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On the Voros coefficient for the Whittaker equation with a large parameter — Some progress around Sato's conjecture in exact WKB analysis

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§1. Introduction

In this article we study a Schrödinger equation

$$\left(-\frac{d^2}{dx^2}+\eta^2 Q(x,\eta)\right)\psi=0 \quad (\eta>0\text{: a large parameter}) \tag{1.1}$$

with the potential $Q(x,\eta)$ of the form

$$Q(x,\eta)=\frac{1}{4}-\frac{\alpha}{x}+\eta^{-2}\frac{\beta}{x^2} \quad (\alpha,\ \beta\text{: complex constants}), \tag{1.2}$$

that is, the Whittaker equation with a large parameter η, from the viewpoint of exact WKB analysis.

In [AKT2] Aoki, Kawai and the second author of this article studied analytic properties of the Borel transform of WKB solutions of an MTP equation, that is, a Schrödinger equation with a merging pair of simple turning points. Making use of the reduction of an MTP equation to a canonical one, we determined the location of "fixed singular points" (i.e., singular points whose relative locations with respect to the reference singular point are unchanged; cf. [DDP], [DP]) of Borel transformed WKB solutions and succeeded in explicitly computing the alien derivative of WKB solutions at fixed singular points for an MTP

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equation. In the case of an MTP equation the canonical equation is given by the Weber equation and what we call Sato's conjecture, that is, the explicit form of the Voros coefficient of the Weber equation in terms of the Bernoulli numbers (cf. [KT1]; see also [SS] and [T] for its proof), played an important role in studying analytic properties of Borel transformed WKB solutions. (See, e.g., (2.14) in Section 2 for the definition of the Voros coefficient.)

On the other hand, as is emphasized in [Ko1] and [Ko2], the behavior of a WKB solution of a Schrödinger equation (1.1) near a simple pole is similar to that near a simple turning point. Inspired by these results, in a joint work [KKKoT1] with Kamimoto and Kawai we extended the analysis developed in [AKT2] to an MPPT equation, that is, a Schrödinger equation with a merging pair of a simple pole and a simple turning point, again by making use of the reduction of an MPPT equation to a canonical one. In the case of an MPPT equation the canonical equation is given by the Whittaker equation and this motivates our interests in the Whittaker equation, and the computer-assisted study of its Voros coefficient indicated that it should be written down again in terms of the Bernoulli numbers. The aim of this paper is to show that it is really the case and to analyze analytic structure of Borel transformed WKB solutions of the Whittaker equation by using the explicit form of the Voros coefficient thus obtained.

The paper is organized as follows: In Section 2 we present our main result, i.e., the precise formulation of the counterpart of Sato's conjecture for the Whittaker equation. In Section 3 we give a proof of the main result presented in Section 2. The proof is done along the idea used in [T]. Finally in Section 4, by using the main result we study analytic structure of the Borel transform of WKB solutions of the Whittaker equation.

In ending this Introduction, we would like to express our sincere gratitude to Professor Takahiro Kawai and Professor Takashi Aoki for their kind encouragements and stimulating discussions with them.

§2. Main Theorem

The equation we discuss in this article is the Whittaker equation with a large parameter η:

$$\left\{-\frac{d^2}{dx^2}+\eta^2\left(\frac{1}{4}-\frac{\alpha}{x}+\eta^{-2}\frac{\beta}{x^2}\right)\right\}\psi=0. \tag{2.1}$$

Here $\alpha\,(\neq 0)$ and β are complex constants. Our main interest consists in the analysis of a WKB solution, that is, a formal solution of (2.1) of the form

$$\psi = \exp\left(\int^x S(x,\eta)dx\right), \tag{2.2}$$

where

$$S(x,\eta) = \eta S_{-1}(x) + S_0(x) + \eta^{-1}S_1(x) + \cdots \tag{2.3}$$

is a formal solution of the Riccati equation

$$S^2 + \frac{dS}{dx} = \eta^2\left(\frac{1}{4} - \frac{\alpha}{x} + \eta^{-2}\frac{\beta}{x^2}\right) \tag{2.4}$$

associated to (2.1). Here we briefly review the construction of WKB solutions (cf. [KT2]). By substituting (2.3) into (2.4), we obtain the following recursion relations:

$$S_{-1}{}^2 = \frac{1}{4} - \frac{\alpha}{x}, \tag{2.5}$$

$$2S_{-1}\,S_0 + \frac{dS_{-1}}{dx} = 0, \tag{2.6}$$

$$2S_{-1}\,S_1 + S_0{}^2 + \frac{dS_0}{dx} = \frac{\beta}{x^2}, \tag{2.7}$$

$$2S_{-1}S_n + \sum_{j=0}^{n-1} S_j S_{n-j-1} + \frac{dS_{n-1}}{dx} = 0 \quad (n \geq 2). \tag{2.8}$$

It follows from (2.5) that the leading term $S_{-1}(x)$ of (2.3) should be

$$S_{-1}(x) = \pm\sqrt{\frac{1}{4} - \frac{\alpha}{x}}. \tag{2.9}$$

Once we fix the sign in (2.9), i.e., the branch of $\sqrt{1/4 - \alpha/x}$, the higher order terms $\{S_n\}_{n\geq 0}$ are determined uniquely and recursively by (2.6) – (2.8). In particular, the sign in (2.9) is inherited only to the odd degree terms. Therefore there exist two formal solutions of (2.4) of the form

$$S^{(\pm)}(x,\eta) = \pm S_{\mathrm{odd}}(x,\eta) + S_{\mathrm{even}}(x,\eta), \tag{2.10}$$

where

$$S_{\mathrm{odd}}(x,\eta) = \sum_{j=0}^{\infty}\eta^{-2j+1}S_{2j-1}(x), \quad S_{\mathrm{even}}(x,\eta) = \sum_{j=0}^{\infty}\eta^{-2j}S_{2j}(x). \tag{2.11}$$

Since

$$S_{\mathrm{even}}(x,\eta) = -\frac{1}{2}\frac{d}{dx}\log S_{\mathrm{odd}}(x,\eta) \tag{2.12}$$

holds (see Remark 2.2 in [KT2]), we thus obtain two WKB solutions of (2.1)

$$\psi_{\pm}(x,\eta) = \frac{1}{\sqrt{S_{\mathrm{odd}}(x,\eta)}} \exp\left(\pm \int_{x_0}^{x} S_{\mathrm{odd}}(x,\eta)dx\right). \tag{2.13}$$

In exact WKB analysis (cf. [V], [DP], [KT2]), we give an analytic meaning to these WKB solutions (2.2) or (2.13) through the Borel resummation method. As was explained in Introduction, the following theorem then plays a crucially important role in studying analytic structure of Borel transformed WKB solutions.

Theorem 2.1. *Let V be the Voros coefficient of the Whittaker equation (2.1),that is,*

$$V := \int_{4\alpha}^{\infty} (S_{\mathrm{odd}}(x,\eta) - \eta S_{-1}(x))dx = \sum_{n=0}^{\infty} \eta^{1-2n} \int_{4\alpha}^{\infty} S_{2n-1}(x)dx. \tag{2.14}$$

Then the following relation holds as a formal power series in η^{-1}:

$$V = \sum_{n=1}^{\infty} (\alpha\eta)^{1-2n} \frac{B_{2n}(-\gamma)}{2n\cdot(2n-1)}. \tag{2.15}$$

Here γ is a constant satisfying $\gamma(\gamma+1) = \beta$ and $B_n(z)$ denotes the Bernoulli polynomial of degree n defined by

$$B_n(z) = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} B_k\, z^{n-k}, \tag{2.16}$$

where $\{B_k\}_{k\geq 0}$ are the Bernoulli numbers defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{2.17}$$

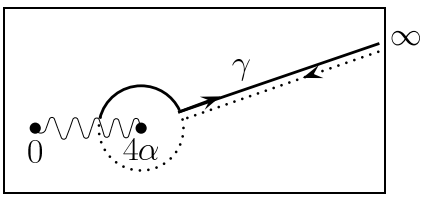

Figure 1

Remark. (i) The lower endpoint $x = 4\alpha$ of the integral in the right-hand side of (2.14) is a simple turning point of (2.1). Since $S_n(x)$ behaves like

$O((x-4\alpha)^{-1-3n/2})$ near $x=4\alpha$, $S_{2n-1}(x)$ $(n\geq 0)$ is not integrable from 4α. We consider the integral in (2.14) as a contour integral along a path γ shown in Figure 1, i.e.,

$$V=\frac{1}{2}\int_{\gamma}(S_{\mathrm{odd}}(x,\eta)-\eta S_{-1}(x))dx. \tag{2.18}$$

Note also that, since we can show

$$S_n(x)=O(|x|^{-1-n/2}) \qquad (|x|\to\infty) \tag{2.19}$$

by an induction on n, $S_n(x)$ with $n\geq 1$ is integrable from the infinity.

(ii) First few Bernoulli numbers are

$$B_0=1, \quad B_1=-\frac{1}{2}, \quad B_{2k+1}=0 \quad (k=1,2,\cdots), \tag{2.20}$$

$$B_2=\frac{1}{6}, \quad B_4=-\frac{1}{30}, \quad B_6=\frac{1}{42}, \quad B_8=-\frac{1}{30}, \quad B_{10}=\frac{5}{66}. \tag{2.21}$$

(iii) The generating function of the Bernoulli polynomials is given by

$$\frac{te^{tz}}{e^t-1}=\sum_{n=0}^{\infty}B_n(z)\frac{t^n}{n!}. \tag{2.22}$$

Formula (2.22) is obtained by using the definition (2.16) of the Bernoulli polynomials and the generating function (2.17) of the Bernoulli numbers.

(iv) It is also known that the Bernoulli polynomial satisfies $B_n(1-z)=(-1)^nB_n(z)$. In view of this relation we can readily find that the $2n$-th Bernoulli polynomial $B_{2n}(z)$ is a function of $z(z-1)$. Therefore the right-hand side of (2.15) is a function of β, not depending on the choice of a solution γ of $\gamma(\gamma+1)=\beta$.

Theorem 2.1 is the counterpart of Sato's conjecture for the Whittaker equation. It is proved in Section 3 and by using Theorem 2.1 we study some analytic properties of the Borel transform of WKB solutions in Section 4. Note that our results play an essential role in analyzing the fixed singular points of an MPPT equation in [KKKoT1] (see also [KKKoT2] for its review).

§3. Proof of Main Theorem

We prove Theorem 2.1 along the same idea as that employed in [T]. Firstly, we derive the difference equation that the Voros coefficient satisfies. In this derivation the raising and lowering operators (with respect to the parameter α) play a crucially important role. Then, as the difference equation determines

a solution uniquely, we compute its solution to verify that it coincides with the right-hand side of (2.15). In this paper we use the Borel transformation technique to compute a unique solution of the difference equation.

§3.1. Derivation of the difference equation

In the discussion of this section the parameter α plays an important role. To show the dependency on α more manifestly, we use the notation $S^{(\pm)}(x,\alpha,\eta)$ for solutions (2.10) of the Riccati equation, $\psi_{\pm}(x,\alpha,\eta)$ for WKB solutions, etc in what follows. We also set

$$I(\alpha,\eta) := \int_{\gamma} (S_{\mathrm{odd}}(x,\alpha,\eta) - \eta S_{-1}(x,\alpha))dx = 2V. \tag{3.1}$$

In this subsection we prove

Proposition 3.1. *$I(\alpha,\eta)$ formally satisfies*

$$\begin{aligned} &I(\alpha+\eta^{-1},\eta) - I(\alpha,\eta) \\ &\quad = 2 + \log\frac{\alpha^2+\eta^{-1}\alpha-\eta^{-2}\beta}{\alpha^2} - 2(\alpha\eta+1)\log(1+\frac{1}{\alpha\eta}) \qquad (3.2)\\ &\quad = 2 + \log\left[\left(1-\frac{\gamma}{\alpha\eta}\right)\left(1+\frac{\gamma+1}{\alpha\eta}\right)\right] - 2(\alpha\eta+1)\log(1+\frac{1}{\alpha\eta}). \qquad (3.3)\end{aligned}$$

In Proposition 3.1 the word "formally" means that the equality holds as a formal power series with respect to η^{-1}. In fact, the right-hand side of (3.2) has a formal power series expansion as

$$-\frac{1+6\beta}{\alpha^2}\eta^{-2} + \frac{1+6\beta}{\alpha^3}\eta^{-3} - \frac{10\beta^2+20\beta+3}{20\alpha^4}\eta^{-4} + \frac{15\beta^2+15\beta+2}{15\alpha^5}\eta^{-5} + \cdots \tag{3.4}$$

by the (formal) use of the Taylor expansion.

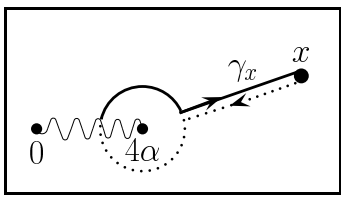

Figure 2

To prove Proposition 3.1, we first note that $S_{2n}(x,\alpha)$ for $n = 1, 2, \cdots$ are single-valued around $x = 4\alpha$ and their residues are zero (cf. (2.12)). Therefore

$$I(\alpha,\eta) = \int_{\gamma} (S^{(+)}(x,\alpha,\eta) - \eta S_{-1}(x,\alpha) - S_0(x,\alpha))dx \tag{3.5}$$

holds. In studying the right-hand side of (3.5), we also consider the following cut-off integrals:

$$I^x(\alpha,\eta)=\int_{\gamma_x} S^{(+)}(x,\alpha,\eta)dx, \tag{3.6}$$

$$I^x_n(\alpha,\eta)=\int_{\gamma_x} S_n(x,\alpha)dx \quad (n=-1,0), \tag{3.7}$$

where γ_x is a contour shown in Figure 2. From (3.5) and the definition of I^x and I^x_n, we have

$$I(\alpha,\eta)=\lim_{x\to\infty}\Big[I^x(\alpha,\eta)-\eta I^x_{-1}(\alpha)-\eta I^x_0(\alpha)\Big]. \tag{3.8}$$

We then determine the asymptotic behavior of $I^x(\alpha+\eta^{-1},\eta)-I^x(\alpha,\eta)$.

Lemma 3.1. *As x tends to infinity, we have*

$$I^x(\alpha+\eta^{-1},\eta)-I^x(\alpha,\eta)=\log\frac{\alpha^2+\eta^{-1}\alpha-\eta^{-2}\beta}{x^2}+O(|x|^{-1}). \tag{3.9}$$

To prove Lemma 3.1, we use the raising and lowering operators of the Whittaker equation with respect to α:

$$\mathcal{A}(\alpha)=\eta^{-1}\frac{d}{dx}+\frac{1}{2}x-\alpha, \qquad \mathcal{A}^\dagger(\alpha)=\eta^{-1}\frac{d}{dx}-\frac{1}{2}x+\alpha. \tag{3.10}$$

We also set

$$L(\alpha)=-\frac{d^2}{dx^2}+\eta^2\left(\frac{1}{4}-\frac{\alpha}{x}+\eta^{-2}\frac{\beta}{x^2}\right). \tag{3.11}$$

Lemma 3.2. *If ψ is a solution of the Whittaker equation* (2.1) (*i.e.* $L(\alpha)\psi=0$), *then $L(\alpha-\eta^{-1})\mathcal{A}(\alpha)\psi=0$ and $L(\alpha+\eta^{-1})\mathcal{A}^\dagger(\alpha)\psi=0$.*

Proof. By direct computations, we have

$$\eta^{-2}x^2L(\alpha)=\mathcal{A}(\alpha+\eta^{-1})\mathcal{A}^\dagger(\alpha)+c(\alpha) \tag{3.12}$$

$$=\mathcal{A}^\dagger(\alpha-\eta^{-1})\mathcal{A}(\alpha)+c(\alpha-\eta^{-1}) \tag{3.13}$$

with

$$c(\alpha)=\alpha^2+\eta^{-1}\alpha-\eta^{-2}\beta. \tag{3.14}$$

Therefore

$$\begin{aligned}\eta^{-2}x^2L(\alpha+\eta^{-1})\mathcal{A}^\dagger(\alpha)\psi&=\Big[\mathcal{A}^\dagger(\alpha)\mathcal{A}(\alpha+\eta^{-1})+c(\alpha)\Big]\mathcal{A}^\dagger(\alpha)\psi\\&=\Big[\mathcal{A}^\dagger(\alpha)\mathcal{A}(\alpha+\eta^{-1})\mathcal{A}^\dagger(\alpha)+c(\alpha)\mathcal{A}^\dagger(\alpha)\Big]\psi\\&=\Big[\mathcal{A}^\dagger(\alpha)\big\{\eta^{-2}x^2L(\alpha)-c(\alpha)\big\}+c(\alpha)\mathcal{A}^\dagger(\alpha)\Big]\psi\\&=0.\end{aligned} \tag{3.15}$$

By the same reasoning we can also prove $\eta^{-2}x^2L(\alpha-\eta^{-1})\mathcal{A}(\alpha)\psi=0$. □

Proof of Lemma 3.1. Lemma 3.2 implies that there exists a constant $C(\eta)$ for which

$$\left(\eta^{-1}x\frac{d}{dx}-\frac{1}{2}x+\alpha\right)\psi_+(x,\alpha,\eta)=C(\eta)\,\psi_+(x,\alpha+\eta^{-1},\eta) \tag{3.16}$$

holds. Since the left-hand side of (3.16) is

$$\left(\eta^{-1}xS^{(+)}-\frac{1}{2}x+\alpha\right)\psi_+(x,\alpha,\eta), \tag{3.17}$$

the logarithmic derivatives of both sides of (3.16) give

$$S^{(+)}(x,\alpha+\eta^{-1},\eta)=S^{(+)}(x,\alpha,\eta)+\frac{d}{dx}\log\left[\eta^{-1}xS^{(+)}(x,\alpha,\eta)-\frac{1}{2}x+\alpha\right]. \tag{3.18}$$

Therefore we obtain

$$\begin{aligned}&\int_{\gamma^x}S^{(+)}(x,\alpha+\eta^{-1},\eta)dx-\int_{\gamma^x}S^{(+)}(x,\alpha,\eta)dx\\&\quad=\log\left[\eta^{-1}xS^{(+)}(x,\alpha,\eta)-\frac{1}{2}x+\alpha\right]-\log\left[\eta^{-1}xS^{(-)}(x,\alpha,\eta)-\frac{1}{2}x+\alpha\right].\end{aligned} \tag{3.19}$$

To determine the asymptotic behavior of the right-hand side of (3.19), we use the following explicit formulas as $x\to\infty$ which are easily obtained from the recursion relations (2.5) – (2.8):

$$S_{-1}(x,\alpha)=\sqrt{\frac{1}{4}-\frac{\alpha}{x}}=\frac{1}{2}-\frac{\alpha}{x}-\frac{\alpha^2}{x^2}-\frac{2\alpha^3}{x^3}+O(|x|^{-4}), \tag{3.20}$$

$$S_0(x,\alpha)=\frac{1}{4}\left(\frac{1}{x}-\frac{1}{x-4\alpha}\right)=-\frac{\alpha}{x^2}+O(|x|^{-3}), \tag{3.21}$$

$$\begin{aligned}S_1(x,\alpha)&=-\frac{5}{16}x^{1/2}(x-4\alpha)^{-5/2}+\frac{1}{8}x^{-1/2}(x-4\alpha)^{-3/2}\\&\qquad+(\beta+\frac{3}{16})x^{-3/2}(x-4\alpha)^{-1/2}\end{aligned} \tag{3.22}$$

$$=\frac{\beta}{x^2}+O(|x|^{-3}). \tag{3.23}$$

These asymptotic formulas together with (2.19) show

$$S^{(+)}(x,\alpha,\eta)=\eta\left(\frac{1}{2}-\frac{\alpha}{x}-\frac{\alpha^2}{x^2}\right)+\frac{-\alpha+\eta^{-1}\beta}{x^2}+O(|x|^{-5/2}), \tag{3.24}$$

$$S^{(-)}(x,\alpha,\eta)=\eta\left(-\frac{1}{2}+\frac{\alpha}{x}+\frac{\alpha^2}{x^2}\right)+\frac{-\alpha-\eta^{-1}\beta}{x^2}+O(|x|^{-5/2}). \tag{3.25}$$

The formula (3.9) readily follows from (3.19), (3.24) and (3.25). This completes the proof of Lemma 3.1. □

Next we determine the asymptotic behavior of $I^x_{-1}(\alpha+\eta^{-1}) - I^x_{-1}(\alpha)$ and $I^x_0(\alpha+\eta^{-1}) - I^x_0(\alpha)$.

Lemma 3.3.

(i) *As x tends to infinity, we have*

$$I^x_{-1}(\alpha+\eta^{-1}) - I^x_{-1}(\alpha) = \eta^{-1}\left[-2 + 2(1+\alpha\eta)\log(1+\frac{1}{\alpha\eta}) + \log\frac{\alpha^2}{x^2}\right] + O(|x|^{-1}). \quad (3.26)$$

(ii) $$I^x_0(\alpha+\eta^{-1}) - I^x_0(\alpha) = 0. \quad (3.27)$$

Proof. Since $S_0(x)$ is single-valued near $x = 4\alpha$, $I^x_0(\alpha)$ is $2\pi i$ multiple of the residue of $S_0(x)$ at $x = 4\alpha$. Hence $I^x_0(\alpha) = -i\pi/2$ and thus we obtain (ii). To prove (i), we compute the integral explicitly:

$$I^x_{-1}(\alpha) = \int_{\gamma_x} S_{-1}dx \quad (3.28)$$

$$= 2x\sqrt{\frac{1}{4}-\frac{\alpha}{x}} - \alpha\log\left[x - 2\alpha + 2x\sqrt{\frac{1}{4}-\frac{\alpha}{x}}\right] + \alpha\log\left[x - 2\alpha - 2x\sqrt{\frac{1}{4}-\frac{\alpha}{x}}\right]. \quad (3.29)$$

As x tends to infinity, we have

$$\sqrt{\frac{1}{4}-\frac{\alpha}{x}} = \frac{1}{2} - \frac{\alpha}{x} + \frac{\alpha^2}{x^2} + O(|x|^{-3}), \quad (3.30)$$

and hence

$$x - 2\alpha + 2x\sqrt{\frac{1}{4}-\frac{\alpha}{x}} = 2x - 4\alpha - \frac{2\alpha^2}{x} + O(|x|^{-2}), \quad (3.31)$$

$$x - 2\alpha - 2x\sqrt{\frac{1}{4}-\frac{\alpha}{x}} = \frac{2\alpha^2}{x} + O(|x|^{-2}). \quad (3.32)$$

Therefore we obtain

$$I^x_{-1}(\alpha) = \int_{\gamma_x} S_{-1}(x,\alpha)dx = x - 2\alpha + \alpha\log\frac{\alpha^2}{x^2} + O(|x|^{-1}). \quad (3.33)$$

This asymptotic behavior of $I^x_{-1}(\alpha)$ verifies (i). □

We now prove Proposition 3.1.

Proof of Proposition 3.1. Lemmas 3.1 and 3.2 give

$$\begin{aligned}
&\Big[I^x(\alpha+\eta^{-1},\eta) - \eta I^x_{-1}(\alpha+\eta^{-1}) - I^x_0(\alpha+\eta^{-1})\Big] \\
&\qquad - \Big[I^x(\alpha,\eta) - \eta I^x_{-1}(\alpha) - I^x_0(\alpha)\Big] \\
&= \Big[I^x(\alpha+\eta^{-1},\eta) - I^x(\alpha,\eta)\Big] - \eta\Big[I^x_{-1}(\alpha+\eta^{-1}) - I^x_{-1}(\alpha)] \qquad (3.34)\\
&= 2 - 2(1+\alpha\eta)\log(1+\frac{1}{\alpha\eta}) + \log\frac{\alpha^2+\eta^{-1}\alpha-\eta^{-2}\beta}{\alpha^2} + O(|x|^{-1}). \qquad (3.35)
\end{aligned}$$

This relation and (3.8) give (3.2). Since $\beta=\gamma(\gamma+1)$, we can also verify

$$\frac{\alpha^2+\eta^{-1}\alpha-\eta^{-2}\beta}{\alpha^2} = \left(1-\frac{\gamma}{\alpha\eta}\right)\left(1+\frac{\gamma+1}{\alpha\eta}\right). \qquad (3.36)$$

This proves (3.3). □

§3.2. Determination of the Voros coefficient

Let $V(\alpha,\eta)$ be the Voros coefficient defined by (2.14). From Proposition 3.1 and (3.1), we have

$$\begin{aligned}
&V(\alpha+\eta^{-1},\eta) - V(\alpha,\eta) \\
&\quad = 1 + \frac{1}{2}\log\left[\left(1-\frac{\gamma}{\alpha\eta}\right)\left(1+\frac{\gamma+1}{\alpha\eta}\right)\right] - (\alpha\eta+1)\log(1+\frac{1}{\alpha\eta}). \qquad (3.37)
\end{aligned}$$

In this subsection we verify that this difference equation determines V uniquely. To this end we convert (3.37) to a difference equation with respect to η by using the homogeneity of $V(\alpha,\eta)$.

Lemma 3.4. *Let*

$$V_{2n-1}(\alpha) = \int_{4\alpha}^{\infty} S_{2n-1}(x,\alpha)dx \quad (n=1,2,3,\cdots). \qquad (3.38)$$

Then

$$V_{2n-1}(\alpha) = \alpha^{1-2n}V_{2n-1}(1) \qquad (3.39)$$

holds for $n=1,2,3,\cdots$.

Proof. Using the recursion relations (2.5) – (2.8), we can verify

$$S_n(\alpha x, \alpha) = \frac{1}{\alpha^{n+1}} S_n(x, 1) \quad (n = -1, 0, 1, 2, \cdots) \tag{3.40}$$

by an induction on n. Therefore

$$\int_{4\alpha}^{\infty} S_{2n-1}(x, \alpha) dx = \alpha \int_{4}^{\infty} S_{2n-1}(\alpha t, \alpha) dt = \alpha^{1-2n} \int_{4}^{\infty} S_{2n-1}(t, 1) dt \tag{3.41}$$

holds for $n = 1, 2, \cdots$. □

Thanks to Lemma 3.4,

$$V(\alpha, \eta) = \sum_{n=1}^{\infty} V_{2n-1}(\alpha) \eta^{1-2n} = \sum_{n=1}^{\infty} V_{2n-1}(1) (\alpha\eta)^{1-2n} \tag{3.42}$$

holds, and

$$(\alpha + \eta^{-1})\eta = \alpha\eta + 1 = \alpha(\eta + \frac{1}{\alpha}) \tag{3.43}$$

entails

$$V(\alpha + \eta^{-1}, \eta) = V(\alpha, \eta + \frac{1}{\alpha}). \tag{3.44}$$

From (3.44) and (3.37) we obtain the following difference equation for V with respect to η.

Proposition 3.2.

$$\begin{aligned} &V(\alpha, \eta + \frac{1}{\alpha}) - V(\alpha, \eta) \\ &\quad = 1 + \frac{1}{2} \log \left[\left(1 - \frac{\gamma}{\alpha\eta} \right) \left(1 + \frac{\gamma + 1}{\alpha\eta} \right) \right] - (\alpha\eta + 1) \log(1 + \frac{1}{\alpha\eta}). \end{aligned} \tag{3.45}$$

We solve this difference equation (3.45) by using the Borel transformation. We first note that (3.45) has a unique (formal) solution of the form

$$V = \sum_{n=1}^{\infty} V_n \eta^{1-2n} = V_1 \eta^{-1} + V_3 \eta^{-3} + V_5 \eta^{-5} + \cdots . \tag{3.46}$$

Hence (3.45) determines V uniquely. We now explicitly compute its Borel transform

$$V_B = \sum_{n=1}^{\infty} \frac{V_n}{(2n-2)!} y^{2n-2} \tag{3.47}$$

by considering the Borel transform of (3.45). The Borel transform of the left-hand side of (3.45) is

$$(e^{-y/\alpha}-1)V_B(\alpha,y). \tag{3.48}$$

To compute the Borel transform of the right-hand side, we use

Lemma 3.5. *Let C be a non-zero complex constant. Then*

$$\text{(i)}\quad \mathcal{B}\Big[\log(1+C\eta^{-1})\Big](y)=\frac{1-e^{-Cy}}{y}. \tag{3.49}$$

$$\text{(ii)}\quad \mathcal{B}\Big[\eta\log(1+C\eta^{-1})-C\Big](y)=\frac{-1+(1+Cy)e^{-Cy}}{y^2}. \tag{3.50}$$

Here $\mathcal{B}$ denotes the Borel transformation with respect to η.

Lemma 3.5 can be proved by straightforward computations. By using Lemma 3.5 we can compute the Borel transform of the right-hand side of (3.45) in the following way:

$$\begin{aligned}
&\mathcal{B}\left[1-(\alpha\eta+1)\log(1+\frac{1}{\alpha\eta})\right]\\
&\quad=-\mathcal{B}\left[\alpha\left(\eta\log(1+\frac{1}{\alpha\eta})-\frac{1}{\alpha}\right)\right]-\mathcal{B}\left[\log(1+\frac{1}{\alpha\eta})\right] &(3.51)\\
&\quad=\frac{\alpha}{y^2}\left[1-(1+\frac{y}{\alpha})e^{-y/\alpha}\right]-\frac{1-e^{-y/\alpha}}{y} &(3.52)\\
&\quad=\alpha\frac{1-e^{-y/\alpha}}{y^2}-\frac{1}{y}, &(3.53)\\
&\mathcal{B}\left[\frac{1}{2}\log(1-\frac{\gamma}{\alpha\eta})(1+\frac{\gamma+1}{\alpha\eta})\right]\\
&\quad=\frac{1}{2}\mathcal{B}\left[\log(1-\frac{\gamma}{\alpha\eta})\right]+\frac{1}{2}\mathcal{B}\left[\log(1+\frac{\gamma+1}{\alpha\eta})\right] &(3.54)\\
&\quad=\frac{1}{2}\cdot\frac{1-e^{\gamma y/\alpha}}{y}+\frac{1}{2}\cdot\frac{1-e^{-(\gamma+1)y/\alpha}}{y} &(3.55)\\
&\quad=\frac{1}{y}-\frac{1}{2}\cdot\frac{e^{\gamma y/\alpha}+e^{-(\gamma+1)y/\alpha}}{y}. &(3.56)
\end{aligned}$$

We conclude from (3.48), (3.53) and (3.56) that the difference equation (3.45) becomes

$$(e^{-y/\alpha}-1)V_B(\alpha,y)=\alpha\frac{1-e^{-y/\alpha}}{y^2}-\frac{1}{2}\cdot\frac{e^{\gamma y/\alpha}+e^{-(\gamma+1)y/\alpha}}{y} \tag{3.57}$$

after the Borel transformation. This equation can be easily solved and we obtain

Theorem 3.1. *The Borel transform V_B of the Voros coefficient V is given by*

$$V_B(\alpha, y) = \frac{1}{2y} \cdot \frac{e^{-\gamma y/\alpha} + e^{(\gamma+1)y/\alpha}}{e^{y/\alpha} - 1} - \frac{\alpha}{y^2}. \tag{3.58}$$

Thus, to prove Theorem 2.1, it suffices to show that the Borel transform of the right-hand side of (2.15) coincides with the right-hand side of (3.58). The Borel transform of the right-hand side of (2.15) is, by its definition,

$$\sum_{n=1}^{\infty} \alpha^{1-2n} \frac{B_{2n}(-\gamma)}{(2n)!} y^{2n-2} = \frac{\alpha}{y^2} \sum_{n=1}^{\infty} \frac{B_{2n}(-\gamma)}{(2n)!} \left(\frac{y}{\alpha}\right)^{2n}. \tag{3.59}$$

We then use (2.22) to obtain

$$\sum_{n=0}^{\infty} \frac{B_{2n}(z)}{(2n)!} t^{2n} = \frac{1}{2} \left[\frac{te^{tz}}{e^t - 1} + \frac{-te^{-tz}}{e^{-t} - 1} \right] = \frac{t}{2} \cdot \frac{e^{tz} + e^{-t(z-1)}}{e^t - 1}. \tag{3.60}$$

From (3.59) and (3.60) we can confirm that the Borel transform of the right-hand side of (2.15) is exactly the same as the right-hand side of (3.58). This proves Theorem 2.1.

Remark. Since

$$\begin{aligned} &e^{-\gamma y/\alpha} + e^{(\gamma+1)y/\alpha} \\ &\quad = \frac{1}{2} \left[(e^{y/\alpha} + 1)(e^{\gamma y/\alpha} + e^{-\gamma y/\alpha}) + (e^{y/\alpha} - 1)(e^{\gamma y/\alpha} - e^{-\gamma y/\alpha}) \right] \qquad (3.61) \\ &\quad = (e^{y/\alpha} + 1) \cosh \frac{\gamma y}{\alpha} + (e^{y/\alpha} - 1) \sinh \frac{\gamma y}{\alpha} \qquad (3.62) \end{aligned}$$

holds, we have the following expression of V_B:

$$V_B = \frac{1}{2} \left[\frac{1}{y} \frac{e^{y/\alpha} + 1}{e^{-y/\alpha} - 1} \cosh \frac{\gamma y}{\alpha} + \frac{1}{y} \sinh \frac{\gamma y}{\alpha} - \frac{2\alpha}{y^2} \right]. \tag{3.63}$$

In [KKKoT1] and [KKKoT2] this expression of V_B was used.

The following lemma will be used in the subsequent section.

Proposition 3.3. *V_B has poles at $y = 2m\pi i\alpha$ ($m \in \mathbb{Z} \setminus \{0\}$) and except for these poles V_B has no other singular points. They are all simple poles and the residues of V_B there are given by*

$$\operatorname*{Res}_{y=2m\pi i\alpha} V_B = \frac{e^{2m\pi i\gamma} + e^{-2m\pi i\gamma}}{4m\pi i}. \tag{3.64}$$

§4. Analytic structure of the Borel transform of WKB solutions for the Whittaker equation

In this section we derive some important analytic properties of the Borel transform of WKB solutions of the Whittaker equation. All of them are obtained as consequences of our main theorems (Theorem 2.1 and Theorem 3.1). For simplicity we restrict our consideration in this section to the case where $\arg\alpha = \pi/2$ or $\arg\alpha$ is sufficiently close to $\pi/2$. The configuration of Stokes curves, i.e., integral curves of the direction field $\Im\left(S_{-1}(x)dx\right) = 0$ emanating from turning points, of the Whittaker equation (2.1) for $\arg\alpha = \pi/2$ is shown in Figure 3. Note that the origin also plays the same role as a turning point (cf. [Ko1] and [Ko2]). We also show in Figure 4 and Figure 6 (resp., Figure 5 and Figure 7) the configuration of Stokes curves for $\arg\alpha = \pi/2 - \varepsilon$ (resp., $\arg\alpha = \pi/2 + \varepsilon$) for sufficiently small positive ε.

As we can observe from these figures, the degeneration occurs in the configuration of Stokes curves for $\arg\alpha = \pi/2$ (Figure 3), i.e., there exists a Stokes curve connecting a turning point 4α and the origin in Figure 3. Although this degeneration is resolved in Figure 4 $\sim$ Figure 7, the topological configuration of Stokes curves is quite different between, for example, Figure 6 and Figure 7; in particular, the configuration of a Stokes curve emanating from 4α abruptly changes at $\arg\alpha = \pi/2$. We will see in §4.2 that this abrupt change of the configuration of Stokes curves is related to Stokes phenomena of WKB solutions that occur when α varies.

§4.1. Fixed singularities of the Borel transform of WKB solutions

Let us consider WKB solutions normalized at a simple turning point 4α:

$$\psi_{\pm} = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left[\pm \int_{4\alpha}^{x} S_{\mathrm{odd}} dx\right]. \tag{4.1}$$

These WKB solutions ψ_+ and ψ_- can be expanded as

$$\psi_{\pm} = e^{\eta y_{\pm}(x)} \sum_{n=0}^{\infty} \psi_{\pm,n}(x) \eta^{-n-1/2} \tag{4.2}$$

with

$$y_{\pm}(x) = \pm \int_{4\alpha}^{x} S_{-1}(x) dx = \pm \int_{4\alpha}^{x} \sqrt{\frac{1}{4} - \frac{\alpha}{x}} dx \tag{4.3}$$

and the Borel transform $\psi_{\pm,B}$ of $\psi_{\pm}$ is given by

$$\psi_{\pm,B}(x,y) = \sum_{n=0}^{\infty} \frac{\psi_{\pm,n}(x)}{\Gamma(n+1/2)} (y + y_{\pm}(x))^{n-1/2}. \tag{4.4}$$

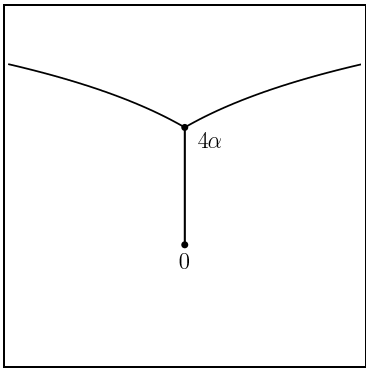

Figure 3: $\arg\alpha = \pi/2$

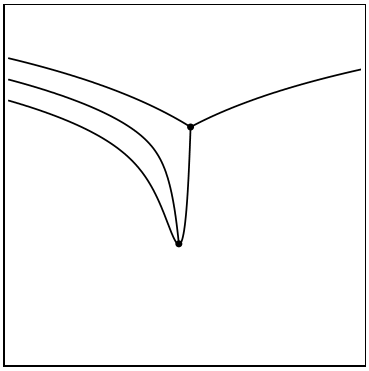

Figure 4: $\arg\alpha = \pi/2 - 0.2$

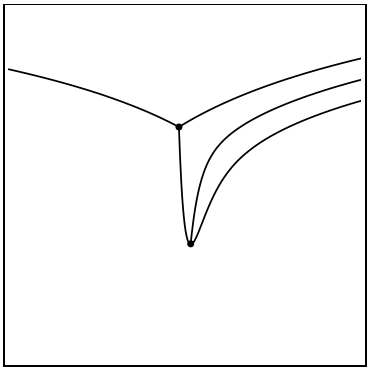

Figure 5: $\arg\alpha = \pi/2 + 0.2$

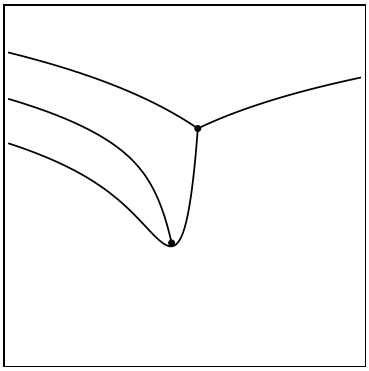

Figure 6: $\arg\alpha = \pi/2 - 0.45$

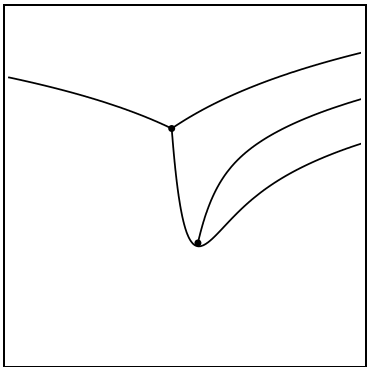

Figure 7: $\arg\alpha = \pi/2 + 0.45$

Theorem 4.1. *We assume that the path of integration in* (4.1) *does not cross any Stokes curve for any* α *with* $|\arg\alpha - \pi/2| < \varepsilon$ $(\varepsilon > 0)$. *Then the Borel transform of the WKB solution* $\psi_{+,B}$ *(resp.,* $\psi_{-,B}$*) has singular points at* $y = -y_+(x) + 2m\pi i\alpha$ *(resp.,* $y = -y_-(x) + 2m\pi i\alpha$*) with* $m \in \mathbb{Z} \setminus \{0\}$. *Furthermore, their alien derivatives there are given by*

$$(\Delta_{y=-y_\pm(x)+2m\pi i\alpha}\psi_{\pm,B})(x,y) = \frac{e^{2m\pi i\gamma} + e^{-2m\pi i\gamma}}{2m}\psi_{+,B}(x, y - 2m\pi i\alpha). \tag{4.5}$$

Note that the singular points described in Theorem 4.1 are fixed singular points (cf. [DDP], [DP]) of ψ_+ (resp., ψ_-) since their relative locations with respect to the reference singular point $y = -y_+(x)$ (resp., $y = -y_-(x)$) are unchanged when x varies.

In this subsection we prove Theorem 4.1. Since the same reasoning holds also for $\psi_{-,B}$, we only consider $\psi_{+,B}$. We refer the reader to [Sa] for the definition of the alien derivative and the relevant alien calculus employed in what follows. See also [AKT2, §3] where similar discussion is given for the Weber equation.

To prove Theorem 4.1 we use the following relation

$$\psi_+ = \exp\left[\int_{4\alpha}^{\infty}(S_{\mathrm{odd}} - \eta S_{-1}(x))dx\right]\psi_+^{(\infty)} = e^V\psi_+^{(\infty)}, \tag{4.6}$$

where $\psi_+^{(\infty)}$ is a WKB solution normalized at infinity:

$$\psi_+^{(\infty)} = \frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp\left[\eta\int_{4\alpha}^{x}S_{-1}(x)dx\right]\exp\left[\int_{\infty}^{x}(S_{\mathrm{odd}}(x,\eta) - \eta S_{-1}(x))dx\right]. \tag{4.7}$$

The formal relation (4.6) becomes

$$\psi_{+,B} = \mathcal{B}\left[e^V\right] * \psi_{+,B}^{(\infty)} \tag{4.8}$$

after the Borel transformation, where $*$ denotes the convolution product for the variable y. Since V_B is singular at $y = 2m\pi i\alpha$ $(m \in \mathbb{Z} \setminus \{0\})$ as we have seen in Section 3, we then find that $y = 2m\pi i\alpha$ is also a singular point of $\psi_{+,B}$.

To compute the alien derivative of ψ_+ there, we use

Lemma 4.1. *The WKB solution* (4.7) *normalized at infinity is Borel summable if the path of integration from infinity to* x *in* (4.7) *does not cross any Stokes curve.*

See [DP, Theorem 1.2.2 (c)] for the corresponding result for the Weber equation. Lemma 4.1 for the Whittaker equation can be verified in a similar manner.

Under our assumption we can choose a path of integration from infinity to x in such a way that it does not cross any Stokes curve (see Remark below). Hence $\psi^{(\infty)}_{+,B}$ is holomorphic at $y = -y_+(x)+2m\pi i\alpha$. This enables us to compute the alien derivative of $\psi_{+,B}$ at $y = -y_+(x) + 2m\pi i\alpha$ as follows:

$$\Delta_{y=-y_+(x)+2m\pi i\alpha}\psi_{+,B} = \Delta_{y=-y_+(x)+2m\pi i\alpha}\Big[\mathcal{B}(e^V) * \psi^{(\infty)}_{+,B}\Big] \tag{4.9}$$

$$= \Big[\Delta_{y=-y_+(x)+2m\pi i\alpha}\mathcal{B}(e^V)\Big] * \psi^{(\infty)}_{+,B} \tag{4.10}$$

$$= \Big[\Delta_{y=-y_+(x)+2m\pi i\alpha}V_B\Big]\mathcal{B}(e^V) * \psi^{(\infty)}_{+,B} \tag{4.11}$$

$$= 2\pi i \operatorname*{Res}_{y=2m\pi i\alpha}[V_B(\alpha, y)] \cdot \psi_{+,B} \tag{4.12}$$

$$= \frac{e^{2m\pi i\gamma} + e^{-2m\pi i\gamma}}{2m}\psi_{+,B}. \tag{4.13}$$

This completes the proof of Theorem 4.1.

Remark. If x does not lie on a Stokes curve for $\arg\alpha = \pi/2$, we can find a path of integration from infinity to x so that it does not cross any Stokes curve for all α with $|\arg\alpha - \pi/2|$ being sufficiently small. Examples of such paths are shown in Figure 8, Figure 9 and Figure 10. For example, for x_j $(j = 1, 2)$ in Figure 8, γ_j gives a path satisfying the above condition.

§4.2. Derivation of the Stokes automorphism for WKB solutions

It follows from Theorem 4.1 that, when $\arg\alpha = \pi/2$, the singular points $y = -y_+(x) - 2m\pi i\alpha$ $(m = 1, 2, 3, \cdots)$ of the Borel transform of the WKB solution $\psi_{+,B}$ are located on

$$\{y;\, \Re y \geq \Re(-y_+(x)), \quad \Im y = \Im(-y_+(x))\}. \tag{4.14}$$

As (4.14) is nothing but the path of integration for the Borel sum of ψ_+, this implies that ψ_+ is not Borel summable when $\arg\alpha = \pi/2$. If we let the parameter α vary from $\arg\alpha = \pi/2 - \varepsilon$ to $\arg\alpha = \pi/2 + \varepsilon$, these singular points cross the path of integration for the Borel sum. Therefore, we conclude that the WKB solution ψ_+ suffers a Stokes phenomenon (with respect to α) at $\arg\alpha = \pi/2$. Such Stokes phenomena are well described by the Stokes automorphism (cf. [Sa]). In this subsection we compute the action of the Stokes automorphism on the WKB solutions $\psi_\pm$.

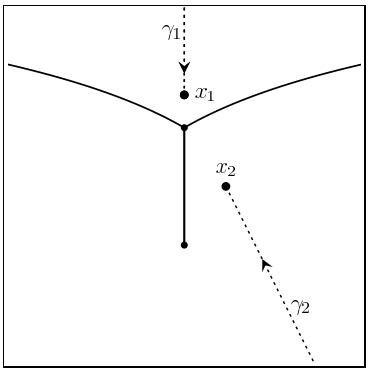

Figure 8: $\arg\alpha = \pi/2$

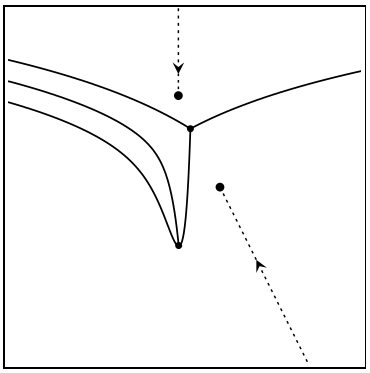

Figure 9: $\arg\alpha = \pi/2 - 0.2$

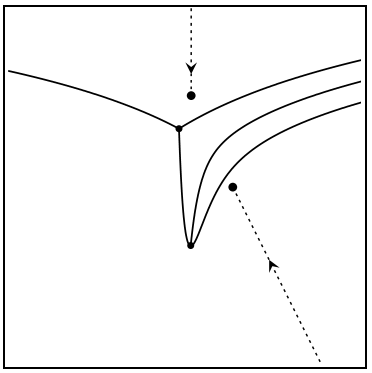

Figure 10: $\arg\alpha = \pi/2 + 0.2$

Let $\mathfrak{S}$ denote the Stokes automorphism. In our case, it is defined by

$$\mathfrak{S}\psi_+ = \exp\left[\sum_{m=1}^{\infty} \Delta_{y=-y_+(x)-2m\pi i\alpha}\right]\psi_+, \tag{4.15}$$

$$\mathfrak{S}\psi_- = \exp\left[\sum_{m=1}^{\infty} \Delta_{y=-y_-(x)-2m\pi i\alpha}\right]\psi_-. \tag{4.16}$$

Theorem 4.2. *When* $\arg\alpha = \pi/2$, $\mathfrak{S}\psi_\pm$ *are explicitly given by*

$$\mathfrak{S}\psi_\pm = (1-e^{2\pi i(\alpha\eta+\gamma)})^{1/2}(1-e^{2\pi i(\alpha\eta-\gamma)})^{1/2}\psi_\pm. \tag{4.17}$$

Proof. We compute the action of the Stokes automorphism in the formal model. Let Δ be the sum of the alien derivatives:

$$\Delta = \sum_{m=1}^{\infty} \Delta_{y=-y_+(x)-2m\pi i\alpha}. \tag{4.18}$$

We then obtain

$$\Delta\psi_+ = \sum_{m=1}^{\infty} \mathfrak{B}^{-1}\left[\Delta_{y=-y_+(x)-2m\pi i\alpha}\psi_{+,B}\right] \tag{4.19}$$

$$= -\sum_{m=1}^{\infty} \frac{e^{2m\pi i\gamma}+e^{-2m\pi i\gamma}}{2m}\mathfrak{B}^{-1}\psi_{+,B}(x,y+2m\pi i\alpha) \tag{4.20}$$

$$= -\sum_{m=1}^{\infty} \frac{e^{2m\pi i\gamma}+e^{-2m\pi i\gamma}}{2m}e^{2m\pi i\alpha\eta}\psi_+ \tag{4.21}$$

$$= \frac{1}{2}\log\left[(1-e^{2\pi i(\alpha\eta+\gamma)})(1-e^{2\pi i(\alpha\eta-\gamma)})\right]\psi_+. \tag{4.22}$$

Here $\mathfrak{B}$ denotes the Borel transformation. Therefore we obtain

$$\mathfrak{S}\psi_+ = e^{\Delta}\psi_+ = (1-e^{2\pi i(\alpha\eta+\gamma)})^{1/2}(1-e^{2\pi i(\alpha\eta-\gamma)})^{1/2}\psi_+. \tag{4.23}$$

The same computation can be done also for ψ_-. □

§4.3. Borel sum of the Voros coefficient

In this final subsection we determine the Borel sum of the Voros coefficient.

Theorem 4.3. *Let* $\epsilon > 0$ *be sufficiently small. Then the Borel sum of the Voros coefficient* V_B *for* $\arg\alpha = \pi/2 - \varepsilon$ *is given by*

$$\frac{1}{2}\log\frac{\Gamma(\alpha\eta-\gamma)}{\sqrt{2\pi}} + \frac{1}{2}\log\frac{\Gamma(\alpha\eta+\gamma)}{\sqrt{2\pi}} + \frac{1}{2}\log(\alpha\eta+\gamma) + \alpha\eta(1-\log(\alpha\eta)), \tag{4.24}$$

and that for $\arg\alpha = \pi/2 + \varepsilon$ *is*

$$-\frac{1}{2}\log\frac{\Gamma(-\alpha\eta-\gamma)}{\sqrt{2\pi}} - \frac{1}{2}\log\frac{\Gamma(-\alpha\eta+\gamma)}{\sqrt{2\pi}} - \frac{1}{2}\log(-\alpha\eta+\gamma) + \alpha\eta(1-\log(\alpha\eta)+\pi i). \tag{4.25}$$

Proof. Making a change of variables $w = y/\alpha$, we obtain

$$\alpha V_B = \frac{1}{2w}\left[\frac{e^w}{e^w-1}e^{\gamma w} + \frac{1}{e^w-1}e^{-\gamma w}\right] - \frac{1}{w^2} \tag{4.26}$$

$$= \frac{1}{2w}\left(\frac{1}{e^w-1}+\frac{1}{2}-\frac{1}{w}\right)e^{\gamma w} + \frac{1}{2w}\left(\frac{1}{e^w-1}+\frac{1}{2}-\frac{1}{w}\right)e^{-\gamma w} + \frac{\sinh(\gamma w)}{2w} + \frac{\cosh(\gamma w)-1}{w^2}. \tag{4.27}$$

Let $\mathcal{S}[V]$ denote the Borel sum of the Voros coefficient V, i.e.,

$$\mathcal{S}[V](\alpha,\eta) := \int_0^\infty e^{-y\eta}V_B(\alpha,y)dy = \int_0^\infty e^{-\alpha w\eta}(\alpha V_B(\alpha,\alpha w))dw. \tag{4.28}$$

To compute $\mathcal{S}[V](\alpha,\eta)$, we make use of the following formula:

$$\int_0^\infty \left(\frac{1}{e^t-1}+\frac{1}{2}-\frac{1}{t}\right)e^{-t\theta}\frac{dt}{t} = \log\frac{\Gamma(\theta)}{\sqrt{2\pi}} - (\theta-\frac{1}{2})\log\theta+\theta \quad (\Re\theta>0) \tag{4.29}$$

(cf. [E], Vol.I, Chapter I, §1.9, (5)). We also use the following formulas for Laplace transforms of hyperbolic functions:

$$\int_0^\infty e^{-pt}\frac{\sinh(at)}{t}dt = \frac{1}{2}\log\frac{p+a}{p-a}, \tag{4.30}$$

$$\int_0^\infty e^{-pt}\frac{\cosh(at)-1}{t^2}dt = \frac{a}{2}\log\frac{p+a}{p-a} + \frac{p}{2}\log(1-\frac{a^2}{p^2}). \tag{4.31}$$

Then, when $\arg\alpha = \pi/2 - \varepsilon$, we find

$$\mathcal{S}[V](\alpha,\eta) = \frac{1}{2}\left[\frac{\Gamma(\eta\alpha-\gamma)}{\sqrt{2\pi}} - \left(\eta\alpha-\gamma-\frac{1}{2}\right)\log(\eta\alpha-\gamma)+\eta\alpha-\gamma\right] + \frac{1}{2}\left[\frac{\Gamma(\eta\alpha+\gamma)}{\sqrt{2\pi}} - \left(\eta\alpha+\gamma-\frac{1}{2}\right)\log(\eta\alpha+\gamma)+\eta\alpha+\gamma\right] + \frac{1}{4}\log\frac{\eta\alpha+\gamma}{\eta\alpha-\gamma} + \frac{\gamma}{2}\log\frac{\eta\alpha+\gamma}{\eta\alpha-\gamma} + \frac{\eta\alpha}{2}\log\left(1-\frac{\gamma^2}{\eta^2\alpha^2}\right) \tag{4.32}$$

$$= \frac{1}{2}\log\frac{\Gamma(\alpha\eta-\gamma)}{\sqrt{2\pi}} + \frac{1}{2}\log\frac{\Gamma(\alpha\eta+\gamma)}{\sqrt{2\pi}} + \frac{1}{2}\log(\alpha\eta+\gamma) + \alpha\eta(1-\log(\alpha\eta)). \tag{4.33}$$

When $\arg\alpha = \pi/2 - \varepsilon$, we can do the computation of $\mathcal{S}[V](\alpha,\eta)$ in a similar manner by using the relation $\mathcal{S}[V](\alpha,\eta) = -\mathcal{S}[V](-\alpha,\eta)$. □

Using the concrete expressions (4.24) and (4.25) for the Borel sum of V given by Theorem 4.3, we can prove Theorem 4.2 without resorting to the alien calculus. We conclude this final section with providing a different proof of Theorem 4.2 which is based on Theorem 4.3.

Since both $\psi_+^{(\infty)}$ and V are Borel summable for $\arg\alpha = \pi/2 \pm \varepsilon$, it follows from (4.6) that

$$\mathcal{S}[\psi_+]\big|_{\arg\alpha=\pi/2-\varepsilon} = \exp\left[\mathcal{S}[V]\big|_{\arg\alpha=\pi/2-\varepsilon}\right] \cdot \mathcal{S}[\psi_+^{(\infty)}]\big|_{\arg\alpha=\pi/2-\varepsilon}, \tag{4.34}$$

$$\mathcal{S}[\psi_+]\big|_{\arg\alpha=\pi/2+\varepsilon} = \exp\left[\mathcal{S}[V]\big|_{\arg\alpha=\pi/2+\varepsilon}\right] \cdot \mathcal{S}[\psi_+^{(\infty)}]\big|_{\arg\alpha=\pi/2+\varepsilon}, \tag{4.35}$$

where $\mathcal{S}$ denotes the Borel sum. Note that $\psi_+^{(\infty)}$ is Borel summable for $\arg\alpha \in [\pi/2-\varepsilon, \pi/2+\varepsilon]$ for a sufficiently small positive ϵ in view of Lemma 4.1. Thus we have

$$\mathcal{S}[\psi_+^{(\infty)}]\big|_{\arg\alpha=\pi/2+\varepsilon} = \mathcal{S}[\psi_+^{(\infty)}]\big|_{\arg\alpha=\pi/2-\varepsilon}. \tag{4.36}$$

Therefore, we obtain

$$\frac{\mathcal{S}[\psi_+]\big|_{\arg\alpha=\pi/2+\varepsilon}}{\mathcal{S}[\psi_+]\big|_{\arg\alpha=\pi/2-\varepsilon}} = \exp\left[\mathcal{S}[V]\big|_{\arg\alpha=\pi/2+\varepsilon} - \mathcal{S}[V]\big|_{\arg\alpha=\pi/2-\varepsilon}\right]. \tag{4.37}$$

The right-hand side of (4.37) can be explicitly computed thanks to Theorem 4.3:

$$\begin{aligned}
&\mathcal{S}[V]\big|_{\arg\alpha=\pi/2+\varepsilon} - \mathcal{S}[V]\big|_{\arg\alpha=\pi/2-\varepsilon} \\
&\quad = -\frac{1}{2}\log\frac{\Gamma(-\alpha\eta-\gamma)\Gamma(-\alpha\eta+\gamma)\Gamma(\alpha\eta-\gamma)\Gamma(\alpha\eta+\gamma)}{(2\pi)^2} \\
&\qquad\qquad - \frac{1}{2}\log(\gamma^2-\alpha^2\eta^2) + \pi i\alpha\eta.
\end{aligned} \tag{4.38}$$

Since

$$\frac{\Gamma(\alpha\eta-\gamma)\Gamma(-\alpha\eta+\gamma)}{2\pi} = \frac{1}{-\alpha\eta+\gamma}\frac{1}{2\sin\pi(\alpha\eta-\gamma)} \tag{4.39}$$

and

$$\frac{\Gamma(\alpha\eta+\gamma)\Gamma(-\alpha\eta-\gamma)}{2\pi} = \frac{1}{-\alpha\eta-\gamma}\frac{1}{2\sin\pi(\alpha\eta+\gamma)} \tag{4.40}$$

hold, the right-hand side of (4.38) becomes

$$\frac{1}{2}\log\frac{4(\alpha^2\eta^2-\gamma^2)\,\sin\pi(\alpha\eta+\gamma)\,\sin\pi(\alpha\eta-\gamma)}{\gamma^2-\alpha^2\eta^2}+\pi i\alpha\eta \tag{4.41}$$

$$=\frac{1}{2}\log\Big[(e^{i\pi(\alpha\eta+\gamma)}-e^{-i\pi(\alpha\eta+\gamma)})(e^{i\pi(\alpha\eta-\gamma)}-e^{-i\pi(\alpha\eta-\gamma)})\Big]+\pi i\alpha\eta \tag{4.42}$$

$$=\frac{1}{2}\log\Big[(1-e^{2i\pi(\alpha\eta+\gamma)})(1-e^{2i\pi(\alpha\eta-\gamma)})\Big]. \tag{4.43}$$

Hence we obtain

$$\mathcal{S}[\psi_+]\big|_{\arg\alpha=\pi/2+\varepsilon}=(1-e^{2i\pi(\alpha\eta+\gamma)})^{1/2}(1-e^{2i\pi(\alpha\eta-\gamma)})^{1/2}\mathcal{S}[\psi_+]\big|_{\arg\alpha=\pi/2-\varepsilon}. \tag{4.44}$$

This proves Theorem 4.2.

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