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Abstract

The turning point problems for instanton-type solutions of Painlevé equations with a large parameter are discussed. Generalizing the main result of [KT2] near a simple turning point, we report in this paper that Painlevé equations can be transformed to the second Painlevé equation and the most degenerate third Painlevé equation near a double turning point and near a simple pole, respectively. An outline of the proof based on the theory of isomonodromic deformations of associated linear differential equations is also explained.

1 Background and main results

The purpose of this report is to discuss the turning point problem for instanton-type solutions of Painlevé equations from the viewpoint of exact WKB analysis.

In our series of papers ([KT1],[AKT],[KT2]) we develop the exact WKB analysis of Painlevé equations (P_J) with a large parameter η (> 0):

$$(P_J) \qquad \frac{d^2\lambda}{dt^2} = G_J\left(\lambda, \frac{d\lambda}{dt}, t\right) + \eta^2 F_J(\lambda, t).$$

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Here J runs over the following set of indices

$$\mathcal{I} = \{\, \mathrm{I},\ \mathrm{II},\ \mathrm{III}',\ \mathrm{III}'(\mathrm{D7}),\ \mathrm{III}'(\mathrm{D8}),\ \mathrm{IV},\ \mathrm{V},\ \mathrm{VI}\,\}, \tag{1.1}$$

and $F_J(\lambda, t)$ and $G_J(\lambda, \mu, t)$ are some rational functions of (λ, t) and (λ, μ, t), respectively. For the concrete form of $F_J(\lambda, t)$ and $G_J(\lambda, \mu, t)$ see Table 1 below. Note that instead of the usual third Painlevé equation (P_{III}) we use $(P_{\mathrm{III}'})$, which is equivalent to (P_{III}), for the sake of convenience in this paper. Note also that it is now considered to be natural to distinguish the degenerate third Painlevé equations of type $(D7)$ and $(D8)$ from the generic third Painlevé equation since the type of their affine Weyl group symmetries is different from that of the generic third Painlevé equation. In this paper, being conformed to this convention, we have listed up $(P_{\mathrm{III}'(\mathrm{D7})})$ and $(P_{\mathrm{III}'(\mathrm{D8})})$ as well in Table 1. These Painlevé equations (P_J) are related to one another according to the so-called coalescence diagram described in Table 2.

As can be readily confirmed, every Painlevé equation (P_J) $(J \in \mathcal{I})$ admits the following formal power series solution (in η^{-1}) called a "0-parameter solution":

$$\lambda_J^{(0)}(t, \eta) = \lambda_0(t) + \eta^{-2}\lambda_2(t) + \eta^{-4}\lambda_4(t) + \cdots, \tag{1.2}$$

where the top term $\lambda_0(t)$ satisfies an algebraic equation

$$F_J(\lambda_0(t), t) = 0 \tag{1.3}$$

and the other terms $\lambda_{2j}(t)$ $(j \geq 1)$ are recursively determined once $\lambda_0(t)$ is fixed. Furthermore, by using the multiple-scale method, we have constructed in [AKT] the following formal solution of (P_J), called a "2-parameter solution" or an "instanton-type solution", containing 2 free complex parameters (α, β):

$$\lambda_J(t, \eta; \alpha, \beta) = \lambda_0(t) + \eta^{-1/2}\lambda_{1/2}(t, \eta) + \eta^{-1}\lambda_1(t, \eta) + \cdots. \tag{1.4}$$

Here the leading term $\lambda_0(t)$ is the same as that of a 0-parameter solution and the other terms $\lambda_{j/2}(t, \eta)$ $(j \geq 1)$ are of the form

$$\lambda_{j/2}(t, \eta) = \sum_{k=0}^{j} b_{j-2k}^{(j/2)}(t) \exp\left((j - 2k)\Phi_J\right), \tag{1.5}$$

$$(P_{\rm I}) \quad \frac{d^2\lambda}{dt^2} = \eta^2(6\lambda^2 + t).$$

$$(P_{\rm II}) \quad \frac{d^2\lambda}{dt^2} = \eta^2(2\lambda^3 + t\lambda + c).$$

$$(P_{\rm III'}) \quad \frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \eta^2\left[\frac{c_\infty\lambda^3}{t^2} - \frac{c'_\infty\lambda^2}{t^2} + \frac{c'_0}{t} - \frac{c_0}{\lambda}\right].$$

$$(P_{\rm III'(D7)}) \quad \frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} - \eta^2\left[\frac{2\lambda^2}{t^2} + \frac{c}{t} + \frac{1}{\lambda}\right].$$

$$(P_{\rm III'(D8)}) \quad \frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \eta^2\left[\frac{\lambda^2}{t^2} - \frac{1}{t}\right].$$

$$(P_{\rm IV}) \quad \frac{d^2\lambda}{dt^2} = \frac{1}{2\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{2}{\lambda} + \eta^2\left[\frac{3}{2}\lambda^3 + 4t\lambda^2 + \left(2t^2 + c_1\right)\lambda - \frac{c_0}{\lambda}\right].$$

$$(P_{\rm V}) \quad \frac{d^2\lambda}{dt^2} = \left(\frac{1}{2\lambda} + \frac{1}{\lambda - 1}\right)\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{(\lambda-1)^2}{t^2}\left(2\lambda - \frac{1}{2\lambda}\right)$$
$$+ \eta^2\frac{2\lambda(\lambda-1)^2}{t^2}\left[c_\infty - \frac{c_0}{\lambda^2} - \frac{c_2 t}{(\lambda-1)^2} - \frac{c_1 t^2(\lambda+1)}{(\lambda-1)^3}\right].$$

$$(P_{\rm VI}) \quad \frac{d^2\lambda}{dt^2} = \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\frac{d\lambda}{dt}$$
$$+ \frac{2\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[1 - \frac{\lambda^2 - 2t\lambda + t}{4\lambda^2(\lambda-1)^2}\right.$$
$$\left. + \eta^2\left\{c_\infty - \frac{c_0 t}{\lambda^2} + \frac{c_1(t-1)}{(\lambda-1)^2} - \frac{c_t t(t-1)}{(\lambda-t)^2}\right\}\right].$$

Table 1: Painlevé equations with a large parameter η. Here λ denotes an unknown function, t is an independent variable and c, c_0 etc. are complex parameters.

$$
\begin{array}{ccccccccc}
(P_{\mathrm{VI}}) & \longrightarrow & (P_{\mathrm{V}}) & \longrightarrow & (P_{\mathrm{III'}}) & \longrightarrow & (P_{\mathrm{III'(D7)}}) & \longrightarrow & (P_{\mathrm{III'(D8)}}) \\
 & & & \searrow & & \searrow & & & \\
 & & & & (P_{\mathrm{IV}}) & \longrightarrow & (P_{\mathrm{II}}) & \longrightarrow & (P_{\mathrm{I}})
\end{array}
$$

Table 2: Coalescence diagram of Painlevé equations.

where $\Phi_J = \Phi_J(t,\eta)$, sometimes called an "instanton", is defined by

$$\Phi_J(t,\eta) = \eta \int^t \sqrt{\frac{\partial F_J}{\partial \lambda}(\lambda_0(s),s)}ds + \alpha\beta \log\left(\eta^2 \theta_J(t)\right) \tag{1.6}$$

with an appropriately defined function $\theta_J(t)$ and $b_l^{(j/2)}(t)$ $(l = j, j-2, \ldots, -j)$ are functions of t depending also on α and β but not on η, that is, the η-dependence of $\lambda_{j/2}(t,\eta)$ comes only from the l-instanton terms $\exp(l\Phi_J)$. In particular, the subleading term $\lambda_{1/2}(t,\eta)$ is of the form

$$\lambda_{1/2}(t,\eta) = \mu_J(t)\left(\alpha \exp(\Phi_J) + \beta \exp(-\Phi_J)\right). \tag{1.7}$$

For the explicit forms of $\theta_J(t)$ and $\mu_J(t)$ we refer the reader to [KT2, Section 1].

The subject of our series of papers ([KT1],[AKT],[KT2]) is the analysis of the structure of $\lambda_J^{(0)}(t,\eta)$ and $\lambda_J(t,\eta;\alpha,\beta)$ near a simple turning point of (P_J). Here a (simple) turning point of (P_J) is defined as follows:

Definition 1.1. Let

$$(\Delta P_J) \quad \frac{d^2}{dt^2}\Delta\lambda = \eta^2 \frac{\partial F_J}{\partial \lambda}(\lambda_J^{(0)},t)\Delta\lambda + \frac{\partial G_J}{\partial \lambda}\left(\lambda_J^{(0)}, \frac{d\lambda_J^{(0)}}{dt}, t\right)\Delta\lambda + \frac{\partial G_J}{\partial \mu}\left(\lambda_J^{(0)}, \frac{d\lambda_J^{(0)}}{dt}, t\right)\frac{d}{dt}\Delta\lambda$$

be the linearized equation (or the Frechét derivative) of (P_J) at its 0-parameter solution $\lambda_J^{(0)}$. Then a turning point of (P_J) is, by definition, a turning point of (ΔP_J). That is, a turning point of (P_J) is a zero of $(\partial F_J/\partial\lambda)(\lambda_0(t),t)$. In particular, a point t satisfying

$$\frac{\partial F_J}{\partial \lambda}(\lambda_0(t),t) = 0 \quad \text{and} \quad \frac{\partial^2 F_J}{\partial \lambda^2}(\lambda_0(t),t) \neq 0 \tag{1.8}$$

is called a simple turning point. Similarly, a Stokes curve (P_J) is defined as a Stokes curve of (ΔP_J), that is, a Stokes curve (P_J) is an integral curve of the direction field $\mathrm{Im}\sqrt{(\partial F_J/\partial\lambda)(\lambda_0(t),t)}\,dt = 0$ emanating from a turning point.

The main result of [KT2] is then described in the following Theorem 1.2 (where we put $\tilde{}$ to the variables relevant to (P_J) to distinguish them from those relevant to the first Painlevé equation (P_{I})).

Theorem 1.2. *Let $\tilde{t} = \tilde{t}_*$ be a simple turning point of (P_J) and $\tilde{\sigma}$ a point on a Stokes curve emanating from $\tilde{t}_*$. Then there exists a neighborhood $\tilde{V}$ of $\tilde{\sigma}$ so that every 2-parameter instanton-type solution $\tilde{\lambda}_J(\tilde{t},\eta;\tilde{\alpha},\tilde{\beta})$ of (P_J) is formally transformed to a 2-parameter instanton-type solution $\lambda_{\mathrm{I}}(t,\eta;\alpha,\beta)$ of (P_{I}) in $\tilde{V}$. To be more specific, there exist a formal transformation $t(\tilde{t},\eta)$ of an independent variable and a formal transformation $x(\tilde{x},\tilde{t},\eta)$ of an unknown function of the form*

$$t(\tilde{t},\eta) = \sum_{j\geq 0} t_{j/2}(\tilde{t},\eta)\eta^{-j/2}, \tag{1.9}$$

$$x(\tilde{x},\tilde{t},\eta) = \sum_{j\geq 0} x_{j/2}(\tilde{x},\tilde{t},\eta)\eta^{-j/2}, \tag{1.10}$$

where $t_{j/2}$ and $x_{j/2}$ are holomorphic in both $\tilde{x}$ and $\tilde{t}$, that satisfy the following relation:

$$x(\tilde{\lambda}_J(\tilde{t},\eta;\tilde{\alpha},\tilde{\beta}),\tilde{t},\eta) = \lambda_{\mathrm{I}}(t(\tilde{t},\eta),\eta;\alpha,\beta). \tag{1.11}$$

Theorem 1.2 implies that the first Painlevé equation (P_{I}) can be thought of as a canonical equation (or a normal form) near a simple turning point of Painlevé equations (P_J). For instanton-type solutions of (P_{I}) we have the following connection formula on its Stokes curve, say, on $\{\arg t = \pi\}$ (cf. [T1]):

$$\frac{\beta'\ 2^{2\alpha'\beta'}}{\Gamma(2\alpha'\beta'+1)} = \frac{\beta\ 2^{2\alpha\beta}}{\Gamma(2\alpha\beta+1)}, \tag{1.12}$$

$$e^{2i\pi\alpha'\beta'}\frac{\alpha'\ 2^{-2\alpha'\beta'}}{\Gamma(-2\alpha'\beta'+1)} = e^{2i\pi\alpha\beta}\frac{\alpha\ 2^{-2\alpha\beta}}{\Gamma(-2\alpha\beta+1)} - ie^{4i\pi\alpha\beta}, \tag{1.13}$$

where $\lambda_{\mathrm{I}}(t,\eta;\alpha,\beta)$ (resp., $\lambda_{\mathrm{I}}(t,\eta;\alpha',\beta')$) is an instanton-type solution of (P_{I}) in $\{\arg t < \pi\}$ (resp., $\{\arg t > \pi\}$). In particular, the analytic continuation across the Stokes curve $\{\arg t = \pi\}$ of a 0-parameter solution $\lambda_{\rm I}^{(0)}(t,\eta) = \lambda_{\rm I}(t,\eta;0,0)$ in $\{\arg t < \pi\}$ is given by $\lambda_{\rm I}(t,\eta;-i/(2\sqrt{\pi}),0)$ in $\{\arg t > \pi\}$. In view of Theorem 1.2 it is expected that the same connection formula as (1.12) and (1.13) should hold also for an instanton-type solution of (P_J) on its Stokes curve emanating from a simple turning point.

The aim of this report is to discuss some generalizations of Theorem 1.2. Now, **what kind of generalizations of Theorem 1.2 is possible?** To consider possible generalizations of Theorem 1.2, we first briefly review a simpler case, that is, the case of second order linear ordinary differential equations

$$\left(-\frac{d^2}{dx^2} + \eta^2 Q(x)\right)\psi = 0. \tag{1.14}$$

It is well-known that at a simple turning point Equation (1.14) can be transformed into the Airy equation (i.e., Equation (1.14) with $Q(x) = x$). In fact, such a transformation is constructed in the framework of exact WKB analysis as well (cf. [KT3, Chapter 2]) and Theorem 1.2 can be regarded as a nonlinear analogue of this result. For linear equations (1.14) several generalizations of this result are also known. For example, at a double turning point (i.e., a double zero of $Q(x)$) (1.14) can be transformed into the Weber equation (i.e., Equation (1.14) with $Q(x) = x^2 + \eta^{-1}E$ with some constant E). Furthermore at a simple pole of $Q(x)$ (1.14) is transformed into the Whittaker equation (i.e., Equation (1.14) with $Q(x) = 1/x + \eta^{-2}\gamma/x^2$ with some constant γ). This fact means that a simple pole of $Q(x)$ also plays a role of turning points for Equation (1.14) and in the framework of exact WKB analysis this fact is verified by Koike in [K].

In parallel to the case of linear equations (1.14) we are then able to consider some generalizations of Theorem 1.2 for Painlevé equations, that is, generalizations to a transformation near a double turning point and that near a simple pole. First, near a double turning point, we can prove the following

Theorem 1.3. *Near a double turning point every 2-parameter instanton-type solution of (P_J) is formally transformed to that of the following second Painlevé equation $(P_{\rm II,deg})$ (in the same sense as in Theorem 1.2):*

$$\frac{d^2\lambda}{dt^2} = \eta^2(2\lambda^3 + t\lambda + \eta^{-1}c). \tag{$P_{\rm II,deg}$}$$

Note that $(P_{\mathrm{II,deg}})$ is different from the ordinary second Painlevé equation (P_{II}) in that the parameter c is multiplied by η^1 in $(P_{\mathrm{II,deg}})$ (while it is multiplied by η^2 in (P_{II})). Next, near a simple pole, we have

Theorem 1.4. *Near a simple pole every* 2*-parameter instanton-type solution of* (P_J) *is formally transformed to that of the most degenerate third Painlevé equation* $(P_{\mathrm{III}'(\mathrm{D8})})$ (*in the same sense as in Theorem 1.2*).

Theorem 1.3 and Theorem 1.4 are the main results of this report. Their precise statements will be given below in Theorem 2.5 and Theorem 3.1, respectively. Theorem 1.3 has been announced also in [T2].

The plan of this report is as follows: In Section 2, after discussing the exact WKB theoretic structure of $(P_{\mathrm{II,deg}})$, we give the definition of a double turning point of (P_J) and explain an outline of the proof of Theorem 1.3. A key idea is to use the relationship between Painlevé equations and the theory of isomonodromic deformations of the associated linear differential equations. Then in Section 3 we review the discussion of [T2], that is, we consider the transformation near a simple pole in a way parallel to Section 2. The details will be discussed in our forthcoming paper(s).

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2 Transformation near a double turning point

2.1 Exact WKB theoretic structure of $(P_{\mathrm{II,deg}})$

In this section we consider transformation near a double turning point. We first investigate the exact WKB theoretic structure of the canonical equation

$$(P_{\mathrm{II,deg}}) \qquad \frac{d^2\lambda}{dt^2} = \eta^2(2\lambda^3 + t\lambda + \eta^{-1}c).$$

As was explained in Section 1, the top term $\lambda_0 = \lambda_0(t)$ of the 0-parameter solution

$$(2.1) \qquad \lambda^{(0)}_{\mathrm{II,deg}}(t,\eta) = \lambda_0(t) + \eta^{-1}\lambda_1(t) + \eta^{-2}\lambda_2(t) + \cdots$$

of $(P_{\mathrm{II,deg}})$ is determined by an algebraic equation

$$F_{\mathrm{II,deg}}(\lambda_0, t) = 2\lambda_0^3 + t\lambda_0 = \lambda_0(2\lambda_0^2 + t) = 0. \tag{2.2}$$

Among the solutions of (2.2) we pick up a solution of $2\lambda_0^2 + t = 0$, i.e.,

$$\lambda_0(t) = \sqrt{-\frac{t}{2}} \quad \left(\text{or} \quad -\sqrt{-\frac{t}{2}}\right), \tag{2.3}$$

when we consider a double turning point. Note that, as $(P_{\mathrm{II,deg}})$ contains an odd order term ηc (with respect to η), $\lambda^{(0)}_{\mathrm{II,deg}}(t,\eta)$ also contains odd order terms $\lambda_1(t)$, $\lambda_3(t)$, ... Then, starting with this top term $\lambda_0(t)$ given by (2.3), we can construct a 2-parameter instanton-type solution of $(P_{\mathrm{II,deg}})$ of the form

$$\begin{aligned}(2.4)\ \lambda_{\mathrm{II,deg}}&(t,\eta;\alpha,\beta)\\ &= \lambda_0(t) + \eta^{-1/2}(6\lambda_0^2+t)^{-1/4}\big(\alpha\exp(\Phi_{\mathrm{II,deg}}) + \beta\exp(-\Phi_{\mathrm{II,deg}})\big) + \cdots\end{aligned}$$

with

$$\Phi_{\mathrm{II,deg}}(t,\eta) = \eta\int_0^t \sqrt{6\lambda_0^2 + s}ds + (2\alpha\beta + c/2)\log\left(\eta^2(6\lambda_0^2+t)^3\right) \tag{2.5}$$

by employing the multiple-scale method. (Since $(P_{\mathrm{II,deg}})$ contains an odd order term ηc, the form of the instanton $\Phi_{\mathrm{II,deg}}$ is slightly different from the general form (1.6) of Φ_J.)

The linearized equation of $(P_{\mathrm{II,deg}})$ at a 0-parameter solution (2.1) is given by

$$\frac{d^2}{dt^2}\Delta\lambda = \eta^2\left(6(\lambda^{(0)}_{\mathrm{II,deg}})^2 + t\right)\Delta\lambda = \eta^2\left(-2t + O(\eta^{-1})\right)\Delta\lambda. \tag{$\Delta P_{\mathrm{II,deg}}$}$$

Hence $(P_{\mathrm{II,deg}})$ has a unique turning point at $t = 0$. Note that this turning point $t = 0$ is also an algebraic branch point of the Riemann surface of $\lambda_0(t)$. The Stokes curves of $(P_{\mathrm{II,deg}})$, i.e., integral curves of the direction field $\mathrm{Im}\sqrt{6\lambda_0^2 + t}\,dt = \mathrm{Im}\sqrt{-2t}\,dt = 0$ emanating from the turning point $t = 0$, thus consist of the following three lines:

$$\{t \in \mathbb{C} \mid \arg t = \pi + 2n\pi/3\ (n \in \mathbb{Z})\}. \tag{2.6}$$

It is expected that a Stokes phenomenon should be observed on each Stokes curve for instanton-type solutions $\lambda_{\mathrm{II,deg}}(t,\eta;\alpha,\beta)$. To analyze the Stokes phenomenon, we make use of the well-known relationship between the Painlevé equation and the theory of isomonodromic deformations of the associated linear differential equation (cf. [O],[JMU]). In the case of $(P_{\mathrm{II,deg}})$ the relationship is formulated as follows: Let $(SL_{\mathrm{II,deg}})$ and $(D_{\mathrm{II,deg}})$ be the following linear differential equations, respectively.

$$(SL_{\mathrm{II,deg}}) \qquad \left(-\frac{\partial^2}{\partial x^2} + \eta^2 Q_{\mathrm{II,deg}}\right)\psi = 0,$$

$$(D_{\mathrm{II,deg}}) \qquad \frac{\partial \psi}{\partial t} = A_{\mathrm{II,deg}}\frac{\partial \psi}{\partial x} - \frac{1}{2}\frac{\partial A_{\mathrm{II,deg}}}{\partial x}\psi,$$

where

$$Q_{\mathrm{II,deg}} = x^4 + tx^2 + 2\eta^{-1}cx + 2K_{\mathrm{II,deg}} - \eta^{-1}\frac{\nu}{x-\lambda} + \eta^{-2}\frac{3}{4(x-\lambda)^2}, \tag{2.7}$$

$$A_{\mathrm{II,deg}} = \frac{1}{2(x-\lambda)}, \tag{2.8}$$

with

$$K_{\mathrm{II,deg}} = \frac{1}{2}\left[\nu^2 - (\lambda^4 + t\lambda^2 + 2\eta^{-1}c\lambda)\right]. \tag{2.9}$$

Then the compatibility condition of $(SL_{\mathrm{II,deg}})$ and $(D_{\mathrm{II,deg}})$ is represented by the Hamiltonian system

$$(H_{\mathrm{II,deg}}) \qquad \frac{d\lambda}{dt} = \eta\frac{\partial K_{\mathrm{II,deg}}}{\partial \nu}, \quad \frac{d\nu}{dt} = -\eta\frac{\partial K_{\mathrm{II,deg}}}{\partial \lambda},$$

which is equivalent to the second order differential equation $(P_{\mathrm{II,deg}})$ for λ. As its consequence, we find that the monodromy data of $(SL_{\mathrm{II,deg}})$ should be independent of the deformation parameter t if a solution of $(H_{\mathrm{II,deg}})$ or $(P_{\mathrm{II,deg}})$ is substituted into the coefficients of $(SL_{\mathrm{II,deg}})$.

To determine the connection formula which describes the Stokes phenomenon for $\lambda_{\mathrm{II,deg}}(t,\eta;\alpha,\beta)$ on a Stokes curve of $(P_{\mathrm{II,deg}})$, we then substitute $\lambda_{\mathrm{II,deg}}(t,\eta;\alpha,\beta)$ into the coefficients of $(SL_{\mathrm{II,deg}})$ and compute its monodromy data by employing the exact WKB analysis. The following is a key proposition in executing the computation of the monodromy data.

Proposition 2.1. *If an instanton-type solution of* $(H_{\mathrm{II,deg}})$ *or* $(P_{\mathrm{II,deg}})$ *is substituted into the coefficients of* $(SL_{\mathrm{II,deg}})$*, the following hold:*
(i) *The top term (with respect to* η^{-1}*)* Q_0 *of the potential* $Q_{\mathrm{II,deg}}$ *of* $(SL_{\mathrm{II,deg}})$ *is factorized as*

$$Q_0 = (x - \lambda_0(t))^2 (x + \lambda_0(t))^2. \tag{2.10}$$

That is, $(SL_{\mathrm{II,deg}})$ *has two double turning points* $x = \lambda_0(t)$ *and* $x = -\lambda_0(t)$.
(ii) *When* t *lies on a Stokes curve* (2.6) *of* $(P_{\mathrm{II,deg}})$*, there exists a Stokes curve of* $(SL_{\mathrm{II,deg}})$ *that connects the two double turning points* $x = \pm\lambda_0(t)$ *of* $(SL_{\mathrm{II,deg}})$. (*Cf.* Figure 1, (ii)*, where the configuration of Stokes curves is shown when* t *lies on a Stokes curve* $\arg t = \pi$.)

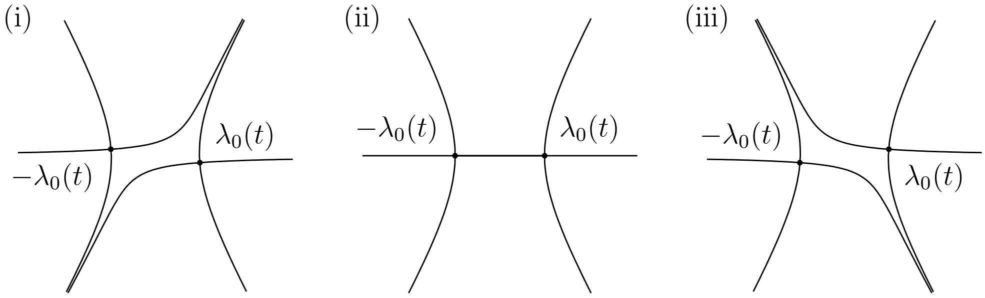

Figure 1: Configuration of Stokes curves of $(SL_{\mathrm{II,deg}})$ in the case of (i) $\arg t < \pi$, (ii) $\arg t = \pi$, and (iii) $\arg t > \pi$.

Proposition 2.1, (ii) implies that a change of the configuration of Stokes curves of $(SL_{\mathrm{II,deg}})$ is observed on each Stokes curve of $(P_{\mathrm{II,deg}})$. For example, the change on a Stokes curve $\arg t = \pi$ is visualized in Figure 1. This change of the configuration causes a Stokes phenomenon for $\lambda_{\mathrm{II,deg}}(t, \eta; \alpha, \beta)$ to occur on a Stokes curve of $(P_{\mathrm{II,deg}})$. As a matter of fact, by substituting an instanton-type solution $\lambda_{\mathrm{II,deg}}(t, \eta; \alpha, \beta)$ into the coefficients of $(SL_{\mathrm{II,deg}})$ and employing the exact WKB analysis for linear equations, we obtain the following

Proposition 2.2. *Suppose that an instanton-type solution of* $(P_{\mathrm{II,deg}})$ *is substituted into the coefficients of* $(SL_{\mathrm{II,deg}})$*. Let* $m_1^{(\pm)}$ *and* $m_2^{(\pm)}$ *be two independent monodromy data (i.e., Stokes multipliers around* $x = \infty$ *in this case) of*

$(SL_{\mathrm{II,deg}})$ *when* t *belongs to the region* $\Omega_\pm = \{t \mid \pm(\arg t - \pi) > 0\}$, *respectively. Then* $(m_1^{(\pm)}, m_2^{(\pm)})$ *can be explicitly computed as follows:*

(*When* $t \in \Omega_+$)

$$m_1^{(+)} = -2\sqrt{\pi}\frac{i\beta\ 2^{2\alpha\beta}}{\Gamma(2\alpha\beta+1)}, \tag{2.11}$$

$$m_2^{(+)} = -2\sqrt{\pi}\ e^{2i\pi\alpha\beta}\frac{\alpha\ 2^{-2\alpha\beta}}{\Gamma(-2\alpha\beta+1)}. \tag{2.12}$$

(*When* $t \in \Omega_-$)

$$m_1^{(-)} = -2\sqrt{\pi}\frac{i\beta\ 2^{2\alpha\beta}}{\Gamma(2\alpha\beta+1)}, \tag{2.13}$$

$$m_2^{(-)} = -2\sqrt{\pi}\left(e^{2i\pi\alpha\beta}\frac{\alpha\ 2^{-2\alpha\beta}}{\Gamma(-2\alpha\beta+1)} - e^{4i\pi\alpha\beta}\frac{i\ 2^{c+2\alpha\beta-1/2}}{\Gamma(c+2\alpha\beta-1/2)}\right). \tag{2.14}$$

Since the computation of monodromy data through the exact WKB analysis heavily depends on the configuration of Stokes curves, the concrete expression of $(m_1^{(+)}, m_2^{(+)})$ becomes different from that of $(m_1^{(-)}, m_2^{(-)})$, as one can readily see in Proposition 2.2, due to the difference of two configurations of Stokes curves shown in Figure 1. On the other hand, thanks to the isomonodromic property, the monodromy data should coincide if two instanton-type solutions in the two regions $\Omega_\pm$ correspond to the same analytic solution. Thus, if an instanton-type solution $\lambda_{\mathrm{II,deg}}(t,\eta;\alpha,\beta)$ in Ω_- corresponds to the same analytic solution with $\lambda_{\mathrm{II,deg}}(t,\eta;\alpha',\beta')$ in Ω_+, we obtain the following relation in view of Proposition 2.2:

$$\frac{\beta'\ 2^{2\alpha'\beta'}}{\Gamma(2\alpha'\beta'+1)} = \frac{\beta\ 2^{2\alpha\beta}}{\Gamma(2\alpha\beta+1)}, \tag{2.15}$$

$$\begin{aligned} &e^{2i\pi\alpha'\beta'}\frac{\alpha'\ 2^{-2\alpha'\beta'}}{\Gamma(-2\alpha'\beta'+1)} \\ &\qquad = e^{2i\pi\alpha\beta}\frac{\alpha\ 2^{-2\alpha\beta}}{\Gamma(-2\alpha\beta+1)} - e^{4i\pi\alpha\beta}\frac{i\ 2^{c+2\alpha\beta-1/2}}{\Gamma(c+2\alpha\beta-1/2)}. \end{aligned} \tag{2.16}$$

In particular, the 0-parameter solution $\lambda^{(0)}_{\mathrm{II,deg}}(t,\eta) = \lambda_{\mathrm{II,deg}}(t,\eta;0,0)$ in the region Ω_- should be analytically continued to $\lambda_{\mathrm{II,deg}}(t,\eta;-i2^{c-1/2}/\Gamma(c+1/2),0)$

in Ω_+ across the Stokes curve $\{\arg t = \pi\}$. This is the mechanism for a Stokes phenomenon to occur for instanton-type solutions of $(P_{\mathrm{II,deg}})$ on its Stokes curve. The above relations (2.15) and (2.16) describe the connection formula on $\{\arg t = \pi\}$.

2.2 Transformation theory to $(P_{\mathrm{II,deg}})$ near a double turning point

As we have observed in Subsection 2.1, in the case of $(P_{\mathrm{II,deg}})$ its linearized equation $(\Delta P_{\mathrm{II,deg}})$ has a unique turning point at $t = 0$ and three Stokes curves emanate from there. We call this kind of turning points a double turning point in general. To be more specific, we define a double turning point of (P_J) as follows:

Definition 2.3. A turning point $t = \tau_d$ of (P_J) is said to be a double turning point if the following two conditions are satisfied.
(i) $t = \tau_d$ is an algebraic branch point of the Riemann surface of $\lambda_0(t)$.
(ii) Near $t = \tau_d$, $(\partial F_J/\partial\lambda)(\lambda_0(t), t)$ has a simple zero, that is,

$$\frac{\partial F_J}{\partial \lambda}(\lambda_0(t), t) = c(t - \tau_d) + \cdots \tag{2.17}$$

holds with a non-zero constant c.

In particular, from each double turning point of (P_J) three Stokes curves emanate thanks to the condition (2.17). Note that at a simple turning point of (P_J) $(\partial F_J/\partial\lambda)(\lambda_0(t), t)$ has a square-root branch point (and hence the condition (i) is automatically satisfied there). Thus a double turning point is a turning point more degenerate than a simple turning point.

A Painlevé equation (P_J) does not always have a double turning point. For example, the first Painlevé equation (P_{I}) has a unique turning point at the origin $t = 0$ which is simple. Similarly, the degenerate third Painlevé equation $(P_{\mathrm{III}'(D7)})$ or $(P_{\mathrm{III}'(D8)})$ does not possess any double turning point. In order that (P_J) may have a double turning point, the parameters contained in (P_J) should satisfy some algebraic condition, the explicit form of which is described in the following

Proposition 2.4. (i) (P_{I}), $(P_{\mathrm{III}'(D7)})$ *and* $(P_{\mathrm{III}'(D8)})$ *have no double turning points.*
(ii) *A double turning point appears for a Painlevé equation* (P_J) $(J =$

$\mathrm{II}, \mathrm{III}', \mathrm{IV}, \mathrm{V}, \mathrm{VI}$) *if and only if the parameters contained in* (P_J) *should satisfy the following relations:*

$$c = 0 \qquad \text{for } J = \mathrm{II}, \tag{2.18}$$

$$c_0(c'_\infty)^2 - c_\infty(c'_0)^2 = 0 \qquad \text{for } J = \mathrm{III}', \tag{2.19}$$

$$2c_0 - c_1^2 = 0 \qquad \text{for } J = \mathrm{IV}, \tag{2.20}$$

$$16c_1^2c_\infty^2 - 8c_0c_1c_2^2 - 8c_\infty c_1c_2^2 + c_2^4 = 0 \qquad \text{for } J = \mathrm{V}, \tag{2.21}$$

$$\begin{aligned} &16(c_0^2c_1^2 + c_1^2c_t^2 + c_t^2c_0^2) - 32(c_0^2c_1c_t + c_0c_1^2c_t + c_0c_1c_t^2) \\ &\quad -64c_0c_1c_t\tilde{c}_\infty - 8(c_0c_1 + c_1c_t + c_tc_0)\tilde{c}_\infty^2 + \tilde{c}_\infty^4 = 0 \qquad \text{for } J = \mathrm{VI}, \end{aligned} \tag{2.22}$$

where $\tilde{c}_\infty = c_\infty - (c_0 + c_1 + c_t)$ *in the case of* $J = \mathrm{VI}$.

Throught this subsection we assume that the conditions (2.18) $\sim$ (2.22) listed in Proposition 2.4, (ii) are satisfied. The problem we want to discuss is to develop transformation theory near a double turning point. Let $t = \tau_d$ be a double turning point of (P_J) ($J = \mathrm{II}, \mathrm{III}', \mathrm{IV}, \mathrm{V}, \mathrm{VI}$). Generalizing the transformation theory (Theorem 1.2) near a simple turning point, we can then prove the following theorem which claims that every 2-parameter instanton-type solution of (P_J) is transformed to that of $(P_{\mathrm{II,deg}})$ near $t = \tau_d$. (In stating Theorem 2.5, we put $\tilde{}$ to the variables relevant to (P_J) to distinguish them from those relevant to $(P_{\mathrm{II,deg}})$.)

Theorem 2.5. *Suppose that the conditions* (2.18) $\sim$ (2.22) *are satisfied. Let* $\tilde{t} = \tilde{\tau}_d$ *be a double turning point of* (P_J) ($J = \mathrm{II}, \mathrm{III}', \mathrm{IV}, \mathrm{V}, \mathrm{VI}$) *and* $\tilde{\sigma}$ *be a point on a Stokes curve emanating from* $\tilde{\tau}_d$*. Then we can find a neighborhood* $\tilde{V}$ *of* $\tilde{\sigma}$ *and a formal power series of* η^{-1} *with constant coefficients*

$$c(\eta) = c_0 + \eta^{-1}c_1 + \eta^{-2}c_2 + \cdots \tag{2.23}$$

such that in $\tilde{V}$ *every* 2*-parameter instanton-type solution* $\tilde{\lambda}_J(\tilde{t}, \eta; \tilde{\alpha}, \tilde{\beta})$ *of* (P_J) *is formally transformed to a* 2*-parameter instanton-type solution* $\lambda_{\mathrm{II,deg}}(t, \eta; \alpha, \beta)$ *of the degenerate second Painlevé equation*

$$\frac{d^2\lambda}{dt^2} = \eta^2(2\lambda^3 + t\lambda + \eta^{-1}c(\eta)) \tag{2.24}$$

with the infinite series $c(\eta)$ *of* (2.23) *being substituted into its coefficient. To be more specific, there exist a formal transformation* $t = t(\tilde{t}, \eta)$ *of an independent variable and a formal transformation* $x = x(\tilde{x}, \tilde{t}, \eta)$ *of an unknown* *function of the form*

$$t(\tilde{t},\eta) = \sum_{j\geq 0} \eta^{-j/2} t_{j/2}(\tilde{t},\eta), \tag{2.25}$$

$$x(\tilde{x},\tilde{t},\eta) = \sum_{j\geq 0} \eta^{-j/2} x_{j/2}(\tilde{x},\tilde{t},\eta), \tag{2.26}$$

where $t_{j/2}$ *and* $x_{j/2}$ *are holomorphic in both* $\tilde{x}$ *and* $\tilde{t}$*, that satisfy the following relation:*

$$x(\tilde{\lambda}_J(\tilde{t},\eta;\tilde{\alpha},\tilde{\beta}),\tilde{t},\eta) = \lambda_{\mathrm{II,deg}}(t(\tilde{t},\eta),\eta;\alpha,\beta). \tag{2.27}$$

Hence the (degenerate) second Painlevé equation ($P_{\mathrm{II,deg}}$) can be regarded as the canonical equation of Painlevé equations near a double turning point. Theorem 2.5 suggests that the connection formula (2.15) and (2.16) for ($P_{\mathrm{II,deg}}$) described in section 2.1 should hold also for an instanton-type solution of (P_J) on its Stokes curve emanating from a double turning point.

Let us explain an outline of the construction of the transformations $t(\tilde{t},\eta)$ and $x(\tilde{x},\tilde{t},\eta)$. It is done in a parallel way to the transformation theory near a simple turning point; we again make use of the relationship between Painlevé equations and the theory of isomonodromic deformations of linear differential equations, that is, we use the fact that (P_J) is equivalent to the compatibility condition of a system of linear differential equations

$$\left(-\frac{\partial^2}{\partial x^2} + \eta^2 Q_J\right)\psi = 0, \tag{SL_J}$$

$$\frac{\partial \psi}{\partial t} = A_J \frac{\partial \psi}{\partial x} - \frac{1}{2}\frac{\partial A_J}{\partial x}\psi. \tag{D_J}$$

(See [KT1] or [KT3, Chapter 4] for the concrete form of Q_J and A_J.) A key proposition in constructing the transformations is then the following Proposition 2.6, which is a generalization of Proposition 2.1 to (P_J).

Proposition 2.6. *Suppose that the conditions* (2.18) $\sim$ (2.22) *are satisfied and let* $t = \tau_d$ *be a double turning point of* (P_J) ($J = \mathrm{II}, \mathrm{III}', \mathrm{IV}, \mathrm{V}, \mathrm{VI}$)*. If an instanton-type solution* $\lambda_J(t,\eta;\alpha,\beta)$ *of* (P_J) *is substituted into the coefficients of* (SL_J)*, then the following hold:*
(i) *The top term* (*with respect to* η^{-1}) Q_0 *of the potential* Q_J *of* (SL_J) *has*

two double zeros, one of which is given by the top term $\lambda_0(t)$ *of the instanton-type solution* $\lambda_J(t,\eta;\alpha,\beta)$*. In what follows the other double zero is denoted by* $\kappa(t)$*. Hence* (SL_J) *has two double turning points* $x=\lambda_0(t)$ *and* $x=\kappa(t)$*.*
(ii) *When* t *lies on a Stokes curve of* (P_J) *emanating from a double turning point* $t=\tau_d$*, there exists a Stokes curve of* (SL_J) *that connects the two double turning points* $x=\lambda_0(t)$ *and* $x=\kappa(t)$ *of* (SL_J)*.*

Using this Proposition 2.6 of geometric character, we construct the transformations in the following manner. (In what follows we again adopt the convention of putting $\tilde{}$ to the variables relevant to (P_J) and (SL_J) to distinguish them from those relevant to $(P_{\mathrm{II,deg}})$ and $(SL_{\mathrm{II,deg}})$.) Let $\tilde{t}=\tilde{\sigma}$ be a point on a Stokes curve of (P_J) emanating from a double turning point $\tilde{t}=\tilde{\tau}_d$ and let $\tilde{\gamma}$ denote a Stokes curve of (SL_J) that connects the two double turning points $\tilde{x}=\tilde{\lambda}_0(\tilde{t})$ and $\tilde{x}=\tilde{\kappa}(\tilde{t})$ at $\tilde{t}=\tilde{\sigma}$ (whose existence is guaranteed by Proposition 2.6, (ii)). Then we can construct an invertible formal transformation $(x(\tilde{x},\tilde{t},\eta),t(\tilde{t},\eta))$ which brings the simultaneous equations (SL_J) and (D_J) into $(SL_{\mathrm{II,deg}})$ and $(D_{\mathrm{II,deg}})$ in a neighborhood of $\tilde{\gamma}\times\{\tilde{\sigma}\}$. That is, we have

Theorem 2.7. *Under the above geometric situation there exist a neighborhood* $\tilde{U}$ *of the Stokes curve* $\tilde{\gamma}$*, a neighborhood* $\tilde{V}$ *of* $\tilde{\sigma}$*, and a formal coordinate transformation*

$$x=x(\tilde{x},\tilde{t},\eta) = \sum_{j\ge 0}\eta^{-j/2}x_{j/2}(\tilde{x},\tilde{t},\eta), \tag{2.28}$$

$$t=t(\tilde{t},\eta) = \sum_{j\ge 0}\eta^{-j/2}t_{j/2}(\tilde{t},\eta) \tag{2.29}$$

with $x_{j/2}(\tilde{x},\tilde{t},\eta)$ *and* $t_{j/2}(\tilde{t},\eta)$ *being holomorphic on* $\tilde{U}\times\tilde{V}$ *and* $\tilde{V}$*, respectively, for which the following conditions* (i) $\sim$ (v) *are satisfied:*
(i) *The function* $x_0(\tilde{x},\tilde{t},\eta)$ *is independent of* η *and* $\partial x_0/\partial\tilde{x}$ *never vanishes on* $\tilde{U}\times\tilde{V}$*.*
(ii) *The function* $t_0(\tilde{t},\eta)$ *is also independent of* η *and* $dt_0/d\tilde{t}$ *never vanishes on* $\tilde{V}$*.*
(iii) $x_0(\tilde{x},\tilde{t})$ *and* $t_0(\tilde{t})$ *satisfy*

$$x_0(\tilde{\lambda}_0(\tilde{t}),\tilde{t},\eta)=\lambda_0(t_0(\tilde{t}))=\sqrt{-\frac{t_0(\tilde{t})}{2}}, \tag{2.30}$$

$$x_0(\tilde{\kappa}_0(\tilde{t}),\tilde{t},\eta)=-\lambda_0(t_0(\tilde{t}))=-\sqrt{-\frac{t_0(\tilde{t})}{2}}. \tag{2.31}$$

(iv) $x_{1/2}$ *and* $t_{1/2}$ *identically vanish.*
(v) *If* $\psi(x,t,\eta)$ *is a WKB solution of* $(SL_{\mathrm{II,deg}})$ *that satisfies* $(D_{\mathrm{II,deg}})$ *also, then* $\tilde{\psi}(\tilde{x},\tilde{t},\eta)$ *defined by*

$$\tilde{\psi}(\tilde{x},\tilde{t},\eta) = \left(\frac{\partial x(\tilde{x},\tilde{t},\eta)}{\partial \tilde{x}}\right)^{-1/2} \psi(x(\tilde{x},\tilde{t},\eta), t(\tilde{t},\eta), \eta) \tag{2.32}$$

satisfies both (SL_J) *and* (D_J) *on* $\tilde{U}\times\tilde{V}$

The transformations (2.25) and (2.26) that provide a local equivalence (2.27) between $\tilde{\lambda}_J(\tilde{t},\eta;\tilde{\alpha},\tilde{\beta})$ and $\lambda_{\mathrm{II,deg}}(t,\eta;\alpha,\beta)$ in Theorem 2.5 are given by the semi-global transformation (2.28) and (2.29) constructed in Theorem 2.7. Otherwise stated, by considering a transformation for the underlying system (SL_J) and (D_J) of linear differential equations, we can find a transformation of the Painlevé equation (P_J). This is a sketch of the proof of Theorem 2.5. The details will be discussed in our forthcoming paper.

3 Transformation near a simple pole

As was outlined in [T2], the transformation theory near a simple pole, i.e., Theorem 1.4, is proved in a parallel way to the case of the transformation theory near a double turning point discussed in Section 2. In this section we briefly review the discussion of [T2] to explain the transformation near a simple pole.

In view of the list of Painlevé equations (Table 1) we readily find that the Painlevé equations (P_J) have the following singular points:

$$\begin{array}{lcl} (P_{\mathrm{I}}),\ (P_{\mathrm{II}}),\ (P_{\mathrm{IV}}) & : & \{\infty\}, \\ (P_{\mathrm{III}'}),\ (P_{\mathrm{III}'(\mathrm{D7})}),\ (P_{\mathrm{III}'(\mathrm{D8})}),\ (P_{\mathrm{V}}) & : & \{0,\infty\}, \\ (P_{\mathrm{VI}}) & : & \{0,1,\infty\}. \end{array} \tag{3.1}$$

Among them a pair of a Painlevé equation and its singular point contained in the following list is of “the first kind” or of “regular singular type”.

$$\begin{array}{l} ((P_{\mathrm{III}'}),0),\ \ ((P_{\mathrm{III}'(\mathrm{D7})}),0),\ \ ((P_{\mathrm{III}'(\mathrm{D8})}),0),\ \ ((P_{\mathrm{V}}),0), \\ ((P_{\mathrm{VI}}),0),\ \ ((P_{\mathrm{VI}}),1),\ \ ((P_{\mathrm{VI}}),\infty). \end{array} \tag{3.2}$$

At a singular point of the first kind, in addition to a double pole type 0-parameter solution, there exists a simple pole type 0-parameter solution, that is, for any pair $((P_J), \tau_s)$ in (3.2), there exists a 0-parameter solution whose top term $\lambda_0(t)$ has a branch point at $t = \tau_s$ and satisfies

$$\frac{\partial F_J}{\partial \lambda}(\lambda_0(t), t) = O((t - \tau_s)^{-3/2}) \quad \text{as} \quad t \to \tau_s, \tag{3.3}$$

where $F_J(\lambda, t)$ denotes the coefficient of η^2 in the expression of (P_J). Note that the condition (3.3) guarantees that the corresponding linearized equation (ΔP_J) of (P_J) at the 0-parameter solution in question has a simple pole type singularity at $t = \tau_s$ after a new independent variable $\tilde{t} = (t - \tau_s)^{1/2}$, which is a local parameter of the Riemann surface of $\lambda_0(t)$ at $t = \tau_s$, is introduced. Consequently, if $((P_J), \tau_s)$ is a simple pole, only one Stokes curve of (P_J) emanates from $t = \tau_s$.

Using the top term $\lambda_0(t)$ of a simple pole type 0-parameter solution, we can also construct a 2-parameter instanton-type solution $\lambda_J(t, \eta; \alpha, \beta)$ of simple pole type for each pair $((P_J), \tau_s)$ listed in (3.3). The problem we want to discuss is then to develop transformation theory for these instanton-type solutions $\lambda_J(t, \eta; \alpha, \beta)$ of simple pole type. The precise formulation of the main result (i.e., Theorem 1.4) in this case is the following theorem (where we again adopt the convention of putting $\tilde{}$ to the variables relevant to (P_J) to distinguish them from those relevant to $(P_{\mathrm{III}'(\mathrm{D}8)})$).

Theorem 3.1. *Let $\tilde{\lambda}_J(\tilde{t}, \eta; \tilde{\alpha}, \tilde{\beta})$ be a 2-parameter instanton-type solution of simple pole type for one of the pairs $((P_J), \tilde{\tau}_s)$ of a Painlevé equation and its singular point listed in (3.2). Let $\tilde{\sigma}$ be a point on a Stokes curve emanating from $\tilde{\tau}_s$. Then we can find a neighborhood $\tilde{V}$ of $\tilde{\sigma}$ and a 2-parameter instanton-type solution $\lambda_{\mathrm{III}'(\mathrm{D}8)}(t, \eta; \alpha, \beta)$ of $(P_{\mathrm{III}'(\mathrm{D}8)})$ such that $\tilde{\lambda}_J(\tilde{t}, \eta; \tilde{\alpha}, \tilde{\beta})$ is formally transformed to $\lambda_{\mathrm{III}'(\mathrm{D}8)}(t, \eta; \alpha, \beta)$ in $\tilde{V}$. To be more specific, there exist a formal transformation $t = t(\tilde{t}, \eta)$ of an independent variable and a formal transformation $x = x(\tilde{x}, \tilde{t}, \eta)$ of an unknown function of the form*

$$t(\tilde{t}, \eta) = \sum_{j \geq 0} \eta^{-j/2} t_{j/2}(\tilde{t}, \eta), \tag{3.4}$$

$$x(\tilde{x}, \tilde{t}, \eta) = \sum_{j \geq 0} \eta^{-j/2} x_{j/2}(\tilde{x}, \tilde{t}, \eta), \tag{3.5}$$

where $t_{j/2}$ and $x_{j/2}$ are holomorphic in both $\tilde{x}$ and $\tilde{t}$, that satisfy the following relation:

$$x(\tilde{\lambda}_J(\tilde{t}, \eta; \tilde{\alpha}, \tilde{\beta}), \tilde{t}, \eta) = \lambda_{\mathrm{III}'(\mathrm{D}8)}(t(\tilde{t}, \eta), \eta; \alpha, \beta). \tag{3.6}$$

Thus ($P_{\mathrm{III'(D8)}}$) can be thought of as a canonical equation of Painlevé equations near a simple pole.

The proof of Theorem 3.1 is done in a parallel way to that of Theorem 2.5. We again make use of the fact that a Painlevé equation (P_J) is equivalent to the compatibility condition of (SL_J) and (D_J) given in Section 2. A key geometric proposition in this case is the following

Proposition 3.2. *Suppose that an instanton-type solution $\lambda_J(t, \eta; \alpha, \beta)$ of simple pole type of (P_J) is substituted into the coefficients of (SL_J). Then the following hold:*
(i) *The top term (with respect to η^{-1}) Q_0 of the potential Q_J of (SL_J) has a double zero at $x = \lambda_0(t)$, that is, (SL_J) has a double turning point at $x = \lambda_0(t)$.*
(ii) *When t lies on a Stokes curve of (P_J) emanating from a simple pole type singular point τ_s in question, there exists a Stokes curve of (SL_J) that starts from $\lambda_0(t)$ and returns to $\lambda_0(t)$ after encircling several singular points and/or turning points of (SL_J).*

For example, in the case of the canonical equation, i.e., the most degenerate third Painlevé equation ($P_{\mathrm{III'(D8)}}$),

$$\lambda^{(0)}_{\mathrm{III'(D8)}}(t, \eta) = \sqrt{t} \tag{3.7}$$

is a 0-parameter solution and the linearized equation of ($P_{\mathrm{III'(D8)}}$) at this 0-parameter solution is given by

$$\frac{d^2}{dt^2}\Delta\lambda = \eta^2 \left(\frac{2}{t^{3/2}} - \eta^{-2}\frac{1}{4t^2} \right) \Delta\lambda. \tag{$\Delta P_{\mathrm{III'(D8)}}$}$$

Hence $t = 0$ is a simple pole type singularity (and a unique turning point) of ($P_{\mathrm{III'(D8)}}$) and only one Stokes curve

$$\{t \in \mathbb{C} \mid \arg t = 4n\pi \ (n \in \mathbb{Z})\} \tag{3.8}$$

(i.e., the positive real axis) emanates from $t = 0$. Since the potential $Q_{\mathrm{III'(D8)}}$ of the associated linear equation ($SL_{\mathrm{III'(D8)}}$) has the form

$$\begin{aligned} Q_{\mathrm{III'(D8)}} \quad = \quad & \frac{t}{2x^3} + \frac{1}{2x} + \frac{\lambda^2}{x^2}\left[\nu^2 - \left(\frac{t}{2\lambda^3} + \frac{1}{2\lambda}\right)\right] \\ & -\eta^{-1}\lambda\nu\left[\frac{1}{x^2} + \frac{1}{x(x-\lambda)}\right] + \eta^{-2}\frac{3}{4(x-\lambda)^2} \end{aligned} \tag{3.9}$$

(where $\nu = \eta^{-1}(t d\lambda/dt + \lambda)/(2\lambda^2)$), its top term $Q_0(x,t)$ becomes

$$Q_0(x,t) = \frac{(x-\sqrt{t})^2}{2x^3} \tag{3.10}$$

after the substitution of an instanton-type solution of simple pole type of $(P_{\mathrm{III}'(\mathrm{D8})})$ beginning with the leading term $\lambda_0(t) = \sqrt{t}$. Using (3.10), we thus find that when t lies on a Stokes curve of $(P_{\mathrm{III}'(\mathrm{D8})})$, i.e., when $t > 0$, a circle $\{|x| = \sqrt{t}\}$ is a Stokes curve of $(SL_{\mathrm{III}'(\mathrm{D8})})$ that starts from $\sqrt{t}$, encircles the simple pole $t = 0$, and returns to $\sqrt{t}$, as is indicated in Figure 2, (ii).

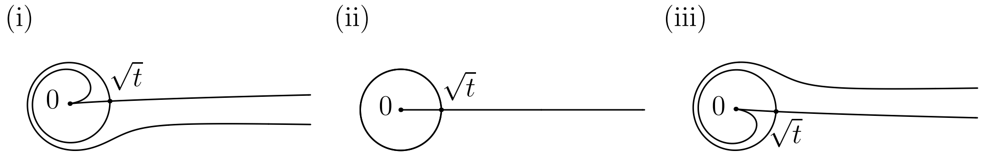

Figure 2: Configuration of Stokes curves of $(SL_{\mathrm{III}'(\mathrm{D8})})$ in the case of (i) $\arg t > 0$, (ii) $\arg t = 0$, and (iii) $\arg t < 0$.

In parallel to Theorem 2.7, near a Stokes curve γ of (SL_J) that starts from a double turning point $\lambda_0(t)$ and returns to $\lambda_0(t)$ at a point $t = \sigma$ on a Stokes curve of (P_J) emanating from a simple pole whose existence is guaranteed by Proposition 3.2, (ii), we can construct an invertible formal transformation which brings (SL_J) and (D_J) into $(SL_{\mathrm{III}'(\mathrm{D8})})$ and $(D_{\mathrm{III}'(\mathrm{D8})})$ in a neighborhood of $\gamma \times \{\sigma\}$. That is, we have

Theorem 3.3. *Let $((P_J), \tilde{\tau}_s)$ be one of the pairs in the list* (3.2) *and $\tilde{\sigma}$ a point on a Stokes curve emanating from $\tilde{\tau}_s$. Suppose that an instanton-type solution $\tilde{\lambda}_J(\tilde{t}, \eta; \tilde{\alpha}, \tilde{\beta})$ of simple pole type of (P_J) is substituted into the coefficients of (SL_J). Then there exist a neighborhood $\tilde{U}$ of a Stokes curve $\tilde{\gamma}$ of (SL_J) that starts from and returns to $\tilde{\lambda}_0(\tilde{t})$ at $\tilde{t} = \tilde{\sigma}$, a neighborhood $\tilde{V}$ of $\tilde{\sigma}$, and a formal coordinate transformation of the form*

$$x = x(\tilde{x}, \tilde{t}, \eta) = \sum_{j\geq 0} \eta^{-j/2} x_{j/2}(\tilde{x}, \tilde{t}, \eta), \tag{3.11}$$

$$t = t(\tilde{t}, \eta) = \sum_{j\geq 0} \eta^{-j/2} t_{j/2}(\tilde{t}, \eta) \tag{3.12}$$

with $x_{j/2}(\tilde{x},\tilde{t},\eta)$ *and* $t_{j/2}(\tilde{t},\eta)$ *being holomorphic on* $\tilde{U}\times\tilde{V}$ *and* $\tilde{V}$, *respectively, for which the following conditions* (i) $\sim$ (v) *are satisfied:*
(i) *The function* $x_0(\tilde{x},\tilde{t},\eta)$ *is independent of* η *and* $\partial x_0/\partial\tilde{x}$ *never vanishes on* $\tilde{U}\times\tilde{V}$.
(ii) *The function* $t_0(\tilde{t},\eta)$ *is also independent of* η *and* $dt_0/d\tilde{t}$ *never vanishes on* $\tilde{V}$.
(iii) $x_0(\tilde{x},\tilde{t})$ *and* $t_0(\tilde{t})$ *satisfy*

$$x_0(\tilde{\lambda}_0(\tilde{t}),\tilde{t},\eta)=\lambda_0(t_0(\tilde{t}))=\sqrt{t_0(\tilde{t})}. \tag{3.13}$$

(iv) $x_{1/2}$ *and* $t_{1/2}$ *identically vanish.*
(v) *If* $\psi(x,t,\eta)$ *is a WKB solution of* $(SL_{\mathrm{III}'(\mathrm{D8})})$ *that satisfies* $(D_{\mathrm{III}'(\mathrm{D8})})$ *also, then* $\tilde{\psi}(\tilde{x},\tilde{t},\eta)$ *defined by*

$$\tilde{\psi}(\tilde{x},\tilde{t},\eta)=\left(\frac{\partial x(\tilde{x},\tilde{t},\eta)}{\partial\tilde{x}}\right)^{-1/2}\psi(x(\tilde{x},\tilde{t},\eta),t(\tilde{t},\eta),\eta) \tag{3.14}$$

satisfies both (SL_J) *and* (D_J) *on* $\tilde{U}\times\tilde{V}$.

The semi-global transformation (3.11) and (3.12) constructed in Theorem 3.3 again provides a local equivalence (3.6) between $\tilde{\lambda}_J(\tilde{t},\eta;\tilde{\alpha},\tilde{\beta})$ and $\lambda_{\mathrm{III}'(\mathrm{D8})}(t,\eta;\alpha,\beta)$ in Theorem 3.1. This is a sketch of the proof of Theorem 3.1. The details will be discussed in our forthcoming paper.

In the case of the canonical equation $(P_{\mathrm{III}'(\mathrm{D8})})$, we can explicitly compute the monodromy data of the associated linear equation $(SL_{\mathrm{III}'(\mathrm{D8})})$ by using exact WKB analysis for linear differential equations. Combining this computation with Proposition 3.2, we obtain the following connection formula for instanton-type solutions of $(P_{\mathrm{III}'(\mathrm{D8})})$ on its Stokes curve $\arg t=0$: Let $\lambda(t,\eta;\alpha,\beta)$ and $\lambda(t,\eta;\alpha',\beta')$ be instanton-type solutions of $(P_{\mathrm{III}'(\mathrm{D8})})$ in the region $\Omega_-=\{\arg t<0\}$ and $\Omega_+=\{\arg t>0\}$, respectively. If $\lambda(t,\eta;\alpha',\beta')$ is the analytic continuation of $\lambda(t,\eta;\alpha,\beta)$ across the Stokes curve $\arg t=0$, then we have

$$\frac{\alpha'\,2^{-2\alpha'\beta'}}{\Gamma(-2\alpha'\beta'+1)}=\frac{\alpha\,2^{-2\alpha\beta}}{\Gamma(-2\alpha\beta+1)}, \tag{3.15}$$

$$\begin{aligned}\frac{i\beta'\,2^{2\alpha'\beta'}}{\Gamma(2\alpha'\beta'+1)}&+e^{2i\pi\alpha'\beta'}\frac{\alpha'\,2^{-2\alpha'\beta'}}{\Gamma(-2\alpha'\beta'+1)}\\&=\frac{i\beta\,2^{2\alpha\beta}}{\Gamma(2\alpha\beta+1)}-e^{-2i\pi\alpha\beta}\frac{\alpha\,2^{-2\alpha\beta}}{\Gamma(-2\alpha\beta+1)}.\end{aligned} \tag{3.16}$$

See [TW, Section 5] for the computation of the monodromy data of ($SL_{\mathrm{III}'(\mathrm{D8})}$). Theorem 3.1 then suggests that the same connection formula as (3.15) and (3.16) should hold also for instanton-type solutions of simple pole type of (P_J) listed in (3.2) on its Stokes curve emanating from a simple pole.

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