right vertical arrow in (50) is given as follows:

$$\begin{aligned} R\pi_*R\mathcal{H}om_{\mathbb{C}_{\widetilde{X}(D)}}\bigl(\Omega^{\bullet,\bullet\,<D_2\leq D_1}_{\widetilde{X}(D)}\otimes\pi^{-1}\mathcal{M},\ \Omega^{\bullet,\bullet\,<D}_{\widetilde{X}(D)}\bigr) &\longrightarrow R\mathcal{H}om_{\mathbb{C}_X}\bigl(R\pi_*\Omega^{\bullet,\bullet\,<D_2\leq D_1}_{\widetilde{X}(D)}\otimes\mathcal{M},\ R\pi_*\Omega^{\bullet,\bullet\,<D}_{\widetilde{X}(D)}\bigr) \\ \simeq R\mathcal{H}om_{\mathbb{C}_X}\bigl(\Omega^{\bullet,\bullet\,<D_2\leq D_1}_{X}\otimes\mathcal{M},\ \Omega^{\bullet,\bullet\,<D}_{X}\bigr) &\longrightarrow R\mathcal{H}om_{\mathbb{C}_X}\bigl(\Omega^{\bullet,\bullet\,<D_2\leq D_1}_{X}\otimes\mathcal{M},\ \Omega^{\bullet,\bullet}_{X}\bigr) \end{aligned} \tag{52}$$

Then, it is easy to check the commutativity of (50). ∎

5.2.2 The case of good meromorphic flat bundle

Let us consider the case $\mathcal{M}$ is a good meromorphic flat bundle V on (X,D).

Theorem 5.6 *The duality morphism* $\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}\mathbb{D}V\longrightarrow\mathbb{D}\,\mathrm{DR}^{<D_2\leq D_1}_{\widetilde{X}(D)}V$ *is an isomorphism.*

Proof We begin with elementary preparations. Let $\mathbb{R}^2=S_0\cup S_1\cup S_2$ be a decomposition given as follows:

$$S_0:=\bigl\{(x,y)\,\big|\,y\geq 0\bigr\}\qquad S_1:=\bigl\{(x,y)\,\big|\,y\leq 0,\ x\leq 0\bigr\}\qquad S_2:=\bigl\{(x,y)\,\big|\,y\leq 0,\ x\geq 0\bigr\}$$

We put $X_1:=(\mathbb{R}\times S_1)\cup(\mathbb{R}_{\geq 0}\times S_0)$ and $X_2:=(\mathbb{R}\times S_2)\cup(\mathbb{R}_{\leq 0}\times S_0)$. The following lemma is easy to see.

Lemma 5.7 $X_i\subset\mathbb{R}^3$ $(i=1,2)$ *are closed* C^0*-submanifolds with boundaries. We have* $X_1\cup X_2=\mathbb{R}^3$ *and* $X_1\cap X_2=\partial X_i$*.* ∎

We put $\mathcal{J}:=]-1,1[$, $\mathcal{J}_+:=[0,1[$, $\mathcal{J}_-:=]-1,0]$, and $\mathcal{I}_i:=[0,1[$ $(i=1,2,3)$. We have a homeomorphism $\partial(\mathcal{I}_1\times\mathcal{I}_2\times\mathcal{I}_3)\simeq\mathbb{R}^2$, and we can identify the decomposition

$$\partial(\mathcal{I}_1\times\mathcal{I}_2\times\mathcal{I}_3)=\bigl(\partial\mathcal{I}_1\times\mathcal{I}_2\times\mathcal{I}_3\bigr)\cup\bigl(\mathcal{I}_1\times\partial\mathcal{I}_2\times\mathcal{I}_3\bigr)\cup\bigl(\mathcal{I}_1\times\mathcal{I}_2\times\partial\mathcal{I}_3\bigr)$$

with $\mathbb{R}^2=S_0\cup S_1\cup S_2$. We put

$$X_1':=\Bigl(\mathcal{J}\times\mathcal{I}_1\times\partial\mathcal{I}_2\times\mathcal{I}_3\Bigr)\cup\Bigl(\mathcal{J}_+\times\partial\mathcal{I}_1\times\mathcal{I}_2\times\mathcal{I}_3\Bigr),\quad X_2':=\Bigl(\mathcal{J}\times\mathcal{I}_1\times\mathcal{I}_2\times\partial\mathcal{I}_3\Bigr)\cup\Bigl(\mathcal{J}_-\times\partial\mathcal{I}_1\times\mathcal{I}_2\times\mathcal{I}_3\Bigr)$$

They are closed subsets of $\mathcal{J}\times\partial(\mathcal{I}_1\times\mathcal{I}_2\times\mathcal{I}_3)$. We obtain the following lemma from Lemma 5.7.

Lemma 5.8 *$X'_i \subset \mathcal{J} \times \partial(\mathcal{I}_1 \times \mathcal{I}_2 \times \mathcal{I}_3)$ are C^0-submanifolds with boundaries. We have $X'_1 \cup X'_2 = \mathcal{J} \times \partial\big(\mathcal{I}_1 \times \mathcal{I}_2 \times \mathcal{I}_3\big)$ and $X'_1 \cap X'_2 = \partial X'_i$.* ∎

We recall some elementary facts on constructible sheaves. Let Y be an ℓ-dimensional C^0-manifold with the boundary ∂Y. For a closed C^0-submanifold $W \subset \partial Y$ with boundary such that $\dim W = \ell$, let j_W denote the inclusion $Y - W \longrightarrow Y$. We have the following natural isomorphisms:

$$R\mathcal{H}om_{\mathbb{C}_Y}\big(j_{W!}\mathbb{C}_{Y-W}, K\big) \simeq Rj_{W*}R\mathcal{H}om_{\mathbb{C}_{Y-W}}(\mathbb{C}_{Y-W}, Rj_W^! K) \simeq Rj_{W*}j_W^* K$$

The dualizing complex of Y is given by $j_{\partial Y!}\mathbb{C}_{Y-\partial Y}[\ell]$.

Lemma 5.9 *Let $Y_i \subset \partial Y$ be closed C^0-submanifolds with boundaries such that $Y_1 \cup Y_2 = Y$ and $Y_1 \cap Y_2 = \partial Y_i$. Then, we have $\mathbb{D}j_{Y_1!}\mathbb{C}_{Y-Y_1} \simeq j_{Y_2!}\mathbb{C}_{Y-Y_2}$.*

Proof The left hand side is naturally isomorphic to $j_{Y_1*}j_{Y_1}^*\omega_Y \simeq j_{Y_1*}j_{0!}\mathbb{C}_{Y-\partial Y}[\ell]$, where j_0 denotes the inclusion $Y - \partial Y \longrightarrow Y - Y_1$. Then, we can check the claim directly. ∎

Let us return to the proof of Theorem 5.6. We have only to consider the case $X = \Delta^n$ and $D = \bigcup_{i=1}^{\ell}\{z_i = 0\}$. As in the proof of Proposition 5.3, we can reduce the issue to the case that $V = \mathcal{O}_X(*D)\,v$ with the meromorphic flat connection $\nabla v = v\,d\mathfrak{a}$, where $\mathfrak{a} = \prod_{i=1}^m z_i^{-m_i}$ $(m_i > 0)$. We put $F_{\mathfrak{a}} := -|\mathfrak{a}|^{-1}\operatorname{Re}\mathfrak{a}$. We have the decomposition $I_1 \sqcup I_2 = \underline{\ell}$ such that $D_j = \bigcup_{i \in I_j}\{z_i = 0\}$ $(j = 1, 2)$. We set $I_j(> m) := \big\{i \in I_j \,\big|\, i > m\big\}$. We also put $D(> m) := \bigcup_{i=m+1}^{\ell}\{z_i = 0\}$ and $D(\leq m) := \bigcup_{i=1}^{m}\{z_i = 0\}$. The closed subsets $W_i \subset \pi^{-1}(D)$ $(i = 1, 2)$ are given as follows:

$$W_1 := \pi^{-1}\Big(D_1 \cap D(> m)\Big) \cup \Big[\pi^{-1}\big(D(\leq m)\big) \cap \{F_{\mathfrak{a}} \geq 0\}\Big]$$

$$W_2 := \pi^{-1}\Big(D_2 \cap D(> m)\Big) \cup \Big[\pi^{-1}\big(D(\leq m)\big) \cap \{F_{\mathfrak{a}} \leq 0\}\Big]$$

Lemma 5.10 *$W_i \subset \pi^{-1}(D)$ are closed C^0-submanifolds with boundaries, and we have $W_1 \cup W_2 = \pi^{-1}(D)$ and $W_1 \cap W_2 = \partial W_i$.*

Proof It is easy to observe that we have only to consider the case $n = \ell$. We have the natural identification $\widetilde{X}(D) \simeq (S^1)^{\ell} \times \mathbb{R}^{\ell}_{\geq 0}$. By the decomposition $\underline{\ell} = \underline{m} \sqcup I_1(> m) \sqcup I_2(> m)$, we identify $\mathbb{R}^{\ell}_{\geq 0} = \mathbb{R}^{m}_{\geq 0} \times \mathbb{R}^{I_1(>m)}_{\geq 0} \times \mathbb{R}^{I_2(>m)}_{\geq 0}$. We fix homeomorphisms

$$\mathbb{R}^m_{\geq 0} \simeq \mathcal{I}_1 \times \mathbb{R}^{m-1}, \quad \mathbb{R}^{I_1(>m)}_{\geq 0} \simeq \mathcal{I}_2 \times \mathbb{R}^{|I_1(>m)|-1}, \quad \mathbb{R}^{I_2(>m)}_{\geq 0} \simeq \mathcal{I}_3 \times \mathbb{R}^{|I_2(>m)|-1}.$$

We put $N := m + |I_1(> m)| + |I_2(> m)| - 3$. Let $H_{\pm}$ be the subsets of $(S^1)^{\ell}$ given as follows:

$$H_+ := \Big\{\cos\Big(\sum m_i\theta_i\Big) \geq 0\Big\} \qquad H_- := \Big\{\cos\Big(\sum m_i\theta_i\Big) \leq 0\Big\}$$

Then, $\pi^{-1}(D)$ is identified with $(S^1)^{\ell} \times \partial(\mathcal{I}_1 \times \mathcal{I}_2 \times \mathcal{I}_3) \times \mathbb{R}^N$, under which we have

$$W_1 \simeq \Big(\big((S^1)^{\ell} \times \mathcal{I}_1 \times \partial\mathcal{I}_2 \times \mathcal{I}_3\big) \cup \big(H_- \times \partial\mathcal{I}_1 \times \mathcal{I}_2 \times \mathcal{I}_3\big)\Big) \times \mathbb{R}^N$$

$$W_2 = \Big(\big((S^1)^{\ell} \times \mathcal{I}_1 \times \mathcal{I}_2 \times \partial\mathcal{I}_3\big) \cup \big(H_+ \times \partial\mathcal{I}_1 \times \mathcal{I}_2 \times \mathcal{I}_3\big)\Big) \times \mathbb{R}^N$$

For a point $Q \in H_+ \cap H_-$, we can take a neighbourhood U_Q such that $U \simeq \mathcal{J} \times \mathbb{R}^{\ell-1}$ under which $H_{\pm} \cap U_Q = \mathcal{J}_{\pm} \times \mathbb{R}^{\ell-1}$. Then, we obtain Lemma 5.10 from Lemma 5.8. ∎

Let j_{W_i} be the inclusion $\widetilde{X}(D) \setminus W_i \longrightarrow \widetilde{X}(D)$. Let $\mathcal{L}$ and $\mathcal{L}^{\vee}$ be the local systems on $\widetilde{X}(D)$ associated to V and $V^{\vee}$, respectively. According to the description of $\mathcal{L}^{<D_1 \leq D_2}$ and $\mathcal{L}^{\vee <D_2 \leq D_1}$, we have the following natural isomorphisms:

$$\mathcal{L}^{<D_1 \leq D_2} \simeq j_{W_1!}\big(\mathcal{L}_{\widetilde{X}(D)\setminus W_1}\big) \qquad \mathcal{L}^{\vee <D_2 \leq D_1} \simeq j_{W_2!}\big(\mathcal{L}^{\vee}_{\widetilde{X}(D)\setminus W_2}\big)$$

By applying Lemma 5.9, we obtain an isomorphism $\mathbb{D}\mathcal{L}^{<D_1 \leq D_2} \simeq \mathcal{L}^{\vee\,<D_2 \leq D_1}$. It is uniquely determined by its restriction to $X - D$. Then, we can deduce that $\mathrm{DR}^{<D_1 \leq D_2}_{\widetilde{X}(D)}\mathbb{D}V \longrightarrow \mathbb{D}\,\mathrm{DR}^{<D_2 \leq D_1}_{\widetilde{X}(D)} V$ is an isomorphism. Thus, the proof of Theorem 5.6 is finished. ∎

Corollary 5.11 *For a good meromorphic flat bundle V on (X,D), we have the following commutative diagram:*

$$\begin{array}{ccc}
R\pi_*\,\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}\,\mathbb{D}V & \xrightarrow{\simeq} & R\pi_*\mathbb{D}\,\mathrm{DR}^{<D_2\leq D_1}_{\widetilde{X}(D)}\,V \\
\simeq\big\downarrow & & \simeq\big\downarrow \\
\mathrm{DR}_X\,V^{\vee}(!D_1) & \xrightarrow{\simeq} & \mathbb{D}\,\mathrm{DR}_X\,V(!D_2)
\end{array}$$

Proof It follows from Theorem 3.18, Proposition 5.5 and Theorem 5.6. ∎

5.3 Functoriality

Let X be a complex manifold, and let D be a normal crossing hypersurface with a decomposition $D=D_1\cup D_2$. Let D_3 be a hypersurface of X. Let $\varphi: X'\longrightarrow X$ be a proper birational morphism such that (i) $D':=\varphi^{-1}(D\cup D_3)$ is normal crossing, (ii) $X'\setminus D'\simeq X\setminus(D\cup D_3)$. Let $\widetilde{X}(D)\longrightarrow X$ and $\widetilde{X}'(D')\longrightarrow X'$ be the real blow up. Both the projections are denoted by π. Let $\widetilde{\varphi}:\widetilde{X}'(D')\longrightarrow\widetilde{X}(D)$ be the induced map. We put $D_1':=\varphi^{-1}(D_1)$. We have $D_2'\subset D'$ such that $D'=D_1'\cup D_2'$ is a decomposition. Let V be a meromorphic flat bundle on (X,D). We set $(V',\nabla'):=\varphi^*(V,\nabla)\otimes\mathcal{O}_X(*D')$.

Theorem 5.12 *We have a morphism $\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(V)\longrightarrow R\widetilde{\varphi}_*\,\mathrm{DR}^{<D_1'\leq D_2'}_{\widetilde{X}'(D')}(V')$ in the derived category of cohomologically constructible sheaves, such that the following diagram of perverse sheaves is commutative:*

$$\begin{array}{ccc}
R\pi_*\,\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(V) & \longrightarrow & R\pi_*R\widetilde{\varphi}_*\,\mathrm{DR}^{<D_1'\leq D_2'}_{\widetilde{X}(D')}(V') \\
\simeq\big\downarrow & & \simeq\big\downarrow \\
\mathrm{DR}_X\big(V(!D_1)\big) & \longrightarrow & R\varphi_*\,\mathrm{DR}_{X'}\big(V'(!D_1')\big)
\end{array}\tag{53}$$

Here, the vertical isomorphisms are given by (45) *and Corollary* 3.16, *and the lower horizontal arrow is induced by the morphism of $\mathcal{D}$-modules $V(!D_1)\longrightarrow\varphi_\dagger V'(!D_1')$.*

Similarly, we have a morphism $R\widetilde{\varphi}_*\,\mathrm{DR}^{<D_2'\leq D_1'}_{\widetilde{X}'(D')}(V')\longrightarrow\mathrm{DR}^{<D_2\leq D_1}_{\widetilde{X}(D)}(V)$ such that the following diagram of perverse sheaves is commutative:

$$\begin{array}{ccc}
R\pi_*R\widetilde{\varphi}_*\,\mathrm{DR}^{<D_2'\leq D_1'}_{\widetilde{X}'(D')}(V') & \longrightarrow & R\pi_*\,\mathrm{DR}^{<D_2\leq D_1}_{\widetilde{X}(D)}(V) \\
\simeq\big\downarrow & & \simeq\big\downarrow \\
R\varphi_*\,\mathrm{DR}_X(V'(!D_2')) & \longrightarrow & \mathrm{DR}_X\big(V(!D_2)\big)
\end{array}\tag{54}$$

Proof We have a naturally induced morphism:

$$\widetilde{\varphi}^{-1}\big(\Omega^{\bullet,\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)}\otimes\pi^{-1}V\big)\longrightarrow\Omega^{\bullet,\bullet\,<D_1'\leq D_2'}_{\widetilde{X}'(D')}\otimes\pi^{-1}V'.\tag{55}$$

It induces a morphism of cohomologically constructible complexes:

$$\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(V)\longrightarrow\widetilde{\varphi}_*\,\mathrm{DR}^{<D_1'\leq D_2'}_{\widetilde{X}'(D')}(V')\tag{56}$$

We can directly check the commutativity of the following diagram:

$$\begin{array}{ccc}
\Omega^{\bullet,\bullet\,<D_1\leq D_2}_X\otimes V & \longrightarrow & \varphi_*\Big(\Omega^{\bullet,\bullet\,<D_1'\leq D_2'}_{X'}\otimes V'\Big) \\
\big\downarrow & & \big\downarrow \\
\pi_*\Big(\Omega^{\bullet,\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)}\otimes\pi^{-1}V\Big) & \longrightarrow & \pi_*\Big(\widetilde{\varphi}_*\Omega^{\bullet,\bullet\,<D_1'\leq D_2'}_{\widetilde{X}'(D')}\otimes\pi^{-1}V'\Big)
\end{array}$$

It implies the commutativity of the following diagram:

$$
\begin{tikzcd}
R\pi_* \operatorname{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(V) \arrow[r] \arrow[d, "\simeq"'] & R\pi_* R\widetilde{\varphi}_* \operatorname{DR}^{<D'_1\leq D'_2}_{\widetilde{X}'(D')}(V') \arrow[d, "\simeq"'] \\
\operatorname{DR}^{<D_1\leq D_2}_{X}(V) \arrow[r] & R\varphi_* \operatorname{DR}^{<D'_1\leq D'_2}_{X'}(V')
\end{tikzcd}
\tag{57}
$$

Then, we obtain the commutativity of (53) from Theorem 3.20.

Considering the dual of (56) with $V^\vee$ (see Theorem 5.6), we obtain the following morphism:

$$
R\widetilde{\varphi}_* \operatorname{DR}^{\leq D'_1<D'_3}_{\widetilde{X}'(D')}(V') \longrightarrow \operatorname{DR}^{\leq D_1<D_2}_{\widetilde{X}(D)}(V)
\tag{58}
$$

Let us show the commutativity of the diagram (54). From (57) for $V^\vee$, we obtain the following commutative diagram:

$$
\begin{tikzcd}
\mathbb{D}R\pi_* R\widetilde{\varphi}_* \operatorname{DR}^{<D'_1\leq D'_2}_{\widetilde{X}'(D')}(V'^\vee) \arrow[r] \arrow[d, "\simeq"'] & \mathbb{D}R\pi_* \operatorname{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(V^\vee) \arrow[d, "\simeq"'] \\
\mathbb{D}R\varphi_* \operatorname{DR}^{<D'_1\leq D'_2}_{X'}(V'^\vee) \arrow[r] & \mathbb{D}\operatorname{DR}^{<D_1\leq D_2}_{X}(V^\vee)
\end{tikzcd}
$$

By Proposition 5.5 and Theorem 5.6, we have the following commutative diagram:

$$
\begin{tikzcd}
\mathbb{D}R\pi_* \operatorname{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(V^\vee) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & R\pi_* \operatorname{DR}^{\leq D_1<D_2}_{\widetilde{X}(D)}(V) \arrow[d, "\simeq"'] \\
\mathbb{D}\operatorname{DR}^{<D_1\leq D_2}(V^\vee) \arrow[r, "\simeq"] & \operatorname{DR}^{<D_2\leq D_1}_{X}(V)
\end{tikzcd}
$$

We have a similar diagram for V'. Then, we obtain the commutativity of (54) from the constructions of (58) and (24). ∎

5.4 A relation between de Rham complexes on real blow up

5.4.1 Sheaves of functions of Nilsson types

Let X be a complex manifold with a normal crossing hypersurface D. Let g be a holomorphic function on X such that $g^{-1}(0) = D$. The image of $\mathrm{id} \times g : X \longrightarrow X \times \mathbb{C}$ is denoted by Γ_g. Let $\pi_1 : \widetilde{X}(D) \longrightarrow X$ and $\pi_2 : X \times \widetilde{\mathbb{C}} \longrightarrow X \times \mathbb{C}$ be the real blow up. We set $\widetilde{X} := \Gamma_g \times_{(X\times\mathbb{C})} (X \times \widetilde{\mathbb{C}})$. We obtain the following commutative diagram:

$$
\begin{tikzcd}
\widetilde{X}(D) \arrow[r, "\widetilde{\rho}"] \arrow[d, "\pi_1"'] & \widetilde{X} \arrow[r, "\widetilde{\iota}_g"] \arrow[d, "\pi_3"'] & X \times \widetilde{\mathbb{C}} \arrow[d, "\pi_2"'] \\
X \arrow[r, "\rho", "\simeq"'] & \Gamma_g \arrow[r, "\iota_g"] & X \times \mathbb{C}
\end{tikzcd}
$$

We set $\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}} := \mathcal{A}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}} \otimes_{\mathcal{O}_{X\times\mathbb{C}}} \mathcal{O}_{\Gamma_g}$ on $\widetilde{X}$. Let t be the coordinate of $\mathbb{C}$. Because t is invertible in $\mathcal{A}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}$, we have $\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}} = \mathcal{A}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}(*g) \otimes_{\mathcal{O}_{X\times\mathbb{C}}} \mathcal{O}_{\Gamma_g}$.

Proposition 5.13 *A naturally defined morphism $\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}} \longrightarrow \widetilde{\rho}_* \mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)}$ induces an isomorphism $\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}} \simeq R\widetilde{\rho}_* \mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)}$.*

Proof Let $\nu_1 : \widetilde{X}(D) \times \widetilde{\mathbb{C}} \longrightarrow \widetilde{X}(D) \times \mathbb{C}$ and $\nu_2 : \widetilde{X}(D) \times \widetilde{\mathbb{C}} \longrightarrow X \times \widetilde{\mathbb{C}}$ be the naturally induced morphisms. According to Theorem 4.9, we have

$$
R\nu_{1*} \mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)\times\widetilde{\mathbb{C}}} = \mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)\times\mathbb{C}}(*t), \qquad R\nu_{2*} \mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)\times\widetilde{\mathbb{C}}} = \mathcal{A}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}(*g)
\tag{59}
$$

We put $\widetilde{X}_1 := \Gamma_g \times_{(X\times\mathbb{C})} \big(\widetilde{X}(D)\times\mathbb{C}\big)$ and $\widetilde{X}_2 := \Gamma_g \times_{(X\times\mathbb{C})} \big(\widetilde{X}(D)\times\widetilde{\mathbb{C}}\big)$. We set

$$\mathcal{A}^{\rm nil}_{\widetilde{X}_1} := \mathcal{A}^{\rm nil}_{\widetilde{X}(D)\times\mathbb{C}}(*t)\otimes_{\mathcal{O}_{X\times\mathbb{C}}}\mathcal{O}_{\Gamma_g}, \qquad \mathcal{A}^{\rm nil}_{\widetilde{X}_2} := \mathcal{A}^{\rm nil}_{\widetilde{X}(D)\times\widetilde{\mathbb{C}}}\otimes_{\mathcal{O}_{X\times\mathbb{C}}}\mathcal{O}_{\Gamma_g}.$$

We have the following naturally given commutative diagram:

$$\begin{array}{ccc}
\widetilde{X}_2 & \xrightarrow{\kappa_1} & \widetilde{X}_1 \\
{\scriptstyle \kappa_2}\big\downarrow & & \big\downarrow \\
\widetilde{X} & \longrightarrow & \Gamma_g
\end{array}$$

From (59), we obtain $R\kappa_{1*}\mathcal{A}^{\rm nil}_{\widetilde{X}_2} = \mathcal{A}^{\rm nil}_{\widetilde{X}_1}$ and $R\kappa_{2*}\mathcal{A}^{\rm nil}_{\widetilde{X}_2} = \mathcal{A}^{\rm nil}_{\widetilde{X}}$. We have a natural identification $\widetilde{X}_1 \simeq \widetilde{X}(D)$ and $\mathcal{A}^{\rm nil}_{\widetilde{X}_1} \simeq \mathcal{A}^{\rm nil}_{\widetilde{X}(D)}$. We also have $\widetilde{\rho}\circ\kappa_1 = \kappa_2$, and hence $R\widetilde{\rho}_*\mathcal{A}^{\rm nil}_{\widetilde{X}(D)} \simeq R\kappa_{2*}\mathcal{A}^{\rm nil}_{\widetilde{X}_2} \simeq \mathcal{A}^{\rm nil}_{\widetilde{X}}$. Thus, we are done. ∎

5.4.2 De Rham complexes for good meromorphic flat bundles

Let X be a complex manifold with a normal crossing hypersurface D. Let $\pi_1 : \widetilde{X}(D) \longrightarrow X$ be the real blow up. Let $\widetilde{\mathbb{C}}$ denote the real blow up of $\mathbb{C}$ along 0. Let g be a holomorphic function on X such that $g^{-1}(0) = D$. The induced inclusion $\mathrm{id}\times g : X \longrightarrow X\times\mathbb{C}$ is denoted by ι_g, and the image is denoted by Γ_g.

$$\begin{array}{ccc}
\widetilde{X}(D) & \xrightarrow{\widetilde{\iota}_g} & X\times\widetilde{\mathbb{C}} \\
{\scriptstyle \pi_1}\big\downarrow & & {\scriptstyle \pi_2}\big\downarrow \\
X & \xrightarrow{\iota_g} & X\times\mathbb{C}
\end{array}$$

Proposition 5.14 *We have a natural quasi-isomorphism* $\mathrm{DR}^{\rm nil}_{X\times\widetilde{\mathbb{C}}}\big(\iota_{g\dagger}\mathcal{M}\big) \simeq R\widetilde{\iota}_{g*}\,\mathrm{DR}^{\rm nil}_{\widetilde{X}(D)}\big(\mathcal{M}\big)$ *on* $X\times\widetilde{\mathbb{C}}$.

Proof Note that $\iota_{g\dagger}\mathcal{M} = i_{g*}\mathcal{M}[\partial_t]$. By using Proposition 5.13, we obtain

$$\pi_2^{-1}\iota_{g\dagger}\mathcal{M}\otimes\mathcal{A}^{\rm nil}_{X\times\widetilde{\mathbb{C}}} \simeq R\widetilde{\iota}_{g*}\Big(\pi_1^{-1}\mathcal{M}[\partial_t]\otimes\mathcal{A}^{\rm nil}_{\widetilde{X}(D)}\Big)$$

Here, a tangent vector v of X and ∂_t acts on $\mathcal{M}[\partial_t]$ by $v(\partial_t^n\otimes m) = \partial_t^n\otimes(vm) - \partial_t^{n+1}\otimes(vg)m$ and $\partial_t(\partial_t^n\otimes m) = \partial_t^{n+1}\otimes m$. Then, we obtain the following natural quasi-isomorphisms:

$$\begin{aligned}
&\pi_2^{-1}\Omega^{\bullet,0}_{X\times\mathbb{C}}\otimes_{\pi_2^{-1}\mathcal{O}_{X\times\mathbb{C}}} R\widetilde{\iota}_{g*}\Big(\pi_1^{-1}\mathcal{M}[\partial_t]\otimes\mathcal{A}^{\rm nil}_{\widetilde{X}(D)}\Big) \xrightarrow{\simeq} \\
&\qquad R\widetilde{\iota}_{g*}\Big(\pi_1^{-1}\big(\Omega^{0,\bullet}_X\otimes\mathcal{M}[\partial_t]\big)\otimes\mathcal{A}^{\rm nil}_{\widetilde{X}(D)} \xrightarrow{dt\,\partial_t} \pi_1^{-1}\big(\Omega^{0,\bullet}_X\otimes\mathcal{M}[\partial_t]\big)\otimes\mathcal{A}^{\rm nil}_{\widetilde{X}(D)}\,dt\Big) \\
&\qquad\qquad \xleftarrow{\simeq} R\widetilde{\iota}_{g*}\Big(0\longrightarrow\pi_1^{-1}\big(\Omega^{\bullet,0}_X\otimes\mathcal{M}\big)\otimes\mathcal{A}^{\rm nil}_{\widetilde{X}(D)}\,dt\Big) \qquad (60)
\end{aligned}$$

Thus, Proposition 5.14 is proved. ∎

5.4.3 Complement

Let $F_0 : X_0 \longrightarrow Y_0$ be a proper morphism of complex manifolds. Let $\widetilde{\mathbb{C}}$ denote the real blow up of $\mathbb{C}$ along 0. We set $X := X_0\times\mathbb{C}$ and $\widetilde{X} := X_0\times\widetilde{\mathbb{C}}$. We use the symbols Y and $\widetilde{Y}$ in similar meanings. Let $F : X \longrightarrow Y$ and $\widetilde{F} : \widetilde{X} \longrightarrow \widetilde{Y}$ be induced by F_0. Put $D_X := X_0\times\{0\}$ and $D_Y := Y_0\times\{0\}$. Let $\pi_X : \widetilde{X} \longrightarrow X$ and $\pi_Y : \widetilde{Y} \longrightarrow Y$ be the projections.

Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module on X such that $\mathcal{M}(*D_X) = \mathcal{M}$. We set

$$\widetilde{F}_\dagger\big(\pi_X^{-1}\mathcal{M}\otimes_{\pi_X^{-1}\mathcal{O}_X}\mathcal{A}^{\rm nil}_{\widetilde{X}}\big) := R\widetilde{F}_*\Big(\pi_X^{-1}\big(\mathcal{D}_{Y\leftarrow X}\otimes^L_{\mathcal{D}_X}\mathcal{M}\big)\otimes_{\pi_X^{-1}\mathcal{O}_X}\mathcal{A}^{\rm nil}_{\widetilde{X}}\Big)$$

By using $\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}} = \widetilde{F}^{-1}\mathcal{A}^{\mathrm{nil}}_{\widetilde{Y}} \otimes_{(F\circ\pi_X)^{-1}\mathcal{O}_Y} \pi_X^{-1}\mathcal{O}_X$, we obtain a natural isomorphism:

$$\widetilde{F}_{\dagger}\bigl(\pi_X^{-1}\mathcal{M}\otimes_{\pi_X^{-1}\mathcal{O}_X}\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}}\bigr) \simeq \pi_Y^{-1}\bigl(F_{\dagger}\mathcal{M}\bigr)\otimes_{\pi_Y^{-1}\mathcal{O}_Y}\mathcal{A}^{\mathrm{nil}}_{\widetilde{Y}}$$

We set $\mathrm{DR}^{\mathrm{nil}}_{\widetilde{X}}(\mathcal{M}) := \pi_X^{-1}\bigl(\Omega_X\otimes^L_{\mathcal{D}_X}\mathcal{M}\bigr)\otimes_{\pi_X^{-1}\mathcal{O}_X}\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}}$. By a formal argument for the compatibility between the de Rham functor and the push-forward, we obtain a natural isomorphism:

$$R\widetilde{F}_*\circ\mathrm{DR}^{\mathrm{nil}}_{\widetilde{X}} \simeq \mathrm{DR}^{\mathrm{nil}}_{\widetilde{Y}}\circ F_{\dagger} \tag{61}$$

5.5 Some rigidity

5.5.1 Statement

We set $X := \Delta^n$ and $D := \bigcup_{i=1}^{\ell}\{z_i = 0\}$. We put $D^{[m]} := \bigcup_{\substack{I\subset\underline{\ell}\\ |I|=m}} D_I$. Let (V,∇) be a good meromorphic flat bundle on (X,D). Let $\mathcal{L}$ be the associated local system on $\widetilde{X}(D)$. Let g be a holomorphic function on X such that $g^{-1}(0)=D$. Let $\Gamma_g\subset X\times\mathbb{C}$ be the image of the graph of g. We put $\widetilde{X} := \Gamma_g\times_{(X\times\mathbb{C})}(X\times\widetilde{\mathbb{C}})$. We have the naturally defined morphisms:

$$\widetilde{X}(D) \xrightarrow{\pi_1} \widetilde{X} \xrightarrow{\pi_0} X$$

We put $\pi_2 := \pi_0\circ\pi_1$. We set $\mathcal{K} := R\pi_{1*}\mathcal{L}^{\leq D}$. In this subsection, we will work on the derived category of cohomologically constructible sheaves.

Theorem 5.15 *Let $n\geq 2$. The restriction* $\mathrm{Hom}(\mathcal{K},\mathcal{K}) \longrightarrow \mathrm{Hom}\bigl(\mathcal{K}_{|\pi_0^{-1}(X-D^{[2]})},\mathcal{K}_{|\pi_0^{-1}(X-D^{[2]})}\bigr)$ *is injective.*

We will give a consequence in Subsection 5.5.6.

5.5.2 Reduction

We have only to show the injectivity of the following morphisms for $m\geq 2$:

$$\mathrm{Hom}\bigl(\mathcal{K}_{|\pi_0^{-1}(X-D^{[m+1]})},\mathcal{K}_{|\pi_0^{-1}(X-D^{[m+1]})}\bigr) \longrightarrow \mathrm{Hom}\bigl(\mathcal{K}_{|\pi_0^{-1}(X-D^{[m]})},\mathcal{K}_{|\pi_0^{-1}(X-D^{[m]})}\bigr)$$

Then, it is easy to observe that we have only to consider the case $\ell=n$ and the following morphism:

$$\mathrm{Hom}(\mathcal{K},\mathcal{K}) \longrightarrow \mathrm{Hom}\bigl(\mathcal{K}_{|\pi_0^{-1}(X-O)},\mathcal{K}_{|\pi_0^{-1}(X-O)}\bigr)$$

By the adjunction $\mathrm{Hom}\bigl(\pi_1^*\mathcal{K},\mathcal{L}^{\leq D}\bigr)\simeq\mathrm{Hom}(\mathcal{K},\mathcal{K})$, we have only to show the injectivity of the following morphism:

$$\mathrm{Hom}\bigl(\pi_1^*\mathcal{K},\,\mathcal{L}^{\leq D}\bigr) \longrightarrow \mathrm{Hom}\bigl(\pi_1^*\mathcal{K}_{|\pi_2^{-1}(X-O)},\,\mathcal{L}^{\leq D}_{|\pi_2^{-1}(X-O)}\bigr)$$

We have $R^i\pi_{1*}\mathcal{L}^{\leq D}=0$ unless $0\leq i\leq n-1$, because the real dimension of the fiber is less than $n-1$. We set

$$\mathcal{K}^i := \pi_1^*R^i\pi_{1*}\mathcal{L}^{\leq D}.$$

Let $\mathrm{j}:\pi_2^{-1}(X-O)\longrightarrow\widetilde{X}(D)$ and $\mathrm{i}:\pi_2^{-1}(O)\longrightarrow\widetilde{X}(D)$.

Lemma 5.16 *To show Theorem* 5.15*, we have only to show*

$$\mathcal{E}xt^j\bigl(\mathrm{i}_*\mathrm{i}^*\mathcal{K}^i,\mathcal{L}^{\leq D}\bigr)=0\quad i,j\leq n-1 \tag{62}$$

Proof From the distinguished triangle $\mathcal{K}^i[-i]\longrightarrow\tau_{\geq i}\pi_1^*\mathcal{K}\longrightarrow\tau_{\geq i+1}\pi_1^*\mathcal{K}\xrightarrow{+1}$, we obtain the long exact sequence:

$$\mathrm{Ext}^{i-1}(\mathcal{K}^i,\mathcal{L}^{\leq D})\longrightarrow\mathrm{Hom}\bigl(\tau_{\geq i+1}\pi_1^*\mathcal{K},\mathcal{L}^{\leq D}\bigr)\longrightarrow\mathrm{Hom}\bigl(\tau_{\geq i}\pi_1^*\mathcal{K},\mathcal{L}^{\leq D}\bigr)\longrightarrow\mathrm{Ext}^i\bigl(\mathcal{K}^i,\mathcal{L}^{\leq D}\bigr)$$

We have the corresponding long exact sequences for the restrictions to $\pi_2^{-1}(X-O)$. The injectivity of $\mathrm{Hom}\big(\tau_{\geq i}\pi_1^*\mathcal{K},\,\mathcal{L}^{\leq D}\big)\longrightarrow \mathrm{Hom}\big(\tau_{\geq i}\pi_1^*\mathcal{K}_{|\pi_2^{-1}(X-O)},\mathcal{L}^{\leq D}_{|\pi_2^{-1}(X-O)}\big)$ can follow from the injectivity of

$$\mathrm{Ext}^i\big(\mathcal{K}^i,\mathcal{L}^{\leq D}\big)\longrightarrow \mathrm{Ext}^i\big(\mathcal{K}^i_{|\pi_2^{-1}(X-O)},\,\mathcal{L}^{\leq D}_{|\pi_2^{-1}(X-O)}\big), \tag{63}$$

$$\mathrm{Hom}\big(\tau_{\geq\, i+1}\pi_1^*\mathcal{K},\,\mathcal{L}^{\leq D}\big)\longrightarrow \mathrm{Hom}\big(\tau_{\geq i+1}\pi_1^*\mathcal{K}_{|\pi_2^{-1}(X-O)},\mathcal{L}^{\leq D}_{|\pi_2^{-1}(X-O)}\big), \tag{64}$$

and the surjectivity of

$$\mathrm{Ext}^{i-1}(\mathcal{K}^i,\mathcal{L}^{\leq D})\longrightarrow \mathrm{Ext}^{i-1}(\mathcal{K}^i_{|\pi_2^{-1}(X-O)},\mathcal{L}^{\leq D}_{|\pi_2^{-1}(X-O)}). \tag{65}$$

By an easy inductive argument, we can reduce Theorem 5.15 to the injectivity of (63) and the surjectivity of (65) for any $i\leq n-1$.

From the exact sequence $0\longrightarrow \mathfrak{j}_!\mathfrak{j}^*\mathcal{K}^i\longrightarrow\mathcal{K}^i\longrightarrow \mathfrak{i}_*\mathfrak{i}^*\mathcal{K}^i\longrightarrow 0$ and the adjunction $\mathrm{Ext}^i\big(\mathfrak{j}_!\mathfrak{j}^*\mathcal{K}^i,\mathcal{L}^{\leq D}\big)\simeq \mathrm{Ext}^i\big(\mathfrak{j}^*\mathcal{K}^i,\,\mathfrak{j}^*\mathcal{L}^{\leq D}\big)$, we obtain the following exact sequence:

$$\begin{aligned}\mathrm{Ext}^{i-1}\big(\mathcal{K}^i,\,\mathcal{L}^{\leq D}\big)\longrightarrow \mathrm{Ext}^{i-1}\big(\mathfrak{j}^*\mathcal{K}^i,\,\mathfrak{j}^*\mathcal{L}^{\leq D}\big)\longrightarrow \mathrm{Ext}^i\big(\mathfrak{i}_*\mathfrak{i}^*\mathcal{K}^i,\,\mathcal{L}^{\leq D}\big)\\ \longrightarrow \mathrm{Ext}^i\big(\mathcal{K}^i,\mathcal{L}^{\leq D}\big)\longrightarrow \mathrm{Ext}^i\big(\mathfrak{j}^*\mathcal{K}^i,\,\mathfrak{j}^*\mathcal{L}^{\leq D}\big)\end{aligned} \tag{66}$$

Hence, the proof of Theorem 5.15 is reduced to the vanishing $\mathrm{Ext}^i\big(\mathfrak{i}_*\mathfrak{i}^*\mathcal{K}^i,\,\mathcal{L}^{\leq D}\big)=0$ for any $0\leq i\leq n-1$. For that purpose, we have only to show (62). Thus, the proof of Lemma 5.16 is finished. ∎

In the following, we will show $\mathcal{E}xt^i\big(\pi_1^{-1}(I),\mathcal{L}^{\leq D}\big)=0$ $(i=0,\ldots,n-1)$ for any constructible sheaf I on $\pi_0^{-1}(O)\simeq S^1$.

5.5.3 Local form of $\pi_1^{-1}(I)$

Let $(z_1,\ldots,z_n)$ be a coordinate with $z_i^{-1}(0)=D_i$. It induces a coordinate $(\theta_1,\ldots,\theta_n)$ of $\pi_2^{-1}(O)$, which is independent of the choice of $(z_1,\ldots,z_n)$ up to parallel transport. We take a coordinate t of $\mathbb{C}$, which induces a coordinate θ of $\pi_0^{-1}(O)$. The induced map $\pi_2^{-1}(O)\longrightarrow\pi_0^{-1}(O)$ is affine with respect to the coordinates $(\theta_1,\ldots,\theta_n)$ and θ.

Let us consider the behaviour of $\pi_1^{-1}(I)$ around $P\in\pi_2^{-1}(O)$, where I is a constructible sheaf on $\pi_0^{-1}(O)$. We may assume $P=(0,\ldots,0)$. The map $\pi_2^{-1}(O)\longrightarrow\pi_0^{-1}(O)$ is of the form $(\theta_1,\ldots,\theta_n)\longmapsto\sum\alpha_i\,\theta_i+\beta$, where $\beta=\pi_1(P)$. The sheaf I is the direct sum of sheaves of the following forms:

- The constant sheaf around β.
- $j_!\mathbb{C}_J$ or $j_*\mathbb{C}_J$, where J is an open interval such that one of the end points is β, and j denotes the inclusion $J\longrightarrow\pi^{-1}(O)$.

Hence, $\pi_1^{-1}(I)$ around P is described as the direct sum of sheaves of the following forms:

- The constant sheaf $\mathbb{C}_{\pi_0^{-1}(O)}$.
- $j_*\mathbb{C}_H$ or $j_!\mathbb{C}_H$, where H is an open half space such that $\partial H\ni P$, and $j:H\longrightarrow\pi_0^{-1}(O)$. They are denoted by $\mathbb{C}_{H*}$ and $\mathbb{C}_{H!}$.

5.5.4 Local form of $\mathcal{L}^{\leq D}$ and $\mathcal{L}/\mathcal{L}^{\leq D}$

Let $P\in\pi_0^{-1}(O)$. We have a decomposition around P:

$$\mathcal{L}=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\nabla)}\mathcal{L}_{\mathfrak{a}}\qquad \mathcal{L}^{\leq D}=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\nabla)}\mathcal{L}^{\leq D}_{\mathfrak{a}}$$

Let us describe $\mathcal{L}_{\mathfrak{a}}$ and $\mathcal{L}/\mathcal{L}_{\mathfrak{a}}^{\leq D}$ around P. For an appropriate coordinate, $\mathfrak{a} = z_1^{-m_1} \cdots z_n^{-m_n}$ for some $m_i \geq 0$. Let $q_{\mathfrak{a}} : \Delta^n \longrightarrow \Delta$ be given by $(z_1, \ldots, z_n) \longmapsto \prod z_i^{m_i}$. Let $\pi_{\Delta} : \widetilde{\Delta}(0) \longrightarrow \Delta$ be the real blow up. We have the induced map:

$$q_{\mathfrak{a}} : \widetilde{X}(D) \longrightarrow \widetilde{\Delta}(0), \quad (r_i, \theta_i) \longmapsto \Bigl(\prod_{i=1}^{n} r_i^{m_i}, \sum m_i\theta_i\Bigr)$$

Let $\mathcal{Q}$ be the local system on $\widetilde{\Delta}(0)$ with Stokes structure, corresponding to $\bigl(\mathcal{O}_{\Delta}(*0),\, d + d(1/z)\bigr)$. Note that $\mathcal{Q}/\mathcal{Q}^{\leq 0}$ is the constructible sheaf $j_*\mathbb{C}_J$ on $\pi_{\Delta}^{-1}(0)$, where $j : J = (-\pi, \pi) \longrightarrow \pi_{\Delta}^{-1}(0)$. Let $r(\mathfrak{a})$ be the rank of $\mathcal{L}_{\mathfrak{a}}$. We have isomorphisms:

$$\mathcal{L}_{\mathfrak{a}} \simeq q_{\mathfrak{a}}^* \mathcal{Q}^{\oplus\, r(\mathfrak{a})} \qquad \mathcal{L}_{\mathfrak{a}}^{\leq D} \simeq q_{\mathfrak{a}}^*\bigl(\mathcal{Q}^{\leq 0}\bigr)^{\oplus\, r(\mathfrak{a})} \qquad \mathcal{L}_{\mathfrak{a}}/\mathcal{L}_{\mathfrak{a}}^{\leq D} \simeq q_{\mathfrak{a}}^*\bigl(\mathcal{Q}/\mathcal{Q}^{\leq 0}\bigr)^{\oplus\, r(\mathfrak{a})}$$

Around P, we have an isomorphism $q_{\mathfrak{a}}^*\bigl(\mathcal{Q}/\mathcal{Q}^{\leq 0}\bigr) \simeq \iota_*\mathbb{C}$, where $Z := q_{\mathfrak{a}}^{-1}(J)$ and $\iota : Z \longrightarrow (S^1)^n \times \mathbb{R}^n_{\geq 0}$. Note that Z is of the form $Z_0 \times \partial\mathbb{R}^n_{\geq 0}$, where Z_0 is the inverse image of J via the induced map $(S^1)^n \times \{0\} \longrightarrow S^1 \times \{0\}$. Hence, $q_{\mathfrak{a}}^*\bigl(\mathcal{Q}/\mathcal{Q}^{\leq 0}\bigr)$ is isomorphic to one of the following, around P:

- The constant sheaf $\mathbb{C}_{(S^1)^n \times \partial\mathbb{R}^n_{\geq 0}}$.
- $j_{K*}\mathbb{C}_{K \times \partial\mathbb{R}^n_{\geq 0}}$, where K is an open half space such that $\partial K \ni P$, and $j_K : K \times \partial\mathbb{R}^n_{\geq 0} \longrightarrow (S^1)^n \times \mathbb{R}^n_{\geq 0}$. It is denoted by $\mathbb{C}_{K \times \partial\mathbb{R}^n_{\geq 0} *}$.

5.5.5 Proof of Theorem 5.15

We reduce the proof of the theorem to the computation of $\mathcal{E}xt^i\bigl(\pi_1^{-1}I, q_{\mathfrak{a}}^{-1}(\mathcal{Q}/\mathcal{Q}^{\leq 0})\bigr)$ for $i \leq n-2$.

Lemma 5.17 *We have $\mathcal{E}xt^i(\pi_1^{-1}I, q_{\mathfrak{a}}^{-1}\mathcal{Q}) = 0$ for any i. In particular, we have isomorphisms:*

$$\mathcal{E}xt^i\bigl(\pi_1^{-1}I,\, q_{\mathfrak{a}}^{-1}\mathcal{Q}^{\leq 0}\bigr) \simeq \mathcal{E}xt^{i-1}\bigl(\pi_1^{-1}I, q_{\mathfrak{a}}^{-1}(\mathcal{Q}/\mathcal{Q}^{\leq 0})\bigr).$$

Proof Let $\iota : (S^1)^n \times \{0\} \longrightarrow (S^1)^n \times \partial\mathbb{R}^n_{\geq 0}$ denote the inclusion. There exists a constructible sheaf $\mathcal{F}$ on $(S^1)^n$ such that $\pi_1^{-1}I \simeq \iota_*\mathcal{F}$. We have the adjunction $\mathcal{E}xt^i\bigl(\iota_*\mathcal{F},\, q_{\mathfrak{a}}^{-1}\mathcal{Q}\bigr) = \iota_*\mathcal{E}xt^i(\mathcal{F}, i^! q_{\mathfrak{a}}^{-1}\mathcal{Q})$. Note $\iota^! q_{\mathfrak{a}}^{-1}\mathcal{Q} = \mathbb{D}\iota^{-1}\mathbb{D}\bigl(q_{\mathfrak{a}}^{-1}\mathcal{Q}\bigr) = 0$, because $\mathbb{D}q_{\mathfrak{a}}^{-1}\mathcal{Q}$ is 0-extension of a constant sheaf on $(S^1)^n \times \mathbb{R}^n_{>0}$ by $(S^1)^n \times \mathbb{R}^n_{>0} \longrightarrow (S^1)^n \times \mathbb{R}^n_{\geq 0}$. Hence, we obtain $\mathcal{E}xt^i\bigl(\iota_*\mathcal{F}, q_{\mathfrak{a}}^{-1}\mathcal{Q}\bigr) = 0$, and the proof of Lemma 5.17 is finished. ∎

Now, let us show the following vanishing of the stalks at P:

$$\mathcal{E}xt^j\bigl(\pi_1^{-1}I, q_{\mathfrak{a}}^{-1}(\mathcal{Q}/\mathcal{Q}^{\leq 0})\bigr)_P = 0, \quad (j \leq n-2) \tag{67}$$

It can be computed on $(S^1)^n \times \partial\mathbb{R}^n_{\geq 0}$. We have the following cases, divided by the local forms of $\pi_1^{-1}(I)$ and $q_{\mathfrak{a}}^{-1}(\mathcal{Q}/\mathcal{Q}^{\leq 0})$ around P:

(I) $\pi_1^{-1}I \simeq \mathbb{C}_{(S^1)^n}$ and $q_{\mathfrak{a}}^{-1}\bigl(\mathcal{Q}/\mathcal{Q}^{\leq 0}\bigr) \simeq \mathbb{C}_{(S^1)^n \times \partial\mathbb{R}^n_{\geq 0}}$.

(II) $\pi_1^{-1}I \simeq \mathbb{C}_{(S^1)^n}$ and $q_{\mathfrak{a}}^{-1}\bigl(\mathcal{Q}/\mathcal{Q}^{\leq 0}\bigr) \simeq \mathbb{C}_{K \times \partial\mathbb{R}^n_{\geq 0} *}$.

(III) $\pi_1^{-1}I = \mathbb{C}_{H\star}$ and $q_{\mathfrak{a}}^{-1}\bigl(\mathcal{Q}/\mathcal{Q}^{\leq 0}\bigr) \simeq \mathbb{C}_{(S^1)^n \times \partial\mathbb{R}^n_{\geq 0}}$, where $\star = *, !$.

(IV) $\pi_1^{-1}I \simeq \mathbb{C}_{H\star}$ and $q_{\mathfrak{a}}^{-1}\bigl(\mathcal{Q}/\mathcal{Q}^{\leq 0}\bigr) \simeq \mathbb{C}_{K \times \partial\mathbb{R}^n_{\geq 0} *}$, where $\star = *, !$. Moreover, this is divided into three cases (IV-1) ∂H and ∂K are transversal, (IV-2) $H = K$, (IV-3) $H = -K$.

In the following, for a given $i : Y_1 \subset Y_2$ and $\star = *, !$, let $\mathbb{C}_{Y_1\star} := i_\star\mathbb{C}_{Y_1}$ on Y_2. It is also denoted just by $\mathbb{C}_{Y_1}$, if there is no risk of confusion.

The case (I) Instead of $(S^1)^n \times \{0\} \longrightarrow (S^1)^n \times \partial\mathbb{R}^n_{\geq 0}$, we have only to consider the inclusion $\{0\} \longrightarrow \partial\mathbb{R}^n_{\geq 0} \simeq \mathbb{R}^{n-1}$. We obtain (67) from the following standard result:

$$\mathcal{E}xt^j\big(\mathbb{C}_0, \mathbb{C}_{\mathbb{R}^{n-1}}\big)_0 \simeq \begin{cases} 0 & (j \leq n-2) \\ \mathbb{C} & (j = n-1) \end{cases}$$

The case (II) We have the exact sequence $0 \longrightarrow \mathbb{C}_{(S^1)^n \setminus K!} \longrightarrow \mathbb{C}_{(S^1)^n} \longrightarrow \mathbb{C}_{K*} \longrightarrow 0$. Let ι denote the inclusion $\big((S^1)^n \setminus K\big) \times \partial\mathbb{R}^n_{\geq 0} \longrightarrow (S^1)^n \times \partial\mathbb{R}^n_{\geq 0}$. Note $\iota^* = \iota^!$, and hence $\iota^!\mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}*} = 0$. We have

$$\mathcal{E}xt^j\Big(\mathbb{C}_{\big((S^1)^n\setminus K\big)\times\{0\}\,!}, \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}\,*}\Big)_P \simeq \iota_*\mathcal{E}xt^j\Big(\mathbb{C}_{\big((S^1)^n\setminus K\big)\times\{0\}}, \iota^!\mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}*}\Big)_P = 0$$

Hence, we obtain

$$\mathcal{E}xt^j\Big(\mathbb{C}_{(S^1)^n}, \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}\,*}\Big)_P \simeq \mathcal{E}xt^j\Big(\mathbb{C}_{K*}, \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}*}\Big)_P = \begin{cases} 0 & (j \leq n-2) \\ \mathbb{C} & (j = n-1) \end{cases}$$

The case (III) Let us consider the case $\star = *$. We have the exact sequence:

$$0 \longrightarrow \mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}\setminus H\times\{0\}\,!} \longrightarrow \mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}} \longrightarrow \mathbb{C}_{H*} \longrightarrow 0$$

Let k_1 denote the inclusion $H\times\{0\} \longrightarrow (S^1)^n\times\partial\mathbb{R}^n_{\geq 0}$, and let k_2 denote the open embedding of the complement. Because $k_1^*\mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}\setminus H\times\{0\}\,!} = 0$, we have the following isomorphisms:

$$\begin{aligned} R\mathcal{H}om\big(\mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}\setminus H\times\{0\}\,!}, \mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}}\big)_P &\simeq R\mathcal{H}om\big(\mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}\setminus H\times\{0\}!}, \mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}\setminus H\times\{0\}!}\big)_P \\ &\simeq k_{2*}\big(\mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}\setminus H\times\{0\}}\big)_P \simeq \big(\mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}}\big)_P \end{aligned} \tag{68}$$

Hence, we obtain $R\mathcal{H}om\big(\mathbb{C}_{H*}, \mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}}\big)_P = 0$. In particular, $\mathcal{E}xt^j(\mathbb{C}_{H*}, \mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}})_P = 0$ for any j.

Let us consider the case $\star =!$. We have the exact sequence $0 \longrightarrow \mathbb{C}_{H!} \longrightarrow \mathbb{C}_{(S^1)^n} \longrightarrow \mathbb{C}_{(S^1)^n\setminus H\,*} \longrightarrow 0$. Hence, we obtain the following isomorphisms:

$$\mathcal{E}xt^j\big(\mathbb{C}_{H!}, \mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}}\big)_P = \mathcal{E}xt^j\big(\mathbb{C}_{(S^1)^n}, \mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}}\big)_P = \begin{cases} 0 & (j \leq n-2) \\ \mathbb{C} & (j = n-1) \end{cases}$$

The case (IV-1) Let us consider the case $\star = *$. Let $\mathcal{N}$ be the kernel of $\mathbb{C}_{H*} \longrightarrow \mathbb{C}_{H\cap K*}$.

Lemma 5.18 *We have* $R\mathcal{H}om\big(\mathcal{N}, \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}*}\big)_P = 0$.

Proof Let ι be the inclusion $\big((S^1)^n \setminus K\big) \times \partial\mathbb{R}^n_{\geq 0} \longrightarrow (S^1)^n \times \partial\mathbb{R}^n_{\geq 0}$. Then, $\mathcal{N}$ is of the form $\iota_!\mathcal{N}_1$. Then, the claim follows from $\iota^!\mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}*}$. ∎

We have the exact sequence: $0 \longrightarrow \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}\setminus(H\cap K)\times\{0\}!} \longrightarrow \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}} \longrightarrow \mathbb{C}_{(H\cap K)\times\{0\}*} \longrightarrow 0$. Let k denote the inclusion $K\times\partial\mathbb{R}^n_{\geq 0} \setminus (H\cap K)\times\{0\} \longrightarrow K\times\partial\mathbb{R}^n_{\geq 0}$. We have the following isomorphisms:

$$\begin{aligned} R\mathcal{H}om\big(\mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}\setminus(H\cap K)\times\{0\}!}, \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}}\big)_P &\simeq Rk_*R\mathcal{H}om\big(\mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}\setminus(H\cap K)\times\{0\}}, \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}\setminus(H\cap K)\times\{0\}}\big)_P \\ &\simeq \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0},P} \end{aligned} \tag{69}$$

Hence, we obtain $R\mathcal{H}om\big(\mathbb{C}_{(H\cap K)\times\{0\}\,*}, \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}\,*}\big)_P = 0$, and $\mathcal{E}xt^j\big(\mathbb{C}_{H*}, \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}\,*}\big)_P = 0$ for any j.

Let us consider the case $\star =!$. We have an exact sequence $0 \longrightarrow \mathbb{C}_{H!} \longrightarrow \mathbb{C}_{(S^1)^n} \longrightarrow \mathbb{C}_{(S^1)^n\setminus H*} \longrightarrow 0$ on $(S^1)^n$. By using the previous results, we obtain

$$\mathcal{E}xt^j\big(\mathbb{C}_{H!}, \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}*}\big)_P = \begin{cases} 0 & (j \leq n-2) \\ \mathbb{C} & (j = n-1) \end{cases}$$

The case (IV-2) Let us consider the case $\star = *$. By considering $0 \longrightarrow \partial\mathbb{R}^n_{\geq 0}$, we obtain

$$\mathcal{E}xt^j\big(\mathbb{C}_{H*}, \mathbb{C}_{H\times\partial\mathbb{R}^n_{\geq 0}*}\big)_P \simeq \begin{cases} 0 & (j \leq n-2) \\ \mathbb{C} & (j = n-1) \end{cases}$$

Let us consider the case $\star =!$. We have an exact sequence $0 \longrightarrow \mathbb{C}_{H!} \longrightarrow \mathbb{C}_{H*} \longrightarrow \mathbb{C}_{\partial H*} \longrightarrow 0$. Let us look at $\mathcal{E}xt^j\big(\mathbb{C}_{\partial H*},\, \mathbb{C}_{H\times\partial\mathbb{R}^n_{\geq 0}}\big)$. For $0 \longrightarrow [0,1[\times\mathbb{R}^{n-1}$, we have $\mathcal{E}xt^j\big(\mathbb{C}_0, \mathbb{C}_{[0,1[\times\mathbb{R}^{n-1}}\big) = 0$ for any j. Hence, we obtain

$$\mathcal{E}xt^j\big(\mathbb{C}_{H!},\, \mathbb{C}_{H\times\partial\mathbb{R}^n_{\geq 0}}\big) = \begin{cases} 0 & (j \leq n-2) \\ \mathbb{C} & (j = n-1) \end{cases}$$

The case (IV-3) It is easy to show $\mathcal{E}xt^j\big(\mathbb{C}_{H!}, \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}}\big) = 0$ for any j. By using the argument in (IV-2), we can show $\mathcal{E}xt^j\big(\mathbb{C}_{H*}, \mathbb{C}_{K\times\partial\mathbb{R}^n}\big) = 0$ for any j. Thus, the proof of Theorem 5.15 is finished. ∎

5.5.6 Some uniqueness of K-structure

We use the notation in Subsection 5.5.1. Let V be a good meromorphic flat bundle on (X,D). Let g be a holomorphic function on X such that $g^{-1}(0) = D$, and let i_g be the graph $X \longrightarrow X \times \mathbb{C}$. We regard $\mathrm{DR}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}(i_{g\dagger}V)$ as a cohomologically constructible sheaf on $\widetilde{X}$.

Let K be a subfield of $\mathbb{C}$. A K-structure of $\mathrm{DR}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}\big(i_{g\dagger}V\big)$ is defined to be a K-cohomologically constructible complex $\mathcal{F}$ on $\widetilde{X}$ with an isomorphism $\alpha : \mathcal{F}\otimes\mathbb{C} \simeq \mathrm{DR}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}(i_{g\dagger}V)$ in the derived category. Two K-structures $(\mathcal{F}_i, \alpha_i)$ $(i = 1,2)$ are called equivalent, if there exists an isomorphism $\beta : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ for which the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{F}_1\otimes\mathbb{C} & \xrightarrow{\beta\otimes\mathbb{C}} & \mathcal{F}_2\otimes\mathbb{C} \\ {\scriptstyle\alpha_1}\big\downarrow & & {\scriptstyle\alpha_2}\big\downarrow \\ \mathrm{DR}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}(i_{g\dagger}V) & \xrightarrow{=} & \mathrm{DR}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}(i_{g\dagger}V) \end{array}$$

Lemma 5.19 *Let $(\mathcal{F}_i, \alpha_i)$ $(i = 1,2)$ be K-structures of $\mathrm{DR}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}\big(i_{g\dagger}V\big)$. If their restriction to $\pi_1^{-1}(X - D^{[2]})$ are equivalent, then they are equivalent on $\widetilde{X}$.*

Proof We put $\mathcal{F}_i^{\mathbb{C}} := \mathcal{F}_i\otimes\mathbb{C}$. We have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{F}_1,\mathcal{F}_2)\otimes\mathbb{C} & \longrightarrow & \mathrm{Hom}\big(\mathcal{F}_{1|\pi_1^{-1}(X-D^{[2]})}, \mathcal{F}_{2|\pi_1^{-1}(X-D^{[2]})}\big)\otimes\mathbb{C} \\ \big\downarrow\simeq & & \big\downarrow\simeq \\ \mathrm{Hom}\big(\mathcal{F}_1^{\mathbb{C}},\mathcal{F}_2^{\mathbb{C}}\big) & \longrightarrow & \mathrm{Hom}\big(\mathcal{F}^{\mathbb{C}}_{1|\pi_1^{-1}(X-D^{[2]})}, \mathcal{F}^{\mathbb{C}}_{2|\pi_1^{-1}(X-D^{[2]})}\big) \end{array}$$

According to Theorem 5.15, the horizontal arrows are injective. Hence, $\mathrm{Hom}(\mathcal{F}_1,\mathcal{F}_2)$ is the intersection of $\mathrm{Hom}\big(\mathcal{F}_{1|\pi_1^{-1}(X-D^{[2]})}, \mathcal{F}_{2|\pi_1^{-1}(X-D^{[2]})}\big)$ and $\mathrm{Hom}\big(\mathcal{F}_1^{\mathbb{C}},\mathcal{F}_2^{\mathbb{C}}\big)$ in $\mathrm{Hom}\big(\mathcal{F}^{\mathbb{C}}_{1|\pi_1^{-1}(X-D^{[2]})}, \mathcal{F}^{\mathbb{C}}_{2|\pi_1^{-1}(X-D^{[2]})}\big)$. Then, the element of $\mathrm{Hom}(\mathcal{F}_1^{\mathbb{C}},\mathcal{F}_2^{\mathbb{C}})$ corresponding to the identity of $\mathrm{DR}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}(i_{g\dagger}V)$ comes from $\mathrm{Hom}(\mathcal{F}_1,\mathcal{F}_2)$. ∎

6 Good pre-K-holonomic $\mathcal{D}$-modules

6.1 Good K-structure and the associated pre-K-Betti structure

6.1.1 Good meromorphic flat bundle with good K-structure

Let $K \subset \mathbb{C}$ be a subfield. Let X be a complex manifold with a normal crossing hypersurface D. Let V be a good meromorphic flat bundle on (X,D).

Definition 6.1 *We say that V has a good K-structure, if the flat bundle $V_{|X-D}$ has a pre-K-Betti structure such that any Stokes filtrations are defined over K.* ∎

Let $D = D_1 \cup D_2$ be a decomposition. Recall that the complex $\mathrm{DR}^{<D_1 \le D_2}_{\widetilde{X}(D)}\big(\mathcal{M}\big)$ is quasi-isomorphic to its 0-th cohomology sheaf $\mathcal{L}^{<D_1 \le D_2}$. (See Subsection 5.1.2.) It is naturally equipped with a K-structure $\mathcal{L}^{<D_1 \le D_2}_K$, if V has a good K-structure. By Corollary 3.16 and (45), we obtain a pre-K-Betti structure

$$\mathcal{F}_V^{<D_1 \le D_2} := R\pi_* \mathcal{L}_K^{<D_1 \le D_2}$$

of the holonomic $\mathcal{D}$-module $V\big(!D_1\big)$. These pre-K-Betti structures are called canonical. Let $D_1' \cup D_2' = D$ be another decomposition such that $D_1 \subset D_1'$. The natural morphism $V\big(!D_1'\big) \longrightarrow V\big(!D_1\big)$ is compatible with the pre-K-Betti structures. We also use the symbols $\mathcal{F}_{V*}$ and $\mathcal{F}_{V!}$ to denote $\mathcal{F}_V^{\le D}$ and $\mathcal{F}_V^{<D}$, respectively. We also use the symbol $\mathcal{F}_V$ to denote $\mathcal{F}_{V*}$ for simplicity.

More generally, let $\iota : Z \subset X$ be a complex submanifold with a normal crossing hypersurface D_Z. Let V_Z be a meromorphic flat bundle on (Z, D_Z). We say $\iota_\dagger V_Z$ has a good K-structure if V_Z has a good K-structure in the above sense. The canonical pre-K-Betti structures for $\iota_\dagger V_Z(!D_{Z,1})$ are also defined in a similar way for a decomposition $D_Z = D_{Z,1} \cup D_{Z,2}$.

6.1.2 Induced pre K-Betti structures on the nearby cycle functor and the maximal functor

We set $X := \Delta^n$ and $D := \bigcup_{i=1}^{\ell} \{z_i = 0\}$. Let V be a good meromorphic flat bundle on (X, D) with a good K-structure. For each $I \subset \underline{\ell}$, we set $I_{!i} := I \cup \{i\}$ and $I_{*i} := I \setminus \{i\}$. The $\mathcal{D}$-module

$$\Pi^{a,b}_{i\star}\big(V(!D(I))\big) = \Big(V \otimes \mathfrak{I}^{a,b}_{z_i}\Big)(!D(I_{\star i}))$$

has the canonical pre-K-Betti structure, where $\star = *, !$. Hence, $\psi_i\big(V(!D(I))\big)$ and $\Xi_i\big(V(!D(I))\big)$ have the induced pre-K-Betti structures.

Lemma 6.2 *The induced pre-K-Betti structure of $\psi_i(V)_{|D_i \setminus \partial D_i}$ is good, i.e., it is compatible with the Stokes filtrations. Moreover, the induced pre-K-Betti structure of $\psi_i\big(V(!D(I))\big)$ is canonical for each $I \subset \underline{\ell}$.*

Proof We have only to consider the case $i = 1$. We give a preparation. By Lemma 3.17, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{DR}_X\Big(\Pi^{-\infty,0}_{1!}\big(V(!D(I))\big)\Big) & \longrightarrow & \mathrm{DR}_X\Big(\Pi^{-\infty,0}_{1*}\big(V(!D(I))\big)\Big) \\
\simeq\big\uparrow & & \simeq\big\uparrow \\
\mathrm{DR}_X^{<D(I_{*1})}\big(\Pi^{-\infty,0}_{1!} V\big) & \longrightarrow & \mathrm{DR}_X^{<D(I_{*1})}\big(\Pi^{-\infty,0}_{1*} V\big) \\
\simeq\big\uparrow & & \simeq\big\uparrow \\
\mathrm{DR}_X^{<D(I_{!1})}\big(V \otimes \mathfrak{I}^{-\infty,0}_{z_1}\big) & \longrightarrow & \mathrm{DR}_X^{<D(I_{*1})}\big(V \otimes \mathfrak{I}^{-\infty,0}_{z_1}\big)
\end{array} \tag{70}$$

By the upper square, the induced K-structure of $\mathrm{DR}_X\,\psi_1\big(V(!D(I))\big)$ can be identified with the K-structure of the following:

$$\mathrm{DR}_X^{<D(I_{*1})}\,\psi_1(V) \simeq \mathrm{Cone}\Big(\mathrm{DR}_X^{<D(I_{*1})}\big(\Pi^{-\infty,0}_{1!} V\big) \longrightarrow \mathrm{DR}_X^{<D(I_{*1})}\big(\Pi^{-\infty,0}_{1*} V\big)\Big) \tag{71}$$

We set $D' := \bigcup_{i=2}^{\ell} D_i$. Let $\pi_1 : \widetilde{X}(D') \longrightarrow X$ be the real blow up. We obtain (71) as the push-forward of the following on $\widetilde{X}(D')$:

$$\mathrm{DR}^{<D(I_{*1}) \le D(\underline{\ell} - I_{!1})}_{\widetilde{X}(D')}\,\psi_1(V) \simeq \mathrm{Cone}\Big(\mathrm{DR}^{<D(I_{*1}) \le D(\underline{\ell} - I_{!1})}_{\widetilde{X}(D')}\big(\Pi^{-\infty,0}_{1!} V\big) \longrightarrow \mathrm{DR}^{<D(I_{*1}) \le D(\underline{\ell} - I_{!1})}_{\widetilde{X}(D')}\big(\Pi^{-\infty,0}_{1*} V\big)\Big) \tag{72}$$

We prepare some commutative diagram in a general setting. For a holonomic $\mathcal{D}_X$-module $\mathcal{M}$, we put

$$\mathrm{DR}^{<D(I_{!1}) \le D(\underline{\ell} - I_{!1})}_{\widetilde{X}(D')}\,\mathcal{M} := \Omega^{<D(I_{!1}) \le D(\underline{\ell} - I_{!1})}_{\widetilde{X}(D')} \otimes_{\pi_1^{-1}\mathcal{O}_X} \pi_1^{-1}\mathcal{M}[\dim X]$$

$$\mathrm{DR}^{<D(I_{*1})\leq D(\underline{\ell}-I_{*1})}_{\widetilde{X}(D')}\mathcal{M}:=\Omega^{<D(I_{*1})\leq D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D')}(*D_1)\otimes_{\pi_1^{-1}\mathcal{O}_X}\pi_1^{-1}\mathcal{M}[\dim X]$$

We have the following commutative diagram obtained from a commutative diagram similar to (14):

$$\begin{tikzcd}
\mathrm{DR}^{<D(I_{*1})\leq D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D')}\mathcal{M}(!D_1) \arrow[r] & \mathrm{DR}^{<D(I_{*1})\leq D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D')}\mathcal{M}(*D_1) \\
\mathrm{DR}^{<D(I_{!1})\leq D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D')}\mathcal{M} \arrow[r] \arrow[u] & \mathrm{DR}^{<D(I_{*1})\leq D(\underline{\ell}-I_{*1})}_{\widetilde{X}(D')}\mathcal{M} \arrow[u, "="']
\end{tikzcd}$$

If $\mathcal{M}$ is a good meromorphic flat bundle, the left vertical arrow is also quasi-isomorphism, which can be shown by the argument in the proof of Proposition 3.15.

Let $\rho:\widetilde{X}(D)\longrightarrow\widetilde{X}(D')$ be the induced map. We have the following natural commutative diagram, where the vertical arrows are quasi-isomorphisms by Theorem 4.9:

$$\begin{tikzcd}
\mathrm{DR}^{<D(I_{!1})\leq D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D')}\mathcal{M} \arrow[r] \arrow[d, "\simeq"'] & \mathrm{DR}^{<D(I_{*1})\leq D(\underline{\ell}-I_{*1})}_{\widetilde{X}(D')}\mathcal{M} \arrow[d, "\simeq"'] \\
\rho_*\,\mathrm{DR}^{<D(I_{!1})\leq D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D)}\mathcal{M} \arrow[r] & \rho_*\,\mathrm{DR}^{<D(I_{*1})\leq D(\underline{\ell}-I_{*1})}_{\widetilde{X}(D)}\mathcal{M}
\end{tikzcd}$$

Thus, we obtain the following commutative diagram, in which the vertical arrows are quasi-isomorphisms:

$$\begin{tikzcd}
\mathrm{DR}^{<D(I_{*1})\leq D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D')}\big(\Pi^{-\infty,0}_{1!}V\big) \arrow[r] & \mathrm{DR}^{<D(I_{*1})\leq D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D')}\big(\Pi^{-\infty,0}_{1*}V\big) \\
\rho_*\,\mathrm{DR}^{<D(I_{!1})\leq D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D)}\big(V\otimes\mathfrak{I}^{-\infty,0}_{z_1}\big) \arrow[r] \arrow[u, "\simeq"] & \rho_*\,\mathrm{DR}^{<D(I_{*1})\leq D(\underline{\ell}-I_{*1})}_{\widetilde{X}(D)}\big(V\otimes\mathfrak{I}^{-\infty,0}_{z_1}\big) \arrow[u, "\simeq"]
\end{tikzcd} \tag{73}$$

Because $\mathrm{DR}^{<D(I_{!1})\leq D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D)}\big(V\otimes\mathfrak{I}^{-\infty,0}_{z_1}\big)$ and $\mathrm{DR}^{<D(I_{*1})\leq D(\underline{\ell}-I_{*1})}_{\widetilde{X}(D)}\big(V\otimes\mathfrak{I}^{-\infty,0}_{z_1}\big)$ are equipped with K-structures compatible with the morphism, we obtain a K-structure of $\mathrm{DR}^{<D(I_{*1})\leq D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D')}\psi_1(V)$ from (72) and (73). Moreover, the lower square in (70) is obtained as the push-forward of (73). Hence, the K-structure of $\mathrm{DR}_X\,\psi_1\big(V(!D(I))\big)$ is obtained as the push-forward of the K-structure of $\mathrm{DR}^{<D(I_{*1})\leq D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D')}\psi_1(V)$.

Let us consider the case $I=\emptyset$. By the above consideration, we obtain that $\mathcal{F}^P_{\leq 0}$ is compatible with the K-structure, where $\mathcal{F}^P$ denotes the Stokes filtration of $\psi_1(V)$ at each point $P\in\pi_1^{-1}(\partial D_1)$. By considering the tensor product with meromorphic flat bundles with rank one, we can deduce that $\mathcal{F}^P$ is defined over K. Since the pre-K-Betti structure of $\psi_1\big(V(!D(I))\big)$ comes from the K-structure of $\mathrm{DR}^{<D(I_{*1})\leq D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D')}\psi_1(V)$, it is canonical. ∎

6.1.3 Good holonomic D-module with good K-structure

Let $\mathcal{M}$ be a good holonomic $\mathcal{D}$-module on (X,D).

Definition 6.3 *We say that $\mathcal{M}$ has a good K-structure, if (i) $\phi_I(\mathcal{M})\big(*D(I^c)\big)$ has a good K-structure, (put $\phi_\emptyset(\mathcal{M}):=\mathcal{M}$), (ii) the induced morphisms*

$$\psi_i\phi_I(\mathcal{M})(*D(I^c))\longrightarrow\phi_i\phi_I(\mathcal{M})(*D(I^c))\longrightarrow\psi_i\phi_I(\mathcal{M})(*D(I^c))\qquad(i\not\in I)$$

are compatible with the K-structures. ∎

A morphism of good holonomic D-module with good K-structures $f:\mathcal{M}_1\longrightarrow\mathcal{M}_2$ is defined to be a morphism of D-modules such that $\phi_I(f)$ are compatible with K-structures for any $I\subset\underline{\ell}$. Let $\mathrm{Hol}^{\mathrm{good}}(X,D,K)$ denote the category of good holonomic D-modules with good K-structures on (X,D). It is an abelian category.

Let V be a good meromorphic flat bundle on X with a good K-structure. Then, we naturally have $\phi_J\big(V\big(!D(I)\big)\big)\big(*D(J^c)\big)\simeq\psi_J(V)$ for any $I,J\subset\underline{\ell}$, which are equipped with good K-structures. Via these K-structures, we regard $V(!D(I))\in\mathrm{Hol}^{\mathrm{good}}(X,D,K)$.

Lemma 6.4 *Let $\mathcal{M} \in \mathrm{Hol}^{\mathrm{good}}(X, D, K)$ such that $\mathcal{M}\big(*D(J)\big) = \mathcal{M}$. Let $I \subset \underline{\ell}$ such that $|I| = n - \dim \operatorname{Supp} \mathcal{M}$ and $V_I = \phi_I(\mathcal{M})\big(*D(I^c)\big) \neq 0$. Then, the morphisms $i_\dagger V_I\big(!D(I^c) * D(J)\big) \longrightarrow \mathcal{M} \longrightarrow i_\dagger V_I$ are compatible with good K-structures.*

Proof We set $\mathcal{M}_1 := i_\dagger V_I\big(!D(I^c) * D(J)\big)$ and $\mathcal{M}_2 := i_\dagger V_I$. Let us show that $\phi_L(\mathcal{M}_1)\big(*D(L^c)\big) \longrightarrow \phi_L(\mathcal{M})\big(*D(L^c)\big) \longrightarrow \phi_L(\mathcal{M}_2)\big(*D(L^c)\big)$ are compatible with K-structures for any $L \subset \underline{\ell}$. We have only to consider the case $L \supset I$. In the case $L = I$, it is clear. Assume that we have already known the compatibility for L. We set $L_j := L \sqcup \{j\} \subset \underline{\ell}$. We obtain the following morphisms compatible with K-structures:

$$\begin{aligned}\psi_j\Big(\phi_L\big(\mathcal{M}_1\big)(*D(L^c))\Big) \longrightarrow \psi_j\Big(\phi_L(\mathcal{M})(*D(L^c))\Big) \longrightarrow \phi_j\phi_L(\mathcal{M})\big(*D(L_j^c)\big) \\ \longrightarrow \psi_j\Big(\phi_L(\mathcal{M})\big(*D(L^c)\big)\Big) \longrightarrow \psi_j\Big(\phi_L(\mathcal{M}_2)\big(*D(L^c)\big)\Big) \quad (74)\end{aligned}$$

It implies the compatibility for L_j. ∎

6.1.4 Pre-K-Betti structures for $\Xi_I\psi_J(\iota_\dagger V_I)$

Let $K \sqcup J \sqcup I = L \subset \underline{\ell}$. Let V_I be a good meromorphic flat bundle on $(D_I, \partial D_I)$. Let $\iota : D_I \longrightarrow X$. For a map $f : K \sqcup J \longrightarrow \{0, 1\}$, we set $K_0(f) := f^{-1}(0) \cap K$. We put

$$\mathcal{C}_f(J, K, \iota_\dagger V_I) := \Big(\iota_\dagger V_I \otimes \bigotimes_{k \in K_0(f)} \mathfrak{I}_{z_k}^{-\infty,1} \otimes \bigotimes_{k \notin K_0(f)} \mathfrak{I}_{z_k}^{-\infty,0}\Big)\big(!D(f^{-1}(0))\big).$$

Let $\mathbf{0}$ denote the constant map valued in $\{0\}$. Let $\boldsymbol{\delta}_i$ denote the map such that $\boldsymbol{\delta}_i(j) = 0$ $(j \neq i)$ and $\boldsymbol{\delta}_i(i) = 1$. We can identify $\Xi_K\psi_J\big(\iota_\dagger V_I\big)$ as the kernel of the following morphism:

$$\mathcal{C}_{\mathbf{0}}\big(J, K, \iota_\dagger V_I\big) \longrightarrow \bigoplus_{i \in K \sqcup J} \mathcal{C}_{\boldsymbol{\delta}_i}\big(J, K, \iota_\dagger V_I\big) \tag{75}$$

If V_I has a good K-structure, we obtain a pre-K-Betti structure of $\Xi_K\psi_J(\iota_\dagger V_I)$ by (75).

Lemma 6.5 *For $i \notin L$, we set $Ki := K \cup \{i\}$ and $Ji := J \cup \{i\}$. The following morphisms are compatible with the pre-K-Betti structures:*

$$\Xi_K\psi_{Ji}\big(\iota_\dagger V_I\big) \longrightarrow \Xi_{Ki}\psi_J\big(\iota_\dagger V_I\big) \longrightarrow \Xi_K\psi_{Ji}\big(\iota_\dagger V_I\big)$$

Proof It is clear by construction. ∎

Recall that we have the naturally induced good K-structure on $\psi_i\big(\iota_\dagger V_I\big)$ for $i \notin I$ (Lemma 6.2).

Lemma 6.6 *For $i \notin L$, the natural isomorphism $\Xi_K\psi_{Ji}\big(\iota_\dagger V_I\big) \simeq \Xi_K\psi_J\Big(\psi_i\big(\iota_\dagger V_I\big)\Big)$ is compatible with the induced K-structures.*

Proof Both the K-structures are obtained as the kernel of the morphism (75) for (Ji, K). ∎

6.1.5 Functoriality

Let $D_1 \cup D_2$ be a decomposition of D. Let D_3 be a hypersurface of X. Let $\varphi : X' \longrightarrow X$ be a proper birational morphism such that (i) $D' := \varphi^{-1}(D_3 \cup D)$ is normal crossing, (ii) $X' - D' \simeq X - (D_3 \cup D)$. Let V be a good meromorphic flat bundle on (X, D) with a good K-structure. We put $V' := \varphi^* V \otimes \mathcal{O}_{X'}(*D')$. We set $D_1' := \varphi^{-1}(D_1)$. We take $D_2' \subset D'$ such that $D_1' \cup D_2'$ is a decomposition.

Proposition 6.7 *V' is equipped with an induced K-good structure. Moreover, the natural morphisms*

$$V(!D_1) \longrightarrow \varphi_\dagger V'(!D_1'), \qquad \varphi_\dagger V'(!D_2') \longrightarrow V(!D_2)$$

are compatible with the canonical pre-K-Betti structures.

Proof The first claim is easy to see. Let us show the second claim. We use the notation in Subsection 5.3. Let $\widetilde{\varphi} : \widetilde{X}'(D') \longrightarrow \widetilde{X}(D)$ be the induced map. By construction, it is easy to see that the morphisms $\mathrm{DR}^{<D_1 \leq D_2}_{\widetilde{X}(D)}(V) \longrightarrow R\widetilde{\varphi}_* \mathrm{DR}^{<D'_1 \leq D'_2}_{\widetilde{X}'(D')}(V')$ and $R\widetilde{\varphi}_* \mathrm{DR}^{<D'_2 \leq D'_1}_{\widetilde{X}'(D')}(V') \longrightarrow \mathrm{DR}^{<D_2 \leq D_1}_{\widetilde{X}(D)}(V)$ are compatible with the induced K-structures. Then, the second claim follows from Theorem 5.12. ∎

6.2 The associated pre-K-Betti structure

6.2.1 ℓ-squares of complexes

For small categories A_i $(i = 1, \ldots, \ell)$, let $\prod_{i=1}^{\ell} A_i$ denote their product, i.e., the category whose objects and morphisms are given by $\mathrm{ob}\Bigl(\prod_{i=1}^{\ell} A_i\Bigr) = \prod_{i=1}^{\ell} \mathrm{ob}(A_i)$ and $\mathrm{Mor}(\boldsymbol{a}, \boldsymbol{b}) = \prod \mathrm{Mor}(a_i, b_i)$. Let Γ be a small category given by the following commutative diagram:

$$\begin{array}{ccc} (0,0) & \xrightarrow{a} & (0,1) \\ {\scriptstyle b}\big\downarrow & & {\scriptstyle c}\big\downarrow \\ (1,0) & \xrightarrow{d} & (1,1) \end{array} \qquad c \circ a = d \circ b$$

Let A be an abelian category. Let $C(A)$ be the category of complexes in A. A square in $C(A)$ is a functor $F : \Gamma \longrightarrow C(A)$. For a given square F, let $H(F)$ be the total complex of the following double complex:

$$F(0,0)[1] \xrightarrow{F(a)+F(b)} F(0,1) \oplus F(1,0) \xrightarrow{F(c)-F(d)} F(1,1)[-1]$$

An ℓ-square in $C(A)$ is a functor $F : \Gamma^{\ell} \longrightarrow C(A)$. Let $\pi_i : \Gamma^{\ell} \longrightarrow \Gamma^{\ell-1}$ be the projection forgetting the i-th component. For a given ℓ-square F, we obtain an $(\ell-1)$-square $\pi_{i*}F$ by $\pi_{i*}F(\boldsymbol{a}) = H\bigl(F_{|\pi_i^{-1}(\boldsymbol{a})}\bigr)$.

Lemma 6.8 *For $i < j$, we have an isomorphism $\pi_{i*}\pi_{j*}F \simeq \pi_{j-1*}\pi_{i*}F$.*

Proof We have only to consider the case $\ell = 2$, $(i,j) = (1,2)$. The i-th terms of the both complexes are given by

$$\bigoplus_{a_1+a_2+b_1+b_2=i-2} F(a_1, a_2, b_1, b_2).$$

The multiplication of -1 on $F(0,0,0,0) \oplus F(1,1,0,0) \oplus F(0,0,1,1) \oplus F(1,1,1,1)$ interpolates the differentials for $\pi_{i*}\pi_{j*}F$ and $\pi_{j-1*}\pi_{i*}F$. ∎

For any subset $I \subset \underline{\ell}$, let $\pi_I : \Gamma^{\ell} \longrightarrow \Gamma^I$ be the naturally defined projection. We take $I = I_0 \subset I_1 \subset \cdots \subset I_m = \underline{\ell}$, which induces the factorization $\pi_I = \pi^{(1)} \circ \pi^{(2)} \circ \cdots \circ \pi^{(m)}$, where $\pi^{(i)} : \Gamma^{I_i} \longrightarrow \Gamma^{I_{i-1}}$. Then, we set $\pi_{I*}F := \pi^{(1)}_* \circ \cdots \circ \pi^{(m)}_* F$. It is well defined up to isomorphisms as above.

6.2.2 The associated pre-K-Betti structure

Let $\mathcal{M}$ be a good holonomic $\mathcal{D}$-module on (X,D). Let $H \subset \underline{\ell}$. Let us construct an H-cube in the category of good holonomic $\mathcal{D}$-modules on (X,D). For $(\boldsymbol{i}, \boldsymbol{j}) = \bigl((i_k, j_k) \,\big|\, k \in H\bigr) \in \mathrm{ob}\,\Gamma^H$, we have the following subsets of H:

$$I(\boldsymbol{i},\boldsymbol{j}) = \bigl\{k \,\big|\, (i_k,j_k) = (0,1)\bigr\}, \quad J(\boldsymbol{i},\boldsymbol{j}) = \bigl\{k \,\big|\, (i_k,j_k) = (0,0), \text{ or } (1,1)\bigr\}, \quad K(\boldsymbol{i},\boldsymbol{j}) = \bigl\{k \,\big|\, (i_k,j_k) = (1,0)\bigr\}$$

Then, we put $\mathcal{Q}^H(\mathcal{M}, \boldsymbol{i}, \boldsymbol{j}) := \Xi_{I(\boldsymbol{i},\boldsymbol{j})}\psi_{J(\boldsymbol{i},\boldsymbol{j})}\phi_{K(\boldsymbol{i},\boldsymbol{j})}\mathcal{M}$. For $k_0 \notin H$, we have the following naturally induced diagram:

$$\begin{array}{ccc} \psi_{k_0}\Xi_I\psi_J\phi_K\mathcal{M} & \longrightarrow & \Xi_{k_0}\Xi_I\psi_J\phi_K\mathcal{M} \\ \big\downarrow & & \big\downarrow \\ \phi_{k_0}\Xi_I\psi_J\phi_K\mathcal{M} & \longrightarrow & \psi_{k_0}\Xi_I\psi_J\phi_K\mathcal{M} \end{array} \tag{76}$$

For each decomposition $H = \{h\} \cup (H - \{h\})$, we have a similar diagram. Thus, we obtain an H-cube $\mathcal{Q}^H(\mathcal{M})$ of good holonomic $\mathcal{D}$-modules. The cohomology associated to (76) is naturally isomorphic to $\Xi_I \psi_J \phi_K \mathcal{M}$. Hence, we have a natural quasi-isomorphism $\pi_{H*} \mathcal{Q}^{\underline{\ell}}(\mathcal{M}) \simeq \mathcal{Q}^H(\mathcal{M})$. In particular, we have a quasi-isomorphism $\pi_{\underline{\ell}*} \mathcal{Q}^{\underline{\ell}}(\mathcal{M}) \simeq \mathcal{M}$.

If $\mathcal{M}$ has a good K-structure, each $\mathcal{Q}^{\underline{\ell}}(\mathcal{M}, \boldsymbol{i}, \boldsymbol{j})$ is equipped with the pre-K-Betti structure $\mathcal{F}^{\underline{\ell}}_{\mathcal{M}}(\boldsymbol{i}, \boldsymbol{j})$, given as in Subsection 6.1.4.

Lemma 6.9 *The morphisms are compatible with the pre-K-Betti structures.*

Proof Let us consider the morphisms in the diagram (76). The morphisms

$$\psi_{k_0} \Xi_I \psi_J \phi_K \mathcal{M} \longrightarrow \Xi_{k_0} \Xi_I \psi_J \phi_K \mathcal{M} \longrightarrow \psi_{k_0} \Xi_I \psi_J \phi_K \mathcal{M}$$

are compatible with the pre-K-Betti structures by construction, as remarked in Lemma 6.5. Let $K' := \underline{\ell} - (K \sqcup k_0)$. By definition, the morphisms

$$\psi_{k_0} \phi_K \mathcal{M}(*D(K')) \longrightarrow \phi_{k_0} \phi_K \mathcal{M}(*D(K')) \longrightarrow \psi_{k_0} \phi_K \mathcal{M}(*D(K'))$$

are compatible with the K-structures. We remark Lemma 6.6, and then it follows that

$$\psi_{k_0} \Xi_I \psi_J \phi_K \mathcal{M} \longrightarrow \phi_{k_0} \Xi_I \psi_J \phi_K \mathcal{M} \longrightarrow \psi_{k_0} \Xi_I \psi_J \phi_K \mathcal{M}$$

are compatible with the pre-K-Betti structures. ∎

Thus, we obtain a pre-K-Betti structure of $\pi_{\underline{\ell}*} \mathcal{Q}^{\underline{\ell}}(\mathcal{M}) \simeq \mathrm{DR}\, \mathcal{M}$, which is independent of the choice of a factorization of $\pi_{\underline{\ell}}$. It is called the pre-K-Betti structure of $\mathcal{M}$ associated to a good K-structure.

More generally, if $\mathcal{M}\bigl(*D(H^c)\bigr) = \mathcal{M}$, any $\mathcal{Q}^H(\mathcal{M}, \boldsymbol{i}, \boldsymbol{j})$ are equipped with the pre-K-Betti structures, which induce a pre-K-Betti structure of $\mathcal{M}$. The naturally defined morphisms $\Xi_{H^c} \Xi_K \psi_J \phi_I(\mathcal{M}) \longrightarrow \Xi_K \psi_J \phi_I(\mathcal{M})$ induce the quasi-isomorphism $\pi_{\underline{\ell}*} \mathcal{Q}^{\underline{\ell}}(\mathcal{M}) \longrightarrow \pi_{H*} \mathcal{Q}^H(\mathcal{M})$, which is compatible with the pre-K-Betti structures. Namely, the associated pre-K-Betti structures of $\mathcal{M}$ are the same.

Lemma 6.10 *The canonical pre-K-Betti structures of $V\bigl(!D(H)\bigr)$ is equal to the pre-K-Betti structure associated to the good K-structure.*

Proof By the above consideration, the following isomorphisms are compatible with the pre-K-Betti structures:

$$V\bigl(!D(H)\bigr) \xrightarrow{\simeq} \mathcal{Q}^H(V(!D(H))) \xleftarrow{\simeq} \mathcal{Q}^{\underline{\ell}}(V(!D(L)))$$

Then, the claim of the lemma follows. ∎

6.2.3 The induced pre-K-Betti structures on the functors along a monomial function

Let g be a meromorphic function on (X, D) such that $g^{-1}(0) \subset D$. Let $D = D_1 \cup D_2$ be a decomposition such that $D_1 \supset g^{-1}(\infty)$ and $D_2 \subset g^{-1}(0)$. (Note that D_i are not necessarily irreducible.) We have the pre-K-Betti structure of $\Xi_g(V, *D_1)$ and $\psi_g(V, *D_1)$ as the kernel of $V \otimes \mathfrak{I}_g^{-\infty,a}(!D_2 * D_1) \longrightarrow V \otimes \mathfrak{I}_g^{-\infty,0}(*D)$ for $a = 1, 0$. Since the canonical pre-K-Betti structures of $V \otimes \mathfrak{I}_g^{-\infty,a}(!D_2 * D_1)$ and $V \otimes \mathfrak{I}_g^{-\infty,0}(*D)$ are associated to the good K-structures, the induced pre-K-Betti structure of $\Xi_g(V, *D_1)$ and $\psi_g(V, *D_1)$ are also associated to the good K-structures.

Let $\mathcal{M} \in \mathrm{Hol}^{\mathrm{good}}(X, D, K)$ be such that $\mathcal{M} = \mathcal{M}(*D_1)$ and $\mathcal{M}(*D) = V$. By Lemma 6.4, we obtain the following complex in $\mathrm{Hol}^{\mathrm{good}}(X, D, K)$:

$$\mathcal{M}(!D_2 * D_1) \longrightarrow \mathcal{M} \oplus \Xi_g\bigl(V, *D_1\bigr) \longrightarrow \mathcal{M}(*D) \tag{77}$$

Hence, we obtain that $\phi_g\bigl(\mathcal{M}, *D_1\bigr) \in \mathrm{Hol}^{\mathrm{good}}(X, D, K)$. The pre-$K$-Betti structure induced by (77) is the same as the one associated to the good K-structures of $\phi_g\bigl(\mathcal{M}, *D_1\bigr)$. Similarly, we have the description of pre-$K(*D_1)$-holonomic $\mathcal{D}(*D_1)$-module $\mathcal{M}$ as the cohomology of

$$\psi_g(\mathcal{M}, *D_1) \longrightarrow \Xi_g(\mathcal{M}, *D_1) \oplus \phi_g(\mathcal{M}, *D_1) \longrightarrow \psi_g(\mathcal{M}, *D_1).$$

6.2.4 Globalization

Let $\mathcal{M}_i$ $(i = 1, 2)$ be good on (X, D) with a good K-structure. Let $\mathcal{F}_i$ be the associated pre-K-Betti structure of $\mathrm{DR}\,\mathcal{M}_i$.

Lemma 6.11 *Let $\varphi : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ be a morphism of D-modules. If it is compatible with the associated pre-K-Betti structures $\mathcal{F}_i$, then it preserves good K-structure of $\mathcal{M}_i$, i.e., $\phi_I(\mathcal{M}_1)\big(*D(I^c)\big) \longrightarrow \phi_I(\mathcal{M}_2)\big(*D(I^c)\big)$ are compatible with K-structures for any I.*

Proof We use an induction on $\rho(\mathcal{M}_1 \oplus \mathcal{M}_2)$. (See Subsection 3.1.2 for ρ.) We take a subset $J \subset \underline{\ell}$ such that $|J| = n - \dim \mathrm{Supp}(\mathcal{M}_1 \oplus \mathcal{M}_2)$ and $(\mathcal{M}_1 \oplus \mathcal{M}_2)\big(*D(J^c)\big) \neq 0$. Let g be a holomorphic function such that $g^{-1}(0) = D(J^c)$. We have the induced good K-structures of $\Xi_g\big(\mathcal{M}_i(*D(J^c))\big)$ and $\mathcal{M}_i(\star g)$ for $i = 1, 2$ and $\star = *, !$. By the assumption, the morphism $\mathcal{M}_{1|X-D(J^c)} \longrightarrow \mathcal{M}_{2|X-D(J^c)}$ is compatible with the K-structures. Hence, $\mathcal{M}_1(\star g) \longrightarrow \mathcal{M}_2(\star g)$ and $\Xi_g\big(\mathcal{M}_1(*g)\big) \longrightarrow \Xi_g\big(\mathcal{M}_2(*g)\big)$ are morphisms in $\mathrm{Hol}^{\mathrm{good}}(X, D, K)$. Moreover, we obtain the following diagram of the pre-K-holonomic $\mathcal{D}$-modules:

$$\begin{array}{ccccc}
\mathcal{M}_1(!g) & \longrightarrow & \Xi_g(\mathcal{M}_1(*g)) \oplus \mathcal{M}_1 & \longrightarrow & \mathcal{M}_1(*g) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M}_2(!g) & \longrightarrow & \Xi_g(\mathcal{M}_2(*g)) \oplus \mathcal{M}_2 & \longrightarrow & \mathcal{M}_2(*g)
\end{array}$$

Hence, the induced morphism $\phi_g(\mathcal{M}_1) \longrightarrow \phi_g(\mathcal{M}_2)$ is also compatible with the pre-K-Betti structures. By using the hypothesis of the induction, we obtain that $\phi_g(\mathcal{M}_1) \longrightarrow \phi_g(\mathcal{M}_2)$ is a morphism in $\mathrm{Hol}^{\mathrm{good}}(X, D, K)$. Therefore, we obtain that $\mathcal{M}_1 \longrightarrow \mathcal{M}_2$ is also a morphism in $\mathrm{Hol}^{\mathrm{good}}(X, D, K)$. ∎

Lemma 6.11 means that a good K-structure can be recovered from the associated pre-K-Betti structure.

Definition 6.12 *A pre-K-Betti structure $\mathcal{F}$ of $\mathcal{M}$ is called good, if it is the pre-K-Betti structures associated to a good K-structure of $\mathcal{M}$.* ∎

Let $(w_1, \ldots, w_n)$ be another holomorphic coordinate such that $w_i^{-1}(0) = z_i^{-1}(0)$.

Lemma 6.13 *If $\mathcal{M}$ has a good K-structure with respect to the coordinate $(z_1, \ldots, z_n)$, it has an induced good K-structure with respect to $(w_1, \ldots, w_n)$ such that the associated pre-K-Betti structures are the same. In this sense, Definition 6.12 is independent of the choice of a coordinate.*

Proof We use symbols $\phi_{\boldsymbol{z},I}$ and $\phi_{\boldsymbol{w},I}$ to distinguish the dependence on the coordinates. As remarked in Subsection 2.2.7, we have the natural isomorphisms (11). They induce isomorphisms $\phi_{\boldsymbol{z},I}(\mathcal{M}) \simeq \phi_{\boldsymbol{w},I}(\mathcal{M})$ and $\psi_i\phi_{\boldsymbol{z},I}(\mathcal{M}) \simeq \psi_i\phi_{\boldsymbol{z},I}(\mathcal{M})$. Hence, we obtain good K-structure of $\mathcal{M}$ with respect to $(w_1, \ldots, w_n)$. Let $\mathcal{Q}^{\underline{\ell}}_{\boldsymbol{z}}(\mathcal{M})$ and $\mathcal{Q}^{\underline{\ell}}_{\boldsymbol{w}}(\mathcal{M})$ denote the $\underline{\ell}$-cube associated to $\mathcal{M}$ with respect to the coordinates $(z_1, \ldots, z_n)$ and $(w_1, \ldots, w_n)$, respectively. It is easy to observe that isomorphisms (11) induce $\pi_{\underline{\ell}*}\mathcal{Q}^{\underline{\ell}}_{\boldsymbol{z}}(\mathcal{M}) \simeq \pi_{\underline{\ell}*}\mathcal{Q}^{\underline{\ell}}_{\boldsymbol{w}}(\mathcal{M})$ compatible with pre-K-Betti structures, and they induce the identity on $\mathcal{M}$. Hence, the associated pre-K-Betti structures on $\mathcal{M}$ are the same. ∎

In particular, the notion makes sense in a global situation.

Definition 6.14 *Let Y be a complex manifold with a normal crossing hypersurface D_Y. Let $\mathcal{M}$ be a good holonomic $\mathcal{D}$-module on (Y, D_Y). A pre-K-Betti structure $\mathcal{F}$ of $\mathcal{M}$ is called good, if it is the pre-K-Betti structure associated to a good K-structure on any coordinate neighbourhood.* ∎

6.3 Preliminary for functoriality via push-forward

6.3.1 Statement

We put $X := \Delta^n$ and $D := \bigcup_{i=1}^{\ell}\{z_i = 0\}$. Let $G : Y \longrightarrow X$ be a proper morphism of complex manifolds. Let D_Y be a simply normal crossing hypersurface of Y with a decomposition $D_Y = D_{Y1} \cup D_{Y2}$ such that $G^{-1}(D) \subset D_{Y2}$.

Let V be a good meromorphic flat bundle on (Y, D_Y) with a good K-structure. Put $\mathcal{M} := V(!D_{Y1})$. Let $\mathcal{F}_{\mathcal{M}}$ be the canonical pre-K-Betti structure. Assume the following:

- $G^i_\dagger\mathcal{M}=0$ for any $i\neq 0$, and $G^0_\dagger\mathcal{M}$ is a good meromorphic flat bundle on (X,D).

We put $\mathcal{G}:=RG_*(\mathcal{F}_{\mathcal{M}})_{|X-D}$, which gives a pre-$K$-Betti structure of $G^0_\dagger(\mathcal{M})_{|X-D}$. The following proposition will be used in the proof of Theorem 8.1. (See Subsection 8.4.1.)

Proposition 6.15 *$\mathcal{G}$ is good, i.e., it is compatible with the Stokes filtrations. Moreover, $RG_*\mathcal{F}_{\mathcal{M}}$ is the canonical K-Betti structure of $G^0_\dagger(\mathcal{M})$.*

Corollary 6.16 *Under the assumption, the induced pre-K-Betti structure of $RG_*\mathbb{D}\mathcal{F}_{\mathcal{M}}$ is the canonical K-Betti structure of $G^0_\dagger\mathbb{D}\mathcal{M}\simeq\mathbb{D}G^0_\dagger\mathcal{M}$.* ∎

Remark 6.17 *The special case* $\dim X=1$ *of Proposition* 6.15 *essentially appeared in* [29] *and* [35]. ∎

6.3.2 A characterization of compatibility with Stokes filtrations

Let g be a holomorphic function on X such that $g^{-1}(0)=D$. Let $i_g:X\longrightarrow X\times\mathbb{C}$ be the graph, and Γ_g be the image. We put $\widetilde{X}:=\Gamma_g\times_{X\times\mathbb{C}}(X\times\widetilde{\mathbb{C}})$. The induced map $\widetilde{X}(D)\longrightarrow\widetilde{X}$ is denoted by ρ.

Let V_1 be an unramifiedly good meromorphic flat bundle on (X,D). Its good set of irregular values is denoted by $\mathrm{Irr}(V_1)$. For each $\mathfrak{a}\in\mathrm{Irr}(V_1)$, put $L(-\mathfrak{a})=\mathcal{O}_X(*D)\,e$ with the meromorphic flat connection $\nabla e=e\,d(-\mathfrak{a})$. We fix a K-structure of $L(-\mathfrak{a})$ by the trivialization $\exp(\mathfrak{a})\,e$. We put $V_1(-\mathfrak{a}):=V_1\otimes L(-\mathfrak{a})$. We regard $\mathrm{DR}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}\bigl(i_{g\dagger}V_1(-\mathfrak{a})\bigr)$ as a constructible sheaf on $\widetilde{X}$.

Lemma 6.18 *Assume that $V_{1|X-D}$ has a K-structure with the following property:*

- *For each $\mathfrak{a}\in\mathrm{Irr}(V_1)$, $\mathrm{DR}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}\bigl(i_{g\dagger}V_1(-\mathfrak{a})\bigr)$ has a K-structure whose restriction to $X-D$ is equal to the one induced by the K-structure of V_1 and $L(-\mathfrak{a})$.*

Then, the K-structure of $V_{1|X-D}$ is good. Moreover, the K-structure of $\mathrm{DR}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}(i_{g\dagger}V_1)$ is equivalent to $R\rho_*\mathcal{L}^{\leq D}_K$.

Proof As for the first claim, the general case can be reduced to the case that D is smooth, which is easy to see. The second claim follows from Lemma 5.19 ∎

6.3.3 Proof of Proposition 6.15

Let $\pi_X:\widetilde{X}\longrightarrow X$ be the induced map.

Lemma 6.19 $\mathrm{DR}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}\bigl(i_{g\dagger}G^0_\dagger\mathcal{M}\bigr)$ *has a K-structure $\mathcal{K}$ whose restriction to $X-D$ is equal to $\mathcal{G}$. Moreover, we have $R\pi_{X*}\mathcal{K}=RG_*\mathcal{F}_{\mathcal{M}}$.*

Proof We put $g_Y:=G^{-1}(g)$. Let $i_{g_Y}:Y\longrightarrow Y\times\mathbb{C}$ denote the graph of g_Y, and Γ_{g_Y} be the image. We put $\widetilde{Y}:=\Gamma_{g_Y}\times_{Y\times\mathbb{C}}(Y\times\widetilde{\mathbb{C}})$. Let π_Y and ρ_Y denote the induced maps $\widetilde{Y}\longrightarrow Y$ and $\widetilde{Y}(D_Y)\longrightarrow\widetilde{Y}$. Let $\widetilde{G}:\widetilde{Y}\longrightarrow\widetilde{X}$ be the induced map.

We have the K-structure $\mathcal{L}^{\leq D_Y}_K$ of $\mathrm{DR}^{\mathrm{nil}}_{\widetilde{Y}(D_Y)}(\mathcal{M})$. According to Proposition 5.14, it induces a K-structure $R\rho_{Y*}\mathcal{L}^{\leq D_Y}_K$ of $\mathrm{DR}^{\mathrm{nil}}_{Y\times\widetilde{\mathbb{C}}}(i_{g_Y\dagger}\mathcal{M})$. By a general compatibility, we have $R\widetilde{G}_*\,\mathrm{DR}^{\mathrm{nil}}_{Y\times\widetilde{\mathbb{C}}}(i_{g_Y\dagger}\mathcal{M})\simeq\mathrm{DR}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}(i_{g\dagger}G^0_\dagger\mathcal{M})$ as remarked in Subsection 5.4.3. Hence, $\mathcal{K}:=R\widetilde{G}_*R\rho_{Y*}\mathcal{L}^{\leq D_Y}_K$ gives a K-structure of $\mathrm{DR}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}\bigl(i_{g\dagger}G^0_\dagger\mathcal{M}\bigr)$ with the desired property. ∎

Let $\kappa:X'\longrightarrow X$ be a ramified covering such that $\kappa^{-1}G^0_\dagger\mathcal{M}$ is unramified with the good set of irregular values $\mathcal{I}$. We put $D':=\kappa^{-1}(D)$. We take a projective birational map $\mu:Y'\longrightarrow Y\times_X X'$ such that (i) Y' is smooth, (ii) $Y'-\mu^{-1}\bigl(Y\times_X D'\bigr)\simeq Y-(Y\times_X D')$, (iii) $D'_Y:=\mu^{-1}(D_Y\times_X X')$ is simply normal crossing. Let $\mu_1:Y'\longrightarrow Y$ be the induced map. Let $G':Y'\longrightarrow X'$ be the induced morphism. For each $\mathfrak{a}\in\mathcal{I}$, we have the induced meromorphic flat bundle $V'(-\mathfrak{a}):=\mu_1^*V\otimes G'^*L(-\mathfrak{a})$ on (Y',D'_Y). We have the decomposition $D'_Y=D'_{Y1}\cup D'_{Y2}$ such that $D'_{Y2}:=\mu_1^{-1}(D_{Y2})$. We put $\mathcal{M}'(-\mathfrak{a}):=\bigl(V'(-\mathfrak{a})\bigr)(!D'_{Y1})$. We have a natural isomorphism $G'^{0}_\dagger(\mathcal{M}'(-\mathfrak{a}))\simeq\kappa^*G^0_\dagger(\mathcal{M})(-\mathfrak{a})$. By applying Lemma 6.18 and Lemma 6.19, we obtain that the first claim of Proposition 6.15. By using Lemma 5.19 and Lemma 6.19, we obtain the second claim of Proposition 6.15. ∎

7 K-holonomic $\mathcal{D}$-modules

7.1 Preliminary

7.1.1 Cell and cell function

Let X be a complex manifold or a smooth complex algebraic variety. In the complex analytic case, we use ordinary topology. In the algebraic case, we consider Zariski topology. In the algebraic setting, $\mathcal{D}$-modules are assumed to be algebraic. An open subset $\mathcal{U}$ is called principal, if it is the complement of a hypersurface. Let P be a point of X. For any closed subvariety W of X, let $\dim_P W$ denote the dimension of W at P. Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module on X with $\dim_P \operatorname{Supp} \mathcal{M} \leq n$. If X is algebraic, we assume that $\mathcal{M}$ is also algebraic. An n-dimensional cell of $\mathcal{M}$ at P is a tuple (Z, U, φ, V) as follows:

(Cell 1) $\varphi : Z \longrightarrow X$ is a morphism of complex manifolds or smooth complex algebraic varieties, such that $P \in \varphi(Z)$ and $\dim Z = n$. We assume that there exists a neighbourhood of X_P of P in X such that $\varphi : \varphi^{-1}(X_P) \longrightarrow X_P$ is projective. We permit that Z may be non-connected or empty.

(Cell 2) $U \subset Z$ is the complement of a simply normal crossing hypersurface D_Z. The restriction $\varphi_{|U}$ is an immersion. Moreover, there exists a hypersurface H of X_P such that $\varphi^{-1}(H) = D_Z \cap \varphi^{-1}(X_P)$.

(Cell 3) V is a meromorphic flat bundle on (Z, D_Z). We have a morphism $\varphi_\dagger(V_!)_P \longrightarrow \mathcal{M}_P \longrightarrow \varphi_\dagger(V)_P$ such that $\mathcal{M}_P(*H) \simeq \varphi_\dagger(V)_P$ and $\mathcal{M}_P(!H) \simeq \varphi_\dagger(V_!)_P$ for a hypersurface H in (Cell 2), where we put $V_! := V(!D_Z)$ and subscript "P" means the restriction to X_P. The restriction of V to some connected components may be 0.

If V is good on (Z, D_Z), $\mathcal{C}$ is called good. For a given holonomic $\mathcal{D}_X$-module $\mathcal{M}$ and $P \in \operatorname{Supp} \mathcal{M}$, there always exists a cell for $\mathcal{M}$ at P. If $\dim_P \mathcal{M} = 1$, any cell is good. If $\dim_P \mathcal{M} = 2$, there always exists a good cell for $\mathcal{M}$ at P, due to Kedlaya [20]. (See also [31] for the algebraic case.) In the algebraic case, there always exists a good cell for $\mathcal{M}$ at P ([31] and [32]).

Remark 7.1 *Let (Z, U, φ) be a tuple satisfying* (Cell 1) *and* (Cell 2). *If we are given a meromorphic flat bundle V on (Z, D_Z), the tuple (Z, U, φ, V) is called a cell at P.* ∎

Let g be a holomorphic or algebraic function on X_P. It is called a cell function for $\mathcal{C}$, if $U = \operatorname{Supp} \mathcal{M}_P \setminus g^{-1}(0)$. For such g, we obtain a description of $\mathcal{M}_P$ as the cohomology of the complex in the category of analytic or algebraic holonomic $\mathcal{D}_{X_P}$-modules:

$$\psi_g\big(\varphi_\dagger(V)_P\big) \longrightarrow \Xi_g\big(\varphi_\dagger(V)_P\big) \oplus \phi_g(\mathcal{M}_P) \longrightarrow \psi_g\big(\varphi_\dagger(V)_P\big)$$

For a given cell, a cell function always exists after X_P and Z are shrinked.

Remark 7.2 *Let $\mathcal{C}$ be a cell of $\mathcal{M}$ at P. If we have a neighbourhood X_P of P satisfying* (Cell 1–3), *any neighbourhood $X'_P \subset X_P$ also satisfies* (Cell 1–3). *Hence, we do not have to be careful with a choice of X_P.* ∎

7.1.2 Refinement and enhancement

Let $\mathcal{C}' = (Z', \varphi', U', V')$ and $\mathcal{C} = (Z, \varphi, U, V)$ be n-cells of $\mathcal{M}$ at P. We say that $\mathcal{C}'$ is a refinement of $\mathcal{C}$, and denote $\mathcal{C}' \prec \mathcal{C}$, if the following holds:

- φ' factors through φ in the sense that there exists $\varphi_1 : Z' \longrightarrow Z$ such that (i) $\varphi' = \varphi \circ \varphi_1$, (ii) $\varphi_1(U') \subset U$.
- $V' = \varphi_1^* V \otimes \mathcal{O}_{Z'}(*D_{Z'})$, where $D_{Z'} := Z' - U'$.

In that situation, there exist naturally induced morphisms:

$$\varphi'_\dagger(V'_!)_P \longrightarrow \varphi_\dagger(V_!)_P \longrightarrow \mathcal{M}_P \longrightarrow \varphi_\dagger(V)_P \longrightarrow \varphi'_\dagger(V')_P \tag{78}$$

We say that $\mathcal{C}'$ is a dominant refinement of $\mathcal{C}$, if U' is dense in U.

Let $\mathcal{C}=(Z,U,\varphi,V)$ be an n-cell of $\mathcal{M}$ at P. We take an n-dimensional closed subvariety $Z'\subset X$ such that $\dim\bigl(\varphi(Z)\cap Z'\bigr)<n$. We take a refinement of $\mathcal{C}$ such that $U\cap Z'=\emptyset$. Let Z_1 be a complex manifold with a projective birational morphism $\varphi_1:Z_1\longrightarrow Z'$ and a smooth open subset $U_1\subset Z_1$ such that (i) $\varphi_{1|U_1}$ is an immersion, (ii) Z_1-U_1 is normal crossing and the pull back of a hypersurface in X around P. We set $\widetilde{Z}:=Z\sqcup Z_1$ and $\widetilde{U}:=U\sqcup U_1$. We have the induced map $\widetilde{\varphi}:\widetilde{Z}\longrightarrow X$. Let $\widetilde{V}$ be a meromorphic flat bundle on $\widetilde{Z}$ such that $\widetilde{V}_{0|Z}=V$ and $\widetilde{V}_{0|Z_1}=0$. Then, it is easy to observe that $\widetilde{\mathcal{C}}:=(\widetilde{Z},\widetilde{U},\widetilde{\varphi},\widetilde{V})$ is an n-cell of $\mathcal{M}$, which is called an enhancement of $\mathcal{C}$.

In the following, for a cell $\mathcal{C}=(Z,U,\varphi,V)$, we implicitly assume $\varphi^{-1}(X_P)=Z$ by taking a refinement of $\mathcal{C}$. So we omit the subscript "P" in $\varphi_\dagger(V_!)_P$ and $\varphi_\dagger(V)_P$.

7.1.3 K-cell and the induced pre-K-Betti structure on the nearby cycle

Let $\mathcal{F}$ be a pre-K-Betti structure of $\mathcal{M}$. Let $\mathcal{C}=(Z,U,\varphi,V)$ be a good n-cell of $\mathcal{M}$ at P. We say that $\mathcal{F}$ and $\mathcal{C}$ are compatible, if the following holds:

- The induced K-structure of $V_{|U}$ is good, i.e., compatible with the Stokes filtrations along D_Z.
- The induced morphisms $\varphi_\dagger(V_!)\longrightarrow\mathcal{M}_P\longrightarrow\varphi_\dagger(V)$ are compatible with the pre-K-Betti structures. (See Subsection 6.1.1 for the canonical pre-K-Betti structures of $V_!$ and V.)

Such a cell $\mathcal{C}$ is called a good K-cell of $(\mathcal{M},\mathcal{F})$. It is not difficult to construct an example of a pre-K-holonomic $\mathcal{D}$-module, for which there does not exist a good K-cell at some point.

Lemma 7.3 *Let $\mathcal{C}=(Z,U,\varphi,V)$ be a good K-cell of $(\mathcal{M},\mathcal{F})$ at P. Let $\mathcal{C}'=(Z',U',\varphi',V')$ be a refinement of $\mathcal{C}$. Then, $\mathcal{C}'$ is also a good K-cell. Moreover, the induced morphisms in* (78) *are compatible with pre-K-Betti structures.*

Proof It follows from Proposition 6.7. ∎

Let g be a cell function for a good K-cell $\mathcal{C}$. Let us observe that pre-K-Betti structures of $\Xi_g\bigl(\varphi_\dagger(V)\bigr)$, $\psi_g\bigl(\varphi_\dagger(V)\bigr)$ and $\phi_g(\mathcal{M}_P)$ are induced. We set $V^{a,b}_{g\star}:=\Pi^{a,b}_{\varphi^{-1}(g)\star}V$ for $\star=*,!$. Note that $\varphi_\dagger\bigl(V^{a,b}_{g\star}\bigr)$ have the canonical pre-K-Betti structures. Since $\Xi_g\bigl(\varphi_\dagger V\bigr)$ and $\psi_g\bigl(\varphi_\dagger V\bigr)$ are of the form $\mathrm{Ker}\Bigl(\varphi_\dagger\bigl(V^{a,b}_{g!}\bigr)\longrightarrow\varphi_\dagger\bigl(V^{a',b'}_{g*}\bigr)\Bigr)$, they are equipped with induced pre-K-Betti structures, denoted by ${}^D\Xi_g(\varphi_*\mathcal{F}_V)$ and ${}^D\psi_g(\varphi_*\mathcal{F}_V)$. We will use the following obvious lemma implicitly.

Lemma 7.4 *The natural isomorphisms $\Xi_g\bigl(\varphi_\dagger(V)\bigr)\simeq\varphi_\dagger\bigl(\Xi_g(V)\bigr)$ and $\psi_g\bigl(\varphi_\dagger V\bigr)\simeq\varphi_\dagger\psi_g(V)$ are compatible with the induced pre-K-Betti structures.* ∎

Since $\phi_g(\mathcal{M}_P)$ is the cohomology of the complex $\varphi_\dagger V_!\longrightarrow\Xi_g(\varphi_\dagger V)\oplus\mathcal{M}\longrightarrow\varphi_\dagger V$, we obtain a pre-$K$-Betti structure of $\phi_g(\mathcal{M}_P)$, denoted by ${}^D\phi_g(\mathcal{F})$. The tuples $\bigl(\Xi_g(\varphi_\dagger V),\Xi_g(\varphi_*\mathcal{F}_V)\bigr)$, $\bigl(\psi_g(\varphi_\dagger V),\psi_g(\varphi_*\mathcal{F}_V)\bigr)$ and $\bigl(\phi_g(\mathcal{M}),{}^D\phi_g(\mathcal{F})\bigr)$ are also denoted by $\Xi_g\varphi_\dagger(V,\mathcal{F}_V)$, $\psi_g\varphi_\dagger(V,\mathcal{F}_V)$ and $\phi_g(\mathcal{M},\mathcal{F})$.

7.2 K-holonomic $\mathcal{D}$-modules

7.2.1 Definition of K-Betti structure

Let X be a complex manifold, and P be a point of X. Let $(\mathcal{M},\mathcal{F})$ be a pre-K-holonomic $\mathcal{D}$-module on X. Let us define the notion of K-Betti structure of $\mathcal{M}$ at P, inductively on the dimension of $\mathrm{Supp}\,\mathcal{M}$.

Definition 7.5 *In the case $\dim_P\mathrm{Supp}\,\mathcal{M}=0$, a K-Betti structure is defined to be a pre-K-Betti structure.*

Let us consider the case $\dim_P\mathrm{Supp}\,\mathcal{M}\leq n$. We say that $\mathcal{F}$ is a K-Betti structure of $\mathcal{M}$ at P, if there exists an n-dimensional good K-cell $\mathcal{C}_0=(Z_0,\varphi_0,U_0,V_0)$ at P with the following property:

- $\dim_P\Bigl(\bigl(\mathrm{Supp}\,\mathcal{M}\cap X_P\bigr)\setminus\varphi_0(Z_0)\Bigr)<n$ *for some neighbourhood X_P of P in X.*

- *For any dominant refinement $\mathcal{C} \prec \mathcal{C}_0$ and any cell function g for $\mathcal{C}$, the induced pre-K-Betti structure ${}^D\phi_g(\mathcal{F})$ is a K-Betti structure of $\phi_g(\mathcal{M}_P)$ at P. Note that* $\dim_P \phi_g(\mathcal{F}) < n$.

Such an n-cell $\mathcal{C}_0$ is called a bounding n-cell of $\mathcal{M}$ at P. ∎

If $\mathcal{C}_0$ is a bounding n-cell of $\mathcal{M}$, its dominant refinement and enhancement are also bounding n-cells of $\mathcal{M}$.

Definition 7.6 *If $\mathcal{F}$ is a K-Betti structure of $\mathcal{M}$ at any point of X, it is called a K-Betti structure of $\mathcal{M}$. A holonomic $\mathcal{D}$-module with a K-Betti structure is called a K-holonomic $\mathcal{D}$-module.* ∎

A morphism of K-holonomic $\mathcal{D}$-modules $(\mathcal{M}_1, \mathcal{F}_1) \longrightarrow (\mathcal{M}_2, \mathcal{F}_2)$ is defined to be a morphism of pre-K-holonomic $\mathcal{D}$-modules.

Proposition 7.7 *The category of K-holonomic $\mathcal{D}$-modules is abelian.*

Proof Let P be any point of X. We use an induction on the dimension of $\operatorname{Supp}_P \mathcal{M}$. Let $(f_{\mathcal{D}}, f_{\mathcal{P}}) : (\mathcal{M}_1, \mathcal{F}_1) \longrightarrow (\mathcal{M}_2, \mathcal{F}_2)$ be a morphism of K-holonomic $\mathcal{D}$-modules. Let us show that $\operatorname{Ker}(f_{\mathcal{P}})$ is a K-Betti structure of $\operatorname{Ker} f_{\mathcal{D}}$.

Let $n \geq \max\bigl\{\dim \operatorname{Supp}_P \mathcal{M}_i\bigr\}$. Let $\mathcal{C}_{i,0} = (Z_{i,0}, U_{i,0}, \varphi_{i,0}, V_{i,0})$ $(i = 1, 2)$ be bounding n-cells for $\mathcal{M}_i$ at P. By considering refinement and enhancement, we may assume that $(Z_{1,0}, U_{1,0}, \varphi_{1,0}) = (Z_{2,0}, U_{2,0}, \varphi_{2,0})$, which is denoted by (Z_0, U_0, φ_0). We may also assume that the union of the irregular values of $V_{i,0}$ are good at each point of the pole. We have an induced morphism $f_{Z_0} : V_{1,0} \longrightarrow V_{2,0}$. We obtain a cell $\mathcal{C}_0(\operatorname{Ker}) = \bigl(Z_0, U_0, \varphi_0, \operatorname{Ker} f_{Z_0}\bigr)$ of $\operatorname{Ker} f_{\mathcal{D}}$.

Let $\mathcal{C}(\operatorname{Ker}) = (Z, U, \varphi, K_Z)$ be a dominant refinement of $\mathcal{C}_0(\operatorname{Ker})$. We have refinements $\mathcal{C}_i = (Z, U, \varphi, V_i)$ of $\mathcal{C}_{i,0}$ with the induced morphism $f_Z : V_1 \longrightarrow V_2$. We have $\operatorname{Ker} f_Z \simeq K_Z$. We obtain the following commutative diagram of pre-K-holonomic $\mathcal{D}$-modules:

$$\begin{array}{ccccc}
\varphi_\dagger V_{1!} & \longrightarrow & \mathcal{M}_{1\,P} & \longrightarrow & \varphi_\dagger V_1 \\
\downarrow & & \downarrow & & \downarrow \\
\varphi_\dagger V_{2!} & \longrightarrow & \mathcal{M}_{2\,P} & \longrightarrow & \varphi_\dagger V_2
\end{array}$$

Hence, the induced morphisms $\varphi_* K_{Z!} \longrightarrow \operatorname{Ker}(f_{\mathcal{D}})_P \longrightarrow \varphi_* K_Z$ are compatible with the pre-K-Betti structures. We have the following commutative diagram of pre-K-holonomic $\mathcal{D}$-modules:

$$\begin{array}{ccc}
\varphi_\dagger\bigl(V^{a,b}_{1,g!}\bigr) & \longrightarrow & \varphi_\dagger\bigl(V^{a,b}_{1,g*}\bigr) \\
\downarrow & & \downarrow \\
\varphi_\dagger\bigl(V^{a,b}_{2,g!}\bigr) & \longrightarrow & \varphi_\dagger\bigl(V^{a,b}_{2,g*}\bigr)
\end{array}$$

Hence, the induced morphisms $\Xi_g\bigl(\varphi_\dagger V_1\bigr) \longrightarrow \Xi_g\bigl(\varphi_\dagger V_2\bigr)$ and $\psi_g\bigl(\varphi_\dagger V_1\bigr) \longrightarrow \psi_g\bigl(\varphi_\dagger V_2\bigr)$ preserve the pre-K-Betti structures. Therefore, $\phi_g(f_{\mathcal{D}})$ preserves the pre-K-Betti structures, i.e., ${}^D\phi_g(f_{\mathcal{P}}) : {}^D\phi_g(\mathcal{F}_1) \longrightarrow {}^D\phi_g(\mathcal{F}_2)$ is induced. By the assumption of the induction, $\operatorname{Ker} {}^D\phi_g(f_{\mathcal{P}})$ is a K-Betti structure. It is easy to obtain that ${}^D\phi_g \operatorname{Ker} f_{\mathcal{P}} = \operatorname{Ker} {}^D\phi_g(f_{\mathcal{P}})$. Then, we can conclude that $\bigl(\operatorname{Ker} f_{\mathcal{D}}, \operatorname{Ker} f_{\mathcal{P}}\bigr)$ is a K-holonomic $\mathcal{D}$-module. The claims for the cokernel and the image can be shown similarly. ∎

7.2.2 Dual

Lemma 7.8 *For any K-holonomic $\mathcal{D}$-module $(\mathcal{M}, \mathcal{F})$, its dual $\mathbb{D}(\mathcal{M}, \mathcal{F}) := (\mathbb{D}\mathcal{M}, \mathbb{D}\mathcal{F})$ is also K-holonomic.*

Proof Let P be any point of $\operatorname{Supp} \mathcal{M}$, and let $\mathcal{C}_0$ be a bounding n-cell at P. Let $\mathcal{C} = (Z, U, \varphi, V)$ be any refinement of $\mathcal{C}_0$. Let $\mathcal{F}_V$ and $\mathcal{F}_{V!}$ be the canonical pre-K-Betti structures of V and $V_!$. Let $\mathcal{C}^\vee := (Z, U, \varphi, V^\vee)$. We have the induced K-structure of $V^\vee$. According to Proposition 5.5 and Theorem 5.6, $\mathbb{D}\mathcal{F}_{V!}$ and $\mathbb{D}\mathcal{F}_V$ are the canonical pre-K-Betti structures of $V^\vee$ and $V_!^\vee$. Hence, we obtain that $\mathcal{C}^\vee$ and $\mathbb{D}\mathcal{F}$ are compatible. We also obtain that $\mathbb{D}\,{}^D\Xi_g \varphi_* \mathcal{F}_V$ is equal to the canonical pre-K-Betti structure of $\Xi_g \varphi_* V^\vee$. Moreover, the induced K-structure of $\phi_g(\mathbb{D}\mathcal{M}_P)$ is equal to $\mathbb{D}\,{}^D\phi_g \mathcal{F}$ under the isomorphism $\phi_g \mathbb{D}\mathcal{M}_P \simeq \mathbb{D}\phi_g \mathcal{M}_P$. By the hypothesis of the induction, it is K-Betti structure. Thus, we obtain that $\mathbb{D}(\mathcal{M}, \mathcal{F})$ is K-holonomic. ∎

7.2.3 Sub-quotient

Let $(\mathcal{M}_1,\mathcal{F}_1)\subset(\mathcal{M},\mathcal{F})$ be a pre-K-holonomic $\mathcal{D}$-submodule.

Lemma 7.9 *If $(\mathcal{M},\mathcal{F})$ is K-holonomic, $(\mathcal{M}_1,\mathcal{F}_1)$ is also K-holonomic. Similar claim holds for quotient.*

Proof Let P be any point of X. We use an induction on the dimension of the support of $\mathcal{M}$. Let $n\geq\dim_P\operatorname{Supp}\mathcal{M}$. Let $\mathcal{C}=(Z,U,\varphi,V)$ be a bounding n-cell of $\mathcal{M}$ at P. Let $V_1\subset V$ denote the subbundle induced by $\mathcal{M}_1$. Then, $\mathcal{C}_1=(Z,U,\varphi,V_1)$ is an n-cell of $\mathcal{M}_1$ at P. Let us show that $\mathcal{C}_1$ and $\mathcal{F}_1$ are compatible. Since the K-structure and the Stokes structure for V_1 are the restriction of those for V, they are compatible. Let $\mathcal{F}_*$ and $\mathcal{F}_!$ denote the canonical K-structures of $\varphi_\dagger V$ and $\varphi_\dagger V_!$. Let $\mathcal{F}_{1*}$ and $\mathcal{F}_{1!}$ denote the canonical K-structures of $\varphi_\dagger V_1$ and $\varphi_\dagger V_{1!}$. We have the following morphisms:

$$
\begin{array}{ccccc}
\varphi_\dagger(V_!) & \longrightarrow & \mathcal{M} & \longrightarrow & \varphi_\dagger(V)\\
\uparrow & & \uparrow & & \uparrow\\
\varphi_\dagger(V_{1!}) & \longrightarrow & \mathcal{M}_1 & \longrightarrow & \varphi_\dagger(V_1)
\end{array}
\qquad
\begin{array}{ccccc}
\mathcal{F}_! & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}_*\\
\uparrow & & \uparrow & & \uparrow\\
\mathcal{F}_{1!} & & \mathcal{F}_1 & & \mathcal{F}_{1*}
\end{array}
$$

Because the morphism $\varphi_\dagger(V_{1!})\longrightarrow\mathcal{M}/\mathcal{M}_1$ is 0, the morphism $\mathcal{F}_{1!}\longrightarrow\mathcal{F}/\mathcal{F}_1$ is also 0, i.e., $\mathcal{F}_{1!}\longrightarrow\mathcal{F}$ factors through $\mathcal{F}_1$. Similarly, we obtain that $\mathcal{F}_1\longrightarrow\mathcal{F}_*$ factors through $\mathcal{F}_{1*}$.

Let f be a cell function for $\mathcal{C}$. We have ${}^D\Xi_f(\mathcal{F})\supset{}^D\Xi_f(\mathcal{F}_1)$ and ${}^D\psi_f(\mathcal{F})\supset{}^D\psi_f\mathcal{F}_1$. Hence, we obtain ${}^D\phi_f(\mathcal{F})\supset{}^D\phi_f(\mathcal{F}_1)$, which are pre-$K$-Betti structures of $\phi_f\mathcal{M}$ and $\phi_f\mathcal{M}_1$. By the assumption of the induction, we obtain that ${}^D\phi_f(\mathcal{F}_1)$ is a K-Betti structure of $\phi_f\mathcal{M}_1$. ∎

7.2.4 Twist

Let $(\mathcal{M},\mathcal{F})$ be a K-holonomic $\mathcal{D}$-module on X. Let $\mathcal{V}$ be a flat bundle on X with a K-structure, i.e., we have a K-local system $\mathcal{F}_\mathcal{V}$ such that $\mathcal{F}_\mathcal{V}\otimes\mathbb{C}\simeq\mathrm{DR}_X(V)$. Then, we obtain a pre-$K$-Betti structure $\mathcal{F}\otimes\mathcal{F}_\mathcal{V}$ of $\mathcal{M}\otimes\mathcal{V}$.

Lemma 7.10 *$\mathcal{F}\otimes\mathcal{F}_\mathcal{V}$ is a K-Betti structure of $\mathcal{M}\otimes\mathcal{V}$.*

Proof Let P be any point of X. We use an induction on $\dim_P\operatorname{Supp}\mathcal{M}$. Let $\mathcal{C}=(Z,U,\varphi,V)$ be a K-cell of $\mathcal{M}$ at P. Then, $\mathcal{C}'=\big(Z,U,\varphi,V\otimes\varphi^*\mathcal{V}\big)$ is a K-cell of $\mathcal{M}\otimes\mathcal{V}$ at P. Let g be a cell function of $\mathcal{C}$. Then, we have natural isomorphism of pre-K-holonomic $\mathcal{D}_X$-modules $\psi_g\big(\varphi_*(V\otimes\varphi^*\mathcal{V})\big)\simeq\psi_g\big(\varphi_*(V)\big)\otimes\mathcal{V}$ and $\Xi_g\big(\varphi_*(V\otimes\varphi^*\mathcal{V})\big)\simeq\Xi_g\big(\varphi_*(V)\big)\otimes\mathcal{V}$. Hence, we obtain an isomorphism of pre-K-holonomic $\mathcal{D}$-modules $\phi_g(\mathcal{M}\otimes\mathcal{V})\simeq\phi_g(\mathcal{M})\otimes\mathcal{V}$. By using the hypothesis of the induction, we obtain that $\phi_g(\mathcal{M}\otimes\mathcal{V})$ is K-holonomic. Hence, we obtain that $\mathcal{M}\otimes\mathcal{V}$ is K-holonomic at P. ∎

7.2.5 Complement

Let $(\mathcal{M},\mathcal{F})$ be a K-holonomic $\mathcal{D}$-module.

Lemma 7.11 *Any good cell $\mathcal{C}=(Z,U,\varphi,V)$ of $\mathcal{M}$ is compatible with $\mathcal{F}$, and the morphisms $\varphi_\dagger V_!\longrightarrow\mathcal{M}\longrightarrow\varphi_\dagger V$ are compatible with the K-Betti structures.*

Proof The first claim is easy to see. If we take an appropriate refinement $\mathcal{C}'=(Z',U',\varphi',V')$ of $\mathcal{C}$, the induced morphisms $\mathcal{M}\longrightarrow\varphi'_*V'$ and $\varphi_\dagger V\longrightarrow\varphi'_\dagger V'$ are compatible with K-Betti structures. Because $\varphi_\dagger V\longrightarrow\varphi'_\dagger V'$ is a monomorphism, we obtain that $\mathcal{M}\longrightarrow\varphi_\dagger V$ is also compatible with K-Betti structures. We can show that $\varphi_!V\longrightarrow\mathcal{M}$ is also compatible with K-Betti structures with a similar argument. ∎

7.3 $K(*D)$-holonomic $\mathcal{D}(*D)$-modules

We introduce some auxiliary notion of $K(*D)$-Betti structure on $\mathcal{D}_{X(*D)}$-modules, where D is a hypersurface. Although we do not need it eventually, it will be useful in the argument in Section 8.

7.3.1 Cell and cell function for holonomic $\mathcal{D}_{X(*D)}$-modules

Let X be a complex manifold or smooth complex algebraic variety, and let D be a hypersurface of X. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_{X(*D)}$-module, i.e., $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module such that $\mathcal{M}(*D)=\mathcal{M}$. Let $P\in D\cap \operatorname{Supp}\mathcal{M}$. A cell of a holonomic $\mathcal{D}_{X(*D)}$-module $\mathcal{M}$ is defined to be a cell of a holonomic $\mathcal{D}_X$-module $\mathcal{M}$. The notions of refinement and enhancement of a cell of a holonomic $\mathcal{D}_{X(*D)}$-module are defined as those for cells of a holonomic $\mathcal{D}_X$-modules. However, we will be interested in the morphisms $\varphi_\dagger(V_!)(*D)\longrightarrow \mathcal{M}_P\longrightarrow \varphi_\dagger V$.

We make a modification for the notion of cell function. Let $\mathcal{C}=(Z,U,\varphi,V)$ be a cell of a holonomic $\mathcal{D}_{X(*D)}$-module $\mathcal{M}$. A cell function g of $\mathcal{C}$ is a meromorphic function on (X,D) such that $U=\operatorname{Supp}\mathcal{M}\setminus\big(g^{-1}(0)\cup D\big)$.

7.3.2 $K(*D)$-cell

Let $\mathcal{F}$ be a pre-$K(*D)$-Betti structure of $\mathcal{M}$. Let $\mathcal{C}=(Z,U,\varphi,V)$ be a good n-cell of $\mathcal{M}$ at P. We say that $\mathcal{F}$ and $\mathcal{C}$ are compatible if (i) the induced K-structure of $V_{|U}$ is compatible with the Stokes filtrations, (ii) the induced morphisms $\varphi_\dagger(V_!)(*D)\longrightarrow\mathcal{M}_P\longrightarrow\varphi_\dagger(V)$ are compatible with the pre-K-Betti structures. Such a cell $\mathcal{C}$ is called a $K(*D)$-cell of $(\mathcal{M},\mathcal{F})$.

Let g be a cell function for a good $K(*D)$-cell $\mathcal{C}$. We set $V^{a,b}_{g\star}(*D):=\big(V\otimes\mathfrak{I}^{a,b}_{\varphi^{-1}(g)}\big)(*\varphi^{-1}D)$ for $\star=*,!$. Note that $\varphi_\dagger\big(V^{a,b}_{g\star}(*D)\big)$ have the canonical pre-K-Betti structures. Since $\Xi_g\big(\varphi_\dagger V,*D\big)$ and $\psi_g\big(\varphi_\dagger V,*D\big)$ are of the form $\operatorname{Ker}\Big(\varphi_\dagger\big(V^{a,b}_{g!}(*D)\big)\longrightarrow\varphi_\dagger\big(V^{a',b'}_{g*}(*D)\big)\Big)$, they are equipped with the induced pre-$K(*D)$-Betti structures, denoted by ${}^D\Xi_g(\varphi_*\mathcal{F}_V,*D)$ and ${}^D\psi_g(\varphi_*\mathcal{F}_V,*D)$. The tuples $\big(\Xi_g(\varphi_\dagger V,*D),{}^D\Xi_g(\varphi_*\mathcal{F}_V,*D)\big)$ and $\big(\psi_g(\varphi_\dagger V,*D),{}^D\psi_g(\varphi_*\mathcal{F}_V,*D)\big)$ are also denoted by $\Xi_g\varphi_\dagger(V,\mathcal{F}_V,*D)$ and $\psi_g\varphi_\dagger(V,\mathcal{F}_V,*D)$. We will use the following obvious lemma implicitly.

Lemma 7.12 *The natural isomorphisms*

$$\Xi_g\big(\varphi_\dagger V,*D\big)\simeq\varphi_\dagger\Xi_g(V,*\varphi^{-1}D),\qquad \psi_g\big(\varphi_\dagger V,*D\big)\simeq\varphi_\dagger\psi_g(V,*\varphi^{-1}D)$$

are compatible with the induced pre-K-Betti structures. ∎

Since $\phi_g(\mathcal{M}_P,*D)$ is the cohomology of the complex of pre-$K(*D)$-holonomic $\mathcal{D}_{X(*D)}$-modules

$$\varphi_\dagger(V_!)(*D)\longrightarrow\Xi_g(\varphi_\dagger V,*D)\oplus\mathcal{M}_P\longrightarrow\varphi_\dagger(V)(*D),$$

we obtain a pre-$K(*D)$-Betti structure of $\phi_g(\mathcal{M}_P,*D)$, denoted by ${}^D\phi_g(\mathcal{F},*D)$. The pre-$K(*D)$-holonomic $\mathcal{D}_{X(*D)}$-module $\big(\phi_g(\mathcal{M}_P,*D),{}^D\phi_g(\mathcal{F},*D)\big)$ is also denoted by $\phi_g(\mathcal{M}_P,\mathcal{F},*D)$.

7.3.3 Definition of $K(*D)$-Betti structure

Let P be a point of D. Let $(\mathcal{M},\mathcal{F})$ be a pre-$K(*D)$-holonomic $\mathcal{D}_{X(*D)}$-module. Let us define the notion of $K(*D)$-Betti structure of $\mathcal{M}$ at P, inductively on the dimension of $\operatorname{Supp}\mathcal{M}$. Note that we have $\mathcal{M}=0$ around P in the case $\dim_P\operatorname{Supp}\mathcal{M}=0$.

Definition 7.13 *Let us consider the case* $\dim_P\operatorname{Supp}\mathcal{M}\leq n$. *We say that $\mathcal{F}$ is a $K(*D)$-Betti structure of $\mathcal{M}$ at P, if there exists an n-dimensional good $K(*D)$-cell $\mathcal{C}_0=(Z_0,\varphi_0,U_0,V_0)$ at P with the following property:*

- $\dim_P\Big(\big(\operatorname{Supp}\mathcal{M}\cap X_P\big)\setminus\varphi_0(Z_0)\Big)<n$ *for some neighbourhood X_P of P in X.*
- *For any dominant refinement $\mathcal{C}\prec\mathcal{C}_0$ and any cell function g for $\mathcal{C}$ as a $\mathcal{D}_{X(*D)}$-module, the induced pre-$K(*D)$-Betti structure ${}^D\phi_g(\mathcal{F},*D)$ is a $K(*D)$-Betti structure at P.*

Such an n-cell $\mathcal{C}_0$ is called a bounding n-cell of $\mathcal{M}$ at P. ∎

If $\mathcal{C}_0$ is a bounding n-cell of $\mathcal{M}$, its dominant refinement and enhancement are also bounding n-cells of $\mathcal{M}$.

Definition 7.14 *If $\mathcal{F}$ is K-Betti structure of $\mathcal{M}$ at any point of $X-D$, and $K(*D)$-Betti structure of $\mathcal{M}$ at any point of D, it is called a $K(*D)$-Betti structure of $\mathcal{M}$. A holonomic $\mathcal{D}_{X(*D)}$-module with a $K(*D)$-Betti structure is called a $K(*D)$-holonomic $\mathcal{D}_{X(*D)}$-module.* ∎

A morphism of $K(*D)$-holonomic $\mathcal{D}_{X(*D)}$-modules $(\mathcal{M}_1,\mathcal{F}_1)\longrightarrow(\mathcal{M}_2,\mathcal{F}_2)$ is defined to be a morphism of pre-$K(*D)$-holonomic $\mathcal{D}_{X(*D)}$-modules. Let $\mathrm{Hol}\bigl(X(*D),K\bigr)$ denote the category of $K(*D)$-holonomic $\mathcal{D}_{X(*D)}$-modules. The following lemma is similar to Proposition 7.7.

Lemma 7.15 *The category* $\mathrm{Hol}\bigl(X(*D),K\bigr)$ *is abelian.* ∎

The following lemma is similar to Lemma 7.9.

Lemma 7.16 *Let* $(\mathcal{M}_1,\mathcal{F}_1)\subset(\mathcal{M},\mathcal{F})$ *be pre-*$K(*D)$*-holonomic* $\mathcal{D}_{X(*D)}$*-submodules. If* $(\mathcal{M},\mathcal{F})$ *is* $K(*D)$*-holonomic, then* $(\mathcal{M}_1,\mathcal{F}_1)$ *is also* $K(*D)$*-holonomic. Similar claim holds for quotient.* ∎

7.3.4 Uniqueness

We have the following uniqueness.

Proposition 7.17 *Let* $\mathcal{M}$ *be a holonomic* $\mathcal{D}_{X(*D)}$*-module. Let* $\mathcal{F}_i$ $(i=1,2)$ *be* $K(*D)$*-Betti structures of* $\mathcal{M}$*. If* $\mathcal{F}_{1|X-D}=\mathcal{F}_{2|X-D}$*, then we have* $\mathcal{F}_1=\mathcal{F}_2$*.*

Proof The claim is local. Let $P\in D$. We use an induction on $\dim_P\operatorname{Supp}\mathcal{M}$. In the case $\dim_P\operatorname{Supp}\mathcal{M}=0$, the claim is clear. Let $\dim_P\operatorname{Supp}\mathcal{M}\leq n$. Let $\mathcal{C}$ be any bounding cell at P, and let g be any cell function of $\mathcal{C}$. Let ${}^D\phi_g(\mathcal{F}_i,*D)$ be the induced pre-$K(*D)$-Betti structures of $\phi_g(\mathcal{M},*D)$. By the assumption of the induction, we have ${}^D\phi_g(\mathcal{F}_1,*D)={}^D\phi_g(\mathcal{F}_2,*D)$. Because $\mathcal{F}_i$ can be reconstructed from ${}^D\phi_g(\mathcal{F}_i,*D)$ and the canonical pre-$K(*D)$-Betti structures of $\psi_g(\varphi_*V,*D)$ and $\Xi_g(\varphi_*V,*D)$, we obtain $\mathcal{F}_1=\mathcal{F}_2$. ∎

7.3.5 Independence of compactification

Let $F:X'\longrightarrow X$ be a proper birational morphism of complex manifolds such that $X'-D'\simeq X-D$, where $D':=F^{-1}(D)$.

Proposition 7.18 *Let* $\mathcal{M}'$ *be a holonomic* $\mathcal{D}_{X'(*D')}$*-module, and we set* $\mathcal{M}:=F_\dagger\mathcal{M}'$*.*

- *Let* $\mathcal{F}'$ *be a* $K(*D')$*-Betti structure of* $\mathcal{M}'$*. Then,* $F_*\mathcal{F}'$ *is a* $K(*D)$*-Betti structure of* $\mathcal{M}$*.*
- *Let* $\mathcal{F}$ *be a* $K(*D)$*-Betti structure of* $\mathcal{M}$*. Then,* $\mathcal{M}'(*D')$ *is equipped with a* K*-Betti structure* $\mathcal{F}'$ *such that* $\mathcal{F}'_{|X'-D'}=\mathcal{F}_{|X-D}$ *under the isomorphism* $\mathcal{M}'_{|X'-D'}\simeq\mathcal{M}_{|X-D}$*. It is functorial.*

Proof We have only to check the claims locally around D. Let P be any point of D. We use an induction on $\dim_P\operatorname{Supp}\mathcal{M}$. Let $\mathcal{C}=(Z,U,\varphi,V)$ be a good cell of $\mathcal{M}$ at P. By taking a refinement, we may assume that φ factors through F, i.e., $\varphi=F\circ\varphi'$, and that $\mathcal{C}'=(Z,U,\varphi',V)$ is a good cell of $\mathcal{M}'$. Let g be a cell function for $\mathcal{C}$ as $\mathcal{D}_{X(*D)}$-module. Note that $g'=g\circ F$ is a cell function for $\mathcal{C}'$. We have a description of $\mathcal{M}'$ as the cohomology of the following complex:

$$\psi_{g'}(\varphi'_\dagger V,*D')\longrightarrow\Xi_{g'}(\varphi'_\dagger V,*D')\oplus\phi_{g'}(\mathcal{M}',*D')\longrightarrow\psi_{g'}(\varphi'_\dagger V,*D') \tag{79}$$

By the push-forward $F_\dagger$, it induces a description of $\mathcal{M}$ as the cohomology of the following complex:

$$\psi_g(\varphi_\dagger V,*D)\longrightarrow\Xi_g(\varphi_\dagger V,*D)\oplus\phi_g(\mathcal{M},*D)\longrightarrow\psi_g(\varphi_\dagger V,*D) \tag{80}$$

Let us show the first claim. By the assumption of the induction, the induced pre-$K(*D)$-Betti structure of $\phi_g(\mathcal{M},*D)$ is a $K(*D)$-Betti structure. Hence, $\mathcal{F}$ is also a $K(*D)$-Betti structure. Let us show the second claim. By the hypothesis of the induction, the $K(*D)$-Betti structure of $\psi_g\bigl(\varphi_\dagger(V),*D\bigr)$ and $\phi_g(\mathcal{M},*D)$ induce the $K(*D)$-Betti structures of $\psi_{g'}\bigl(\varphi'_\dagger(V),*D'\bigr)$ and $\phi_{g'}(\mathcal{M}',*D')$, which are compatible with the natural morphisms. We also have the canonical K-Betti structures of $\psi_{g'}\bigl(\varphi'_\dagger(V),*D'\bigr)$ and $\Xi_{g'}\bigl(\varphi'_\dagger V,*D'\bigr)$. By Proposition 7.17, the induced $K(*D)$-Betti structures on $\psi_{g'}\bigl(\varphi'_\dagger(V),*D'\bigr)$ are the same. Hence, (79) is a complex of $K(*D)$-holonomic $\mathcal{D}(*D)$-modules. Hence, we have an induced $K(*D)$-Betti structure of $\mathcal{M}'$. The functoriality is clear from the above construction. ∎

Corollary 7.19 *The functor* $F_\dagger$ *gives the equivalence of the categories* $\mathrm{Hol}\bigl(X(*D),K\bigr)$ *and* $\mathrm{Hol}\bigl(X'(*D'),K\bigr)$*.* ∎

8 Some functoriality

8.1 Statements

In the following, $\mathcal{D}$-modules are assumed to be algebraic unless otherwise indicated. We give several statements.

Theorem 8.1 *Let $F : X \longrightarrow Y$ be a projective morphism of smooth algebraic varieties. Let $(\mathcal{M}, \mathcal{F})$ be a K-holonomic $\mathcal{D}_X$-module. Then, $F_\dagger^i(\mathcal{M}, \mathcal{F}) := \bigl(F_\dagger^i \mathcal{M}, F_\dagger^i \mathcal{F}\bigr)$ are K-holonomic for any i.*

Here, $F_\dagger^i \mathcal{F}$ is the i-th cohomology of $RF_* \mathcal{F}$ with respect to the middle perversity.

Theorem 8.2 *Let X be a smooth complex algebraic variety with a normal crossing hypersurface D. Let $\mathcal{M}$ be a good holonomic $\mathcal{D}$-module on (X, D) with a good K-structure. The associated pre-K-Betti structure $\mathcal{F}$ is a K-Betti structure of $\mathcal{M}$.*

Theorem 8.3 *Let X be a smooth complex algebraic variety with a hypersurface D, and let $(\mathcal{M}, \mathcal{F})$ be a K-holonomic $\mathcal{D}$-module on X.*

- *There exists a unique $K(*D)$-Betti structure $\mathcal{F}(*D)$ of $\mathcal{M}(*D)$ such that the natural morphism $\mathcal{M} \longrightarrow \mathcal{M}(*D)$ is compatible with the pre-K-Betti structures.*
- *For a morphism of K-holonomic $\mathcal{D}$-modules $(\mathcal{M}_1, \mathcal{F}_1) \longrightarrow (\mathcal{M}_2, \mathcal{F}_2)$, the morphism $\mathcal{M}_1(*D) \longrightarrow \mathcal{M}_2(*D)$ is compatible with the induced pre-K-Betti structures.*

We will use an induction on the dimension of the support of $\mathcal{M}$ for the proof. Let $SI(<n)$, $GOOD(<n)$ and $LOC(<n)$ denote the statements of Theorems 8.1, 8.2 and 8.3 in the case $\dim \operatorname{Supp} \mathcal{M} < n$, respectively. Our induction will proceed as follows:

- $SI(<n) + GOOD(<n) \Longrightarrow GOOD(\leq n)$ (Subsection 8.2.3).
- $SI(<n) + GOOD(<n) + LOC(<n) \Longrightarrow LOC(\leq n)$ (Subsection 8.3.3).
- $SI(<n) + GOOD(<n) + LOC(<n) \Longrightarrow SI(\leq n)$ (Subsection 8.4).

Remark 8.4 *In the proof, we will observe the equivalence of $K(*D)$-Betti structure and K-Betti structure. (See Lemma 8.8.)* ∎

Remark 8.5 *The arguments in Subsections 8.2 and 8.3 can work even in the analytic situation. Although most of Subsection 8.4 can also work even in the analytic situation, we need the existence of resolution of turning points for any meromorphic flat bundle.* ∎

8.2 Step 1

8.2.1 K-Cell

Let $\varphi : Z \longrightarrow X$ be a projective morphism of smooth complex algebraic varieties such that $\dim Z = n$. Let D be a normal crossing hypersurface of Z such that $\varphi_{|Z-D}$ is immersive. Let (V, ∇) be a good meromorphic flat bundle on (Z, D) with K-structure compatible with the Stokes filtrations. Let $\mathcal{F}_*$ be the associated K-structure of $\mathrm{DR}(V)$. Let $\mathcal{F}_!$ be the associated K-structure of $\mathrm{DR}(V_!)$.

Proposition 8.6 *Assume that $SI(<n)$ and $GOOD(<n)$. Then, $\varphi_* \mathcal{F}_V$ is a K-Betti structure of $\varphi_\dagger V$, and $\varphi_\dagger \mathcal{F}_{V!}$ is a K-Betti structure of $\varphi_\dagger V_!$.*

Proof Note that $\mathcal{C}_0 = (Z, U, \varphi, V)$ is an n-cell of $\varphi_\dagger V$, where $U = Z - D$. Let us show that it is a bounding n-cell. Let $\mathcal{C}' = (Z', U', \varphi', V')$ be a dominant refinement. Let g be a cell function for $\mathcal{C}'$. We have a factorization $\varphi' = \varphi \circ \varphi_1$, where $\varphi_1 : Z' \longrightarrow Z$. We put $g' := g \circ \varphi$. We have $V' = \varphi_1^{-1} V \otimes \mathcal{O}_{Z'}(*g')$. We have the canonical pre-K-Betti structures $\mathcal{F}_{V'}$ and $\mathcal{F}_{V'!}$ of V' and $V'_!$, respectively. According to Theorem 5.12, the morphisms $\varphi_{1\dagger}V'_! \longrightarrow \varphi_\dagger V \longrightarrow \varphi_{1\dagger}V'$ are compatible with pre-K-Betti structures. We have the induced pre-K-Betti structures ${}^D\Xi_g(\varphi_*\mathcal{F}_V)$, ${}^D\psi_g(\varphi_*\mathcal{F}_V)$ and ${}^D\phi_g(\varphi_*\mathcal{F}_V)$ of $\Xi_g(\varphi_\dagger V)$, $\psi_g(\varphi_\dagger V)$ and $\phi_g(\varphi_\dagger V)$, respectively. We also obtain pre-K-Betti structures ${}^D\Xi_{g'}(\mathcal{F}_{V'})$, ${}^D\psi_{g'}(\mathcal{F}_{V'})$ and ${}^D\phi_{g'}(\mathcal{F}_{V'})$ of $\Xi_{g'}(V')$, $\psi_{g'}(V')$ and $\phi_{g'}(V')$ on Z'. Note that $\bigl(\phi_{g'}(V'), {}^D\phi_{g'}(\mathcal{F}_{V'})\bigr)$ is good on (Z',D'). (See Subsection 6.2.3). Hence, it should be K-holonomic according to the assumption $GOOD(<n)$. Then, $\varphi_\dagger\bigl(\phi_{g'}(V'),\, {}^D\phi_{g'}(\mathcal{F}_{V'})\bigr)$ is K-holonomic by the assumption $SI(<n)$. Because $\Bigl(\phi_g(\varphi_\dagger V),\, {}^D\phi_g(\varphi_*\mathcal{F})\Bigr) \subset \varphi'_\dagger\bigl(\phi_{g'}(V'),\, {}^D\phi_{g'}(\mathcal{F}_{V'})\bigr)$, we obtain that ${}^D\phi_g(\varphi_*\mathcal{F})$ is a K-Betti structure of $\phi_g(\varphi_\dagger V)$ by Lemma 7.9. Another claim can be shown similarly, or we can deduce it as dual. ∎

Corollary 8.7 *Assume that $SI(<n)$ and $GOOD(<n)$. Let f be a cell function of $\mathcal{C}=(Z,U,\varphi,V)$. Then, $\psi_f(\varphi_\dagger V)$ and $\Xi_f(\varphi_\dagger V)$ with the canonical pre-K-Betti structures are K-holonomic.*

Proof Applying the previous results to $\varphi_\dagger\bigl(\Pi^{a,b}_{f\star}V\bigr)$ $(\star=!,*)$, we obtain that they are K-holonomic. Then, we obtain the corollary. ∎

8.2.2 Gluing

According to Corollary 8.7, we obtain the gluing construction of K-holonomic $\mathcal{D}$-module. Let X be a complex manifold, $\mathcal{C}=(Z,U,\varphi,V)$ be a K-cell as in Subsection 8.2.1. Let f be a cell function for $\mathcal{C}$ on X. Let $\mathcal{Q}$ be a K-holonomic $\mathcal{D}$-module whose support is contained in $f^{-1}(0)$. Assume that we are given morphisms of K-holonomic $\mathcal{D}$-modules

$$\psi_f(\varphi_\dagger V) \longrightarrow \mathcal{Q} \longrightarrow \psi_f(\varphi_\dagger V),$$

such that the composite is equal to the nilpotent map N on $\psi_f(\varphi_\dagger V)$. Then, we obtain a K-holonomic $\mathcal{D}$-module as the cohomology of the following complex:

$$\psi_f(\varphi_\dagger V) \longrightarrow \Xi_f(\varphi_\dagger V)\oplus\mathcal{Q} \longrightarrow \psi_f(\varphi_\dagger V)$$

8.2.3 Good holonomic $\mathcal{D}$-module with good K-structure

Let us show $GOOD(\leq n)$ by assuming $SI(<n)$ and $GOOD(<n)$. Let X be a smooth complex algebraic variety with a simply normal crossing hypersurface D. Let $\mathcal{M}$ be a good holonomic $\mathcal{D}$-module on (Z,D) with a good K-structure such that $\dim\operatorname{Supp}\mathcal{M}=n$. Let $\mathcal{F}$ be the associated pre-K-Betti structure. We would like to show that $\mathcal{F}$ is a K-Betti structure. Let $D=\bigcup_{i=1}^{\ell}D_i$ be the irreducible decomposition. We may assume that X is affine and that each D_i is given as $g_i^{-1}(0)$ for an algebraic function. Let $\rho(\mathcal{M})\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{>0}$ denote the pair of $\dim\operatorname{Supp}\mathcal{M}$ and the irreducible components of $\operatorname{Supp}\mathcal{M}$ with the maximal dimension. We use the lexicographic order on $\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{>0}$. For a good holonomic $\mathcal{D}$-module $\mathcal{M}$ on (X,D), there exists $J\subset\underline{\ell}$ with $n=\dim Z-|J|$ such that $V:=\mathcal{M}(*g)\neq 0$ comes from a meromorphic flat bundle on D_J, where $g:=\prod_{j\notin J}g_j$. We have a description of $\mathcal{M}$ as the cohomology of the complex of pre-K-holonomic $\mathcal{D}$-modules $\psi_g(V)\longrightarrow\Xi_g(V)\oplus\phi_g(\mathcal{M})\longrightarrow\psi_g(V)$. By Corollary 8.7, $\psi_g(V)$ and $\Xi_g(V)$ are K-holonomic. Because $\rho(\phi_g(\mathcal{M}))<\rho(\mathcal{M})$, we obtain that $\phi_g(\mathcal{M})$ is K-holonomic. Hence, we obtain that $\mathcal{M}$ is also K-holonomic, and we obtain $GOOD(\leq n)$.

8.3 Step 2

8.3.1 Equivalence of $K(*D)$-Betti structure and K-Betti structure

Let X be a smooth complex algebraic variety with a hypersurface D. Let $(\mathcal{M},\mathcal{F})$ be a pre-K-holonomic $\mathcal{D}_{X(*D)}$-module with $\dim\operatorname{Supp}\mathcal{M}\leq n$.

Lemma 8.8

- *Assume $SI(<n)$ and $GOOD(<n)$. If $\mathcal{F}$ is a $K(*D)$-Betti structure, then it is a K-Betti structure.*
- *Assume $LOC(\leq n)$. If $\mathcal{F}$ is a K-Betti structure, then it is a $K(*D)$-Betti structure.*

Proof Let us show the first claim. We use an induction on the dimension of the support. Let P be any point of $D \cap \operatorname{Supp} \mathcal{M}$. We take a bounding cell $\mathcal{C} = (Z, U, \varphi, V)$ of $(\mathcal{M}, \mathcal{F})$ at P, and a cell function g of $\mathcal{C}$ as $\mathcal{D}_{X(*D)}$-module. We have a description of $\mathcal{M}$ as the cohomology of the following complex of $K(*D)$-holonomic $\mathcal{D}_{X(*D)}$-modules:

$$\psi_g\big(\varphi_\dagger(V_!),\, *D\big) \longrightarrow \Xi_g(\varphi_\dagger V, *D) \oplus \phi_g(\mathcal{M}, *D) \longrightarrow \psi_g\big(\varphi_\dagger(V), *D\big)$$

By the hypothesis of the induction, $\phi_g(\mathcal{M}, *D)$ is K-holonomic. According to Corollary 8.7, $\psi_g\big(\varphi_\dagger(V_!), *D\big)$ and $\Xi_g(\mathcal{M}, *D)$ are K-holonomic. Hence, we obtain that $\mathcal{M}$ is also K-holonomic.

Let us show the second claim. By the assumption $LOC(\leq n)$, we obtain a $K(*D)$-holonomic $\mathcal{D}_{X(*D)}$-module $\big(\mathcal{M}(*D), \mathcal{F}(*D)\big)$ with a morphism $(\mathcal{M}, \mathcal{F}) \longrightarrow \big(\mathcal{M}(*D), \mathcal{F}(*D)\big)$ of pre-K-holonomic $\mathcal{D}$-modules. Because $\mathcal{M} = \mathcal{M}(*D)$, we obtain $\mathcal{F} = \mathcal{F}(*D)$, and hence $\mathcal{F}$ is a $K(*D)$-Betti structure. ∎

We reformulate the uniqueness (Proposition 7.17) as follows.

Corollary 8.9 *Let $\star$ be $*$ or $!$. Assume $SI(< n)$, $GOOD(< n)$ and $LOC(\leq n)$. Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module on X such that $\mathcal{M}(\star D) = \mathcal{M}$. Let $\mathcal{F}_i$ $(i = 1, 2)$ be K-Betti structures on $\mathcal{M}$. If $\mathcal{F}_{1|X-D} = \mathcal{F}_{2|X-D}$, then $\mathcal{F}_1 = \mathcal{F}_2$.*

Proof The claim for $\star = *$ follows from Lemma 8.8 and Proposition 7.17. We obtain the claim for $\star =!$ by using the dual with Lemma 7.8. ∎

Corollary 8.10 *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Assume that one of the following holds: (i) $\mathcal{M}(!D) \longrightarrow \mathcal{M}$ is surjective, (ii) $\mathcal{M} \longrightarrow \mathcal{M}(*D)$ is injective. Let $\mathcal{F}_i$ $(i = 1, 2)$ be K-Betti structures on $\mathcal{M}$. If $\mathcal{F}_{1|X-D} = \mathcal{F}_{2|X-D}$, then $\mathcal{F}_1 = \mathcal{F}_2$.* ∎

We reformulate the independence of compactification (Proposition 7.18). Let $F : X' \longrightarrow X$ be a projective birational morphism of complex manifolds. Let D be a hypersurface, and we put $D' := F^{-1}(D)$. Assume $X' - D' \simeq X - D$.

Proposition 8.11 *Assume $SI(< n)$, $GOOD(< n)$ and $LOC(\leq n)$. Let $\mathcal{M}'$ be a holonomic $\mathcal{D}_{X'(*D')}$-module. We set $\mathcal{M} := F_\dagger \mathcal{M}'$.*

- *Let $\mathcal{F}'$ be a K-Betti structure of $\mathcal{M}'$. Then, $F_* \mathcal{F}'$ is a K-Betti structure of $\mathcal{M}$.*
- *Let $\mathcal{F}$ be a K-Betti structure of $\mathcal{M}$. Then, $\mathcal{M}'$ is equipped with a K-Betti structure $\mathcal{F}'$ such that $\mathcal{F}'_{|X'-D'} = \mathcal{F}_{|X-D}$ under the isomorphism $\mathcal{M}'_{|X'-D'} \simeq \mathcal{M}_{|X-D}$. It is functorial.* ∎

8.3.2 $K(*D)$-cell

Let $\varphi : Z \longrightarrow X$ be a morphism of smooth complex algebraic varieties such that $\dim Z = n$. Let D_Z be a normal crossing hypersurface of Z such that $\varphi_{|Z-D_Z}$ is immersive, and that $D_1 := \varphi^{-1}(D) \subset D_Z$. Let V be a good meromorphic flat bundle on (Z, D_Z) with K-structure compatible with the Stokes filtrations. Let $\mathcal{F}_V$ be the associated pre-K-Betti structure of V. Let $\mathcal{F}_{V!}(*D_1)$ be the associated pre-K-Betti structure of $V_!(*D_1)$.

Proposition 8.12 *Assume $SI(<n)$, $GOOD(<n)$ and $LOC(<n)$. Then, $\varphi_\dagger\big(V_!(*D_1), \mathcal{F}_{V!}(*D_1)\big)$ and $\varphi_\dagger\big(V, \mathcal{F}_V\big)$ are $K(*D)$-holonomic.*

Proof Let us show that $\mathcal{C}_0 = (Z, U, \varphi, V)$ is a bounding n-cell. Let $\mathcal{C}' = (Z', U', \varphi', V')$ be a dominant refinement. Let g be a cell function for $\mathcal{C}'$ as $\mathcal{D}(*D)$-modules. We have a factorization $\varphi' = \varphi \circ \varphi_1$, where $\varphi_1 : Z' \longrightarrow Z$. We put $g' := g \circ \varphi$ and $D'_1 := \varphi^{-1} D$. We have $V' = \varphi_1^{-1} V \otimes \mathcal{O}_{Z'}(*g')$. According to Proposition 6.7, the morphisms $\varphi'_\dagger(V'_!)(*D) \longrightarrow \varphi_\dagger(V_!)(*D) \longrightarrow \varphi_\dagger V \longrightarrow \varphi'_\dagger V'$ are compatible with the canonical pre-$K(*D)$-Betti structures. We obtain the induced pre-$K(*D)$-Betti structures of $\phi_g\big(\varphi_\dagger(V), *D\big)$ and $\phi_g\big(\varphi_\dagger(V_!), *D\big)$.

We obtain pre-K-holonomic $\mathcal{D}$-modules $\phi_{g'}\big(V'_!, *D'_1\big)$ and $\phi_{g'}\big(V', *D'_1\big)$ on Z'. Because they are good on (Z', D'), they are K-holonomic by $GOOD(<n)$. We obtain that $\phi_g\big(\varphi'_\dagger V', *D\big)$ and $\phi_g\big(\varphi'_\dagger\big(V'_!\big), *D\big)$ are K-holonomic by the assumption $SI(< n)$. By Lemma 8.8 we obtain that $\phi_g(\varphi'_\dagger V', *D)$ and $\phi_g(\varphi'_\dagger V'_!, *D)$ are

$K(*D)$-holonomic. Because $\phi_g(\varphi_\dagger V, *D) \subset \phi_g\big(\varphi'_\dagger V', *D\big)$ is compatible with the pre-K-Betti structures, we obtain that $\phi_g(\varphi_\dagger V, *D)$ is also a $K(*D)$-holonomic by Lemma 7.16. Since the surjection $\phi_g\big(\varphi'_\dagger V'_!, *D\big) \longrightarrow \phi_g\big(\varphi_\dagger V_!, *D\big)$ is compatible with the pre-K-Betti structures, $\phi_g\big(\varphi_\dagger V_!, *D\big)$ is also $K(*D)$-holonomic by Lemma 7.16. ∎

Corollary 8.13 *Assume SI(<n), GOOD(<n) and LOC(<n). Let f be a cell function of an n-dimensional cell $\mathcal{C} = (Z, U, \varphi, V)$ as $\mathcal{D}_{X(*D)}$-module. Then, $\psi_f(\varphi_\dagger V, *D)$ and $\Xi_f(\varphi_\dagger V, *D)$ with the canonical pre-K-Betti structures are $K(*D)$-holonomic.*

Proof Applying the previous results to $\Pi^{a,b}_{f\star}\big(\varphi_\dagger V, *D\big)$ for $\star = *, !$, we obtain that they are $K(*D)$-holonomic. Then, we obtain the corollary. ∎

8.3.3 Localization

Let us show $LOC(\leq n)$ by assuming $SI(<n)$, $GOOD(<n)$ and $LOC(<n)$. By Proposition 7.17, the problem is local. Let $(\mathcal{M}, \mathcal{F})$ be a K-holonomic $\mathcal{D}_X$-module with $\dim \operatorname{Supp} \mathcal{M} \leq n$.

Let P be any point of D. Let (Z, U, φ, V) be a bounding cell of $\mathcal{M}$ at P. By taking a refinement, we may assume $U \cap D = \emptyset$. Let g be a cell function of $\mathcal{M}$ as $\mathcal{D}$-modules. We put $g_1 := \varphi^{-1}(g)$ and $D_1 := \varphi^{-1}(D)$. We have the expression of $\mathcal{M}$ as the cohomology of the following complex of the K-holonomic $\mathcal{D}$-modules:

$$\psi_g \varphi_\dagger(V_!) \longrightarrow \Xi_g \varphi_\dagger(V) \oplus \phi_g(\mathcal{M}) \longrightarrow \psi_g \varphi_\dagger(V) \tag{81}$$

By the assumption of the induction, $\psi_g\big(\varphi_\dagger V_!, *D\big)$ and $\phi_g(\mathcal{M}, *D)$ are equipped with the induced $K(*D)$-Betti structures. We also have the following commutative diagram of pre-K-holonomic $\mathcal{D}$-modules:

$$\begin{array}{ccccc}
\psi_g(V) & \longrightarrow & \phi_g(\mathcal{M}) & \longrightarrow & \psi_g(V) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\psi_g\big(\varphi_\dagger V_!, *D\big) & \longrightarrow & \phi_g(\mathcal{M}, *D) & \longrightarrow & \psi_g\big(\varphi_\dagger V_!, *D\big)
\end{array}$$

We have the canonical pre-K-Betti structures of $\psi_{g_1}\big(V, *D_1\big)$ and $\Xi_{g_1}\big(V, *D_1\big)$. According to Corollary 8.13, their push-forward $\varphi_\dagger \psi_{g_1}\big(V, *D_1\big)$ and $\varphi_\dagger \Xi_{g_1}\big(V, *D_1\big)$ are $K(*D)$-holonomic. We also have the following commutative diagram of pre-K-holonomic $\mathcal{D}$-modules:

$$\begin{array}{ccccc}
\varphi_\dagger \psi_{g_1}(V) & \longrightarrow & \varphi_\dagger \Xi_{g_1}(V) & \longrightarrow & \varphi_\dagger \psi_{g_1}(V) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\varphi_\dagger \psi_{g_1}\big(V, *D_1\big) & \longrightarrow & \varphi_\dagger \Xi_{g_1}\big(V, *D_1\big) & \longrightarrow & \varphi_\dagger \psi_{g_1}\big(V, *D_1\big)
\end{array}$$

By Proposition 7.17, the identification $\varphi_\dagger \psi_{g_1}\big(V, *D_1\big) \simeq \psi_g\big(\varphi_\dagger V, *D\big)$ is compatible with the pre-K-Betti structures. Hence, we obtain a $K(*D)$-Betti structure of $\mathcal{M}(*D)$ with a morphism of pre-K-holonomic $\mathcal{D}$-modules $\mathcal{M} \longrightarrow \mathcal{M}(*D)$ whose restriction to $X - D$ is an isomorphism. The functoriality is clear from the above construction. ∎

8.3.4 Twist

Let $(\mathcal{M}, \mathcal{F})$ be a $K(*D)$-holonomic $\mathcal{D}(*D)$-module with $\dim \operatorname{Supp} \mathcal{M} \leq n$. Let $\mathcal{V}$ be a meromorphic flat bundle on (X, D) with a K-Betti structure $\mathcal{F}_{\mathcal{V}}$. According to Lemma 7.10, $\mathcal{F}_{\mathcal{M}|X-D} \otimes \mathcal{F}_{\mathcal{V}|X-D}$ is a K-Betti structure of $(\mathcal{M} \otimes \mathcal{V})_{|X-D}$.

Lemma 8.14 *Assume SI(< n), GOOD(< n) and LOC(< n). There exists a $K(*D)$-Betti structure $\mathcal{F}_{\mathcal{M}\otimes\mathcal{V}}$ of $\mathcal{M} \otimes \mathcal{V}$ such that*

$$\mathcal{F}_{\mathcal{M}\otimes\mathcal{V}|X-D} \simeq \mathcal{F}_{\mathcal{M}|X-D} \otimes \mathcal{F}_{\mathcal{V}|X-D}.$$

It is functorial with respect to $\mathcal{M}$ and $\mathcal{V}$.

Proof Let $P \in D$. We have only to consider the issue locally around P. We use an induction on $\dim_P \operatorname{Supp} \mathcal{M}$. Let $\mathcal{C} = (Z, U, \varphi, V)$ be a dominating cell of $\mathcal{M}$ at P. By considering an appropriate refinement, we may assume that $V \otimes \varphi^* \mathcal{V}$ is good on (Z, D_Z), where $D_Z = Z - U$. Let g be a cell function for $\mathcal{C}$ as $\mathcal{D}_{X(*D)}$-module. By the hypothesis of the induction, we have the $K(*D)$-Betti structure of $\psi_g(\varphi_\dagger V, *D) \otimes \mathcal{V}$ and $\phi_g(\varphi_\dagger V, *D) \otimes \mathcal{V}$. According to Proposition 8.12, we have the $K(*D)$-Betti structures of $\psi_g(\varphi_\dagger V, *D) \otimes \mathcal{V}$ and $\Xi_g(\varphi_\dagger V, *D) \otimes \mathcal{V}$ induced by the isomorphisms $\psi_g(\mathcal{M}, *D) \otimes \mathcal{V} \simeq \psi_g(\mathcal{M} \otimes \mathcal{V}, *D)$ and $\Xi_g(\mathcal{M}, *D) \otimes \mathcal{V} \simeq \Xi_g(\mathcal{M} \otimes \mathcal{V}, *D)$. By the uniqueness, the induced $K(*D)$-Betti structures on $\psi_g(\mathcal{M}, *D) \otimes \mathcal{V}$ are equal. Because $\mathcal{M} \otimes \mathcal{V}$ is expressed as the cohomology of the complex

$$\psi_g(\mathcal{M}, *D) \otimes \mathcal{V} \longrightarrow \Xi_g(\mathcal{M}, *D) \otimes \mathcal{V} \oplus \phi_g(\mathcal{M}, *D) \otimes \mathcal{V} \longrightarrow \psi_g(\mathcal{M}, *D) \otimes \mathcal{V},$$

we obtain a $K(*D)$-Betti structure on $\mathcal{M} \otimes \mathcal{V}$ with the desired property. ∎

8.4 Step 3

Let us show that $SI(<n)$, $GOOD(<n)$ and $LOC(<n)$ imply $SI(\leq n)$. The following argument is inspired by [2].

8.4.1 Special case I

Let $G : X \longrightarrow Y$ be a projective morphism of complex algebraic varieties. Let D be a normal crossing hypersurface of X. Let V be a meromorphic flat bundle on (X, D) with a K-Betti structure. Let $D = D_1 \cup D_2$ be a decomposition of D. We have the holonomic $\mathcal{D}$-module $\mathcal{M} := V(*D_1 !D_2)$ with the induced K-Betti structure, denoted by $\mathcal{F}$.

Proposition 8.15 *If $G^i_\dagger \mathcal{M} = 0$ for $i \neq 0$, $RG_* \mathcal{F}$ is a K-Betti structure of $G^0_\dagger \mathcal{M}$.*

Proof Since the claim is local, we may assume that Y is affine. Let us consider the case $\operatorname{Supp} G^0_\dagger \mathcal{M} \subsetneq G(X)$. We take a function f such that $\operatorname{Supp} G^0_\dagger \mathcal{M} \subset f^{-1}(0)$ and $G(X) \not\subset f^{-1}(0)$. We set $f_X := G^{-1}(f)$. We have a description of the K-holonomic $\mathcal{D}$-module $\phi_{f_X} \mathcal{M}$ as the cohomology of the following:

$$\mathcal{M}(!f_X) \longrightarrow \Xi_{f_X} \mathcal{M}(*f_X) \oplus \mathcal{M} \longrightarrow \mathcal{M}(*f_X)$$

By the assumption, we obtain that $G_\dagger \mathcal{M}(!f_X) = G_\dagger \mathcal{M}(*f_X) = G_\dagger \Xi_{f_X} \mathcal{M}(*f_X) = 0$. Hence, we obtain that $G_\dagger(\mathcal{M}, \mathcal{F}) \simeq G_\dagger \phi_{f_X}(\mathcal{M}, \mathcal{F})$. By the assumption $SI(<n)$, we obtain that $RG_* \mathcal{F}$ is a K-Betti structure of $G^0_\dagger \mathcal{M}$.

Let us consider the case $G(X) = \operatorname{Supp} \mathcal{M}$. Let $P \in \operatorname{Supp} G^0_\dagger \mathcal{M}$. As remarked in Subsection 7.1.1, there exists a good cell $\mathcal{C} = (Z, U, \varphi, E)$ of $G^0_\dagger \mathcal{M}$ at P, according to [32]. Let g be a cell function of $\mathcal{C}$. We set $g_Z := \varphi^{-1} g$ and $g_X := G^{-1} g$. We have the K-Betti structures $\mathcal{F}(*g_X)$ of $\mathcal{M}(*g_X)$, obtained as the localization. (See Subsection 8.3.3.) By considering the dual, we obtain the K-Betti structure $\mathcal{F}(!g_X)$ of $\mathcal{M}(!g_X)$.

Lemma 8.16

- *The K-structure of E is compatible with the Stokes structure.*
- *For $\star = *, !$, the natural isomorphisms $\varphi_\dagger E_\star \simeq G_\dagger(\mathcal{M})(\star g)$ are compatible with the pre-K-Betti structures.*

Proof Let us consider the case $\star = *$. The case $\star = !$ can be argued similarly. We take a projective birational morphism $\kappa : X_1 \longrightarrow X$ such that (i) X_1 is smooth, (ii) $X_1 - (g_X \circ \kappa)^{-1}(0) \simeq X - g_X^{-1}(0)$, (iii) $(g_X \circ \kappa)^{-1}(0)$ is normal crossing, (iv) the induced morphism $X' \longrightarrow Y$ factors into $X' \xrightarrow{G_Z} Z \xrightarrow{\varphi} Y$.

We set $D'_1 := \kappa^{-1}\big(D_1 \cup g^{-1}(0)\big)$. Let D'_2 be the complement of D'_1 in $D' := \kappa^{-1}(D \cup g^{-1}(0))$. We set $V' := \kappa^{-1} V \otimes \mathcal{O}(*D')$. We set $\mathcal{M}' := V'(*D'_1 !D'_2)$. Note that $\kappa_\dagger \mathcal{M}' \simeq \mathcal{M}(*g_X)$ and $G_{Z\dagger} \mathcal{M}' = E$.

According to Proposition 8.11, we have the induced K-Betti structure $\mathcal{F}'$ of $\mathcal{M}'$ such that $R\kappa_* \mathcal{F}' = \mathcal{F}(*g_X)$. By Proposition 6.15, we obtain that the K-structure of E is compatible with the Stokes structures, and that $RG_{Z*} \mathcal{F}'$ is the canonical K-Betti structure of $G_{Z\dagger} \mathcal{M}'$. Hence, we obtain that $RG_* \mathcal{F}(*g_X)$ is the canonical K-Betti structure of $G_\dagger(\mathcal{M})(*g) = \varphi_\dagger E$. Thus, we obtain Lemma 8.16. ∎

Lemma 8.17 *The natural isomorphisms $G_\dagger \Xi_{g_X}\big(\mathcal{M}(*g_X)\big) \simeq \Xi_g\big(\varphi_\dagger E\big)$ and $G_\dagger \psi_{g_X}\big(\mathcal{M}(*g_X)\big) \simeq \psi_g\big(\varphi_\dagger E\big)$ are compatible with the induced pre-K-Betti structures.*

Proof By Lemma 8.16, we obtain that the natural isomorphisms $G_\dagger\big(\mathcal{M}(*g_X)\otimes \mathfrak{I}^{a,b}_{g_X}\big)(\star g_X) \simeq \varphi_\dagger E \otimes \mathfrak{I}^{a,b}_{g_X}(\star g_X)$ are compatible with the induced pre-K-Betti structures. Hence, we obtain Lemma 8.17. ∎

By Lemma 8.16, the morphisms $\varphi_\dagger E_! \longrightarrow G_\dagger \mathcal{M} \longrightarrow \varphi_\dagger E$ are compatible with the induced pre-K-Betti structures. Hence, we have an induced pre-K-Betti structure ${}^D\phi_g(RG_*\mathcal{F})$ of $\phi_g(G^0_\dagger\mathcal{M})$. We also have the induced K-Betti structure ${}^D\phi_{g_X}(\mathcal{F})$ of $\phi_{g_X}\mathcal{M}$. By using Lemma 8.17, we obtain ${}^D\phi_g(RG_*\mathcal{F}) = RG_*{}^D\phi_{g_X}(\mathcal{F})$ under the isomorphism $\phi_g(G^0_\dagger\mathcal{M}) \simeq G^0_\dagger \phi_{g_X}\mathcal{M}$. By the assumption $SI(<\dim X)$, we obtain that ${}^D\phi_g\big(RG_*\mathcal{F}\big)$ is a K-Betti structure of $\phi_g\big(G_\dagger\mathcal{M}\big)$. Thus, we obtain Proposition 8.15. ∎

8.4.2 Special case II

Let $G : X \longrightarrow Y$ be a projective morphism of complex algebraic varieties. Let $\varphi : Z \longrightarrow X$ be a projective morphism. Let D_Z be a normal crossing hypersurface of Z. Let V be a good meromorphic flat bundle on (Z, D_Z) with a K-Betti structure. Assume that $\varphi_{|Z-D_Z}$ is an immersion. Let $D_Z = D_{Z,1}\cup D_{Z,2}$ be a decomposition. We have the holonomic $\mathcal{D}$-module $V\big(*D_{Z,1}!D_{Z,2}\big)$ on Z, with the canonical K-Betti structure $\mathcal{F}_V(*D_{Z,1}!D_{Z,2})$. We set $\mathcal{M} := \varphi_\dagger V(*D_{Z,1}!D_{Z,2})$ on X, with the canonical K-Betti structure $\mathcal{F} := \varphi_\dagger \mathcal{F}_V(*D_{Z,1}!D_{Z,2})$.

Lemma 8.18 *If $G^i_\dagger\mathcal{M} = 0$ for any $i \neq 0$, then the induced pre-K-Betti structure of $G^0_\dagger\mathcal{M}$ is a K-Betti structure.*

Proof It follows from Proposition 8.15. ∎

8.4.3 Special case III

Let $\mathcal{E}$ be a locally free sheaf on a smooth complex algebraic variety Y. We put $X := \mathbb{P}(\mathcal{E})$. Let H_i $(i = 1, 2)$ be hyperplane subbundles. Let $\mathcal{N}$ be a K-holonomic $\mathcal{D}$-module on X such that $\mathcal{N}(*H_1) = \mathcal{N}$. By shrinking Y, we take a meromorphic function g on X such that (i) $g^{-1}(\infty) \subset H_1$, (ii) $\mathcal{N}(*g)$ is a cell. Assume the following:

- H_2 is non-characteristic to $\mathcal{N}$, $\psi_g(\mathcal{N}, *H_1)$, $\Xi_g(\mathcal{N}, *H_1)$ and $\phi_g(\mathcal{N}, *H_1)$.

Lemma 8.19 *The induced pre-K-Betti structure of $G^0_\dagger\mathcal{N}(!H_2)$ is a K-Betti structure.*

Proof We have the K-holonomic $\mathcal{D}$-modules

$$\mathcal{N}(*g)\otimes \mathfrak{I}^{a,b}_g(!g*H_1!H_2), \quad \big(\mathcal{N}(*g)\otimes \mathfrak{I}^{a',b'}_g\big)(!H_2).$$

Note that $G^i_\dagger\Big(\big(\mathcal{N}(*g)\otimes\mathfrak{I}^{a,b}_g(!g*H_1!H_2),\big)\Big) = 0$ and $G^i_\dagger\Big(\big(\mathcal{N}(*g)\otimes\mathfrak{I}^{a',b'}_g\big)(!H_2)\Big) = 0$ unless $i = 0$. According to Lemma 8.18, the induced pre-K-Betti structures of

$$G^0_\dagger\Big(\big(\mathcal{N}(*g)\otimes\mathfrak{I}^{a,b}_g\big)(!g*H_1!H_2)\Big), \quad G^0_\dagger\Big(\big(\mathcal{N}(*g)\otimes\mathfrak{I}^{a',b'}_g\big)(!H_2)\Big)$$

are K-Betti structures. Hence, we obtain that the induced pre-K-Betti structure of

$$G^0_\dagger\Big(\Xi_g(\mathcal{N}(*g), *H_1)(!H_2)\Big), \quad G^0_\dagger\Big(\psi_g(\mathcal{N}(*g), *H_1)(!H_2)\Big)$$

are K-Betti structures. Because we have a description of $G^0_\dagger\mathcal{N}(!H_2)$ as the cohomology of the following complex

$$G^0_\dagger\psi_g\big(\mathcal{N}(*g), *H_1\big)(!H_2) \longrightarrow G^0_\dagger\Xi_g\big(\mathcal{N}(*g), *H_1\big)(!H_2)\oplus G^0_\dagger\phi_g(\mathcal{N}, *H_1)(!H_2) \longrightarrow G^0_\dagger\psi_g\big(\mathcal{N}(*g), *H_1\big)(!H_2),$$

we obtain Lemma 8.19. ∎

8.4.4 Proof of Theorem 8.1

We have only to consider the case $X = \mathbb{P}(\mathcal{E})$ for some locally free sheaf $\mathcal{E}$ on Y. We use an induction on the dimension of the support of $\mathcal{M}$. We take a resolution $\mathrm{Tot}(\mathcal{Q}_{\bullet,\bullet})$ of $\mathcal{M}$ as in Subsection 2.3.4. It is naturally equipped with the K-Betti structure $\mathrm{Tot}\big(\mathcal{F}^{\mathcal{Q}}_{\bullet,\bullet}\big)$. Then, $F^i_{\dagger}(\mathcal{M},\mathcal{F})$ is described as the i-th cohomology of $\mathrm{Tot}\Big(F^0_{\dagger}\big(\mathcal{Q}_{\bullet,\bullet},\mathcal{F}^{\mathcal{Q}}_{\bullet,\bullet}\big)\Big)$. Hence, we have only to show that $F^0_{\dagger}\big(\mathcal{Q}_{\bullet,\bullet},\mathcal{F}^{\mathcal{Q}}_{\bullet,\bullet}\big)$ are K-holonomic. By the construction, we have $\dim\mathrm{Supp}\,\mathcal{Q}_{i,j}<\dim\mathrm{Supp}\,\mathcal{M}$ for $(i,j)\neq(0,0)$, to which we can apply the hypothesis of the induction. Hence, we have only to show that $F^0_{\dagger}\big(\mathcal{Q}_{0,0},\mathcal{F}^{\mathcal{Q}}_{0,0}\big)$ is K-holonomic, which follows from Lemma 8.19. Thus, the proof of Theorem 8.1 is finished. ∎

9 Derived category of algebraic K-holonomic $\mathcal{D}$-modules

We study the standard functors on the derived category of algebraic K-holonomic $\mathcal{D}$-modules. We have only to follow very closely the argument due to Beilinson [2], [3] and Saito [38]. This section is included for a rather expository purpose.

9.1 Standard exact functors

Let X be a smooth complex quasi-projective variety. We take a smooth projective completion $X\subset\overline{X}$ such that $D=\overline{X}-X$ is a normal crossing hypersurface. We set $\mathrm{Hol}(X,K):=\mathrm{Hol}\big(\overline{X}(*D),K\big)$, which is independent of the choice of a completion $\overline{X}$ (Proposition 8.11). Let $D^b_{\mathrm{hol}}(X,K)$ be the derived category of $\mathrm{Hol}(X,K)$. We will implicitly use the following obvious lemma. (Later, we will prove the stronger version in Theorem 9.14.)

Lemma 9.1 *The forgetful functors* $\mathrm{Hol}(X,K)\longrightarrow\mathrm{Hol}(X)$ *is faithful.* ∎

Dual Let $(\mathcal{M},\mathcal{F})\in\mathrm{Hol}\big(\overline{X}(*D),K\big)$. We put $\mathbb{D}_X\mathcal{M}:=\mathbb{D}_{\overline{X}}(\mathcal{M})(*D)$. It is naturally equipped with the induced K-Betti structure $\mathbb{D}_{\overline{X}}\mathcal{F}(*D)$. Thus, we obtain $\mathbb{D}_X(\mathcal{M},\mathcal{F}):=\big(\mathbb{D}_{\overline{X}}(\mathcal{M})(*D),\mathbb{D}_{\overline{X}}\mathcal{F}(*D)\big)$.

Lemma 9.2 $\mathbb{D}_X(\mathcal{M},\mathcal{F})$ *is well defined in* $\mathrm{Hol}(X,K)$.

Proof Let $\overline{X}'$ be another smooth projective compactification of X. Put $D':=\overline{X}'-X$. We assume to have a projective morphism $\varphi:\overline{X}'\longrightarrow\overline{X}$ such that $\varphi_{|X}=\mathrm{id}_X$. Let $(\mathcal{M}',\mathcal{F}')$ be a K-holonomic $\mathcal{D}_{\overline{X}'(*D')}$-module such that $\varphi_{\dagger}\mathcal{M}'=\mathcal{M}$ and $\mathcal{F}'_{|X}=\mathcal{F}_{|X}$. We have $\big(\mathbb{D}_{\overline{X}'}\mathcal{F}'(*D')\big)_{|X}=\big(\mathbb{D}_{\overline{X}}\mathcal{F}(*D)\big)_{|X}$ under the natural isomorphism $\mathbb{D}_{\overline{X}'}\mathcal{M}'(*D')_{|X}\simeq\mathbb{D}_X\mathcal{M}(*D)_{|X}$. It implies the claim of the lemma. ∎

Corollary 9.3 *There exists a functor* $\mathbb{D}_X$ *on* $\mathrm{Hol}(X,K)$ *which is compatible with the standard duality functors on* $\mathrm{Hol}(X)$ *and the category of K-perverse sheaves. We also have a functor* $\mathbb{D}_X$ *on* $D^b_{\mathrm{hol}}(X,K)$*, compatible with the standard duality functors on* $D^b_{\mathrm{hol}}(X)$ *and* $D^b_c(K_X)$*. They are unique up to natural equivalence.* ∎

We use the symbol ${}^K\mathbb{D}_X$, if we would like to emphasize that it is a functor for K-holonomic $\mathcal{D}$-modules.

Lemma 9.4 *For* $\mathcal{M},\mathcal{N}\in\mathrm{Hol}(X,K)$*, we have a natural isomorphism:*

$$\mathrm{Ext}^i_{\mathrm{Hol}(X,K)}(\mathcal{M},\mathcal{N})\simeq\mathrm{Ext}^i_{\mathrm{Hol}(X,K)}\big({}^K\mathbb{D}_X\mathcal{N},{}^K\mathbb{D}_X\mathcal{M}\big)$$

Proof It follows from the comparison of Yoneda extensions. ∎

Localization Let H be a hypersurface of X. As is shown in Theorem 8.3 and Proposition 8.11, we have the localization:

$$*H:\mathrm{Hol}(X,K)\longrightarrow\mathrm{Hol}(X,K),\quad(\mathcal{M},\mathcal{F})\longmapsto\big(\mathcal{M}(*H),\mathcal{F}(*H)\big)$$

It is an exact functor. By considering the dual, we obtain an exact functor:

$$!H:\mathrm{Hol}(X,K)\longmapsto\mathrm{Hol}(X,K),\quad(\mathcal{M},\mathcal{F})\longmapsto\big(\mathcal{M}(!H),\mathcal{F}(!H)\big)$$

They induce exact functors $*H$ and $!H$ on $D^b_{\mathrm{hol}}(X,K)$.

Lemma 9.5 *For $\mathcal{M},\mathcal{N}\in \mathrm{Hol}(X,K)$, we have the following natural isomorphisms:*

$$\mathrm{Ext}^i_{\mathrm{Hol}(X,K)}\bigl(\mathcal{M},\mathcal{N}(*D)\bigr)\simeq \mathrm{Ext}^i_{\mathrm{Hol}(X,K)}\bigl(\mathcal{M}(*D),\mathcal{N}(*D)\bigr)$$

$$\mathrm{Ext}^i_{\mathrm{Hol}(X,K)}\bigl(\mathcal{M}(!D),\mathcal{N}\bigr)\simeq \mathrm{Ext}^i_{\mathrm{Hol}(X,K)}\bigl(\mathcal{M}(!D),\mathcal{N}(!D)\bigr)$$

Proof It follows from comparisons of Yoneda extensions. ∎

Nearby cycle, vanishing cycle and maximal functors Let g be an algebraic function on X. By Lemma 8.14, we have the exact functor $\Pi^{a,b}_{g\star}$ ($\star=*,!$) on $\mathrm{Hol}(X,K)$ given by $\Pi^{a,b}_{g\star}(\mathcal{M},\mathcal{F}):=\bigl((\mathcal{M},\mathcal{F})\otimes\mathfrak{I}^{a,b}_g\bigr)(\star g)$ and $a,b\in\mathbb{Z}$. Hence, we obtain the exact functors Ξ_g, ψ_g and ϕ_g on $\mathrm{Hol}(X,K)$. They induce the corresponding exact functors on $D^b_{\mathrm{hol}}(X,K)$. We use the symbols ${}^K\Xi_g$, ${}^K\psi_g$ and ${}^K\phi_g$, when we would like to emphasize that they are functors for K-holonomic $\mathcal{D}$-modules.

9.2 Push-forward and pull-back

9.2.1 Statement

Let $f:X\longrightarrow Y$ be an algebraic morphism of quasi-projective varieties. We take a factorization $X\subset\overline{X}\xrightarrow{f'}Y$ such that (i) f' is projective, (ii) $H=\overline{X}-X$ is normal crossing. We have a natural equivalence between $\mathrm{Hol}\bigl(\overline{X}(*H),K\bigr)$ and $\mathrm{Hol}(X,K)$. Let $(\overline{\mathcal{M}},\overline{\mathcal{F}})\in\mathrm{Hol}\bigl(\overline{X}(*H),K\bigr)$ correspond to $(\mathcal{M},\mathcal{F})\in\mathrm{Hol}(X,K)$. According to Theorem 8.1, we have

$${}^Kf^i_*(\mathcal{M},\mathcal{F}):=\bigl(f^i_\dagger\overline{\mathcal{M}},f^i_\dagger\overline{\mathcal{F}}\bigr)\in\mathrm{Hol}(Y,K),\quad {}^Kf^i_!(\mathcal{M},\mathcal{F}):=\bigl(f^i_\dagger\overline{\mathcal{M}}(!H),f^i_\dagger\overline{\mathcal{F}}(!H)\bigr)\in\mathrm{Hol}(Y,K)$$

They are independent of the choice of $\overline{X}$ up to natural isomorphisms. Thus, we obtain the cohomological functors we have the cohomological functor ${}^Kf^i_*,{}^Kf^i_!:\mathrm{Hol}(X,K)\longrightarrow\mathrm{Hol}(Y,K)$ for $i\in\mathbb{Z}$.

Proposition 9.6 *For $\star=!,*$, there exists a functor of triangulated categories*

$${}^Kf_\star:D^b_{\mathrm{hol}}(X,K)\longrightarrow D^b_{\mathrm{hol}}(Y,K)$$

such that (i) it is compatible with the standard functor $f_\star:D^b_{\mathrm{hol}}(X)\longrightarrow D^b_{\mathrm{hol}}(Y)$, (ii) the induced functor $H^i({}^Kf_\star):\mathrm{Hol}(X,K)\longrightarrow\mathrm{Hol}(Y,K)$ is isomorphic to ${}^Kf^i_\star$. It is characterized by the property (i) and (ii) up to natural equivalence.

As in [38], the pull back is defined to be the adjoint of the push-forward.

Proposition 9.7 *${}^Kf_!$ has the right adjoint ${}^Kf^!$, and ${}^Kf_*$ has the left adjoint ${}^Kf^*$. Thus, we obtain the following functors:*

$${}^Kf^\star:D^b_{\mathrm{hol}}(Y,K)\longrightarrow D^b_{\mathrm{hol}}(X,K)\qquad(\star=!,*)$$

They are compatible with the corresponding functors of holonomic $\mathcal{D}$-modules with respect to the forgetful functor.

Let us consider the case that f is a closed immersion, via which X is regarded as a submanifold of Y. Let $D^b_{\mathrm{hol},X}(Y,K)$ be the full subcategory of $D^b_{\mathrm{hol}}(Y,K)$ which consists of the objects $\mathcal{M}^\bullet$ such that the supports of the cohomology $\bigoplus_i\mathcal{H}^i\mathcal{M}^\bullet$ are contained in X.

Proposition 9.8 *The natural functor ${}^Kf_!:D^b_{\mathrm{hol}}(X,K)\longrightarrow D^b_{\mathrm{hol},X}(Y,K)$ is an equivalence.*

9.2.2 Preliminary

Let X be a smooth complex projective variety with a hypersurface D. Let $D^b_{\mathrm{hol}}\bigl(X(*D),K\bigr)$ denote the derived category of $\mathrm{Hol}\bigl(X(*D),K\bigr)$. Similarly, let $D^b_{\mathrm{hol}}\bigl(X(*D)\bigr)$ denote the derived category of $\mathrm{Hol}\bigl(X(*D)\bigr)$.

Let $f:X\longrightarrow Y$ be a morphism of smooth *projective* varieties. Let D_X and D_Y be hypersurfaces of X and Y respectively, such that $D_X\supset f^{-1}(D_Y)$. We have the functor ${}^Kf^i_*:\mathrm{Hol}\bigl(X(*D_X),K\bigr)\longrightarrow \mathrm{Hol}\bigl(Y(*D_Y),K\bigr)$, naturally given by $f^i_{\dagger}$. We have the decomposition $D_X=D_{X1}\cup D_{X2}$ such that $D_{X2}=f^{-1}(D_Y)$. We have the functor ${}^Kf^i_!:\mathrm{Hol}\bigl(X(*D_X),K\bigr)\longrightarrow\mathrm{Hol}\bigl(Y(*D_Y),K\bigr)$ given by

$$
{}^Kf^i_!(\mathcal{M},\mathcal{F})=\bigl(f^i_{\dagger}\mathcal{M}(!D_{X1}*D_{X2}),\,f^i_{\dagger}\mathcal{F}(!D_{X1}*D_{X2})\bigr).
$$

Lemma 9.9 *For $\star=*,!$, there exists a functor ${}^Kf_{\star}:D^b_{\mathrm{hol}}\bigl(X(*D_X),K\bigr)\longrightarrow D^b_{\mathrm{hol}}\bigl(Y(*D_Y),K\bigr)$ such that (i) it is compatible with the standard functor $f_{\star}:D^b_{\mathrm{hol}}\bigl(X(*D_X)\bigr)\longrightarrow D^b_{\mathrm{hol}}\bigl(Y(*D_Y)\bigr)$, (ii) the induced functor $H^i({}^Kf_{\star}):\mathrm{Hol}\bigl(X(*D_X),K\bigr)\longrightarrow\mathrm{Hol}\bigl(Y(*D_Y),K\bigr)$ are isomorphic to ${}^Kf^i_{\star}$. It is characterized by (i) and (ii) up to natural equivalence.*

Proof Let us consider the case $\star=*$. Let $\mathcal{M}^{\bullet}$ be a complex of K-holonomic $\mathcal{D}_{X(*D_X)}$-modules. We take sufficiently generic ample hypersurfaces H_i $(i=1,\ldots,M)$ and H'_j $(j=1,\ldots,N)$ such that $\bigcap_{i=1}^M H_i=\emptyset$ and $\bigcap_{i=1}^N H'_j=\emptyset$. We put $H_I:=\bigcup_{i\in I}H_i$ and $H'_J:=\bigcup_{j\in I}H'_j$. We have $f^i_{\dagger}\mathcal{M}(*H_I!H'_J*D_X)=0$ for $i\neq 0$, and we have K-holonomic $\mathcal{D}_Y(*D_Y)$-modules ${}^Kf^0_{\dagger}\mathcal{M}^{\bullet}(*H_I!H'_J*D_X)$. For $m,n\geq 0$, we put

$$
\mathcal{C}^{m,n}(\mathcal{M}^p,\boldsymbol{H},\boldsymbol{H}'):=\bigoplus_{|I|=m+1,\,|J|=n+1}\mathcal{M}^p(*H_I!H'_J*D_X).
$$

Let $\mathrm{Tot}\bigl(\mathcal{C}^{\bullet,\bullet}(\mathcal{M}^{\bullet},\boldsymbol{H},\boldsymbol{H}')\bigr)$ be the total complex. It is naturally quasi-isomorphic to $\mathcal{M}^{\bullet}$.

Let $(\boldsymbol{H}_i,\boldsymbol{H}'_i)$ $(i=1,2)$ be tuples of sufficiently generic ample hypersurfaces as above for $\mathcal{M}^{\bullet}$. We say that we have a morphism $(\boldsymbol{H}_1,\boldsymbol{H}'_1)\longrightarrow(\boldsymbol{H}_2,\boldsymbol{H}'_2)$, if either $\boldsymbol{H}_1\subset\boldsymbol{H}_2$ or $\boldsymbol{H}'_1\supset\boldsymbol{H}'_2$ is satisfied. In that case, we have a naturally induced morphism $\mathcal{C}^{\bullet,\bullet}(\mathcal{M}^{\bullet},\boldsymbol{H}_1,\boldsymbol{H}'_1)\longrightarrow\mathcal{C}^{\bullet,\bullet}(\mathcal{M}^{\bullet},\boldsymbol{H}_2,\boldsymbol{H}'_2)$. For given tuples of sufficiently generic ample hypersurfaces $(\boldsymbol{H}_i,\boldsymbol{H}'_i)$ $(i=1,2)$, we can find a sequence of tuples of hypersurfaces $(\boldsymbol{H}^{(j)},\boldsymbol{H}'^{(j)})$ $(j=1,\ldots,2L)$ such that (i) $(\boldsymbol{H}^{(1)},\boldsymbol{H}'^{(1)})=(\boldsymbol{H}_1,\boldsymbol{H}'_1)$ and $(\boldsymbol{H}^{(2L)},\boldsymbol{H}'^{(2L)})=(\boldsymbol{H}_2,\boldsymbol{H}'_2)$, (ii) we have morphisms

$$
(\boldsymbol{H}^{(2m-1)},\boldsymbol{H}'^{(2m-1)})\longleftarrow(\boldsymbol{H}^{(2m)},\boldsymbol{H}'^{(2m)})\longrightarrow(\boldsymbol{H}^{(2m+1)},\boldsymbol{H}'^{(2m+1)}).
$$

Let $\mathcal{M}^{\bullet}_1\longrightarrow\mathcal{M}^{\bullet}_2$ be a morphism of complexes of K-holonomic $\mathcal{D}_{X(*D_X)}$-modules. We can take a tuple of ample hypersurfaces $(\boldsymbol{H},\boldsymbol{H}')$ which are sufficiently generic with respect to both $\mathcal{M}^{\bullet}_i$ $(i=1,2)$. For such a $(\boldsymbol{H},\boldsymbol{H}')$, we obtain an induced morphism $\mathcal{C}^{\bullet,\bullet}(\mathcal{M}^{\bullet}_1,\boldsymbol{H},\boldsymbol{H}')\longrightarrow\mathcal{C}^{\bullet,\bullet}(\mathcal{M}^{\bullet}_2,\boldsymbol{H},\boldsymbol{H}')$.

For each $\mathcal{M}^{\bullet}$, we take a tuple $(\boldsymbol{H},\boldsymbol{H}')$ as above, and we put

$$
{}^Kf_*\mathcal{M}^{\bullet}:={}^Kf^0_{\dagger}\,\mathrm{Tot}\,\mathcal{C}^{\bullet,\bullet}\bigl(\mathcal{M}^{\bullet},\boldsymbol{H},\boldsymbol{H}'\bigr)
$$

in $D^b_{\mathrm{hol}}(Y,K)$. By using the above considerations, we obtain the map

$$
\mathrm{Hom}_{D^b_{\mathrm{hol}}(X,K)}\bigl(\mathcal{M}^{\bullet}_1,\mathcal{M}^{\bullet}_2\bigr)\longrightarrow\mathrm{Hom}_{D^b_{\mathrm{hol}}(Y,K)}\bigl({}^Kf_*\mathcal{M}^{\bullet}_1,\,{}^Kf_*\mathcal{M}^{\bullet}_2\bigr),
$$

which is compatible with $\mathrm{Hom}_{D^b_{\mathrm{hol}}(X)}\bigl(\mathcal{M}^{\bullet}_1,\mathcal{M}^{\bullet}_2\bigr)\longrightarrow\mathrm{Hom}_{D^b_{\mathrm{hol}}(Y)}\bigl(f_{\dagger}\mathcal{M}^{\bullet}_1,\,f_{\dagger}\mathcal{M}^{\bullet}_2\bigr)$. Thus, we obtain the functor ${}^Kf_*:D^b_{\mathrm{hol}}(X,K)\longrightarrow D^b_{\mathrm{hol}}(Y,K)$. By construction, it satisfies the conditions (i) and (ii). We set ${}^Kf_!:={}^K\mathbb{D}_Y\circ{}^Kf_*\circ{}^K\mathbb{D}_X$. It satisfies the conditions (i) and (ii). The uniqueness follows from the existence of a resolution by K-holonomic $\mathcal{D}$-modules $\mathcal{N}$ such that $f^i_{\dagger}\mathcal{N}=0$ unless $i=0$. ∎

9.2.3 Proof of Proposition 9.6

We take projective completions $X\subset\overline{X}$ and $Y\subset\overline{Y}$ with the following commutative diagram:

$$
\begin{tikzcd}
X \arrow[r, "\subset"] \arrow[d, "f"'] & \overline{X} \arrow[d, "\overline{f}"'] & D_X \arrow[l, "\supset"'] \arrow[d, "f_D"'] \\
Y \arrow[r, "\subset"] & \overline{Y} & D_Y \arrow[l, "\supset"']
\end{tikzcd}
\tag{82}
$$

Here, $D_X := \overline{X} - X$ and $D_Y := \overline{Y} - Y$. We have the functors ${}^K f_\star : D^b_{\mathrm{hol}}\big(\overline{X}(*D_X), K\big) \longrightarrow D^b_{\mathrm{hol}}\big(\overline{Y}(*D_Y), K\big)$, which induce ${}^K f_\star : D^b_{\mathrm{hol}}(X, K) \longrightarrow D^b_{\mathrm{hol}}(Y, K)$.

Let $X \subset \overline{X}'$ and $Y \subset \overline{Y}'$ be another projective completions with a commutative diagram as in (82). We set $D'_X := \overline{X}' - X$ and $D'_Y := \overline{Y} - Y$. Let us show that the induced morphism ${}^K f_\star : D^b_{\mathrm{hol}}(X, K) \longrightarrow D^b_{\mathrm{hol}}(Y, K)$ are equal up to equivalence. We have only to consider the case that we have the following commutative diagram:

$$\begin{array}{ccc} \overline{X}' & \xrightarrow{\overline{f}'} & \overline{Y}' \\ {\scriptstyle \varphi_X}\big\downarrow & & {\scriptstyle \varphi_Y}\big\downarrow \\ \overline{X} & \xrightarrow{\overline{f}} & \overline{Y} \end{array}$$

Here, φ_X and φ_Y are projective and birational such that $\varphi_X^{-1}(D_X) = D'_X$ and $\varphi_Y^{-1}(D_Y) = D'_Y$. We have the following diagrams which are commutative up to equivalences:

$$\begin{array}{ccc} D^b_{\mathrm{hol}}\big(\overline{X}'(*D'_X), K\big) & \xrightarrow{{}^K f_\star} & D^b_{\mathrm{hol}}\big(\overline{Y}'(*D'_Y), K\big) \\ {\scriptstyle {}^K\varphi_{X\star}}\big\downarrow & & {\scriptstyle {}^K\varphi_{Y\star}}\big\downarrow \\ D^b_{\mathrm{hol}}\big(\overline{X}(*D_X), K\big) & \xrightarrow{{}^K f_\star} & D^b_{\mathrm{hol}}\big(\overline{Y}(*D_Y), K\big) \end{array}$$

It implies that ${}^K f_\star : D^b_{\mathrm{hol}}(X, K) \longrightarrow D^b_{\mathrm{hol}}(Y, K)$ are independent of the choice of projective completions up to equivalence. Thus, the proof of Proposition 9.6 is finished. ∎

9.2.4 Proof of Proposition 9.8

Let $\mathcal{M}, \mathcal{N} \in \mathrm{Hol}(X, K)$. According to [4], we have only to check the following effaceability:

- For any $f \in \mathrm{Ext}^i_{\mathrm{Hol}(Y,K)}(\mathcal{M}, \mathcal{N})$, there exists a monomorphism with a monomorphism $\mathcal{N} \longrightarrow \mathcal{N}'$ in $\mathrm{Hol}(X, K)$ such that the image of f in $\mathrm{Ext}^i_{\mathrm{Hol}(Y,K)}(\mathcal{M}, \mathcal{N}')$ is 0.

We can show it by using the arguments in Sections 2.2.1 and 2.2.2 in [2].

9.2.5 Proof of Proposition 9.7

We have only to consider the cases (i) f is a closed immersion, (ii) f is a projection $X \times Y \longrightarrow Y$. We closely follow the arguments in Subsections 2.19 and 4.4 of [38].

Closed immersion Let $f : X \longrightarrow Y$ be a closed immersion. The open immersion $X - Y \longrightarrow X$ is denoted by j. Let $\mathcal{M}^\bullet$ be a complex of K-holonomic $\mathcal{D}_Y$-modules. Let H_i $(i = 1, \ldots, N)$ be sufficiently general ample hypersurfaces such that (i) $H_i \supset X$, (ii) $\mathcal{M}^\bullet \longrightarrow \mathcal{M}^\bullet(*H_i)$ are injective, (iii) $\bigcap_{i=1}^N H_i = X$. For any subset $I = (i_1, \ldots, i_m) \subset \{1, \ldots, N\}$, let $\mathbb{C}_I$ be the subspace of $\bigwedge^m \mathbb{C}^N$ generated by $e_{i_1} \wedge \cdots \wedge e_{i_m}$, where $e_i \in \mathbb{C}^N$ denotes an element whose j-th entry is 1 $(j = i)$ or 0 $(j \neq i)$. For $I = I_0 \sqcup \{i\}$, the inclusion $\mathcal{M}^p(*H_{I_0}) \longrightarrow \mathcal{M}^p(*H_I)$ and the multiplication of e_i induces $\mathcal{M}^p(*H_{I_0}) \otimes \mathbb{C}_{I_0} \longrightarrow \mathcal{M}^p(*H_I) \otimes \mathbb{C}_I$. For $m \geq 0$, we put $\mathcal{C}^m(\mathcal{M}^p, *\boldsymbol{H}) := \bigoplus_{|I|=m} \mathcal{M}^p(*H_I) \otimes \mathbb{C}_I$, and we obtain the double complex $\mathcal{C}^\bullet(\mathcal{M}^\bullet, *\boldsymbol{H})$. The total complex is denoted by $\mathrm{Tot}\, \mathcal{C}^\bullet(\mathcal{M}^\bullet, *\boldsymbol{H})$. It is easy to observe that the support of the cohomology of $\mathrm{Tot}\, \mathcal{C}^\bullet(\mathcal{M}^\bullet, *\boldsymbol{H})$ is contained in X. According to Proposition 9.8, we obtain ${}^K f^! \mathcal{M}^\bullet := \mathrm{Tot}\, \mathcal{C}^\bullet(\mathcal{M}^\bullet, *\boldsymbol{H})$ in $D^b_{\mathrm{hol}}(X, K)$. We obtain a functor ${}^K f^! : D^b_{\mathrm{hol}}(Y, K) \longrightarrow D^b_{\mathrm{hol}}(X, K)$ as in Lemma 9.9. Note that the underlying $\mathcal{D}_Y$-complex is naturally quasi-isomorphic to $f^! \mathcal{M}^\bullet$, where $f^!$ is the left adjoint of $f_\dagger : D^b_{\mathrm{hol}}(X) \longrightarrow D^b_{\mathrm{hol}}(Y)$.

We have the naturally defined morphism $\alpha : \mathrm{Tot}\, \mathcal{C}^\bullet(\mathcal{M}^\bullet, *\boldsymbol{H}) \longrightarrow \mathcal{M}^\bullet$. We put $\mathcal{K}^\bullet := \mathrm{Cone}(\alpha)$. We have another description. For $m \geq 0$, we put $\overline{\mathcal{C}}^m(\mathcal{M}^p, *\boldsymbol{H}) := \bigoplus_{|I|=m+1} \mathcal{M}^p(*H_I) \otimes \mathbb{C}_I$, and we obtain the double complex $\overline{\mathcal{C}}^\bullet(\mathcal{M}^\bullet, *\boldsymbol{H})$. We have a natural quasi-isomorphism $\mathcal{K}^\bullet \simeq \mathrm{Tot}\, \overline{\mathcal{C}}^\bullet(\mathcal{M}^\bullet, *\boldsymbol{H})$. By using the second description and Lemma 9.5, we obtain the following vanishing for any $\mathcal{N}^\bullet \in D^b_{\mathrm{hol}}(X, K)$:

$$\mathrm{Hom}_{D^b_{\mathrm{hol}}(Y,K)}\big({}^K f_! \mathcal{N}^\bullet, \mathcal{K}^\bullet\big) = 0$$

Hence, we have the following isomorphisms for any K-holonomic $\mathcal{D}_X$-complex $\mathcal{N}^\bullet$:

$$\mathrm{Hom}_{D^b_{\mathrm{hol}}(Y,K)}\big({}^Kf_!\mathcal{N}^\bullet,\ \mathcal{M}^\bullet\big) \simeq \mathrm{Hom}_{D^b_{\mathrm{hol}}(Y,K)}\big({}^Kf_!\mathcal{N}^\bullet,\ {}^Kf_!{}^Kf^!\mathcal{M}^\bullet\big) \simeq \mathrm{Hom}_{D^b_{\mathrm{hol}}(X,K)}\big(\mathcal{N}^\bullet,\ {}^Kf^!\mathcal{M}^\bullet\big)$$

Hence, we obtain that the above functor ${}^Kf^!$ is the right adjoint of ${}^Kf_!$. By taking the dual, we obtain the left adjoint ${}^Kf^*$ of ${}^Kf_*$.

Projection Let $f : Z\times Y \longrightarrow Y$ be the natural projection. Let $(\mathcal{M},\mathcal{F})$ be a K-holonomic $\mathcal{D}_Y$-module. We put ${}^Kf^*(\mathcal{M},\mathcal{F}) := \big(\mathcal{O}_Z\boxtimes\mathcal{M}[-\dim Z], K_Z\boxtimes\mathcal{C}\big)$. It is easy to check that ${}^Kf^*(\mathcal{M},\mathcal{F})$ is K-holonomic. Thus, we obtain the exact functor ${}^Kf^* : D^b_{\mathrm{hol}}(Y,K) \longrightarrow D^b_{\mathrm{hol}}(Z\times Y,K)$. Let us show that ${}^Kf^*$ is the left adjoint of ${}^Kf_*$. We have only to repeat the argument in Subsection 4.4 of [38], which we include for the convenience of readers. We have only to construct natural transformations $\alpha : \mathrm{id} \longrightarrow {}^Kf_*{}^Kf^*$ and $\beta : {}^Kf^*{}^Kf_* \longrightarrow \mathrm{id}$ such that

$$\beta\circ{}^Kf^*\alpha : {}^Kf^*\mathcal{M}^\bullet \longrightarrow {}^Kf^*{}^Kf_*{}^Kf^*\mathcal{M}^\bullet \longrightarrow {}^Kf^*\mathcal{M}^\bullet, \qquad {}^Kf_*\beta\circ\alpha : {}^Kf_*\mathcal{N}^\bullet \longrightarrow {}^Kf_*{}^Kf^*{}^Kf_*\mathcal{N}^\bullet \longrightarrow {}^Kf_*\mathcal{N}^\bullet$$

are the identities. We define α as the external product with $(\mathbb{C},K) \longrightarrow \big(H_{DR}(Z), H^0(Z,K)\big)$. For the construction of β, the following diagram is used:

$$\begin{array}{ccccc}
Z\times Y & \xrightarrow{\ i\ } & Z\times Z\times Y & \xrightarrow{\ q_1\ } & Z\times Y\\
 & & {\scriptstyle q_2}\big\downarrow & & {\scriptstyle p_1}\big\downarrow\\
 & & Z\times Y & \xrightarrow{\ p_2\ } & Y
\end{array}$$

Here, i is induced by the diagonal $Z \longrightarrow Z\times Z$, q_j are induced by the projection $Z\times Z \longrightarrow Z$ onto the j-th component, and p_j are the projections. We have the following morphisms of $\mathcal{D}$-complexes, compatible with the K-Betti structures:

$$ {}^Kf^*{}^Kf_*\mathcal{M}^\bullet = {}^Kp_2^*{}^Kp_{1*}\mathcal{M}^\bullet \simeq {}^Kq_{2*}{}^Kq_1^*\mathcal{M}^\bullet \longrightarrow {}^Kq_{2*}\big({}^Ki_*{}^Ki^*{}^Kq_1^*\mathcal{M}^\bullet\big) \simeq {}^Ki^*{}^Kq_1^*\mathcal{M}^\bullet \tag{83}$$

Lemma 9.10 *We have a natural isomorphism ${}^Ki^*{}^Kq_1^*\mathcal{M}^\bullet \simeq \mathcal{M}^\bullet$ in $D^b_{\mathrm{hol}}(Z\times Y,K)$.*

Proof We have a natural isomorphism of the underlying $\mathcal{D}$-complexes. We have only to check that it is compatible with K-Betti structures. Since the composite ${}^Ki^*{}^Kq_1^* : D^b_{\mathrm{hol}}(Z\times Y,K) \longrightarrow D^b_{\mathrm{hol}}(Z\times Y,K)$ is exact, we have only to consider the compatibility for any K-holonomic $\mathcal{D}_{Z\times Y}$-module $\mathcal{M}$. Moreover, we have only to check it locally on $Z\times Y$. Then, it can be done directly from the construction. ∎

We define β as the composite of (83) with the isomorphism in Lemma 9.10. Let us look at ${}^Kf_*\beta\circ\alpha$, which is the composite of the following morphisms:

$${}^Kf_*\mathcal{N} = {}^Kp_{1*}\mathcal{N} \longrightarrow {}^Kp_{2*}{}^Kp_2^*{}^Kp_{1*}\mathcal{N} \longrightarrow {}^Kp_{2*}{}^Kq_{2*}{}^Kq_1^*\mathcal{N} \longrightarrow {}^Kp_{2*}{}^Kq_{2*}{}^Ki_*{}^Ki^*{}^Kq_1^*\mathcal{N} \longrightarrow {}^Kf_*{}^Ki^*{}^Kq_1^*\mathcal{N} \simeq {}^Kf_*\mathcal{N}$$

Hence, we can observe that it is equivalent to the identity. As for $\beta\circ{}^Kf^*\alpha$, it is expressed as follows:

$${}^Kf^*\mathcal{N} = {}^Kp_2^*\mathcal{N} \longrightarrow {}^Kp_2^*{}^Kp_{1*}{}^Kp_1^*\mathcal{N} \longrightarrow {}^Kq_{2*}{}^Kq_1^*{}^Kp_1^*\mathcal{N} \longrightarrow {}^Kq_{2*}{}^Ki_*{}^Ki^*{}^Kq_2^*{}^Kp_2^*\mathcal{N} \simeq {}^Kp_2^*\mathcal{N} = {}^Kf^*\mathcal{N}$$

Hence, it is equivalent to the identity. Thus, the proof of Proposition 9.7 is finished. ∎

9.3 Tensor product and inner homomorphism

9.3.1 Statement

Let $(\mathcal{M}_i,\mathcal{F}_i)$ $(i=1,2)$ be K-holonomic $\mathcal{D}$-modules on X_i.

Proposition 9.11 *$\mathcal{F}_1\boxtimes\mathcal{F}_2$ is a K-Betti structure of $\mathcal{M}_1\boxtimes\mathcal{M}_2$. As a result, we obtain a natural functor $\boxtimes : \mathrm{Hol}(X_1,K)\times\mathrm{Hol}(X_2,K) \longrightarrow \mathrm{Hol}(X_1\times X_2,K)$, compatible with the standard external products $\boxtimes : \mathrm{Hol}(X_1)\times\mathrm{Hol}(X_2) \longrightarrow \mathrm{Hol}(X_1\times X_2)$ and $D^b_c(K_{X_1})\times D^b_c(K_{X_2}) \longrightarrow D^b_c(K_{X_1\times X_2})$.*

Before going into a proof of Proposition 9.11, we give a consequence. Let X be an algebraic variety. Let $\delta_X : X \longrightarrow X\times X$ be the diagonal morphism. We obtain the functors $\otimes$ and $R\mathcal{H}om$ on $D^b_{\mathrm{hol}}(X,K)$ in standard ways:

$$\mathcal{M}\otimes\mathcal{N} := {}^K\delta_X^*\big(\mathcal{M}\boxtimes\mathcal{N}\big), \quad R\mathcal{H}om(\mathcal{M},\mathcal{N}) := {}^K\delta_X^!\big(\mathbb{D}_X\mathcal{M}\boxtimes\mathcal{N}\big)$$

They are compatible with the corresponding functors on $D^b_{\mathrm{hol}}(X)$.

9.3.2 Preliminary

Let $(\mathcal{M}, \mathcal{F}_{\mathcal{M}})$ be a K-holonomic $\mathcal{D}_X$-module. Let $\mathcal{V}$ be a good meromorphic flat bundle on (Y, D_Y) with a K-Betti structure $\mathcal{F}_{\mathcal{V}}$. Let $\mathcal{F}_{\mathcal{V}!}$ be the canonical K-Betti structure of $\mathcal{V}_!$.

Lemma 9.12 *$\mathcal{F}_{\mathcal{V}} \boxtimes \mathcal{F}_{\mathcal{M}}$ and $\mathcal{F}_{\mathcal{V}!} \boxtimes \mathcal{F}_{\mathcal{M}}$ are K-Betti structures of $\mathcal{V} \boxtimes \mathcal{M}$ and $\mathcal{V}_! \boxtimes \mathcal{M}$, respectively.*

Proof We use an induction on the dimension of the support of $\mathcal{M}$. Let P be any point of X. We have only to consider locally around $Y \times \{P\}$. Let $\mathcal{C} = (Z, U, \varphi, V)$ be a K-cell of $\mathcal{M}$ at P. Let g be a cell function of $\mathcal{C}$. The pre-K-holonomic $\mathcal{D}$-module $\mathcal{V} \otimes \mathcal{M}$ is expressed as the cohomology of the following complex of pre-K-holonomic $\mathcal{D}$-modules:

$$\mathcal{V} \boxtimes \psi_g\bigl(\varphi_\dagger V\bigr) \longrightarrow \mathcal{V} \boxtimes \Xi_g\bigl(\varphi_\dagger V\bigr) \oplus \mathcal{V} \boxtimes \phi_g(\mathcal{M}) \longrightarrow \mathcal{V} \boxtimes \psi_g\bigl(\varphi_\dagger V\bigr)$$

By the hypothesis of the induction, $\mathcal{F}_{\mathcal{V}} \boxtimes {}^D\psi_g(\varphi_* \mathcal{F}_V)$ and $\mathcal{F}_{\mathcal{V}} \boxtimes {}^D\phi_g(\varphi_* \mathcal{F}_V)$ are K-Betti structures of $\mathcal{V} \boxtimes \psi_g(\varphi_\dagger V)$ and $\mathcal{V} \boxtimes \phi_g(\varphi_\dagger V)$, respectively. We put $g_Z := \varphi^* g$. By using Theorem 8.2, we obtain that $\mathcal{F}_{\mathcal{V}} \boxtimes {}^D\Xi_{g_Z}\bigl(\mathcal{F}_V\bigr)$ and $\mathcal{F}_{\mathcal{V}} \boxtimes {}^D\psi_{g_Z}\bigl(\mathcal{F}_V\bigr)$ are K-Betti structures of $\mathcal{V} \boxtimes \Xi_{g_Z}(V)$ and $\mathcal{V} \boxtimes \psi_{g_Z}(V)$, respectively. By construction, the isomorphism $\mathcal{V} \boxtimes \varphi_\dagger\bigl(\psi_{g_Z}(V)\bigr) \simeq \mathcal{V} \boxtimes \psi_g\bigl(\varphi_\dagger V\bigr)$ preserves K-Betti structures. Hence, we obtain that $\mathcal{F}_{\mathcal{M}} \boxtimes \mathcal{F}_{\mathcal{V}}$ is a K-Betti structure. Thus, we obtain the first claim. By considering the dual, we obtain the second claim. ∎

Let g be a holomorphic function on Y such that $g^{-1}(0) = D_Y$. We obtain the following corollary from Lemma 9.12.

Corollary 9.13 *${}^D\psi_g(\mathcal{F}_{\mathcal{V}}) \boxtimes \mathcal{F}_{\mathcal{M}}$ and ${}^D\Xi_g(\mathcal{F}_{\mathcal{V}}) \boxtimes \mathcal{F}_{\mathcal{M}}$ are K-Betti structures of $\psi_g(\mathcal{V}) \boxtimes \mathcal{M}$ and $\Xi_g(\mathcal{V}) \boxtimes \mathcal{M}$, respectively.* ∎

9.3.3 Proof of Proposition 9.11

Let P be any point of X_1. We have only to consider locally around $\{P\} \times X_2$. We use an induction on $\dim_P \operatorname{Supp} \mathcal{M}_1$. Let $\mathcal{C} = (Z, U, \varphi, V)$ be a K-cell of $\mathcal{M}_1$. The pre-K-holonomic $\mathcal{D}$-module $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ is expressed as the cohomology of the following complex:

$$\psi_g(\varphi_\dagger V) \boxtimes \mathcal{M}_2 \longrightarrow \Xi_g(\varphi_\dagger V) \boxtimes \mathcal{M}_2 \oplus \phi_g(\mathcal{M}_1) \boxtimes \mathcal{M}_2 \longrightarrow \psi_g(\varphi_\dagger V) \boxtimes \mathcal{M}_2$$

By the hypothesis of the induction, $\psi_g(\varphi_\dagger V) \boxtimes \mathcal{M}_2$ and $\phi_g(\varphi_\dagger V) \boxtimes \mathcal{M}_2$ are K-holonomic. According to Theorem 8.1 and Corollary 9.13, $\Xi_g(\varphi_\dagger V) \boxtimes \mathcal{M}_2$ is K-holonomic. Hence, we obtain that $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ is also K-holonomic. Thus, we obtain Proposition 9.11. ∎

9.4 K-structure of the space of morphisms

9.4.1 Statement

Theorem 9.14 *For $M^\bullet, N^\bullet \in D^b_{\mathrm{hol}}(X, K)$, the induced morphism*

$$\operatorname{Hom}_{D^b_{\mathrm{hol}}(X,K)}(M^\bullet, N^\bullet) \otimes \mathbb{C} \longrightarrow \operatorname{Hom}_{D^b_{\mathrm{hol}}(X)}(M^\bullet, N^\bullet)$$

is an isomorphism. In other words, $D^b_{\mathrm{hol}}(X, K) \otimes \mathbb{C} \longrightarrow D^b_{\mathrm{hol}}(X)$ is fully faithful.

We closely follow Beilinson's argument in [2] for the proof.

Theorem 9.15 *We have the following natural isomorphism*

$$\operatorname{Hom}_{D^b_{\mathrm{hol}}(X,K)}(M^\bullet, N^\bullet) \simeq \operatorname{Hom}_{D^b_{\mathrm{hol}}(X,K)}\bigl(\mathcal{O}_X,\ R\mathcal{H}om(M^\bullet, N^\bullet)[d_X]\bigr)$$

We essentially use a commutative diagram due to Saito in [39].

9.4.2 Homomorphisms and extensions for good meromorphic flat bundles

Let X be a complex manifold with a normal crossing hypersurface D. Let V be a good meromorphic flat bundle on (X, D) with K-good structure, and let $\mathcal{L}(V)$ be the associated local system with the Stokes structure on $\widetilde{X}(D)$. It is naturally equipped with a K-structure $\mathcal{L}_K(V)$. If we are given an extension $0 \longrightarrow V \longrightarrow P \longrightarrow \mathcal{O}_X(*D) \longrightarrow 0$ as K-holonomic $\mathcal{D}_X$-modules, P is also a good meromorphic flat bundle with a good K-structure, and it induces an extension $0 \longrightarrow \mathcal{L}_K(V)^{\leq D} \longrightarrow \mathcal{L}_K(P)^{\leq D} \longrightarrow K_{\widetilde{X}(D)} \longrightarrow 0$ of K-constructible sheaves. Conversely, assume that we are given an extension of K-constructible sheaves $0 \longrightarrow \mathcal{L}_K(V)^{\leq D} \longrightarrow \mathcal{G}_K \longrightarrow \mathbb{C}_{\widetilde{X}(D)} \longrightarrow 0$. We obtain a K-local system $\widetilde{\mathcal{G}}_K := \widetilde{\iota}_* \mathcal{G}_{|X \setminus D}$, where $\iota : X \setminus D \longrightarrow X$. The $\mathbb{C}$-local system $\widetilde{\mathcal{G}}_K \otimes \mathbb{C}$ is naturally equipped with a Stokes structure compatible with the K-structure. Hence, we obtain an extension of K-holonomic $\mathcal{D}_X$-modules $0 \longrightarrow V \longrightarrow P \longrightarrow \mathcal{O}_X(*D) \longrightarrow 0$. The above procedures are mutually inverse. Thus, we obtain a bijection $\mathrm{Ext}^1_{\mathrm{Hol}(X,K)}\big(\mathcal{O}_X(*D), V\big) \simeq \mathrm{Ext}^1_{K_{\widetilde{X}(D)}}\big(K_{\widetilde{X}(D)}, \mathcal{L}_K(V)\big) \simeq H^1\big(X, \mathcal{F}_V\big)$. Similarly, we have a natural bijection $\mathrm{Ext}^0_{\mathrm{Hol}(X,K)}\big(\mathcal{O}_X(*D), V\big) \simeq H^0(X, \mathcal{F}_V)$.

Let V, W be good meromorphic flat bundles on (X, D) with good K-structures. We have a natural bijection $\mathrm{Ext}^i_{\mathrm{Hol}(X,K)}(W, V) \simeq \mathrm{Ext}^i_{\mathrm{Hol}(X,K)}\big(\mathcal{O}_X(*D), W^\vee \otimes V\big)$ for any i. Hence, we obtain the natural isomorphisms $\mathrm{Ext}^i_{\mathrm{Hol}(X,K)}\big(W, V\big) \simeq H^i\big(X, \mathcal{F}_{W^\vee \otimes V}\big)$ for $i = 0, 1$. Because

$$H^i\big(X, \mathcal{F}_{W^\vee \otimes V}\big) \otimes_K \mathbb{C} \simeq H^i\big(X, \mathrm{DR}_X(W^\vee \otimes V)\big) =: H^i_{\mathrm{DR}}(X, W^\vee \otimes V),$$

the vector spaces $H^i_{\mathrm{DR}}(X, W^\vee \otimes V)$ have the natural K-structure. We say that an element $f \in H^i_{\mathrm{DR}}(X, W^\vee \otimes V)$ is compatible with K-structure, if it comes from $H^i\big(X, \mathcal{F}_{W^\vee \otimes V}\big)$. An element $f \in H^1_{\mathrm{DR}}(X, W^\vee \otimes V)$ induces an extension $0 \longrightarrow V \longrightarrow P \longrightarrow W \longrightarrow 0$ in $\mathrm{Hol}(X, K)$ as observed above.

9.4.3 Some extension

Let X be a smooth complex quasi-projective variety. Let V_i $(i = 1, 2)$ be flat bundles on X with a good K-structure, i.e., there exists a projective variety $\overline{X} \supset X$ such that (i) $D := \overline{X} - X$ is normal crossing, (ii)V_i are good meromorphic flat bundle on $(\overline{X}, D)$ with a good K-structure. According to [2], we have $\mathrm{Ext}^i_{\mathrm{Hol}(X)}(V_1, V_2) \simeq H^i\big(X, V_1^\vee \otimes V_2\big)$.

Lemma 9.16 *There exist an open subset $U \subset X$ and an extension $V_3 \supset V_{2|U}$ on U of algebraic flat bundles with a good K-structure, such that the induced morphisms $\mathrm{Ext}^i_{\mathrm{Hol}(X)}(V_1, V_2) \longrightarrow \mathrm{Ext}^i_{\mathrm{Hol}(U)}(V_{1|U}, V_3)$ are 0 for $i > 0$.*

Proof We use an induction on $\dim X$. In the case $\dim X = 0$, the claim is trivial. Let us consider the case $\dim X > 0$. We take a Zariski open subset $X_1 \subset X$ with a smooth affine fibration $\rho : X_1 \longrightarrow Z_1$ such that the relative dimension is 1. For any meromorphic flat bundle $\mathcal{V}$ on X_1, we put $\rho^q_*(\mathcal{V}) := R^q \rho_*\big(\mathcal{V} \otimes \Omega^\bullet_{X_1/Z_1}\big)$. For a Zariski open subset $Z'_1 \subset Z_1$, the induced morphism $\rho^{-1}(Z'_1) \longrightarrow Z'_1$ is also denoted by ρ.

We may assume that $L_q := \rho^q_*(V_1^\vee \otimes V_2)$ are meromorphic flat bundles on Z_1 with a good K-structure. We have $L_q = 0$ unless $q = 0, 1$. It is easy to reduce Lemma 9.16 to Lemma 9.17 below which is Lemma 2.1.2 of [2] with a minor enhancement.

Lemma 9.17

(a) *There exist a Zariski open subset $Z_2 \subset Z_1$ and an extension $P \supset V_{2|X_2}$ of algebraic flat bundles with good K-structures on $X_2 := \rho^{-1}(Z_2)$, such that the induced morphism $\rho^1_*(V_1^\vee \otimes V_{2|X_2}) \longrightarrow \rho^1_*(V_1^\vee \otimes P)$ is 0.*

(b) *There exists a Zariski open subset $Z_3 \subset Z_1$ and an extension $Q \supset V_{2|X_3}$ of algebraic flat bundles with good K-structures on $X_3 := \rho^{-1}(Z_3)$, such that the induced maps*

$$H^p_{\mathrm{DR}}\big(Z_3, \rho^0_*(V_1^\vee \otimes V_{2|X_3})\big) \longrightarrow H^p_{\mathrm{DR}}\big(Z_3, \rho^0_*(V_1^\vee \otimes Q)\big)$$

are 0 for any $p > 0$.

Proof We have only to use the argument in the proof of Lemma 2.1.2 of [2]. We give only an indication. Let $\alpha \in H^0_{\mathrm{DR}}\big(Z_1, L_1^\vee \otimes L_1\big) = H^0_{\mathrm{DR}}\big(Z_1, \rho^1_*((\rho^* L_1 \otimes V_1)^\vee \otimes V_2)\big)$ be the element corresponding to the identity of L_1, which is compatible with K-structure. We have the following exact sequence compatible with K-structures:

$$H^1_{\mathrm{DR}}\Big(X_1, \big(\rho^* L_1 \otimes V_1\big)^\vee \otimes V_2\Big) \longrightarrow H^0_{\mathrm{DR}}\Big(Z_1, \rho^1_*\big((\rho^* L_1 \otimes V_1)^\vee \otimes V_2\big)\Big)$$
$$\xrightarrow{\partial} H^2_{\mathrm{DR}}\Big(Z_1, \rho^0_*\big((\rho^* L_1 \otimes V_1)^\vee \otimes V_2\big)\Big) = H^2_{\mathrm{DR}}(Z_1, L_1^\vee \otimes L_0)$$

Applying the hypothesis of the induction to $L_0^\vee$ and $L_1^\vee$, we have a Zariski open subset $Z_2 \subset Z_1$ and an extension $\varphi : L_1^\vee \subset R$ of algebraic flat bundles with a good K-structures on Z_2, such that the induced morphism $H^2\big(Z, L_1^\vee \otimes L_0\big) \longrightarrow H^2(Z_1, R \otimes L_0)$ is 0. In particular, $\varphi(\partial\alpha) = 0$. We obtain the element

$$\varphi(\alpha) \in H^0_{\mathrm{DR}}\big(Z_1, R \otimes L_1\big) = H^0_{\mathrm{DR}}\Big(Z_1, \rho^1_*\big((\rho^* R^\vee \otimes V_1)^\vee \otimes V_2\big)\Big)$$

which is compatible with K-structure. By construction, we have a lift $\widetilde{\varphi(\alpha)} \in H^1_{\mathrm{DR}}\Big(X, (\rho^* R^\vee \otimes V_1)^\vee \otimes V_2\Big)$ compatible with K-structure. It induces an extension $0 \longrightarrow V_{2|X_2} \longrightarrow P \longrightarrow \rho^* R^\vee \otimes V_{1|X_2} \longrightarrow 0$ of algebraic flat bundles with good K-structures on X_2. (See Subsection 9.4.2.) It is easy to observe that P is the desired one. Thus, we obtain the claim (a). The claim (b) can also be shown by the argument in [2]. ∎

9.4.4 Proof of Theorem 9.14

We put $C_1(X) := \mathrm{Hol}(X)$ and $C_2(X) := \mathrm{Hol}(X, K) \otimes \mathbb{C}$. Let V_i $(i = 1, 2)$ be algebraic flat bundles on X with good K-structures. Let us consider the natural morphism:

$$g_X : \mathrm{Ext}^i_{C_2(X)}(V_1, V_2) \longrightarrow \mathrm{Ext}^i_{C_1(X)}(V_1, V_2)$$

It is an isomorphism in the case $i = 0, 1$.

Lemma 9.18 *Let $i > 0$.*

- *Let $a \in \mathrm{Ext}^i_{C_2(X)}(V_1, V_2)$ such that $g_X(a) = 0$. There exists $U \subset X$ such that $a = 0$ in $\mathrm{Ext}^i_{C_2(U)}(V_{1|U}, V_{2|U})$.*
- *Let $a \in \mathrm{Ext}^i_{C_1(X)}(V_1, V_2)$. There exist $U \subset X$ and $b \in \mathrm{Ext}^i_{C_2(U)}(V_{1|U}, V_{2|U})$ such that $a_{|U} = g_U(b)$.*

Proof We give only an outline. We use an induction on i. We have already known the case $i = 1$. Let $a \in \mathrm{Ext}^i_{C_2(X)}(V_1, V_2)$ such that $g_X(a) = 0$. We have an extension $V_2 \subset V_3$ of a meromorphic flat bundle with a good K-structure such that the image of a is mapped to 0 via $\mathrm{Ext}^i_{C_2(X)}(V_1, V_2) \longrightarrow \mathrm{Ext}^i_{C_2(X)}(V_1, V_3)$. Let $K := V_3/V_2$. We have $c \in \mathrm{Ext}^{i-1}_{C_2(X)}(V_1, K)$ which is mapped to a via $\mathrm{Ext}^{i-1}_{C_2(X)}(V_1, K) \longrightarrow \mathrm{Ext}^i_{C_2(X)}(V_1, V_2)$. We have $d \in \mathrm{Ext}^{i-1}_{C_1(X)}(V_1, V_3)$ which is mapped to $g_X(c)$ via $\mathrm{Ext}^{i-1}_{C_1(X)}(V_1, V_3) \longrightarrow \mathrm{Ext}^{i-1}_{C_1(X)}(V_1, K)$. By using the hypothesis of the induction, we can find $U \subset X$ and $e \in \mathrm{Ext}^{i-1}_{C_2(U)}(V_1, K)$ such that $g_U(e) = d_{|U}$. By using the hypothesis of the induction, and by shrinking U, we may assume e is mapped to $c_{|U}$ via $\mathrm{Ext}^{i-1}_{C_2(X)}(V_1, V_3) \longrightarrow \mathrm{Ext}^{i-1}_{C_2(X)}(V_1, K)$. Hence, we obtain $a_{|U} = 0$.

Let $a \in \mathrm{Ext}^i_{C_1(X)}(V_1, V_2)$. According to Lemma 9.16, we can find $U \subset X$ and an extension $V_{2|U} \subset V_3$ of meromorphic flat bundles with good K-structures such that the induced map $\mathrm{Ext}^j_{C_1(U)}(V_{1|U}, V_{2|U}) \longrightarrow \mathrm{Ext}^j_{C_1(U)}(V_{1|U}, V_3)$ is 0 for any $j > 0$. We put $K := V_3/V_{2|U}$ We can find $c \in \mathrm{Ext}^{i-1}_{C_1(U)}(V_{1|U}, K)$ which is mapped to a via $\mathrm{Ext}^{i-1}_{C_1(U)}(V_{1|U}, K) \longrightarrow \mathrm{Ext}^i_{C_1(U)}(V_{1|U}, V_{2|U})$. By using the hypothesis of the induction and by shrinking U, we can find $d \in \mathrm{Ext}^{i-1}_{C_2(U)}(V_{1|U}, K)$ such that $g_U(d) = c$. Let b be the image of d via $\mathrm{Ext}^{i-1}_{C_2(U)}(V_{1|U}, K) \longrightarrow \mathrm{Ext}^i_{C_2(U)}(V_{1|U}, V_{2|U})$. Then, it has the desired property. ∎

Let $M, N \in C_2(X)$. We would like to show that $\mathrm{Ext}^i_{C_2(X)}(M, N) \longrightarrow \mathrm{Ext}^i_{C_1(X)}(M, N)$ is an isomorphism. We use an induction on the dimension of the support of $M \oplus N$. We take a hypersurface $D \subset X$ such that (i) $M(*D)$

and $N(*D)$ are cells, (ii) $X - D$ is affine. We have the distinguished triangles ${}^K i_* {}^K i^! N \longrightarrow N \longrightarrow N(*D) \xrightarrow{+1}$ and $M(!D) \longrightarrow M \longrightarrow {}^K i_* {}^K i^* M \xrightarrow{+1}$. For $j = 1, 2$, we obtain the following exact sequence:

$$\operatorname{Ext}^{i-1}_{C_j}\big(M(!D), N(*D)\big) \longrightarrow \operatorname{Ext}^i_{C_j}({}^K i_* {}^K i^* M,\ {}^K i_* {}^K i^! N) \longrightarrow \operatorname{Ext}^i_{C_j}(M, N) \\ \longrightarrow \operatorname{Ext}^i_{C_j}\big(M(!D), N(*D)\big) \longrightarrow \operatorname{Ext}^{i+1}_{C_j}({}^K i_* {}^K i^* M,\ {}^K i_* {}^K i^! N) \quad (84)$$

By the hypothesis of the induction, $\operatorname{Ext}^i_{C_2}({}^K i_* {}^K i^* M, {}^K i_* {}^K i^! N) \longrightarrow \operatorname{Ext}^i_{C_1}({}^K i_* {}^K i^* M, {}^K i_* {}^K i^! N)$ is an isomorphism. We have the natural isomorphisms $\operatorname{Ext}^i_{C_j}(M(!D), N(*D)) \simeq \operatorname{Ext}^i_{C_j}(M(*D), N(*D))$, as remarked in Lemma 9.5. Let Z be the support of $M(*D)$ and $N(*D)$. By Beilinson's argument using the functors Ξ, ϕ and ψ (see Subsection 2.2.1 of [2]), we have natural isomorphisms

$$\operatorname{Ext}^i_{C_j(X)}\big(M(*D), N(*D)\big) \simeq \operatorname{Ext}^i_{C_j(Z)}\big(M(*D), N(*D)\big).$$

For $D_1 \subset D_2$, we have the following commutative diagram:

$$\begin{array}{ccccccc} M & \longrightarrow & M(*D_1) & \qquad & N(!D_1) & \longrightarrow & N \\ {\scriptstyle =}\downarrow & & \downarrow & & \uparrow & & {\scriptstyle =}\uparrow \\ M & \longrightarrow & M(*D_2) & & N(!D_2) & \longrightarrow & N \end{array}$$

Hence, we have the following commutative diagram:

$$\begin{array}{ccccc} \operatorname{Ext}^i_{C_j}({}^K i_{1*} {}^K i_1^* M, {}^K i_{1*} {}^K i_1^! N) & \longrightarrow & \operatorname{Ext}^i_{C_j}(M, N) & \longrightarrow & \operatorname{Ext}^i_{C_j}(M(!D_1), N(*D_1)) \\ \downarrow & & {\scriptstyle =}\downarrow & & \downarrow \\ \operatorname{Ext}^i_{C_j}({}^K i_{2*} {}^K i_2^* M, {}^K i_{2*} {}^K i_2^! N) & \longrightarrow & \operatorname{Ext}^i_{C_j}(M, N) & \longrightarrow & \operatorname{Ext}^i_{C_j}(M(!D_2), N(*D_2)) \end{array}$$

Then, it is easy to show that $\operatorname{Ext}^i_{C_2}(M, N) \longrightarrow \operatorname{Ext}^i_{C_1}(M, N)$ is an isomorphism by using Lemma 9.18. ∎

9.4.5 Proof of Theorem 9.15

Recall a commutative diagram in [39]. For $M^\bullet, N^\bullet \in D(\mathcal{D}_X)$, we have the following commutative diagram:

$$\begin{array}{ccc} \operatorname{Hom}_{D(\mathcal{D}_X)}(M^\bullet, N^\bullet) & \xrightarrow{\simeq} & \operatorname{Hom}_{D(\mathcal{D}_{X\times X})}\big(M^\bullet \boxtimes \mathbb{D}N^\bullet,\ \delta_\dagger \mathcal{O}_X[d_X]\big) \\ \downarrow & & \downarrow \\ \operatorname{Hom}_{D(\mathbb{C}_X)}\big(\mathrm{DR}_X\, M^\bullet,\ \mathrm{DR}_X\, N^\bullet\big) & \xrightarrow{\simeq} & \operatorname{Hom}_{D(\mathbb{C}_X)}\big(\mathrm{DR}_X\, M^\bullet \otimes \mathbb{D}\,\mathrm{DR}_X\, N^\bullet,\ \delta_* \mathbb{C}_X[2d_X]\big) \end{array} \quad (85)$$

Let M be a holonomic $\mathcal{D}_X$-module with a K-Betti structure $\mathcal{F}$. We have

$$\operatorname{Hom}_{D(\mathcal{D}_X)}(M, M) \simeq \operatorname{Hom}_{\mathrm{Hol}(X)}(M, M) \simeq \operatorname{Hom}_{\mathrm{Hol}(X,K)}(M, M) \otimes \mathbb{C}$$

We have similar isomorphisms for $\operatorname{Hom}_{D(\mathcal{D}_X)}\big(M \boxtimes \mathbb{D}M,\ \delta_\dagger \mathcal{O}_X[d_X]\big)$. Hence, we obtain the following diagram from (85):

$$\begin{array}{ccc} \operatorname{Hom}_{\mathrm{Hol}(X,K)}(M, M) \otimes \mathbb{C} & \underset{\simeq}{\xrightarrow{c}} & \operatorname{Hom}_{\mathrm{Hol}(X\times X,K)}\big(M \boxtimes \mathbb{D}M,\ \delta_\dagger \mathcal{O}_X[d_X]\big) \otimes \mathbb{C} \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow \\ \operatorname{Hom}_{D(\mathbb{C}_X)}\big(\mathrm{DR}_X\, M,\ \mathrm{DR}_X\, M\big) & \xrightarrow{\simeq} & \operatorname{Hom}_{D(\mathbb{C}_X)}\big(\mathrm{DR}_X\, M \otimes \mathbb{D}\,\mathrm{DR}_X\, M,\ \delta_* \mathbb{C}_X[2d_X]\big) \\ {\scriptstyle \simeq}\uparrow & & {\scriptstyle \simeq}\uparrow \\ \operatorname{Hom}_{D(K_X)}\big(\mathcal{F}, \mathcal{F}\big) \otimes \mathbb{C} & \xrightarrow{\simeq} & \operatorname{Hom}_{D(K_X)}\big(\mathcal{F} \boxtimes \mathbb{D}\mathcal{F},\ \delta_* K_X[2d_X]\big) \otimes \mathbb{C} \end{array}$$

Note that a is injective. Hence, b is also injective. Since a and b are compatible with K-structures, c is also compatible with K-structures. Let $C : M \otimes \mathbb{D}M \longrightarrow \delta_* \mathcal{O}_X[d_X]$ correspond to $1 : M \longrightarrow M$. It is compatible with K-Betti structures.

For $M^\bullet \in D^b_{\mathrm{hol}}(X, K)$, let $C : M^\bullet \boxtimes \mathbb{D}M^\bullet \longrightarrow \delta_\dagger \mathcal{O}_X[d_X]$ correspond to $1 : M^\bullet \longrightarrow M^\bullet$. We obtain that C is compatible with K-Betti structures. Then, we obtain that the isomorphism

$$\mathrm{Hom}_{D(\mathcal{D}_X)}(M^\bullet, N^\bullet) \longrightarrow \mathrm{Hom}_{D(\mathcal{D}_{X\times X})}\big(M^\bullet \boxtimes \mathbb{D}N^\bullet,\ \delta_\dagger \mathcal{O}_X[d_X]\big)$$

is compatible with K-Betti structures for any $M^\bullet, N^\bullet \in D_{\mathrm{hol}}(X, K)$. By taking the dual, we obtain Theorem 9.15. ∎

References

[1] C. Banica, *Le complété formel d'un espace analytique le long d'un sous-espace: Un théoréme de comparaison*, Manuscripta Math. **6**, (1972), 207–244.

[2] A. Beilinson, *On the derived category of perverse sheaves.* In *K-theory, arithmetic and geometry (Moscow, 1984–1986)*, Lecture Notes in Math., **1289**, Springer, Berlin, (1987), 27–41.

[3] A. Beilinson, *How to glue perverse sheaves*, In *K-theory, arithmetic and geometry (Moscow, 1984–1986)*, Lecture Notes in Math., **1289**, Springer, Berlin, (1987), 42–51.

[4] A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, **100**, (1982), 5–171.

[5] A. Beilinson, S. Bloch, P. Deligne, H. Esnault, *Periods for irregular connections on curves*, preprint

[6] J. Bingener, *Über Formale Komplexe Räume*, Manuscripta Math. **24**, (1978), 253–293.

[7] S. Bloch, H. Esnault, *Homology for irregular connections*, J. Théor. Nombres Bordeaux **16** (2004), 357–371.

[8] P. Deligne, *Équation différentielles à points singuliers réguliers,* Lectures Notes in Maths., vol. **163**, Springer, 1970.

[9] P. Deligne, B. Malgrange, J-P. Ramis, *Singularités Irrégulières*, Documents Mathématiques **5**, Société Mathématique de France, 2007.

[10] R. Hartshorne, *On the De Rham cohomology of algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. **45** (1975), 5–99.

[11] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, (1977).

[12] M. Hien, *Periods for flat algebraic connections*, Invent. Math. **178**, (2009), 1–22.

[13] R. Hotta, K. Takeuchi and T. Tanisaki, *D-modules, perverse sheaves, and representation theory*, Progress in Mathematics, **236**. Birkhäuser Boston, Inc., Boston, MA, 2008.

[14] B. Iversen, *Cohomology of sheaves*, Springer-Verlag, Berlin, 1986.

[15] M. Kashiwara, *On the maximally overdetermined system of linear differential equations. I*, Publ. Res. Inst. Math. Sci. **10** (1974/75), 563–579.

[16] M. Kashiwara, *The Riemann-Hilbert problem for holonomic systems*, Publ. Res. Inst. Math. Sci. **20** (1984), 319–365.

[17] M. Kashiwara, *Vanishing cycle sheaves and holonomic systems of differential equations*, IN: *Algebraic geometry (Tokyo/Kyoto, 1982)* Lecture Notes in Math., **1016**, Springer, Berlin, (1983). 134–142.

[18] M. Kashiwara, *D-modules and microlocal calculus*, Translations of Mathematical Monographs, 217. Iwanami Series in Modern Mathematics. American Mathematical Society, 2003

[19] M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Springer-Verlag, Berlin, 1990

[20] K. Kedlaya, *Good formal structures for flat meromorphic connections, I; Surfaces*, arXiv:0811.0190

[21] B. A. Krasnov, *Formal Modifications. Existence Theorems for modifications of complex manifolds*, Math. USSR. Izvestija **7**, (1973), 847–881.

[22] G. Laumon, *Transformation de Fourier, constantes d'équations fonctionnelles et conjecture de Weil.* Inst. Hautes Études Sci. Publ. Math. **65**, (1987), 131–210.

[23] A. Levelt, *Jordan decomosition for a class of singular differential operators*, Ark. Math. **13**, (1975), 1–27

[24] R. MacPherson, K. Vilonen, *Elementary construction of perverse sheaves,* Invent. Math. **84** (1986), 403–435.

[25] H. Majima, *Asymptotic analysis for integrable connections with irregular singular points*, Lecture Notes in Mathematics, **1075**, Springer-Verlag, Berlin, 1984.

[26] B. Malgrange, *Ideals of differentiable functions*, Tata Institute of Fundamental Research Studies in Mathematics **3**, Tata Institute of Fundamental Research, Bombay, Oxford University Press, London, 1967.

[27] B. Malgrange, *La classification des connexions irrégulières à une variable*, In *Mathematics and physics (Paris, 1979/1982),* Progr. Math., 37, Birkhäuser Boston, Boston, MA, (1983), 381–399.

[28] B. Malgrange, *Polynômes de Bernstein-Sato et cohomologie évanescente*, IN: *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, Astérisque, **101-102**, (1983), 243–267.

[29] B. Malgrange, *Équations différentielles à coefficients polynomiaux,* Progress in Mathematics, **96**, Birkhäuser, Boston, 1991.

[30] Z. Mebkhout, *Une équivalence de catégories.* Compositio Math. **51** (1984), 51–62.

[31] T. Mochizuki, *Good formal structure for meromorphic flat connections on smooth projective surfaces*, In *Algebraic Analysis and Around*, Advanced Studies in Pure Mathematics **54**, (2009), 223–253. math:0803.1346

[32] T. Mochizuki, *Wild harmonic bundles and wild pure twistor D-modules*, math:0803.1344, Version 3

[33] T. Mochizuki, *Asymptotic behaviour of variation of pure polarized TERP structures*, math:0811.1384

[34] C. Sabbah, *Équations différentielles à points singuliers irréguliers et phénomène de Stokes en dimension* 2, Astérisque, **263**, Société Mathématique de France, Paris, 2000.

[35] C. Sabbah, *Introduction to Stokes structures*, arXiv:0912.2762

[36] M. Saito, *Modules de Hodge polarisables*, Publ. RIMS., **24** (1988), 849–995.

[37] M. Saito, *Duality for vanishing cycle functors*, Publ. Res. Inst. Math. Sci. **25** (1989), 889–921.

[38] M. Saito, *Mixed Hodge modules*, Publ. RIMS., **26**, (1990), 221-333.

[39] M. Saito, *Induced D-modules and differential complexes*, Bull. Soc. Math. France **117** (1989), 361–387.

[40] J-L. Verdier, *Extension of a perverse sheaf over a closed subspace.* IN: *Differential systems and singularities (Luminy, 1983).* Astérisque **130** (1985), 210–217.

Address
Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan,
takuro@kurims.kyoto-u.ac.jp