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RICCI CURVATURE AND CONVERGENCE OF LIPSCHITZ FUNCTIONS

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Abstract

We give a definition of convergence of differential of Lipschitz functions with respect to measured Gromov-Hausdorff topology. As their applications, we give a characterization of harmonic functions with polynomial growth on asymptotic cones of manifolds with nonnegative Ricci curvature and Euclidean volume growth, and distributional Laplacian comparison theorem on limit spaces of Riemannian manifolds.

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1 Introduction

Let $\{(M_i, m_i)\}$ be a sequence of pointed n-dimensional complete Riemannian manifolds $(n \geq 2)$ with $\mathrm{Ric}_{M_i} \geq -(n-1)$ and (Y, y, υ) a pointed proper metric space (i.e. every bounded subset of Y is relatively compact) with Radon measure υ on Y satisfying $(M_i, m_i, \underline{\mathrm{vol}})$ converges to (Y, y, υ) in the sense of measured Gromov-Hasdorff topology. Here $\underline{\mathrm{vol}}$ is the renormalized Riemannian volume of (M_i, m_i): $\underline{\mathrm{vol}} = \mathrm{vol}/\mathrm{vol}\, B_1(m_i)$. We fix $R > 0$, a sequence of Lipschitz functions f_i on $B_R(m_i) = \{w \in M_i; \overline{w, m_i} < R\}$ and a Lipschitz function f_∞ on $B_R(y)$ satisfying $\sup_i \mathbf{Lip} f_i < \infty$. Here $\overline{w, m_i}$ is the distance between w and m_i, $\mathbf{Lip} f_i$ is the Lipschitz constant of f_i. Then we say that f_i *converges to* f_∞ if $f_i(x_i) \to f_\infty(x_\infty)$ for every $x_i \in B_R(m_i)$ and $x_\infty \in B_R(y)$ satisfying that x_i converges to x_∞. See section 2 for these precise definitions. Assume $\{f_i\}$ converges to f_∞ below.

The purpose of this paper is to give a definition: *differential* df_i *of* f_i *converges to differential* df_∞ *of* f_∞ in this setting. To give the definition below, we shall recall celebrated works for limit spaces of Riemannian manifolds by Cheeger-Colding. By [5] and [9], we can construct the cotangent bundle T^*Y of Y, a fiber $T^*_w Y$ is a finite dimensional real vector space with canonical inner product $\langle \cdot, \cdot \rangle(w)$ for a.e. $w \in Y$. Moreover, every

Lipschitz function g on $B_R(y)$ have canonical differential section: $dg(w) \in T^*_w Y$ for a.e. $w \in B_R(y)$. See section 4 in [5] and section 6 in [9] for the details.

We shall give a definition of convergence of differential of Lipschitz functions (see Definition 4.18):

DEFINITION 1.1 (Convergence of differential of Lipschitz functions). We say that df_i *converges to* df_∞ *on* $B_R(y)$ if for every $\epsilon > 0$, $x_\infty \in B_R(y)$ $z_\infty \in Y$, $x_i \in B_R(m_i)$ and $z_i \in M_i$ satisfying that x_i converges to x_∞ and that z_i converges to z_∞, there exists $r > 0$ such that

$$\limsup_{i\to\infty} \left| \frac{1}{\underline{\mathrm{vol}}\, B_t(x_i)} \int_{B_t(x_i)} \langle dr_{z_i}, df_i \rangle d\underline{\mathrm{vol}} - \frac{1}{\upsilon(B_t(x_\infty))} \int_{B_t(x_\infty)} \langle dr_{z_\infty}, df_\infty \rangle d\upsilon \right| < \epsilon$$

and

$$\limsup_{i\to\infty} \frac{1}{\underline{\mathrm{vol}}\, B_t(x_i)} \int_{B_t(x_i)} |df_i|^2 d\underline{\mathrm{vol}} \le \frac{1}{\upsilon(B_t(x_\infty))} \int_{B_t(x_\infty)} |df_\infty|^2 d\upsilon + \epsilon$$

for every $0 < t < r$.

If df_i converges to df_∞ on $B_R(y)$, then we denote it by $(f_i, df_i) \to (f_\infty, df_\infty)$ on $B_R(y)$. Assume $(f_i, df_i) \to (f_\infty, df_\infty)$ and $(g_i, dg_i) \to (g_\infty, dg_\infty)$ on $B_R(y)$ below.

In the paper, we will study several properties of the convergence and give their applications. For example, we will give the following in section 4:

THEOREM 1.2. *We have*

$$\lim_{i\to\infty} \int_{B_R(m_i)} F_i(\langle df_i, dg_i \rangle) d\underline{\mathrm{vol}} = \int_{B_R(y)} F_\infty(\langle df_\infty, dg_\infty \rangle) d\upsilon$$

for every sequence of continuous functions $\{F_i\}_{i=1,2,\cdots,\infty}$ *on* $\mathbf{R}^k$ *satisfying that* F_i *converges to* F_∞ *uniformly on each compact subsets of* $\mathbf{R}$. *Especially, if* $f_\infty = g_\infty$, *then we have*

$$\lim_{i\to\infty} \frac{1}{\underline{\mathrm{vol}}\, B_R(m_i)} \int_{B_R(m_i)} F_i(|df_i - dg_i|) d\underline{\mathrm{vol}} = F_\infty(0).$$

See Proposition 4.5 and Theorem 4.20 for the proof. We will also give the following in the section:

THEOREM 1.3. *Let* h_i *be a harmonic function on* $B_R(m_i)$ *and* h_∞ *a Lipschitz function on* $B_R(y)$ *satisfying that* $\sup_i \mathbf{Lip} h_i < \infty$ *and that* h_i *converges to* h_∞ *on* $B_R(y)$. *Then* h_∞ *is harmonic function on* $B_R(y)$, $(h_i, dh_i) \to (h_\infty, dh_\infty)$ *on* $B_R(y)$.

We remark that the harmonicity of h_∞ in Theorem 1.3 is given already in [24] by Ding. We will give an alternative proof of it in section 4 (see Corollary 4.37).

The organization of this paper is as follows:

In the next section, we will give several important notions and propeties for metric spaces and manifolds to understand this paper. Most of statements in the section do not have the proof, we will give a reference for them only.

In section 3, we will give results of rectifiability for limit spaces of Riemannian manifolds (Theorem 3.17 and Theorem 3.54). It is important that we can take functions which give a rectitfiability of limit spaces, by *distance functions* in these theorem. As a corollary, we will give an explicit geometric formula of radial derivative for Lipschitz functions (Theorem 3.33). These results are used in section 4 essentially. In [45], we will also give a geometric application of results in this section 3 to limit spaces of Riemannian manifolds with Ricci curvature bounded below.

In section 4, we will give a definition of convergences of L^∞-functions associated to measured Gromov-Hausdorff convergence and give the definition of convergence of differential of Lipschitz functions again via the definition of convergence of L^∞-functions. After that, we will give several properties of the convergence. Main properties of them are Theorem 4.20, Theorem 4.27 and Corollary 4.35.

In section 5, as an application of results in section 4, we will study harmonic functions on asymptotic cones of manifolds with nonnegative Ricci curvature and Euclidean volume growth via Colding-Minicozzi big theory ([17, 18, 19, 20, 21, 22]). See Definition 5.3 for the definition of asymptotic cones. It is important that we can replace most of statements for harmonic functions on manifolds in [18] with one on asymptotic cones via Ding's important works [23, 24] and Theorem 4.20. For instance, we will prove that the space of harmonic functions with polynomial growth of a fixed rate is finite dimensional vector space (Theorem 5.34). We can regard it as *asymptotic cones version* of finite dimensionality conjecture on manifolds by Yau (see for instance Conjecture 0.1 in [17]). We remark that most of important essential ideas to prove these statements given in [18, 22]. Roughly speaking, we can get these results by "taking limit of most of results in [18] via Theorem 4.20". As an application of them to manifolds, we will prove the following Liouville type theorem:

THEOREM 1.4. *Let M be an n-dimensional ($n \geq 3$) complete Riemannian manifold with nonnegative Ricci curvature and Euclidean volume growth. Then, there exists unique $d_1 \geq 1$ satisfying the following properties:*

1. *For every asymptotic cone M_∞ of M and $0 < d < d_1$, we have*

$$H^d(M_\infty) = \{\text{Constant functions}\}.$$

Here $H^d(M_\infty)$ is the linear space of harmonic functions on M_∞ with order of growth at most d.

2. There exists an asymptotic cone M_∞ of M such that

$$H^{d_1}(M_\infty) \neq \{\text{Constant functions}\}.$$

3. For every $0 < d < d_1$, we have

$$H^d(M) = \{\text{Constant functions}\}.$$

See Corollary 5.48 for the proof.

In section 6, as another application of results in section 4, we will give (distributional) Laplacian comparison theorem on limit spaces of Riemannian manifolds by using several results in [42]. See Theorem 6.1. This formulation is given in [53] by Kuwae-Shioya on weighted Alexandrov spaces. Roughly speaking, this Laplacian comparison theorem implies that limit spaces of Riemannian manifolds have "definite lower bound of Ricci curvature in some sense." In fact, we can get a stability result of lower bound of Ricci curvature with respect to Gromov-Hausdorff topology (Corollary 6.3). The corollary is well known in the setting of metric measure spaces. See for instance [65, 66, 72, 88, 89, 92, 93]. We will give an alternative proof of it via the Laplacian comparison theorem.

In section 7, we will give proofs of several propositions used in previous sections.

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2 Preliminaries

Our aim in this section is to introduce important notions and properties for metric spaces and manifolds to understand statements in this paper.

2.1 Metric measure spaces

For a positive number $\epsilon > 0$, we use following notation:

$$a = b \pm \epsilon \Longleftrightarrow |a - b| < \epsilon.$$

We denote by $\Psi(\epsilon_1, \epsilon_2, \dots, \epsilon_k; c_1, c_2, \dots, c_l)$ (more simply, Ψ) some positive function on $\mathbf{R}^k_{>0} \times \mathbf{R}^l$ satisfying

$$\lim_{\epsilon_1, \epsilon_2, \dots, \epsilon_k \to 0} \Psi(\epsilon_1, \epsilon_2, \dots, \epsilon_k; c_1, c_2, \dots, c_l) = 0$$

for each fixed real numbers $c_1, c_2, \dots, c_l$. We often denote by $C(c_1, c_2, \dots, c_l)$ some (positive) constant depending only on fixed real numbers $c_1, c_2, \dots, c_l$.

For a metric space Z, a point $z \in Z$ and a positive number $r > 0$, we use the following notation:

$$B_r(z) = \{x \in Z; \overline{z,x} < r\}, \overline{B}_r(z) = \{x \in Z; \overline{z,x} \leq r\}, \partial B_r(z) = \{x \in Z; \overline{z,x} = r\}.$$

Here $\overline{y,x}$ is the distance between y and x, we often denote the distance by $d_Z(y,x)$. For $r < R$, we put $A_{r,R}(z) = \overline{B}_R(z) \setminus B_r(z)$. For every $A \subset Z$, we also put $B_r(A) = \{x \in Z; \overline{A,w} < r\}$ and $\overline{B}_r(A) = \{x \in Z; \overline{A,x} \leq r\}$. For an open subset U of Z and $\eta > 0$, we put $U_\eta = \{w \in U; B_\eta(w) \subset U\}$. It is easy to check that U_η is closed subset of Z. For $z \in Z$, we define 1-Lipschitz function r_z on Z by $r_z(w) = \overline{z,w}$.

For a Lipschitz function f on Z and a point $z \in Z$, we will use the following notations:

1. If z is not an isolated point in Z, then we put

$$lipf(z) = \liminf_{r \to 0} \left(\sup_{x \in B_r(z) \setminus \{z\}} \frac{|f(x) - f(z)|}{\overline{x,z}} \right),$$

   if z is an isolated point in Z, then we put $lipf(z) = 0$.

2. If z is not an isolated point in Z, then we put

$$\mathrm{Lip}f(z) = \limsup_{r \to 0} \left(\sup_{x \in B_r(z) \setminus \{z\}} \frac{|f(x) - f(z)|}{\overline{x,z}} \right),$$

   if z is an isolated point in Z, then we put $\mathrm{Lip}f(z) = 0$.

3. If Z is not single point, then we put

$$\mathbf{Lip}f = \sup_{w_1 \neq w_2} \frac{|f(w_1) - f(w_2)|}{\overline{w_1, w_2}} < \infty,$$

   if Z is a single point, then we put $\mathbf{Lip}f = 0$.

We shall remark that for every subset $A \subset Z$ and Lipschitz function f on A, there exists a Lipschitz function f^* on Z such that $f^*|_A = f$ and $\mathbf{Lip}f^* = \mathbf{Lip}f$. In fact, if we define a function f^* on Z by $f^*(z) = \inf_{a \in A}(f(a) + \mathbf{Lip}f\overline{z,a})$, then it is easy to check that $f^*|_A = f$ and $\mathbf{Lip}f^* = \mathbf{Lip}f$.

For a Borel subset A of Z, an extended real valued Borel function f on A and an extended nonnegative real valued Borel function g on A, we say that g is an upper gradient for f if for every $a_1, a_2 \in A$ and continuous rectifiable curve $\gamma : [0, l] \to A$ parametrized by arclength with $\gamma(0) = a_1, \gamma(l) = a_2$, we have

$$|f(a_1) - f(a_2)| \leq \int_0^l g(\gamma(s))ds.$$

For an open subset $U \subset Z$ and a Lipschitz function f on U, $lipf$ is an upper gradient for f on U. See [5, Proposition 1.11].

We say that Z is *proper* if every bounded subbsets of Z are relatively compact. We also say that Z is a *geodesic space* if for every $x_1, x_2 \in Z$, there exists an isometric embedding γ from $[0, \overline{x_1, x_2}]$ to Z such that $\gamma(0) = x_1, \gamma(\overline{x_1, x_2}) = x_2$. We say that γ is a *minimal geodesic from x_1 to x_2*. For a proper geodesic space W and $w \in W$, we put $C_w = \{z \in W;$ For every $x \in W \setminus \{z\}$, we have $\overline{w,z} + \overline{z,x} > \overline{w,x}\}$ (if W is a single point, then we put $C_w = \emptyset$). We call C_w *cut locus of W at w*.

For a proper metric space Z and a Borel measure υ on Z, we say that υ *is Radon measure* if $\upsilon(K) < \infty$ for every compact set K,

$$\upsilon(A) = \sup_{K \subset A:\text{compact}} \upsilon(K) = \inf_{A \subset O:\text{open}} \upsilon(O)$$

for every Borel subset A of Z. Then we say that a pair (Z, υ) is a *metric measure space* in this paper. For a metric measure space (Z, υ), a point $z \in Z$ and $k \in \mathbf{R}_{\geq 0}$, we say that υ *is Ahlfors k-regular at z* if there exist $r > 0$ and $C \geq 1$ such that $C^{-1} \leq \upsilon(B_t(z))/t^k \leq C$ for every $0 < t < r$. We shall introduce the notion of *υ-rectifiability* for metric measure spaces by Cheeger-Colding. See [9, Definition 5.3] and [9, Theorem 5.7]. For metric spaces X_1, X_2, $0 < \delta < 1$ and a bijection map f from X_1 to X_2, we say that f *gives $(1 \pm \delta)$-bi-Lipschitz equivalent to X_2* if f and f^{-1} are $(1 + \delta)$-Lipschitz map.

Definition 2.1 (Rectifiability for metric measure spaces). For a metric measure space (Z, υ) and a Borel subset $A \subset Z$, we say that A *is υ-rectifiable* if there exists a positive integer m, a collection of Borel subset $\{C_{k,i}\}_{1 \leq k \leq m, i \in \mathbf{N}}$ of A and a collection of bi-Lipschitz embedding map $\{\phi_{k,i} : C_{k,i} \to \mathbf{R}^k\}$ satisfying the following properties:

1. $\upsilon(A \setminus \bigcup_{k,i} C_{k,i}) = 0$
2. υ is Ahlfors k-regular at each $x \in C_{k,i}$.
3. For every k, $x \in \bigcup_{i \in \mathbf{N}} C_{k,i}$ and $0 < \delta < 1$, there exists $C_{k,i}$ such that $x \in C_{k,i}$ and that the map $\phi_{k,i}$ gives $(1 \pm \delta)$-bi-Lipschitz equivalent to the image $\phi_{k,i}(C_{k,i})$.

We shall recall the definition of Sobolev spaces on metric measure spaces (see [4] and [41]). We fix a metric measure space (Z, υ) satisfying that Z is a geodesic space and that (Z, υ) satisfies doubling condition below: For every $r > 0$, there exists $K = K(r) \geq 1$ such that $0 < \upsilon(B_{2s}(x)) \leq 2^K \upsilon(B_s(x))$ for every $x \in Z$ and $0 < s < r$. We fix an open set $U \subset Z$. For functions $f, g \in L^2(U)$, we say that g *is a generalized upper gradient for f* if there exists sequences of extended real valued functions f_i on U and upper gradient g_i for f_i on U such that $f_i \to f$ and $g_i \to g$ in $L^2(U)$. Let $H_{1,2}(U)$ be the subspace of $L^2(U)$

consisting functions f satisfying that there exists a generalized upper gradient g for f on U. By [5, Theorem 2.10], for every $f \in H_{1,2}(U)$, there exists unique $g_f \in L^2(U)$ satisfying that $|g_f|_{L^2(U)} \leq |g|_{L^2(U)}$ for every generalized upper gradient g for f. We define a norm $|\cdot|_{1,2}$ on $H_{1,2}(U)$ by $|f|_{1,2} = |f|_{L^2(U)} + |g_f|_{L^2(U)}$. We call $(H_{1,2}(U), |\cdot|_{1,2})$ *the Sobolev space*. We put $K(U) = \{k \in H_{1,2}(U);$ There exists $\eta > 0$ such that $\upsilon(\{k \neq 0\} \cap (U \setminus U_\eta)) = 0\}$.

We recall the definition of (2-)harmonic function on metric measure spaces by Cheeger. For a Borel function f on U, we say that f *is harmonic on* U if $f|_V \in H_{1,2}(V)$ for every bounded subset $V \subset U$ and $|g_{f+k}|_{L^2(V)} \geq |g_f|_{L^2(V)}$ for every $k \in K(U)$.

We shall recall the definition of *weak Poincaré inequality of type* $(1,2)$ for metric measure spaces. We say that (Z, υ) satisfies *a weak Poincaré inequality of type* $(1,2)$ if for every $R > 0$, there exist $\tau \geq 1$ and $C \geq 1$ such that

$$\frac{1}{\upsilon(B_r(x))} \int_{B_r(x)} \left| f - \frac{1}{\upsilon(B_r(x))} \int_{B_r(x)} f d\upsilon \right| d\upsilon \leq Cr \sqrt{\frac{1}{\upsilon(B_{\tau r}(x))} \int_{B_{\tau r}(x)} g_f^2 d\upsilon}$$

for every $x \in Z$, $0 < r < R$ and $f \in H_{1,2}(B_{\tau r}(x))$. We remark that if (Z, υ) satisfies a weak Poincaré inequality of type $(1,2)$, then for every $R > 0$, there exist $C_1 \geq 1$ such that

$$\frac{1}{\upsilon(B_r(x))} \int_{B_r(x)} \left| f - \frac{1}{\upsilon(B_r(x))} \int_{B_r(x)} f d\upsilon \right| d\upsilon \leq C_1 r \sqrt{\frac{1}{\upsilon(B_r(x))} \int_{B_r(x)} g_f^2 d\upsilon}$$

for every $x \in Z$, $0 < r < R$ and $f \in H_{1,2}(B_r(x))$. See for instance (4.4) in [5] or [37].

We shall give a short review of important results about differentiability of Lipschitz functions on metric measure spaces by Cheeger. We assume that (Z, υ) satisfies weak Poincaré inequality of type $(1,2)$ below. Then, by section 4 in [5], we can construct the cotangent bundle T^*Z of Z. See [5, Definition 4.42] for the construction. We will give several fundamental properties of the cotangent bundle only:

1. T^*Z is a topological space.

2. There exists a Borel map $\pi : T^*Z \to Z$ such that $\upsilon(Z \setminus \pi(T^*Z)) = 0$.

3. For every $w \in \pi(T^*Z)$, $\pi^{-1}(w)$ is finite dimensional real vector space with canonical norm $|\cdot|(w)$.

4. For every open set $U \subset Z$ and $f \in H_{1,2}(U)$, there exists a Borel set $V \subset U$ and a Borel map df (called differential section of f) from V to T^*Z such that $\upsilon(U \setminus V) = 0$ and that $\pi \circ df(w) = w$, $|df|(w) = g_f(w)$ for every $w \in V$. Moreover, if f is Lipschitz, then $|df|(w) = \mathrm{Lip} f(w) = lip f(w)$.

5. For every open set $U \subset Z$ and Lipschitz functions f_1, f_2 on U, Leibnitz rule hold:

$$d(f_1 f_2)(w) = f_2(w) df_1(w) + f_1(w) df_2(w)$$

for a.e. $w \in U$.

See section 4 and 5 in [5] for the details.

In addition, we assume that Z is υ-rectifiable below. Then, by section 6 in [9], for a.e. $w \in Z$, each norms $|\cdot|(w)$ defines the inner product $\langle \cdot, \cdot \rangle(w)$, i.e. $|v|(w) = \sqrt{\langle v, v \rangle(w)}$ for every $v \in \pi^{-1}(w)$. We call $\{\langle \cdot, \cdot \rangle(w)\}_{w \in Y}$ *Riemannian metric of* Y and denote it by $\langle \cdot, \cdot \rangle$. Moreover, the following bilinear form

$$\int_Z \langle df_1, df_2 \rangle d\upsilon$$

on $H_{1,2}(Z)$ is closable (see [9, Theorem 6.25]). Therefore this bilinear form determines a canonical (positive definite) self-adjoint operator Δ_Z on $L^2(Z)$. We call Δ_Z *Laplace operator of* (Z, υ) *or Laplacian of* (Z, υ) Moreover, if Z is compact, then $(1 + \Delta_Z)^{-1}$ is compact operator (see [9, Theorem 6.27]).

2.2 Gromov-Hausdorff convergence

For compact metric spaces X_1, X_2, we define *Gromov-Hausdorff distance between* X_1 *and* X_2 by

$$d_{GH}(X_1, X_2) = \inf\{d_H^W(\phi_1(X_1), \phi_2(X_2)); \text{There exist a metric space } W \text{ and isometric embeddings } \phi_i \text{ from } X_i \text{ to } W (i = 1, 2)\}.$$

Here d_H^W is the Hausdroff distance and the infimum above runs over all W, ϕ_i satisfying conditions above. We remark that d_{GH} is a distance on the set of isometry class of compact metric spaces. On the other hand, for compact metric spaces X_1, X_2, a positive number $\epsilon > 0$ and a map ϕ from X_1 to X_2, we say that ϕ is an ϵ-*Gromov-Hausdorff approximation* if $B_\epsilon(\text{Image}\phi) = X_1$ and $|\overline{x, y} - \overline{\phi(x), \phi(y)}| < \epsilon$ for every $x, y \in X_1$. It is easy to check that if $d_{GH}(X_1, X_2) \leq \epsilon$, then there exists an 3ϵ-Gromov-Hausdorff approximation from X_1 to X_2 and that if there exists an ϵ-Gromov-Hausdorff approximation from X_1 to X_2, then $d_{GH}(X_1, X_2) \leq 9\epsilon$. For a sequence of compact metric spaces X_i, we say that X_i *converges to* X_∞ if $d_{GH}(X_i, X_\infty)$ converges to 0. Then we denote it by $X_i \to X_\infty$. Similarly, for pointed compact metric spaces $(X_1, x_1), (X_2, x_2)$, we can define the *pointed* Gromov-Hausdorff distance $d_{GH}((X_1, x_1), (X_2, x_2))$.

Moreover, for a sequence of pointed proper geodesic spaces, (Z_i, z_i), we say that (Z_i, z_i) converges to (Z_∞, z_∞) if there exist sequences of positive numbers ϵ_i, R_i and a (Borel) map ϕ_i from $(B_{R_i}(z_i), z_i)$ to $(B_{R_i}(z_\infty), z_\infty)$ such that $\epsilon_i \to 0$, $R_i \to \infty$ as $i \to \infty$, $B_{R_i}(z_\infty) \subset B_{\epsilon_i}(\text{Image}\phi_i)$ and $|\overline{x_1, x_2} - \overline{\phi_i(x_1), \phi_i(x_2)}| \leq \epsilon_i$ for every $x_1, x_2 \in B_{R_i}(x_i)$. We denote it by $(Z_i, z_i) \overset{(\phi_1, R_i, \epsilon_i)}{\to} (Z_\infty, z_\infty)$, or more simply $(Z_i, z_i) \to (Z_\infty, z_\infty)$. For every

$x_\infty \in Z_\infty$ and $x_i \in Z_i$, we say that x_i converges to x_∞ if $\overline{\phi_i(x_i), x_\infty} \to 0$. Then, we denote it by $x_i \to x_\infty$.

Let $(Z_i, z_i) \to (Z_\infty, z_\infty)$. For a sequence of sets $A_i \subset Z_i$ satisfying that there exists $R > 0$ such that $A_i \subset B_R(z_i)$ for every i, we say that A_i *is included by* A_∞ *asymptotically* if for every $\epsilon > 0$, there exists i_0 such that for every $i \geq i_0$, $\phi_i(A_i) \subset B_\epsilon(A_\infty)$. Then we denote it by $\limsup_{i\to\infty} A_i \subset A_\infty$. (If $A_\infty = \emptyset$, then $\limsup_{i\to\infty} A_i \subset A_\infty$ implies $A_i = \emptyset$ for every sufficiently large i.) Similarly, we also say that A_∞ *is included by* A_i *asymptotically* if for every $\epsilon > 0$, there exists i_0 such that for every $i \geq i_0$, $A_\infty \subset A_\epsilon(\phi_i(A_i))$. Then we denote it by $A_\infty \subset \liminf_{i\to\infty} A_i$. Let $C_\infty \subset \liminf_{i\to\infty} C_i$. For a sequence of Lipschitz function f_i on C_i satisfying $\sup_i \mathbf{Lip} f_i < \infty$, we say that f_∞ *is a restriction of* f_i *asymptotically* if for every $w \in C_\infty$, subsequence $\{n(i)\}$ of $\mathbf{N}$ and $w_{n(i)} \in C_{n(i)}$ satisfying $\overline{\phi_{n(i)}(w_{n(i)}), w} \to 0$, we have

$$\lim_{i\to\infty} f_{n(i)}(w_{n(i)}) = f_\infty(w).$$

Let $\limsup_{i\to\infty} D_i \subset D_\infty$ and D_∞ be compact. For a sequence of Lipschitz function g_i on D_i satisfying $\sup_i \mathbf{Lip} g_i < \infty$, we say that g_∞ *is an extension of* g_i *asymptotically* if for every $w \in D_\infty$, subsequence $\{n(i)\}$ of $\mathbf{N}$ and $w_{n(i)} \in D_{n(i)}$ satisfying $\overline{\phi_{n(i)}(w_{n(i)}), w} \to 0$, we have

$$\lim_{i\to\infty} g_{n(i)}(w_{n(i)}) = g_\infty(w).$$

For a sequence of compact set $K_i \subset Z_i$, we say that (Z_i, z_i, K_i) converges to $(Z_\infty, z_\infty, K_\infty)$ if there exists $\tau_i > 0$ such that $\tau_i \to 0$, $\phi_i(K_i) \subset B_{\epsilon_i+\tau_i}(K_\infty)$ and $K_\infty \subset B_{\epsilon_i+\tau_i}(\phi_i(K_i))$. Then we denote it by $(Z_i, z_i, K_i) \overset{(\phi_1, R_i, \epsilon_i)}{\to} (Z_\infty, z_\infty, K_\infty)$ or, more simply, $(Z_i, z_i, K_i) \to (Z_\infty, z_\infty, K_\infty)$ or $K_i \to K_\infty$. It is easy to check that $(Z_i, z_i, K_i) \to (Z_\infty, z_\infty, K_\infty)$ holds if and only if $\limsup_{i\to\infty} K_i \subset K_\infty$ and $K_\infty \subset \liminf_{i\to\infty} K_i$ hold.

Let $(Z_i, z_i, K_i) \to (Z_\infty, z_\infty, K_\infty)$. For a sequence of Lipschitz functions, $f_i^1, f_i^2, \dots, f_i^k$ on K_i satisfying $\sup_{i,l}(\mathbf{Lip} f_i^l + |f_i^l|_{L^\infty}) < \infty$, we say that $(Z_i, z_i, K_i, f_i^1, \dots, f_i^k)$ converges to $(Z_\infty, z_\infty, K_\infty, f_\infty^1, \dots, f_\infty^k)$ if

$$\lim_{i\to\infty} f_i^l(x_i) = f_\infty^l(x_\infty)$$

for every $x_i \in K_i$ and $x_\infty \in K_\infty$ satisfying $x_i \to x_\infty$. It is easy to check that this condition holds if and only if f_∞^l is an extension (or a restriction) of $\{f_i^l\}$ asymptotically for every l. We denote it by $(Z_i, z_i, K_i, f_i^1, \dots, f_i^k) \to (Z_\infty, z_\infty, K_\infty, f_\infty^1, \dots, f_\infty^k)$, or more simply, $f_i^l \to f_\infty^l$ for every l. Then we can also check that

$$\lim_{i\to\infty} |f_i^l - f_\infty^l \circ \phi_i|_{L^\infty(K_i)} = 0$$

easily.

EXAMPLE 2.2. Let $(Z_i, z_i) \to (Z_\infty, z_\infty)$. Then it is easy to check that $\limsup_{i\to\infty} \overline{B}_R(z_i) \subset B_R(z_\infty)$ and $\overline{B}_R(z_\infty) \subset \liminf_{i\to\infty} B_R(z_i)$.

EXAMPLE 2.3. Let $(Z_i, z_i) \to (Z_\infty, z_\infty)$. Then for every $A \subset Z_\infty$ and $\tau_i \to 0$, we have $\limsup_{i\to\infty} B_{\tau_i}((\phi_i)^{-1}(A_i)) \subset A$ and $A \subset \liminf_{i\to\infty}(\phi_i)^{-1}(A_i)$.

It is not difficult to check the following proposition:

PROPOSITION 2.4. *Let $(Z_i, z_i) \to (Z_\infty, z_\infty)$, A_i^1, A_i^2 bounded subsets of Z_i. Then we have the following:*

1. *If $\limsup_{i\to\infty} A_i^j \subset A_\infty^j$ for $j = 1, 2$, then $\limsup_{i\to\infty}(A_i^1 \cup A_i^2) \subset A_\infty^1 \cup A_\infty^2$.*
2. *If $A_\infty^j \subset \liminf_{i\to\infty} A_i^j$ for $j = 1, 2$, then $\liminf_{i\to\infty}(A_i^1 \cup A_i^2) \subset A_\infty^1 \cup A_\infty^2$.*
3. *If $X, Y \subset Z_\infty$ satisfies $\limsup_{i\to\infty} A_i^1 \subset X$, $\limsup_{i\to\infty} A_i^1 \subset Y$ and $X \cup Y \subset \liminf_{i\to\infty} A_i^1$, then $\overline{X} = \overline{Y}$. Here, $\overline{X}$ is the closure of X in Z_∞.*

We shall give a proof of the next proposition:

PROPOSITION 2.5. *Let (Z_i, z_i) be a sequence of proper geodesic spaces, Λ a set and $\{A_i^\lambda\}_{\lambda\in\Lambda}$ a collection of bounded subsets of Z_i. We assume that (Z_i, z_i) converges to (Z_∞, z_∞), A_∞^λ is compact for every $\lambda \in \Lambda$ and that $\limsup_{i\to\infty} A_i^\lambda \subset A_\infty^\lambda$ for every $\lambda \in \Lambda$. Then, $\limsup_{i\to\infty} \bigcap_{\lambda\in\Lambda} A_i^\lambda \subset \bigcap_{\lambda\in\Lambda} A_\infty^\lambda$.*

PROOF. The proof is done by a contradiction. We assume that the assertion is false. Then, there exists $\tau > 0$ such that for every i, there exist $N_i \geq i$ and $w_i \in \phi_{N_i}(\bigcap_{\lambda\in\Lambda} A_{N_i}^\lambda) \setminus B_\tau(\bigcap_{\lambda\in\Lambda} A_\infty^\lambda)$. Without loss of generality, we can assume that there exists $w_\infty \in Z_\infty$ such that $w_i \to w_\infty$. By the assumption, we have $w_\infty \in \overline{A_\infty^\lambda} = A_\infty^\lambda$ for every $\lambda \in \Lambda$. Thus, $w_\infty \in \bigcap_{\lambda\in\Lambda} A_\infty^\lambda$. Especially we have $w_i \in B_\tau(\bigcap_{\lambda\in\Lambda} A_\infty^\lambda)$ for every sufficiently large i. This is a contradiction. □

We shall consider convergence of a sequence of complement of open balls:

PROPOSITION 2.6. *Let (Z_i, z_i) be a sequence of proper geodesic spaces and A_i a bounded subset of Z_i. We assume that (Z_i, z_i) converges to (Z_∞, z_∞), A_∞ is compact and that $\limsup_{i\to\infty} A_i \subset A_\infty$. Then for every $r > 0$ and $x_i \to x_\infty \in Z_\infty$, we have $\limsup_{i\to\infty}(A_i \setminus B_r(x_i)) \subset A_\infty \setminus B_r(x_\infty)$.*

PROOF. We assume that the assertion is false. Then there exists $\tau > 0$ such that for every i, there exist $N_i \geq i$ and $w_i \in \phi_{N_i}(A_{N_i} \setminus B_r(x_{N_i})) \setminus B_\tau(A_\infty \setminus B_r(x_\infty))$. Without loss of generality, we can assume that there exists $w_\infty \in Z_\infty$ such that $w_i \to w_\infty$. By the assumption, we have $w_\infty \in \overline{A_\infty} = A_\infty$. We take $\alpha_i \in A_{N_i} \setminus B_r(x_{N_i})$ satisfying $w_i = \phi_{N_i}(\alpha_i)$. Then, since $\overline{\alpha_i, x_{N_i}} \geq r$, we have $\overline{w_\infty, x_\infty} \geq r$. Therefore, $w_\infty \in A_\infty \setminus B_r(x_\infty)$. Thus, we have $w_i \in B_\tau(A_\infty \setminus B_r(x_\infty))$ for every sufficiently large i. This is a contradiction. □

EXAMPLE 2.7. Let $(Z_i, z_i) \to (Z_\infty, z_\infty)$. Then, for every $r > 0$, we have $\limsup_{i\to\infty} \partial B_r(z_i) \subset \partial B_r(z_\infty)$.

The proof of next proposition is done by a contradiction similar to the proof of Proposition 2.5 or 2.6.

PROPOSITION 2.8. *Let (Z_i, z_i) be a sequence of proper geodesic spaces and η_i a positive numbers. We assume that (Z_i, z_i) converges to (Z_∞, z_∞) and $\eta_i \to \eta_\infty$. Then for every $r > 0$, we have $\limsup_{i\to\infty}(B_r(z_i))_{\eta_i} \subset (B_r(z_\infty))_{\eta_\infty}$.*

We will give the following fundamental result by Gromov for precompactness of Gromov-Hausdorff topology. See [35] for the proof.

PROPOSITION 2.9. *Let $\{(Z_i, z_i)\}_i$ be a sequence of pointed proper geodesic spaces. We assume that for every $\epsilon > 0$ and $R \geq 1$, there exists N such that for every i, there exists a finite covering $\{B_\epsilon(x_j)\}_{j=1,\dots,N}$ of $B_R(z_i)$. Then, there exist a subsequence $\{(Z_{n(i)}, z_{n(i)})\}$ and a pointed proper geodesic space (Z_∞, z_∞) such that $(Z_{n(i)}, z_{n(i)})$ converges to (Z_∞, z_∞).*

We will give a result of precompactness for a sequence of compact sets;

PROPOSITION 2.10. *Let (Z_i, z_i) be a sequence of proper geodesic spaces and K_i a sequence of compact subset of Z_i. We assume that (Z_i, z_i) converges to (Z_∞, z_∞) and that there exists $R > 0$ such that $K_i \subset B_R(z_i)$ for every i. Then, there exist a subsequence $\{n(i)\}$ and a compact subset K_∞ of Z_∞ such that $(Z_{n(i)}, z_{n(i)}, K_{n(i)})$ converges to $(Z_\infty, z_\infty, K_\infty)$.*

PROOF. By the assumption, for every k, there exists N_k such that for every i, there exists $x_1(i,k),\dots,x_{N_k}(i,k) \in B_R(z_i)$ such that $K_i \subset B_R(z_i) \subset \bigcup_{j=1}^{N_k} B_{k^{-1}}(x_j(i,k))$. Since Z_∞ is proper, by diagonal argument, there exists a subsequence $\{n(i)\}$ such that $\{\phi_{n(i)}(x_j(n(i),k))\}$ is Cauchy sequence for every j, k. We put $x_j(k) = \lim_{i\to\infty} \phi_{n(i)}(x_j(n(i),k))$ and $K_\infty = \overline{\{x_j(k)\}}$. It is easy to check that $(Z_{n(i)}, z_{n(i)}, K_{n(i)})$ converges to $(Z_\infty, z_\infty, K_\infty)$. □

We will give a result of precompactness for a sequence of Lipschitz functions.

PROPOSITION 2.11. *Let (Z_i, z_i) be a sequence of proper geodesic spaces, K_i a sequence of compact subset of Z_i and f_i a sequence of Lipschitz function on K_i. We assume that (Z_i, z_i, K_i) converges to $(Z_\infty, z_\infty, K_\infty)$ and that $\sup_i(\mathbf{Lip} f_i + |f_i|_{L^\infty}) < \infty$. Then there exist a Lipschitz function f_∞ on K_∞ and a subsequence $\{n(i)\}$ such that $(Z_{n(i)}, z_{n(i)}, K_{n(i)}, f_{n(i)})$ converges to $(Z_\infty, z_\infty, K_\infty, f_\infty)$.*

PROOF. We take a countable dense subset $\{x_j\}$ of K_∞. For every x_j, we take $x_j(i) \in K_i$ satisfying that $x_j(i)$ converges to x_j. Then, there exists a subsequence $\{n(i)\}$ of $\mathbf{N}$

such that the sequence $\{f_{n(i)}(x_j(n(i)))\}$ is Cauchy sequence. We define a function F_∞ on $\{x_j\}$ by

$$F_\infty(x_j) = \lim_{i\to\infty} f_{n(i)}(x_j(n(i))).$$

It is easy to check that the function F_∞ is $\sup_i \mathbf{Lip} f_i$-Lipschitz function. Therefore there exists unique Lipschitz function f_∞ on K_∞ such that $F_\infty(x_j) = f_\infty(x_j)$. It is easy to check that f_∞ satisfies the assertion. □

We shall give a fundamental covering lemma (for proper metric spaces). See chapter 1 in [81] for the proof.

PROPOSITION 2.12. *Let X be a proper metric space, A a subset of X, Λ a set, $\{x_\lambda\}_{\lambda\in\Lambda}$ a collection of points in X and $\{r_\lambda\}_{\lambda\in\Lambda}$ a collection of positive numbers. We assume that for every $x \in A$ and $\epsilon > 0$, there exists $\lambda \in \Lambda$ such that $x \in \overline{B}_{r_\lambda}(x_\lambda)$ and $\mathrm{diam}\overline{B}_{r_\lambda}(x_\lambda) < \epsilon$. Then, there exists a countable subset $\Lambda_1 \subset \Lambda$ satisfying the following properties:*

1. *$\{\overline{B}_{r_{\lambda_1}}(x_{\lambda_1})\}_{\lambda_1\in\Lambda_1}$ are pairwise disjoint collection.*
2. *For every finite subset $\Lambda_2 \subset \Lambda_1$, we have*

$$A \setminus \bigcup_{\lambda_2\in\Lambda_2} \overline{B}_{r_{\lambda_2}}(x_{\lambda_2}) \subset \bigcup_{\lambda\in\Lambda_1\setminus\Lambda_2} \overline{B}_{5r_\lambda}(x_\lambda).$$

We shall recall the definition of measured Gromov-Hausdorff convergence by Fukaya, first. Let $(Z, z_i) \to (Z_\infty, z_\infty)$. For a sequence of Radon measure υ_i on Z_i, we say that (Z_i, z_i, υ_i) *converges to* $(Z_\infty, z_\infty, \upsilon_\infty)$ *in the sense of measured Gromov-Hausdorff topology* if

$$\lim_{i\to\infty} \upsilon_i(B_r(x_i)) = \upsilon_\infty(B_r(x_\infty))$$

for every $r > 0$, $x_\infty \in Z_\infty$ and sequence $x_i \in Z_i$ satisfying $x_i \to x_\infty$. Then we denote it by $(Z_i, z_i, \upsilon_i) \to (Z_\infty, z_\infty, \upsilon_\infty)$. We introduce a following fundamental result for precompactness of measured Gromov-Hausdorff topology. See [7, Theorem 1.6] or [30].

PROPOSITION 2.13. *Let $\{(Z_i, z_i, \upsilon_i)\}_i$ be a sequence of pointed proper geodesic spaces with Radon measure υ_i. We assume that $\upsilon_i(B_1(z_i)) = 1$ and that for every $R > 0$ there exists $K = K(R) \geq 1$ such that $\upsilon_i(B_{2r}(x_i)) \leq 2^K \upsilon_i(B_r(x_i))$ for every $0 < r < R$, $i \in \mathbf{N}$ and $x_i \in Z_i$. Then, there exist a subsequence $\{(Z_{n(i)}, z_{n(i)}, \upsilon_i)\}$ and a pointed proper geodesic space with Radon measure $(Z_\infty, z_\infty, \upsilon_\infty)$ such that $(Z_{n(i)}, z_{n(i)}, \upsilon_i)$ converges to $(Z_\infty, z_\infty, \upsilon_\infty)$.*

Next, we will give a relation between the measure of limit set and the limit of measures of sets:

PROPOSITION 2.14. *Let $\{(Z_i, z_i, \upsilon_i)\}_i$ be a sequence of pointed proper geodesic spaces with Radon measure υ_i and A_i a Borel subset of Z_i. We assume that $\upsilon_i(B_1(z_i)) = 1$, A_∞ is compact, $(Z_i, z_i, \upsilon_i) \to (Z_\infty, z_\infty, \upsilon_\infty)$, $\limsup_{i\to\infty} A_i \subset A_\infty$ and that for every $R > 0$ there exist $K = K(R) \geq 1$ such that $\upsilon_i(B_{2r}(x_i)) \leq 2^K \upsilon_i(B_r(x_i))$ for every $0 < r < R$, $i \in \mathbf{N}$ and $x_i \in Z_i$. Then we have*

$$\limsup_{i\to\infty} \upsilon_i(A_i) \leq \upsilon_\infty(A_\infty).$$

PROOF. By Proposition 2.12, there exists a pairwise disjoint collection $\{\overline{B}_{r_j}(x_j)\}_{j\in\mathbf{N}}$ such that $x_j \in A_\infty$, $0 < r_j << 1$ and $A_\infty \setminus \bigcup_{i=1}^N \overline{B}_{r_i}(x_i) \subset \bigcup_{i=N+1}^\infty \overline{B}_{5r_i}(x_i)$ for every N. We fix $\epsilon > 0$. We take N satisfying $\sum_{i=N+1}^\infty \upsilon_\infty(B_{r_i}(x_i)) < \epsilon$. By the assumption, we have $\sum_{i=N+1}^\infty \upsilon(B_{5r_i}(x_i)) < 2^{5K(1)}\epsilon$. We consider an open covering $\{B_{s_i}(y_i)\} = \{B_{(1+\epsilon)r_i}(x_i)\}_{i=1,\dots,N} \cup \{B_{5(1+\epsilon)r_i}(x_i)\}_{i\geq N+1}$ of A_∞. By compactness of A_∞, there exists a finite subcollection $\{B_{t_i}(z_i)\}_{i=1,\dots,l}$ of $\{B_{s_i}(y_i)\}$, such that $A_\infty \subset \bigcup_{i=1}^l B_{t_i}(z_i)$ and $|\upsilon_\infty(A_\infty) - \sum_{i=1}^l \upsilon_\infty(B_{t_i}(z_i))| < \Psi(\epsilon; K)$. There exists $\tau_0 > 0$ such that $\tau_0 << \min\{t_j; 1 \leq j \leq l\}$ and that $B_{\tau_0}(A_\infty) \subset \bigcup_{i=1}^l B_{t_i}(z_i)$. We take $\tau > 0$ and a sequence $z_i(j) \in Z_j$ satisfying that $\tau < \tau_0$ and that $z_i(j) \to z_i$. Then since $\phi_i(A_i) \subset B_{\tau_0}(A_\infty)$ for every sufficiently large i, it is easy to check that $A_i \subset \bigcup_{j=1}^l B_{t_j+\tau}(z_j(i))$ for every sufficiently large i. Therefore we have $\upsilon_i(A_i) \leq \sum_{j=1}^l \upsilon_i(B_{t_j+\tau}(z_j(i)))$. Thus,

$$\limsup_{i\to\infty} \upsilon_\infty(A_i) \leq \sum_{j=1}^l \upsilon_\infty(B_{t_j+\tau}(z_j)).$$

By letting $\tau \to 0$ and $\epsilon \to 0$, we have the assertion. □

PROPOSITION 2.15. *Let $\{(Z_i, z_i, \upsilon_i)\}_i$ be a sequence of pointed proper geodesic spaces with Radon measure υ_i. We assume that $\upsilon_i(B_1(z_i)) = 1$, $\mathrm{diam} Z_\infty > 0$, $(Z_i, z_i, \upsilon_i) \overset{(\phi_i, R_i, \epsilon_i)}{\to} (Z_\infty, z_\infty, \upsilon_\infty)$ and that for every $R > 0$, there exist $K = K(R) \geq 1$ such that $\upsilon_i(B_{2r}(x_i)) \leq 2^K \upsilon_i(B_r(x_i))$ for every $0 < r < R$, $i \in \mathbf{N}$ and $x_i \in Z_i$. Then, we have*

$$\lim_{i\to\infty} \sup_{x_i \in B_R(z_i), 0<r<R} |\upsilon_i(B_r(x_i)) - \upsilon_\infty(B_r(\phi_i(x_i)))| = 0$$

for every $R \geq 1$.

PROOF. By the assumption, it is easy to check that $\mathrm{rad} Z_\infty > 0$. Here $\mathrm{rad} X = \inf_{x_2\in X}(\sup_{x_1\in X} \overline{x_1, x_2})$ for metric space X. We put $K = K(100R)$. We take $0 < \tau << \mathrm{rad} Z_\infty$. Then, by the definition, there exists N satisfying that for every $N \leq i \leq \infty$ and $w \in Z_i$, there exists $\hat{w} \in Z_i$ such that $\overline{w, \hat{w}} = \tau$. Since $B_{\hat{\tau}}(w) \subset B_{\tau+\hat{\tau}}(\hat{w}) \setminus B_{\tau-\hat{\tau}}(\hat{w})$, by [21, Lemma 3.3] (or [4, Proposition 6.12]), there exists $0 < \hat{\tau} << \tau$ such that for every $N \leq i \leq \infty$, $w \in Z_i$ and $0 < t < \hat{\tau}$, we have

$$\upsilon_i(B_t(w)) \leq \Psi(t; K, R)\upsilon_i(B_{10\tau}(w)).$$

Therefore, for every $\epsilon > 0$, there exists $N_1 \in \mathbf{N}$ and $0 < r_1 << \min\{R, \hat{\tau}, \epsilon, 1\}$ such that for every $N_1 \leq i \leq \infty$, $0 < s < r_1$ and $z \in B_R(z_i)$, we have $\upsilon_i(B_s(z)) \leq \epsilon$. We take $\{x_j\}_{j=1,\dots,l} \subset B_R(z_\infty)$ and $\{t_j\}_{j=1,\dots,\hat{l}} \subset [0,R]$ satisfying $B_{\hat{R}}(z_\infty) \subset \bigcup_{j=1}^{l} B_{\epsilon r_1}(x_j)$ and $[0,R] \subset \bigcup_{j=1}^{\hat{l}} B_{\epsilon r_1}(t_j)$. We take $x_j(i) \in B_R(z_i)$ satisfying that $x_j(i) \to x_j$. There exists $N_2 \geq N_1$ such that $|\upsilon_i(B_{t_{\hat{j}}}(x_j(i))) - \upsilon_\infty(B_{t_{\hat{j}}}(x_j))| < \epsilon$. for every $i \geq N_2$, $j = 1,\dots,l$ and $\hat{j} = 1,\dots,\hat{l}$. Then, for every $z \in B_R(z_\infty)$ and $s \in [r_1, R]$, we take $j \in \{1,\dots,l\}$ and $\hat{j} \in \{1,\dots,\hat{l}\}$ satisfying $\overline{z, z_j} < \epsilon r_1$ and $|s - t_{\hat{j}}| < \epsilon r_1$. Then by [21, Lemma 3.3],

$$|\upsilon_\infty(B_s(z)) - \upsilon_\infty(B_{t_{\hat{j}}}(x_j))| \leq \upsilon_\infty(B_{s+5\epsilon r_1}(z)) - \upsilon_\infty(B_{s-5\epsilon r_1}(z)) \tag{1}$$

$$\leq \Psi(\epsilon; K, R, \tau)\upsilon_\infty(B_R(z_\infty)). \tag{2}$$

On the other hand, for a sequence $z(i) \in B_R(z_i)$ satisfying $z(i) \to z$,

$$|\upsilon_i(B_s(z(i))) - \upsilon_i(B_{t_{\hat{j}}}(x_j(i)))| \leq \upsilon_i(B_{s+10\epsilon r_1}(z(i))) - \upsilon_i(B_{s-10\epsilon r_1}(z(i))) \tag{3}$$

$$\leq \Psi(\epsilon; K, R, \tau)\upsilon_i(B_R(z_i)) \tag{4}$$

$$\leq \Psi(\epsilon; K, R, \tau)\upsilon_\infty(B_R(z_\infty)) \tag{5}$$

for every $i \geq N_2$. Thus, we have

$$|\upsilon_i(B_s(z(i))) - \upsilon_\infty(B_s(z))| < \Psi(\epsilon; K, R, \tau)\upsilon_\infty(B_R(z_\infty)).$$

Therefore, we have the assertion. □

We remark that an assumption $\mathrm{diam} Z_\infty > 0$ of Proposition 2.15 is necessary. For example, consider a sequence $\mathbf{S}^n(r) \to \{p\}$ as $r \to 0$. Here $\mathbf{S}^n(r) = \{x \in \mathbf{R}^{n+1}; |x| = r\}$.

PROPOSITION 2.16. *Let $\{(Z_i, z_i, \upsilon_i)\}_i$ be a sequence of pointed proper geodesic spaces with Radon measure υ_i. We assume that $\upsilon_i(B_1(z_i)) = 1$, $(Z_i, z_i, \upsilon_i) \overset{(\phi_i, R_i, \epsilon_i)}{\to} (Z_\infty, z_\infty, \upsilon_\infty)$ and that for every $R > 0$, there exist $K = K(R) \geq 1$ such that $\upsilon_i(B_{2r}(x_i)) \leq 2^K \upsilon_i(B_r(x_i))$ for every $0 < r < R$, $i \in \mathbf{N}$ and $x_i \in Z_i$ Then we have*

$$\lim_{i \to \infty} \int_{Z_i} f \circ \phi_i d\upsilon_i = \int_{Z_\infty} f d\upsilon_\infty$$

for every $f \in C_c^0(Z_\infty)$.

PROOF. We put $A = \mathrm{supp} f$ and fix $\epsilon > 0$. We take $R \geq 1$ satisfying $A \subset B_R(z_\infty)$ and put $K = K(100R)$. For every $x \in Z_\infty$, we take $r(x) > 0$ satisfying that for every $w \in B_{r(x)}(x)$, we have $f(w) = f(x) \pm \epsilon$. By Proposition 2.12, there exists a pairwise disjoint collection $\{\overline{B}_{r_i}(x_i)\}_i$ such that $x_i \in A$, $0 < r_i << \min\{r(x_i), \epsilon\}$ and $K \setminus \bigcup_{i=1}^{N} \overline{B}_{r_i}(x_i) \subset \bigcup_{i=N+1}^{\infty} \overline{B}_{5r_i}(x_i)$ for every N. We take N satisfying $\sum_{i=N+1}^{\infty} \upsilon_\infty(B_{r_i}(x_i)) < \epsilon$. By the

assumption, we have $\sum_{i=N+1}^{\infty} \upsilon_\infty(B_{5r_i}(x_i)) < \Psi(\epsilon; K)$. We take $x_j(i) \in Z_i$ satisfying that $x_j(i) \to x_j$. Then we have

$$\int_{Z_i} f \circ \phi_i d\upsilon_i = \sum_{j=1}^{N} \int_{B_{r_j}(x_j(i))} f \circ \phi_i d\upsilon_i \pm \left| \int_{Z_i \setminus \bigcup_{j=1}^{N} B_{r_j}(x_j(i))} f \circ \phi_i d\upsilon_i \right|.$$

We also have

$$\left| \int_{Z_i \setminus \bigcup_{j=1}^{N} B_{r_j}(x_j(i))} f \circ \phi_i d\upsilon_i \right| = \left| \int_{\phi_i^{-1}(A) \setminus \bigcup_{j=1}^{N} B_{r_j}(x_j(i))} f \circ \phi_i d\upsilon_i \right| \tag{6}$$

$$\leq \sup |f| \upsilon_i(\phi_i^{-1}(A) \setminus \bigcup_{j=1}^{N} B_{r_j}(x_j(i)) \tag{7}$$

$$\leq \sup |f| \upsilon_i(\overline{\phi_i^{-1}(A)} \setminus \bigcup_{j=1}^{N} B_{r_j}(x_j(i))). \tag{8}$$

By Proposition 2.14, we have

$$\limsup_{i \to \infty} \upsilon_i(\overline{\phi_i^{-1}(A)} \setminus \bigcup_{j=1}^{N} B_{r_j}(x_j(i))) \leq \upsilon_\infty(A \setminus \bigcup_{j=1}^{N} B_{r_j}(x_j)) \tag{9}$$

$$\leq \sum_{i=N+1}^{\infty} \upsilon_\infty(\overline{B}_{5r_j}(x_j)) \leq \Psi(\epsilon; K). \tag{10}$$

Therefore for every sufficiently large i, we have

$$\int_{Z_i} f \circ \phi_i d\upsilon_i = \sum_{j=1}^{N} (f(x_j) \pm \epsilon) \upsilon_i(B_{r_j}(x_j(i))) \pm \Psi(\epsilon; K, \sup |f|) \tag{11}$$

$$= \sum_{j=1}^{N} (f(x_j) \pm \epsilon) \upsilon_\infty(B_{r_j}(x_j)) \pm \Psi(\epsilon; K, \sup |f|, R) \tag{12}$$

$$= \int_{\bigcup_{j=1}^{N} B_{r_j}(x_j)} f d\upsilon_\infty \pm \Psi(\epsilon; K, \sup |f|) \tag{13}$$

$$= \int_{Z_i} f d\upsilon_\infty \pm \left(\int_{A \setminus \bigcup_{j=1}^{N} B_{r_j}(x_j)} |f| d\upsilon_\infty + \Psi(\epsilon; K, \sup |f|) \right) \tag{14}$$

$$= \int_{Z_i} f d\upsilon_\infty \pm \Psi(\epsilon; K, \sup |f|). \tag{15}$$

Therefore we have the assertion. □

In section 4, we will generalize Proposition 2.16. See Proposition 4.13.

2.3 Riemannian manifolds and its limit space

First, we shall introduce a very powerful gradient estimates for harmonic functions on manifolds by Cheng-Yau. This estimate is used in this paper many times. We fix $n \geq 2$.

THEOREM 2.17 (Cheng-Yau, [13]). *Let K be a real number, R a positive number, M a complete n-dimensional Riemannian manifold with $\mathrm{Ric}_M \geq K(n-1)$, m a point in M and f a nonnegative valued harmonic function on $B_R(m)$. Then, we have*

$$|\nabla f|(x) \leq C(n) f(x) \frac{R(R|K(n-1)|+1)}{R^2 - \overline{m,x}^2}$$

for every $x \in B_R(m)$.

Next theorem is a fundamental result for the study of Gromov-Hausdorff convergence of Riemannian manifolds:

THEOREM 2.18 (Bishop-Gromov, [35]). *Let K be a real number, M a complete n-dimensional Riemannian manifold with $\mathrm{Ric}_M \geq K(n-1)$ and m a point in M. Then we have*

$$\frac{\mathrm{vol}\, B_r(m)}{\mathrm{vol}\, B_r(\underline{p})} \geq \frac{\mathrm{vol}\, B_s(m)}{\mathrm{vol}\, B_s(\underline{p})}$$

for every $0 < r < s$. Here, $\underline{p}$ is a point in the n-dimensional space form $\underline{M}^n_K$ whose sectional curvature is equal to K.

As a corollary of Theorem 2.18, if a sequence of pointed n-dimensional complete Riemannian manifolds with renormalized volume $\{(M_i, m_i, \underline{\mathrm{vol}})\}$ satisfy $\mathrm{Ric}_{M_i} \geq K(n-1)$, then the sequence satisfies the assumption of Proposition 2.13. Here renormalized volume means

$$\underline{\mathrm{vol}} = \frac{\mathrm{vol}}{\mathrm{vol}\, B_1(m_i)}.$$

For a real number K and a pointed proper geodesic space (Y, y), in this paper, we say that (Y, y) *is (n, K)-Ricci limit space* if there exist a sequence of real numbers $\{K_i\}$ and a sequence of pointed n-dimensional complete Riemannian manifolds $\{(M_i, m_i)\}$ with $\mathrm{Ric}_{M_i} \geq K_i(n-1)$ such that $K_i \to K$ and $(M_i, m_i) \to (Y, y)$. Then, we often call (Y, y) a Ricci limit space of $\{(M_i, m_i)\}$. Similarly, for a pointed proper geodesic space with Radon measure (Y, y, υ), we also say that (Y, y, υ) is (n, K)-Ricci limit space if there exist a sequence of real numbers $\{K_i\}$ and a sequence of pointed n-dimensional complete Riemannian manifolds $\{(M_i, m_i)\}$ with $\mathrm{Ric}_{M_i} \geq K_i(n-1)$ such that $K_i \to K$ and $(M_i, m_i, \underline{\mathrm{vol}}) \to (Y, y, \upsilon)$. More simply, for $(n, -1)$-Ricci limit space (Y, y) (or (Y, y, υ)), we say that (Y, y) is Ricci limit space. See section 4.1 in [65]. We shall fix a Ricci limit space (Y, y, υ) in this subsection and give a very short review of structure theory of Ricci limit spaces developed by Cheeger-Colding below. See [7, 8, 9] for the details.

We shall give an important notion called *tangent cone* to study Ricci limit spaces: For pointed proper geodesic spaces (Z,z) and (X,x), we say that (Z,z) *is a tangent cone of* X *at* x if there exists a sequence of positive numbers $\{r_i\}$ such that $r_i \to 0$ and $(X,x,r_i^{-1}d_X) \to (Z,z)$. For $k \geq 1$, we put $\mathcal{R}_k(Y) = \{x \in Y;$ All tangent cones at x are isometric to $\mathbf{R}^k\}$ and call it *k-dimensional regular set*. More simply, we shall denote it by $\mathcal{R}_k$. We also put $\mathcal{R} = \bigcup_{1 \leq k \leq n} \mathcal{R}_k$ and call it *regular set*. Next theorem is an important properties for Ricci limit spaces:

THEOREM 2.19 (Cheeger-Colding, [7]). *We have* $v(Y \setminus \mathcal{R}) = 0$.

For $\delta, r > 0$ and $0 < \alpha < 1$, we put $(\mathcal{R}_k)_{\delta,r} = \{x \in Y; d_{GH}((\overline{B}_s(x),x),(\overline{B}_s(0_k),0_k)) \leq \delta s$ for every $0 < s \leq r\}$ and $(\mathcal{R}_{k;\alpha})_r = \{x \in Y; d_{GH}((\overline{B}_s(x),x),(\overline{B}_s(0_k),0_k)) \leq s^{1+\alpha}$ for every $0 < s \leq r\}$. Here $0_k \in \mathbf{R}^k$. By the definition, we remark that these set are closed. It is easy to check that $\bigcap_{\delta>0}(\bigcup_{r>0}(\mathcal{R}_k)_{\delta,r}) = \mathcal{R}_k$. We also put $\mathcal{R}_{k;\alpha} = \bigcup_{r>0}(\mathcal{R}_{k;\alpha})_r$. By [7, Theorem 3.23] and [7, Theorem 4.6], there exists $0 < \alpha(n) < 1$ such that $v(\mathcal{R}_k \setminus \mathcal{R}_{k;\alpha(n)}) = 0$, v is Ahlfors k-regular at each point in $\mathcal{R}_{k;\alpha(n)}$ for every k. Next, we shall introduce an important result for rectifiability and Poincaré inequality on Ricci limit spaces:

THEOREM 2.20 (Cheeegr-Colding, [9]). *Y is v-rectifiable, (Y,v) satisfies weak $(1,2)$-Poincaré inequality.*

More strongly, they proved that *segment inequality* on Ricci limit spaces holds. (We do not give the definition here. See [9, Theorem 2.15].) Therefore we can construct the cotangent bundle T^*Y of Y. Finally, for cut loci on Ricci limit spaces, we also remark that $v(C_x) = 0$ for every $x \in Y$. See [42, Theorem 3.2]. These results above are used in section 3, essentially.

3 Rectifiability on limit spaces

In this section, we shall study a rectifiability of Ricci limit spaces. These results given in this section are used in section 4, essentially.

3.1 Radial rectifiability

The main result in this subsection is Theorem 3.17.

LEMMA 3.1. *Let Z be a proper geodesic space, z a point in Z, s, δ positive numbers, v a Radon measure on Z and F a nonnegative valued Borel function on $B_s(m)$. We assume that there exists $K \geq 1$ such that for every $w \in B_s(z)$ and $0 < t \leq s$, we have*

$0 < \upsilon(B_{2t}(w)) \le 2^K \upsilon(B_t(w))$,

$$\frac{1}{\upsilon(B_s(z))}\int_{B_s(z)} F d\upsilon \le \delta.$$

Then, there exists a compact set $K \subset \overline{B}_{s/10^2}(z)$ *such that* $\upsilon(K)/\upsilon(B_{s/10^2}(z)) \ge 1-\Psi(\delta;K)$ *and that for every* $x \in K$ *and* $0 < t \le s/10^2$,

$$\frac{1}{\upsilon(B_t(x))}\int_{B_t(x)} F d\upsilon \le \Psi(\delta;K).$$

Proof. Without loss of generality, we can assume that F is a nonnegative valued Borel function on Z by $F \equiv 0$ on $Z \setminus B_s(z)$. We fix $C > 0$. We put $A_1(C) = \{w \in B_s(z); \int_{B_{s/10^2}(w)} F d\upsilon \ge C\upsilon(B_{s/10^2}(w))\}$ and take $x_1^1,\dots,x_{k_1}^1 \in A_1(C)$ which are an $s/10$-maximal separated subset of $A_1(C)$. We also put $A_2(C) = \{w \in B_s(m) \setminus \bigcup_{i=1}^{k_1} B_s(x_i^1); \int_{B_{s/10^3}(w)} F d\upsilon \ge C\upsilon(B_{s/10^3}(w))\}$ and take $x_1^2,\dots,x_{k_2}^2 \in A_2(C)$ which are $s/10^2$-maximal separated subset of $A_1(C)$. By iterating this argument, we put $A_l(C) = \{w \in B_s(m) \setminus \bigcup_{1\le j\le l-1,\ 1\le i\le k_j} B_{s/10^{l-2}}(x_i^{l-1}); \int_{B_{s/10^{l+1}}(w)} F d\mathrm{vol} \ge C\upsilon(B_{s/10^{l+1}}(w))\}$ and take $x_1^l,\dots,x_{k_l}^l \in A_l(C)$ which are $s/10^l$-maximal separated subset of $A_l(C)$.

Claim 3.2. *The collection* $\{\overline{B}_{s/10^{l+1}}(x_i^l)\}$ *are pairwise disjoint.*

We take $w \in \overline{B}_{s/10^{\hat{l}+1}}(x_{\hat{i}}^{\hat{l}}) \cap \overline{B}_{s/10^{l+1}}(x_i^l)$. We assume that $l < \hat{l}$. Then, by the definition, we have $x_{\hat{i}}^{\hat{l}} \in M \setminus \bigcup_{j=1}^{k_l} B_{s/10^{l-1}}(x_j^l)$. Especially, we have $\overline{x_{\hat{i}}^{\hat{l}}, x_i^l} \ge s/10^{l-1}$. Therefore, we have $\overline{B}_{s/10^{\hat{l}+1}}(x_{\hat{i}}^{\hat{l}}) \cap \overline{B}_{s/10^{l+1}}(x_i^l) = \emptyset$. This is a contradiction. Therefore, we have $l = \hat{l}$. By the definition, we have $i = \hat{i}$. Thus, we have Claim 3.2.

It is easy to check the following claim.

Claim 3.3. *We have* $\bigcup_{i\in\mathbf{N}} A_i(C) \subset \bigcup_{l\in\mathbf{N},1\le i\le k_l} \overline{B}_{s/10^{l-2}}(x_i^l)$

We have

(16)
$$\sum_{l\in\mathbf{N},1\le i\le k_l}\int_{B_{\frac{s}{10^{l+1}}}(x_i^l)} F d\upsilon \ge C\sum_{l\in\mathbf{N},1\le i\le k_l}\upsilon(B_{\frac{s}{10^{l+1}}}(x_i^l))$$

(17)
$$\ge CC(n)\sum_{l\in\mathbf{N},1\le i\le k_l}\upsilon(B_{\frac{s}{10^{l-2}}}(x_i^l)) \ge CC(n)\upsilon(\bigcup_{l\in\mathbf{N},1\le i\le k_l} B_{\frac{s}{10^{l-2}}}(x_i^l)).$$

On the other hand,

(18)
$$\sum_{l\in\mathbf{N},1\le i\le k_l}\int_{B_{\frac{s}{10^{l+1}}}(x_i^l)} F d\upsilon = \int_{\bigcup_{l\in\mathbf{N},1\le i\le k_l} B_{\frac{s}{10^{l+1}}}(x_i^l)} F d\upsilon$$

(19)
$$\le \int_{B_s(z)} F d\upsilon \le C(n)\upsilon(B_s(z))\delta.$$

Therefore, we have

$$\frac{\upsilon(\bigcup_{l\in\mathbf{N},1\le i\le k_l} B_{\frac{s}{10^{l-2}}}(x_i^l))}{\upsilon(B_s(m))} \le \frac{\delta}{C}C(n).$$

By taking $C=\sqrt{\delta}, K=\overline{B}_{s/10^2}(z)\setminus\bigcup_{l\in\mathbf{N},1\le i\le k_l} B_{\frac{s}{10^{l-2}}}(x_i^l)$, we have the assertion. □

Definition 3.4. let (Y,y,υ) be a Ricci limit space, k an integer satisfying $k\le n$ and r,δ positive numbers satisfying $r<1$ and $\delta<1$. Let $(\mathcal{R}_k)^y_{\delta,r}$, denote the set of points, $w\in Y$ such that for every $0<s\le r$, there exists a map Φ from $\overline{B}_s(w)$ to $\mathbf{R}^k$ such that $\pi_1\circ\Phi=r_y$ and that Φ gives an δs-Gromov-Hausdorff approximation to $\overline{B}_s(\Phi(w))$. Here, π_1 is the projection from $\mathbf{R}^k=\mathbf{R}\times\mathbf{R}^{k-1}$ to $\mathbf{R}$.

Lemma 3.5. *We have*

$$\bigcap_{\delta>0}\left(\bigcup_{r>0}\left((\mathcal{R}_k)^x_{\delta,r}\setminus C_x\right)\right)=\mathcal{R}_k\setminus C_x.$$

Proof. It is easy to check that

$$\bigcap_{\delta>0}\left(\bigcup_{r>0}\left((\mathcal{R}_k)^x_{\delta,r}\setminus C_x\right)\right)\subset\mathcal{R}_k\setminus C_x.$$

We take $w\in\mathcal{R}_k\setminus C_x$. By the definition, for every $\delta>0$, there exists $r>0$ such that for every $0<s<r$, there exists an δs-Gromov-Hausdorff approximation from $(\overline{B}_s(0_k),0_k)$ to $(\overline{B}_s(w),w)$. Here, $\overline{B}_s(0_k)\subset\mathbf{R}^k$. On the other hand, by splitting theorem (see [4, Theorem 9.27]), there exist a pointed proper geodesic space (W_s,w_s) and a map $\hat{\Phi}$ from $(\overline{B}_s(w),w)$ to $(\overline{B}_s(0,w_s),(0,w_s))$ such that $\pi_{\mathbf{R}}\circ\hat{\Phi}=r_x-\overline{x,w}$ and that $\hat{\Phi}$ gives an δs-Gromov-Hausdorff approximation. Here, $\overline{B}_s(0,w_s)\subset\mathbf{R}\times W_s$ with the product metric $\sqrt{d^2_{\mathbf{R}}+d^2_{W_s}}$, $\pi_{\mathbf{R}}$ is the projection from $\mathbf{R}\times W_s$ to $\mathbf{R}$. By rescaling $s^{-1}d_{\mathbf{R}^k}$ and [44, Claim 4.4], there exists an $\Psi(\delta;n)s$-Gromov-Hausdorff approximation f from $(\overline{B}_s(w_s),w_s)$ to $(\overline{B}_s(0_{k-1}),0_{k-1})$. We define a map g from $\overline{B}_s(w)$ to $\mathbf{R}^k$ by $g(z)=(\overline{x,z},f\circ\hat{\Phi})$. Let π_s be the canonical retraction from $\mathbf{R}^k$ to $\overline{B}_s(g(w))$. We put $\hat{g}=\pi_s\circ g$. Then, it is easy to check that $\hat{g}$ gives a $\Psi(\delta;n)s$-Gromov-Hausdorff approximation to $(\overline{B}_s(\hat{g}(w)),\overline{g}(w))$. Since δ is arbitrary, we have the assertion. □

For every proper geodesic space X, a point $x\in X$ and a positive number $\tau>0$, we put

$$\mathcal{D}^\tau_x=\{w\in X;\ \text{There exists } \alpha\in X \text{ such that } \overline{\alpha,w}\ge\tau \text{ and } \overline{x,w}+\overline{w,\alpha}=\overline{x,\alpha}\}.$$

It is easy to check that $\mathcal{D}^\tau_x$ is a closed set. By the definition, we have

$$\bigcup_{\tau>0}\mathcal{D}^\tau_x=X\setminus C_x.$$

LEMMA 3.6. *Let (Y, y, υ) be a Ricci limit space, k an integer satisfying $k \leq n$, δ, r positive numbers satisfying $\delta < 1, r < 1$, x a point in Y and w a point in $(\mathcal{R}_k)^x_{\delta,r} \cap \mathrm{Leb}((\mathcal{R}_k)_{\delta,r}) \backslash (C_x \cup \{x\})$. Then, there exists $\eta(w) > 0$ satisfying the following property: For every $0 < s \leq \eta(w)$, there exist a compact set $L \subset \overline{B}_s(w) \cap (\mathcal{R}_k)_{\delta,r}$ and points $x_2, x_3, \dots, x_k \in Y$ such that $\upsilon(L)/\upsilon(B_s(w)) \geq 1 - \Psi(\delta; n)$ and that the map $\Phi = (r_x, r_{x_2}, \dots, r_{x_k})$ from L to $\mathbf{R}^k$, gives $(1 \pm \Psi(\delta; n))$-bi-Lipschitz equivalent to the image $\Phi(L)$.*

PROOF. There exists $0 < \tau < r$ such that $\upsilon(B_s(w) \cap (\mathcal{R}_k)_{\delta,r})/\upsilon(B_s(w)) \geq 1 - \delta$ for every $0 < s < \tau$ and $w \in \mathcal{D}^\tau_x \setminus B_\tau(x)$. Let $(M_i, m_i, \underline{\mathrm{vol}}) \to (Y, y, \upsilon)$. We take $x_i, w_i \in M_i$ satisfying $w_i \to w$, $x_i \to x$. We fix $0 < s << \min\{\delta, \tau\}$. Then, for every sufficiently large i, there exists an δs-Gromov-Hausdorff approximation $\Phi^i = (\Phi^i_1, \dots, \Phi^i_k)$ from $(\overline{B}_s(w_i), w_i)$ to $(\overline{B}_s(0_k), 0_k)$ such that $\Phi^i_1 = r_{x_i} - r_{x_i}(w_i)$. We put $s_0 = \sqrt{\delta} s$. For convenience, we shall use the following notations for rescaled metrics $s_0^{-1} d_{M_i}, s_0^{-1} d_Y$: $\hat{\mathrm{vol}} = \mathrm{vol}^{s_0^{-1} d_{M_i}}$, $\hat{r}_w(\alpha) = s_0^{-1} r_w(\alpha)$, $\hat{B}_t(\alpha) = B_t^{s_0^{-1} d_{M_i}}(\alpha) = B_{s_0 t}(\alpha)$, $\hat{\upsilon} = \upsilon/\upsilon(B_{s_0}(y))$, $\hat{g} = s_0^{-1} g$ for a Lipschitz function g and so on. We also denote the differential section of g as rescaled manifolds $(M_i, s_0^{-1} d_{M_i})$ by $\hat{dg} : M_i \to T^* M_i$ and denote the Riemannian metric of $(M_i, s_0^{-1} d_{M_i})$ by $\langle \cdot, \cdot \rangle_{s_0} = s_0^{-2} \langle \cdot, \cdot \rangle$. We remark that $(M_i, m_i, s_0^{-1} d_{M_i}, \underline{\mathrm{vol}}^{s_0^{-1} d_{M_i}}) \to (Y, y, s_0^{-1} d_Y, \hat{\upsilon})$. The following claim follows from the proof of splitting theorem (see for instance [4, Lemma 9.8], [4, Lemma 9.10] and [4, Lemma 9.13]).

CLAIM 3.7. *For every sufficiently large i, there exist harmonic functions $\hat{\mathbf{b}}^i_j$ on $\hat{B}_{100^2}(w_i) (j = 1, \dots, k)$, and points $x^i_j \in \hat{B}_{\sqrt{\delta}^{-1}}(w_i), (j = 2, \dots, k))$ such that $|\hat{\mathbf{b}}^i_j - \hat{r}_{x^i_j}|_{L^\infty(\hat{B}_{100^2}(w_i))} \leq \Psi(\delta; n)$,*

$$\frac{1}{\hat{\mathrm{vol}}\, \hat{B}_{100^2}(w_i)} \int_{\hat{B}_{100^2}(w_i)} |\hat{d}\hat{\mathbf{b}}^i_j - \hat{d}\hat{r}_{x^i_j}|^2_{s_0} d\hat{\mathrm{vol}} \leq \Psi(\delta; n),$$

$$\frac{1}{\hat{\mathrm{vol}}\, \hat{B}_{100^2}(w_i)} \int_{\hat{B}_{100^2}(w_i)} |\langle \hat{d}\hat{\mathbf{b}}^i_j, \hat{\mathbf{b}}^i_l \rangle_{s_0}| d\hat{\mathrm{vol}} = \delta_{jl} \pm \Psi(\delta; n)$$

and

$$\frac{1}{\hat{\mathrm{vol}}\, \hat{B}_{100^2}(w_i)} \int_{\hat{B}_{100^2}(w_i)} |\mathrm{Hess}_{\hat{\mathbf{b}}^i_j}|^2_{s_0} d\hat{\mathrm{vol}} \leq \Psi(\delta; n)$$

for $1 \leq j, l \leq k$. Here $x = x^1_i$.

We define a nonnegative Borel function F_i on $\hat{B}_{100^2}(w_i)$ by

$$F_i = \sum_{l=1}^k \hat{\mathrm{Lip}}(\hat{\mathbf{b}}^i_l - \hat{r}_{x^i_l})^2 + \sum_{l \neq j} |\langle \hat{d}\hat{\mathbf{b}}^i_l, \hat{d}\hat{\mathbf{b}}^i_j \rangle_{s_0}| + \sum_{l=1}^k (|\mathrm{Hess}_{\hat{\mathbf{b}}^i_l}|_{s_0})^2.$$

By Lemma 3.1, there exists a compact set $K_i \subset \hat{\overline{B}}_{100}(w_i)$ such that $\hat{\mathrm{vol}}\, K_i / \hat{\mathrm{vol}}\, \hat{B}_{100}(w_i) \geq 1 - \Psi(\delta; n)$ and that for every $\alpha \in K_i$ and $0 < t < 100$, we have

$$\frac{1}{\hat{\mathrm{vol}}\, \hat{B}_t(\alpha)} \int_{\hat{B}_t(\alpha)} F_i d\hat{\mathrm{vol}} \leq \Psi(\delta; n).$$

Claim 3.8. *For every sufficiently large i, $\alpha \in K_i \cap \hat{B}_{50}(w_i)$ and $0 < t < 50$, there exists a constant $C^i_j (j = 1,\dots,k)$ such that $\hat{\mathbf{b}}^i_j = \hat{r}_{x^i_j} + C^i_j \pm \Psi(\delta; n)t$ on $\hat{B}_t(\alpha)$ for $j = 1,\dots,k$.*

The proof is as follows. By Poincaré inequality, we have

$$\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)} \int_{\hat{B}_t(\alpha)} \left| (\hat{\mathbf{b}}^i_j - \hat{r}_{x^i_j}) - \frac{1}{\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)} \int_{\hat{B}_t(\alpha)} (\hat{\mathbf{b}}^i_j - \hat{r}_{x^i_j}) d\hat{\mathrm{vol}} \right| d\hat{\mathrm{vol}} \tag{20}$$

$$\leq tC(n) \sqrt{\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)} \int_{\hat{B}_t(\alpha)} (\hat{\mathrm{Lip}}(\hat{\mathbf{b}}^i_1 - \hat{r}_{x_i}))^2 d\hat{\mathrm{vol}}} \tag{21}$$

$$\leq t\Psi(\delta; n). \tag{22}$$

For $C > 0$, Let $A_j(C)$, denote the set of points $\beta \in \hat{B}_t(\alpha)$, such that

$$\left| (\hat{\mathbf{b}}^i_j(\beta) - \hat{r}_{x^i_j}(\beta)) - \frac{1}{\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)} \int_{\hat{B}_t(\alpha)} (\hat{\mathbf{b}}^i_j - \hat{r}_{x^i_j}) d\hat{\mathrm{vol}} \right| \geq C.$$

Then, we have

$$\Psi(\delta; n)t \geq \frac{1}{\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)} \int_{\hat{B}_t(\alpha)} \left| (\hat{\mathbf{b}}^i_j - \hat{r}_{x^i_j}) - \frac{1}{\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)} \int_{\hat{B}_t(\alpha)} (\hat{\mathbf{b}}^i_j - \hat{r}_{x^i_j}) d\mathrm{vol} \right| d\hat{\mathrm{vol}} \tag{23}$$

$$\geq \frac{1}{\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)} \int_{A_j(C)} \left| (\hat{\mathbf{b}}^i_j - \hat{r}_{x^i_j}) - \frac{1}{\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)} \int_{\hat{B}_t(\alpha)} (\hat{\mathbf{b}}^i_j - \hat{r}_{x^i_j}) d\hat{\mathrm{vol}} \right| d\hat{\mathrm{vol}} \tag{24}$$

$$\geq C \frac{\hat{\mathrm{vol}}\, A_j(C)}{\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)}. \tag{25}$$

Therefore, for above $\Psi(\delta; n)$, if we put $C = \sqrt{\Psi(\delta; n)}t$, then we have

$$\frac{\hat{\mathrm{vol}}\, A_j(C)}{\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)} \leq \sqrt{\Psi(\delta; n)}.$$

Here, we assume that $\hat{B}_{\epsilon t}(\beta) \subset A_j(C)$ for some $\beta \in \hat{B}_t(\alpha)$ and $\epsilon > 0$. Then, by Bishop-Gromov volume comparison theorem, we have

$$C(n)\epsilon^n \leq \frac{\hat{\mathrm{vol}}\, B_{\epsilon t}(\beta)}{\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)} \leq \frac{\hat{\mathrm{vol}}\, A_j(C)}{\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)} \leq \sqrt{\Psi(\delta; n)}.$$

Therefore, for $C(n)$ above, if we take $\epsilon = \left(2C(n)^{-1}\sqrt{\Psi(\delta; n)}\right)^{1/n}$, then we have a contradiction. We put $\epsilon = \left(2C(n)^{-1}\sqrt{\Psi(\delta; n)}\right)^{1/n}$. We take $\beta \in \hat{B}_t(\alpha)$. We also take $\hat{\beta} \in \hat{B}_{(1-\epsilon)t}(\alpha)$ satisfying $\hat{r}_\beta(\hat{\beta}) < \epsilon t$. Then, there exists $\gamma \in \hat{B}_{\epsilon t}(\hat{\beta}) \setminus A_j(C)$. Thus, we have $\gamma \in \hat{B}_t(\alpha)$. By the definition of $A_j(C)$, we have

$$\hat{\mathbf{b}}^i_j(\gamma) = \hat{r}_{x^i_j}(\gamma) + \frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{100}(\alpha)} \int_{\hat{B}_{100}(\alpha)} (\hat{\mathbf{b}}^i_j - \hat{r}_{x^i_j}) d\hat{\mathrm{vol}} \pm \sqrt{\Psi(\delta; n)}t.$$

By Cheng-Yau's gradient estimate, we have $|\hat{\nabla}\hat{\mathbf{b}}^i_j|_{s_0} \leq C(n)$. Thus, we have

$$\hat{\mathbf{b}}^i_j(\beta) = \hat{r}_{x^i_j}(\beta) + \frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{100}(\alpha)} \int_{\hat{B}_{100}(\alpha)} (\hat{\mathbf{b}}^i_j - \hat{r}_{x^i_j}) d\hat{\mathrm{vol}} \pm \Psi(\epsilon; n)t.$$

Therefore we have Claim 3.8.

By an argument similar to the proof of [9, Theorem 3.3], we have the following:

Claim 3.9. *For every sufficiently large i, $\alpha \in K_i \cap \hat{B}_{50}(w_i)$ and $0 < t \leq 10^{-5}$, there exist a compact set $Z_t \subset M_i$, a point $z_t \in Z_t$ and a map ϕ from $(\hat{\overline{B}}_t(\alpha), \alpha)$ to $(\hat{\overline{B}}_t(z_t), z_t)$ such that the map $\Phi = (\hat{\mathbf{b}}^i_1, \dots, \hat{\mathbf{b}}^i_k, \phi)$ from $\hat{\overline{B}}_t(\alpha)$ to $\hat{\overline{B}}_{t+\Psi(\delta;n)t)}(\Phi(\alpha)) \subset (\mathbf{R}^k \times Z_t, \sqrt{d^2_{\mathbf{R}^k} + (s_0{}^{-1}d_{M_i})^2})$, gives $\Psi(\delta; n)t$-Gromov-Hausdorff approximation.*

We put $\hat{K}_i = K_i \cap \hat{\overline{B}}_{40}(w_i)$. Then, we have $\hat{\mathrm{vol}}\,K_i/\hat{\mathrm{vol}}\,\hat{B}_{40}(w_i) \geq 1 - \Psi(\delta; n)$. By Proposition 2.10, without loss of generality, we can assume that there exist a compact set $K_\infty \subset \hat{\overline{B}}_{40}(w)$ and points $x^\infty_j \in Y (2 \leq j \leq k)$ such that $x^i_j \to x^\infty_j$ and $K_i \to K_\infty$. By Proposition 2.14, we have $\hat{v}(K_\infty)/\hat{v}(\hat{B}_{40}(w)) \geq 1 - \Psi(\delta; n)$.

On the other hand, by Proposition 2.9, Claim 3.8 and 3.9, for every $\alpha \in K_\infty$ and $0 < t \leq 10^{-5}$, there exist a compact metric space Z_∞, a point $z_\infty \in Z_\infty$ and a map ϕ from $(\hat{\overline{B}}_t(\alpha), \alpha)$ to $(\overline{B}_t(z_\infty), z_\infty)$ such that the map $\hat{\phi} = (\hat{r}_x, \hat{r}_{x^\infty_2}, \dots, \hat{r}_{x^\infty_k}, \phi)$ from $\hat{\overline{B}}_t(\alpha)$ to $\hat{\overline{B}}_{t+\Psi(\delta;n)t}(\hat{\phi}(\alpha))$, gives an $\Psi(\delta; n)t$-Gromov-Hausdorff approximation.

We put $\hat{K_\infty} = K_\infty \cap (\mathcal{R}_k)_{\delta,r} \cap \overline{B}_{10^{-10}s_0}(w)$. Then, we have $\upsilon(\hat{K_\infty})/\upsilon(\overline{B}_{10^{-10}s_0}(w)) \geq 1 - \Psi(\delta; n)$. On the other hand, for every $\alpha \in \hat{K_\infty}$ and $0 < t \leq 10^{-5}$, if we take ϕ, Z_∞, z_∞ as above, then, since $\alpha \in (\mathcal{R}_k)_{\delta,r}$, we have $\mathrm{diam} Z_\infty \leq \Psi(\delta; n)t$. Especially, the map $f = (\hat{r}_x, \hat{r}_{x^\infty_2}, \dots, \hat{r}_{x^\infty_k})$ from $\hat{\overline{B}}_t(\alpha)$ to $\overline{B}_{t+\Psi(\delta;n)t}(f(\alpha))$, gives an $\Psi(\delta; n)t$-Gromov-Hausdorff approximation. Especially, for every $\alpha, \beta \in \hat{K_\infty}$ satisfying $\alpha \neq \beta$, if we put $0 < t = \hat{r}_\alpha(\beta) \leq 10^{-5}$, then we have

$$\sqrt{(\overline{x, \alpha}^{s_0{}^{-1}d_Y} - \overline{x, \beta}^{s_0{}^{-1}d_Y})^2 + \sum_{l=2}^{k} (\overline{x^\infty_l, \alpha}^{s_0{}^{-1}d_Y} - \overline{x^\infty_l, \beta}^{s_0{}^{-1}d_Y})^2} = \overline{\alpha, \beta}^{s_0{}^{-1}d_Y} \pm \Psi(\delta; n)t$$
$$= (1 \pm \Psi(\delta; n))\overline{\alpha, \beta}^{s_0{}^{-1}d_Y}.$$

Therefore, we have the assertion. □

Lemma 3.10. *Let (Y, y, υ) be a Ricci limit space and x a point in Y. Then, there exist a collection of compact subsets $\{C^x_{k,i}\}_{1 \leq k \leq n, i \in \mathbf{N}}$ of Y and a collection of points $\{x^l_{k,i}\}_{2 \leq l \leq k \leq n, i \in \mathbf{N}} \in Y$ satisfying the following properties:*

1. *$\bigcup_{i \in \mathbf{N}} C^x_{k,i} \subset \mathcal{R}_k$ for every k.*
2. *$\upsilon(\mathcal{R}_k \setminus \bigcup_{i \in \mathbf{N}} C^x_{k,i}) = 0$ for every k.*

3. *For every $z \in \bigcup_{i\in\mathbf{N}} C^x_{k,i}$ and $0 < \delta < 1$, there exists $C^x_{k,i}$ such that $z \in C^x_{k,i}$ and the map $\Phi^x_{k,i} = (r_x, r_{x^2_{k,i}}, \dots, r_{x^k_{k,i}})$ from $C^x_{k,i}$ to $\mathbf{R}^k$ gives $(1 \pm \delta)$-bi-Lipschitz equivalent to the image $\Phi^x_{k,i}(C^x_{k,i})$.*

Proof. We put

$$A_k = \bigcap_{m_1\in\mathbf{N}} \left(\bigcup_{m_2\in\mathbf{N}} (\mathcal{R}_k)^x_{1/m_1,1/m_2} \cap \mathrm{Leb}((\mathcal{R}_k)_{1/m_1,1/m_2}) \setminus (C_x \cup \{x\}) \right).$$

Claim 3.11. *We have $A_k \subset \mathcal{R}_k$ and $\upsilon(\mathcal{R}_k \setminus A_k) = 0$.*

The proof is as follows. For

$$B_k = \bigcap_{m_1\in\mathbf{N}} \left(\bigcup_{m_2\in\mathbf{N}} (\mathcal{R}_k)^x_{1/m_1,1/m_2} \cap (\mathcal{R}_k)_{1/m_1,1/m_2} \setminus (C_x \cup \{x\}) \right),$$

by Proposition 7.5, we have, $A_k \subset B_k$, $\upsilon(B_k \setminus A_k) = 0$. On the other hand, by Lemma 3.5, we have $B_k = \mathcal{R}_k \setminus \{C_x \cup \{x\}\}$. Since $\upsilon(C_x) = 0$, we have Claim 3.11.

For every $z \in A_k$ and $N \in \mathbf{N}$, we take $m_2 = m_2(z, N)$ satisfying $z \in (\mathcal{R}_k)^x_{1/N,1/m_2} \cap \mathrm{Leb}((\mathcal{R}_k)_{1/N,1/m_2}) \setminus (C_x \cup \{x\})$. By Lemma 3.6, there exists $\eta(z, N) > 0$ such that for every $0 < s \le \eta(z, N)$, there exist a compact set $L(z, s, N) \subset \overline{B}_s(z) \cap (\mathcal{R}_k)_{1/N,1/m_2}$ and points $x_2(z, s, N), \dots, x_k(z, s, N) \in Y$ such that $\upsilon(L(z, s, N))/\upsilon(\overline{B}_s(z)) \ge 1 - \Psi(N^{-1}; n)$ and that the map $\Phi_{z,s,N}(w) = (\overline{x, w}, \overline{x_2(z, s, N), w}, \dots, \overline{x_k(z, s, N), w})$ from $L(z, s, N)$ to $\mathbf{R}^k$, gives $(1 \pm \Psi(N^{-1}; n))$-bi-Lipschitz equivalent to the image $\Phi_{z,s,N}(L(z, s, N))$. We fix $R > 1$. By Lemma 2.12, there exists pairwise disjoint collection $\{\overline{B}_{s_i^{N.R}}(z_i^{N,R})\}_{i\in\mathbf{N}}$ such that $z_i^{N,R} \in A_k \cap \overline{B}_R(y)$, $0 < s_i^{N,R} \le \eta(z_i^{N,R}, N)/100$ and that $A_k \cap \overline{B}_R(y) \setminus \bigcup_{i=1}^m \overline{B}_{s_i^{N,R}}(z_i^{N,R}) \subset \bigcup_{i=m+1}^\infty \overline{B}_{5s_i^{N,R}}(z_i^{N,R})$ for every m. We put $\hat{L}(i, N, R) = L(z_i^{N,R}, 5s_i^{N,R}, N) \cap A_k \cap \overline{B}_R(y) \subset A_k \cap \overline{B}_R(y)$.

Claim 3.12. $\upsilon(A_k \cap \overline{B}_R(y) \setminus \bigcup_{N\ge N_0, i\in\mathbf{N}} \hat{L}(i, N, R)) = 0$ *for every* $N_0 \in \mathbf{N}$.

Because, for every $N \geq N_0$, we have

$$(26)\quad \upsilon\left(A_k \cap \overline{B}_R(y) \setminus \bigcup_{i\in\mathbf{N}} \hat{L}(i,N,R)\right)$$

$$(27)\quad \leq \upsilon\left(\bigcup_{i\in\mathbf{N}}\left(\overline{B}_{5s_i^{N,R}}(z_i^{N,R}) \cap A_k \cap \overline{B}_R(y)\right) \setminus \bigcup_{i\in\mathbf{N}}\left(L(z_i^{N,R}, 5s_i^{N,R}, N) \cap A_k \cap \overline{B}_R(y)\right)\right)$$

$$(28)\quad \leq \sum_{i\in\mathbf{N}} \upsilon(\overline{B}_{5s_i^{N,R}}(z_i^{N,R}) \setminus L(z_i^{N,R}, 5s_i^{N,R}, N))$$

$$(29)\quad \leq \Psi(N^{-1};n) \sum_{i\in\mathbf{N}} \upsilon(\overline{B}_{5s_i^{N,R}}(z_i^{N,R}))$$

$$(30)\quad \leq \Psi(N^{-1};n) \sum_{i\in\mathbf{N}} \upsilon(B_{s_i^{N,R}}(z_i^{N,R}))$$

$$(31)\quad \leq \Psi(N^{-1};n)\upsilon(B_{2R}(y)).$$

Therefore, by letting $N \to \infty$, we have Claim 3.12.

By Claim 3.12, we have $\upsilon(A_k \cap \overline{B}_R(y) \setminus \bigcap_{N_0}(\bigcup_{N\geq N_0, i\in\mathbf{N}} \hat{L}(i,N,R))) = 0$. We put $E(i,N,R) = \hat{L}(i,N,R) \cap \bigcap_{N_0\in\mathbf{N}}(\bigcup_{N\geq N_0, j\in\mathbf{N}} \hat{L}(j,N,R))$. Then, we have $\upsilon(A_k \cap \overline{B}_R(y) \setminus \bigcup_{i,N\in\mathbf{N}} E(i,N,R)) = 0$. For every $z \in \bigcup_{i,N\in\mathbf{N}} E(i,N,R)$ and $0 < \delta < 1$, we take $i, N \in \mathbf{N}$ satisfying $z \in E(i,N,R)$. We also take $N_0 \in \mathbf{N}$ satisfying $N_0^{-1} << \delta$. Then there exist $\hat{N} \geq N_0$ and $\hat{i} \in \mathbf{N}$ such that $z \in \hat{L}(\hat{i},\hat{N},R)$. Then, the map $\phi(w) = (\overline{x,w}, \overline{x_2(z_{\hat{i}}^{\hat{N},R}, s_{\hat{i}}^{\hat{N},R}), w}, \dots, \overline{x_k(z_{\hat{i}}^{\hat{N},R}, s_{\hat{i}}^{\hat{N},R}), w})$ from $L(z_{\hat{i}}^{\hat{N},R}, s_{\hat{i}}^{\hat{N},R}, \hat{N})$ to $\mathbf{R}^k$, gives $\Psi(N^{-1},n)$-bi-Lipschitz equivalent to the image. Especially, the map gives $(1\pm\delta)$-bi-Lipschitz equivalent to the image. We remark that $\hat{L}(\hat{i},\hat{N},R) \subset L(z_{\hat{i}}^{\hat{N},R}, s_{\hat{i}}^{\hat{N},R}, \hat{N})$ and $z \in \hat{L}(\hat{i},\hat{N},R) \cap \bigcap_{l\in\mathbf{N}}(\bigcup_{j\geq l, p\in\mathbf{N}} \hat{L}(p,j,R)) = E(\hat{i},\hat{N},R)$. Therefore, if we put $x_2(i,N,R) = x_2(z_i^{N,R}, s_i^{N,R}, R), \dots, x_k(i,N,R) = x_k(z_i^{N,R}, s_i^{N,R}, R)$, then we have the following claim:

Claim 3.13. *For every $z \in \bigcup_{i,N\in\mathbf{N}} E(i,N,R)$ and $0 < \delta < 1$, there exists $E(i,N,R)$ such that $z \in E(i,N,R)$ and that the map $\phi(w) = (\overline{x,w}, \overline{x_2(i,N,R),w}, \dots, \overline{x_k(i,N,R),w})$ from $E(i,N,R)$ to $\mathbf{R}^k$, gives $(1\pm\delta)$-bi-Lipschitz equivalent to the image.*

By Claim 3.13, it is easy to check the assertion. □

Lemma 3.14. *With same notaion as in Lemma 3.10, let $\{\mathcal{D}_{k,i,j}^x\}_{j\in\mathbf{N}}$ be a collection of Borel subsets of $C_{k,i}^x$ satisfying $\upsilon(C_{k,i}^x \setminus \bigcup_{j\in\mathbf{N}} \mathcal{D}_{k,i,j}^x) = 0$. Then, there exists a collection of Borel subsets $\{\mathcal{E}_{k,i,j}^x\}$ such that $\mathcal{E}_{k,i,j}^x \subset \mathcal{D}_{k,i,j}^x$, $\upsilon(\mathcal{D}_{k,i,j}^x \setminus \mathcal{E}_{k,i,j}^x) = 0$ and that for every k, $z \in \bigcup_{i,j\in\mathbf{N}} \mathcal{E}_{k,i,j}^x$ and $0 < \delta < 1$, there exists $\mathcal{E}_{k,i,j}^x$ such that $z \in \mathcal{E}_{k,i,j}^x$ and that the map $\Phi_{k,i,j}^x = (r_x, r_{x_{k,i}^2}, \dots, r_{x_{k,i}^k})$ from $\mathcal{E}_{k,i,j}^x$ to $\mathbf{R}^k$ gives $(1\pm\delta)$-bi-Lipschitz equivalent to the image $\Phi_{k,i,j}^x(\mathcal{E}_{k,i,j}^x)$.*

PROOF. We fix $1 \leq k \leq n$. For every $M \in \mathbf{N}$, we put $\mathcal{B}_M = \{i \in \mathbf{N}$; the map $\phi = (r_x, r_{x^2_{k,i}}, \cdots, r_{x^k_{k,i}})$ from $C^x_{k,i}$ to $\mathbf{R}^k$, gives $(1 \pm M^{-1})$-bi-Lipschitz equivalent to the image $\}$ and $\mathcal{E}^x_{k,i,j} = \mathcal{D}^x_{k,i,j} \cap \bigcap_{M \in \mathbf{N}} (\bigcup_{i \in \mathcal{B}_M, j \in \mathbf{N}} \mathcal{D}^x_{k,i,j})$.

CLAIM 3.15. $\upsilon(\mathcal{D}^x_{k,i,j} \setminus \mathcal{E}^x_{k,i,j}) = 0$.

Because, by Lemma 3.10, we have $\bigcup_{i \in \mathbf{N}} C^x_{k,i} \subset \bigcap_{M \in \mathbf{N}} (\bigcup_{i \in \mathcal{B}_M} C^x_{k,i}))$. On the other hand, it is easy to check that $\bigcap_{M \in \mathbf{N}} (\bigcup_{i \in \mathcal{B}_M} C^x_{k,i})) \subset \bigcup_{i \in \mathbf{N}} C^x_{k,i}$. Therefore, we have $\bigcap_{M \in \mathbf{N}} (\bigcup_{i \in \mathcal{B}_M} C^x_{k,i})) = \bigcup_{i \in \mathbf{N}} C^x_{k,i}$. Thus, $\upsilon(\mathcal{D}^x_{k,i,j} \setminus \mathcal{E}^x_{k,i,j}) = \upsilon(\mathcal{D}^x_{k,i,j} \cap \bigcup_{l \in \mathbf{N}} C^x_{k,l} \setminus \mathcal{E}^x_{k,i,j}) = \upsilon(\mathcal{D}^x_{k,i,j} \cap \bigcap_{M \in \mathbf{N}} (\bigcup_{l \in \mathcal{B}_M} C^x_{k,l}) \setminus \mathcal{E}^x_{k,i,j}) = \upsilon(\mathcal{D}^x_{k,i,j} \cap \bigcap_{M \in \mathbf{N}} (\bigcup_{l \in \mathcal{B}_M, j \in \mathbf{N}} \mathcal{D}^x_{k,l,j}) \setminus \mathcal{E}^x_{k,i,j}) = 0$. Therefore we have Claim 3.15.

CLAIM 3.16. *For every $z \in \bigcup_{i,j \in \mathbf{N}} \mathcal{E}^x_{k,i,j}$ and $0 < \delta < 1$, there exists $\mathcal{E}^x_{k,i,j}$ such that $z \in \mathcal{E}^x_{k,i,j}$ and that the map ϕ from $\mathcal{E}^x_{k,i,j}$ to $\mathbf{R}^k$ defined by $\phi = (r_x, r_{x^2_{k,i}}, \cdots, r_{x^k_{k,i}})$ gives $(1 \pm \delta)$-bi-Lipschitz equivalent to the image.*

Because, we take $M \in \mathbf{N}$ and $i, j \in \mathbf{N}$ satisfying $M^{-1} << \delta$ and $z \in \mathcal{E}^x_{k,i,j}$. By the definition, there exist $N_0 \in \mathcal{B}_M$ and $N_1 \in \mathbf{N}$ such that $z \in \mathcal{D}^x_{k,N_0,N_1}$. Therefore, we have $z \in \mathcal{D}^x_{k,N_0,N_1} \cap \bigcap_{\hat{M} \in \mathbf{N}} (\bigcup_{\hat{i} \in \mathcal{B}_{\hat{M}}, \hat{j} \in \mathbf{N}} \mathcal{D}^x_{k,\hat{i},\hat{j}}) = \mathcal{E}^x_{k,N_0,N_1}$ and the map $\phi = (r_x, r_{x^2_{k,j}}, \cdots, r_{x^k_{k,j}})$ from $\mathcal{E}^x_{k,N_0,N_1}$ to $\mathbf{R}^k$, gives $(1 \pm M^{-1})$-bi-Lipschitz equivalent to the image. Therefore, we have Claim 3.16.

Thus, we have the assertion. □

The following theorem is the main result in this subsection. See Appendix 7.4 or (2.2) in [8] or [42, Definition 4.1] for the definition of the measure υ_{-1}.

THEOREM 3.17 (Radial rectifiability). *Let (Y, y, υ) be a Ricci limit space satisfying $Y \neq \{y\}$ and x a point in Y. Then, there exist a collection of Borel subsets $\{C^x_{k,i}\}_{1 \leq k \leq n, i \in \mathbf{N}}$ of Y, a collection of points $\{x^l_{k,i}\}_{2 \leq l \leq k \leq n, i \in \mathbf{N}}$ of Y, a positive number $0 < \alpha(n) < 1$ and a Borel subset A of $[0, \mathrm{diam} Y)$ such that the following properties hold:*

1. $\bigcup_{i \in \mathbf{N}} C^x_{k,i} \subset \mathcal{R}_{k,\alpha(n)} \setminus C_x$.
2. $\upsilon(\mathcal{R}_k \setminus \bigcup_{i \in \mathbf{N}} C^x_{k,i}) = 0$.
3. *For every $C^x_{k,i}$ and $z \in C^x_{k,i}$, we have $\lim_{r \to 0} \upsilon(B_r(z) \cap C^x_{k,i}) / \upsilon(B_r(z)) = 1$.*
4. *For every $C^x_{k,i}$, there exists $A^x_{k,i} > 1$ such that $(A^x_{k,i})^{-1} \leq \upsilon(B_r(z))/r^k \leq A^x_{k,i}$ holds for every $z \in C^x_{k,i}$ and $0 < r < 1$.*
5. *The limit measure υ and k-dimensional Hausdorff measure H^k are mutually absolutely continuous on $C^x_{k,i}$.*

6. *For every $z \in \bigcup_{i\in\mathbf{N}} C^x_{k,i}$ and $0 < \delta < 1$, there exists $C^x_{k,i}$ such that $z \in C^x_{k,i}$ and that the map $\Phi^x_{k,i} = (r_x, r_{x^2_{k,i}}, \cdots, r_{x^k_{k,i}})$ from $C^x_{k,i}$ to $\mathbf{R}^k$ gives $(1\pm\delta)$-bi-Lipschitz equivalent to the image $\Phi^x_{k,i}(C^x_{k,i})$.*

7. *$H^1([0, \mathrm{diam}Y) \setminus A) = 0$.*

8. *For every $R \in A$, the collection $\{\partial B_R(x) \cap C^x_{k,i}\} \subset \partial B_R(x) \setminus C_x$ satisfies following properties:*

   (a) *$\upsilon_{-1}\left((\partial B_R(x) \setminus C_x) \setminus \bigcup_{1\le k\le n, i\in\mathbf{N}} C^x_{k,i}\right) = 0$.*

   (b) *For every $\partial B_R(x) \cap C^x_{k,i}$, there exist $B^x_{k,i} > 1$ and $\tau^x_{k,i} > 0$ such that $(B^x_{k,i})^{-1} \le \upsilon_{-1}(\partial B_R(x) \cap B_r(z) \setminus C_x)/r^{k-1} \le \upsilon_{-1}(\partial B_R(x) \cap \overline{B}_r(z))/r^{k-1} \le B^x_{k,i}$ for every $z \in \partial B_R(x) \cap C^x_{k,i}$ and $0 < r < \tau^x_{k,i}$.*

   (c) *For every $z \in \bigcup_{i\in\mathbf{N}}(\partial B_R(x) \cap C^x_{k,i})$ and $0 < \delta < 1$, there exists $\partial B_R(x) \cap C^x_{k,i}$ such that $z \in \partial B_R(x) \cap C^x_{k,i}$ and that the map $\hat{\Phi}^x_{k,i} = (r_{x^2_{k,i}}, \cdots, r_{x^k_{k,i}})$ from $\partial B_R(x) \cap C^x_{k,i}$ to $\mathbf{R}^{k-1}$, gives $(1 \pm \delta)$-bi-Lipschitz equivalent to the image $\hat{\Phi}^x_{k,i}(\partial B_R(x) \cap C^x_{k,i})$.*

   Especially, $\partial B_R(x) \setminus C_x$ is υ_{-1}-rectifiable.

Proof. First, we shall prove the following claim:

Claim 3.18. *For every $R > 0$, $z \in \overline{B}_R(x) \setminus \{x\}$ and $0 < \epsilon < \min\{\overline{z,x}/100, 1\}$, we have $\upsilon_{-1}(\partial B_{\overline{x,z}}(x) \cap \overline{B}_\epsilon(z)) \le C(n)\upsilon(B_\epsilon(z))/\epsilon$.*

Because, by [43, Corollary 5.7], we have

$$\frac{\upsilon_{-1}(\partial B_{\overline{x,z}}(x) \cap \overline{B}_\epsilon(z))}{\mathrm{vol}\,\partial B_{\overline{x,z}}(\underline{p})} \le C(n)\frac{\upsilon(C_x(\partial B_{\overline{x,z}}(x) \cap \overline{B}_\epsilon(z)) \cap A_{\overline{x,z}-2\epsilon,\overline{x,z}}(x))}{\mathrm{vol}\,A_{\overline{x,z}-2\epsilon,\overline{x,z}}(\underline{p})}.$$

Here $C_x(A) = \{z \in Y;$ there exists $a \in A$ such that $\overline{x,z} + \overline{z,a} = \overline{z,a}\}$ for every subset A of Y. On the other hand, by triangle inequality, we have $C_x(\partial B_{\overline{x,z}}(x) \cap \overline{B}_\epsilon(z)) \cap A_{\overline{x,z}-2\epsilon,\overline{x,z}}(x) \subset \overline{B}_{100\epsilon}(z)$. Thus, we have

$$\upsilon_{-1}(\partial B_{\overline{x,z}}(x) \cap \overline{B}_\epsilon(z)) \le \frac{\mathrm{vol}\,\partial B_{\overline{x,z}}(\underline{p})}{\mathrm{vol}\,A_{\overline{x,z}-2\epsilon,\overline{x,z}}(\underline{p})}\upsilon(B_{100\epsilon}(z))C(n) \le C(n,R)\frac{1}{\epsilon}\upsilon(B_\epsilon(z)).$$

Therefore, we have Claim 3.18.

We take collections of Borel sets $\{C^x_{k,i}\}$ and of points $\{x^l_{k,i}\}$ as in Lemma 3.10. By Lemma 3.14, without loss of generality, we can assume that for every $C^x_{k,i}$, there exists $\tau > 0$ such that $C^x_{k,i} \subset \mathcal{D}^\tau_x \setminus B_\tau(x)$. Moreover, by [9, Theorem 3.23] and [9, Theorem 4.6], we can assume that for every $C^x_{k,i}$, there exists $A^x_{k,i} > 1$ such that for every $0 < r < 1$ and $z \in C^x_{k,i}$, we have $(A^x_{k,i})^{-1} \le \upsilon(B_r(z))/r^k \le A^x_{k,i}$. By Proposition 7.5, we can also assume that for every $C^x_{k,i}$ and $z \in C^x_{k,i}$, we have $\lim_{r\to 0} \upsilon(B_r(z) \cap C^x_{k,i})/\upsilon(B_r(z)) = 1$.

Claim 3.19. *Let (Y, y, υ) be a Ricci limit space, x a point in Y , τ, R positive numbers satisfying $0 < \tau < 1$, $R > 1$, and z a point in $\mathcal{D}_x^\tau \cap B_R(x) \setminus B_\tau(x)$. Then, we have $\upsilon(\partial B_{\overline{x,z}}(x) \cap B_\epsilon(z) \setminus C_x) \geq C(n,R)\upsilon(B_\epsilon(z))/\epsilon$ for every $0 < \epsilon < \tau/100$.*

The proof is as follows. We take $w \in Y$ satisfying $\overline{z,w} = \epsilon/100$ and $\overline{x,z} + \overline{z,w} = \overline{x,w}$. By [43, Theorem 4.6], we have

$$\frac{\upsilon(B_{\frac{\epsilon}{1000}}(w))}{\mathrm{vol}\, A_{\overline{x,z},\overline{x,z}+\epsilon}(\underline{p})} \leq C(n)\frac{\upsilon_{-1}\left(C_x(B_{\frac{\epsilon}{1000}}(w)) \cap \partial B_{\overline{x,z}}(x)\right)}{\mathrm{vol}\, \partial B_{\overline{x,z}}(\underline{p})}.$$

By triangle inequality, we have $C_x(B_{\epsilon/1000}(w)) \cap \partial B_{\overline{x,z}}(x) \subset \partial B_{\overline{x,z}}(x) \cap B_\epsilon(z)$. Thus, by Bishop-Gromov volume comparison theorem for υ,

$$\upsilon_{-1}(\partial B_{\overline{x,z}}(x) \cap B_\epsilon(z) \setminus C_x) \geq C(n)\frac{\mathrm{vol}\, \partial B_{\overline{x,z}}(\underline{p})}{\mathrm{vol}\, A_{\overline{x,z},\overline{x,z}+\epsilon}(\underline{p})}\upsilon(B_{\epsilon/1000}(w)) \tag{32}$$

$$\geq C(n,R)\frac{1}{\epsilon}\upsilon(B_{\frac{\epsilon}{1000}}(w)) \tag{33}$$

$$\geq C(n,R)\frac{1}{\epsilon}\upsilon(B_{5\epsilon}(w)) \tag{34}$$

$$\geq C(n,R)\frac{\upsilon(B_\epsilon(z))}{\epsilon}. \tag{35}$$

Therefore we have Claim 3.19.

By Claim 3.18 and 3.19, for every $C_{k,i}^x$, there exist $B_{k,i}^x > 1$ and $\tau_{k,i}^x > 0$ such that for every $z \in C_{k,i}^x$ and $0 < r < \tau_{k,i}^x$, we have $(B_{k,i}^x)^{-1} \leq \upsilon(\partial B_{\overline{x,z}}(x) \cap B_r(z) \setminus C_x)/r^k \leq B_{k,i}^x$. We put $\hat{A} = \{t \in [0, \mathrm{diam}Y); \upsilon_{-1}(\partial B_t(x) \setminus \bigcup C_{k,i}^x) = 0\}$. Since $\upsilon(Y \setminus \bigcup C_{k,i}^x) = 0$, by [43, Proposition 5.1] and [43, Theorem 5.2], we have, $\hat{A}$ is H^1-Lebesgue measurable, $H^1([0, \mathrm{diam}Y) \setminus \hat{A}) = 0$. Since υ is a Radon measure, there exists a Borel set $A \subset \hat{A}$ such that $H^1(\hat{A} \setminus A) = 0$. Thus we have the assertion. □

3.2 Calculation of radial derivative for Lipschitz functions

The purpose in this subsection is to calculate the radial derivative of Lipschitz functions: $\langle dr_x, df \rangle$ explicitly. The main result in this subsection is Theorem 3.33.

Lemma 3.20. *Let (Y, y) be a Ricci limit space satisfying $Y \neq \{y\}$, z a point in $Y \setminus C_y$, f a Lipschitz function on Y, τ a positive number and γ_i an isometric embedding from $[0, \overline{y,z} + \tau]$ to Y satisfying $\gamma_i(0) = y$ and $\gamma_i(\overline{y,z}) = z (i = 1, 2)$. We put $f_i = f \circ \gamma_i$. Then, we have $lipf_1(\overline{y,z}) = lipf_2(\overline{y,z})$ and $\mathrm{Lip}f_1(\overline{y,z}) = \mathrm{Lip}f_2(\overline{y,z})$.*

Proof. For every real number ϵ satisfying $0 < |\epsilon| << \tau$, by splitting theorem (see [4, Theorem 9.25] or [6, Theorem 6.64]), we have $\overline{\gamma_1(\overline{x,z}+\epsilon), \gamma_2(\overline{x,z}+\epsilon)} \leq \Psi(|\epsilon|; n)|\epsilon|$. Therefore, we have

$$\frac{|f \circ \gamma_1(\overline{x,z}+\epsilon) - f \circ \gamma_1(\overline{x,z})|}{|\epsilon|} \leq \frac{|f \circ \gamma_2(\overline{x,z}+\epsilon) - f \circ \gamma_2(\overline{x,z})|}{|\epsilon|} + \mathbf{Lip}f\Psi(|\epsilon|; n).$$

Thus, we have $\mathrm{Lip}f_1(\overline{y,z}) \leq \mathrm{Lip}f_2(\overline{y,z})$ and $lipf_1(\overline{y,z}) \leq lipf_2(\overline{y,z})$. Therefore we have $\mathrm{Lip}f_1(\overline{y,z}) = \mathrm{Lip}f_2(\overline{y,z})$ and $lipf_1(\overline{y,z}) = lipf_2(\overline{y,z})$. □

We shall give the following definition:

DEFINITION 3.21. Let (Y, y) be a Ricci limit space, z a point in $Y \setminus C_y$, τ a positive number, γ an isometric embedding from $[0, \overline{y,z}+\tau]$ to Y satisfying $\gamma(0) = y$ and $\gamma(\overline{y,z}) = z$. We put $F = f \circ \gamma$. Then, we put $lip_y^{\mathrm{rad}} f(z) = lipF(\overline{y,z})$ and $\mathrm{Lip}_y^{\mathrm{rad}} f(z) = \mathrm{Lip}F(\overline{y,z})$.

THEOREM 3.22. *Let (Y, y, υ) be a Ricci limit space, x a point in Y and f a Lipschitz function on Y. Then, we have the following:*

1. $lipf(z)^2 = lip_x^{\mathrm{rad}} f(z)^2 + lip(f|_{\partial B_{\overline{x,z}}(x)})(z)^2$ *for a.e.* $z \in Y$.
2. $\mathrm{Lip}f(z)^2 = \mathrm{Lip}_x^{\mathrm{rad}} f(z)^2 + \mathrm{Lip}(f|_{\partial B_{\overline{x,z}}(x)})(z)^2$ *for a.e.* $z \in Y$.
3. $\mathrm{Lip}(f|_{\partial B_{\overline{x,z}}(x)})(z) = lip(f|_{\partial B_{\overline{x,z}}(x)\setminus C_x})(z)$ *for a.e.* $z \in Y \setminus C_x$.

PROOF. First we shall remark the following:

CLAIM 3.23. *Let f be a Lipschitz function on $\mathbf{R}^k$. Then, we have* $\mathrm{Lip}f(z)^2 = (\mathrm{Lip}(f|_{\mathbf{R}\times\{z_2,\cdots,z_k\}})(z))^2 + (\mathrm{Lip}(f|_{\{z_1\}\times\mathbf{R}^{k-1}})(z))^2 = (lip(f|_{\mathbf{R}\times\{z_2,\cdots,z_k\}})(z))^2 + (lip(f|_{\{z_1\}\times\mathbf{R}^{k-1}})(z))^2 = lipf(z)^2$ *for a.e* $z = (z_1,\cdots, z_k) \in \mathbf{R}^k$.

Because, by Rademacher's theorem for Lipschitz functions on $\mathbf{R}^k$, the function f is totally differentiable at a.e $z \in \mathbf{R}^k$. Therefore we have Claim 3.23.

The next claim is clear:

CLAIM 3.24. *Let Z_i be metric spaces $(i = 1, 2)$, δ a positive number with $0 < \delta < 1$, and Φ a map from Z_1 to Z_2 satisfying $\Phi(Z_i) = Z_2$ and $(1-\delta)\overline{x_1,x_2} \leq \overline{\Phi(x_1),\Phi(x_2)} \leq (1+\delta)\overline{x_1,x_2}$ for every $x_1, x_2 \in Z_1$. Then, for every Lipschitz function f on Z_2, we have, $(1-\Psi(\delta))\mathrm{Lip}f(\Phi(z_1)) \leq \mathrm{Lip}(f\circ\Phi)(z_1) \leq (1+\Psi(\delta))\mathrm{Lip}f(z_1)$, $(1-\Psi(\delta))lipf(\Phi(z_1)) \leq lip(f\circ\Phi)(z_1) \leq (1+\Psi(\delta))lipf(\Phi(z_1))$ for every $z_1 \in Z_1$.*

We will give a proof of the following claim in appendix.

CLAIM 3.25. *For every Lebesgue measurable $A \subset \mathbf{R}^k$, we put $sl_1 - \mathrm{Leb}A = \{a = (a_1,\cdots,a_k) \in A; \lim_{r\to 0} H^{k-1}(\{a_1\}\times\overline{B}_r(a_2,\cdots,a_k)\cap A)/H^{k-1}(\{a_1\}\times\overline{B}_r(a_2,\cdots,a_k)) = 1\}$. Then we have the following:*

1. *The set $sl_1 - \mathrm{Leb}A$ is a Lebesgue measurable set.*
2. *For every $t \in \mathbf{R}$, $H^{k-1}(A \cap \{t\}\times\mathbf{R}^{k-1} \setminus sl_1 - \mathrm{Leb}A) = 0$.*
3. *$H^k(A \setminus sl_1 - \mathrm{Leb}A) = 0$.*

We put $L = \mathbf{Lip}f$. We take collections of Borel sets $\{C^x_{k,i}\}_{1\leq k\leq n, i\in\mathbf{N}}$ and of points $\{x^l_{k,i}\}_{2\leq k\leq n, i\in\mathbf{N}, 2\leq l\leq k}$ as in Theorem 3.17. We fix a sufficiently small $\delta > 0$ and $C_{k,i}$ satisfying that the map $\Phi^x_{k,i} = (r_x, r_{x^2_{k,i}}, \dots, r_{x^k_{k,i}})$ from $C^x_{k,i}$ to $\mathbf{R}^k$, gives $(1\pm\delta)$-bi-Lipschitz equivalent to the image. Then we put a function $f^x_{k,i} = f\circ(\Phi^x_{k,i})^{-1}$ on $\Phi^x_{k,i}(C^x_{k,i})$. and take a Lipschitz function $F^x_{k,i}$ on $\mathbf{R}^k$ satisfying $F^x_{k,i}|_{\Phi^x_{k,i}(C^x_{k,i})} = f^x_{k,i}$ and $\mathbf{Lip}F^x_{k,i} = \mathbf{Lip}f^x_{k,i}$.

Claim 3.26. *With notation as above, we have the following:*

1. $(1-\Psi(\delta;n))\mathrm{Lip}F^x_{k,i}(w) \leq \mathrm{Lip}f((\Phi^x_{k,i})^{-1}(w)) \leq (1+\Psi(\delta;n))\mathrm{Lip}F^x_{k,i}(w)$ *for a.e* $w \in \Phi^x_{k,i}(C^x_{k,i})$.

2. $(1-\Psi(\delta;n))lipF^x_{k,i}(w) \leq lipf((\Phi^x_{k,i})^{-1}(w)) \leq (1+\Psi(\delta;n))lipF^x_{k,i}(w)$ *for a.e* $w \in \Phi^x_{k,i}(C^x_{k,i})$.

3. $\mathrm{Lip}(F^x_{k,i}|_{\mathbf{R}\times\{w_2,\dots,w_k\}})(w) - L\Psi(\delta;n) \leq \mathrm{Lip}^{\mathrm{rad}}_x f((\Phi^x_{k,i})^{-1}(w)) \leq \mathrm{Lip}(F^x_{k,i}|_{\mathbf{R}\times\{w_2,\dots,w_k\}})(w) + L\Psi(\delta;n)$ *for a.e* $w = (w_1,\dots,w_k) \in \Phi^x_{k,i}(C^x_{k,i})$.

4. $lip(F^x_{k,i}|_{\mathbf{R}\times\{w_2,\dots,w_k\}})(w) - L\Psi(\delta;n) \leq lip^{\mathrm{rad}}_x f((\Phi^x_{k,i})^{-1}(w)) \leq lip(F^x_{k,i}|_{\mathbf{R}\times\{w_2,\dots,w_k\}})(w) + L\Psi(\delta;n)$ *for a.e* $w = (w_1,\dots,w_k) \in \Phi^x_{k,i}(C^x_{k,i})$.

5. $(1-\Psi(\delta;n))\mathrm{Lip}(F^x_{k,i}|_{\{w_1\}\times\mathbf{R}^{k-1}})(w) \leq \mathrm{Lip}(f|_{\partial B_{\overline{x,(\Phi^x_{k,i})^{-1}(w)}}(x)\cap C^x_{k,i}})((\Phi^x_{k,i})^{-1}(w)) \leq (1+\Psi(\delta;n))\mathrm{Lip}(F^x_{k,i}|_{\{w_1\}\times\mathbf{R}^{k-1}})(w)$ *for a.e.* $w = (w_1,\dots,w_k) \in \Phi^x_{k,i}(C^x_{k,i})$.

6. $(1-\Psi(\delta;n))lip(F^x_{k,i}|_{\{w_1\}\times\mathbf{R}^{k-1}})(w) \leq lip(f|_{\partial B_{\overline{x,(\Phi^x_{k,i})^{-1}(w)}}(x)\cap C^x_{k,i}})((\Phi^x_{k,i})^{-1}(w)) \leq (1+\Psi(\delta;n))lip(F^x_{k,i}|_{\{w_1\}\times\mathbf{R}^{k-1}})(w)$ *for a.e.* $w = (w_1,\dots,w_k) \in \Phi^x_{k,i}(C^x_{k,i})$.

The proof is as follows. First, we shall check the statement 1. We put $\mathbf{C}^x_{k,i} = \mathrm{Leb}(\Phi^x_{k,i}(C^x_{k,i}))\cap\Phi^x_{k,i}(\mathrm{Leb}C^x_{k,i})$. Then, we have $H^k(\Phi^x_{k,i}(C^x_{k,i})\setminus\mathbf{C}^x_{k,i}) = 0$. By Claim 3.24 and Proposition 7.5, we have $(1-\Psi(\delta))\mathrm{Lip}(F^x_{k,i}|_{\Phi_{k,i}(C^x_{k,i})})(w) \leq \mathrm{Lip}(f|_{C^x_{k,i}})((\Phi^x_{k,i})^{-1}(w)) \leq (1+\Psi(\delta))\mathrm{Lip}(F^x_{k,i}|_{\Phi^x_{k,i}(C^x_{k,i})})(w)$, $\mathrm{Lip}(F^x_{k,i}|_{\Phi^x_{k,i}(C^x_{k,i})})(w) = \mathrm{Lip}F^x_{k,i}(w)$ and $\mathrm{Lip}(f|_{C^x_{k,i}})((\Phi^x_{k,i})^{-1}(w)) = \mathrm{Lip}f((\Phi^x_{k,i})^{-1}(w))$ for every $w \in \mathbf{C}^x_{k,i}$. Therefore we have the statement 1. Similarly, we have the statement 2.

Next, we shall give a proof of statement 3. We put $\mathbf{C}^{x,f}_{k,i} = sl_1-\mathrm{Leb}\mathbf{C}^x_{k,i}\cap\{w\in\mathbf{R}^k; F^x_{k,i}$ is totally differentiable at $w.\}$. Then, by Claim 3.25, we have $H^k(\mathbf{C}^x_{k,i}\setminus\mathbf{C}^{x,f}_{k,i}) = 0$.

We take a point $w \in \mathbf{C}^{x,f}_{k,i}$ and put $w_\epsilon = w + (\epsilon, 0, \dots, 0)$ for every $\epsilon > 0$. Since $w \in \mathrm{Leb}\mathbf{C}^x_{k,i}$, there exists $\hat{w}_\epsilon \in \mathbf{C}^x_{k,i}$ such that $\overline{w_\epsilon, \hat{w}_\epsilon} \leq a(\epsilon)\epsilon (a(\tau)\to 0$ as $\tau\to 0)$. Clearly, $(1-\delta)(\epsilon - a(\epsilon)\epsilon) \leq (1-\delta)\overline{w,\hat{w}_\epsilon} \leq \overline{(\Phi^x_{k,i})^{-1}(w), (\Phi^x_{k,i})^{-1}(\hat{w}_\epsilon)} \leq (1+\delta)\overline{w,\hat{w}_\epsilon} \leq (1+\delta)(\epsilon + a(\epsilon)\epsilon)$. We define the projection π_1 from $\mathbf{R}^k$ to $\mathbf{R}$ by $\pi_1(w) = w_1$. Then we have $\overline{x, (\Phi^x_{k,i})^{-1}(\hat{w}_\epsilon)} = \pi_1(\hat{w}_\epsilon) = \pi_1(w_\epsilon) \pm a(\epsilon)\epsilon = \pi_1(w) + \epsilon \pm a(\epsilon)\epsilon = \overline{x, (\Phi^x_{k,i})^{-1}(w)} + \overline{(\Phi^x_{k,i})^{-1}(w), (\Phi^x_{k,i})^{-1}(\hat{w}_\epsilon)} \pm (\delta + a(\epsilon))\epsilon$. By Lemma 3.14, without loss of generality, we can assume that there exists $\tau_0 > 0$ such that $C_{k,i} \subset \mathcal{D}^{\tau_0}_x$. We take an isometric embedding γ from $[0, \overline{x, (\Phi^x_{k,i})^{-1}(w)} + \tau_0]$

to Y satisfying $\gamma(0) = x$ and $\gamma(\overline{x, (\Phi^x_{k,i})^{-1}(w)}) = (\Phi^x_{k,i})^{-1}(w)$. Then, by rescaling $\epsilon^{-1}d_Y$ and splitting theorem, we have $\overline{(\Phi^x_{k,i})^{-1}(\hat{w}_\epsilon), \gamma(\overline{x, (\Phi^x_{k,i})^{-1}(w) + \epsilon})} \leq \Psi(a(\epsilon), \delta; n)\epsilon$. For $\epsilon << \tau_0$, we have

$$\begin{aligned}\frac{|F^x_{k,i}(w) - F^x_{k,i}(w_\epsilon)|}{\epsilon} &\leq \frac{|F^x_{k,i}(w) - F^x_{k,i}(\hat{w}_\epsilon)|}{\epsilon} + La(\epsilon) \\ &\leq \frac{|f((\Phi^x_{k,i})^{-1}(w)) - f(\gamma(\overline{x, (\Phi^x_{k,i})^{-1}(w)} + \epsilon))|}{\epsilon} + L\Psi(a(\epsilon), \delta; n).\end{aligned}$$

By letting $\epsilon \to 0$, we have $\mathrm{Lip}(F^x_{k,i}|_{\mathbf{R}\times\{w_2,\dots,w_k\}})(w) \leq \mathrm{Lip}^{\mathrm{rad}}_x f((\Phi^x_{k,i})^{-1}(w)) + L\Psi(\delta; n)$. We take a sequence $\{\epsilon_i\}$ such that $\epsilon_j \to 0$ and that

$$\lim_{j\to\infty} \frac{|f \circ (\Phi^x_{k,i})^{-1}(w) - f(\gamma(\overline{x, (\Phi^x_{k,i})^{-1}(w)} + \epsilon_j))|}{|\epsilon_j|} = \mathrm{Lip}^{\mathrm{rad}}_x f((\Phi^x_{k,i})^{-1}(w)).$$

We fix $j \in \mathbf{N}$. We assume that $\epsilon_j > 0$. Since $(\Phi^x_{k,i})^{-1}(w) \in \mathrm{Leb}C^x_{k,i}$, there exists $\hat{w}(j) \in C^x_{k,i}$ such that $\overline{\hat{w}(j), \gamma(\overline{x, (\Phi^x_{k,i})^{-1}(w)} + \epsilon_j)} \leq \tau_j\epsilon_j (\tau_j \to 0 \text{ as } j \to \infty)$. Then, we have

$$\pi_1(\hat{w}(j)) - \pi_1(w) = \overline{x, \hat{w}(j)} - \overline{x, (\Phi^x_{k,i})^{-1}(w)} \tag{36}$$

$$= \overline{x, \gamma(\overline{x, (\Phi^x_{k,i})^{-1}(w)} + \epsilon_j)} \pm \tau_j\epsilon_j \tag{37}$$

$$= \epsilon_j \pm \tau_j\epsilon_j \tag{38}$$

$$= \overline{\gamma(\overline{x, (\Phi^x_{k,i})^{-1}(w)} + \epsilon_j), (\Phi^x_{k,i})^{-1}(w)} \pm \tau_j\epsilon_j \tag{39}$$

$$\geq (1-\delta)\overline{\Phi^x_{k,i}(\hat{w}(j)), w} - \tau_j\epsilon_j. \tag{40}$$

On the other hand, since $\overline{\Phi^x_{k,i}(\hat{w}(j)), w} \leq (1+\delta)\epsilon_j + \tau_j\epsilon_j$, we have $\overline{w + (\epsilon_j, 0, \dots, 0), \Phi^x_{k,i}(\hat{w}(j))} \leq \Psi(|\epsilon_j|, \delta; n)|\epsilon_j|$. Similarly, we have the inequality above in the case $\epsilon_j < 0$. We put $w(j) = w + (\epsilon_j, 0, \dots, 0)$. Then, we have

$$\begin{aligned}\frac{|f((\Phi^x_{k,i})^{-1}(w)) - f(\gamma(\overline{x, (\Phi^x_{k,i})^{-1}(w)}) + \epsilon_j))}{|\epsilon_j|} &\leq \frac{|F^x_{k,i}(w) - F^x_{k,i}(\Phi^x_{k,i}(\hat{w}(j)))|}{|\epsilon_j|} + L\tau_j \\ &\leq \frac{|F^x_{k,i}(w) - F^x_{k,i}(w(j))|}{|\epsilon_j|} + L\Psi(|\epsilon_j|, \tau_j, \delta; n).\end{aligned}$$

By letting $j \to \infty$, we have the statement 3. Similarly, we have the statement 4.

We shall give a proof of the statement 5. we take $w \in \mathbf{C}^{xf}_{k,i}$. By Claim 3.24, we have

$$(1 - \Psi(\delta))\mathrm{Lip}(F^x_{k,i}|_{\{w_1\}\times\mathbf{R}^{k-1}\cap\mathbf{C}^x_{k,i}})(w) \leq \mathrm{Lip}(f|_{(\Phi^x_{k,i})^{-1}(\{w_1\}\times\mathbf{R}^{k-1}\cap\mathbf{C}^x_{k,i})})(\Phi^x_{k,i})^{-1}(w) \tag{41}$$

$$\leq (1 + \Psi(\delta))\mathrm{Lip}(F^x_{k,i}|_{\{w_1\}\times\mathbf{R}^{k-1}\cap\mathbf{C}^x_{k,i}})(w). \tag{42}$$

We remark that $(\Phi^x_{k,i})^{-1}(\{w_1\}\times\mathbf{R}^{k-1}\cap\mathbf{C}^x_{k,i})=\partial B_{\overline{x,(\Phi^x_{k,i})^{-1}(w)}}(x)\cap\mathbf{C}^x_{k,i}$. By Proposition 7.7, we have $\mathrm{Lip}(F^x_{k,i}|_{\{w_1\}\times\mathbf{R}^{k-1}\cap\mathbf{C}^x_{k,i}})(w)=\mathrm{Lip}(F^x_{k,i}|_{\{w_1\}\times\mathbf{R}^{k-1}})(w)$. Therefore, by Claim 3.24, we have

$$(1-\Psi(\delta))\mathrm{Lip}(F^x_{k,i}|_{\{w_1\}\times\mathbf{R}^{k-1}})(w)\leq \mathrm{Lip}f|_{\partial B_{\overline{x,(\Phi^x_{k,i})^{-1}(w)}}(x)\cap\mathbf{C}^x_{k,i}})((\Phi^x_{k,i})^{-1}(w)) \tag{43}$$

$$\leq \mathrm{Lip}(f|_{\partial B_{\overline{x,(\Phi^x_{k,i})^{-1}(w)}}(x)\cap C^x_{k,i}})(\Phi^x_{k,i})^{-1}(w)) \tag{44}$$

$$\leq (1+\Psi(\delta))\mathrm{Lip}(F^x_{k,i}|_{\{w_1\}\times\mathbf{R}^{k-1}\cap\Phi^x_{k,i}(C^x_{k,i})})(w) \tag{45}$$

$$\leq (1+\Psi(\delta))\mathrm{Lip}(F^x_{k,i}|_{\{w_1\}\times\mathbf{R}^{k-1}})(w). \tag{46}$$

Thus we have the statement 5. Similarly, we have the statement 6.

Therefore we have Claim 3.26.

Claim 3.27. *With same notation as in Claim 3.26, we have*

$$lip(f|_{\partial B_{\overline{x,(\Phi^x_{k,i})^{-1}(w)}}(x)\cap C^x_{k,i}})((\Phi^x_{k,i})^{-1}(w))\geq \mathrm{Lip}(f_{\partial B_{\overline{x,(\Phi^x_{k,i})^{-1}}}(x)})((\Phi^x_{k,i})^{-1}(w))-\Psi(\delta;n,L)$$

for a.e $w\in\Phi^x_{k,i}(C^x_{k,i})$.

The proof is as follows. We will use same notaion as in the proof of Claim 3.26. We take $w\in\Phi^x_{k,i}(\mathrm{Leb}(\Phi^x_{k,i})^{-1}(\mathbf{C}^{x,f}_{k,i}))$ and put $z=(\Phi^x_{k,i})^{-1}(w)$. First, we assume $k\geq 2$. We shall prove that $z\in\partial B_{\overline{x,z}}(x)$ is not an isolated point in $\partial B_{\overline{x,z}}(x)\backslash C_x$. Because, by the definition of $sl_1-\mathrm{Leb}(\mathbf{C}^x_{k,i})$, there exists a sequence $\{\beta(j)\}\in\mathbf{C}^x_{k,i}$ such that $\pi_1(\beta(j))=\pi_1(w)$, $\beta(j)\neq w$ and $\beta(j)\to w$. Then, we have $(\Phi^x_{k,i})^{-1}(\beta(j))\neq z$, $(\Phi^x_{k,i})^{-1}(\beta(j))\in\partial B_{\overline{x,z}}(x)\backslash C_x$ and $(\Phi^x_{k,i})^{-1}(\beta(j))\to z$. Therefore, z is not an isolated point in $\partial B_{\overline{x,z}}(x)\setminus C_x$.

We take a sequence $\{z(j)\}\in\partial B_{\overline{x,z}}(x)\setminus\{z\}$ such that $z(j)\to z$ and that $|f(z(j))-f(z)|/\overline{z(j),z}\to\mathrm{Lip}(f|_{\partial B_{\overline{x,z}}(x)})(z)$. We put $\eta_j=\overline{z(j),z}>0$. Since $z\in\mathrm{Leb}(\Phi^x_{k,i})^{-1}(\mathbf{C}^{x,f}_{k,i})$, there exists $\hat{z}(j)\in(\Phi^x_{k,i})^{-1}(\mathbf{C}^{x,f}_{k,i})$ such that $\overline{z(j),\hat{z}(j)}\leq\hat{\tau}_j\eta_j(\hat{\tau}_j\to 0$ as $j\to\infty)$. We put $\alpha(j)=\Phi^x_{k,i}(\hat{z}(j))$. Thus, we have $|\pi_1(\alpha(j))-\pi_1(w)|\leq(1+\delta)\hat{\tau}_j\eta_j$. Therefore, there exists $\hat{\alpha}(j)\in\{w_1\}\times\mathbf{R}^{k-1}$ such that $\overline{w(j),\hat{\alpha}(j)}\leq\Psi(\hat{\tau}_j;n)\eta_j$. Then, we have

$$\frac{|f(z(j))-f(z)|}{\overline{z(j),z}}\leq\frac{|f(\hat{z}(j))-f(z)|}{\eta_j}+L\hat{\tau}_j \tag{47}$$

$$\leq\frac{|F^x_{k,i}(w(j))-F^x_{k,i}(w)|}{\eta_j}+\Psi(\hat{\tau}_j;n,L) \tag{48}$$

$$\leq\frac{|F^x_{k,i}(\hat{\alpha}(j))-F^x_{k,i}(w)|}{\overline{\hat{\alpha}(j),w}}\frac{\overline{\hat{\alpha}(j),w}}{\eta_j}+L\Psi(\hat{\tau}_j;n,L). \tag{49}$$

By letting $j\to\infty$, we have Claim 3.27 for the case $k\geq 2$. Next, we assume $k=1$. It suffices to check that z is an isolated point in $\partial B_{\overline{x,z}}(x)$. We assume that z is not an isolated

point in $\partial B_{\overline{x,z}}(x)$. Then, there exists a sequence $\{z(i)\} \in \partial B_{\overline{x,z}}(x) \backslash \{z\}$ such that $z(i) \to z$. We take an isometric embedding γ from $[0, \overline{x,z} + \tau_0]$ to Y such that $\gamma(0) = x, \gamma(\overline{x,z}) = z$. Here τ_0 is a positive constant. We put $\epsilon(i) = \overline{z, z(i)}$. Then we have $\overline{z(i), \gamma(\overline{x,z} - \epsilon_i)} \geq \overline{x, z(i)} - \overline{x, \gamma(\overline{x,z} - \epsilon_i)} = \epsilon_i$, $\overline{z(i), \gamma(\overline{x,z} + \epsilon_i)} \geq \overline{x, \gamma(\overline{x,z} + \epsilon_i)} - \overline{x, z(i)} = \epsilon_i$. On the other hand, by Proposition 2.9, without loss of generality, we can assume that $(Y, \epsilon_i^{-1} d_Y, z)$ converges to some tangent cone $(T_z Y, 0_z)$ at z. By the argument above and splitting theorem, there exists a pointed proper geodesic space (W, w) such that $T_z Y = \mathbf{R} \times W$ and that $W \neq \{w\}$. On the other hand, $z \in C_{1,i} \subset \mathcal{R}_1$. This is a contradiction. Therefore we have the Claim 3.27 for the case $k = 1$.

By Claim 3.23, 3.26 and 3.27, for every $N \in \mathbf{N}$, we have $\mathrm{Lip} f(z)^2 = \mathrm{Lip}_x^{\mathrm{rad}} f(z)^2 + \mathrm{Lip}(f|_{\partial B_{\overline{x,z}(x)}})(z)^2 \pm N^{-1} = lip_x^{\mathrm{rad}} f(z)^2 + lip(f|_{\partial B_{\overline{x,z}}(x) \backslash C_x})(z)^2 \pm N^{-1} = lip f(z)^2 \pm N^{-1}$ for a.e. $z \in Y \setminus C_x$. Therefore, we have the assertion. □

Remark 3.28. For every Ricci limit space (Y, y, v) and Lipschitz function f on Y, by [4, Corollary 6.36], we have $lip f(x) = \mathrm{Lip} f(x)$ for a.e. $x \in Y$.

By an argument similar to the proof of Lemma 3.20, we have the following:

Lemma 3.29. *Let (Y, y) be a Ricci limit space satisfying $Y \neq \{y\}$, z a point in $Y \setminus C_y$, f a Lipschitz function on Y, τ a positive number and γ an isometric embedding from $[0, \overline{y,z} + \tau]$ to Y satisfying $\gamma(0) = y$ and $\gamma(\overline{y,z}) = z$. We assume that the limit $\lim_{r \to 0}(f \circ \gamma(\overline{y,z} + r) - f(z))/r$ exists. Then, for every isometric embedding $\hat{\gamma} : [0, \overline{y,z} + \tau] \to Y$ such that $\gamma(0) = y$ and that $\gamma(\overline{y,z}) = z$, we have $\lim_{r \to 0}(f \circ \hat{\gamma}(\overline{y,z} + r) - f(z))/r = \lim_{r \to 0}(f \circ \gamma(\overline{y,z} + r) - f(z))/r$.*

We shall give the following definition:

Definition 3.30. Let (Y, y) be a Ricci limit space satisfying $Y \neq \{y\}$, f a Lipschitz function on Y. We put

$$A_y = \left\{ x \in Y \setminus C_y ; \text{The limit } \lim_{r \to 0} \frac{f \circ \gamma(\overline{x,y} + r) - f(x)}{r} \text{ exists} \right\}.$$

Here γ is an isometric embedding from $[0, \overline{y,x} + \tau]$ $(\tau > 0)$ to Y satisfying $\gamma(0) = y$ and $\gamma(\overline{y,x}) = x$. For $x \in A_y$, we put

$$\frac{df}{dr_y}(x) = \lim_{r \to 0} \frac{f \circ \gamma(\overline{x,y} + r) - f(x)}{r}.$$

Similarly, we have the following lemma:

Lemma 3.31. *Let (Y, y) be a Ricci limit space satisfying $Y \neq \{y\}$, z a point in $Y \setminus C_y$, f a Lipschitz function on Y, τ a positive number and $\gamma_i (i = 1, 2)$ isometric embeddings from $[0, \overline{y,z} + \tau]$ to Y satisfying $\gamma(0) = y$ and $\gamma(\overline{y,z}) = z$. Then, we have $\liminf_{r \to 0} |f \circ \gamma_1(\overline{y,z} + r) - f(z)|/|r| = \liminf_{r \to 0} |f \circ \gamma_2(\overline{y,z} + r) - f(z)|/|r|$.*

With same notaion as in Lemma 3.31, we put $\underline{\mathrm{Lip}}_x^{\mathrm{rad}} f(z) = \liminf_{r\to 0} |f\circ\gamma_1(\overline{y,z}+r)-f(z)|/|r|$.

LEMMA 3.32. *Let (Y,y,υ) be a Ricci limit space, x a point in Y and f a Lipschitz function on Y. Then, we have $\underline{\mathrm{Lip}}_x^{\mathrm{rad}} f(z) = \mathrm{Lip}_x^{\mathrm{rad}} f(z)$ for a.e. $z\in Y$.*

PROOF. We will use same notaion as in the proof of Claim 3.26. We put $L = \mathbf{Lip}f$. We take a sufficiently small $0<\delta<1$ and a Borel set $C^x_{k,i}$ such that the map $\Phi^x_{k,i} = (r_x, r_{x^2_{k,i}},\cdots, r_{x^k_{k,i}})$ from $C^x_{k,i}$ to $\mathbf{R}^k$, gives a $(1\pm\delta)$-bi-Lipschitz equivalent to the image. We take $w\in \mathbf{C}^{x,f}_{k,i}$ and put $z=(\Phi^x_{k,i})^{-1}(w)$. We choose an isometric embedding γ from $[0,\overline{x,z}+\tau]$ to Y such that $\gamma(0)=x, \gamma(\overline{x,z})=z$. Here, τ is a positive constant. We take a sequence of real number, $\{\epsilon_i\}$ such that $\epsilon_i\to 0$ and $\lim_{i\to\infty}|f\circ\gamma(\overline{x,z}+\epsilon_i)-f(z)|/|\epsilon_i| = \underline{\mathrm{Lip}}_x^{\mathrm{rad}} f(z)$. By an argument similar to the proof of Claim 3.13, there exists $\hat{w}(j)\in C^x_{k,i}$ such that $\overline{\hat{w}(j),\gamma(\overline{x,z}+\epsilon_j)}\le \tau_j|\epsilon_j|(\tau_j\to 0$ as $j\to\infty)$ and that

$$\frac{|f(z)-f(\gamma(\overline{x,z}+\epsilon_j))|}{|\epsilon_j|} = \frac{|F^x_{k,i}(w)-F^x_{k,i}(\Phi^x_{k,i}(\hat{w}(j)))|}{|\epsilon_j|} - 2L\tau_j \tag{50}$$

$$\ge \frac{|F^x_{k,i}(w)-F^x_{k,i}(w_j)|}{|\epsilon_j|} - \Psi(\tau_j,\delta;n,L). \tag{51}$$

By letting $j\to\infty$, we have $\underline{\mathrm{Lip}}_x^{\mathrm{rad}} f(z)\ge \mathrm{Lip}(F^x_{k,i}|_{\mathbf{R}\times\{w_2,\cdots,w_k\}})(w) - \Psi(\delta;n,L) \ge \mathrm{Lip}_x^{\mathrm{rad}} f(z) - \Psi(\delta;n,L)$. Therefore, we have the assertion. □

Thus, we have

$$\mathrm{Lip}_x^{\mathrm{rad}} f(z) = \lim_{h\to 0}\frac{|f\circ\gamma(\overline{x,z}+h)-f(z)|}{|h|}$$

for a.e. $z\in Y\setminus C_x$.

THEOREM 3.33 (Radial derivative for Lipschitz functions). *Let (Y,y,υ) be a Ricci limit space satisfying $Y\neq\{y\}$, x a point in Y and f a Lipschitz function on Y. Then, we have $\upsilon(Y\setminus A_x)=0$ and*

$$\frac{df}{dr_x}(z) = \langle df, dr_x\rangle(z)$$

for a.e. $z\in A_x$.

PROOF. For every $w\in Y\setminus C_x$, there exist $\tau>0$ and an isometric embedding γ from $[0,\overline{x,z}+\tau]$ to Y such that $\gamma(0)=x$ and $\gamma(\overline{x,w})=w$. Then, by Theorem 3.22 and Lemma

3.32, for a.e. $w \in Y \setminus C_x$, we have

$$\begin{aligned}
\langle dr_x, df\rangle(w) &= \frac{1}{2}(\mathrm{Lip}(r_x+f)(w)^2 - \mathrm{Lip}f(w)^2 - \mathrm{Lip}r_x(w)^2)\\
&= \frac{1}{2}(\mathrm{Lip}_x^{\mathrm{rad}}(r_x+f)(w)^2 + \mathrm{Lip}((r_x+f)|_{\partial B_{\overline{x,z}}(x)\setminus C_x})(w)^2\\
&\quad - \mathrm{Lip}_x^{\mathrm{rad}}f(w)^2 - \mathrm{Lip}(f|_{\partial B_{\overline{x,z}}\setminus C_x})(w)^2 - 1)\\
&= \frac{1}{2}(\mathrm{Lip}_x^{\mathrm{rad}}(r_x+f)(w)^2 + \mathrm{Lip}(f|_{\partial B_{\overline{x,z}}(x)\setminus C_x})(w)^2\\
&\quad - \mathrm{Lip}_x^{\mathrm{rad}}f(w)^2 - \mathrm{Lip}(f|_{\partial B_{\overline{x,z}}\setminus C_x})(w)^2 - 1)\\
&= \frac{1}{2}(\mathrm{Lip}_x^{\mathrm{rad}}(r_x+f)(w)^2 - \mathrm{Lip}_x^{\mathrm{rad}}f(w)^2 - 1)\\
&= \frac{1}{2}\left(\lim_{h\to 0}\frac{|(r_x+f)\circ\gamma(\overline{x,w}+h) - (r_x+f)(w)|^2}{|h|^2} - \lim_{h\to 0}\frac{|f\circ\gamma(\overline{x,w}+h)-f(w)|^2}{|h|^2} - 1\right)\\
&= \frac{1}{2}\left(\lim_{h\to 0}\left|1+\frac{f\circ\gamma(\overline{x,w}+h)-f(w)}{h}\right|^2 - \lim_{h\to 0}\frac{|f\circ\gamma(\overline{x,w}+h)-f(w)|^2}{|h|^2} - 1\right)\\
&\left(\text{Here, we have the existence of the limit } \lim_{h\to 0}\frac{f\circ\gamma(\overline{x,w}+h)-f(w)}{h}.\right)\\
&= \frac{1}{2}\left(1 + 2\lim_{h\to 0}\frac{f\circ\gamma(\overline{x,w}+h)-f(w)}{h} + \lim_{h\to 0}\frac{|f\circ\gamma(\overline{x,w}+h)-f(w)|^2}{|h|^2}\right.\\
&\quad \left. - \lim_{h\to 0}\frac{|f\circ\gamma(\overline{x,w}+h)-f(w)|^2}{|h|^2} - 1\right)\\
&= \lim_{h\to 0}\frac{f\circ\gamma(\overline{x,w}+h)-f(w)}{h} = \frac{df}{dr_x}(w).
\end{aligned}$$

□

3.3 Rectifiability associated with Lipschitz functions

In this section, we will give a generalization of Theorem 3.17. First, we shall state the following lemma:

LEMMA 3.34. *Let δ be a positive number, $\{(M_i, m_i)\}_i$ a sequence of n-dimensional complete Riemannian manifolds with $\mathrm{Ric}_{M_i} \geq -\delta(n-1)$, (Y, y, υ) an $(n,-\delta)$-Ricci limit space of $\{(M_i, m_i, \underline{\mathrm{vol}})\}_i$, x, x_1, x_2 points in Y, $x(i), x_1(i), x_2(i)$ points in M_i, $\mathbf{b}_1^i$ a harmonic function on $B_{100}(x(i))$ and $\mathbf{b}_1^\infty$ a Lipschitz function on $B_{100}(x)$. We assume that $\overline{x,x_1} \geq \delta^{-1}$, $\overline{x,x_2} \geq \delta^{-1}$, $\overline{x,x_1} + \overline{x,x_2} - \overline{x_1,x_2} \leq \delta$, $x(i) \to x$, $x_j(i) \to x_j(i) (j = 1,2)$, $\sup_i \mathbf{Lip}\mathbf{b}_1^i < \infty$, $\mathbf{b}_1^i \to \mathbf{b}_1^\infty$ on $B_{100}(x)$,*

$$|\mathbf{b}_1^i - r_{x_1(i)}|_{L_\infty(B_{100}(x(i)))} \leq \delta,$$

$$\frac{1}{\mathrm{vol}\, B_{100}(x(i))}\int_{B_{100}(x(i))}|\nabla \mathbf{b}_1^i-\nabla r_{x_1(i)}|^2 d\mathrm{vol}\le\delta$$

and

$$\frac{1}{\mathrm{vol}\, B_{100}(x(i))}\int_{B_{100}(x(i))}|\mathrm{Hess}_{\mathbf{b}_1^i}|^2 d\mathrm{vol}\le\delta.$$

Then, we have

$$\frac{1}{\upsilon(B_1(x))}\int_{B_1(x)}|d\mathbf{b}_1^\infty-dr_{x_1}|^2 d\upsilon<\Psi(\delta;n).$$

We remark that Lemma 3.34 does *not* follows from [4, Lemma 9.10] directly. We shall give a proof of Lemma 3.34 in the proof of the following Lemma 3.35.

LEMMA 3.35. *Let δ be a positive number, $\{(M_i,m_i)\}_i$ a sequence of n-dimensional complete Riemannian manifolds with $\mathrm{Ric}_{M_i}\ge-\delta(n-1)$, (Y,y,υ) an $(n,-\delta)$-Ricci limit space of $\{(M_i,m_i,\underline{\mathrm{vol}})\}_i$, $x,x_j(j=1,2,3,4)$ points in Y and $x(i),x_j(i)(j=1,2,3,4)$ points in M_i. We assume that $x(i)\to x$, $x_j(i)\to x_j(j=1,2,3,4))$, $\overline{x,x_j}\ge\delta^{-1}$, $\overline{x,x_1}+\overline{x,x_2}-\overline{x_1,x_2}\le\delta$ and $\overline{x,x_3}+\overline{x,x_4}-\overline{x_3,x_4}\le\delta$. Then, for every sufficiently large i, we have*

$$\frac{1}{\upsilon(B_1(x))}\int_{B_1(x)}\langle dr_{x_1},dr_{x_2}\rangle d\upsilon=\frac{1}{\mathrm{vol}\, B_1(x(i))}\int_{B_1(x(i))}\langle dr_{x_1(i)},dr_{x_2(i)}\rangle d\mathrm{vol}\pm\Psi(\delta;n)$$

$$\frac{1}{\upsilon(B_1(x))}\int_{B_1(x)}\left|\langle dr_{x_1},dr_{x_2}\rangle d\upsilon-\frac{1}{\upsilon(B_1(x))}\int_{B_1(x)}\langle dr_{x_1},dr_{x_2}\rangle d\upsilon\right|d\upsilon<\Psi(\delta;n)$$

and

$$\frac{1}{\mathrm{vol}\, B_1(x(i))}\int_{B_1(x(i))}\left|\langle dr_{x_1(i)},dr_{x_2(i)}\rangle-\frac{1}{\upsilon(B_1(x))}\int_{B_1(x)}\langle dr_{x_1},dr_{x_2}\rangle d\upsilon\right|d\mathrm{vol}<\Psi(\delta;n)$$

PROOF. First, we remark the following claim:

CLAIM 3.36. *For every sufficiently large i, there exist harmonic functions $\mathbf{b}_1^i,\mathbf{b}_3^i$ on $B_{100}(x(i))$ such that $\mathbf{Lipb}_j^i\le C(n)$, $|\mathbf{b}_j^i-r_{x_j(i)}|_{L^\infty(B_{100}(x(i)))}\le\Psi(\delta;n)$,*

$$\frac{1}{\mathrm{vol}\, B_{100}(x(i))}\int_{B_{100}(x(i))}|d\mathbf{b}_j^i-dr_{x_j(i)}|^2 d\mathrm{vol}\le\Psi(\delta;n)$$

and

$$\frac{1}{\mathrm{vol}\, B_{100}(x(i))}\int_{B_{100}(x(i))}|\mathrm{Hess}_{\mathbf{b}_j^i}|^2 d\mathrm{vol}\le\Psi(\delta;n)$$

for $j=1,3$.

See [4, Lemma 9.8, Lemma 9.10, Lemma 9.13] or [6, Lemma 6.15, Lemma 6.22, Proposition 6.60] for the proof of Claim 3.36.

Since $C(n)(|\mathrm{Hess}_{\mathbf{b}_1^i}|^2 + |\mathrm{Hess}_{\mathbf{b}_3^i}|^2)$ is an upper gradient of $\langle d\mathbf{b}_1^i, d\mathbf{b}_3^i\rangle$, by Poincaré inequality, we have

$$\frac{1}{\mathrm{vol}\, B_{100}(x(i))}\int_{B_{100}(x(i))}\left|\langle d\mathbf{b}_1^i, d\mathbf{b}_3^i\rangle - \frac{1}{\mathrm{vol}\, B_{100}(x(i))}\int_{B_{100}(x(i))}\langle d\mathbf{b}_1^i, d\mathbf{b}_3^i\rangle d\mathrm{vol}\right| d\mathrm{vol}$$
$$\leq C(n)\frac{1}{\mathrm{vol}\, B_{100}(x(i))}\int_{B_{100}(x(i))}\left(|\mathrm{Hess}_{\mathbf{b}_1^i}|^2 + |\mathrm{Hess}_{\mathbf{b}_3^i}|^2\right) d\mathrm{vol} \leq \Psi(\delta; n).$$

Therefore, we have

$$\frac{1}{\mathrm{vol}\, B_{100}(x(i))}\int_{B_{100}(x(i))}\left|\langle d\mathbf{b}_3^i, dr_{x_1(i)}\rangle - \frac{1}{\mathrm{vol}\, B_{100}(x(i))}\int_{B_{100}(x(i))}\langle d\mathbf{b}_3^i, dr_{x_1(i)}\rangle d\mathrm{vol}\right| d\mathrm{vol}$$
$$\leq \Psi(\delta; n).$$

By Proposition 2.11, without loss of generality, we can assume that there exists Lipschitz functions $\mathbf{b}_1^\infty, \mathbf{b}_3^\infty$ on $B_{100}(x)$ such that $\mathbf{b}_j^i \to \mathbf{b}_j^\infty$ on $B_{100}(x)$. By Theorem 3.33, there exists a Borel set $A \subset B_{100}(x) \setminus C_{x_1}$ such that $\upsilon(B_{100}(x) \setminus A) = 0$ and that

$$\lim_{h\to 0}\frac{f\circ\gamma(\overline{x_1, a} + h) - f(a)}{h} = \langle dr_{x_1}, d\mathbf{b}_3^\infty\rangle(a)$$

for every $a \in A$ and minimal geodesic γ from x_1 to a. By Lusin's theorem, there exists a Borel set $A(\delta) \subset A$ such that $\upsilon(A \setminus A(\delta)) < \delta\upsilon(B_1(x))$ and that the function $\langle dr_{x_1}, df\rangle|_{A(\delta)}$ is continuous. For every $0 < \eta < \delta$, we put a function f_η^δ on $A(\delta) \setminus B_{2\delta}(x)$ by

$$f_\eta^\delta(z) = \sup_{w\in C_z(\{x_1\})\cap\overline{B}_\eta(z)}\left|\frac{f(z) - f(w)}{\overline{z, w}} - \langle dr_{x_1}, df\rangle(z)\right|.$$

It is easy to check that f_η^δ is an upper semi-continuous function. Especially, f_η^δ is a Borel function. By the definition of A, for every $a \in A$, we have $\lim_{\eta\to 0} f_\eta^\delta(a) = 0$. Thus, by Egoroff's theorem, there exists a Borel set $X = X(\delta) \subset A(\delta)$ such that $\upsilon(A(\delta) \setminus X(\delta)) < \delta\upsilon(B_1(x))$ and that

$$\lim_{\eta\to 0}(\sup_{a\in X} f_\eta^\delta(a)) = 0.$$

We take $\eta = \eta(\delta) << \delta$ satisfying $\sup_{a\in X} f_{\eta_0}^\delta(a) < \delta$ for every $\eta_0 \leq \eta$. For every i, let X_i denote the set of points $w \in B_1(x(i))$ such that

$$\left|\langle d\mathbf{b}_3^i, dr_{x_1(i)}\rangle(w) - \frac{1}{\mathrm{vol}\, B_{100}(x(i))}\int_{B_{100}(x(i))}\langle d\mathbf{b}_3^i, dr_{x_1(i)}\rangle d\mathrm{vol}\right| \leq \Psi(\delta; n)$$

. Then, we have $\mathrm{vol}(B_1(x(i)) \setminus X_i)/\mathrm{vol}\, B_1(x(i)) \leq \Psi(\delta; n)$ for every sufficiently large i. For every i, we define a Borel function F_i on $B_{100}(x(i)) \setminus C_{x_1(i)}$,

$$F_i(w) = \frac{\mathbf{b}_3^i(\gamma(\overline{x_1(i), w} - \eta^2)) - \mathbf{b}_3^i(w)}{-\eta^2}.$$

Here, γ is the minimal geodesic from $x_1(i)$ to w.

Claim 3.37. *For every sufficiently large i, we have*

$$\frac{1}{\operatorname{vol} B_{10}(x(i))}\int_{B_{10}(x(i))\setminus C_{x_1(i)}}|\langle d\mathbf{b}_3^i, dr_{x_1(i)}\rangle - F_i(w)|d\mathrm{vol} \le \Psi(\delta;n).$$

The proof is as follows. It is easy to check that for every $a<b$, smooth function f on (a,b) and $c\in(a,b)$, we have

$$f(t) = f(c) + f'(t)(t-c) - \int_c^t (s-c)f''(s)ds.$$

Therefore, we have

$$\frac{\mathbf{b}_3^i(\gamma(\overline{x_1(i),w}-\eta^2)) - \mathbf{b}_3^i(w)}{-\eta^2} = \frac{d\mathbf{b}_3^i}{dr_{x_1(i)}}(w) - \frac{1}{\eta^2}\int_{\overline{x_1(i),w}-\eta^2}^{\overline{x_1(i),w}} \left(s - (\overline{x_1(i),w}-\eta^2)\right)\frac{d^2\mathbf{b}_3^i}{dr_{x_1(i)}^2}(\gamma(s))ds.$$

Thus, by an argument similar to the proof of [44, Estimate 2.6], we have

$$\frac{1}{\operatorname{vol} B_{10}(x(i))}\int_{B_{10}(x(i))\setminus C_{x_1(i)}}\left|\langle d\mathbf{b}_3^i, dr_{x_1(i)}\rangle - F_i(w)\right| d\mathrm{vol} \tag{52}$$

$$\le \frac{1}{\eta^2}\frac{1}{\operatorname{vol} B_{10}(x(i))}\int_{B_{10}(x(i))}\int_{\overline{x_1(i),w}-\eta^2}^{\overline{x_1(i),w}}\eta^2|\mathrm{Hess}_{\mathbf{b}_3^i}|(\gamma(s))ds\underline{d\mathrm{vol}} \tag{53}$$

$$\le \eta^2 C(n)\frac{1}{\operatorname{vol} B_{100}(x(i))}\int_{B_{100}(x(i))}|\mathrm{Hess}_{\mathbf{b}_3^i}|d\mathrm{vol} \tag{54}$$

$$\le \eta^2 C(n)\sqrt{\frac{1}{\operatorname{vol} B_{100}(x(i))}\int_{B_{100}(x(i))}|\mathrm{Hess}_{\mathbf{b}_3^i}|^2 d\mathrm{vol}} \tag{55}$$

$$\le \eta^2 C(n)\Psi(\delta;n). \tag{56}$$

Therefore, we have Claim 3.37

Claim 3.38. *For sufficiently large i, we have*

$$\frac{1}{\upsilon(B_1(x))}\int_{B_1(x)}\left|\langle d\mathbf{b}_3^\infty, dr_{x_1}\rangle - \frac{1}{\operatorname{vol} B_1(x(i))}\int_{B_1(x(i))}\langle d\mathbf{b}_3^i, dr_{x_1(i)}\rangle d\mathrm{vol}\right| d\upsilon \le \Psi(\delta;n).$$

The proof is as follows. Let $Y_i = \{w\in \overline{B}_1(x(i))\setminus C_{x_1(i)}; |\langle d\mathbf{b}_3^i, dr_{x_1(i)}\rangle(w) - F_i(w)| \le \Psi(\delta;n)\}$. By Claim 3.37, we have $\operatorname{vol}(\overline{B}_1(x(i))\setminus Y_i)/\operatorname{vol}\overline{B}_1(x(i)) \le \Psi(\delta;n)$ for every sufficiently large i. We put $Z_i = X_i\cap Y_i$. We take a compact set $W_i\subset Z_i$ satisfying $\operatorname{vol}(Z_i\setminus W_i)/\operatorname{vol}\overline{B}_1(x(i)) \le \Psi(\delta;n)$. Thus, we have $\operatorname{vol}(\overline{B}_1(x(i))\setminus W_i)/\operatorname{vol}\overline{B}_1(x(i)) \le \Psi(\delta;n)$ for every sufficiently large i. By Proposition 2.10, without loss of generality, we can assume that there exists a compact set $W_\infty\subset \overline{B}_1(x)$ such that $W_j\to W_\infty$. By Lemma 2.14, we have $\upsilon(W_\infty)/\upsilon(\overline{B}_1(x)) \ge 1-\Psi(\delta;n)$. We put $E = W_\infty\cap X$, then $\upsilon(\overline{B}_1(x)\setminus E)\le \Psi(\delta;n)\upsilon(\overline{B}_1(x))$. For every $w_i\in W_i$ and $w\in E$, we take the minimal

geodesic γ_{w_i} from $x_1(i)$ to w_i and a minimal geodesic γ_w from x_1 to w. Then, there exists i_0 such that for every $i \geq i_0$, $w \in E$ and $w_i \in W_i$ satisfying $w_i \to w$, we have $\epsilon_i << \eta$,

$$\left| \langle d\mathbf{b}_3^i, dr_{x_1(i)} \rangle(w) - \frac{\mathbf{b}_3^i(\gamma_i(\overline{x_1(i), w_i} - \eta^2)) - \mathbf{b}_3^i(w_i)}{-\eta^2} \right| \leq \Psi(\delta; n)$$

and

$$\left| \langle d\mathbf{b}_3^i, dr_{x_1(i)} \rangle(w_i) - \frac{1}{\operatorname{vol} B_{100}(x(i))} \int_{B_{100}(x(i))} \langle d\mathbf{b}_3^i, dr_{x_1(i)} \rangle d\mathrm{vol} \right| \leq \Psi(\delta; n).$$

On the other hand, by rescaling $\eta^{-2} d_Y$, since

$$\overline{x_1, \phi_i(\gamma_i(\overline{x_1(i), w_i} - \eta^2))}^{\eta^{-2} d_Y} \geq \eta^{-1}, \ \overline{\phi_i(\gamma_i(\overline{x_1(i), w_i} - \eta^2)), w}^{\eta^{-2} d_Y} \geq \eta^{-1}$$

and

$$\overline{x_1, \phi_i(\gamma_i(\overline{x_1(i), w_i} - \eta^2))}^{\eta^{-2} d_Y} + \overline{\phi_i(\gamma_i(\overline{x_1(i), w_i} - \eta^2)), w}^{\eta^{-2} d_Y} - \overline{x_1, w}^{\eta^{-2} d_Y} \leq \eta,$$

by splitting theorem, we have

$$\overline{\phi_i(\gamma_i(\overline{x_1(i), w_i} - \eta^2)), \gamma(\overline{x_1, w} - \eta^2)}^{\eta^{-2} d_Y} \leq \Psi(\delta; n).$$

Therefore, we have

$$\left| \frac{\mathbf{b}_3^i(\gamma_i(\overline{x_1(i), w_i} - \eta^2)) - \mathbf{b}_3^i(w_i)}{-\eta^2} - \frac{\mathbf{b}_3^\infty(\gamma(\overline{x_1, w} - \eta^2)) - \mathbf{b}_3^\infty(w)}{-\eta^2} \right| \leq \Psi(\delta; n).$$

Thus, we have

$$\left| \langle d\mathbf{b}_3^\infty, dr_{x_1} \rangle(w) - \frac{1}{\operatorname{vol} B_{100}(x(i))} \int_{B_{100}(x(i))} \langle d\mathbf{b}_3^i, dr_{x_1(i)} \rangle d\mathrm{vol} \right| \leq \Psi(\delta; n).$$

We put

$$C_i = \frac{1}{\operatorname{vol} B_{100}(x(i))} \int_{B_{100}(x(i))} \langle d\mathbf{b}_3^i, dr_{x_1(i)} \rangle d\mathrm{vol}.$$

Then

$$\frac{1}{v(B_1(x))} \int_{B_1(x)} |\langle d\mathbf{b}_3^\infty, dr_{x_1} \rangle - C_i| \, dv \tag{57}$$

$$= \frac{1}{v(B_1(x))} \int_{B_1(x) \setminus E} |\langle d\mathbf{b}_3^\infty, dr_{x_1} \rangle - C_i| \, dv + \frac{1}{v(B_1(x))} \int_E |\langle d\mathbf{b}_3^\infty, dr_{x_1} \rangle - C_i| \, dv \tag{58}$$

$$\leq \frac{C(n) v(B_1(x) \setminus E)}{v(B_1(x))} + \frac{v(E)}{v(B_1(x))} \Psi(\delta; n) \leq \Psi(\delta; n). \tag{59}$$

Therefore, we have Claim 3.38.

Claim 3.39. *We have*

$$\frac{1}{v(B_1(x))}\int_{B_1(x)}|d\mathbf{b}_3^\infty|^2dv \le 1+\Psi(\delta;n).$$

This proof is as follows. Since

$$\frac{1}{\operatorname{vol}B_1(x(i)}\int_{B_1(x(i))}||d\mathbf{b}_3^i|-1|d\mathrm{vol}\le\Psi(\delta;n)$$

for every sufficiently large i, by [5, Lemma 16.2], there exists a compact set $K_i\subset\overline{B}_1(x(i))$ such that $\underline{\mathrm{vol}}(B_1(x(i))\setminus K_i)/\underline{\mathrm{vol}}B_1(x(i))\le\Psi(\delta;n)$ and that $\mathbf{Lip}(\mathbf{b}_3^i|_{K_i})\le 1+\Psi(\delta;n)$. By Proposition 2.10, without loss of generality, we can assume that there exists a compact set $K_\infty\subset\overline{B}_1(x)$ such that $K_i\to K_\infty$. By Lemma 2.14, we have $v(K_\infty)/v(B_1(x))\ge 1-\Psi(\delta;n)$. By the definition, we have $\mathbf{Lip}(\mathbf{b}_3^\infty|_{K_\infty})\le 1+\Psi(\delta;n)$. We put $\hat{K}_\infty=\mathrm{Leb}K_\infty$. Then by Proposition 7.7, we have

$$\begin{aligned}\frac{1}{v(B_1(x))}\int_{B_1(x)}|d\mathbf{b}_3^\infty|^2dv &= \frac{1}{v(B_1(x))}\int_{\hat{K}_\infty}|d\mathbf{b}_3^\infty|^2dv+\frac{1}{v(B_1(x))}\int_{B_1(x)\setminus K_\infty}|d\mathbf{b}_3^\infty|^2dv\\ &\le \frac{1}{v(B_1(x))}\int_{\hat{K}_\infty}(\mathrm{Lip}\mathbf{b}_3^\infty)^2dv+C(n)\frac{v(B_1(x)\setminus K_\infty)}{v(B_1(x))}\\ &\le \frac{1}{v(B_1(x))}\int_{\hat{K}_\infty}(\mathrm{Lip}(\mathbf{b}_3^\infty|_{K_\infty}))^2dv+\Psi(\delta;n)\\ &\le \frac{1}{v(B_1(x))}\int_{\hat{K}_\infty}(1+\Psi(\delta;n))dv+\Psi(\delta;n)\le 1+\Psi(\delta;n).\end{aligned}$$

Therefore, we have Claim 3.39.

If we consider the case $x_1=x_3, x_2=x_4$, then, by Claim 3.36, 3.38 and 3.39, we have

(60)
$$\frac{1}{v(B_1(x))}\int_{B_1(x)}|d\mathbf{b}_3^\infty-dr_{x_3}|^2dv$$

(61)
$$=\frac{1}{v(B_1(x))}\int_{B_1(x)}|d\mathbf{b}_3^\infty|^2dv-2\frac{1}{v(B_1(x))}\int_{B_1(x)}\langle d\mathbf{b}_3^\infty,dr_{x_3}\rangle dv+\frac{1}{v(B_1(x))}\int_{B_1(x)}|dr_{x_3}|^2dv$$

(62)
$$\le 1+\Psi(\delta;n)-2(1-\Psi(\delta;n))+1\le\Psi(\delta;n)$$

for every sufficiently large i. Therefore, we have Lemma 3.34. On the other hand, Lemma 3.35 follows from Lemma 3.34 and Claim 3.38. □

Lemma 3.40. *Let $\{(M_i,m_i)\}_i$ be a sequence of n-dimensional complete Riemannian manifolds with $\mathrm{Ric}_{M_i}\ge -(n-1)$, (Y,y,v) a Ricci limit space of $\{(M_i,m_i,\underline{\mathrm{vol}})\}_i$, τ*

a positive number, x, x_1, x_2 points in Y, $x(i), x_1(i), x_2(i)$ points in M_i. We assume that $x \in \bigcap_{j=1,2}(\mathcal{D}^{\tau}_{x_j(i)} \setminus B_{\tau}(x_j))$, $x(i) \to x$ and $x_j(i) \to x_j (j = 1, 2)$. Then, for every sufficiently large i, we have

$$\frac{1}{v(B_r(x))}\int_{B_r(x)} \langle dr_{x_1}, dr_{x_2}\rangle dv = \frac{1}{\mathrm{vol}\, B_r(x(i))}\int_{B_r(x)} \langle dr_{x_1(i)}, dr_{x_2(i)}\rangle \mathrm{dvol} \pm \Psi(r, \frac{r}{\tau}; n),$$

$$\frac{1}{v(B_r(x))}\int_{B_r(x)} \left| \langle dr_{x_1}, dr_{x_2}\rangle dv - \frac{1}{v(B_r(x))}\int_{B_r(x)} \langle dr_{x_1}, dr_{x_2}\rangle dv \right| dv \le \Psi(r, \frac{r}{\tau}; n)$$

and

$$\frac{1}{\mathrm{vol}\, B_r(x(i))}\int_{B_r(x)} \left| \langle dr_{x_1(i)}, dr_{x_2(i)}\rangle - \frac{1}{\mathrm{vol}\, B_r(x(i))}\int_{B_r(x)} \langle dr_{x_1(i)}, dr_{x_2(i)}\rangle \mathrm{dvol} \right| \mathrm{dvol} \le \Psi(r, \frac{r}{\tau}; n)$$

Proof. By rescaling $r^{-1}d_Y$ and Lemma 3.35, it is easy to check the assertion. □

Next corollary follows from Lemma 3.35, 3.40 directly:

Corollary 3.41. *Let $\{(M_i, m_i)\}_i$ be a sequence of n-dimensional complete Riemannian manifolds with $\mathrm{Ric}_{M_i} \ge -(n-1)$, (Y, y) a Ricci limit space of $\{(M_i, m_i, \underline{\mathrm{vol}})\}_i$, τ, L positive numbers, $x, x_1, \dots, x_k, z_1, \dots, z_l$ points in Y, $x(i), x_1(i), \dots, x_k(i), z_1(i), \dots, z_l(i)$ points in M_i and $a_1, \dots, a_k, b_1, \dots, b_l$ real numbers. We assume that $x \in \bigcap_{i=1}^{k}(\mathcal{D}^{\tau}_{x_i} \setminus B_{\tau}(x_i)) \cap \bigcap_{i=1}^{l}(\mathcal{D}^{\tau}_{z_i} \setminus B_{\tau}(z_i))$, $x(i) \to x$, $x_j(i) \to x_j (j = 1, \dots, k)$, $z_m(i) \to z_m (m = 1, \dots, l)$ and $\sum_{i=1}^{k} a_i^2 + \sum_{i=1}^{l} b_i^2 \le L$. We put $f = \sum_{j=1}^{k} a_j r_{x_j}, g = \sum_{j=1}^{l} b_j r_{z_j}, f_i = \sum_{j=1}^{k} a_j r_{x_j(i)}$ and $g_i = \sum_{j=1}^{l} b_j r_{z_j(i)}$. Then, for every sufficiently large i, we have,*

$$\frac{1}{\mathrm{vol}\, B_r(x(i))}\int_{B_r(x(i))} \left| \langle df_i, dg_i\rangle - \frac{1}{v(B_r(x))}\int_{B_r(x)} \langle df, dg\rangle dv \right| \mathrm{dvol} \le \Psi(r, \frac{r}{\tau}; n, L),$$

$$\frac{1}{v(B_r(x))}\int_{B_r(x)} \left| \langle df, dg\rangle - \frac{1}{\mathrm{vol}\, B_r(x(i))}\int_{B_r(x(i))} \langle df_i, dg_i\rangle \mathrm{dvol} \right| dv \le \Psi(r, \frac{r}{\tau}; n, L).$$

Lemma 3.42. *Let $\{(M_i, m_i)\}_i$ be a sequence of n-dimensional complete Riemannian manifolds with $\mathrm{Ric}_{M_i} \ge -(n-1)$, (Y, y, v) a Ricci limit space of $\{(M_i, m_i, \underline{\mathrm{vol}})\}_i$, $l, k_\alpha (1 \le \alpha \le l)$ positive integers, r, ϵ, τ, L positive numbers, $x, x_t^s (1 \le s \le l, 1 \le t \le k_l)$ points in Y, $x(i), x_t^s(i)$ points in M_i and $a_t^s (1 \le s \le l, 1 \le t \le k_l)$ real numbers. We put $f_j = \sum_{m=1}^{k_j} a_m^j r_{x_m^j}, f_j^i = \sum_{m=1}^{k_j} a_m^j r_{x_m^j(i)}$. We assume that $l \le n$, $k_i \le n (1 \le i \le l)$, $x \in \bigcap_{1\le i\le l, 1\le j\le k_i}^{k}(\mathcal{D}^{\tau}_{x_j^i} \setminus B_{\tau}(x_j^i))$, $x(i) \to x$, $x_t^s(i) \to x_t^s$, $\sum_{i,j}(a_j^i)^2 \le L$ and*

$$\frac{1}{v(B_r(x))}\int_{B_r(x)} \langle df_j, df_i\rangle dv = \delta_{ij} \pm \epsilon.$$

Then, for every sufficiently large i, there exists a compact set $K_r^i \subset \overline{B}_{r/10}(x(i))$ satisfying the following properties:

1. $\mathrm{vol}(B_{r/10}(x(i)) \setminus K^i_r)/\mathrm{vol}\, B_{r/10}(x(i)) \le \Psi(r, r/\tau, \epsilon; n, L)$

2. *For every $w \in K^i_r$ and $0 < s < r/10^6$, there exist a compact set $Z \subset \overline{B}_s(w)$, $z \in Z$ and a map $\phi : (\overline{B}_s(w), w) \to (Z, z)$ such that the map $\Phi = (f^i_1, f^i_2, \dots, f^i_l, \phi)$ from $\overline{B}_s(w)$ to $\overline{B}_{s+\Psi(r,r/\tau,\epsilon;n,L)s}(f^i_1(w), \dots, f^i_l(w), \phi(w))$, gives $\Psi(r, r/\tau, \epsilon; n, L)s$-Gromov-Hausdorff approximation.*

3. *For every $w \in K^i_r$ and $0 < s < r/10^6$, we have*

$$\frac{1}{\mathrm{vol}\, B_s(w)} \int_{B_s(w)} |\langle df^i_\alpha, df^i_\beta \rangle - \delta_{\alpha\beta}| d\underline{\mathrm{vol}} < \Psi(r, \frac{r}{\tau}, \epsilon; n, L).$$

Proof. By Lemma 3.42, we have

$$\frac{1}{\mathrm{vol}\, B_r(x(i))} \int_{B_r(x(i))} |\langle df^i_j, df^i_{\hat{l}} \rangle - \delta_{j,\hat{l}}| d\mathrm{vol} \le \Psi(r, \frac{r}{\tau}, \epsilon; n, L)$$

for every sufficiently large i. We consider rescaled distances $r^{-1}d_Y$ and $r^{-1}d_{M_i}$. For convenience, we shall use the following notations: $\hat{\mathrm{vol}} = \mathrm{vol}^{r^{-1}d_{M_i}}$, $\hat{\upsilon} = \upsilon/\upsilon(B_r(y))$, $\hat{r}_z(w) = r^{-1}\overline{w,z}^{d_Y}$, $\hat{B}_s(w) = B^{r^{-1}d_Y}_s(w) = B_{sr}(w)$, $\hat{g} = r^{-1}g$ for Lipschitz function g and so on. We remark that $(M_i, m_i, r^{-1}d_{M_i}, \underline{\mathrm{vol}}^{r^{-1}d_{M_i}}) \to (Y, y, r^{-1}d_Y, \hat{\upsilon})$. We also denote the differential section of Lipschitz function f on Y as metric measure space $(Y, \hat{\upsilon})$ by $\hat{d}f : Y \to T^*Y$ and denote the Riemannian metric of rescaled Ricci limit space $(Y, y, r^{-1}d_Y, \hat{\upsilon})$ by $\langle \cdot, \cdot \rangle_r$. By the definition, we have $\langle \cdot, \cdot \rangle_r = r^{-2}\langle \cdot, \cdot \rangle$. Then we have

$$\frac{1}{\hat{\mathrm{vol}}\, \hat{B}_1(x(i))} \int_{\hat{B}_1(x(i))} |\langle \hat{d}\hat{f}^i_j, \hat{d}\hat{f}^i_{\hat{l}} \rangle_r - \delta_{j,\hat{l}}| d\hat{\mathrm{vol}} \le \Psi(r, \frac{r}{\tau}, \epsilon; n, L).$$

On the other hand, by [4, Lemma 9.8, Lemma 9.10, Lemma 9.13] (or [6, Lemma 6.15, Lemma 6.22, Proposition 6.60]), there exist harmonic functions $\hat{\mathbf{b}}^{m,i}_j (1 \le m \le l, 1 \le j \le k_m)$ on $\hat{B}_{100}(x(i))$ such that $|\hat{\mathbf{b}}^{m,i}_j - \hat{r}_{x^m_j(i)}|_{L^\infty(\hat{B}_{100}(x(i)))} \le \Psi(r, r/\tau; n)$,

$$\frac{1}{\hat{\mathrm{vol}}\, \hat{B}_{100}(x(i))} \int_{\hat{B}_{100}(x(i))} |\hat{d}\hat{\mathbf{b}}^{m,i}_j - \hat{d}\hat{r}_{x^m_j(i)}|^2_r d\hat{\mathrm{vol}} \le \Psi(r, \frac{r}{\tau}; n),$$

and

$$\frac{1}{\hat{\mathrm{vol}}\, \hat{B}_{100}(x(i))} \int_{\hat{B}_{100}(x(i))} |\mathrm{Hess}_{\hat{\mathbf{b}}^{m,i}_j}|^2_r d\hat{\mathrm{vol}} \le \Psi(r, \frac{r}{\tau}; n).$$

We put $\hat{\mathbf{b}}^i_j = \sum_{m=1}^{k_j} a^j_m \hat{\mathbf{b}}^{m,i}_j$. Then, we have

$$|\hat{f}^i_j - \hat{\mathbf{b}}^i_j|_{L^\infty(\hat{B}_{100}(x(i)))} \le \Psi(r, \frac{r}{\tau}; n, L),$$

$$\frac{1}{\hat{\mathrm{vol}}\, \hat{B}_{100}(x(i))} \int_{\hat{B}_{100}(x(i))} |\hat{d}\hat{\mathbf{b}}^i_j - \hat{d}\hat{f}^i_j|^2_r d\hat{\mathrm{vol}} \le \Psi(r, \frac{r}{\tau}; n, L)$$

and

$$\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{100}(x(i))}\int_{\hat{B}_{100}(x(i))}|\mathrm{Hess}_{\hat{\mathbf{b}}^i_j}|^2_r d\hat{\mathrm{vol}} \le \Psi(r,\frac{r}{\tau};n,L).$$

Especially, we have

$$\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{100}(x(i))}\int_{\hat{B}_{100}(x(i))}|\langle \hat{d}\hat{\mathbf{b}}^i_j, \hat{d}\hat{\mathbf{b}}^i_l\rangle_r - \delta_{j,l}|d\hat{\mathrm{vol}} \le \Psi(r,\frac{r}{\tau},\epsilon;n,L)$$

We put

$$\hat{F}_i = \sum_{j=1}^{l}|\hat{d}\hat{\mathbf{b}}^i_j - \hat{d}\hat{f}^i_j|^2_r + \sum_{j=1}^{l}||\hat{d}\hat{\mathbf{b}}^i_j|^2_r - 1| + \sum_{j<\hat{l}}|\langle \hat{d}\mathbf{b}^i_j, \hat{d}\mathbf{b}^i_{\hat{l}}\rangle_r| + \sum_{j=1}^{l}|\mathrm{Hess}_{\hat{\mathbf{b}}^i_j}|^2_r.$$

By Lemma 3.1, we have the following:

Claim 3.43. *For every sufficiently large i, there exists a compact set $K^i_r \subset \hat{B}_{1/10}(x(i))$ such that*

$$\frac{\hat{\mathrm{vol}}(\hat{B}_{\frac{1}{10}}(x(i)) \setminus K^i_r)}{\hat{\mathrm{vol}}\,\hat{B}_{\frac{1}{10}}(x(i))} \le \Psi(r,\frac{r}{\tau},\epsilon;n,L),$$

and that

$$\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{5s}(w)}\int_{\hat{B}_{5s}(w)}\hat{F}_i d\hat{\mathrm{vol}} \le \Psi(r,\frac{r}{\tau},\epsilon;n,L)$$

for every $w \in K^i_r$ and $0 < s < 1/10$.

We fix $w \in K^i_r$ and $0 < s \le 1/10$. By an argument same to the proof of [9, Theorem 3.3], we have the following:

Claim 3.44. *There exist a compact set $Z \subset \hat{\overline{B}}_s(w)$, a point $z \in Z$ and a map ϕ from $\hat{\overline{B}}_{s/10^5}(w)$ to Z, such that the map $\Phi(\alpha) = (\hat{\mathbf{b}}^i_1(\alpha),\dots,\hat{\mathbf{b}}^i_l(\alpha),\phi(\alpha))$ from $\hat{\overline{B}}_{s/10^5}(w)$ to $\overline{B}_{s/10^5+\Psi s}(\hat{\mathbf{b}}^i_1(w),\dots,\hat{\mathbf{b}}^i_l(w),\phi(w)) \subset \mathbf{R}^k \times Z$, gives Ψs-Gromov-Hausdorff approximation. Here, $\Psi = \Psi(r, r/\tau, \epsilon; n, L)$.*

Since

$$\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{5s}(w)}\int_{\hat{B}_{5s}(w)}|\hat{d}\hat{\mathbf{b}}^i_j - \hat{d}\hat{f}^i_j|^2_r d\hat{\mathrm{vol}} \le \Psi(r,\frac{r}{\tau},\epsilon;n,L),$$

by segment inequality (see [9, Theorem 2.15]), for every $z_1 \in \hat{\overline{B}}_s(w)$, there exist $\hat{z}_1 \in \hat{\overline{B}}_{5s}(w)$, $\hat{w} \in \hat{\overline{B}}_{5s}(w)$ and a minimal geodesic γ from $\hat{z}_1$ to $\hat{w}$ such that $\overline{z_1,\hat{z}_1} \le \Psi(r, r/\tau, \epsilon; n, L)$, $\overline{w,\hat{w}} \le \Psi(r, r/\tau, \epsilon; n, L)$, and that

$$\int_0^{\overline{\hat{z}_1,\hat{w}}}\hat{\mathrm{Lip}}(\hat{\mathbf{b}}^i_j - \hat{f}^i_j)(\gamma(t))dt \le \Psi(r,\frac{r}{\tau},\epsilon;n,L)s.$$

Therefore, we have

$$|\hat{\mathbf{b}}^i_j(\hat{z}_1) - \hat{f}^i_j(\hat{z}_1) - (\hat{\mathbf{b}}^i_j(\hat{w}) - \hat{f}^i_j(\hat{w}))| \le \int_0^{\overline{\hat{z}_1,\hat{w}}}\hat{\mathrm{Lip}}(\hat{\mathbf{b}}^i_j - \hat{f}^i_j)(\gamma(t))dt \le \Psi(r,\frac{r}{\tau},\epsilon;n,L)s.$$

By Cheng-Yau's gradient estimate, we have $\hat{\mathbf{Lip}}(\hat{\mathbf{b}}^i_j|_{\hat{B}_{2s}(w)}) \leq C(n,L)$. Thus, we have $|\hat{\mathbf{b}}^i_j(z_1) - \hat{f}^i_j(z_1) - (\hat{\mathbf{b}}^i_j(w) - \hat{f}^i_j(w))| \leq \Psi(r, r/\tau, \epsilon; n, L)s$. Therefore, if we put $C = \hat{\mathbf{b}}^i_j(w) - \hat{f}^i_j(w)$, then we have

$$\hat{\mathbf{b}}^i_j = \hat{f}^i_j + C \pm \Psi(r, \frac{r}{\tau}, \epsilon; n, L)s$$

on $\hat{\overline{B}}_s(w)$.

Thus, the map $\hat{\Phi}(\alpha) = (\hat{f}^i_1(\alpha),\cdots, \hat{f}^i_l(\alpha), \phi(\alpha))$ from $\hat{\overline{B}}_{s/10^5}(w)$ to $\overline{B}_{s/10^5+\Psi s}(\hat{f}^i_1(w),\cdots, \hat{f}^i_l(w), \phi(w))$, gives Ψs-Gromov-Hausdorff approximation. Therefore we have the assertion. □

Lemma 3.45. *Let (Y, y, υ) be a Ricci limit space, $\tau, \epsilon, \delta, L$ positive numbers, $l, m, k_s (1 \leq s \leq l \leq m)$ positive integers, $x, x^s_t (1 \leq s \leq l, 1 \leq t \leq k_s)$ points in Y and a^s_t real numbers. We put $f_j = \sum_{m=1}^{k_j} a^j_m r_{x^j_m}$. We assume that $x \in \mathrm{Leb}\left(\bigcap_{1\leq i\leq l, 1\leq j\leq k_i} (\mathcal{D}^\tau_{x^i_j} \setminus \{x^i_j\}) \cap (\mathcal{R}_m)_{\delta,\tau}\right)$, $\sum_{i,j}(a^i_j)^2 \leq L$ and*

$$\limsup_{r\to 0} \frac{1}{\upsilon(B_r(x))} \int_{B_r(x)} |\langle df_j, df_i\rangle - \delta_{ij}| d\upsilon \leq \epsilon.$$

Then, for every sufficiently small $s > 0$, there exists a compact set $K_s \subset \overline{B}_s(x)$ satisfying the following properties:

1. *$\upsilon(K_s)/\upsilon(B_s(x)) \geq 1 - \Psi(\epsilon, \delta; n, L)$.*
2. *For every $\alpha \in K_s$ and every sufficiently small $t > 0$, there exist points $w^t_1(\alpha),\cdots, w^t_{m-l}(\alpha) \in Y$ and a compact set $U_t \subset \overline{B}_t(\alpha)$ such that $\upsilon(U_t)/\upsilon(B_t(\alpha)) \geq 1 - \Psi(\epsilon, \delta; n, L)$ and that the map $\Phi_t = (f_1,\cdots, f_l, r_{w^t_1(\alpha)},\cdots, r_{w^t_{m-l}(\alpha)})$ from U_t to $\mathbf{R}^m$, gives $(1 \pm \Psi(\epsilon, \delta; n, L))$-bi-Lipschitz equivalent to the image $\Phi_t(U_t)$.*

Proof. Let $(M_i, m_i, \underline{\mathrm{vol}}) \to (Y, y, \upsilon)$. We take $x^s_t(i) \in M_i$ satisfying $x^s_t(i) \to x^s_t$ and put $f^i_j = \sum_{m=1}^{k_j} a^j_m r_{x^j_m(i)}$. There exists $s_1 > 0$ such that $s_1 << \tau$,

$$\frac{1}{\upsilon(B_{10^{10}s}(x))} \int_{B_{10^{10}s}(x)} |\langle df_j, df_i\rangle - \delta_{ij}| d\upsilon + \frac{\upsilon\left(B_{10^{10}s}(x) \cap \bigcap_{1\leq i\leq l, 1\leq j\leq k_i} (\mathcal{D}^\tau_{x^i_j} \cap (\mathcal{R}_m)_{\delta,r})\right)}{\upsilon(B_{10^{10}s}(x))} \leq 3\epsilon$$

for every $0 < s < s_1$. By Proposition 2.10 and Lemma 3.42, for every $0 < s < s_1$, there exists a compact set $K_s \subset \overline{B}_{10^9 s}(x)$ satisfying the following properties:

1. $\upsilon(K_s)/\upsilon(B_{10^9 s}(x)) \geq 1 - \Psi(\epsilon; n, L)$.
2. For every $w \in K_s$ and $0 < t < 10^4 s$, there exist a compact set $Z^w_t \subset \overline{B}_t(w)$ and a map ϕ^w_t from $\overline{B}_t(w)$ to Z^w_t such that the map $\Phi^w_t = (f_1,\cdots, f_l, \phi^w_t)$ from $\overline{B}_t(w)$ to $\overline{B}_{10^9(t+\Psi t)}(f_1(w),\cdots, f_l(w), \phi^w_t(w))$, gives Ψt-Gromov-Hausdorff approximation. Here $\Psi = \Psi(\epsilon; n, L)$

3. For every $w \in K_s$ and $0 < t < 10^4 s$, we have

$$\frac{1}{\upsilon(B_t(w))} \int_{B_t(w)} |\langle df_j, df_i \rangle - \delta_{ij}| d\upsilon \leq \Psi(\epsilon; n, L).$$

Here, with same notation as in Lemma 3.42, we used Proposition 4.13 as

$$\lim_{k \to \infty} \frac{1}{\underline{\mathrm{vol}}\, B_t(w(k))} \int_{B_t(w(k))} |\langle df_j^k, df_i^k \rangle - \delta_{ij}| d\underline{\mathrm{vol}} = \frac{1}{\upsilon(B_t(w))} \int_{B_t(w)} |\langle df_j, df_i \rangle - \delta_{ij}| d\upsilon.$$

for $w(k) \to w$. We fix $0 < s < s_1$ and take K_s, $w \in K_s \cap \mathrm{Leb}(\bigcap_{1 \leq i \leq l, 1 \leq j \leq k_i} (\mathcal{D}^{\tau}_{x_j^i} \setminus \{x_j^i\}) \cap (\mathcal{R}_m)_{\delta,r}), 0 < t < 10^4 s, Z_t^w, \phi_t^w, \Phi_t^w$ as above. We remark that $\upsilon(K_s \cap \mathrm{Leb}(\bigcap_{1 \leq i \leq l, 1 \leq j \leq k_i} (\mathcal{D}^{\tau}_{x_j^i} \setminus \{x_j^i\}) \cap (\mathcal{R}_m)_{\delta,r}))/\upsilon(B_{10^9 s}(x)) \geq 1 - \Psi(\epsilon; n, L)$. We assume that t is sufficiently small and that

$$\frac{\upsilon\left(B_{\hat{t}}(w) \cap \bigcap_{1 \leq i \leq l, 1 \leq j \leq k_i} (\mathcal{D}^{\tau}_{x_j^i} \setminus \{x_j^i\}) \cap (\mathcal{R}_m)_{\delta,r}\right)}{\upsilon(B_{\hat{t}}(w))} \geq 1 - \epsilon$$

for every $0 < \hat{t} \leq t$. There exist points $y_i^+, y_j^- \in \overline{B}_t(w) (1 \leq i, j \leq l)$ such that $\overline{\Phi_t^w(y_i^+), (\underbrace{0, \dots, 0, t}_{i}, 0, \dots, 0, \phi_t^w(w))} \leq \Psi t$ and $\overline{\Phi_t^w(y_j^-), (\underbrace{0, \dots, 0, -t}_{j}, 0, \dots, 0, \phi_t^w(w))} \leq \Psi t$. We also take an Ψt-Gromov-Hausdorff approximation $\hat{\Phi}_t^w$ from $\overline{B}_{10^9(t+\Psi t)}(f_1(w), \dots, f_l(w), \phi_t^w(w))$ to $\overline{B}_t(w)$ satisfying $\overline{\Phi_t^w \circ \hat{\Phi}_t^w(\alpha), \alpha} \leq \Psi t$ for every $\alpha \in \overline{B}_{10^9(t+\Psi t)}(f_1(w), \dots, f_l(w), \phi_t^w(w))$ and $\overline{\hat{\Phi}_t^w \circ \Phi_t^w(\beta), \beta} \leq \Psi t$ for every $\beta \in \overline{B}_t(w)$. On the other hand, we can take δt-Gromov-Hausdorff approximation ψ_t^w from $(\overline{B}_t(w), w)$ to $(\overline{B}_t(0_m), 0_m)$ and $\hat{\psi}_t^w$ from $(\overline{B}_t(0_m), 0_m)$ to $(\overline{B}_t(w), w)$ satisfying that $\psi_t^w \circ \hat{\psi}_t^w(\alpha), \alpha \leq 5\delta t$ for every $\alpha \in \overline{B}_t(0_m)$ and $\hat{\psi}_t^w \circ \psi_t^w(\beta), \beta \leq 5\delta t$ for every $\beta \in \overline{B}_t(w)$. Especially, there exists an Ψt-Gromov-Hausdorff approximation $\hat{h}_t^w$ from $(\overline{B}_t(0_{m-l}), 0_{m-l})$ to $(Z_t^w, \phi_t^w(w))$ such that $\overline{(0, \dots, 0, \alpha), \psi_t^w \circ \hat{\Phi}_t^w(f_1(w), \dots, f_l(w), \hat{h}_t^w(\alpha))} \leq \Psi t$ for every $\alpha \in Z_t^w$. Here $\Psi = \Psi(\epsilon, \delta; n, L)$. Without loss of generality, we can assume that $\overline{\psi_t^w(y_i^+), (\underbrace{0, \dots, 0, t}_{i}, 0, \dots, 0)} \leq \Psi t$. There exist points $z_i^+, z_j^- \in \overline{B}_t(w) (l+1 \leq i, j \leq m)$ such that $\overline{\psi_t^w(z_i^+), (\underbrace{0, \dots, 0, t}_{i}, 0, \dots, 0)} \leq \Psi t$ and $\overline{\psi_t^w(z_j^-), (\underbrace{0, \dots, 0, -t}_{j}, 0, \dots, 0)} \leq \Psi t$. We put $F_i = f_i - f_i(w)$ and define a function G_i on $(\overline{B}_t(0_m), 0_m)$ by $G_i = F_i \circ \psi_t^w$. Since $\pi_{\mathbf{R}^{m-l}}(\psi_t^w \circ \hat{\Phi}_t^w(f_1(w), \dots, f_l(w), \hat{h}_t^w(\alpha))), \alpha \leq \Psi t$, the map $G = (G_1, \dots, G_l, \pi_{l+1}, \dots, \pi_m)$ from $(\overline{B}_t(0_m), 0_m)$ to $(\overline{B}_{t+\Psi t}(0_m), 0_m)$ gives Ψt-Gromov-Hausdorff approximation and satisfies $\overline{G(\underbrace{0, \dots, 0, \pm t}_{i}, 0, \dots, 0), (\underbrace{0, \dots, 0, \pm t}_{i}, 0, \dots, 0)} \leq \Psi t$. Here $\pi_{\mathbf{R}^{m-l}}$ is the canonical projection $\mathbf{R}^m = \mathbf{R}^l \times \mathbf{R}^{m-l} \to \mathbf{R}^{m-l}$ and π_i is the i-th projection $\mathbf{R}^m \to \mathbf{R}$. Thus, we have $\overline{\alpha, G(\alpha)} \leq \Psi t$ for every $\alpha \in \overline{B}_t(0_m)$. Especially, we have the following claim:

Claim 3.46. *We have*

$$|G_i - \pi_i| \leq \Psi(\epsilon, \delta; n, L) t$$

on $B_t(0_m)$.

We fix $0 < \hat{t} < t$. By rescaling $\hat{t}^{-1}d_Y$, $\hat{t}^{-1}d_{\mathbf{R}^m}$, Claim 3.46 and the definition of Busemann function, we have the following:

Claim 3.47. *We have*

$$|F_i(\alpha) - (r_{y_i^-}(\alpha) - r_{y_i^-}(w))| \leq \Psi\left(\epsilon, \delta, \frac{\hat{t}}{t}, \frac{\Psi(\epsilon,\delta;n,L)t}{\hat{t}}; n, L\right)\hat{t}$$

on $\overline{B}_{\hat{t}}(w)$.

We take $y_j^-(k), z_j^-(k), w(k) \in M_k$ such that $y_j^-(k) \to y_j^-, z_j^-(k) \to z_j^-$ and $w(k) \to w$. For $\Psi = \Psi(\epsilon, \delta; n, L)$ in Claim 3.47, we put $r = \sqrt{\Psi}t$.

For convenience, for rescaled distances $r^{-1}d_Y$ and $r^{-1}d_{M_i}$, we shall use the same notation as in the proof of Lemma 3.42 below: $f_i^k, \hat{d}f, \hat{\mathrm{vol}}$ and so on.

Claim 3.48. *For every sufficiently large k, we have*

$$\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{100}(w(k))}\int_{\hat{B}_{100}(w(k))} |\hat{d}\hat{f}_i^k - \hat{d}\hat{r}_{y_i^-(k)}|_r^2 d\hat{\mathrm{vol}} \leq \Psi(\epsilon, \delta; n, L).$$

This proof is as follows. By the assumption and Proposition 4.13, for every sufficiently large k, we have

$$\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{1000}(x(k))}\int_{\hat{B}_{1000}(x(k))} ||\hat{d}\hat{f}_i^k|_r^2 - 1| d\hat{\mathrm{vol}} \leq \Psi(\epsilon, \delta; n, L).$$

By an argument similar to the proof of Lemma 3.42, for every sufficiently large k, there exist harmonic functions $\hat{\mathbf{b}}_i^k$ on $\hat{B}_{100}(w(k))$, such that $\mathbf{Lip}\hat{\mathbf{b}}_i^k \leq C(n)$, $|\hat{\mathbf{b}}_i^k - \hat{f}_i^k|_{L^\infty(\hat{B}_{100}(w(k)))} \leq \Psi(r, r/\tau; n, L)$,

$$\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{1000}(w(k))}\int_{\hat{B}_{1000}(w(k))} |\hat{d}\hat{\mathbf{b}}_i^k - \hat{d}\hat{f}_i^k|_r^2 d\hat{\mathrm{vol}} \leq \Psi(r, r/\tau; n, L)$$

and

$$\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{1000}(w(k))}\int_{\hat{B}_{1000}(w(k))} |\mathrm{Hess}_{\hat{\mathbf{b}}_i^k}|_r^2 d\hat{\mathrm{vol}} \leq \Psi(r, r/\tau; n, L).$$

For every $\alpha \in \hat{B}_{1000}(w(k)) \setminus C_{y_i^-(k)}$, we take the minimal geodesic γ_i^α from $y_i^-(k)$ to α on $(M_i, r^{-1}d_{M_i})$. We fix $0 < h < 1$. By Claim 3.47, there exists k_0 such that for every $k \geq k_0$

and $\alpha \in \hat{B}_{1000}(w(k)) \setminus C_{y_i^-(k)}$, we have

$$\frac{\hat{\mathbf{b}}_i^k(\alpha) - \hat{\mathbf{b}}_i^k(\gamma_i^\alpha(\overline{y_i^-(k), \alpha}^{r^{-1}d_{M_k}} - h))}{h} \tag{63}$$

$$= \frac{\hat{f}_i^k(\alpha) - \hat{f}_i^k(\gamma_i^\alpha(\overline{y_i^-(k), \alpha}^{r^{-1}d_{M_k}} - h))}{h} \pm \frac{\Psi(\epsilon, \delta; n, L)}{h} \tag{64}$$

$$= \frac{\hat{f}_i(\phi_k(\alpha)) - \hat{f}_i(\phi_k(\gamma_i^\alpha(\overline{y_i^-(k), \alpha}^{r^{-1}d_{M_k}} - h)))}{h} \pm \frac{\Psi(\epsilon, \delta; n, L)}{h} \tag{65}$$

$$= \frac{\overline{y_i^-, \phi_k(\alpha)}^{r^{-1}d_Y} - \overline{y_i^-, \phi_k(\gamma_i^\alpha(\overline{y_i^-(k), \alpha}^{r^{-1}d_{M_k}} - h)}^{r^{-1}d_Y}}{h} \pm \frac{\Psi(\epsilon, \delta; n, L)}{h} \tag{66}$$

$$= \frac{\overline{y_i^-(k), \alpha}^{r^{-1}d_{M_k}} - \overline{y_i^-(k), \gamma_i^\alpha(\overline{y_i^-(k), \alpha}^{r^{-1}d_{M_k}} - h)}^{r^{-1}d_{M_k}}}{h} \pm \frac{\Psi(\epsilon, \delta; n, L)}{h} \tag{67}$$

$$= 1 \pm \frac{\Psi(\epsilon, \delta; n, L)}{h}. \tag{68}$$

On the other hand, by an argument similar to the proof of Claim 3.37, we have

$$\left| \frac{1}{\hat{\text{vol}}\, \hat{B}_{100}(w(k))} \int_{\hat{B}_{100}(w(k))} \frac{1}{h} \int_{\overline{y_i^-(k), \alpha}^{r^{-1}d_{M_k}} - h}^{\overline{y_i^-(k), \alpha}^{r^{-1}d_{M_k}}} \left(s - (\overline{y_i^-(k), \alpha}^{r^{-1}d_{M_k}} - h) \right) \frac{d^2 \hat{\mathbf{b}}_i^k \circ \gamma_i^\alpha}{ds^2} ds d\hat{\text{vol}} \right| \tag{69}$$

$$\leq C(n) \frac{h}{\hat{\text{vol}}\, \hat{B}_{1000}(w(k))} \int_{\hat{B}_{1000}(w(k))} |\text{Hess}_{\hat{\mathbf{b}}_i^k}|_r d\hat{\text{vol}} \leq \Psi(\epsilon, \delta; n, L). \tag{70}$$

Since

$$\hat{\mathbf{b}}_i^k(\alpha) = \hat{\mathbf{b}}_i^k(\gamma_i^\alpha(\overline{y_i^-(k), \alpha}^{r^{-1}d_{M_k}} - h)) + \frac{d\hat{\mathbf{b}}_i^k}{d\hat{r}_{y_i^-(k)}}(\alpha) h \tag{71}$$

$$- \int_{\overline{y_i^-(k), \alpha}^{r^{-1}d_{M_k}} - h}^{\overline{y_i^-(k), \alpha}^{r^{-1}d_{M_k}}} \left(s - (\overline{y_i^-(k), \alpha}^{r^{-1}d_{M_k}} - h) \right) \frac{d^2 \hat{\mathbf{b}}_i^k \circ \gamma_i^\alpha}{ds^2} ds, \tag{72}$$

for every $\alpha \in \hat{B}_{100}(w(k)) \setminus C_{y_i^-(k)}$, we have

$$\frac{1}{\hat{\text{vol}}\, \hat{B}_{100}(w(k))} \int_{\hat{B}_{100}(w(k))} \langle d\hat{\mathbf{b}}_i^k, d\hat{r}_{y_i^-(k)} \rangle_r d\hat{\text{vol}} = 1 \pm \frac{\Psi(\epsilon, \delta; n, L)}{h}.$$

Therefore, we have

$$\begin{aligned}
&\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{100}(w(k))}\int_{\hat{B}_{100}(x(k))}|\hat{d}\hat{f}_i^k-\hat{d}\hat{r}_{y_i^-(k)}|_r^2 d\hat{\mathrm{vol}}\\
&=\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{100}(w(k))}\int_{\hat{B}_{100}(w(k))}|\hat{d}\hat{f}_i^k|_r^2 d\hat{\mathrm{vol}}-\frac{2}{\hat{\mathrm{vol}}\,\hat{B}_{100}(w(k))}\int_{\hat{B}_{100}(w(k))}\langle\hat{d}\hat{f}_i^k,\hat{d}\hat{r}_{y_i^-(k)}\rangle_r d\hat{\mathrm{vol}}+1\\
&=1-2\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{100}(w(k))}\int_{\hat{B}_{100}(w(k))}\langle\hat{d}\hat{\mathbf{b}}_i^k,\hat{d}\hat{r}_{y_i^-(k)}\rangle_r d\hat{\mathrm{vol}}+1\pm\Psi(\epsilon,\delta;n,L)\\
&=2-2(1\pm\frac{\Psi(\epsilon,\delta;n,L)}{h})\pm\Psi(\epsilon,\delta;n,L)=\frac{\Psi(\epsilon,\delta;n,L)}{h}.
\end{aligned}$$

Therefore, we have Claim 3.48.

Next claim follows from Claim 3.48 and [4, Theorem 9.29] directly:

Claim 3.49. *For every sufficiently large k, we have*

$$\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{100}(w(k))}\int_{\hat{B}_1(w(k))}|\langle\hat{d}\hat{f}_i^k,\hat{d}\hat{r}_{z_j^-(k)}\rangle_r|d\hat{\mathrm{vol}}\le\Psi(\epsilon,\delta;n,L)$$

for every $1\le i\le l$ and $l+1\le j\le m$. Moreover we have

$$\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{100}(w(k))}\int_{\hat{B}_1(w(k))}|\langle\hat{d}\hat{f}_i^k,\hat{d}\hat{f}_{\hat{i}}^k\rangle_r|d\hat{\mathrm{vol}}\le\Psi(\epsilon,\delta;n,L)$$

for every $1\le i<\hat{i}\le l$.

There exist harmonic functions $\hat{\mathbf{b}}_i^k(l+1\le i\le m)$ on $\hat{B}_{1000}(w(k))$ such that $|\hat{r}_{z_i^-}-\hat{\mathbf{b}}_i^k|_{L^\infty(\hat{B}_{1000}(w(k)))}\le\Psi(\epsilon,\delta;n,L)$,

$$\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{1000}(w(k))}\int_{\hat{B}_{1000}(w(k))}|\hat{d}\hat{\mathbf{b}}_i^k-\hat{d}\hat{r}_{z_i^-(k)}|_r^2 d\hat{\mathrm{vol}}\le\Psi(\epsilon,\delta;n,L)$$

and

$$\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{1000}(w(k))}\int_{\hat{B}_{1000}(w(k))}|\mathrm{Hess}_{\hat{\mathbf{b}}_i^k}|_r^2 d\hat{\mathrm{vol}}\le\Psi(\epsilon,\delta;n,L).$$

We put

$$\hat{F}_k=\sum_{1\le i,j\le m}|\langle\hat{d}\hat{\mathbf{b}}_i^k,\hat{d}\hat{\mathbf{b}}_j^k\rangle_r-\delta_{i,j}|+\sum_{1\le i\le m}|\mathrm{Hess}_{\hat{\mathbf{b}}_i^k}|_r^2+\sum_{i=1}^{l}|\hat{d}\hat{\mathbf{b}}_i^k-\hat{d}\hat{f}_i^k|_r^2+\sum_{i=l+1}^{m}|\hat{d}\hat{\mathbf{b}}_i^k-\hat{d}\hat{r}_{z_i^-}|_r^2.$$

Then, by Lemma 3.1, for every sufficiently large k, there exists a compact set $C(k)\subset\overline{\hat{B}}_1(w(k))$ such that $\hat{\mathrm{vol}}(\hat{B}_1(w(k))\setminus C(k))/\hat{\mathrm{vol}}\,\hat{B}_1(w(k))\le\Psi(\epsilon,\delta;n,L)$ and that for every $\alpha\in C(k)$ and $0<\hat{s}<10$, we have

$$\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{\hat{s}}(\alpha)}\int_{\hat{B}_{\hat{s}}(\alpha)}\hat{F}_k d\hat{\mathrm{vol}}\le\Psi(\epsilon,\delta;n,L).$$

Thus, by an argument similar to the proof of [9, Theorem 3.3], for every $\alpha \in C(k)$ and $0 < \hat{s} < 1$, there exist a compact set $P^\alpha_s \subset \hat{\overline{B}}_{\hat{s}}(\alpha)$, a point $p^\alpha_{\hat{s}} \in P^\alpha_{\hat{s}}$ and a map $q^\alpha_{\hat{s}}$ from $(\hat{\overline{B}}_{\hat{s}}(\alpha), \alpha)$ to $(\overline{B}_{\hat{s}}(p^\alpha_{\hat{s}}), p^\alpha_{\hat{s}})$ such that the map $Q^\alpha_{\hat{s}} = (\hat{\mathbf{b}}^k_1,\dots,\hat{\mathbf{b}}^k_m, q^\alpha_{\hat{s}})$ from $\hat{\overline{B}}_{\hat{s}}(\alpha)$ to $\hat{\overline{B}}_{\hat{s}+\Psi\hat{s}}(\hat{\mathbf{b}}^k_1(\alpha),\dots,\hat{\mathbf{b}}^k_m(\alpha), p^\alpha_{\hat{s}})$, gives $\Psi\hat{s}$-Gromov-Hausdorff approximation. For every $\alpha \in C(k)$ and $0 < \hat{s} < 1$. by an argument similar to the proof of Claim 3.44, we have

$$\hat{\mathbf{b}}^k_i = \hat{f}^k_i + \text{constant} \pm \Psi\hat{s}$$

on $\hat{B}_{\hat{s}}(\alpha)$ for every $1 \le i \le l$, and

$$\hat{\mathbf{b}}^k_i = \hat{r}_{z^-_i(k)} + \text{constant} \pm \Psi\hat{s}$$

on $\hat{B}_{\hat{s}}(\alpha)$ for $l+1 \le i \le m$. Therefore, the map $\hat{Q}^\alpha_{\hat{s}} = (\hat{f}^k_1,\dots,\hat{f}^k_l, \hat{r}_{z^-_{l+1}(k)},\dots,\hat{r}_{z^-_m(k)}, q^\alpha_{\hat{s}})$ from $\hat{\overline{B}}_{\hat{s}}(\alpha)$ to $\hat{\overline{B}}_{\hat{s}+\Psi\hat{s}}(\hat{f}^k_1(\alpha),\dots,\hat{f}^k_l(\alpha), \hat{r}_{z^-_{l+1}(k)}(\alpha),\dots,\hat{r}_{z^-_m(k)}(\alpha), p^\alpha_{\hat{s}})$, gives $\Psi\hat{s}$-Gromov-Hausdorff approximation.

By Proposition 2.10, without loss of generality, we can assume that there exists a compact set $C(\infty) \subset \hat{\overline{B}}_1(w)$ such that $C(k) \to C(\infty)$. We put $U = C(\infty) \cap \bigcap_{1\le i\le l, 1\le j\le k_i} (\mathcal{D}^\tau_{x^i_j} \setminus \{x^i_j\}) \cap (\mathcal{R}_m)_{\delta,r}$. By Proposition 2.14, we have $\hat{v}(\hat{B}_1(w) \cap U)/\hat{v}(\hat{B}_1(w)) \ge 1 - \Psi$. Since $\alpha \in (\mathcal{R}_m)_{\tau,\delta}$, by the argument above, the map $T^\alpha_{\hat{s}} = (\hat{f}_1,\dots,\hat{f}_l, \hat{r}_{z^-_{l+1}},\dots,\hat{r}_{z^-_m})$ from $\hat{\overline{B}}_{\hat{s}}(\alpha)$ to $\overline{B}_{\hat{s}}(T^\alpha_{\hat{s}}(\alpha))$, gives $\Psi\hat{s}$-Gromov-Hausdorff approximation for every $\alpha \in U$ and $0 < \hat{s} < 1$. Therefore for every $\alpha, \beta \in U \cap \hat{B}_{1/2}(w)$ satisfying $\alpha \ne \beta$, if we put $\hat{s} = \overline{\alpha,\beta}^{r^{-1}d_Y} < 1$, then we have

$$\overline{(\hat{f}_1(\alpha),\dots,\hat{f}_l(\alpha), \hat{r}_{z^-_{l+1}}(\alpha),\dots,\hat{r}_{z^-_m}(\alpha)), (\hat{f}_1(\beta),\dots,\hat{f}_l(\beta), \hat{r}_{z^-_{l+1}}(\beta),\dots,\hat{r}_{z^-_m}(\beta))} \tag{73}$$

$$= \overline{\alpha,\beta}^{r^{-1}d_Y} \pm \Psi\hat{s} \tag{74}$$

$$= (1 \pm \Psi)\overline{\alpha,\beta}^{r^{-1}d_Y}. \tag{75}$$

Therefore we have the assertion. □

LEMMA 3.50. *Let (Y, y, v) be a Ricci limit space, $l, k, m (1 \le l \le m \le n)$ positive integers, x a point in Y, $h_i (1 \le i \le l)$ Lipschitz functions on Y, τ a positive number, $x_i (1 \le i \le k)$ points in Y and $a^j_i (1 \le i \le k, 1 \le j \le l)$ real numbers We put $f_j = \sum_{i=1}^k a^j_i r_{x_i}$. We assume that*

$$\lim_{r\to 0} \frac{1}{v(B_r(x))} \int_{B_r(x)} |df_j - dh_j| dv = 0$$

for every j,

$$x \in \bigcap_{\delta>0} \left(\bigcup_{r>0} \mathrm{Leb} \left(\bigcap_{i,j} (\mathcal{D}^\tau_{x^j_i} \setminus \{x^j_i\}) \cap (\mathcal{R}_m)_{\delta,r} \right) \right),$$

the limit

$$\lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\langle dh_i, dh_j\rangle d\upsilon$$

exists for every i, j, and

$$\det\left(\lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\langle dh_i, dh_j\rangle d\upsilon\right)_{i,j}\neq 0.$$

Then, for every $0<\delta<1$, there exists $r_0>0$ such that for every $0<s<r_0$, there exists compact set $K_s\subset \overline{B}_s(x)$ satisfying the following properties:

1. *$\upsilon(K_s)/\upsilon(B_s(x))\geq 1-\delta$.*
2. *For every $\alpha\in K_s$ and every sufficiently small $t>0$, there exist points $w_1^t(\alpha),\dots,w_{m-l}^t(\alpha)\in Y$ and a compact set $U_t\subset \overline{B}_t(\alpha)$ such that $\upsilon(U_t)/\upsilon(B_t(\alpha))\geq 1-\delta$ and that the map $\Phi_t=((h_1,\dots,h_l)A, r_{w_1^t(\alpha)},\dots,r_{w_{m-l}^t(\alpha)})$ from U_t to $\mathbf{R}^m$, gives $(1\pm\delta)$-bi-Lipschitz equivalent to the image $\Phi_t(U_t)$. Here,*

$$A=\sqrt{\left(\lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\langle dh_i, dh_j\rangle d\upsilon\right)_{i,j}}^{-1}.$$

Proof. We define Lipschitz functions g_i on Y by $(g_1,\dots,g_l)=(h_1,\dots,h_l)A$. By the definition, we have

$$\lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\langle g_i, g_j\rangle d\upsilon=\delta_{i,j}.$$

By Corollary 3.41, we have

$$\lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}|\langle g_i, g_j\rangle-\delta_{i,j}|d\upsilon=0.$$

We put $(F_1,\dots,F_l)=(\sum_{i=1}^k b_i^1 r_{x_i},\dots,\sum_{i=1}^k b_i^l r_{x_i})=(\sum_{i=1}^k a_i^1 r_{x_i},\dots,\sum_{i=1}^k a_i^l r_{x_i})A$ and take $L\geq 1$ such that $|A|+\sum_{i,j}(b_i^j)^2\leq L$. We fix $0<\delta<1$. By Lemma 3.45, we have the following claim:

Claim 3.51. *There exists $r_1>0$ such that for every $0<s\leq r_1$, there exist a compact set $K_s\subset\overline{B}_s(x)$ satisfying the following properties:*

1. *$\upsilon(K_s)/\upsilon(B_s(x))\geq 1-\delta$.*
2. *For every $\alpha\in K_s$ and every sufficiently small $t>0$, there exist points $w_1^t(\alpha),\dots,w_{m-l}^t(\alpha)\in Y$ and a compact set $E_t\subset\overline{B}_t(\alpha)$ such that $\upsilon(E_t)/\upsilon(B_t(\alpha))\geq 1-\delta$ and that the map $\Phi_t=(F_1,\dots,F_l,r_{w_1^t(\alpha)},\dots,r_{w_{m-l}^t(\alpha)})$ from E_t to $\mathbf{R}^m$, gives $(1\pm\delta)$-bi-Lipschitz equivalent to the image.*

On the other hand, there exists $r_0 > 0$ such that

$$\frac{1}{\upsilon(B_s(x))}\int_{B_s(x)}\sum_j |dF_j - dg_j|d\upsilon \le \delta$$

for every $0 < s < r_0$. Thus, by Lemma 3.1, we have the following;

Claim 3.52. *For every $0 < s < r_0/100$, there exists a compact set $X_s \subset \overline{B}_s(x)$ such that $\upsilon(X_s)/\upsilon(\overline{B}_s(x)) \ge 1 - \Psi(\delta; n)$ and that*

$$\frac{1}{\upsilon(B_{5\hat{s}}(\alpha))}\int_{B_{5\hat{s}}(\alpha)}\sum_j |dF_j - dg_j|d\upsilon \le \Psi(\delta; n)$$

for every $\alpha \in X_s$ and $0 < \hat{s} \le s$.

We put $V_s = K_s \cap X_s$ for $0 < s < \min\{r_0, r_1\}/1000$. Then we have $\upsilon(V_s)/\upsilon(B_s(x)) \ge 1 - \Psi(\delta; n)$. We fix $0 < s < \min\{r_0, r_1\}/1000$. We also take $\alpha \in V_s$ and sufficiently small $t > 0$. By an argument similar to the proof of Claim 3.44, we have

$$F_j = f_j + \text{constant} \pm \Psi(\delta; n)t$$

on $\overline{B}_t(\alpha)$. We put $U_t = B_{t/2}(\alpha) \cap E_t$. Then we have $\upsilon(U_t)/\upsilon(B_{t/2}(\alpha)) \ge 1 - \Psi(\delta; n)$. For $p_1, p_2 \in B_{t/2}(\alpha) \cap E_t$ satisfying $p_1 \ne p_2$, if we put $\hat{t} = \overline{p_1, p_2} > 0$, then we have

(76)
$$\overline{(f_1(p_1),\dots, f_l(p_1), r_{w^t_1(\alpha)},\dots, r_{w^t_{m-l}(\alpha)}(p_1)), (f_1(p_2),\dots, f_l(p_2), r_{w^t_1(\alpha)}(p_2),\dots, r_{w^t_{m-l}(\alpha)}(p_2))}$$

(77)
$$= \overline{(F_1(p_1),\dots, F_l(p_1), r_{w^t_1(\alpha)},\dots, r_{w^t_{m-l}(\alpha)}(p_1)), (F_1(p_2),\dots, F_l(p_2), r_{w^t_1(\alpha)}(p_2),\dots, r_{w^t_{m-l}(\alpha)}(p_2))} \pm \Psi\hat{t}$$

(78)
$$= (1 \pm \delta)\overline{p_1, p_2} \pm \Psi\hat{t} = (1 \pm \Psi)\overline{p_1, p_2}.$$

Therefore we have the assertion. □

Lemma 3.53. *Let (Y, y, υ) be a Ricci limit space, l a positive integer integer, $f_i, f (1 \le i \le l)$ Lipschitz functions on Y and A a Borel subset of Y. We assume that for a.e. $x \in A$, $\mathrm{span}\{df_1(x),\dots, df_l(x)\} = T^*_xY$. Then, for a.e. $x \in A$, there exists $b_1(x),\dots, b_l(x) \in \mathbf{R}$ such that*

$$\lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\left|df - \sum_{i=1}^{l} b_i(x)df_i\right|^2 d\upsilon = 0.$$

Proof. Without loss of generality, we can assume that for every $x \in A$, $\{df_i(x)\}$ is a base of T^*_xY. For every $x \in A$, we put

$$(b_1(x), \dots, b_l(x)) = (\langle df, df_1\rangle(x), \dots, \langle df, df_l\rangle(x))\sqrt{(\langle df_i, df_j\rangle(x))_{i,j}}^{-1}.$$

By Corollary 7.6, for a.e. $x \in A$, we have

$$\lim_{r\to 0} \frac{1}{\upsilon(B_r(x))} \int_{B_r(x)} |df|^2 d\upsilon = |df|^2(x),$$

$$\lim_{r\to 0} \frac{1}{\upsilon(B_r(x))} \int_{B_r(x)} \langle df, df_i\rangle d\upsilon = \langle df, df_i\rangle(x)$$

for every i and

$$\lim_{r\to 0} \frac{1}{\upsilon(B_r(x))} \int_{B_r(x)} \langle df_i, df_j\rangle d\upsilon = \langle df_i, df_j\rangle(x)$$

for every i, j. Therefore, for a.e. $x \in A$, since

$$\lim_{r\to 0} \frac{1}{\upsilon(B_r(x))} \int_{B_r(x)} |df|^2 d\upsilon = |df|^2(x) = \left|\sum_{i=1}^{l} b_i(x) df_i(x)\right|^2,$$

$$\lim_{r\to 0} \frac{1}{\upsilon(B_r(x))} \int_{B_r(x)} \left\langle df, \sum_{i=1}^{l} b_i(x) df_i \right\rangle d\upsilon = \sum_{i=1}^{l} b_i(x) \langle df, df_i\rangle(x) \tag{79}$$

$$= \sum_{i=1}^{l} b_i(x) \left\langle \sum_{j=1}^{l} b_j(x) df_j, df_i \right\rangle (x) \tag{80}$$

$$= \left|\sum_{i=1}^{l} b_i(a) df_i(x)\right|^2 \tag{81}$$

and

$$\lim_{r\to 0} \frac{1}{\upsilon(B_r(x))} \int_{B_r(x)} \left|\sum_{i=1}^{l} b_i(a) df_i\right|^2 d\upsilon = \sum_{i,j}^{l} b_i(x) b_j(x) \langle df_i, df_j\rangle(x) = \left|\sum_{i=1}^{l} b_i(x) df_i(x)\right|^2,$$

we have

$$\lim_{r\to 0} \frac{1}{\upsilon(B_r(x))} \int_{B_r(x)} \left|df - \sum_{i=1}^{l} b_i(x) df_i\right|^2 d\upsilon \tag{82}$$

$$= \lim_{r\to 0} \frac{1}{\upsilon(B_r(x))} \int_{B_r(x)} |df|^2 d\upsilon - 2 \lim_{r\to 0} \frac{1}{\upsilon(B_r(x))} \int_{B_r(x)} \left\langle df, \sum_{i=1}^{l} b_i(x) df_i \right\rangle d\upsilon \tag{83}$$

$$+ \lim_{r\to 0} \frac{1}{\upsilon(B_r(x))} \int_{B_r(x)} \left|\sum_{i=1}^{l} b_i(a) df_i\right|^2 d\upsilon = 0. \tag{84}$$

□

THEOREM 3.54 (Rectifiability associated with Lipschitz functions). *Let (Y, y, υ) be a Ricci limit space, l a positive integer, $f_i (1 \le i \le l)$ Lipschitz functions on Y, A a Borel subset of Y. We assume that $\{f_1(x), \dots, f_l(x)\}$ are linearly independent for a.e. $x \in A$. Then, there exist $0 < \alpha(n) < 1$, a collection of compact sets $\{C_{k,i}\}_{l\le k\le n, i\in \mathbf{N}} \subset A$, points $\{x_{k,i}\} \in A$ and $\{x^s_{k,i}\}_{1\le s\le k-l} \in Y$ satisfying the following properties:*

1. $\upsilon(A \setminus \bigcup_{l\le k\le n, i\in\mathbf{N}} C_{k,i}) = 0$.

2. *For every* $l \le k \le n$, $x \in \bigcup_{i\in\mathbf{N}} C_{k,i}$ *and* $0 < \delta < 1$, *there exists* $i \in \mathbf{N}$ *such that* $x \in C_{k,i}$ *and that the map* $\phi_{k,i} = ((f_1(z),\dots, f_l(z))\sqrt{(\langle df_i, df_j\rangle)_{i,j}(x_{k,i})}^{-1}, r_{x^1_{k,i}},\dots, r_{x^{k-l}_{k,i}})$ *gives a* $(1\pm\delta)$*-bi-Lipschitz equivalent to the image* $\phi_{k,i}(C_{k,i})$.

3. $C_{k,i} \subset \mathcal{R}_{k,\alpha(n)} \cap \bigcap_{j=1}^{k-l}(Y \setminus (C_{x^j_{k,i}} \cup \{x^j_{k,i}\}))$.

4. *The limit measure* υ *and* k*-dimensional Hausdorff measure* H^k *are mutually absolutely continuous on* $C_{k,i}$. *Moreover,* υ *is Ahlfors* k*-regular at every* $x \in C_{k,i}$.

Proof. We take a collection of Borel subset $\{C^y_{k,i}\}$ of Y and a collection of points $\{x^{\hat{l}}_{k,i}\}$ in Y as in Theorem 3.17. For convenience, we put $x^1_{k,i} = y, C_{k,i} = C^y_{k,i}$. By Lemma 3.14, we can assume that $C_{k,i}$ is bounded for every i, k. By the definition of T^*Y (see section 4 in [4] or section 6 in [9] for the detail), we have $\mathrm{span}\{dr_{x^1_{k,i}}(x),\dots, dr_{x^k_{k,i}}(x)\} = T^*_xY$ for a.e. $x \in C^y_{k,i}$. Therefore, by the assumption, we have $\upsilon(A \cap C_{k,i}) = 0$ for $k < l$. Since

$$\upsilon\left(\mathcal{R}_k \setminus \bigcup_{\tau>0}\left(\bigcap_{\delta>0}\left(\bigcup_{r>0}\mathrm{Leb}\left(\bigcap_{i,j}(\mathcal{D}^\tau_{x^j_i} \setminus \{x^j_i\}) \cap (\mathcal{R}_k)_{\delta,r}\right)\right)\right)\right) = 0,$$

by Lemma 3.50 and Lemma 3.53, we have the following claim:

Claim 3.55. *For every* $k \ge l$ *and* $i \in \mathbf{N}$, *there exists a Borel set* $A_{k,i} \subset A \cap C_{k,i}$ *satisfying the following properties:*

1. $\upsilon(A \cap C_{k,i} \setminus A_{k,i}) = 0$.

2. *For every* $x \in A_{k,i}$ *and* $0 < \delta < 1$, *there exists* $r^\delta_x > 0$ *such that for every* $0 < s < r^\delta_x$, *there exists a compact set* $K(x,\delta,s) \subset \overline{B}_s(x)$ *satisfying the following properties:*

   (a) $\upsilon(K(x,\delta,s))/\upsilon(B_s(x)) \ge 1-\delta$.

   (b) *For every* $\alpha \in K(x,\delta,s)$ *and every sufficiently small* $t > 0$, *there exist points* $w(i,x,\delta,s,\alpha,t) \in Y (1 \le i \le k-l)$ *and a compact set* $U(x,\delta,s,\alpha,t) \subset \overline{B}_t(\alpha)$ *such that the map*

   $$\Phi^{x,\delta,s,\alpha,t} = ((f_1,\dots, f_l)A(x), r_{w(1,x,\delta,s,\alpha,t)},\dots, r_{w(k-l,x,\delta,s,\alpha,t)})$$

   from $U(x,\delta,s,\alpha,t)$ *to* $\mathbf{R}^k$, *gives* $(1\pm\delta)$*-bi-Lipschitz equivalent to the image. Here,*

   $$A(x) = \sqrt{\left(\lim_{r\to0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\langle df_s, df_t\rangle d\upsilon\right)_{s,t}}^{-1} \tag{85}$$

   $$= \sqrt{(\langle df_s, df_t\rangle(x))_{s,t}}^{-1}. \tag{86}$$

We put $\hat{A}_{k,i} = \mathrm{Leb}(A_{k,i})$. For every $N \in \mathbf{N}$ and $x \in \hat{A}_{k,i}$, we take $0 < s_x^N < \min\{r_x^{1/N}, N^{-1}\}$ satisfying

$$\frac{\upsilon(B_{s_x^N}(x) \cap A_{k,i})}{\upsilon(B_{s_x^N}(x))} \geq 1 - N^{-1}.$$

We take $K(x, N^{-1}, s_x^N)$ as in Claim 3.55. We put $\hat{K}(x, N^{-1}, s_x^N) = K(x, N^{-1}, s_x^N) \cap \hat{A}_{k,i}$. Thus, we have

$$\frac{\upsilon\left(B_{s_x^N}(x) \cap \hat{K}(x, N^{-1}, s_x^N)\right)}{\upsilon(B_{s_x^N}(x))} \geq 1 - 100N^{-1}.$$

For every $\alpha \in \hat{K}(x, N^{-1}, s_x^N)$, there exists a sufficiently small $0 < t = t(\alpha) < N^{-1}$ such that

$$\frac{\upsilon(B_{\hat{t}}(\alpha) \cap A_{k,i})}{\upsilon(B_{\hat{t}}(\alpha))} \geq 1 - N^{-1}$$

for every $0 < \hat{t} < t$. We take $w(i, x, N^{-1}, s_x^N, \alpha, \hat{t})$ and $U(x, N^{-1}, s_x^N, \alpha, \hat{t})$ as in Claim 3.55. We put $\hat{U}(x, N^{-1}, s_x^N, \alpha, \hat{t}) = U(x, N^{-1}, s_x^N, \alpha, \hat{t}) \cap \hat{A}_{k,i}$. Then we have

$$\frac{\upsilon\left(B_{\hat{t}}(\alpha) \cap \hat{U}(x, N^{-1}, s_x^N, \alpha, \hat{t})\right)}{\upsilon(B_{\hat{t}}(\alpha))} \geq 1 - 1000N^{-1}.$$

By Lemma 2.12, it is not difficult to check that the following claim:

Claim 3.56. *With same notation as above, there exist* $x_j^N \in \hat{A}_{k,i}$, $\alpha_j^N \in \hat{K}(x_j^N, N^{-1}, s_{x_j^N}^N)$ *and* $0 < t_j^N < t(\alpha_j^N)$ *such that*

$$\upsilon\left(A_{k,i} \setminus \bigcup_{j \in \mathbf{N}} \hat{U}(x_j^N, N^{-1}, s_{x_j^N}^N, \alpha_j^N, t_j^N)\right) \leq \Psi(N^{-1}; n)\upsilon(B_{10}(A_{k,i})).$$

We put $\hat{U}(j, N) = \hat{U}(x_j^N, N^{-1}, s_{x_j^N}^N, \alpha_j^N, t(\alpha_j^N))$, $w(i, j, N) = w(i, x_j^N, N^{-1}, s_{x_j^N}^N, \alpha_j^N, t(\alpha_j^N))$, $U(j) = \bigcap_{N_0 \in \mathbf{N}} \left(\bigcup_{N_1 \geq N_0} \hat{U}(j, N_1)\right)$ and $U(j, N) = \hat{U}(j, N) \cap U(j)$. Then we have $\upsilon(A_{k,i} \setminus \bigcup_{j \in \mathbf{N}} U(j)) = 0$ and $\bigcup_{N \in \mathbf{N}} U(j, N) = U(j)$. We fix j. We take $w \in \bigcup_{N \in \mathbf{N}} U(j, N)$ and $0 < \delta < 1$. There exists N_0 such that $w \in U(j, N_0)$. We take N_1 satisfying $N_1^{-1} << \delta$. Since $w \in \bigcup_{N_2 \geq N_1} \hat{U}(j, N_2)$, there exists $N_2 \geq N_1$ such that $w \in \hat{U}(j, N_2)$. Especially we have $w \in U(j, N_2)$. Thus the map $G_{j,N_2} = ((f_1, \dots, f_l)A(x_j^{N_2}), r_{w(1,j,N_2)}, \dots, r_{w(k-l,j,N_2)})$ from $U(j, N_2)$ to $\mathbf{R}^k$, gives $(1 \pm N_2^{-1})$-bi-Lipschitz equivalent to the image. Especially, G_{j,N_2} gives $(1 \pm \delta)$-bi-Lipschitz equivalent to the image. Therefore, we have the assertion. □

Remark 3.57. Radial rectifiability theorem (Theorem 3.17) corresponds to Theorem 3.54 for a distance function r_x.

We shall give two corollaries of Theorem 3.54. For metric space X, we define a distance on $\mathbf{R}_{\geq 0} \times X/\{0\} \times X$ by

$$\overline{(t_1, x_1), (t_2, x_2)} = \sqrt{t_1^2 + t_2^2 - 2t_1 t_2 \cos \min\{\overline{x_1, x_2}, \pi\}}.$$

Let $C(X)$ denote this metric space and $p = [(0, x)] \in C(X)$.

COROLLARY 3.58. *Let X be a compact geodesic space, l a nonnegative integer. We assume that $l \leq n$, $\dim_H X = n - l - 1$, $(\mathbf{R}^l \times C(X), (0_l, p))$ is an $(n, -1)$-Ricci limit space. Here $p \in C(X)$ is the pole. Then, X is H^{n-l-1}-rectifiable.*

PROOF. We define 1-Lipschitz functions $\pi_j (1 \leq j \leq l)$ and g on $\mathbf{R}^k \times C(X)$ by $\pi_j(t_1, \dots, t_l, w) = t_j$ and $g(t_1, \dots, t_l, w) = \overline{p, w}$. By Theorem 3.33, we have $\langle d\pi_i, d\pi_j \rangle(\alpha) = \delta_{i,j}, \langle d\pi_i, dg \rangle(\alpha) = 0, |dg|(\alpha) = 1$ for a.e. $\alpha \in \mathbf{R}^k \times C(X)$. Therefore, we can take a collection of $\{C_{k,i}\}_{l+1 \leq k \leq n}$ as in Theorem 3.54 for Lipschitz functions $\pi_1, \dots, \pi_l, g$ and $A = \mathbf{R}^l \times C(X)$. By an argument similar to the proof of Lemma 7.21, the product measure $H^l \times H^{n-l}$ on $\mathbf{R}^l \times C(X)$ is equal to H^n. Therefore by Fubini's theorem, we have

$$0 = H^n(\mathbf{R}^l \times C(X) \setminus \bigcup_{k,i} C_{k,i}) = \int_{\mathbf{R}^l} H^{n-l}(\{t_1, \dots, t_l\} \times C(X) \setminus \bigcup_{k,i} C_{k,i}) dH^l.$$

Especially, we can take $(t_1, \dots, t_l) \in \mathbf{R}^l$ satisfying $H^{n-l}(\{t_1, \dots, t_l\} \times C(X) \setminus \bigcup C_{k,i}) = 0$. We put $\hat{C}_{k,i} = \{t_1, \dots, t_l\} \times C(X) \cap C_{k,i}$ and regard it as a subset of $C(X)$. By an argument similar to the proof of Proposition 7.22, we have

$$\int_{C(X)} f dH^{n-l} = \int_0^\infty \int_{\partial B_t(p)} f dH^{n-l-1} dt$$

for every $f \in L^1(C(X))$. (This is *co-area formula for distance function from the pole on* $C(X)$). Especially, we have

$$H^{n-l-1}(\partial B_t(p) \cap C(X) \setminus \bigcup_{k,i} \hat{C}_{k,i}) = 0$$

for a.e. $t > 0$. Then it is not difficult to check the assertion. □

REMARK 3.59. With same notation as in Corollary 3.58, for every $x \in X$ and $r > 0$, we have $0 < H^{n-l-1}(B_r(x)) < \infty$. It follows from [7, Theorem 5.9], [9, Theorem 4.6] and co-area formula for distance function from the pole on $C(X)$. Since it is not difficult to check it, we skipped the proof.

Similarly, we have the following:

COROLLARY 3.60. *Let (X, x) be a pointed proper geodesic space, l a nonnegative integer. We assume that $l \leq n$, $\dim_H X = n - l$, $(\mathbf{R}^l \times X, (0_l, x))$ is $(n, -1)$-Ricci limit space. Then, X is H^{n-l}-rectifiable.*

4 Convergence of Borel functions and Lipschitz functions

In this section, we will give several notions of convergence of a sequence of Borel functions. By using these notions, we will define a notion of convergence of differential of Lipschitz functions (see Definition 4.18). Moreover, by using results in section 3, we will discuss convergence of harmonic functions. Throughout subsections 4.1 and 4.2, we shall consider the following situation: Let (Z_i, z_i) be a sequence of pointed proper geodesic spaces, υ_i a Radon measure on Z_i satisfying $\upsilon_i(B_1(z_i)) = 1$, and for every $R \geq 1$, there exists $K = K(R) \geq 1$ such that for every $1 \leq i \leq \infty$, $z \in Z_i$ and $0 < s \leq R$, we have $\upsilon_i(B_{2s}(z)) \leq 2^K \upsilon_i(B_s(z))$. We assume that $(Z_i, z_i, \upsilon_i) \overset{(\phi_i, R_i, \epsilon_i)}{\to} (Z_\infty, z_\infty, \upsilon_\infty)$. We fix $x_i \in Z_i$ satisfying $x_i \to x_\infty$.

4.1 Infinitesimal constant convergence property

Our aims in this subsection are to define the following notion of *infinitesimal constant convergence* and to give several fundamental properties of it:

Definition 4.1 (Infinitesimal constant convergence property). Let R be a positive number, w a point in $B_R(x_\infty)$ and f_i a Borel function on $B_R(x_i)(1 \leq i \leq \infty)$ satisfying $\sup_i |f_i|_{L^\infty(B_R(x_i))} + |f_\infty|_{L^\infty(B_R(x_\infty))} < \infty$. We say that $\{f_i\}_i$ has *infinitesimal constant convergence property to f_∞ at w* if for every $\epsilon > 0$, there exists $r > 0$ such that

$$\limsup_{i\to\infty} \frac{1}{\upsilon_i(B_t(w_i))} \int_{B_t(w_i)} \left| f_i - \frac{1}{\upsilon_\infty(B_t(w))} \int_{B_t(w)} f_\infty d\upsilon_\infty \right| d\upsilon_i \leq \epsilon$$

and

$$\limsup_{i\to\infty} \frac{1}{\upsilon_\infty(B_t(w))} \int_{B_t(w)} \left| f - \frac{1}{\upsilon_i(B_t(w_i))} \int_{B_t(w_i)} f_i d\upsilon_i \right| d\upsilon_\infty \leq \epsilon$$

for every $0 < t < r$ and $w_i \to w$.

Example 4.2. It is easy to check that for every $f \in C^0(B_R(x_\infty))$, if we put $f_i = f \circ \phi_i$ and $f_\infty = f$, then, $\{f_i\}$ has infinitesimal constant convergence property to f_∞ at every $w \in B_R(x_\infty)$.

Example 4.3. If f_i is Lipschitz function with $\sup_i \mathbf{Lip} f_i < \infty$, and $f_i \to f_\infty$, then for every $w \in B_R(x_\infty)$, $\{f_i\}_i$ has infinitesimal constant convergence property to f_∞ at w.

Example 4.4. Let $w_i \to w \in B_R(x_\infty)$, $r > 0$ satisfying $B_r(w) \subset B_R(x_\infty)$. Then, $\{1_{B_R(x_i) \setminus \overline{B}_r(w_i)}\}_i$ has infinitesimal constant convergence property to $1_{B_R(x_\infty) \setminus \overline{B}_r(w_\infty)}$ at every $\alpha \in B_R(x_\infty) \setminus \partial B_r(w)$.

We shall give a fundamental result for infinitesimal constant convergence property:

PROPOSITION 4.5. *Let k be a positive integer, R a positive number, f_i^l Borel functions on $B_R(x_i)(1 \leq l \leq k, 1 \leq i \leq \infty)$ satisfying $\sup_{i,l}(|f_i^l|_{L^\infty(B_R(x_i))} + |f_\infty^l|_{L^\infty(B_R(x_\infty))}) < \infty$, w a point in $B_R(x_\infty)$ and $\{F_i\}_{1 \leq i \leq \infty}$ a sequence of continuous functions on $\mathbf{R}^k$. We assume that $\{f_i^l\}_{1 \leq i \leq \infty}$ has infinitesimal constant convergence property to f_∞^l at w for every l and that F_i converges to F_∞ in the sense of compact uniformly topology. Then, the sequence $\{F_i(f_i^1, \ldots, f_i^k)\}$ has infinitesimal constant convergence property to $F_\infty(f_\infty^1, \ldots, f_\infty^k)$ at w.*

PROOF. We fix $\epsilon > 0$. We take $\hat{R}, L \geq 1$ satisfying that $\bigcup_i \mathrm{Image}(f_i^1, \ldots, f_i^l) \subset B_{\hat{R}}(0_k)$, $\sup_{i,l}(|f_i^l|_{L^\infty(B_R(x_i))} + |f_\infty^l|_{L^\infty(B_R(x_\infty))}) \leq \hat{R}$ and $\sup_i |F_i|_{L^\infty(B_{\hat{R}}(0_k))} \leq L$. There exists a nonnegative valued function b on $\mathbf{R}_{>0}$ such that $b(t) \to 0$ as $t \to 0$ and that for every $t > 0$, there exists i_t such that $F_\infty(\alpha) = F_i(\beta) \pm b(t)$ for every $\alpha \in B_{\hat{R}}(0_k)$, $i \geq i_t$ and $\beta \in B_t(\alpha)$. On the other hand, there exists $\tau_1 > 0$ satisfying the following properties: For every $0 < s < \tau_1$, there exists j_s such that

$$\frac{1}{\upsilon_i(B_s(w_i))} \int_{B_s(w_i)} \left| f_i^l - \frac{1}{\upsilon_\infty(B_s(w))} \int_{B_s(w)} f_\infty^l d\upsilon_\infty \right| \upsilon_i \leq \epsilon$$

and

$$\frac{1}{\upsilon_\infty(B_s(w))} \int_{B_s(w)} \left| f_\infty^l - \frac{1}{\upsilon_i(B_s(w_i))} \int_{B_s(w_i)} f_i^l d\upsilon_i \right| \upsilon_\infty \leq \epsilon$$

for every $1 \leq l \leq k$, $i \geq j_s$ and $w_i \to w$. Especially, we have

$$\frac{1}{\upsilon_\infty(B_s(w))} \int_{B_s(w)} f_\infty^l d\upsilon_\infty = \frac{1}{\upsilon_i(B_s(w_i))} \int_{B_s(w_i)} f_i^l d\upsilon_i \pm \epsilon.$$

We fix $0 < s < \tau_1$. Therefore, there exist a sequence of compact sets $K_i \subset B_s(w_i)$ and a compact set $K_\infty \subset B_s(w)$ such that $\upsilon_i(K_i)/\upsilon_i(B_s(w_i)) \geq 1 - \Psi(\epsilon; K(1))$, $\upsilon_\infty(K_\infty)/\upsilon_\infty(B_s(w)) \geq 1 - \Psi(\epsilon; K(1))$ and that

$$\left| f_i^l(\alpha) - \frac{1}{\upsilon_\infty(B_s(w))} \int_{B_s(w)} f_\infty^l d\upsilon_\infty \right| < \Psi(\epsilon; K(1))$$

and

$$\left| f_\infty^l(\beta) - \frac{1}{\upsilon_i(B_s(w_i))} \int_{B_s(w_i)} f_i^l d\upsilon_i \right| < \Psi(\epsilon; K(1))$$

for every $j_s \leq i < \infty$, $1 \leq l \leq k$, $\alpha \in K_i$ and $\beta \in K_\infty$. Without loss of generality, we can assume that there exists a compact set $\hat{K} \subset \overline{B}_s(w)$ such that $K_i \to \hat{K}$. We put $\hat{K}_\infty = \hat{K} \cap K_\infty$. By Proposition 2.14, we have $\upsilon_\infty(\hat{K}_\infty)/\upsilon_\infty(B_s(w)) \geq 1 - \Psi(\epsilon; K(1))$. We put

$$a_i^l = \frac{1}{\upsilon_i(B_s(w_i))} \int_{B_s(w_i)} f_i^l d\upsilon_i.$$

Then, there exists $k_s \geq j_s$ such that

$$F_\infty(f^1_\infty(\alpha),\dots,f^k_\infty(\alpha)) = F_\infty(a^1_\infty,\dots,a^k_\infty) \pm b(\Psi(\epsilon;K(1))) \tag{87}$$

$$= F_i(a^1_i,\dots,a^k_i) \pm 2b(\Psi(\epsilon;K(1))) \tag{88}$$

$$= F_i(f^1_i(\alpha_i),\dots,f^k_i(\alpha_i)) \pm 3b(\Psi(\epsilon;K(1))) \tag{89}$$

for every $i \geq k_s$, $\alpha \in \hat{K}_\infty$ and $\alpha_i \in K_i$ with $\alpha_i \to \alpha$. Thus, we have

$$\frac{1}{\upsilon_\infty(B_s(w))}\int_{B_s(w)} \left|F_\infty(f^1_\infty,\dots,f^k_\infty) - F_\infty(a^1_\infty,\dots,a^k_\infty)\right| d\upsilon_\infty \tag{90}$$

$$= \frac{1}{\upsilon_\infty(B_s(w))}\int_{\hat{K}_\infty} \left|F_\infty(f^1_\infty,\dots,f^k_\infty) - F_\infty(a^1_\infty,\dots,a^k_\infty)\right| d\upsilon_\infty \pm \Psi(\epsilon;K(1),L) \tag{91}$$

$$< 3b(\Psi(\epsilon;K(1))) + \Psi(\epsilon;K(1),L) \tag{92}$$

and

$$\frac{1}{\upsilon_i(B_s(w_i))}\int_{B_s(w_i)} \left|F_i(f^1_i,\dots,f^k_i) - F_i(a^1_i,\dots,a^k_i)\right| d\upsilon_i \tag{93}$$

$$= \frac{1}{\upsilon_i(B_s(w_i))}\int_{K_i} \left|F_i(f^1_i,\dots,f^k_i) - F_i(a^1_i,\dots,a^k_i)\right| d\upsilon_i \pm \Psi(\epsilon;K(1),L) \tag{94}$$

$$< 3b(\Psi(\epsilon;K(1))) + \Psi(\epsilon;K(1),L) \tag{95}$$

for $i \geq k_s$. Moreover, we have

$$\frac{1}{\upsilon_\infty(B_s(w))}\int_{B_s(w)} F_\infty(f^1_\infty,\dots,f^k_\infty) d\upsilon_\infty \tag{96}$$

$$= \frac{1}{\upsilon_\infty(B_s(w))}\int_{\hat{K}_\infty} F_\infty(f^1_\infty,\dots,f^k_\infty) d\upsilon_\infty \pm \Psi(\epsilon;K(1),L) \tag{97}$$

$$= (1 \pm \Psi(\epsilon;K(1)))(F_\infty(a^1_\infty,\dots,a^k_\infty) \pm b(\Psi(\epsilon;K(1))) \pm \Psi(\epsilon;K(1)) \tag{98}$$

$$= (1 \pm \Psi(\epsilon;K(1)))(F_i(a^1_i,\dots,a^k_i) \pm b(\Psi(\epsilon;K(1))) \pm \Psi(\epsilon;K(1),L) \tag{99}$$

$$= (1 \pm \Psi(\epsilon;K(1)))\left(\frac{1}{\upsilon_i(B_s(w_i))}\int_{K_i} F_i(f^1_i,\dots,f^k_i) d\upsilon_i \pm 3b(\Psi(\epsilon;K(1)))\right) \pm \Psi(\epsilon;K(1),L) \tag{100}$$

$$= (1 \pm \Psi(\epsilon;K(1)))\left(\frac{1}{\upsilon_i(B_s(w_i))}\int_{B_s(w_i)} F_i(f^1_i,\dots,f^k_i) d\upsilon_i \pm 3b(\Psi(\epsilon;K(1)))\right) \pm \Psi(\epsilon;K(1),L) \tag{101}$$

for $i \geq k_s$. Therefore, we have the assertion. □

REMARK 4.6. By the proof of Proposition 4.5, we also have the following: Let k be a positive integer, f_i^l Borel functions on $B_R(x_i)(1 \leq l \leq k, 1 \leq i \leq \infty)$ satisfying $\sup_{i,l}(|f_i^l|_{L^\infty(B_R(x_i))}+|f_\infty^l|_{L^\infty(B_R(x_\infty))}) < \infty$, w a point in $B_R(x_\infty)$ and $\{F_i\}_{1\leq i\leq\infty}$ a sequence of locally L^∞ functions on $\mathbf{R}^k$. Assume the following:

1. $\{f_i^l\}_{1\leq i\leq\infty}$ has infinitesimal constant convergence property to f_∞^l at w for every l.
2. The limits
$$a^l = \lim_{r\to 0} \frac{1}{\upsilon_\infty(B_r(w))} \int_{B_r(w)} f_\infty^l d\upsilon_\infty$$
exist for every l.
3. There exists an open neighborhood U at $(a^1, \cdots, a^k) \in \mathbf{R}^k$ such that F_i is continuous on U for every $1 \leq i \leq \infty$ and that F_i converges to F_∞ on U uniformly.

Then, the sequence $\{F_i(f_i^1,\cdots, f_i^k)\}$ has infinitesimal constant convergence property to $F_\infty(f_\infty^1,\cdots, f_\infty^k)$ at w.

For Ricci limit spaces, we shall give a sufficient condition to satisfy infinitesimal constant convergence property for radial derivative of Lipschitz functions:

PROPOSITION 4.7. *Let $\{(M_i, m_i, \underline{\mathrm{vol}})\}_i$ be a sequence of pointed connected n-dimensional complete Riemannian manifolds with $\mathrm{Ric}_{M_i} \geq -(n-1)$, (Y, y, υ) be a pointed proper geodesic space with Radon measure υ, R a positive number, x_∞ a point in Y, x_i a point in M_i, f_i a C^2-function on $B_R(x_i)$ and f_∞ a Lipschitz function on $B_R(x)$. We assume that $\sup_i \mathbf{Lip} f_i < \infty$, $(M_i, m_i, x_i, f_i, \underline{\mathrm{vol}}) \overset{(\phi_i, R_i, \epsilon_i)}{\to} (Y, y, x_\infty, f_\infty, \upsilon)$ and that*
$$\sup_i \int_{B_R(x_i)} |\mathrm{Hess}_{f_i}|^2 d\underline{\mathrm{vol}} < \infty.$$
Then, there exists a Borel subset $A \subset B_R(x_\infty)$ such that $\upsilon(B_R(x_\infty) \setminus A) = 0$ and that for every $z \in A$ and $w_i \to w \in Y$, the sequence $\{\langle dr_{w_i}, df_i\rangle\}$ has infinitesimal constant convergence property to $\langle dr_w, df_\infty\rangle$ at z.

PROOF. We fix $\epsilon > 0$ and take $L \geq 1$ satisfying
$$\sup_i \left(\frac{1}{\underline{\mathrm{vol}}\, B_R(x_i)} \int_{B_R(x_i)} |\mathrm{Hess}_{f_i}|^2 d\underline{\mathrm{vol}} + \mathbf{Lip} f_i \right) \leq L.$$
By Theorem 3.33, there exist $0 < \eta << \epsilon$ and a Borel subset $X(\epsilon) \subset B_R(x_\infty) \cap \mathcal{D}_z^\eta \setminus B_\eta(z)$ such that
$$\frac{\upsilon(B_R(x_\infty) \setminus X(\epsilon))}{\upsilon(B_R(x_\infty))} \leq \epsilon$$

and that

$$\left|\frac{f_\infty \circ \gamma(\overline{z,\alpha}+h) - f_\infty(\alpha)}{h} - \langle dr_z, df_\infty\rangle(\alpha)\right| \le \epsilon$$

for every $\alpha \in X(\epsilon)$, h satisfying $0 < |h| < \eta$ and isometric embedding γ from $[0, \overline{z,\alpha}+\eta]$ to Y with $\gamma(0) = z$ and $\gamma(\overline{z,\alpha}) = \alpha$. By Corollary 7.6, there exists Borel set $\hat{X}(\epsilon) \subset X(\epsilon)$ such that $\upsilon(X(\epsilon) \setminus \hat{X}(\epsilon)) = 0$ and that

$$\lim_{t\to 0} \frac{1}{\upsilon(B_t(\alpha))} \int_{B_t(\alpha)} |\langle dr_z, df_\infty\rangle - \langle dr_z, df_\infty\rangle(\alpha)| d\upsilon = 0$$

for every $\alpha \in \hat{X}(\epsilon)$. For every $\alpha \in \hat{X}(\epsilon)$, there exists $r(\alpha) > 0$ such that

$$\frac{1}{\upsilon(B_t(\alpha))} \int_{B_t(\alpha)} |\langle dr_z, df_\infty\rangle - \langle dr_z, df_\infty\rangle(\alpha)| d\upsilon < \epsilon$$

for every $0 < t < r(x)$. We put $l = \eta^{-1/4}$. By an argument similar to the proof of Proposition 3.1, for every i, there exists a compact subset $K_i \subset B_{R-\epsilon}(x_i)$ such that

$$\frac{\underline{\mathrm{vol}}(B_{R-\epsilon}(x_i) \setminus K_i)}{\underline{\mathrm{vol}}\, B_{R-\epsilon}(x_i)} \le \Psi(l^{-1}; n, R, L)$$

and that

$$\frac{1}{\underline{\mathrm{vol}}\, B_t(w)} \int_{B_t(w)} |\mathrm{Hess}_{f_i}|^2 d\underline{\mathrm{vol}} \le l$$

for every $w \in K_i$ and $0 < t < \epsilon/100$. Without loss of generality, we can assume that there exists a compact set $K_\infty \subset \overline{B}_R(x_\infty)$ such that $K_i \to K_\infty$. We put $W(\epsilon) = K_\infty \cap X(\epsilon)$. By Proposition 2.14, we have

$$\frac{\upsilon(W(\epsilon))}{\upsilon(B_R(x_\infty))} \ge 1 - \Psi(\epsilon; n, R, L).$$

We fix $\alpha \in W(\epsilon)$, $0 < t << \min\{\eta, r(\alpha)\}$ and an isometric embedding γ from $[0, \overline{z,\alpha}+\eta]$ to Y satisfying $\gamma(0) = z$ and $\gamma(\overline{z,\alpha}) = \alpha$. We take $\alpha_i \in K_i$ satisfying $\alpha_i \to \alpha$. We define a Borel function F_i on $B_t(\alpha_i) \setminus (C_{z_i} \cup \{z_i\})$ by

$$F_i(\beta) = \frac{f_i \circ \gamma_\beta(\overline{z_i,\beta} - \eta^2) - f_i(\beta)}{-\eta^2}.$$

Here γ_β is the minimal geodesic from z_i to β. By an argument similar to the proof of Claim 3.37, we have

$$\frac{1}{\underline{\mathrm{vol}}\, B_t(\alpha_i)} \int_{B_t(\alpha_i)} |\langle df_i, dr_{z_i}\rangle - F_i| d\underline{\mathrm{vol}} \tag{102}$$

$$\le \eta^2 \frac{C(n)}{\underline{\mathrm{vol}}\, B_{10t}(\alpha_i)} \int_{B_{10t}(\alpha_i)} |\mathrm{Hess}_{f_i}|^2 d\underline{\mathrm{vol}} \le \eta^2 C(n) l \le \Psi(\epsilon; n) \tag{103}$$

for every i. We take i_0 satisfying that $\epsilon_i << t$ for every $i \geq i_0$. For every $i \geq i_0$ and $\beta_i \in B_t(\alpha_i)$, we remark that $\overline{\phi_i(\beta_i), \alpha} \leq t + \epsilon_i \leq \eta^3$. Then, since

$$\overline{z, \phi_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2))}^{\eta^{-2}d_Y} + \overline{\phi_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2)), \phi_i(\beta_i)}^{\eta^{-2}d_Y} - \overline{z, \phi_i(\beta_i)}^{\eta^{-2}d_Y} < 3\epsilon_i,$$

we have

$$\overline{z, \phi_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2))}^{\eta^{-2}d_Y} + \overline{\phi_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2)), \alpha}^{\eta^{-2}d_Y} - \overline{z, \alpha}^{\eta^{-2}d_Y} < 5\eta.$$

Similarly, we have

$$\overline{z, \phi_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2))}^{\eta^{-2}d_Y} + \overline{\phi_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2)), \gamma(\overline{z, \alpha} + \eta)}^{\eta^{-2}d_Y} - \overline{z, \gamma(\overline{z, \alpha} + \eta)}^{\eta^{-2}d_Y} < 5\eta,$$

$$\overline{\phi_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2)), \gamma(\overline{z, \alpha} + \eta)}^{\eta^{-2}d_Y} \geq \eta^{-1} - \eta,$$

$$\overline{\phi_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2)), z}^{\eta^{-2}d_Y} \geq \eta^{-1} - \eta$$

and

$$\overline{\phi_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2)), \alpha}^{\eta^{-2}d_Y} = 1 \pm 5\eta.$$

Therefore, by splitting theorem, we have

$$\overline{\phi_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2)), \gamma(\overline{z, \alpha} - \eta^2)}^{\eta^{-2}d_Y} \leq \Psi(\eta; n).$$

Thus we have

$$\frac{f_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2)) - f_i(\beta_i)}{-\eta^2} = \frac{f_\infty(\phi_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2))) - f_\infty(\phi_i(\beta_i))}{-\eta^2} \pm \frac{\epsilon_i}{\eta^2} \tag{104}$$

$$= \frac{f_\infty(\gamma(\overline{z, \alpha} - \eta^2))) - f_\infty(\alpha)}{-\eta^2} \pm \Psi(\eta; n, L) \tag{105}$$

$$= \langle dr_z, df_\infty \rangle(\alpha) \pm \Psi(\eta; n, L). \tag{106}$$

Especially, we have

$$\frac{1}{\underline{\mathrm{vol}}\, B_t(\alpha_i)} \int_{B_t(\alpha_i)} |F_i - \langle dr_z, df_\infty \rangle(\alpha)| d\underline{\mathrm{vol}} \leq \Psi(\eta; n, L)$$

for $i \geq i_0$. Therefore if we put $W = \bigcap_{N_1 \in \mathbf{N}} (\bigcup_{N_2 \geq N_1} W(N_2^{-1}))$, then $\upsilon(B_R(x_\infty) \setminus W) = 0$, $\{\langle dr_{z_i}, df_i \rangle\}$ has infinitesimal constant convergence property to $\langle dr_w, df_\infty \rangle$ at every $w \in W$. □

Remark 4.8. We shall introduce the following important method to get some uniformly Hessian estimates by using cut-off functions with good properties by Cheeger-Colding: Let $(M, m, \underline{\mathrm{vol}})$ be a pointed connected n-dimensional complete Riemannian

manifold with renormalized measure satisfying $\mathrm{Ric}_M \geq -(n-1)$, R a positive number and f a C^2-function on $B_R(m)$. We assume that there exists $L \geq 1$ such that

$$|\nabla f|_{L^\infty(B_R(m))} + \frac{1}{\underline{\mathrm{vol}}\, B_R(m)} \int_{B_R(m)} |\Delta f|^2 d\underline{\mathrm{vol}} \leq L$$

Then, we have

$$\frac{1}{\underline{\mathrm{vol}}\, B_r(m)} \int_{B_r(m)} |\mathrm{Hess}_f|^2 d\underline{\mathrm{vol}} < C(n, r, R, L)$$

for every $0 < r < R$. The proof is as follows. By standard smoothing argument, without loss of generality, we can assume that f is a smooth function. There exists a smooth function ϕ on M such that $0 \leq \phi \leq 1$, $\phi|_{B_r(m)} = 1$, $\mathrm{supp}\phi \subset B_R(m)$, $|\nabla\phi| \leq C(n, r, R)$ and $|\Delta\phi| \leq C(n, r, R)$ (see for instance [4, Theorem 8.16]). By Bochner's formula, we have

$$-\frac{1}{2}\Delta|\nabla(\phi f)|^2 \geq |\mathrm{Hess}_{\phi f}|^2 - \langle \nabla\Delta(\phi f), \nabla(\phi f)\rangle - (n-1)|\nabla(\phi f)|^2.$$

Thus, we have

$$\frac{1}{\underline{\mathrm{vol}}\, B_r(m)} \int_{B_r(m)} |\mathrm{Hess}_f|^2 d\underline{\mathrm{vol}} \tag{107}$$

$$\leq \frac{C(n, r, R)}{\underline{\mathrm{vol}}\, B_R(m)} \int_{B_R(m)} |\mathrm{Hess}_{\phi f}|^2 d\underline{\mathrm{vol}} \tag{108}$$

$$\leq \frac{C(n, r, R)}{\underline{\mathrm{vol}}\, B_R(m)} \int_{B_R(m)} (\Delta(\phi f))^2 \, d\underline{\mathrm{vol}} + C(n, R, L) \tag{109}$$

$$\leq \frac{2C(n, r, R)}{\underline{\mathrm{vol}}\, B_R(m)} \int_{B_R(m)} (f\Delta\phi)^2 + (\phi\Delta f)^2 + |\langle \nabla f, \nabla\phi\rangle|^2 d\underline{\mathrm{vol}} + C(n, R, L) \tag{110}$$

$$\leq C(n, r, R, L). \tag{111}$$

This observation performs a crucial role to study limit functions of harmonic functions.

The following proposition follows from Lemma 3.40 directly.

PROPOSITION 4.9. *Let $\{(M_i, m_i, \underline{\mathrm{vol}})\}$ be a sequence of pointed connected n-dimensional complete Riemannian manifolds with renormalized measure satisfying $\mathrm{Ric}_{M_i} \geq -(n-1)$, (Y, y, υ) be a Ricci limit space of $\{(M_i, m_i, \underline{\mathrm{vol}})\}_i$. Then for every $w^1, w^2 \in Y$, $z \in Y \setminus (C_{w^1} \cup C_{w^2} \cup \{w^1, w^2\})$ and $w_i^j \to w^j \in Y (j = 1, 2)$, the sequence $\{\langle dr_{w_i^1}, dr_{w_i^2}\rangle\}$ has infinitesimal constant convergence property to $\langle dr_{w_\infty^1}, dr_{w_\infty^2}\rangle$ at z.*

4.2 Infinitesimal convergence property

In this subsection, we will give a notion of *infinitesimal convergence property* and its fundamental properties.

DEFINITION 4.10 (Infinitesimal convergence property). Let R be a positive number, w a point in $B_R(x_\infty)$ and f_i a Borel function on $B_R(x_i)(1 \leq i \leq \infty)$ satisfying $\sup_i |f_i|_{L^\infty(B_R(x_i))} + |f_\infty|_{L^\infty(B_R(x_\infty))} < \infty$. We say that $\{f_i\}_i$ has *infinitesimal convergence property to f_∞ at w* if for every $\epsilon > 0$, there exists $r > 0$ such that

$$\limsup_{i\to\infty} \left| \frac{1}{v_i(B_t(w_i))} \int_{B_t(w_i)} f_i dv_i - \frac{1}{v_\infty(B_t(w))} \int_{B_t(w)} f_\infty dv_\infty \right| \leq \epsilon$$

for every $0 < t < r$ and $w_i \to w$.

It is clear that if the sequence $\{f_i\}_i$ has infinitesimal constant convergence property to f_∞ at w, then $\{f_i\}_i$ has infinitesimal convergence property to f_∞ at w. We skip the proof of the next proposition because it is not difficult.

PROPOSITION 4.11 (Linearlity of infinitesimal convergence property). *Let R be a positive number, a_i, b_i, c_i, d_i Borel functions on $B_R(x_i)(1 \leq i \leq \infty)$, w a point in $B_R(x_\infty)$. We assume that $\sup_i(|a_i| + |b_i| + |c_i| + |d_i|)_{L^\infty(B_R(x_i))} < \infty$ and that $\{a_i\}_i$, $\{b_i\}_i$ have infinitesimal constant convergence property to a_∞, b_∞ at w, respectively and $\{c_i\}_i$, $\{d_i\}_i$ have infinitesimal convergence property to c_∞, d_∞ at w, respectively. Then $\{a_i c_i + b_i d_i\}$ has infinitesimal convergence property to $a_\infty c_\infty + b_\infty d_\infty$ at w.*

The next proposition follows from an argument similar to the proof of Proposition 2.14:

PROPOSITION 4.12. *Let R be a positive number, K_i a Borel subset of $\overline{B}_R(x_i)$ and f_i a nonnegative valued Borel function on $\overline{B}_R(x_i)(1 \leq i \leq \infty)$ satisfying $\sup_i |f_i|_{L^\infty(B_R(x_i))} + |f_\infty|_{L^\infty(B_R(x_\infty))} < \infty$. We assume that K_∞ is compact, $\limsup_{i\to\infty} K_i \subset K_\infty$ and that for a.e. $w \in K_\infty$, $\{f_i\}$ has infinitesimal convergence property to f_∞ at w. Then we have*

$$\limsup_{i\to\infty} \int_{K_i} f_i dv_i \leq \int_{K_\infty} f_\infty dv_\infty.$$

We shall state a fundamental result for infinitesimal convergence property:

PROPOSITION 4.13. *Let R be a positive number, K_i a Borel subset of $\overline{B}_R(x_i)$ and $\{f_i\}_i$ a Borel function on $\overline{B}_R(x_i)(1 \leq i \leq \infty)$ satisfying $\sup_i |f_i|_{L^\infty(B_R(x_i))} + |f_\infty|_{L^\infty(B_R(x_\infty))} < \infty$. We assume that K_∞ is compact, $\limsup_{i\to\infty} K_i \subset K_\infty$ and that for a.e. $w \in K_\infty$, $\{1_{K_i}\}_i$ and $\{f_i\}_i$ have infinitesimal convergence property to 1_{K_∞}, f_∞ at w, respectively. Then, we have*

$$\lim_{i\to\infty} \int_{K_i} f_i dv_i = \int_{K_\infty} f_\infty dv_\infty.$$

PROOF. We fix $\epsilon > 0$. We take $L \geq 1$ satisfying $\sup_i |f_i|_{L^\infty} + |f_\infty| + \upsilon_\infty(B_R(x_\infty)) < L$. There exists a Borel subset $\hat{K}_\infty \subset K_\infty$ satisfying the following properties: For every $w \in \hat{K}_\infty$, there exists $t_w > 0$ such that $\overline{B}_{10t_w}(w) \subset B_R(x)$ and that

$$\limsup_{i\to\infty} \left| \frac{1}{\upsilon_i(B_s(w_i))} \int_{B_s(w_i)} f_i d\upsilon_i - \frac{1}{\upsilon_\infty(B_s(w))} \int_{B_s(w)} f_\infty d\upsilon_\infty \right| < \epsilon,$$

$$\frac{\upsilon_\infty(B_s(w) \cap K_\infty)}{\upsilon_\infty(B_s(w))} \geq 1 - \epsilon$$

and

$$\limsup_{i\to\infty} \left| \frac{1}{\upsilon_i(B_s(w_i))} \int_{B_s(w_i)} 1_{K_i} d\upsilon_i - \frac{1}{\upsilon_\infty(B_s(w))} \int_{B_s(w)} 1_{K_\infty} d\upsilon_\infty \right| < \epsilon$$

for every $0 < s < t_w$ and $w_i \to w$. By Lemma 2.12, there exists a pairwise disjoint collection $\{\overline{B}_{r_i}(x_i)\}_i$ such that $x_i \in K_\infty$, $r_i << t_{x_i}$, and that $K_\infty \setminus \bigcup_{i=1}^N \overline{B}_{r_i}(x_i) \subset \bigcup_{i=N+1}^\infty \overline{B}_{5r_i}(x_i)$ for every N. We take N satisfying $\sum_{i=N+1}^\infty \upsilon_\infty(B_{r_i}(x_i)) < \epsilon$. Then, we have $\sum_{i=N+1}^\infty \upsilon_\infty(B_{5r_i}(x_i)) < 2^{5K(1)}\epsilon$. We take $x_i(j) \in Z_j$ satisfying $x_i(j) \to x_i$. Then we have

$$\int_{K_\infty} f_\infty d\upsilon_\infty = \sum_{i=1}^N \int_{B_{r_i}(x_i)\cap K_\infty} f_\infty d\upsilon_\infty \pm \int_{\bigcup_{i=N+1}^\infty \overline{B}_{5r_i}(x_i)} |f_\infty| d\upsilon_\infty \tag{112}$$

$$= \sum_{i=1}^N \int_{B_{r_i}(x_i)} f_\infty d\upsilon_\infty \pm \Psi(\epsilon; K(1), L) \tag{113}$$

$$= \sum_{i=1}^N \int_{B_{r_i}(x_i(j))} f_j d\upsilon_j \pm \Psi(\epsilon; K(1), L) \tag{114}$$

$$= \sum_{i=1}^N \int_{B_{r_i}(x_i(j))\cap K_j} f_j d\upsilon_j \pm \Psi(\epsilon; K(1), L) \tag{115}$$

$$= \int_{K_j} f_j d\upsilon_j \pm \left(\int_{K_j \setminus \bigcup_{i=1}^N \overline{B}_{r_i}(x_i(j))} |f_j| d\upsilon_j + \Psi(\epsilon; K(1), L) \right). \tag{116}$$

for every sufficiently large j. On the other hand, by Proposition 2.5, Proposition 2.6 and Proposition 2.14, we have

$$\limsup_{j\to\infty} \int_{K_j \setminus \bigcup_{i=1}^N \overline{B}_{r_i}(x_i(j))} |f_j| d\upsilon_j \leq L \limsup_{j\to\infty} \upsilon_j(K_j \setminus \bigcup_{i=1}^N B_{r_i}(x_i(j))) \tag{117}$$

$$\leq L\upsilon_\infty(K_\infty \setminus \bigcup_{i=1}^N B_{r_i}(x_i)) \tag{118}$$

$$\leq \Psi(\epsilon; K(1), L). \tag{119}$$

Therefore, we have the assertion. □

REMARK 4.14. Proposition 2.16 also follows from Example 4.2, 4.4 and Proposition 4.13 directly.

Next corollary follows from Proposition 4.13 directly.

COROLLARY 4.15. *Let R, r_i be positive numbers, N a positive integer, $\{z_j\}_{1\leq j\leq N}$ points in Y and f_i a Borel function on $B_R(x_j)(1 \leq i \leq \infty)$ satisfying $\sup_i |f_i|_{L^\infty(B_R(x_i))} + |f_\infty|_{L^\infty(B_R(x_\infty))} < \infty$. We assume that for a.e. $w \in B_R(x_\infty) \setminus \bigcup_{i=1}^N B_{r_i}(z_i)$, $\{f_i\}_i$ have infinitesimal convergence property to f_∞ at w. Then, we have*

$$\lim_{j\to\infty} \int_{B_R(x_j)\setminus\bigcup_{i=1}^N B_{r_i}(z_i(j))} f_j d\upsilon_j = \int_{B_R(x_\infty)\setminus\bigcup_{i=1}^N B_{r_i}(z_i)} f_\infty d\upsilon_\infty$$

for every $z_i(j) \to z_i$.

We end this subsection by giving the following proposition:

PROPOSITION 4.16. *Let A_i be a Borel subset of $B_R(x_i)$ and $w \in \mathrm{Leb}A_\infty$. We assume that $\{1_{A_i}\}_i$ has infinitesimal convergence property to 1_{A_∞} at w. Then $\{1_{A_i}\}$ has infinitesimal constant convergence property to 1_{A_∞} at w.*

PROOF. We fix $\epsilon > 0$ and take a sequence $w_i \to w$. There exists $r > 0$ such that

$$\frac{\upsilon_\infty(B_t(w) \cap A_\infty)}{\upsilon_\infty(B_t(w))} \geq 1 - \epsilon$$

and

$$\limsup_{i\to\infty} \left| \frac{1}{\upsilon_i(B_t(w_i))} \int_{B_t(w_i)} 1_{A_i} d\upsilon_i - \frac{1}{\upsilon_\infty(B_t(w_\infty))} \int_{B_t(w_\infty)} 1_{A_\infty} d\upsilon_\infty \right| < \epsilon$$

for every $0 < t < r$. We fix $0 < t < r$. Then we have

$$\begin{aligned}
&\frac{1}{\upsilon_i(B_t(w_i))} \int_{B_t(w_i)} \left| 1_{A_i} - \frac{1}{\upsilon_\infty(B_t(w_\infty))} \int_{B_t(w_\infty)} 1_{A_\infty} d\upsilon_\infty \right| d\upsilon_i && (120)\\
&\leq \frac{1}{\upsilon_i(B_t(w_i))} \int_{B_t(w_i)} \left| 1_{A_i} - \frac{1}{\upsilon_\infty(B_t(w_\infty))} \int_{B_t(w_i)} 1_{A_i} d\upsilon_\infty \right| d\upsilon_i + \epsilon && (121)\\
&= \frac{1}{\upsilon_i(B_t(w_i))} \int_{B_t(w_i)} \left| 1_{A_i} - \frac{\upsilon_i(A_i)}{\upsilon_i(B_t(w_i))} \right| d\upsilon_i + \epsilon && (122)\\
&= \frac{1}{\upsilon_i(B_t(w_i))} \int_{A_i} \frac{\upsilon_i(B_t(w_i) \setminus A_i)}{\upsilon_i(B_t(w_i))} d\upsilon_i + \frac{1}{\upsilon_i(B_t(w_i))} \int_{B_t(w_i)\setminus A_i} \frac{\upsilon_i(A_i)}{\upsilon_i(B_t(w_i))} d\upsilon_i + \epsilon && (123)\\
&\leq 2\frac{\upsilon_i(B_t(w_i) \setminus A_i)}{\upsilon_i(B_t(w_i))} + \epsilon < 3\epsilon + 2\epsilon < 5\epsilon. && (124)
\end{aligned}$$

for every sufficiently large i. Similarly, we have

$$\frac{1}{v_\infty(B_t(w_\infty))}\int_{B_t(w_\infty)}\left|1_{A_\infty}-\frac{1}{v_i(B_t(w_i))}\int_{B_t(w_i)}1_{A_i}dv_i\right|dv_\infty \tag{125}$$

$$\leq \frac{1}{v_\infty(B_t(w_\infty))}\int_{B_t(w_\infty)}\left|1_{A_\infty}-\frac{1}{v_\infty(B_t(w_\infty))}\int_{B_t(w_\infty)}1_{A_\infty}dv_\infty\right|dv_\infty+\epsilon \tag{126}$$

$$\leq 2\frac{v_\infty(B_t(w_\infty)\setminus A_\infty)}{v_\infty(B_t(w_\infty))}+\epsilon<3\epsilon \tag{127}$$

for every sufficiently large i. Thus, we have the assertion. □

4.3 Convergence of differential of Lipschitz functions

The purpose of this subsection is to give a definition of convergence: $df_i \to df_\infty$. See Definition 1.1 or Definition 4.18. Throughout this subsection, we fix the following situation: Let $\{(M_i, m_i, \underline{\mathrm{vol}})\}_i$ be a sequence of pointed, connected n-dimensional complete Riemannian manifolds with renormalized measure satisfying $\mathrm{Ric}_{M_i} \geq -(n-1)$, (Y, y, v) a Ricci limit space of $\{(M_i, m_i, \underline{\mathrm{vol}})\}_i$, R a positive number, x_i a point in M_i, x_∞ a point in Y, f_i a Lipschitz function on $B_R(x_i)$ and f_∞ a Lipschitz function on $B_R(x_\infty)$. We assume that $\sup_i(\mathbf{Lip}f_i + |f_i|_{L^\infty}) < \infty$ and that $x_i \to x_\infty$.

For $w \in B_R(x_\infty)$, we say that f_i *converges to* f_∞ at w if $f_i(w_i) \to f_\infty(w)$ holds for every $w_i \to w$. We denote it by $f_i \to f_\infty$ at w. It is easy to check that the following conditions are equivalent:

1. $\{f_i\}$ has infinitesimal convergence property to f_∞ at w.
2. $f_i \to f_\infty$ at w.
3. $\{f_i\}$ has infinitesimal constant convergence property to f_∞ at w.

We shall consider a convergence of energy of Lipschitz functions. See also [5, Corollary 10.17].

Definition 4.17 (Infinitesimal upper semicontinuity of energy). We say that $\{f_i\}_i$ has *infinitesimal upper semicontinuity of energy to* f_∞ *at* $w \in B_R(x_\infty)$ if for every $\epsilon > 0$ and $w_i \to w$, there exists $r > 0$ such that

$$\limsup_{i\to\infty}\frac{1}{\underline{\mathrm{vol}}\, B_t(w_i)}\int_{B_t(w_i)}(\mathrm{Lip}f_i)^2 d\underline{\mathrm{vol}} \leq \frac{1}{v(B_t(w))}\int_{B_t(w)}(\mathrm{Lip}f_\infty)^2 dv+\epsilon$$

for every $0 < t < r$.

By the definition, if $\{(\mathrm{Lip}f_i)^2\}_i$ has infinitesimal convergence property to $(\mathrm{Lip}f_\infty)^2$ at w, then $\{f_i\}_i$ has infinitesimal upper semicontinuity of energy to f_∞ at w. Next, we shall give a definition of convergence of differential of Lipschitz functions:

DEFINITION 4.18 (Convergence of differential of Lipschitz functions). We say that df_i converges to df_∞ at $w \in B_R(x_\infty)$ if $\{\langle dr_{z_i}, df_i\rangle\}_i$ has infinitesimal convergence property to $\langle df_\infty, dg_\infty\rangle$ at w for every $z_i \to z \in Y$ and $\{f_i\}_i$ has infinitesimal upper semicontinuity of energy to f_∞ at w. Then we denote it by $df_i \to df_\infty$ at w. Moreover, for a subset A of $B_R(x_\infty)$, if $f_i \to f_\infty$ and $df_i \to df_\infty$ at every $a \in A$, then we denote it by $(f_i, df_i) \to (f_\infty, df_\infty)$ on A.

PROPOSITION 4.19. *For every $w_i \to w \in Y$, we have $(r_{w_i}, dr_{w_i}) \to (r_w, dr_w)$ on Y.*

PROOF. It follows from Proposition 4.9 and Proposition 4.13 directly. □

The following theorem is the main result in this subsection:

THEOREM 4.20. *Let g_i be a Lipschitz function on $B_R(x_i)$ and A a Borel subset of $B_R(x_\infty)$. We assume that $df_i \to df_\infty$ and $dg_i \to dg_\infty$ on A. Then, for a.e. $w \in A$, the sequence $\{\langle df_i, dg_i\rangle\}_i$ has infinitesimal constant convergence property to $\langle df_\infty, dg_\infty\rangle$ at w.*

PROOF. By Theorem 3.17 and Lemma 3.53, there exist a collection of Borel set $A_j \subset A \setminus \{x_\infty\}$, positive integers $1 \le k_j \le n$ and points $x_1^j, \dots, x_{k_j}^j \in Y$ satisfying the following properties:

1. $\upsilon(A \setminus \bigcup_{j=1}^\infty A_j) = 0$.
2. $A_j \subset Y \setminus \bigcup_{l=1}^{k_j}(C_{x_l^j} \cup \{x_l^j\})$.
3. For every $w \in A_j$, there exists $a_1^j, \dots, a_{k_j}^j, b_1^j, \dots, b_{k_j}^j \in \mathbf{R}$ such that

$$\lim_{r\to 0} \frac{1}{\upsilon(B_r(w))} \int_{B_r(w)} \left| df_\infty - d\left(\sum_{l=1}^{k_j} a_l^j r_{x_l^j}\right)\right|^2 + \left|dg_\infty - d\left(\sum_{l=1}^{k_j} b_l^j r_{x_l^j}\right)\right|^2 d\upsilon = 0.$$

We take $w \in A_j$ and $a_1^j, \dots, a_{k_j}^j, b_1^j, \dots, b_{k_j}^j \in \mathbf{R}$ satisfying equalities above. We also take $L \ge 1$ satisfying $\sup_i(\mathbf{Lip} f_i + \mathbf{Lip} g_i) + \sum_{l=1}^{k_j}((a_l^j)^2 + (b_l^j)^2) \le L$. There exists $\tau > 0$ such that $w \in \bigcup_{l=1}^{k_j}(D^\tau_{x_l^j} \setminus B_\tau(x_l^j))$. We also take sequences $x_l^j(i) \to x_l^j$ and $w_i \to w$. We fix $\epsilon > 0$ satisfying $\epsilon << \tau$. Then, there exists $0 < r << \epsilon$ such that

$$\frac{1}{\upsilon(B_t(w))} \int_{B_t(w)} \left| df_\infty - d\left(\sum_{l=1}^{k_j} a_l^j r_{x_l^j}\right)\right|^2 + \left|dg_\infty - d\left(\sum_{l=1}^{k_j} b_l^j r_{x_l^j}\right)\right|^2 d\upsilon \le \epsilon,$$

$$\limsup_{i\to\infty} \frac{1}{\underline{\mathrm{vol}}\, B_t(w_i)} \int_{B_t(w_i)} (\mathrm{Lip} f_i)^2 d\underline{\mathrm{vol}} \le \frac{1}{\upsilon(B_t(w))} \int_{B_t(w)} (\mathrm{Lip} f_\infty)^2 d\upsilon + \epsilon,$$

$$\limsup_{i\to\infty} \frac{1}{\underline{\mathrm{vol}}\, B_t(w_i)} \int_{B_t(w_i)} (\mathrm{Lip} g_i)^2 d\underline{\mathrm{vol}} \le \frac{1}{\upsilon(B_t(w))} \int_{B_t(w)} (\mathrm{Lip} g_\infty)^2 d\upsilon + \epsilon,$$

$$\limsup_{i\to\infty}\left|\frac{1}{\underline{\mathrm{vol}}\,B_t(w_i)}\int_{B_t(w_i)}\langle df_i, dr_{x_l^j(i)}\rangle d\underline{\mathrm{vol}} - \frac{1}{\upsilon(B_t(w))}\int_{B_t(w)}\langle df_\infty, dr_{x_l^j}\rangle d\upsilon\right| < \epsilon$$

and

$$\limsup_{i\to\infty}\left|\frac{1}{\underline{\mathrm{vol}}\,B_t(w_i)}\int_{B_t(w_i)}\langle dg_i, dr_{x_l^j(i)}\rangle d\underline{\mathrm{vol}} - \frac{1}{\upsilon(B_t(w))}\int_{B_t(w)}\langle dg_\infty, dr_{x_l^j}\rangle d\upsilon\right| < \epsilon$$

for every l and $0 < t < r$ We fix $0 < t < r$ below. Thus, by Lemma 3.40, we have

$$\frac{1}{\upsilon(B_t(w))}\int_{B_t(w)}\left|\langle df_\infty, dg_\infty\rangle - \frac{1}{\upsilon(B_t(w))}\int_{B_t(w)}\left\langle d\left(\sum_{l=1}^{k_j} a_l^j r_{x_l^j}\right), d\left(\sum_{l=1}^{k_j} b_l^j r_{x_l^j}\right)\right\rangle d\upsilon\right| d\upsilon \le \Psi(\epsilon; L)$$

and

$$\frac{1}{\upsilon(B_t(w))}\int_{B_t(w)}\left|\langle df_\infty, dg_\infty\rangle - \frac{1}{\upsilon(B_t(w))}\int_{B_t(w)}\langle df_\infty, dg_\infty\rangle d\upsilon\right| d\upsilon \tag{128}$$

$$= \frac{1}{\upsilon(B_t(w))}\int_{B_t(w)}\left|\left\langle d\left(\sum_{l=1}^{k_j} a_l^j r_{x_l^j}\right), d\left(\sum_{l=1}^{k_j} b_l^j r_{x_l^j}\right)\right\rangle\right. \tag{129}$$

$$\left. - \frac{1}{\upsilon(B_t(w))}\int_{B_t(w)}\left\langle d\left(\sum_{l=1}^{k_j} a_l^j r_{x_l^j}\right), d\left(\sum_{l=1}^{k_j} b_l^j r_{x_l^j}\right)\right\rangle d\upsilon\right| d\upsilon \pm \Psi(\epsilon; n, L) \tag{130}$$

$$= \Psi(\epsilon; n, L). \tag{131}$$

On the other hand, for every sufficiently large i, we have

$$\frac{1}{\underline{\mathrm{vol}}\,B_t(w_i)}\int_{B_t(w_i)}\left|df_i - d\left(\sum_{l=1}^{k_j} a_l^j r_{x_l^j(i)}\right)\right|^2 d\underline{\mathrm{vol}} \tag{132}$$

$$= \frac{1}{\underline{\mathrm{vol}}\,B_t(w_i)}\int_{B_t(w_i)}|df_i|^2 d\underline{\mathrm{vol}} - \sum_{l=1}^{k_j}\frac{a_l^j}{\underline{\mathrm{vol}}\,B_t(w_i)}\int_{B_t(w_i)}\langle df_i, dr_{x_l^j(i)}\rangle d\underline{\mathrm{vol}} \tag{133}$$

$$+ \sum_{l,\hat{l}}\frac{a_l^j a_{\hat{l}}^j}{\underline{\mathrm{vol}}\,B_t(w_i)}\int_{B_t(w_i)}\langle dr_{x_l^j(i)}, dr_{x_{\hat{l}}^j(i)}\rangle d\underline{\mathrm{vol}} \tag{134}$$

$$\le \frac{1}{\upsilon(B_t(w))}\int_{B_t(w)}|df_\infty|^2 d\upsilon - \sum_{l=1}^{k}\frac{a_l^j}{\upsilon(B_t(w))}\int_{B_t(w)}\langle df_\infty, dr_{x_l^j}\rangle d\upsilon \tag{135}$$

$$+ \sum_{l,\hat{l}}\frac{a_l^j a_{\hat{l}}^j}{\upsilon(B_t(w))}\int_{B_t(w)}\langle dr_{x_l^j}, dr_{x_{\hat{l}}^j}\rangle d\upsilon + \Psi(\epsilon; n, L) \tag{136}$$

$$= \frac{1}{\upsilon(B_t(w))}\int_{B_t(w)}\left|df_\infty - d\left(\sum_{l=1}^{k_j} a_l^j r_{x_l^j}\right)\right|^2 d\upsilon + \Psi(\epsilon; n, L) \le \Psi(\epsilon; n, L). \tag{137}$$

Similarly, we have

$$\frac{1}{\underline{\mathrm{vol}}\, B_t(w_i)} \int_{B_t(w_i)} \left| dg_i - d\left(\sum_{l=1}^{k_j} b_l^j r_{x_l^j(i)} \right) \right|^2 d\underline{\mathrm{vol}} \leq \Psi(\epsilon; n, L)$$

for every sufficiently large i. Especially, we have

(138)

$$\frac{1}{\underline{\mathrm{vol}}\, B_t(w_i)} \int_{B_t(w_i)} \left| \langle df_i, dg_i \rangle - \frac{1}{\underline{\mathrm{vol}}\, B_t(w_i)} \int_{B_t(w_i)} \left\langle d\left(\sum_{l=1}^{k_j} a_l^j r_{x_l^j(i)} \right), d\left(\sum_{l=1}^{k_j} b_l^j r_{x_l^j(i)} \right) \right\rangle d\underline{\mathrm{vol}} \right| d\underline{\mathrm{vol}}$$

(139)

$$\leq \Psi(\epsilon; n, L).$$

Therefore, we have the assertion. □

We remark that Theorem 1.2 follows from Theorem 4.20 directly.

Corollary 4.21. *Let Ω be a non-empty open subset of $B_R(x_\infty)$. We assume that for a.e. $w \in \Omega$, $df_i \to df_\infty$ at w. Then $df_i \to df_\infty$ on Ω.*

Proof. The assertion follows from Example 4.4, Proposition 4.13 and Theorem 4.20. □

Corollary 4.22. *Let g_i be a Lipschitz function on $B_R(x_i)$ satisfying $\sup_i(\mathbf{Lip} g_i + |g_i|_{L^\infty}) < \infty$ and A a Borel subset of $B_R(x_\infty)$. We assume that $(f_i, df_i) \to (f_\infty, df_\infty)$ and $(g_i, dg_i) \to (g_\infty, dg_\infty)$ on A. Then, there exists a Borel subset $\hat{A}$ of A such that $v(A \backslash \hat{A}) = 0$ and that $(f_i+g_i, d(f_i+g_i)) \to (f_\infty+g_\infty, d(f_\infty+g_\infty))$ and $(f_i g_i, d(f_i g_i)) \to (f_\infty g_\infty, d(f_\infty g_\infty))$ on $\hat{A}$.*

Proof. By Theorem 4.20, there exists a Borel subset $\hat{A}$ of A such that $v(A \setminus \hat{A}) = 0$ and that $\{|df_i|^2\}_i, \{\langle df_i, dg_i \rangle\}_i$ and $\{|dg_i|^2\}_i$ have infinitesimal constant convergence property to $|df_\infty|^2, \langle df_\infty, dg_\infty \rangle$ and $|dg_\infty|^2$ on $\hat{A}$, respectively. Since $|d(f_i g_i)|^2 = f_i^2 |dg_i|^2 + 2 f_i g_i \langle df_i, dg_i \rangle + g_i |df_i|^2$, by Proposition 4.5, we have, $\{|d(f_i g_i)|^2\}_i$ has infinitesimal constant convergence property to $f_\infty^2 |dg_\infty|^2 + 2 f_\infty g_\infty \langle df_\infty, dg_\infty \rangle + g_\infty^2 |df_\infty|^2 = |d(f_\infty g_\infty)|^2$ on $\hat{A}$. On the other hand, since $d(f_i g_i) = g_i df_i + f_i dg_i$, by Proposition 4.11, for every $z_i \to z$, we have, $\{\langle dr_{z_i}, d(f_i g_i) \rangle\}_i$ has infinitesimal convergence property to $g_\infty \langle dr_{z_\infty}, df_\infty \rangle + f_\infty \langle dr_{z_\infty}, dg_\infty \rangle = \langle dr_{z_\infty}, d(f_\infty g_\infty) \rangle$ on $\hat{A}$. Therefore we have $(f_i g_i, d(f_i g_i)) \to (f_\infty g_\infty, d(f_\infty g_\infty))$ on $\hat{A}$. Similarly, we have $(f_i + g_i, d(f_i + g_i)) \to (f_\infty + g_\infty, d(f_i + g_i))$ on $\hat{A}$. □

Corollary 4.23. *Let K_i be a Borel subset of $\overline{B}_R(x_i)$ and g_i a Lipschitz function on $B_R(x_i)$ satisfying $\sup_i(\mathbf{Lip} g_i + |g_i|_{L^\infty}) < \infty$. We assume that K_∞ is compact, $\limsup_{i \to \infty} \subset$*

K_∞ and that for a.e. $w \in K_\infty$, 1_{K_i} has infinitesimal convergence property to 1_{K_∞} at w, $dg_i \to df_\infty$ and $df_i \to df_\infty$ at w. Then for every sequence of continuous functions F_i on $\mathbf{R}$ satisfying that F_i converges to F_∞ in the sense of compact uniformly topology, we have

$$\lim_{i\to\infty} \int_{K_i} F_i(|df_i - dg_i|) d\underline{\mathrm{vol}} = F_\infty(0)\upsilon(K_\infty).$$

PROOF. The assertion follows from Proposition 4.5, Proposition 4.16 and Theorem 4.20. □

REMARK 4.24. By several arguments in section 3 and the proof of Theorem 4.20, we can also prove the following: If $\{f_i\}_i$ satisfies,

1. $\{f_i\}_i$ has infinitesimal upper semicontinuity of energy to f_∞ at every $\alpha \in B_R(x_\infty)$,
2. there exists a dense subset A of $B_R(x_\infty)$ and a Borel subset $\hat{A}$ of $B_R(x_\infty)$ such that $\upsilon(B_R(x_\infty) \setminus \hat{A}) = 0$ and that for every $w \in A$ and $w_i \to w$, $\{\langle dr_{w_i}, df_i\rangle\}_i$ has infinitesimal convergence property to $\langle dr_w, df_\infty\rangle$ at every $\alpha \in \hat{A}$,

then, $df_i \to df_\infty$ on $B_R(x_\infty)$.

REMARK 4.25. Similarly, for a sequence of Ricci limit spaces $\{(Y_i, y_i, \upsilon_i)\}_i$ and a sequence of Lipschitz function f_i on $B_R(y_i)$, we can also define a notion of convergence: $df_i \to df_\infty$ and prove several properties as above.

REMARK 4.26. For fixed Ricci limit space (Y, y, υ), a sequence of Lipschitz functions f_i on $B_R(y)$ satisfying $\sup_i \mathbf{Lip} f_i < \infty$, we have, $df_i \to df_\infty$ on $B_R(y)$ (in the sense of the convergence $(Y, y, \upsilon) \overset{(\mathrm{id}_Y, R_i, \epsilon_i)}{\to} (Y, y, \upsilon)$) if and only if $|\mathrm{Lip}(f_i - f_\infty)|_{L^2(B_R(y))} \to 0$. We shall check it. By Corollary 4.23, it suffices to check that 'if' part. We assume that $|\mathrm{Lip}(f_i - f_\infty)|_{L^2(B_R(y))} \to 0$. Then, especially, for every $w \in B_R(y)$, $\{f_i\}_i$ has infinitesimal upper semicontinuity of energy to f_∞ at w. On the other hand, by Proposition 4.19, we have

$$\lim_{i\to\infty} \int_{B_R(y)} |dr_{x_i} - dr_{x_\infty}|^2 d\upsilon = 0$$

for $x_i \to x_\infty \in Y$. Therefore, $\{\langle dr_{x_i}, df_i\rangle\}$ has infinitesimal convergence property to $\langle dr_{x_\infty}, df_\infty\rangle$ at every $w \in B_R(y)$. Thus, $df_i \to df_\infty$ on $B_R(y)$.

We will give a sufficient condition to satisfy infinitesimal upper semicontinuity of energy in the next subsection. See Proposition 4.33.

4.4 Approximation theorem

Throughout this subsection, we shall use the following notation (same to previous subsection): Let $\{(M_i, m_i, \underline{\mathrm{vol}})\}_i$ be a sequence of pointed, connected n-dimensional complete Riemannian manifolds with renormalized measure satisfying $\mathrm{Ric}_{M_i} \geq -(n-1)$, (Y, y, υ) a Ricci limit space of $\{(M_i, m_i, \underline{\mathrm{vol}})\}$, R a positive number, x_i a point in M_i, x_∞ a point in Y satisfying $(M_i, m_i, x_i, \underline{\mathrm{vol}}) \overset{(\phi_i, R_i, \epsilon_i)}{\to} (Y, y, x_\infty, \upsilon)$. The purpose in this subsection is to give an approximation theorem (Theorem 4.27). Roughly speaking, it means that for given Lipschitz function on $B_R(x_\infty)$, there exists a sequence of Lipschitz function on $B_R(x_i)$ approximating the function in the sense of the topology: $(f_i, df_i) \to (f_\infty, df_\infty)$.

THEOREM 4.27 (Approximation theorem). *Let L, R be positive numbers, f_∞ a L-Lipschitz function on $\overline{B}_R(x_\infty)$, A_i a Borel subset of $\overline{B}_R(x_i)$, A_∞ a compact subset of $\overline{B}_R(x_\infty)$ and f_i a L-Lipschitz function on A_i. We assume that $\limsup_{i\to\infty} A_i \subset A_\infty$ and that $f_\infty|_{A_\infty}$ is an extension of $\{f_i\}_i$ asymptotically. Then, for every $\epsilon > 0$, there exist an open set $\Omega_\epsilon \subset B_R(x_\infty) \setminus A_\infty$, $C(n, L)$-Lipschitz function f^ϵ_∞ on $B_R(x_\infty)$ and a sequence of $C(n, L)$-Lipschitz function f^ϵ_i on $B_R(x_i)$ such that $(f^\epsilon_i, df^\epsilon_i) \to (f^\epsilon_\infty, df^\epsilon_\infty)$ on Ω_ϵ, $f^\epsilon_\infty|_{A_\infty} = f|_{A_\infty}$, $f^\epsilon_i|_{A_i} = f_i|_{A_i}$ and that*

$$\frac{\upsilon(B_R(x_\infty) \setminus (\Omega_\epsilon \cup A_\infty))}{\upsilon(B_R(x_\infty))} + |f_\infty - f^\epsilon_\infty|_{L^\infty(B_R(x_\infty))} + \frac{1}{\upsilon_\infty(B_R(x_\infty))} \int_{B_R(x_\infty)} |df^\epsilon_\infty - df_\infty|^2 d\upsilon < \epsilon.$$

PROOF. We fix sufficiently small $\epsilon > 0$ and $\xi > 0$. (We will decide ξ later.) By Lemma 3.14 and (the proof of) Theorem 3.17, there exist a (pairwise disjoint) collection of Borel set $E_j \subset B_R(x_\infty)$, positive numbers $\tau_j > 0$, positive integers $1 \leq k_j \leq n$ and points $x^j_1, \ldots, x^j_{k_j} \in Y$ satisfying following properties:

1. $\upsilon_\infty(B_R(x_\infty) \setminus \bigcup_j E_j) = 0$.
2. $E_j \subset \bigcap_{l=1}^{k_j} (\mathcal{D}^{\tau_j}_{x^j_l} \setminus B_{\tau_j}(x^j_l))$.
3. For every $w \in E_j$,

$$\langle dr_{x^j_l}, dr_{x^j_{\hat{l}}} \rangle(w) = \lim_{r\to 0} \frac{1}{\upsilon(B_r(w))} \int_{B_r(w)} \langle dr_{x^j_l}, dr_{x^j_{\hat{l}}} \rangle d\upsilon = \delta_{l,\hat{l}} \pm \epsilon$$

4. For every $w \in E_j$, there exist $a^j_1(w), \ldots, a^j_{k_j}(w) \in \mathbf{R}$ such that

$$\lim_{r\to 0} \frac{1}{\upsilon(B_r(w))} \int_{B_r(w)} \left| df - d\left(\sum_{l=1}^{k_j} a^j_l(w) r_{x^j_l} \right) \right|^2 d\upsilon = 0.$$

For every $w \in E_j$, there exists $0 < r_w << \tau_j$ such that $\overline{B}_t(w) \subset B_R(x_\infty)$ and

$$\frac{1}{\upsilon(B_t(w))}\int_{B_t(w)}\left|df - d\left(\sum_{l=1}^{k_j} a_l^j r_{x_l^j}\right)\right|^2 d\upsilon < \epsilon$$

for every $0 < t < r_w$. We put $X = \bigcup_{j=1}^{\infty}(E_j \setminus \overline{B}_{5\xi}(A_\infty))$. By Proposition 2.12, there exists a pairwise disjoint collection $\{\overline{B}_{r_i}(z_i)\}_i \subset B_R(x_\infty)$ such that $z_i \in X$, $r_i << \min\{r_{z_i}, \epsilon, \xi\}$ and $X \setminus \bigcup_{i=1}^{N}\overline{B}_{r_i}(z_i) \subset \bigcup_{i=N+1}^{\infty}\overline{B}_{5r_i}(z_i)$ for every N. For every i, we take $l(i)$ satisfying $z_i \in E_{l(i)}$. We fix N satisfying $\sum_{i=N+1}^{\infty}\upsilon(B_{r_i}(z_i)) < \epsilon$. We take sequences $z_i(j) \to z_i$ and $x_m^l(j) \to x_m^l$. We define a function F_i^j on $B_{r_i}(z_i(j))$ and a function F_i on $B_{r_i}(z_i)$ by

$$F_i^j = \sum_{m=1}^{k_{l(i)}} a_m^{l(i)} r_{x_m^{l(i)}(j)} + C_i,\ F_i = \sum_{m=1}^{k_{l(i)}} a_m^{l(i)} r_{x_m^{l(i)}} + C_i.$$

Here C_i is the constant satisfying $F_i(z_i) = f_\infty(z_i)$.

Claim 4.28. *We have* $\mathbf{Lip}F_i^j + \mathbf{Lip}F_i \leq C(n, L)$ *for every* i, j.

The proof is as follows: Since

$$|df_\infty(z_i)|^2 = \sum_{s,t} a_s^{l(i)} a_t^{l(i)} \langle dr_{x_s^{l(i)}}, r_{x_t^{l(i)}}\rangle(z_i) \tag{140}$$

$$= \sum_{s,t} a_s^{l(i)} a_t^{l(i)} (\delta_{s,t} \pm \epsilon) \tag{141}$$

$$= (1 \pm \epsilon)\sum_{s=1}^{l(i)} (a_s^{l(i)})^2 \pm \epsilon \sum_{s \neq t} |a_s^{l(i)}||a_t^{l(i)}| \tag{142}$$

$$= (1 \pm \epsilon)\sum_{s=1}^{l(i)} (a_s^{l(i)})^2 \pm \Psi(\epsilon; n)\sum_{s=1}^{l(i)} (a_s^{l(i)})^2 \tag{143}$$

$$= (1 \pm \Psi(\epsilon; n))\sum_{s=1}^{l(i)} (a_s^{l(i)})^2 \tag{144}$$

and $|df_\infty|(z_i) \leq L$, we have

$$\sum_{m=1}^{k_i} (a_m^{l(i)})^2 \leq L^2 + \Psi(\epsilon; n, L).$$

Therefore we have Claim 4.28.

Since $\{\overline{B}_{r_i}(z_i(j))\}_{1 \leq i \leq N}$ are pairwise disjoint for every sufficiently large j, we define a function F_j on $\bigcup_{m=1}^{N}\overline{B}_{(1-\xi)r_i}(z_i(j))$ and a function F_∞ on $\bigcup_{m=1}^{N}\overline{B}_{(1-\xi)r_i}(z_i)$ by $F_j|_{B_{(1-\xi)r_i}(z_i(j))} = F_j^i|_{B_{(1-\xi)r_i}(z_i(j))}, F_\infty|_{B_{(1-\xi)r_i}(z_i)} = F_j|_{B_{(1-\xi)r_i}(z_i)}$.

Claim 4.29. *We have* $\mathbf{Lip}F_j, \mathbf{Lip}F_\infty \leq C(n,L) + \xi^{-1}\Psi(\epsilon;n,L)$ *for every sufficiently large* j.

The proof is as follows. By Claim 4.28, for every i,j, we have $\mathbf{Lip}(F_j|_{\overline{B}_{(1-\xi)r_i}(z_i(j))}) + \mathbf{Lip}(F_\infty|_{\overline{B}_{(1-\xi)r_i}(z_i)}) \leq C(n,L)$. There exists j_0 such that $\epsilon_j << \min\{\xi r_1, \dots, \xi r_N\}$ for every $j \geq j_0$. We fix $j \geq j_0$, $1 \leq l < m \leq N$, $w_l(j) \in \overline{B}_{(1-\xi)r_l}(z_l(j))$ and $w_m(j) \in \overline{B}_{(1-\xi)r_m}(z_m(j))$. Since $\overline{B}_{r_l}(z_l(j)) \cap \overline{B}_{r_m}(z_m(j)) = \emptyset$, by taking $\alpha(j) \in \partial B_{r_l}(z_l)$ satisfying $\overline{w_l(j), \alpha(j)} + \overline{\alpha(j), w_m(j)} = \overline{w_l(j), w_m(j)}$, we have $\overline{w_l(j), w_m(j)} \geq \overline{w_l(j), \alpha(j)} \geq \xi r_l$. Similarly, we have $\overline{w_l(j), w_m(j)} \geq \xi r_m$. Thus, we have $\overline{w_l(j), w_m(j)} \geq \xi(r_l + r_m)/2$. On the other hand, since

$$\frac{1}{\upsilon(B_{10r_l}(z_l))} \int_{B_{10r_l}(z_l)} \left| \mathrm{Lip}\left(f_\infty - \sum_{s=1}^{k_l} a_s^{k_l} r_{x_s^{k_l}} \right) \right|^2 d\upsilon < \epsilon,$$

by segment inequality on limit spaces ([9, Theorem 2.6]), there exist $\hat{z}_l, \phi_j(\hat{w_l}(j)) \in B_{r_l}(z_l)$ and a minimal geodesic γ from $\hat{z}_l$ to $\phi_j(\hat{w_l}(j))$ such that $\overline{z_l, \hat{z}_l} + \overline{\phi_j(w_l(j)), \phi_j(\hat{w_l}(j))} < \Psi(\epsilon;n) r_l$ and that

$$\int_0^{\overline{\hat{z}_l, \phi_j(\hat{w_l}(j))}} \mathrm{Lip}\left(f_\infty - \sum_{s=1}^{k_l} a_s^{k_l} r_{x_s^{k_l}} \right)(\gamma(t))dt < \Psi(\epsilon;n) r_l.$$

Therefore we have

$$\left| f_\infty(\hat{z}_l) - \sum_{s=1}^{k_l} a_s^{k_l} r_{x_s^{k_l}}(\hat{z}_l) - \left(f_\infty(\phi_j(\hat{z_l}(j))) - \sum_{s=1}^{k_l} a_s^{k_l} r_{x_s^{k_l}}(\phi_j(\hat{z_l}(j))) \right) \right| \tag{145}$$

$$\leq \int_0^{\overline{\hat{z}_l, \phi_j(\hat{w_l}(j))}} \mathrm{Lip}\left(f_\infty - \sum_{s=1}^{k_l} a_s^{k_l} r_{x_s^{k_l}} \right)(\gamma(t))dt < \Psi(\epsilon;n) r_l. \tag{146}$$

Thus

$$\left| f_\infty(z_l) - \sum_{s=1}^{k_l} a_s^{k_l} r_{x_s^{k_l}}(z_l) - \left(f_\infty(\phi_j(z_l(j))) - \sum_{s=1}^{k_l} a_s^{k_l} r_{x_s^{k_l}}(\phi_j(z_l(j))) \right) \right| \leq \Psi(\epsilon;n,L) r_l.$$

Especially, we have $|F_j(w_l(j)) - f_\infty \circ \phi_j(w_l(j))| \leq \Psi(\epsilon;n,L) r_l$. Similarly, we have $|F_j(w_m(j)) - f_\infty \circ \phi_j(w_m(j))| \leq \Psi(\epsilon;n,L) r_m$ and $|F_\infty - f_\infty| \leq \Psi(\epsilon;n,L) r_l$ on $\overline{B}_{(1-\xi)r_l}(z_l)$. Therefore we have

$$|F_j(w_l(j)) - F_j(w_m(j))| \leq |f_\infty \circ \phi_j(w_l(j)) - f_\infty \circ \phi_j(w_l(j))| + \Psi(\epsilon;n,L)(r_l + r_m) \tag{147}$$

$$\leq L\overline{\phi_j(w_l(j)), \phi_j(w_m(j))} + \Psi(\epsilon;n,L)(r_l + r_m) \tag{148}$$

$$\leq L(\overline{w_l(j), w_m(j)} + \epsilon_j) + \Psi(\epsilon;n,L)(r_l + r_m) \tag{149}$$

$$\leq L\overline{w_l(j), w_m(j)} + \Psi(\epsilon;n,L)(r_l + r_m) \tag{150}$$

$$\leq (L + \xi^{-1}\Psi(\epsilon;n,L))\overline{w_l(j), w_m(j)}. \tag{151}$$

Thus, by Claim 4.28, we have $\mathbf{Lip}F_j \leq C(n,L) + \xi^{-1}\Psi(\epsilon;n,L)$. Similarly, we have $\mathbf{Lip}F_\infty \leq C(n,L) + \xi^{-1}\Psi(\epsilon;n,L)$. Therefore we have Claim 4.29.

Claim 4.30. *For every sufficiently large j, we have $\bigcup_{i=1}^N \overline{B}_{(1-\xi)r_i}(z_i(j)) \subset M_i \setminus B_{2\xi}(A_i)$ and $\bigcup_{i=1}^N \overline{B}_{(1-\xi)r_i}(z_i) \subset Y \setminus B_{2\xi}(A_\infty)$.*

Because, by the definition, we have $\bigcup_{i=1}^N \overline{B}_{r_i}(z_i) \subset Y \setminus B_{2\xi}(A_\infty)$. On the other hand, by the assumption, there exists i_0 such that for every $i \geq i_0$, we have $\phi_i(A_i) \subset B_\xi(A_\infty)$ and $\epsilon_i << \min_{1\leq j\leq N}\{\xi r_j\}$. Thus, since $\phi_i(\bigcup_{i=1}^N \overline{B}_{(1-\xi)r_i}(z_i(j))) \subset \bigcup_{i=1}^N \overline{B}_{r_i}(z_i) \subset Y \setminus B_{4\xi}(A_\infty)$ for every $i \geq i_0$, we have Claim 4.30.

On the other hand, we remark the following claim:

Claim 4.31. *We have*

$$\lim_{i\to\infty} \sup_{A_i} |f_i - f_\infty \circ \phi_i| = 0.$$

The proof is done by a contradiction. We assume that the assertion is false. Then, there exist $\tau > 0$, a subsequence $\{n(i)\}$ of $\mathbf{N}$ and $\alpha_i \in A_{n(i)}$ such that $|f_{n(i)}(\alpha_i) - f_\infty \circ \phi_{n(i)}(\alpha_i)| > \tau$. Without loss of generality, we can assume that there exists $\alpha_\infty \in Y$ such that $\phi_{n(i)}(\alpha_i) \to \alpha_\infty$. Thus, $\liminf_{i\to\infty} |f_{n(i)}(\alpha_i) - f_\infty(\alpha_\infty)| \geq \tau$. On the other hand, by the assumption, we have $\alpha_\infty \in \overline{A_\infty} = A_\infty$. Since $f_\infty|_{A_\infty}$ is an extension of $\{f_i\}$ asymptotically, this is a cotradiction. Therefore we have Claim 4.31.

We put $W_j = \bigcup_{m=1}^N B_{(1-\xi)r_i}(z_i(j))$ and $W_\infty = \bigcup_{m=1}^N B_{(1-\xi)r_i}(z_i)$. By Claim 4.30, we can define a Lipschitz function G_j on $W_j \cup A_j$ and a Lipschitz function G_∞ on $W_\infty \cup A_\infty$ by $G_j|_{W_j} = F_j|_{W_j}$, $G_j|_{A_j} = f_j$, $G_\infty|_{W_\infty} = F_\infty|_{W_\infty}$ and $G_\infty|_{A_\infty} = f_\infty|_{A_\infty}$.

Claim 4.32. *We have $\mathbf{Lip}G_j, \mathbf{Lip}G_\infty \leq C(n,L) + \xi^{-1}\Psi(\epsilon;n,L)$ for every sufficiently large j.*

The proof is as follows. We put $\xi_j = \sup_{A_j} |f_j - f_\infty \circ \phi_j|$. Then by the proof of Claim 4.29, there exists j_0 such that for every $j \geq j_0$, $\alpha_j \in \overline{B}_{(1-\xi)r_i}(z_i(j))$ and $\beta_j \in A_j$, we have

$$|G_j(\alpha_j) - G_j(\beta_j)| = |F_j(\alpha_j) - f_j(\beta_j)| \tag{152}$$

$$\leq |F_\infty \circ \phi_j(\alpha_j) - f_\infty \circ \phi_j(\beta_j)| + \Psi(\epsilon;n,L)r_i + \xi_j \tag{153}$$

$$\leq |f_\infty \circ \phi_j(\alpha_j) - f_\infty \circ \phi_j(\beta_j)| + \Psi(\epsilon;n,L)r_i + \xi_j \tag{154}$$

$$\leq L\overline{\phi_j(\alpha_j), \phi_j(\beta_j)} + \Psi(\epsilon;n,L)r_i \tag{155}$$

$$\leq L(\overline{\alpha_j, \beta_j} + \epsilon_j) + \Psi(\epsilon;n,L)\xi \tag{156}$$

$$\leq (L + \Psi(\epsilon;n,L))\overline{\alpha_j, \beta_j}. \tag{157}$$

Therefore, by Claim 4.29, we have $\mathbf{Lip}G_j \leq C(n,L) + \xi^{-1}\Psi(\epsilon;n,L)$ for every sufficiently large j. Similarly, we have $\mathbf{Lip}G_\infty \leq C(n,L) + \xi^{-1}\Psi(\epsilon;n,L)$. Thus, we have Claim 4.32.

For $\Psi = \Psi(\epsilon; n, L)$ in Claim 4.32, we put $\xi = \sqrt{\Psi}$. We take a Lipschitz function f_j^ϵ on M_i and a Lipschitz function f_∞^ϵ on Y satisfying that $\mathbf{Lip} f_j^\epsilon = \mathbf{Lip} G_j$, $\mathbf{Lip} f_\infty^\epsilon = \mathbf{Lip} G_\infty$, $f_j^\epsilon|_{W_j \cup A_j} = F_j|_{W_j \cup A_j}$ and $f_\infty^\epsilon|_{W_\infty \cup A_\infty} = F_\infty|_{W_\infty \cup A_\infty}$. We put $\Omega_\epsilon = W_\infty$. Then, by the definition, Proposition 4.19 and Corollary 4.22, we have $(f_i^\epsilon, df_i^\epsilon) \to (f_\infty^\epsilon, df_\infty^\epsilon)$ on Ω_ϵ. We have

(158)
$$\int_{B_R(x_\infty)} |df_\infty - df_\infty^\epsilon|^2 dv \leq \int_{X \setminus \overline{B}_{5\xi}(A_\infty)} |df_\infty - df_\infty^\epsilon|^2 dv + \int_{\overline{B}_{5\xi}(A_\infty)} |df_\infty - df_\infty^\epsilon|^2 dv$$

(159)
$$\leq \sum_{i=1}^{N} \int_{B_{(1-\xi)r_i}(z_i)} |df_\infty - df_\infty^\epsilon|^2 dv$$

(160)
$$\pm \left(5L^2 v(B_{5\xi}(A_\infty) \setminus A_\infty) + \int_{A_\infty} |df_\infty^\epsilon - df_\infty|^2 dv + \Psi(\epsilon; n, L) \right)$$

(161)
$$\leq \sum_{i=1}^{N} \epsilon v(B_{(1-\xi)r_i}(z_i)) \pm \left(5L^2 v(B_{5\xi}(A_\infty) \setminus A_\infty) + \Psi(\epsilon; n, L) \right)$$

(162)
$$\leq \epsilon v(B_R(x_\infty)) \pm \left(5L^2 v(B_{5\xi}(A_\infty) \setminus A_\infty) + \Psi(\epsilon; n, L) \right)$$

and

(163)
$$v(B_R(x_\infty) \setminus (\Omega_\epsilon \cup A_\infty)) \leq v(X \setminus (\Omega_\epsilon \cup A_\infty)) + v(\overline{B}_\xi(A_\infty) \setminus A_\infty)$$

(164)
$$\leq \sum_{i=N+1}^{\infty} v(B_{5r_i}(z_i)) + v(\overline{B}_\xi(A_\infty) \setminus A_\infty)$$

(165)
$$\leq C(n)\epsilon + v(\overline{B}_\xi(A_\infty) \setminus A_\infty).$$

Since A_∞ is compact, we remark that $\lim_{r \to 0} v(B_r(A_\infty) \setminus A_\infty) = 0$. We put $\tau(r) = v(B_r(A_\infty) \setminus A_\infty)$. On the other hand, by the proof of Claim 4.29, we have $|f_\infty^\epsilon - f_\infty| < \Psi(\epsilon; n, L)$ on $\Omega_\epsilon \cup A_\infty$. For every $w \in B_R(x_\infty)$, there exists $\hat{w} \in \Omega_\epsilon \cup A_\infty$ such that $\overline{w, \hat{w}} < \Psi(\epsilon, \tau(5\xi); n, L, v(B_R(x_\infty)))$. Therefore, we have $|f_\infty^\epsilon(w) - f_\infty(w)| \leq |f_\infty^\epsilon(\hat{w}) - f_\infty(\hat{w})| + \Psi(\epsilon, \tau(5\xi); n, L, v(B_R(x_\infty))) \leq \Psi(\epsilon, \tau(5\xi); n, L, v(B_R(x_\infty)))$. Thus, we have $|f_\infty^\epsilon - f_\infty| < \Psi(\epsilon, \tau(5\xi); n, L, v(B_R(x_\infty)))$ on $B_R(x_\infty)$. Therefore, we have the assertion. □

As a corollary of Theorem 4.27, we shall give a sufficient condition to satisfy infinitesimal upper semicontinuity of energy:

PROPOSITION 4.33. *Let R be a positive number, f_i a C^2-function on $B_R(x_i)(i \in \mathbf{N})$, f_∞ a Lipschitz function on $\overline{B}_R(x_\infty)$. Assume that*

$$\sup_i \left(\mathbf{Lip} f_i + \int_{B_R(x_i))} |\Delta f_i| d\underline{\mathrm{vol}} \right) < \infty$$

and $f_i \to f_\infty$ on $B_R(x_\infty)$. Then, we have

$$\limsup_{i\to\infty}\int_{B_R(x_i)}(\mathrm{Lip}f_i)^2 d\underline{\mathrm{vol}} \le \int_{B_R(x_\infty)}(\mathrm{Lip}f_\infty)^2 d\upsilon.$$

Especially, the sequence $\{f_i\}_i$ has infinitesimal upper semicontinuity of energy to f_∞ at every $w \in B_R(x_\infty)$.

Proof. We put $g_i = \Delta f_i$. First, we shall remark the following:

Claim 4.34. *For every Lipschitz function k on $B_R(x_i)$ satisfying* $\mathrm{supp}k \subset B_R(x_i)$*, we have*

$$\int_{B_R(x_i)}|d(f_i+k)|^2 d\underline{\mathrm{vol}} - 2\int_{B_R(x_i)} g_i(f_i+k)d\underline{\mathrm{vol}} \ge \int_{B_R(x_i)}|df_i|^2 d\underline{\mathrm{vol}} - 2\int_{B_R(x_i)} g_i f_i d\underline{\mathrm{vol}}.$$

Because, since

$$\begin{aligned}\int_{B_R(x_i)}|d(f_i+k)|^2 d\underline{\mathrm{vol}} - 2\int_{B_R(x_i)} g_i(f_i+k)d\underline{\mathrm{vol}} = &\int_{B_R(x_i)}|df_i|^2 d\underline{\mathrm{vol}} - 2\int_{B_R(x_i)} g_i f_i d\underline{\mathrm{vol}} \\ &+ \int_{B_R(x_i)}|dk|^2 d\underline{\mathrm{vol}},\end{aligned}$$

we have Claim 4.34.

We fix $\epsilon > 0$ and take $L \ge 1$ satisfying

$$\sup_i\left(\mathbf{Lip}f_i + |f_i|_{L^\infty} + \int_{B_R(x_i)}|g_i|d\underline{\mathrm{vol}}\right) < L.$$

Since $\limsup_{i\to} A_{R-\epsilon,R}(x_i) \subset A_{R-\epsilon,R}(x_\infty)$, by Theorem 4.27, there exist a $C(n,L)$-Lipschitz function f_∞^ϵ on $B_R(x_\infty)$, a $C(n,L)$-Lipschitz function f_i^ϵ on $B_R(x_i)$ and an open set $\Omega_\epsilon \subset B_R(x_\infty)\setminus A_{R-\epsilon,R}(x_\infty)$ such that $f_\infty^\epsilon|_{A_{R-\epsilon,R}(x_\infty)} = f_\infty|_{A_{R-\epsilon,R}(x_\infty)}$, $f_i^\epsilon|_{A_{R-\epsilon,R}(x_i)} = f_i|_{A_{R-\epsilon,R}(x_i)}$, $(f_i^\epsilon, df_i^\epsilon) \to (f_\infty^\epsilon, df_\infty^\epsilon)$ on Ω_ϵ and that

$$\frac{\upsilon\left(B_R(x_\infty)\setminus(\Omega_\epsilon \cup A_{R-\epsilon,R}(x_\infty))\right)}{\upsilon(B_R(x_\infty))} + |f_\infty - f_\infty^\epsilon|_{L^\infty(B_R(x_\infty))} + \frac{1}{\upsilon_\infty(B_R(x_\infty))}\int_{B_R(x_\infty)}|df_\infty^\epsilon - df_\infty|^2 d\upsilon$$

$< \epsilon$.

By Claim 4.34, we have

$$\int_{B_R(x_i)}|df_i^\epsilon|^2 d\underline{\mathrm{vol}} - 2\int_{B_R(x_i)} g_i f_i^\epsilon d\underline{\mathrm{vol}} \ge \int_{B_R(x_i)}|df_i|^2 d\underline{\mathrm{vol}} - 2\int_{B_R(x_i)} g_i f_i d\underline{\mathrm{vol}}.$$

By Proposition 2.12, without loss of generality, we can assume that there exists a pairwise disjoint finite collection $\{\overline{B}_{r_i}(z_i)\}_{1\le i\le N}$ such that $\Omega_\epsilon = \bigcup_{i=1}^N B_{r_i}(z_i)$. We take a sequence

$z_i(j) \to z_i$. We put $\Omega_\epsilon(j) = \bigcup_{i=1}^N B_{r_i}(z_i(j))$. Since $\underline{\mathrm{vol}}(\Omega_\epsilon(j) \cup A_{R-\epsilon,R}(x_i))/\underline{\mathrm{vol}}\, B_R(x_i) \geq 1-\epsilon$ for every sufficiently large j, by Proposition 4.13, we have

$$\left| \int_{B_R(x_j)} |df_j^\epsilon|^2 d\underline{\mathrm{vol}} - \int_{B_R(x_\infty)} |df_\infty|^2 d\upsilon \right| < \Psi(\epsilon; n, L, R)\upsilon(B_R(x_\infty)).$$

On the other hand, since $\sup_{B_R(x_j)} |f_j^\epsilon - f_j| \leq C(n,R,L) \sup_{\Omega_\epsilon(j)} |f_j^\epsilon - f_j|$ and $\limsup_{j\to\infty} \sup_{\Omega_\epsilon(j)} |f_j^\epsilon - f_j| \leq \sup_{\Omega_\epsilon} |f_\infty^\epsilon - f_\infty|$, we have

$$\left| \int_{B_R(x_j)} g_j f_j^\epsilon d\underline{\mathrm{vol}} - \int_{B_R(x_j)} g_j f_j d\underline{\mathrm{vol}} \right| \leq \sup_{B_R(x_j)} |f_j^\epsilon - f_j| \int_{B_R(x_j)} |g_j| d\underline{\mathrm{vol}} \leq \Psi(\epsilon; n, R, L)\upsilon(B_R(x_\infty))$$

for every sufficiently large j. Therefore, by Proposition 4.13, we have

$$\limsup_{i\to\infty} \int_{B_R(x_i)} |df_i|^2 d\underline{\mathrm{vol}} \leq \int_{B_R(x_\infty)} |df_\infty^\epsilon|^2 d\upsilon + \Psi(\epsilon; n, L, R)\upsilon(B_R(x_\infty)).$$

Thus, we have

$$\limsup_{i\to\infty} \int_{B_R(x_i)} |df_i|^2 d\underline{\mathrm{vol}} \leq \int_{B_R(x_\infty)} |df_\infty|^2 d\upsilon + \Psi(\epsilon; n, L, R)\upsilon(B_R(x_\infty)).$$

By letting $\epsilon \to 0$, we have the assertion. □

Next corollary follows from Remark 4.8 and Proposition 4.33 directly:

Corollary 4.35. *Let R be a positive number, f_i a C^2-function on $B_R(x_i)$ and f_∞ Lipschitz functions on $B_R(x_\infty)$. Assume that*

$$\sup_i \left(\mathbf{Lip} f_i + \int_{B_R(x_i))} |\Delta f_i|^2 d\underline{\mathrm{vol}} \right) < \infty$$

and $f_i \to f_\infty$ on $B_R(x_\infty)$. Then, we have $(f_i, df_i) \to (f_\infty, df_\infty)$ on $B_R(x_\infty)$.

Corollary 4.36. *Let R be a positive number, f_i a C^2-function on $B_R(x_i)$ and f_∞ a Lipschitz function on $B_R(x_\infty)$ satisfying $\sup_i(\mathbf{Lip} f_i + |\Delta f_i|_{L^\infty(B_R(x_i))}) < \infty$. We assume that $f_i \to f_\infty$ on $B_R(x_\infty)$ and that there exists a L^∞-function g_∞ on $B_R(x_\infty)$ such that $\{\Delta f_i\}_i$ has infinitesimal convergence property to g_∞ at a.e. $w \in B_R(x_\infty)$. Then, for every Lipschitz function k_∞ satisfying $\mathrm{supp} k_\infty \subset B_R(x_\infty)$, we have*

$$\int_{B_R(x_\infty)} \langle df_\infty, dk_\infty \rangle d\upsilon = \int_{B_R(x_\infty)} k_\infty g_\infty d\upsilon.$$

Proof. By Corollary 4.35, we have $(f_i, df_i) \to (f_\infty, df_\infty)$ on $B_R(x_\infty)$. We take $L \geq 1$ satisfying $\sup_i(\mathbf{Lip} f_i + |f_i|_{L^\infty} + |\Delta f_i|_{L^\infty}) < L$. We put $r = \sup_{w \in \mathrm{supp} k_\infty} \overline{x_\infty, w}$ and $g_i = \Delta f_i$. By compactness of $\mathrm{supp} k_\infty$, we have $r < R$. We fix $\epsilon > 0$ satisfying $\epsilon < R - r$. By Theorem

4.27, there exist a $C(n,L)$-Lipschitz function k^ϵ_∞ on $B_R(x_\infty)$, a $C(n,L)$-Lipschitz function k^ϵ_i on $B_R(x_i)$ and an open set $\Omega_\epsilon \subset B_R(x_\infty) \setminus A_{R-\epsilon,R}(x_\infty)$ such that $k^\epsilon_\infty|_{A_{R-\epsilon,R}(x_\infty)} = 0$, $k^\epsilon_i|_{A_{R-\epsilon,R}(x_i)} = 0$, $(k^\epsilon_i, dk^\epsilon_i) \to (k^\epsilon_\infty, dk^\epsilon_\infty)$ on Ω_ϵ and that

$$\frac{\upsilon\left(B_R(x_\infty) \setminus (\Omega_\epsilon \cup A_{R-\epsilon,R}(x_\infty))\right)}{\upsilon(B_R(x_\infty))} + |k_\infty - k^\epsilon_\infty|_{L^\infty(B_R(x_\infty))} + \frac{1}{\upsilon_\infty(B_R(x_\infty))}\int_{B_R(x_\infty)} |dk^\epsilon_\infty - dk_\infty|^2 d\upsilon$$

$< \epsilon$.

By Proposition 4.11, $\{k^\epsilon_i g_i\}_i$ has infinitesimal convergence property to $k^\epsilon_\infty g_\infty$ at a.e. $w \in \Omega_\epsilon$. By an argument similar to the proof of Proposition 4.33, and Proposition 4.13, we have

$$\left|\int_{B_R(x_i)} \langle df_i, dk^\epsilon_i\rangle d\underline{\mathrm{vol}} - \int_{B_R(x_\infty)} \langle df_\infty, dk^\epsilon_\infty\rangle d\upsilon\right| + \left|\int_{B_R(x_i)} g_i k^\epsilon_i d\underline{\mathrm{vol}} - \int_{B_R(x_\infty)} g_\infty k^\epsilon_\infty d\upsilon\right|$$
$$< \Psi(\epsilon; n, L, R)\upsilon(B_R(x_\infty))$$

for every sufficiently large i. Since

$$\int_{B_R(x_i)} \langle df_i, dk^\epsilon_i\rangle d\underline{\mathrm{vol}} = \int_{B_R(x_i)} g_i k^\epsilon_i d\underline{\mathrm{vol}},$$

we have

$$\int_{B_R(x_\infty)} \langle df_\infty, dk_\infty\rangle d\upsilon = \int_{B_R(x_\infty)} g_\infty k_\infty d\upsilon \pm \Psi(\epsilon; n, L, R)\upsilon(B_R(x_\infty)).$$

By letting $\epsilon \to 0$, we have the assertion. □

The following corollary follows from Corollary 4.35 and 4.36 directly. See also [24].

COROLLARY 4.37. *Let R be a positive number, f_i a harmonic function on $B_R(x_i)$ and f_∞ a Lipschitz function on $B_R(x_\infty)$ satisfying $\sup_i \mathbf{Lip} f_i < \infty$. We assume that $f_i \to f_\infty$ on $B_R(x_\infty)$. Then, we have $(f_i, df_i) \to (f_\infty, df_\infty)$ on $B_R(x_\infty)$. Moreover, for every Lipschitz function k_∞ satisfying $\mathrm{supp} k_\infty \subset B_R(x_\infty)$, we have*

$$\int_{B_R(x_\infty)} \langle df_\infty, dk_\infty\rangle d\upsilon = 0.$$

Especially f_∞ is a harmonic function on $B_R(x_\infty)$.

5 Harmonic functions on asymptotic cones

In this section, we will give several applications of results in section 4 to harmonic functions on asymptotic cones of manifolds with nonnegative Ricci curvature and Euclidean volume growth via Colding-Minicozzi theory [17, 18, 19, 20, 21, 22] for harmonic functions on manifolds. Throughout this section, we will always assume that dimensions of all manifolds are greater than 2.

5.1 Convergence of frequency functions

Throughout this section 5, we fix an n-dimensional complete Riemannian manifolds M satisfying $\mathrm{Ric}_M \geq 0$ and *Euclidean volume growth condition*:

$$\lim_{R\to\infty} \frac{\mathrm{vol}^{g_M} B_R(m)}{R^n} > 0.$$

Here m is a point in M and g_M is the Riemannian metric of M. We remark that by Bishop-Gromov volume comparison theorem, the limit above always exists and does not depend on choice of m. We denote the limit by $V_M^{g_M} = \lim_{R\to\infty} \mathrm{vol}^{g_M} B_R(m)/R^n$. It is easy to check that $V_M^{r^{-2}g_M} = V_M^{g_M}$ for $r > 0$. Therefore we shall use the notaiton: $V_M = V_M^{g_M}$. We fix a point $m \in M$ below. Then the global Green's function $G^{g_M}(m, x)$ on M with singularity at m exists. See [79]. First, we shall introduce an important result about asymptotic behavior of G^{g_M} by Colding-Minicozzi:

THEOREM 5.1 (Colding-Minicozzi, [20]). *We have*

$$\lim_{\overline{m,x}\to\infty} \frac{G^{g_M}(m, x)}{\overline{m, x}^{2-n}} = \frac{\mathrm{vol}\, B_1(0_n)}{V_M}.$$

By the definition of Green's function, we have

$$G^{r^{-2}g_M}(m, x) = \frac{G^{g_M}(m, x)}{r^{2-n}}.$$

It is known that there exists $C_1 > 1$ such that $\overline{m, x}^{2-n} \leq G^{g_M}(m, x) \leq C_1\overline{m, x}^{2-n}$ for every $m \neq x$. We define a smooth function $b_m^{g_M}$ on $M \setminus \{m\}$ by

$$b_m^{g_M}(x) = \left(\frac{V_M}{\mathrm{vol}\, B_1(0_n)} G^{g_M}(m, x)\right)^{\frac{1}{2-n}}.$$

Thus we have $b_m^{r^{-2}g_M} = b_m^{g_m}/r$. We shall use the notation $b^{g_M} = b_m^{g_M}$ simply. Then we have

$$\left(\frac{V_M}{\mathrm{vol}\, B_1(0_n)}\right)^{2-n} \overline{m, y}^{r^{-2}g_M} \leq b^{r^{-2}g_M}(y) \leq \left(\frac{C_1 V_M}{\mathrm{vol}\, B_1(0_n)}\right)^{2-n} \overline{m, y}^{r^{-2}g_M}$$

for every $r > 0$. We put $b^{g_M}(m) = 0$. It is easy to check

$$\nabla^{g_M} b^{g_M} = \frac{V_M}{(2-n)\mathrm{vol}\, B_1(0_n)} (b^{g_M})^{n-1} \nabla^{g_M} G^{g_M}(m, \cdot).$$

On the other hand, for every $\epsilon > 0$, there exists $R(\epsilon) > 0$ such that

$$\int_{b^{g_M} \leq R} ||\nabla b^{g_M}|^2 - 1|^2 d\mathrm{vol} \leq \epsilon \mathrm{vol}(\{b^{g_M} \leq R\}),$$

$$\int_{b^{g_M} \leq R} |\mathrm{Hess}_{(b^{g_M})^2} - 2g_M|^2 d\mathrm{vol} \leq \epsilon \mathrm{vol}(\{b^{g_M} \leq R\})$$

for every $R > R(\epsilon)$ and that

$$\left| \frac{b^{g_M}(x)}{\overline{m,x}^{g_M}} - 1 \right| < \epsilon$$

for every $x \in M \setminus B_{R(\epsilon)}(m)$. See (2.23), (2.24) and (2.25) in [19] or section 4 in [20] for proofs of these results.

LEMMA 5.2. *We have*

$$\lim_{R\to\infty} \frac{\mathrm{vol}(\{b^{g_M} \leq R\})}{\mathrm{vol}\, B_R^{g_M}(m)} = 1$$

PROOF. For every $0 < \epsilon < 1$, we take $R(\epsilon) > 0$ as above. We put

$$\hat{R}(\epsilon) = \left(\frac{C_1 V_M}{\mathrm{vol}\, B_1(0_n)} \right)^{2-n} R(\epsilon) + R(\epsilon).$$

We take $R > \hat{R}(\epsilon)$. First, we shall show $B_R(m) \subset \{b^{g_M} \leq (1+\epsilon)R\}$. We take $y \in B_R(m)$. By the definition of b^{g_M}, if $y = m$, then $y \in \{b^{g_M} \leq (1+\epsilon)R\}$. If $y \neq m$ and $\overline{m,y} \leq R(\epsilon)$, then we have

$$b^{g_M}(y) \leq \left(\frac{C_1 V_M}{\mathrm{vol}\, B_1(0_n)} \right)^{2-n} \overline{m,y} \leq \left(\frac{C_1 V_M}{\mathrm{vol}\, B_1(0_n)} \right)^{2-n} R(\epsilon) \leq \hat{R}(\epsilon) \leq R.$$

Especially, we have $y \in \{b^{g_M} \leq (1+\epsilon)R\}$. On the other hand, by the definition of $R(\epsilon)$, if $\overline{m,y} > R(\epsilon)$, then $|b^{g_M}(y) - \overline{m,y}| < \epsilon \overline{m,y}$. Especially, we have $b^{g_M}(y) \leq (1+\epsilon)\overline{m,y} < (1+\epsilon)R$. Thus, we have $B_R(m) \subset \{b^{g_M} \leq (1+\epsilon)R\}$. Next, we shall show $\{b^{g_M} \leq (1+\epsilon)R\} \subset B_{\frac{1+\epsilon}{1-\epsilon}R}(m)$. We take $x \in \{b^{g_M} \leq (1+\epsilon)R\}$ satisfying $\overline{m,x} \geq R(\epsilon)$. Then, we have $(1-\epsilon)\overline{m,x} \leq b^{g_M}(x) \leq (1+\epsilon)R$. Thus, we have $\{b^{g_M} \leq (1+\epsilon)R\} \subset B_{\frac{1+\epsilon}{1-\epsilon}R}(m)$. Therefore, we have $B_{\frac{R}{1+\epsilon}}(m) \subset \{b^{g_M} \leq R\} \subset B_{\frac{R}{1-\epsilon}}(m)$ for every $R > 2\hat{R}(\epsilon)$. Since

$$\lim_{R\to\infty} \frac{\mathrm{vol}\, B_{\frac{R}{1-\epsilon}}(m)}{\mathrm{vol}\, B_{\frac{R}{1+\epsilon}}(m)} = \left(\frac{1+\epsilon}{1-\epsilon} \right)^n,$$

we have the assertion. □

We shall define frequency functions for harmonic functions on M. For $R > 0$, $0 < r < R$ and a harmonic function u on $\{b^{g_M} < R\}$, we put

$$I_u^{g_M}(r) = r^{1-n} \int_{b^{g_M}=r} u^2 |\nabla^{g_M} b^{g_M}| d\mathrm{vol}_{n-1}^{g_M},$$

$$D_u^{g_M}(r) = r^{2-n} \int_{b^{g_M}\leq r} |\nabla^{g_M} u|^2 d\mathrm{vol}^{g_M}$$

and

$$F_u^{g_M}(r) = r^{3-n} \int_{b^{g_M}=r} \left| \frac{\partial u}{\partial n} \right|^2 |\nabla b^{g_M}| d\mathrm{vol}_{n-1}^{g_M}.$$

Here n is the unit outer vector of $\{b^{g_M} = r\}$, $\mathrm{vol}_{n-1}^{g_M}$ is the $(n-1)$-dimensional Hausdorff measure with respect to the Riemannian metric g_M. Moreover, we put

$$U_u^{g_M}(r) = \frac{D_u^{g_M}(r)}{I_u^{g_M}(r)} \text{ if } I_u^{g_M}(r) \neq 0,$$

$$U_u^{g_M}(r) = 0 \text{ if } I_u^{g_M}(r) = 0$$

and call the function $U_u^{g_M}$ on $(0, R)$ *frequency function for* u. We remark that the critical set of b^{g_M} has codimension two. See [12], [38] or [19, Remark 2.11]. By maximum principle on manifolds, $U_u^{g_M}(r) = 0$ for some $0 < r < R$ if and only if u is a constant function. The following fundamental properties of functions above are given in [19]:

$$D_u^{g_M}(r) \leq \left(\frac{r}{s}\right)^{2-n} D_u^{g_M}(s),$$

$$\frac{dI_u^{g_M}}{dr} = 2\frac{D_u^{g_M}(r)}{r},$$

$$I_u^{g_M}(s) = \exp\left(2\int_r^s \frac{U_u^{g_M}(t)}{t}dt\right) I_u^{g_M}(r)$$

for $r < s$ (see (2.10), (2.12), (2.13) and (2.14) in [19]). For every $\tau, r > 0$, $R > r\tau$ and harmonic function u on $\{b^{g_M} < R\}$, we put $u_\tau = u/\tau$. Then we have $D_{u_\tau}^{\tau^{-2}g_M}(r) = \tau^{-2}D_u^{g_M}(r\tau), I_{u_\tau}^{\tau^{-2}g_M}(r) = \tau^{-2}I_u^{g_M}(r\tau), F_{u_\tau}^{\tau^{-2}g_M}(r) = \tau^{-2}F_u^{g_M}(r\tau)$ and $U_{u_\tau}^{\tau^{-2}g_M}(r) = U_u^{g_M}(r\tau)$.

We shall recall the definition of *asymptotic cone (or tangent cone at infinity) of* M by Cheeger-Colding:

Definition 5.3 (Asymptotic cone). For pointed proper geodesic space (M_∞, m_∞), we say that (M_∞, m_∞) *is an asymptotic cone (or tangent cone at infinity) of* M if there exists a sequence $R_i \to \infty$ such that $(M, m, R_i^{-1}d_M) \to (M_\infty, m_\infty)$.

We fix an asymptotic cone (M_∞, m_∞) of M and a sequence $R_i \to \infty$ satisfying $(M, m, R_i^{-1}d_M) \to (M_\infty, m_\infty)$ in this subsection below. We remark that by [7, Theorem 5.9], we have $(M, m, R_i^{-1}d_M, \mathrm{vol}^{R_i^{-2}g_M}) \to (M_\infty, m_\infty, H^n)$. We shall introduce an important result for asymptotic cones by Cheeger-Colding:

Theorem 5.4 (Cheeger-Colding, [6]). *With same notation as above, there exists a compact geodesic space* X *such that* $\mathrm{diam}X \leq \pi$ *and* $(M_\infty, m_\infty) = (C(X), p)$.

See [4, Theorem 9.79] or [6] for the proof. We fix X as in Theorem 5.4. For $R > 0$, $0 < r < R$ and Lipschitz function u on $\overline{B}_R(p)$ satisfying that u is harmonic on $B_R(p)$, we put

$$I_u(r) = r^{1-n}\int_{\partial B_r(p)} u^2 dH^{n-1}$$

and

$$D_u(r) = r^{2-n} \int_{B_r(p)} |du|^2 dH^n.$$

Moreover, we put

$$U_u(r) = \frac{D_u(r)}{I_u(r)} \text{ if } I_u(r) \neq 0$$

and

$$U_u(r) = 0 \text{ if } I_u(r) = 0.$$

We also remark that by Proposition 7.22, the function

$$F_u(r) = r^{3-n} \int_{\partial B_r(p)} \langle dr_p, du \rangle^2 dH^{n-1}.$$

is well defined for a.e. $r \in (0, R)$.

REMARK 5.5. We remark the following: Let R be a positive number, u_i a harmonic function on $B^{g_M}_{RR_i}(m)$. Assume that $\sup_i |(u_i)_{R_i}|_{L^\infty(B_r^{R_i^{-2}g_M}(m))} < \infty$ for every $0 < r < R$. Then we have $\sup_i \mathbf{Lip}\left((u_i)_{R_i}|_{B_r^{R_i^{-2}g_M}(m)}\right) < \infty$ for every $0 < r < R$. The proof is as follows. We fix $\hat{r}$ satisfying $r < \hat{r} < R$. Since $\overline{B}_r(p)$ is convex, it is not difficult to see that there exists i_0 such that for every $i \geq i_0$, $x_1(i), x_2(i) \in \overline{B}_r^{R_i^{-2}g_M}(m)$ and geodesic γ_i from $x_1(i)$ to $x_2(i)$, we have Image$\gamma_i \subset \overline{B}_{\hat{r}}^{R_i^{-2}g_M}(m)$. Therefore, by Cheng-Yau's gradient estimate, we have $\limsup_{i\to\infty} \mathbf{Lip}\left((u_i)_{R_i}|_{B_r^{R_i^{-2}g_M}(m)}\right) < \infty$ for every $0 < r < R$. Thus we have the assertion.

PROPOSITION 5.6. *Let R be a positive number, u_i a harmonic function on $B^{g_M}_{RR_i}(m)$ and u_∞ a Lipschitz function on $B_R(p)$. We assume that $\sup_i |(u_i)_{R_i}|_{L^\infty(B_r^{R_i^{-2}g_M}(m))} < \infty$ and $(u_i)_{R_i} \to u_\infty$ on $B_r(p)$ for every $0 < r < R$. Then, for every $0 < r < s < R$, we have*

$$\lim_{i\to\infty} \sup_{t\in[r,s]} \left| D^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t) - D_{u_\infty}(t) \right| = 0$$

and

$$\lim_{i\to\infty} \sup_{t\in[r,s]} \left| I^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t) - I_{u_\infty}(t) \right| = 0$$

PROOF. We fix $0 < \hat{r} < r < s < \hat{s} < R$. We take $L \geq 1$ such that $|u_\infty|_{L^\infty(B_{\hat{s}}(x_\infty))} + \mathbf{Lip}u_\infty \leq L$. We fix $\epsilon > 0$ satisfying $\epsilon << \min\{\hat{r}, R - \hat{s}\}$. Then, by the proof of Lemma 5.2, there exists $R_1(\epsilon) > 1$ such that

$$B^{g_M}_{(1-\epsilon^2)R}(m) \subset \{b^{g_M} \leq R\} \subset B^{g_M}_{(1+\epsilon^2)R}(m)$$

and

$$\int_{b^{g_M}\leq R}\left||\nabla^{g_M}b^{g_M}|^2-1\right|^2\leq\epsilon^8\mathrm{vol}\{b^{g_M}\leq R\}$$

for every $R>R_1(\epsilon)$. Especially, by Cauchy-Schwartz inequality, we have

$$\int_{b^{g_M}\leq R}\left||\nabla^{g_M}b^{g_M}|^2-1\right|\leq\epsilon^4\mathrm{vol}\{b^{g_M}\leq R\}$$

and

$$\int_{b^{g_M}\leq R}||\nabla^{g_M}b^{g_M}|-1|\leq\epsilon^2\mathrm{vol}\{b^{g_M}\leq R\}.$$

For $0<t<R$, we put

$$F_i(t)=\int_{b^{R_i^{-2}g_M}\leq t}(u_i)_{R_i}^2|\nabla^{R_i^{-2}g_M}b^{R_i^{-2}g_M}|^2d\mathrm{vol}^{R_i^{-2}g_M}.$$

Then, we have

$$\frac{dF_i}{dt}(t)=\int_{b^{R_i^{-2}g_M}=t}(u_i)_{R_i}^2|\nabla^{R_i^{-2}g_M}b^{R_i^{-2}g_M}|d\mathrm{vol}_{n-1}^{R_i^{-2}g_M}=I^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t)t^{n-1}.$$

Thus, we have

$$\begin{aligned}\frac{d^2F_i}{dt^2}(t)&=2t^{n-1}\frac{D^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t)}{t}+(n-1)I^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t)t^{n-2}\\&=2\int_{b^{R_i^{-2}g_M}\leq t}|\nabla^{R_i^{-2}g_M}(u_i)_{R_i}|^2d\mathrm{vol}^{R_i^{-2}g_M}+\frac{(n-1)}{t}\int_{b^{R_i^{-2}g_M}=t}(u_i)_{R_i}^2|\nabla^{R_i^{-2}g_M}b^{R_i^{-2}g_M}|^2d\mathrm{vol}_{n-1}^{R_i^{-2}g_M}.\end{aligned}$$

On the other hand, in general, for every C^2-function f on $\mathbf{R}$, we have

$$f(t)=f(a)+(t-a)f'(a)-\int_a^t(s-t)f''(s)ds$$

for every $a,s,t\in\mathbf{R}$. Therefore, for every $0<t<R$, we have

$$\begin{aligned}&\left|\frac{F_i(t+\epsilon)-F_i(t)}{\epsilon}-\int_{b^{R_i^{-2}g_M}=t}(u_i)_{R_i}^2|\nabla^{R_i^{-2}g_M}b^{R_i^{-2}g_M}|d\mathrm{vol}^{R_i^{-2}g_M}\right|\\&\leq\int_t^{t+\epsilon}2\int_{b^{R_i^{-2}g_M}\leq a}|\nabla^{R_i^{-2}g_M}(u_i)_{R_i}|^2d\mathrm{vol}^{R_i^{-2}g_M}da\\&\quad+(n-1)\int_t^{t+\epsilon}a^{-1}\int_{b^{R_i^{-2}g_M}=a}(u_i)_{R_i}^2|\nabla^{R_i^{-2}g_M}b^{R_i^{-2}g_M}|d\mathrm{vol}^{R_i^{-2}g_M}da\\&\leq2\epsilon\int_{b^{R_i^{-2}g_M}\leq t+\epsilon}|\nabla^{R_i^{-2}g_M}(u_i)_{R_i}|^2d\mathrm{vol}^{R_i^{-2}g_M}+\frac{n-1}{t}\int_{t\leq b^{g_M}\leq t+\epsilon}(u_i)_{R_i}^2|\nabla^{R_i^{-2}g_M}b^{R_i^{-2}g_M}|^2d\mathrm{vol}^{R_i^{-2}g_M}.\end{aligned}$$

By Proposition 2.15, there exists $i_0\in\mathbf{N}$ such that $R_i\hat{r}\geq10R_1(\epsilon)$, $|(u_i)_{R_i}|_{L^\infty(B^{R_i^{-2}g_M}_{\hat{s}}(m))}\leq 10L$ and

$$\sup_{a\in[0,R]}\left|\mathrm{vol}^{R_i^{-2}g_M}B_a^{R_i^{-2}g_M}(m)-H^n(B_a(p))\right|<\epsilon^2$$

for every $i \geq i_0$. Then, by Cheng-Yau's gradient estimate, for every $i \geq i_0$ and $r < t < s$,

$$\int_{b^{R_i^{-2}g_M} \leq t+\epsilon} |\nabla^{R_i^{-2}g_M}(u_i)_{R_i}|^2 d\mathrm{vol}^{R_i^{-2}g_M} \leq \int_{B^{R_i^{-2}g_M}_{(1+\epsilon)(t+\epsilon)}(m)} |\nabla^{R_i^{-2}g_M}(u_i)_{R_i}|^2 d\mathrm{vol}^{R_i^{-2}g_M} \tag{166}$$

$$\leq C(n, L, R). \tag{167}$$

Here, we used $H^n(B_R(p)) = R^n H^n(B_1(p)) \leq R^n C(n)$. Moreover, we have

$$\int_{t \leq b^{R_i^{-2}g_M} \leq t+\epsilon} (u_i)^2_{R_i} |\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}|^2 d\mathrm{vol}^{R_i^{-2}g_M} \tag{168}$$

$$\leq \int_{t \leq b^{R_i^{-2}g_M} \leq t+\epsilon} (u_i)^2_{R_i} d\mathrm{vol}^{R_i^{-2}g_M} + \int_{t \leq b^{R_i^{-2}g_M} \leq t+\epsilon} (u_i)^2_{R_i} \left| |\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}|^2 - 1 \right| d\mathrm{vol}^{R_i^{-2}g_M} \tag{169}$$

$$\leq \int_{t \leq b^{R_i^{-2}g_M} \leq t+\epsilon} (u_i)^2_{R_i} d\mathrm{vol}^{R_i^{-2}g_M} + 100L^2 \mathrm{vol}^{R_i^{-2}g_M}\{t \leq b^{R_i^{-2}g_M} \leq t+\epsilon\} \tag{170}$$

$$\leq 200L^2 \mathrm{vol}^{R_i^{-2}g_M}\{t \leq b^{R_i^{-2}g_M} \leq t+\epsilon\} \tag{171}$$

$$\leq 200L^2 \mathrm{vol}^{R_i^{-2}g_M} A_m^{R_i^{-2}g_M}\left((1-\epsilon^2)t, (1+\epsilon^2)(t+\epsilon)\right) \tag{172}$$

$$\leq 200L^2 H^n\left(A_p\left((1-\epsilon^2)t, (1+\epsilon^2)(t+\epsilon)\right)\right) + 300L^2\epsilon^2. \tag{173}$$

On the other hand, we have

$$\frac{F_i(t+\epsilon) - F_i(t)}{\epsilon} = \frac{1}{\epsilon} \int_{t \leq b^{R_i^{-2}g_M} \leq t+\epsilon} (u_i)^2_{R_i} d\mathrm{vol}^{R_i^{-2}g_M} \tag{174}$$

$$\pm \frac{1}{\epsilon} \int_{t \leq b^{R_i^{-2}g_M} \leq t+\epsilon} (u_i)^2_{R_i} \left| |\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}|^2 - 1 \right| d\mathrm{vol}^{R_i^{-2}g_M}, \tag{175}$$

and

$$\frac{1}{\epsilon} \int_{t \leq b^{R_i^{-2}g_M} \leq t+\epsilon} (u_i)^2_{R_i} \left| |\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}|^2 - 1 \right| d\mathrm{vol}^{R_i^{-2}g_M} \tag{176}$$

$$\leq \frac{100L^2}{\epsilon} \int_{b^{R_i^{-2}g_M} \leq t+\epsilon} \left| |\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}|^2 - 1 \right| d\mathrm{vol}^{R_i^{-2}g_M} \tag{177}$$

$$\leq \frac{100L^2}{\epsilon} \epsilon^2 \mathrm{vol}^{R_i^{-2}g_M}\left(\{b^{R_i^{-2}g_M} \leq t+\epsilon\}\right) \tag{178}$$

$$\leq 100L^2 \epsilon \frac{\mathrm{vol}^{g_M} B^{g_M}_{(1+\epsilon^2)(t+\epsilon)R_i}(m)}{R_i^n} \tag{179}$$

$$\leq \epsilon C(n, L, R). \tag{180}$$

We remark that

$$\left|\int_{t\le b^{R_i^{-2}g_M}\le t+\epsilon}(u_i)^2_{R_i}d\mathrm{vol}^{R_i^{-2}g_M}-\int_{A_m^{R_i^{-2}g_M}(t,t+\epsilon)}(u_i)^2_{R_i}d\mathrm{vol}^{R_i^{-2}g_M}\right| \tag{181}$$

$$\le 100L^2\mathrm{vol}^{R_i^{-2}g_M}\left(\{t\le b^{R_i^{-2}g_M}\le t+\epsilon\}\triangle A_m^{R_i^{-2}g_M}(t,t+\epsilon)\right). \tag{182}$$

Here $A\triangle B=(A\setminus B)\cup(B\setminus A)$.

Claim 5.7. *We have*

$$\{t\le b^{R_i^{-2}g_M}\le t+\epsilon\}\triangle A_m^{R_i^{-2}g_M}(t,t+\epsilon) \tag{183}$$

$$\subset A_m^{R_i^{-2}g_M}\left((1-\epsilon^2)(t+\epsilon),(1+\epsilon^2)(t+\epsilon)\right)\cup A_m^{R_i^{-2}g_M}\left((1-\epsilon^2)t,(1+\epsilon^2)t\right) \tag{184}$$

for every $i\ge i_0$ and $r<t<s$.

The proof is as follows. We put $A_i^\epsilon(t)=\{t\le b^{R_i^{-2}g_M}\le t+\epsilon\}\triangle A_m^{R_i^{-2}g_M}(t,t+\epsilon)$. First, we take $y\in\{t\le b^{R_i^{-2}g_M}\le t+\epsilon/2\}\cap A_i^\epsilon(t)$. Then we have $y\in B^{R_i^{-2}g_M}_{(1+\epsilon^2)(t+\epsilon/2)}(m)$. Especially, we have

$$\overline{m,y}^{R_i^{-2}g_M}\le(1+\epsilon^2)(t+\frac{\epsilon}{2})<t+\epsilon.$$

Since $y\in M\setminus A_m^{R_i^{-2}g_M}(t,t+\epsilon)$, we have $y\in B_t^{R_i^{-2}g_M}(m)$. Thus, we have $\{t\le b^{R_i^{-2}g_M}\le t+\epsilon/2\}\cap A_i^\epsilon(t)\subset B_t^{R_i^{-2}g_M}(m)\setminus B^{R_i^{-2}g_M}_{(1-\epsilon^2)t}(m)$. Similarly, we have $\{t+\epsilon/2\le b^{R_i^{-2}g_M}\le t+\epsilon\}\cap A_i^\epsilon(t)\subset B^{R_i^{-2}g_M}_{(1+\epsilon^2)(t+\epsilon)}(m)\setminus B^{R_i^{-2}g_M}_{t+\epsilon}(m)$. Therefore, we have

$$\{t\le b^{R_i^{-2}g_M}\le t+\epsilon\}\cap A_i^\epsilon(t)\subset A_m^{R_i^{-2}g_M}((1-\epsilon^2)t,t)\cup A_m^{R_i^{-2}g_M}(t+\epsilon,(1+\epsilon^2)(t+\epsilon)).$$

Next, we take $x\in A_i^\epsilon(t)\cap A_m^{R_i^{-2}g_M}(t,t+\epsilon/2)$. Then we have

$$b^{R_i^{-2}g_M}(x)\le(1+\epsilon^2)\overline{m,x}^{R_i^{-2}g_M}\le(1+\epsilon^2)(t+\epsilon/2)<t+\epsilon.$$

Since $x\in M\setminus\{t\le b^{R_i^{-2}g_M}\le t+\epsilon\}$, we have $b^{R_i^{-2}g_M}(x)<t$. Therefore, we have $x\in B^{R_i^{-2}g_M}_{(1+\epsilon^2)t}(m)$. Thus, we have $A_m^{R_i^{-2}g_M}(t,t+\epsilon/2)\cap A_i^\epsilon(t)\subset A_m^{R_i^{-2}g_M}(t,(1+\epsilon^2)t)$. Similarly, we have $A_m^{R_i^{-2}g_M}(t+\epsilon/2,t+\epsilon)\cap A_i^\epsilon(t)\subset A_m^{R_i^{-2}g_M}(t+\epsilon,(1+\epsilon^2)(t+\epsilon))$. Therefore we have Claim 5.7.

By Claim 5.7 and Bishop-Gromov volume comparison theorem, we have

$$\epsilon^{-1}\mathrm{vol}^{R_i^{-2}g_M}\left(\{t\le b^{R_i^{-2}g_M}\le t+\epsilon\}\triangle A_m^{R_i^{-2}g_M}(t,t+\epsilon)\right) \tag{185}$$

$$\le\epsilon^{-1}\mathrm{vol}^{R_i^{-2}g_M}\left(A_m^{R_i^{-2}g_M}\left((1-\epsilon^2)(t+\epsilon),(1+\epsilon^2)(t+\epsilon)\right)\right) \tag{186}$$

$$+\epsilon^{-1}\mathrm{vol}^{R_i^{-2}g_M}\left(A_m^{R_i^{-2}g_M}\left((1-\epsilon^2)t,(1+\epsilon^2)t\right)\right)\Big) \tag{187}$$

$$\le 3\epsilon^{-1}\epsilon^2\mathrm{vol}^{R_i^{-2}g_M}_{n-1}\left(\partial B^{R_i^{-2}g_M}_{(1-\epsilon^2)(t+\epsilon)}(m)\setminus C_m\right)+3\epsilon^{-1}\epsilon^2\mathrm{vol}^{R_i^{-2}g_M}_{n-1}\left(\partial B^{R_i^{-2}g_M}_{(1-\epsilon^2)t}(m)\setminus C_m\right) \tag{188}$$

$$\le 6\epsilon\mathrm{vol}\,\partial B_R(0_n). \tag{189}$$

Therefore we have

$$\left|\int_{t\le b^{R_i^{-2}g_M}\le t+\epsilon}(u_i)_{R_i}^2 d\mathrm{vol}^{R_i^{-2}g_M}-\int_{A_m^{R_i^{-2}g_M}(t,t+\epsilon)}(u_i)_{R_i}^2 d\mathrm{vol}^{R_i^{-2}g_M}\right|\le 600L^2\epsilon\mathrm{vol}\,\partial B_R(0_n).$$

for every $i\ge i_0$ and $r<t<s$. We take the canonical retraction, π_t from $C(X)$ to $\overline{B}_t(p)$ for every $t>0$. It is easy to check that π_t is 1-Lipschitz map. We put $u_\infty^t=(u_\infty)^2\circ\pi_t$. We have $\mathbf{Lip}u_\infty^t\le\mathbf{Lip}(u_\infty)^2$. By Proposition 7.22, we have

$$\left|\int_t^{t+\epsilon}\int_{\partial B_a(p)}(u_\infty)^2dH^{n-1}da-\int_t^{t+\epsilon}\int_{\partial B_a(p)}u_\infty^t dH^{n-1}da\right| \tag{190}$$

$$\le\int_{A_p(t,t+\epsilon)}|(u_\infty)^2-u_\infty^t|dH^n \tag{191}$$

$$\le\mathbf{Lip}(u_\infty)^2\epsilon H^n(A_p(t,t+\epsilon)). \tag{192}$$

for every $r<t<s$. On the other hand,

$$\int_t^{t+\epsilon}\int_{\partial B_a(p)}u_\infty^t dH^{n-1}da=\int_t^{t+\epsilon}\left(\frac{a}{t}\right)^{n-1}\int_{\partial B_t(p)}(u_\infty)^2dH^{n-1}da \tag{193}$$

$$=\int_{\partial B_t(p)}(u_\infty)^2dH^{n-1}\int_t^{t+\epsilon}\left(\frac{a}{t}\right)^{n-1}da \tag{194}$$

$$=I_{u_\infty}(t)\frac{(t+\epsilon)^n-t^n}{n} \tag{195}$$

$$=I_{u_\infty}(t)(\epsilon t^{n-1}\pm\Psi(\epsilon;n,R)\epsilon). \tag{196}$$

Therefore we have

$$\lim_{i\to\infty}\sup_{t\in[r,s]}\left|I^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t)-I_{u_\infty}(t)\right|=0.$$

Next, we shall prove

$$\lim_{i\to\infty}\sup_{t\in[r,s]}\left|D^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t)-D_{u_\infty}(t)\right|=0.$$

We shall use same notations as above. It is clear that

$$t^{2-n}\int_{B^{R_i^{-2}g_M}_{(1-\epsilon^2)t}(m)}|\nabla^{R_i^{-2}g_M}(u_i)_{R_i}|^2d\mathrm{vol}^{R_i^{-2}g_M}\le D^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t) \tag{197}$$

$$\le t^{2-n}\int_{B^{R_i^{-2}g_M}_{(1+\epsilon^2)t}(m)}|\nabla^{R_i^{-2}g_M}(u_i)_{R_i}|^2d\mathrm{vol}^{R_i^{-2}g_M} \tag{198}$$

for every $i\ge i_1$ and $r<t<s$. On the other hand, we have

$$\int_{A_m^{R_i^{-2}g_M}((1-\epsilon^2)t,(1+\epsilon^2)t)}|\nabla^{R_i^{-2}g_M}(u_i)_{R_i}|^2d\mathrm{vol}^{R_i^{-2}g_M}(m) \tag{199}$$

$$\le C(n,L,R)\mathrm{vol}^{R_i^{-2}g_M}\,A_m^{R_i^{-2}g_M}((1-\epsilon^2)t,(1+\epsilon^2)t)) \tag{200}$$

$$\le C(n,L,R)\left(H^n\left(A_p((1-\epsilon^2)t,(1+\epsilon^2)t)\right)+\epsilon\right). \tag{201}$$

Therefore, by Theorem 4.37, we have the assertion. □

For every $0 < r < R$ and harmonic function u on $\{b^{g_M} < R\}$, we put

$$E_u^{g_M}(r) = r^{2-n} \int_{b^{g_M} \le r} |\nabla^{g_M} u|^2 |\nabla^{g_M} b^{g_M}|^2 d\mathrm{vol}^{g_M}.$$

It is easy to check that for every $\tau, r, R > 0$ satisfying $R > r\tau$ and a harmonic function u on $\{b^{g_M} < R\}$, we have $E_{u_\tau}^{\tau^{-2} g_M}(r) = \tau^{-2} E_u^{g_M}(\tau r)$. By an argument similar to the proof of Proposition 5.6 (or [19, Proposition 3.3]), we have the following:

PROPOSITION 5.8. *With same assumption as in Lemma 5.6, we have*

$$\lim_{i \to \infty} \sup_{t \in [r,s]} \left| E_{(u_i)_{R_i}}^{R_i^{-2} g_M}(t) - D_{u_\infty}(t) \right| = 0$$

for every $0 < r < s < R$.

We shall introduce an important result [23, Theorem 2.1] by Ding:

THEOREM 5.9 (Ding, [23]). *For every $0 < r < R$, all harmonic functions on $B_R(p)$ are Lipschitz on $B_r(p)$. Moreover, for every $0 < r < s < R$ and harmonic function v on $B_R(p)$, there exist a subsequence $\{n(i)\}_i$ of $\mathbf{N}$ and a sequence of harmonic functions $v_{n(i)}$ on $B_s^{R_{n(i)}^{-2} g_M}(m)$ such that $v_{n(i)} \to u_\infty$ on $B_r(x_\infty)$.*

PROOF. We shall give an outline of the proof only. First, we shall show that u_∞ is Lipschitz function. By [51, Proposition 5.1], for every $u \in H_{1,2}(M_i)$ and $R > 0$, we have

$$\int_M u(y)^2 H^{R^{-2} g_M}(t, y, x) d\mathrm{vol}_y^{R^{-2} g_M} \le 2t \int_M |d^{R^{-2} g_M} u|^2 d\mathrm{vol}_y^{R^{-2} g_M} \tag{202}$$

$$+ \left(\int_M u(y) H^{R^{-2} g_M}(t, y, x) d\mathrm{vol}_y^{R^{-2} g_M} \right)^2 \tag{203}$$

for a.e. $x \in M$. Here. $H^{R^{-2} g_M}(t, y, x)$ is the heat kernel for rescaled manifold $(M, R^{-2} g_M)$. By [24, Theorem 5.54] and [5, Lemma 10.3] (or Theorem 4.27), for every $u \in \mathcal{K}(C(X))$, we have,

$$\int_{C(X)} u(y)^2 H_\infty(t, y, x) dH^n(y) \le 2t \int_{C(X)} |du|^2 dH^n(y) + \left(\int_{C(X)} u(y) H_\infty(t, y, x) dH^n(y) \right)^2$$

for a.e. $x \in C(X)$. Here H_∞ is as in [24, Theorem 5.54]. Since $\mathcal{K}(C(X))$ is dense in $H_{1,2}(C(X))$, the inequality above holds for every $u \in H_{1,2}(C(X))$. Next, we fix $x \in X$ and $0 < t < R$. Then, by Bishop-Gromov volume comparison theorem, it is easy to check that $H^n(B_t((1, x))) \ge C(n, V_M) t^n$. For every $R > 0$, we define the map ϕ_R from $A_p(R-t, R+t)$ to $A_p(1 - \frac{t}{R}, 1 + \frac{t}{R})$ by $\phi_R((\hat{t}, x)) = (\hat{t}/R, x)$. Since $H^n(\phi_R(A)) = R^n H^n(A)$ for every Borel subset $A \subset A_p(R - t, R + t)$, we have

$$H^n(B_t(R, x)) = R^n H^n(B_{\frac{t}{R}}(1, x)) \ge C(n, V_M) t^n.$$

Therefore, $(C(X), H^n)$ is Ahlfors n-regular metric measure space (see section 1 in [51]). By [24, Theorem 6.1], [24, Theorem 6.20] and [51, Theorem 1.1], u_∞ is locally Lipschitz function on $B_R(p)$. By convexity of $B_s(p)$ and the proof of [51, Theorem 1.1], u_∞ is Lipschitz on $B_s(p)$. Next, we shall take $L \geq 1$ satisfying $\mathbf{Lip}(u_\infty|_{B_s(p)}) + |u_\infty|_{L^\infty(B_s(p))} \leq L$. Without loss of generality, we can assume that there exists a sequence of Lipschitz functions f_i on $\overline{B}_s^{R_i^{-2}g_M}(m)$ such that $\mathbf{Lip} f_i + |f_i|_{L^\infty(B_s(p))} \leq 10L$ and $f_i \to u_\infty$ on $B_s(p)$. We take a harmonic function u_i on $B_s^{R_i^{-2}g_M}(m)$ such that

$$u_i|_{\partial B_s^{R_i^{-2}g_M}(m)} = f_i|_{\partial B_s^{R_i^{-2}g_M}(m)}$$

in the sense of Perron's method for f_i. We shall give a short review of Perron's method of subharmonic functions in this setting below. See for instance section 2.8 in [33]. For $f \in C^0(B_s^{R_i^{-2}g_M}(m))$, we say that f is subharmonic (superharmonic) in $B_s^{R_i^{-2}g_M}(m)$ if for every $w \in B_s^{R_i^{-2}g_M}(m)$, $r_1 > 0$ with $\overline{B}_{r_1}^{R_i^{-2}g_M}(w) \subset B_s^{R_i^{-2}g_M}(m)$, and $h \in C^0(\overline{B}_{r_1}^{R_i^{-2}g_M}(w))$ satisfying $h|_{B_{r_1}^{R_i^{-2}g_M}(w)}$ is harmonic and $h|_{\partial B_{r_1}^{R_i^{-2}g_M}(w)} \leq (\geq) f|_{\partial B_{r_1}^{R_i^{-2}g_M}(w)}$, we also have $h \leq (\geq) f$ on $B_{r_1}^{R_i^{-2}g_M}(w)$. For $g \in C^0(\overline{B}_s^{R_i^{-2}g_M}(m))$, we say that g is a subfunction relative to $f_i|_{B_s^{R_i^{-2}g_M}(m)}$ if $g|_{B_s^{R_i^{-2}g_M}(m)}$ is a subharmonic function and $g|_{\partial B_s^{R_i^{-2}g_M}(m)} \leq f_i|_{\partial B_s^{R_i^{-2}g_M}(m)}$. We also say that g is a superfunction relative to $f_i|_{B_s^{R_i^{-2}g_M}(m)}$ if $g|_{B_s^{R_i^{-2}g_M}(m)}$ is a superharmonic function and $g|_{\partial B_s^{R_i^{-2}g_M}(m)} \geq f_i|_{\partial B_s^{R_i^{-2}g_M}(m)}$. Let S_{f_i} denote the set of subfunctions relative to $f_i|_{B_s^{R_i^{-2}g_M}(m)}$. Then we put a function u_i on $B_s^{R_i^{-2}g_M}(m)$ by

$$u_i(w) = \sup_{v \in S_{f_i}} v(w).$$

By an argument similar to the proof of [33, Theorem 2.12], it is easy to check that u_i is harmonic on $B_s^{R_i^{-2}g_M}(m)$.

We fix $0 < \tau < 3R$, $x \in \partial B_s(p)$ and $z \in \partial B_{2s}(p)$ satisfying $\overline{p,x} + \overline{x,z} = \overline{p,z}$. We take sequences $x(i) \in \partial B_s^{R_i^{-2}g_M}(m)$ and $z(i) \in \partial B_{2s}^{R_i^{-2}g_M}(m)$ such that $x(i) \to x$ and $z(i) \to z$. Then it is easy to check that for every $\alpha \in B_s(p)$, we have

$$C_1(n,R)\overline{x,\alpha}^2 \leq \overline{z,\alpha} - \overline{z,x} \leq \overline{x,\alpha}.$$

We fix $\alpha \in B_r(p)$ and take a sequence of points $\alpha(i) \in B_s^{R_i^{-2}g_M}(m)$ satisfying $\alpha(i) \to \alpha$. We put $b^i = (r_{z(i)}^{R_i^{-2}g_M})^{2-n} - (r_{z(i)}^{R_i^{-2}g_M})^{2-n}(x(i))$ on $B_s^{R_i^{-2}g_M}(m)$. By Laplacian comparison theorem on manifolds (or (4.11) in [4]), we have, a function b^i is a superharmonic, a function $f_i(x(i)) + 100L\tau + C(n,L,R)b^i/\tau^2$ is a superfunction relative to $f_i|_{\partial B_s^{R_i^{-2}g_M}(m)}$ and a function $f_i(x(i)) - 100L\tau - C(n,L,R)b^i/\tau^2$ is a subfunction relative to $f_i|_{\partial B_s^{R_i^{-2}g_M}(m)}$ for every sufficiently large i. By an argument similar to the proof of [33, Lemma 2.13],

we have

$$|f_i(x(i)) - u_i(\alpha(i))| \le C(n,R,L)\tau + \frac{C(n,R,L)}{\tau^2}\overline{x(i),\alpha(i)}^{R_i^{-2}g_M}$$

for every sufficiently large i. On the other hand, by Proposition 2.11 and Corollary 4.37, we can assume that there exists a harmonic function $\hat{u}_\infty$ on $B_s(p)$ such that $\hat{u}_\infty|_{B_{\hat{s}}(p)}$ is a Lipschitz function, $u_i \to u_\infty$ on $B_{\hat{s}}(p)$ for every $0 < \hat{s} < s$. Thus we have

$$|u_\infty(x) - \hat{u}_\infty(\alpha)| \le C(n,R,L)\tau + \frac{C(n,R,L)}{\tau^2}\overline{x,\alpha}$$

for every $\alpha \in B_s(p)$. If we put $\tau = \overline{x,\alpha}^{1/3}$, then we have

$$|u_\infty(x) - \hat{u}_\infty(\alpha)| \le C(n,R,L)\overline{x,\alpha}^{\frac{1}{3}}.$$

for every $x \in \partial B_s(p)$ and $\alpha \in B_s(p)$. Since $\hat{u}_\infty \in H_{1,2}(B_{\hat{s}}(p))$ for every $0 < \hat{s} < s$, and u_∞ is Lipschitz on $\overline{B}_s(p)$, by [80, Cororally 6.6] and an estimate above, we have $\sup_{B_s(p)} |u_\infty - \hat{u}_\infty| = \lim_{\hat{s}\to s} \left(\sup_{B_{\hat{s}}(p)} |u_\infty - \hat{u}_\infty|\right) = 0$. Therefore, we have the assertion. □

We shall remark that the following:

Corollary 5.10. *Let R be a positive number and u_∞, v_∞ harmonic functions on $B_R(p)$. Then $u_\infty + v_\infty$ is a harmonic function on $B_R(p)$.*

From now on, we shall replace most of many important statements in [19] with statements on asymptotic cones:

Proposition 5.11. *For every $0 < r < s < R$ and harmonic function u_∞ on $B_R(p)$, we have*

$$D_{u_\infty}(r) \le \left(\frac{r}{s}\right)^{2-n} D_{u_\infty}(s),$$

$$I_{u_\infty}(s) - I_{u_\infty}(r) = \int_r^s 2\frac{D_{u_\infty}(t)}{t}dt.$$

Moreover, if $I_{u_\infty}(r) > 0$, then we have

$$I_{u_\infty}(s) = \exp\left(2\int_r^s \frac{U_{u_\infty}(t)}{t}dt\right) I_{u_\infty}(r).$$

Proof. By Theorem 5.9, without loss of generality, we can assume that the assumption of Proposition 5.6 holds. Since

$$D^{R_i^{-2}g_M}_{(u_i)_{R_i}}(r) \le \left(\frac{r}{s}\right)^{2-n} D^{R_i^{-2}g_M}_{(u_i)_{R_i}}(s),$$

by letting $i \to \infty$, we have the first assertion. Similarly, since

$$I^{R_i^{-2}g_M}_{(u_i)_{R_i}}(s) - I^{R_i^{-2}g_M}_{(u_i)_{R_i}}(r) = \int_r^s 2\frac{D^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t)}{t}dt,$$

by letting $i \to \infty$ and dominated convergence theorem, we have the second assertion. Especially, we remark that I_{u_∞} is a continuous function and that a monotonicity $I_{u_\infty}(r) \leq I_{u_\infty}(s)$ holds. We shall prove the third assertion. By Proposition 5.6 and the monotonicity of I_{u_∞}, we have $\liminf_{i\to\infty}\left(\inf_{\alpha\in[r,s]} I^{R_i^{-2}g_M}_{(u_i)_{R_i}}(\alpha)\right) > 0$. Therefore, by Cheng-Yau's gradient estimate, we have

$$\limsup_{i\to\infty}\left(\sup_{\alpha\in[r,s]} U^{R_i^{-2}g_M}_{(u_i)_{R_i}}(\alpha)\right) < \infty.$$

On the other hand, since

$$I^{R_i^{-2}g_M}_{(u_i)_{R_i}}(s) = \exp\left(2\int_r^s \frac{U^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t)}{t}dt\right) I^{R_i^{-2}g_M}_{(u_i)_{R_i}}(r),$$

by letting $i \to 0$, dominated convergence theorem and Proposition 5.6, we have the third assertion. □

COROLLARY 5.12. *Let r, R be positive numbers with $r < R$ and u_∞ a harmonic function on $B_R(p)$. If $U_{u_\infty}(r) = 0$, then u_∞ is a constant function on $B_r(p)$.*

PROOF. First, we assume $I_{u_\infty}(r) = 0$. Then, by Proposition 5.11, we have $D_{u_\infty}(t) = 0$ for a.e. $0 < t < r$. Since D_{u_∞} is continuous, we have $D_{u_\infty}(r) = 0$. Thus, by Poincaré inequality on limit spaces, we have

$$\frac{1}{\upsilon(B_r(p))}\int_{B_r(p)}\left|f - \frac{1}{\upsilon(B_r(p))}\int_{B_r(p)} f d\upsilon\right| d\upsilon \leq C(n,R) r\sqrt{\frac{1}{\upsilon(B_r(p))}\int_{B_r(p)}(\mathrm{Lip}f)^2 d\upsilon} = 0.$$

Since f is Lipschitz on $B_r(p)$, f is a constant function on $B_r(p)$. Next, if $U_{u_\infty}(r) = 0$ and $I_{u_\infty}(r) > 0$, then, by the definition, we have $D_{u_\infty}(r) = 0$. Therefore, by an argumetnt above, we have the assertion in this case. □

The following corollary follows from Proposition 5.11 and continuity of the function: $t \mapsto H^n(B_t(p))$, directly.

COROLLARY 5.13. *For every $R > 0$ and harmonic function u_∞ on $B_R(p)$, the function I_{u_∞} is a C^1-function on $(0, R)$ and*

$$\frac{dI_{u_\infty}}{dt}(t) = \frac{2D_{u_\infty}(t)}{t}.$$

For every $0 < r < R$ and harmonic function u on $B^{g_M}_R(m)$ satisfying $u \neq 0$, we put

$$W^{g_M}_u(r) = \frac{E^{g_M}_u(r)}{I^{g_M}_u(r)}$$

With same assumption of Lemma 5.6, if u_∞ is not a constant function on $B_r(p)$, then, by Proposition 5.6 and Proposition 5.8, we have

$$\lim_{i\to\infty} W^{R_i^{-2}g_M}_{(u_i)_{R_i}}(r) = U_{u_\infty}(r).$$

PROPOSITION 5.14. *For every $0 < r < s < R$ and harmonic function u_∞ on $B_{7R}(p)$, we have*

$$U_{u_\infty}(r) \leq U_{u_\infty}(s).$$

PROOF. By Theorem 5.9, there exists a sequence of harmonic functions u_i on $B^{g_M}_{6RR_i}(m)$ such that $\sup_i \mathbf{Lip} u_i < \infty$ and $(u_i)_{R_i} \to u_\infty$ on $B_{6R}(p)$. We fix $\epsilon > 0$. Without loss of generality, we can assume that $U_{u_\infty}(r) > 0$. We shall use same notation as in [19, Proposition 4.11]. We put $\Omega_0 = s/r, \gamma = D_{u_\infty}(2s)/D_{u_\infty}(r)+1$. Then we take $\hat{R} = R(m, \gamma, \epsilon, \Omega_0)$ as in [19, Proposition 4.11]. By Proposition 5.6, there exists i_0 such that $R_i r > \hat{R}$ and

$$\frac{D^{g_M}_{u_i}(2\Omega_0 R_i r)}{D^{g_M}_{u_i}(R_i r)} = \frac{D^{g_M}_{u_i}(2R_i s)}{D^{g_M}_{u_i}(R_i r)} = \frac{D^{R_i^{-2}g_M}_{(u_i)_{R_i}}(2s)}{D^{R_i^{-2}g_M}_{(u_i)_{R_i}}(r)} \leq \gamma$$

for every $i \geq i_0$. Then, by [19, Proposition 4.11], we have

$$\int_{R_i r}^{R_i s} \frac{d \log W^{g_M}_{u_i}}{dt} dt \geq -\epsilon.$$

i.e. we have

$$\log W^{g_M}_{(u_i)}(R_i s) - \log W^{g_M}_{(u_i)}(R_i t) \geq -\epsilon.$$

Since $W^{g_M}_{(u_i)}(R_i s) = W^{R_i^{-2}g_M}_{(u_i)_{R_i}}(s)$, by letting $i \to \infty$, we have

$$\log U_{u_\infty}(s) - \log U_{u_\infty}(r) \geq -\epsilon.$$

Since ϵ is arbitrary, we have the assertion. □

REMARK 5.15. Most of their results in [19] are about global harmonic functions on manifolds. However, by the proof, their results in [19] also hold for harmonic function on a big domain like one used in the proof of Proposition 5.14. We will often use these facts below.

For $d \geq 0$, we put $\mathcal{H}^d(M_\infty) = \{u_\infty : M_\infty \to \mathbf{R}$; u_∞ is a harmonic function and there exists $C > 1$ such that $|u_\infty(x)| \leq C(1 + \overline{m_\infty, x}^d)$ for every $x \in M_\infty\}$.

PROPOSITION 5.16. *We have $U_{u_\infty}(t) \leq d$ for every $t > 0$ and $u_\infty \in \mathcal{H}^d(M_\infty)$.*

PROOF. This proof is done by a contradiction. We assume that there exist $\tau_0, s_0 > 0$ such that $U_{u_\infty}(s_0) \geq d_0 + \tau_0$. By Proposition 5.14, we have $U_{u_\infty}(s) \geq d + \tau_0$ for every $s \geq s_0$. Since $u_\infty \in \mathcal{H}^d(M_\infty)$, there exist $s_1 > s_0$ and $C > 1$ such that

$$I_{u_\infty}(s) = s^{1-n} \int_{\partial B_s(p)} u_\infty^2 dH^{n-1} \leq s^{1-n} s^{2d} \mathrm{vol}\, \partial B_s(p) C \leq C s^{2d} \mathrm{vol}\, B_1(0_n)$$

for every $s \geq s_1$. For $s > s_1$, by Proposition 5.11, we have

$$Cs^{2d}\mathrm{vol}\, B_1(0_n) \geq \exp\left(\int_{s_1}^{s} \frac{2U_{u_\infty}(t)}{t}dt\right) I_{u_\infty}(s_1) \geq \exp\left(\int_{s_1}^{s} \frac{2d+2\tau_0}{t}dt\right) I_{u_\infty}(s_1).$$

Therefore, we have

$$2d + \frac{\log(C\mathrm{vol}\, B_1(0_n))}{\log s} \geq \frac{1}{\log s}\int_{s_1}^{s} \frac{2d+2\tau_0}{t}dt + \frac{\log I_{u_\infty}(s_1)}{\log s}.$$

By letting $s \to \infty$, we have $2d \geq 2d + 2\tau_0$. This is a contradiction. □

Proposition 5.17. *For every $0 < s < t < \alpha < R$ and harmonic function u_∞ on $B_{7R}(p)$, we have*

$$I_{u_\infty}(t) \leq \left(\frac{t}{s}\right)^{2U_{u_\infty}(\alpha)} I_{u_\infty}(s).$$

Proof. First, we assume that u_∞ is not a constant function on $B_s(p)$. By Theorem 5.9, there exists a sequence of harmonic functions u_i on $B^{g_M}_{6RR_i}(m)$ such that $\sup_i \mathbf{Lip} u_i < \infty$ and $(u_i)_{R_i} \to u_\infty$ on $B_{6R}(p)$. We fix $\epsilon > 0$. By the assumption and Corollary 5.12, there exists $0 < r < s$ such that $U_{u_\infty}(r) > 0$. We shall apply [19, Corollary 4.37]. We put $\Omega_0 = 2\alpha/r$, $\Omega = \alpha/r$ and

$$\gamma = \frac{D_{u_\infty}(2\Omega r)}{D_{u_\infty}(r)} + 1.$$

We take $\hat{R} = R(m, \gamma, \epsilon, \Omega_0)$ as in [19, Corollary 4.37]. There exists i_0 such that $R_i r > \hat{R}$ and that

$$\frac{D^{R_i^{-2}g_M}_{(u_i)_{R_i}}(2\Omega r)}{D^{R_i^{-2}g_M}_{(u_i)_{R_i}}(r)} < \gamma$$

for every $i \geq i_0$. Thus, by [19, Corollary 4.37], we have

$$I^{g_M}_{u_i}(R_i t) \leq \left(\frac{R_i t}{R_i s}\right)^{2(1+\epsilon)W^{g_M}_{u_i}(\Omega R_i r)} I^{g_M}_{u_i}(R_i s).$$

Thus by letting $i \to \infty$, we have

$$I_{u_\infty}(t) \leq \left(\frac{t}{s}\right)^{2(1+\epsilon)U_{u_\infty}(\alpha)} I_{u_\infty}(s).$$

Since ϵ is arbitrary, we have the assertion. Next we assume that u_∞ is a constant function on $B_s(p)$. We put $\hat{s} = \sup\{\beta \in [0, R]; u_\infty \text{ is a constant function on } B_\beta(p)\}$. If $\hat{s} \geq t$, then, since $I_{u_\infty}(t) = I_{u_\infty}(s)$, the assertion is clear. We assume $\hat{s} < t$. We take $\hat{s} < \tilde{s} < t$. Then, by an argument above, we have

$$I_{u_\infty}(t) \leq \left(\frac{t}{\tilde{s}}\right)^{2U_{u_\infty}(\alpha)} I_{u_\infty}(\tilde{s}).$$

By $s \leq \hat{s}$, $I_{u_\infty}(s) = I_{u_\infty}(\hat{s})$ and letting $\tilde{s} \to \hat{s}$, we have the assertion. □

COROLLARY 5.18. *Let s, R be positive numbers with $0 < s < R$ and u_∞ a harmonic function on $B_{7R}(p)$. Assume that $U_{u_\infty}(s) = 0$. Then u_∞ is a constant function on $B_R(p)$.*

PROOF. First, we assume that $I_{u_\infty}(s) = 0$. Then, by Proposition 5.17, we have $I_{u_\infty}(t) = 0$ for every $s < t < R$. Therefore, by Proposition 5.12, we have the assertion. Next, we assume that $I_{u_\infty}(s) > 0$ and $U_{u_\infty}(s) = 0$. Then, we put $\hat{u}_\infty = u_\infty - u_\infty(p)$. We remark that $\hat{u}_\infty \equiv 0$ on $B_s(p)$. Since $I_{\hat{u}_\infty}(s) = 0$, we have the assertion. □

PROPOSITION 5.19. *Let R be a positive number and u_∞ a harmonic function on $B_{7R}(p)$ with $u_\infty(p) = 0$. Assume that u_∞ is not a constant function on $B_R(p)$. Then, we have*

$$U_{u_\infty}(s) \geq 1$$

for every $0 < s < R$.

PROOF. By Theorem 5.9, there exists a sequence of harmonic functions u_i on $B^{g_M}_{6RR_i}(m)$ such that $\sup_i \mathbf{Lip} u_i < \infty$ and $(u_i)_{R_i} \to u_\infty$ on $B_{6R}(p)$. Moreover, we can assume that $(u_i)_{R_i}(m) = 0$. We remark that by Proposition 5.18, $U_{u_\infty}(r) > 0$ for every $0 < r < R$. We fix a sufficiently small $\epsilon > 0$. We shall apply [19, Corollary 4.40] and use same notation as in there. We take $\Omega_L = \Omega_L(n, \epsilon) \geq 2$ as in [19, Corollay 4.40] (or [19, Corollary 3.29]). We put $\Omega_0 = 5\Omega_L$, $r = s/2(2\Omega_L)^2 < s$ and

$$\gamma = \frac{D_{u_\infty}(2(2\Omega_L)^2 r)}{D_{u_\infty}(r)} + 1 = \frac{D_{u_\infty}(s)}{D_{u_\infty}(r)} + 1.$$

We take $\hat{R} = R(m, \gamma, \epsilon, \Omega_0)$ as in [19, Corollary 4.40]. Then there exists i_0 such that $R_i r > \hat{R}$ and

$$\frac{D^{g_M}_{u_i}(2(2\Omega_L)^2 R_i r)}{D^{g_M}_{u_i}(R_i r)} = \frac{D^{R_i^{-2} g_M}_{(u_i)_{R_i}}(2(2\Omega_L)^2 r)}{D^{R_i^{-2} g_M}_{(u_i)_{R_i}}(r)} \leq \gamma$$

for every $i \geq i_0$. Then by [19, Corollary 4.40], we have

$$1 - 3\epsilon \leq U^{g_M}_{u_i}(2\Omega_L R_i r) = U^{R_i^{-2} g_M}_{(u_i)_{R_i}}(2\Omega_L r).$$

By letting $i \to \infty$, Proposition 5.6 and Proposition 5.14, we have $1 - 3\epsilon \leq U_{u_\infty}(2\Omega_L r) \leq U_{u_\infty}(s)$. Since ϵ is arbitrary, we have the assertion. □

PROPOSITION 5.20. *Let r, s, R, δ, d_0 be positive numbers with $0 < r < s < R$ and u_∞ a harmonic function on $B_{7R}(p)$. We assume that $U_{u_\infty}(s) \leq d_0$, u_∞ is not a constant function on $B_R(p)$ and*

$$\left| \log \frac{U_{u_\infty}(s)}{U_{u_\infty}(r)} \right| < \delta.$$

Then, we have

$$\int_{A_p(r,s)} r_p^{-n} \left| r_p \langle dr_p, du_\infty \rangle - U_{u_\infty}(r_p) u_\infty \right|^2 dH^n \leq \Psi(\delta; n, d_0) I_{u_\infty}(s)$$

PROOF. By Theorem 5.9, there exists a sequence of harmonic functions u_i on $B^{g_M}_{6RR_i}(m)$ such that $\sup_i \mathbf{Lip} u_i < \infty$ and $(u_i)_{R_i} \to u_\infty$ on $B_{6R}(p)$. We shall apply [19, Proposition 4.50]. We put $\Omega_0 = 2s/r$, $\Omega = s/r$ and

$$\gamma = \frac{D_{u_\infty}(2\Omega r)}{D_{u_\infty}(r)} + 1.$$

Then, by Proposition 5.6, there exists i_0 such that

$$\frac{D^{R_i^{-2}g_M}_{u_\infty}(2\Omega r)}{D^{R_i^{-2}g_M}_{u_\infty}(r)} \le \gamma,$$

$$\max_{r\le t\le \Omega r} U^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t) \le 2d_0$$

and

$$\left|\log \frac{U^{R_i^{-2}g_M}_{(u_i)_{R_i}}(\Omega r)}{U^{R_i^{-2}g_M}_{(u_i)_{R_i}}(r)}\right| \le \delta$$

for every $i \ge i_0$. Thus, by [19, Proposition 4.50], we have,

$$\int_{rR_i\le b^{g_M}\le sR_i} (b^{g_M})^{-n}\left(b^{g_M}\frac{\partial u_i}{\partial n} - U^{g_M}_{u_i}(b^{g_M})|\nabla^{g_M} b^{g_M}|\right)^2 d\mathrm{vol}^{g_M} \le \Psi(\delta;n,d_0) I^{g_M}_{u_i}(R_i s)$$

for every sufficiently large i. On the other hand, by Cheng-Yau's gradient estimate, we have

$$|\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}| = \frac{V_M}{(n-2)\mathrm{vol}\, B_1(0_n)} |b^{R_i^{-2}g_M}|^{n-1} |\nabla^{R_i^{-2}g_M} G^{R_i^{-2}g_M}(m,\cdot)| \tag{204}$$

$$\le \frac{V_M}{(n-2)\mathrm{vol}\, B_1(0_n)} 2(r_m^{R_i^{-2}g_M})^{n-1} C(n) (r_m^{R_i^{-2}g_M})^{-1} |G^{R_i^{-2}g_M}(m,\cdot)| \tag{205}$$

$$\le C(n)(r_m^{R_i^{-2}g_M})^{-1}(r_m^{R_i^{-2}g_M})^{n-1}(r_m^{R_i^{-2}g_M})^{2-n} \tag{206}$$

$$\le C(n). \tag{207}$$

on $A_m^{R_i^{-2}g_M}(r,s)$ for every sufficiently large i. Thus by Corollary 4.37 and Theorem 5.1, we have $(b^{R_i^{-2}g_M}, db^{R_i^{-2}g_M}) \to (r_p, dr_p)$ on $A_p(r,s)$. We also have

$$\int_{r\le b^{R_i^{-2}g_M}\le s} (b^{R_i^{-2}g_M})^{-n}\left(b^{R_i^{-2}g_M}(R_i^{-2}g_M)\left(\nabla^{R_i^{-2}g_M}(u_i)_{R_i}, \nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}\right)\right. \tag{208}$$

$$\left. - U^{R_i^{-2}g_M}_{(u_i)_{R_i}}(b^{R_i^{-2}g_M})|\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}|^2\right)^2 d\mathrm{vol}^{R_i^{-2}g_M} \tag{209}$$

$$= \int_{rR_i\le b^{g_M}\le sR_i} (b^{g_M})^{-n}|\nabla b^{g_M}|^2\left(b^{g_M}\frac{\partial u_i}{\partial n} - U^{g_M}_{u_i}(b^{g_M})|\nabla^{g_M} b^{g_M}|\right)^2 d\mathrm{vol}^{g_M} \tag{210}$$

$$\le C(n)\int_{rR_i\le b^{g_M}\le sR_i} (b^{g_M})^{-n}\left(b^{g_M}\frac{\partial u_i}{\partial n} - U^{g_M}_{u_i}(b^{g_M})|\nabla^{g_M} b^{g_M}|\right)^2 d\mathrm{vol}^{g_M} \tag{211}$$

$$\le \Psi(\delta;n,d_0) I^{g_M}_{u_i}(R_i s) = \Psi(\delta;n,d_0) I^{R_i^{-2}g_M}_{(u_i)_{R_i}}(s) \tag{212}$$

for every sufficiently large i. Therefore, by letting $i \to \infty$, Proposition 4.5 and Proposition 5.6, we have the assertion. □

The following corollary follows from Proposition 5.20 directly.

COROLLARY 5.21. *Let r, s, R be positive numbers with $r < s < R$ and u_∞ be a harmonic function on $B_{7R}(p)$ with $u_\infty(p) = 0$. We assume that $U_{u_\infty}(r) = U_{u_\infty}(s)$. Then we have*

$$r_p(w)\langle du_\infty, dr_p\rangle(w) = U_{u_\infty}(s)u_\infty(w)$$

for a.e $w \in A_p(r, s)$.

PROPOSITION 5.22. *With same assumption as in Corollary 5.21, we have*

$$u_\infty(\hat{t}, x) = \frac{u_\infty(t, x)}{t^C}\hat{t}^C$$

for every $r \le t \le \hat{t} \le s$ and $x \in X$. Here $C = U_{u_\infty}(r)$.

PROOF. We define a Borel function a on $A_p(r, s)$ by

$$a(t, x) = \limsup_{h\to 0} \frac{u_\infty(t + h, x) - u_\infty(t, x)}{h}.$$

By Theorem 3.33 and Corollary 5.21, there exists a Borel set $A \subset A_p(r, s)$ such that $H^n(A_p(r, s) \setminus A) = 0$ and that $\langle dr_p, du_\infty\rangle(z) = a(z) = Cu_\infty(z)/r_p(z)$ for every $z \in A$. On the other hand, for $0 < s \le r_0 \le s_0 \le s$, we put a bi-Lipschitz map $\phi(t, x) = (t, x)$ from $A_p(r_0, s_0)$ to $[r_0, s_0] \times X$. Then we have $H^n([r_0, s_0] \times X \setminus \phi(A)) = 0$. Therefore by Fubini's theorem, there exists a Borel set $\hat{X} \subset X$ such that $H^{n-1}(X \setminus \hat{X}) = 0$ and that $H^1([r_0, s_0]\times\{x\}\setminus\phi(A)) = 0$ for every $x \in X$. Thus we have $H^1(\phi^{-1}([r_0, s_0]\times\{x\}\setminus\phi(A))) = 0$ for $x \in \hat{X}$. For every $x \in \hat{X}$, by Rademacher's theorem for Lipschitz functions on $\mathbf{R}$,

$$u_\infty(s_0, x) - u_\infty(r_0, x) = \int_{r_0}^{s_0} a(t, x)dt \tag{213}$$

$$= \int_{r_p(\phi^{-1}([r_0,s_0]\times\{x\}\cap\phi(A)))} a(t, x)dt \tag{214}$$

$$= \int_{r_p(\phi^{-1}([r_0,s_0]\times\{x\}\cap\phi(A)))} \frac{Cu_\infty(t, x)}{t}dt \tag{215}$$

$$= \int_{r_0}^{s_0} \frac{Cu_\infty(t, x)}{t}dt. \tag{216}$$

For every $x \in X$, by taking a sequence $x_i \in \hat{X}$ satisfying $x_i \to x$ and dominated convergence theorem, we have

$$u_\infty(s_0, x) - u_\infty(r_0, x) = \int_{r_0}^{s_0} \frac{Cu_\infty(t, x)}{t}dt.$$

Thus, for every $x \in X$, the map $f_x(\tilde{t}) = u_\infty(\tilde{t}, x)$ on $[r, s]$ is C^1-function, we have

$$\frac{df_x}{d\tilde{t}}(\tilde{t}) = \frac{Cf_x(\tilde{t})}{\tilde{t}}.$$

Therefore, we have the assertion. □

PROPOSITION 5.23. *Let r, s, δ, R, d_0 be positive numbers with $0 < r < s < R$, and u_∞, v_∞ harmonic functions on $B_{7R}(p)$. We assume that $\max_{r \le t \le s} U_{v_\infty}(t) \le d_0$, v_∞ is not a constant function on $B_R(p)$ and*

$$\left| \log \frac{U_{v_\infty}(s)}{U_{v_\infty}(r)} \right| < \delta.$$

Then, we have

$$\begin{aligned}
&\left| s_0^{1-n} \int_{\partial B_{s_0}(p)} u_\infty v_\infty d\upsilon - \exp\left(2 \int_{r_0}^{s_0} \frac{U_{v_\infty}(\hat{s})}{\hat{s}} d\hat{s} \right) r_0^{1-n} \int_{\partial B_{r_0}(p)} u_\infty v_\infty d\upsilon \right|^2 \\
&\le \Psi(\delta; n, d_0) \left(\frac{s_0}{r_0} \right)^{6d_0+3} I_{u_\infty}(s_0) I_{v_\infty}(s_0).
\end{aligned}$$

for every $r \le r_0 \le s_0 \le s$.

PROOF. By Theorem 5.9, there exists a sequence of harmonic functions u_i, v_i on $B^{g_M}_{6RR_i}(m)$ such that $\sup_i (\mathbf{Lip} u_i + \mathbf{Lip} v_i) < \infty$, $(u_i)_{R_i} \to u_\infty$, $(v_i)_{R_i} \to v_\infty$ on $B_{6R}(p)$. By the proof of Proposition 5.20 (or [19, Proposition 4.50]), there exists i_0 such that

$$\int_{rR_i \le b^{g_M} \le sR_i} (b^{g_M})^{-n} \left(b^{g_M} \frac{\partial v_i}{\partial n} - U^{g_M}_{v_i}(b^{g_M}) |\nabla^{g_M} b^{g_M}| \right)^2 d\mathrm{vol}^{g_M} \le \Psi(\delta; n, d_0) I^{g_M}_{v_i}(R_i s)$$

for every $i \ge i_0$. Thus, by [19, Corollary 5.24], we have

$$\begin{aligned}
&\left| (R_i s_0)^{1-n} \int_{b^{g_M} = R_i s_0} u_i v_i d\mathrm{vol}^{g_M}_{n-1} - \exp\left(2 \int_{r_0 R_i}^{s_0 R_i} \frac{U^{g_M}_{v_i}(\hat{s})}{\hat{s}} d\hat{s} \right) (R_i r_0)^{1-n} \int_{b^{g_M} = R_i r_0} u_i v_i d\mathrm{vol}^{g_M}_{n-1} \right|^2 \\
&\le \Psi(\delta; n, d_0) \left(\frac{s_0}{r_0} \right)^{6d_0+3} I^{g_M}_{u_i}(R_i s_0) I^{g_M}_{v_i}(R_i s_0)
\end{aligned}$$

for $i \ge i_0$. By rescaling $R_i^{-2} g_M$, we have

$$\begin{aligned}
&\left| s_0^{1-n} \int_{b^{R_i^{-2} g_M} = s_0} (u_i)_{R_i} (v_i)_{R_i} d\mathrm{vol}^{R_i^{-2} g_M}_{n-1} \right. \\
&\qquad \left. - \exp\left(2 \int_{r_0}^{s_0} \frac{U^{R_i^{-2} g_M}_{(v_i)_{R_i}}(\hat{s})}{\hat{s}} d\hat{s} \right) r_0^{1-n} \int_{b^{R_i^{-2} g_M} = r_0} (u_i)_{R_i} (v_i)_{R_i} d\mathrm{vol}^{R_i^{-2} g_M}_{n-1} \right|^2 \\
&\le \Psi(\delta; n, d_0) \left(\frac{s_0}{r_0} \right)^{6d_0+3} I^{R_i^{-2} g_M}_{(u_i)_{R_i}}(s_0) I^{R_i^{-2} g_M}_{(v_i)_{R_i}}(s_0).
\end{aligned}$$

On the other hand, by Proposition 5.6, we have

$$\begin{aligned}
&\int_{b^{R_i^{-2}g_M}=s_0} (u_i)_{R_i}(v_i)_{R_i} d\mathrm{vol}_{n-1}^{R_i^{-2}g_M} \\
&= \frac{1}{2}\int_{b^{R_i^{-2}g_M}=s_0} \left((u_i)_{R_i}+(v_i)_{R_i}\right)^2 d\mathrm{vol}_{n-1}^{R_i^{-2}g_M} - \frac{1}{2}\int_{b^{R_i^{-2}g_M}=s_0} (u_i)_{R_i}^2 d\mathrm{vol}_{n-1}^{R_i^{-2}g_M} \\
&\qquad - \frac{1}{2}\int_{b^{R_i^{-2}g_M}=s_0} (v_i)_{R_i}^2 d\mathrm{vol}_{n-1}^{R_i^{-2}g_M} \\
&\overset{i\to\infty}{\to} \frac{1}{2}\int_{\partial B_{s_0}(p)} (u_\infty+v_\infty)^2 dH^{n-1} - \frac{1}{2}\int_{\partial B_{s_0}(p)} u_\infty^2 dH^{n-1} - \frac{1}{2}\int_{\partial B_{s_0}(p)} v_\infty^2 dH^{n-1} \\
&= \int_{\partial B_{s_0}(p)} u_\infty v_\infty dH^{n-1}.
\end{aligned}$$

Therefore we have the assertion. □

The following corollary follows from Proposition 5.23 directly:

COROLLARY 5.24. *Let r, s, R be positive numbers with $0 < r < s < R$ and u_∞, v_∞ harmonic functions on $B_{7R}(p)$. We assume that $U_{v_\infty}(r) = U_{v_\infty}(s)$ and v_∞ is not a constant function on $B_R(p)$. Then, we have*

$$s_0^{1-n}\int_{\partial B_{s_0}(p)} u_\infty v_\infty dH^{n-1} = \left(\frac{s_0}{r_0}\right)^{2C} r_0^{1-n}\int_{\partial B_{r_0}(p)} u_\infty v_\infty dH^{n-1}$$

for every $r \le r_0 \le s_0 \le s$. Here $C = U_{v_\infty}(r)$.

Next proposition follows from Proposition 5.19 directly:

PROPOSITION 5.25. *For every non-constant harmonic function u_∞ on $C(X)$ with $u_\infty(p) = 0$, we have*

$$\mathrm{ord}_0 u_\infty \ge 1.$$

PROPOSITION 5.26. *With same assumption as in Lemma 5.6, for every $0 < r < s < R$, we have*

$$\lim_{i\to\infty}\int_r^s F_{(u_i)_{R_i}}^{R_i^{-2}g_M}(t)dt = \int_r^s F_{u_\infty}(t)dt.$$

PROOF. Since $(b^{R_i^{-2}g_M}, db^{R_i^{-2}g_M}) \to (r_p, dr_p)$ on $A_p(r,s)$, by Corollary 4.37, we have

$$\begin{aligned}
&\int_r^s F^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t)dt \\
&= \int_r^s t^{3-n} \int_{b^{R_i^{-2}g_M}=t} (R^{-2}g_M) \left(\nabla^{R_i^{-2}g_M}(u_i)_{R_i}, \frac{\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}}{|\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}|} \right)^2 |\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}| d\mathrm{vol}^{R_i^{-2}g_M} dt \\
&= \int_{r\le b^{R_i^{-2}g_M}\le s} (R^{-2}g_M) \left(\nabla^{R_i^{-2}g_M}(u_i)_{R_i}, \frac{\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}}{|\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}|} \right)^2 |\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}|^2 (b^{R_i^{-2}g_M})^{3-n} d\mathrm{vol}^{R_i^{-2}g_M} \\
&= \int_{r\le b^{R_i^{-2}g_M}\le s} (R^{-2}g_M)(\nabla^{R_i^{-2}g_M}(u_i)_{R_i}, \nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M})^2 (b^{R_i^{-2}g_M})^{3-n} d\mathrm{vol}^{R_i^{-2}g_M} \\
&\overset{i\to\infty}{\to} \int_{A_p(r,s)} r_p^{3-n} \langle du_\infty, dr_p\rangle^2 dv = \int_r^s F_{u_\infty}(t)dt.
\end{aligned}$$

□

PROPOSITION 5.27. *For every $0 < r < s < R$ and harmonic function u_∞ on $B_R(p)$, we have*

$$D_{u_\infty}(s) - D_{u_\infty}(r) = \int_r^s \frac{2F_{u_\infty}(t)}{t} dt.$$

PROOF. We can assume that the assumption of Proposition 5.6 holds. By (4.3) in [19], we have

$$\begin{aligned}
&E^{R_i^{-2}g_M}_{(u_i)_{R_i}}(s) - E^{R_i^{-2}g_M}_{(u_i)_{R_i}}(r) \\
&= \int_r^s \frac{2F^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t)}{t} dt + \int_r^s \frac{2E^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t)}{t} dt - \int_r^s t^{1-n} \int_{b^{R_i^{-2}g_M}\le t} 2|\nabla^{R_i^{-2}g_M}(u_i)_{R_i}|^2 d\mathrm{vol}^{R_i^{-2}g_M} dt \\
&\pm \int_r^s t^{1-n} \int_{b^{R_i^{-2}g_M}\le t} \Big| \mathrm{Hess}^{R_i^{-2}g_M}_{(b^{R_i^{-2}g_M})^2} \left(\nabla^{R_i^{-2}g_M}(u_i)_{R_i}, \nabla^{R_i^{-2}g_M}(u_i)_{R_i} \right) \\
&\qquad - 2(R_i^{-2}g_M) \left(\nabla^{R_i^{-2}g_M}(u_i)_{R_i}, \nabla^{R_i^{-2}g_M}(u_i)_{R_i} \right) \Big| d\mathrm{vol}^{R_i^{-2}g_M} dt.
\end{aligned}$$

By Corollary 4.37 and Theorem 5.1, we have

$$\lim_{i\to\infty} \int_{b^{R_i^{-2}g_M}\le t} |d(u_i)_{R_i}|^2 d\mathrm{vol}^{R_i^{-2}g_M} = \int_{B_t(p)} |du_\infty|^2 dH^n.$$

By dominated convergence theorem, we have

$$\begin{aligned}
\lim_{i\to\infty} \int_r^s t^{1-n} \int_{b^{R_i^{-2}g_M}\le t} 2|\nabla^{R_i^{-2}g_M}(u_i)_{R_i}|^2 d\mathrm{vol}^{R_i^{-2}g_M} dt &= \int_r^s t^{1-n} \int_{B_t(p)} 2|du_\infty|^2 dH^2 dt \\
&= \int_r^s \frac{2E_{u_\infty}(t)}{t} dt.
\end{aligned}$$

On the other hand, we recall

$$\lim_{R\to\infty}\frac{1}{\mathrm{vol}^{g_M}(\{b^{g_M}\le R\})}\int_{b^{g_M}\le R}|\mathrm{Hess}_{(b^{g_M})^2}-2g_M|d\mathrm{vol}^{g_M}=0.$$

Thus we have

$$\lim_{i\to\infty}\int_{b^{R_i^{-2}g_M}\le t}\Big|\mathrm{Hess}^{R_i^{-2}g_M}_{(b^{R_i^{-2}g_M})^2}\left(\nabla^{R_i^{-2}g_M}(u_i)_{R_i},\nabla^{R_i^{-2}g_M}(u_i)_{R_i}\right)$$
$$-2(R_i^{-2}g_M)\left(\nabla^{R_i^{-2}g_M}(u_i)_{R_i},\nabla^{R_i^{-2}g_M}(u_i)_{R_i}\right)\Big|d\mathrm{vol}^{R_i^{-2}g_M}dt=0.$$

Therefore we have the assertion. □

We shall give a short review of important works by Ding in [23] and [24]. By Corollary 3.58, X is H^{n-1}-rectifiable. By [24, Lemma 4.3], (X,H^{n-1}) satisfies weak Poincaré inequality of type $(1,2)$ locally. Thus, by section 4 in [5] (or section 6 in [9]) and Proposition 7.25, we can define the cotangent bundle T^*X of X. We denote the differential section of a Lipschitz function f on X by $d_Xf:X\to T^*X$. By [9, Theorem 6.25], there exists a unique self-adjoint operator Δ_X on $L^2(X)$ such that

$$\int_X\langle d_Xf,d_Xg\rangle dH=\int_X f\Delta_Xg dH^n$$

for every $f\in H_{1,2}(X)$ and $g\in\mathrm{Domain}(\Delta)$. For every i, we take a i-th eigenfunction ϕ_i on X and the i-th eigenvalue $\lambda_i\ge 0$, i.e. $\Delta_X\phi_i=\lambda_i\phi_i$ $(0=\lambda_0<\lambda_1\le\lambda_2\le\cdots)$. We define the nonnegative number α_i by satisfying $\lambda_i=\alpha_i(\alpha_i+n-2)$. According to [23], the function $v_i(r,x)=r^{\alpha_i}\phi_i(x)$ on $C(X)$ is a harmonic function on $C(X)$. Actually, by [24, Theorem 4.15], for every Lipschitz function $f\in\mathcal{K}(C(X)\setminus\{p\})$, we have

$$\int_{C(X)}\langle df,dv_i\rangle dH^n=\int_0^\infty\int_{\partial B_r(p)}\Big(-\alpha_i(\alpha_i-1)r^{\alpha_i-2}f\phi_i \tag{217}$$
$$-\frac{n-1}{r}\alpha_ir^{\alpha_i-1}+\frac{1}{r^2}\langle d_Xf,d_X\phi_i\rangle\Big)dH^{n-1}dr \tag{218}$$
$$=\int_0^\infty\int_{\partial B_r(p)}\Big(-\alpha_i(\alpha_i-1)r^{\alpha_i-2}f\phi_i \tag{219}$$
$$-(n-1)\alpha_ir^{\alpha_i-2}f\phi_i+\lambda_ir^{\alpha_i-2}f\phi_i\Big)dH^{n-1}dr \tag{220}$$
$$=0. \tag{221}$$

Thus, v_i is a harmonic function on $C(X)\setminus\{p\}$. Moreover, by [24, Corollary 4.25], v_i is a harmonic function on $C(X)$. By Theorem 5.9, v_i is locally Lipschitz. Especially, ϕ_i is Lipschitz. Therefore, we have $\lambda_1\ge n-1$ (see [23, Corollary 2.4] and [23, Corollary 2.5]). On the other hand, it is easy to check

$$U_{v_i}(s)=\alpha_i$$

for every $s > 0$. We say that the function v_i *is a homogeneous harmonic function with growth* α_i. We shall prove that we can apply [24, Theorem 4.15] for every $d \geq 0$ and $u_\infty \in \mathcal{H}^d(M_\infty)$ below. As a corollary, we will give the classification of harmonic functions with polynomial growth on asymptotic cones (see Theorem 5.34).

We put

$$\mathrm{ord}_\infty u_\infty = \lim_{r\to\infty} U_{u_\infty}(r),\ \mathrm{ord}_0 u_\infty = \lim_{r\to 0} U_{u_\infty}(r)$$

for every harmonic function u_∞ on $C(X)$. By an argument similar to the proof of [19, Lemma 1.36], we can prove the following proposition:

PROPOSITION 5.28. *For harmonic functions u_∞, v_∞ on $C(X)$, we have*

$$\mathrm{ord}_\infty(u_\infty + v_\infty) \leq \max\{\mathrm{ord}_\infty u_\infty, \mathrm{ord}_\infty v_\infty\}.$$

DEFINITION 5.29. For harmonic functions u_∞, v_∞ on $C(X)$, we say that u_∞ *and* v_∞ *are orthogonal* if

$$\int_{\partial B_1(p)} u_\infty v_\infty dv = 0.$$

PROPOSITION 5.30. *Let u_∞ be a harmonic function on $C(X)$. We assume that $\mathrm{ord}_\infty u_\infty = d < \infty$ and that v and u_∞ are orthogonal for every homogeneous harmonic function v with growth α satisfying $\alpha < d$. Then, we have*

$$D_{u_\infty}(s) \geq \left(\frac{s}{r}\right)^{2d} D_{u_\infty}(r)$$

for every $0 < r < s < \infty$.

PROOF. For every i, we take the i-th eigenvalue λ_i of Δ_X, a i-th eigenfunction ϕ_i of Δ_X, the nonnegative number α_i satisfying $\lambda_i = \alpha_i(\alpha_i+n-2)$ and a homogeneous harmonic function $v_i(t,x) = r^{\alpha_i}\phi_i(x)$ with growth α_i. By Corollary 5.24 anf the assumption, we have

$$\int_{\partial B_t(p)} v_i u_\infty dH^{n-1} = 0$$

for every $t > 0$ and $\alpha_i < d$. We put $\lambda = d(d+n-2)$. We remark that $\alpha_i < d$ holds if and only if $\lambda_i < \lambda$ holds. We put $i_d = \max\{i \in \mathbf{N} | \alpha_i < d\}$. Thus, we have $\lambda_{i_d} < \lambda \leq \lambda_{i_d+1}$. We also remark

$$\lambda_{i_d+1} = \inf\left\{ \frac{\int_X |d_X u|^2 dH^{n-1}}{\int_X u^2 dH^{n-1}} \middle| u \in H_{1,2}(X),\ u \neq 0,\ \int_X u\phi_j dH^{n-1} = 0 \text{ for every } 1 \leq j \leq i_d \right\}.$$

Since the k-th eigenvalue λ_k^t of $\Delta_{\partial B_t(p)}$ is equal to $t^{-2}\lambda_k$, we have

$$\frac{\int_{\partial B_t(p)} |d_{\partial B_t(p)} u_\infty|^2 dH^{n-1}}{\int_{\partial B_t(p)} (u_\infty)^2 dH^{n-1}} \geq \frac{\lambda}{t^2}.$$

Here $d_{\partial B_t(p)}f$ is differential section: $d_{\partial B_t(p)}f : \partial B_t(p) \to T^*\partial B_t(p)$ of a Lipschitz function f on $\partial B_t(p)$. On the other hand, by Theorem 3.22 and Proposition 7.22, for a.e. $t > 0$, we have $|du_\infty|^2(w) = (\langle dr_p, du_\infty\rangle(w))^2 + |d_{\partial B_r(p)}u_\infty|^2(w)$ for a.e. $w \in \partial B_t(p)$. Therefore, we have

$$\int_{\partial B_t(p)} (|du_\infty|^2 - \langle dr_p, du_\infty\rangle^2) dH^{n-1} \geq \frac{\lambda}{t^2} \int_{\partial B_t(p)} u_\infty^2 dH^{n-1}$$

i.e.

$$t^{3-n} \int_{\partial B_t(p)} |du_\infty|^2 dH^{n-1} - F_{u_\infty}(t) \geq \lambda I_{u_\infty}(t)$$

for a.e. $t > 0$. We shall use the notation: $f' = df/dt$ for locally Lipschitz functions f on $\mathbf{R}$ below. By Proposition 5.27, D_{u_∞} is locally Lipschitz function on $(0, \infty)$. By the definition of D_{u_∞}, Proposition 7.22 and Rademacher's theorem for Lipschitz functions on $\mathbf{R}$, we have

$$D'_{u_\infty}(t) = (2-n)t^{1-n} \int_{B_t(p)} (\mathrm{Lip} u_\infty)^2 dH^n + t^{2-n} \int_{\partial B_t(p)} (\mathrm{Lip} u_\infty)^2 dH^{n-1}$$

for a.e. $t > 0$. Therefore, we have

$$tD'_{u_\infty}(t) - (2-n)D_{u_\infty}(t) - F_{u_\infty}(t) \geq \lambda I_{u_\infty}(t)$$

for a.e. $t > 0$. On the other hand, by Proposition 5.27, we have $D'_{u_\infty}(t) = 2F_{u_\infty}(t)/t$ for every $t > 0$. Therefore, we have

$$\frac{t}{2} D'_{u_\infty}(t) - (2-n)D_{u_\infty}(t) \geq \lambda I_{u_\infty}(t)$$

for a.e. $t > 0$. Thus we have

$$\frac{D'_{u_\infty}(t)}{D_{u_\infty}(t)} - \frac{2(2-n)}{t} \geq \frac{2\lambda I_{u_\infty}(t)}{tD_{u_\infty}(t)} \geq \frac{2\lambda}{dt}$$

for a.e. $t > 0$. Therefore, we have

$$\frac{D'_{u_\infty}(t)}{D_{u_\infty}(t)} \geq \frac{1}{t}\left(\frac{2\lambda}{d} + 2(2-n)\right) \tag{222}$$

$$= \frac{1}{t}\frac{2\lambda + 4d - 2nd}{d} \tag{223}$$

$$= \frac{1}{t}\frac{2d(d+n-2) + 4d - 2nd}{d} \tag{224}$$

$$= \frac{2d}{t}. \tag{225}$$

for a.e $t > 0$. By integrating the inequality above, we have the assertion. □

PROPOSITION 5.31. *Let g be a Lipschitz function on X and f a C^2-function on $\mathbf{R}_{>0}$. We assume that $f(1) = 1$, $\lim_{r\to 0} f(r) = 0$, $g \neq 0$ and that function $u(r,x) = f(r)g(x)$ on $C(X) \setminus \{p\}$ is locally Lipschitz and harmonic. Then, there exists $\lambda \geq n-1$ such that $\Delta_X g = \lambda g$ and that $f(r) = r^p$. Here p is the nonnegative number satisfying $\lambda = p(p+n-2)$.*

PROOF. For every i, we take the i-th eigenvalue λ_i of Δ_X and a i-th eigenfunction ϕ_i of Δ_X. We put $g = \sum_{i=1}^{\infty} a_i\phi_i$ in $H_{1,2}(X)$. For every function h on X, we shall define a function h^r on $\partial B_r(p)$ by $h^r(r,x) = h(x)$. It is easy to check that $g^r = \sum_{i=1}^{\infty} a_i\phi_i^r$ in $H_{1,2}(\partial B_r(p))$. We remark that $\Delta_{\partial B_r(p)}\phi_i^r = \lambda_i^r\phi_i^r$ and $\lambda_i^r = r^{-2}\lambda_i$. By [24, Theorem 4.15] and Corollary 4.37, for every Lipschitz function $\phi \in \mathcal{K}(C(X)\setminus\{p\})$, we have

$$\begin{aligned}
0 &= \int_{C(X)} \langle du, d\phi\rangle dH^n \\
&= \int_0^{\infty}\int_{\partial B_r(p)} \left(\phi\left(-\frac{d^2f}{dr^2}(r)g(x) - \frac{n-1}{r}\frac{df}{dr}(r)g(x)\right) + \langle d_{\partial B_r(p)}\phi, d_{\partial B_r(p)}g^r\rangle f(r)\right) dH^{n-1}dr \\
&= \int_0^{\infty}\int_{\partial B_r(p)} \phi\left(-\frac{d^2f}{dr^2}(r)g(x) - \frac{n-1}{r}\frac{df}{dr}(r)g(x) + f(r)\sum_{i=1}^{\infty} a_i\lambda_i^r\phi_i^r\right) dH^{n-1}dr.
\end{aligned}$$

Especially, for every Lipschitz function $a \in \mathcal{K}(\mathbf{R}_{>0})$ and Lipschitz function b on X, we have

$$\int_0^{\infty} a(r)\int_{\partial B_r(p)} b(x)\left(-\frac{d^2f}{dr^2}(r)g(x) - \frac{n-1}{r}\frac{df}{dr}(r)g(x) + f(r)\sum_{i=1}^{\infty} a_i\lambda_i^r\phi_i^r\right) dH^{n-1}dr = 0.$$

Since

$$\sum_{i=1}^{\infty}(\lambda_i^r)^2 a_i^2 \int_{\partial B_r(p)} (\phi_i^r)^2 dH^{n-1} = \int_{\partial B_r(p)} |d_{\partial B_r(p)}g^r|^2 dH^{n-1} < \infty,$$

the function

$$-\frac{d^2f}{dr^2}(r)g(x) - \frac{n-1}{r}\frac{df}{dr}(r)g(x) + f(r)\sum_{i=1}^{\infty} a_i\lambda_i^r\phi_i^r$$

on $\partial B_r(p)$ is in $L^2(\partial B_r(p))$. Since the space which consist of Lipschitz functions on $\partial B_r(p)$ is dence in $L^2(\partial B_r(p))$, we have

$$\begin{aligned}
0 &= \int_0^{\infty} a(r)\int_{\partial B_r(p)} \left|-\frac{d^2f}{dr^2}(r)g(x) - \frac{n-1}{r}\frac{df}{dr}(r)g(x) + f(r)\sum_{i=1}^{\infty} a_i\lambda_i^r\phi_i^r\right|^2 dH^{n-1}dr \\
&= \int_0^{\infty} a(r)\int_{\partial B_r(p)} \left|-\frac{d^2f}{dr^2}(r)g(x) - \frac{n-1}{r}\frac{df}{dr}(r)g(x) + \frac{f(r)}{r^2}\sum_{i=1}^{\infty} a_i\lambda_i\phi_i(x)\right|^2 dH^{n-1}dr.
\end{aligned}$$

On the other hand, it is easy to check that the function (of r)

$$\int_{\partial B_r(p)} \left|-\frac{d^2f}{dr^2}(r)g(x) - \frac{n-1}{r}\frac{df}{dr}(r)g(x) + \frac{f(r)}{r^2}\sum_{i=1}^{\infty} a_i\lambda_i\phi_i(x)\right|^2 dH^{n-1}$$

is continuous. Therefore for every $r > 0$, there exists $A(r) \subset X$ such that $H^{n-1}(X \setminus A(r)) = 0$ and

$$-\frac{d^2f}{dr^2}(r)g(x) - \frac{n-1}{r}\frac{df}{dr}(r)g(x) + \frac{f(r)}{r^2}\sum_{i=1}^{\infty} a_i\lambda_i\phi_i(x) = 0$$

for every $x \in A(r)$. We put

$$\lambda = \frac{d^2f}{dr^2}(1) + (n-1)\frac{df}{dr}(1).$$

Then, for every Lipschitz function ϕ on X, we have

$$\int_X \lambda g\phi dH^{n-1} = \int_X \phi\sum_{i=1}^{\infty} a_i\lambda_i\phi_i dH^{n-1} = \int_X \langle d_X\phi, d_Xg\rangle dH^{n-1}.$$

Thus, g is a λ-eigenfunction. Therefore, by [23, Corollary 2.5], we have $\lambda \geq n-1$. For every $r > 0$, we have

$$\begin{aligned}
0 &= -\frac{d^2f}{dr^2}(r)\int_X g^2dH^{n-1} - \frac{n-1}{r}\frac{df}{dr}(r)\int_X g^2dH^{n-1} + \frac{f(r)}{r^2}\int_X g\sum_{i=1}^{\infty} a_i\lambda_i\phi_i(x)dH^{n-1} \\
&= -\frac{d^2f}{dr^2}(r)\int_X g^2dH^{n-1} - \frac{n-1}{r}\frac{df}{dr}(r)\int_X g^2dH^{n-1} + \frac{f(r)}{r^2}\int_X |d_Xg|^2dH^{n-1} \\
&= -\frac{d^2f}{dr^2}(r)\int_X g^2dH^{n-1} - \frac{n-1}{r}\frac{df}{dr}(r)\int_X g^2dH^{n-1} + \frac{f(r)}{r^2}\lambda\int_X g^2dH^{n-1}.
\end{aligned}$$

Thus, we have

$$-\frac{d^2f}{dr^2}(r) - \frac{n-1}{r}\frac{df}{dr}(r) + \frac{f(r)}{r^2}\lambda = 0.$$

Therefore, we have the assertion. □

Next corollary follows from Proposition 5.26 and Proposition 5.31 directly:

COROLLARY 5.32. *Let u_∞ be a nonconstant harmonic function on $C(X)$ with $u_\infty(p) = 0$. We assume that $\mathrm{ord}_0u_\infty = \mathrm{ord}_\infty u_\infty = d < \infty$. Then, the function $g(x) = u_\infty(1, x)$ on X is a $d(d+n-2)$-eigenfunction of Δ_X. Moreover, we have $u_\infty(r, x) = r^dg(x)$.*

COROLLARY 5.33. *Let u_∞ be a nonconstant harmonic function on $C(X)$. We assume that $u_\infty(p) = 0$, $\mathrm{ord}_\infty u = d < \infty$ and that v and u_∞ are orthogonal for every homogeneous harmonic function v with growth α satisfying $\alpha < d$. Then, the function $g(x) = u_\infty(1, x)$ on X is a $d(d+n-2)$-eigenfunction of Δ_X. Moreover we have $u_\infty(r, x) = r^dg(x)$.*

PROOF. We fix $0 < r < s < \infty$. By Proposition 5.14 we have $D_{u_\infty}(s)/D_{u_\infty}(r) \leq I_{u_\infty}(s)/I_{u_\infty}(r)$. By Proposition 5.17, we have

$$\frac{I_{u_\infty}(s)}{I_{u_\infty}(r)} \leq \left(\frac{s}{r}\right)^{2d}.$$

On the other hand, by the assumption and Proposition 5.30, we have

$$\frac{D_{u_\infty}(s)}{D_{u_\infty}(r)} \geq \left(\frac{s}{r}\right)^{2d}.$$

Therefore, we have $U_{u_\infty}(s) = U_{u_\infty}(r)$. By Corollary 5.32, we have the assertion. □

For every i, we denote the i-th eigenvalue of Δ_X by $\lambda_i(X)$, $(0 = \lambda_0(X) < \lambda_1(X) \leq \lambda_2(X) \leq \cdots)$. For $\lambda \geq 0$, we put $E_\lambda(X) = \mathrm{span}\{\phi_i; \Delta_X\phi_i = \lambda_i(X)\phi_i,\ \lambda_i \leq \lambda\}$. Then, by an argument similar to the proof of [19, Theorem 1.67], we have the following main theorem in this subsection.

THEOREM 5.34 (Harmonic functions with polynomial growth on asymptotic cones). *For every $d \geq 0$, we have*

$$\mathrm{dim}H^d(C(X)) = \mathrm{dim}E_{d(d+n-2)}(X).$$

Especially, we have $\mathrm{dim}H^d(C(X)) < \infty$.

5.2 Gromov-Hausdorff topology on moduli space of asymptotic cones.

In this subsection, we will study the moduli space of asymptotic cones of a fixed nonnegatively Ricci curved manifold M with Euclidean volume growth. In general, asymptotic cones of M are *not* unique. See [7] and [73] for such examples. Therefore, we shall consider the moduli space of them: $\mathcal{M}_M = \{X$: compact geodesic space ; $(C(X), p)$ is an asymptotic cone of M $\}$. We define a topology on $\mathcal{M}_M$ by Gromov-Hausdorff distance d_{GH}. On the other hand, if we put $\hat{\mathcal{M}}_M = \{(M_\infty, m_\infty)$: an asymptotic cone of $M\}$ and define a topology on $\hat{\mathcal{M}}_M$ by pointed Gromov-Hausdorff topology, then the canonical map $\pi(X) = (C(X), p)$ from $\mathcal{M}_M$ to $\hat{\mathcal{M}}_M$ give a homeomorphism. We remark that if a sequence of asymptotic cones (M^i_∞, m^i_∞) of M converges to some proper geodesic space $(M^\infty_\infty, m^\infty_\infty)$, then $(M^\infty_\infty, m^\infty_\infty)$ is also an asymptotic cone of M. Therefore, by Proposition 2.9, $\hat{\mathcal{M}}_M$ is compact, especially, $\mathcal{M}_M$ is compact. The main result in this subsection is the following theorem. We can regard it as "$\mathcal{M}_M$-version" of [30, (0.4) Theorem] by Fukaya or [9, Theorem 7.9] by Cheeger-Colding.

THEOREM 5.35. *If X_i converges to X_∞ in $\mathcal{M}_M$, then (X_i, H^{n-1}) converges to (X_∞, H^{n-1}). Moreover, we have*

$$\lim_{i\to\infty} \lambda_k(X_i) = \lambda_k(X_\infty)$$

foe every $k \geq 1$. Here, $\lambda_k(X)$ is the k-th eigenvalue of the Laplacian Δ_X on $X \in \mathcal{M}_M$.

PROOF. Let x_i be a point in X_i and x_∞ a point in X_∞ satisfying that $x_i \to x_\infty$. We take $r > 0$ and $\epsilon > 0$. We put $A^r_\epsilon(x_i) = \{(t,x) \in C(X_i); x \in B_r(x_i), 1-\epsilon \le t \le 1+\epsilon.\}$. By Proposition 4.13, we have

$$\lim_{i\to\infty} H^n(A^r_\epsilon(x_i)) = H^n(A^r_\epsilon(x_\infty)).$$

By Proposition 7.22, we have

$$H^n(A^r_\epsilon(x_i)) = \int_{1-\epsilon}^{1+\epsilon} H^{n-1}(\partial B_t(p_i) \cap A^r_\epsilon(x_i))dt = C(n)\epsilon H^{n-1}(B^{X_i}_r(x_i))$$

for every $1 \le i \le \infty$. Here, p_i is the pole of $C(X_i)$. Thus, we have $(X_i, H^{n-1}) \to (X_\infty, H^{n-1})$. We shall give a proof of second assertion by induction for k. We fix a subsequence $\{n(i)\}_i$ of $\mathbf{N}$. We take a Lipschitz function on $X_{n(i)}$ satisfying $f_1^{n(i)} \in E_{\lambda_1(X_{n(i)})}(X_{n(i)})$ and

$$\frac{1}{H^{n-1}(X_{n(i)})}\int_{X_{n(i)}} (f_1^{n(i)})^2 dH^{n-1} = 1.$$

By the definition, we have

$$\frac{1}{H^{n-1}(X_{n(i)})}\int_{X_{n(i)}} |df_1^{n(i)}|^2 dH^{n-1} = \lambda_1(X_{n(i)}).$$

We define a harmonic function $u_1^{n(i)}$ on $C(X_{n(i)})$ by $u_1^{n(i)}(r,x) = r^{\alpha_1^{n(i)}} f_1^{n(i)}(x)$. Here $\alpha_1^{n(i)}$ is the positive number satisfying $\lambda_1(X_{n(i)}) = \alpha_1^{n(i)}(\alpha_1^{n(i)} + n - 2)$. Since $\lambda_1(X_{n(i)}) \ge n-1$, we have $\alpha_1^{n(i)} \ge 1$. Then, by Proposition 3.22, we have

$$\begin{aligned}
&\int_{B_7(p_{n(i)})} (\mathrm{Lip} u_1^{n(i)})^2 dH^n \\
&= \int_0^7 \int_{\partial B_r(p_{n(i)})} (\alpha_1^{n(i)})^2 (r^{\alpha_1^{n(i)}-1})^2 (f_1^{n(i)})^2 dH^{n-1}dr + \int_0^7 \int_{\partial B_r(p_{n(i)})} r^{2\alpha_1^{n(i)}-2} |d_X f_1^{n(i)}|^2 dH^{n-1}dr \\
&= \int_0^7 (\alpha_1^{n(i)})^2 r^{2\alpha_1^{n(i)}-2} r^{n-1} H^{n-1}(X_{n(i)})dr + \int_0^7 r^{2\alpha_1^{n(i)}+n-1-2}\lambda_1(X_{n(i)})H^{n-1}(X_{n(i)})dr \\
&= H^{n-1}(X_{n(i)}) \left(\frac{7^{2\alpha_1^{n(i)}+n-2}(\alpha_1^{n(i)})^2}{2\alpha_1^{n(i)}+n-2} + \frac{7^{\alpha_1^{n(i)}+n}\lambda_1(X_{n(i)})}{2\alpha_1^{n(i)}+n-2} \right).
\end{aligned}$$

By Li-Schoen's gradient estimate (Theorem 5.44) and Theorem 5.9, we have

$$\mathbf{Lip}(u_1^{n(i)}|_{B_2(p_{n(i)})}) \le \frac{C(n)}{H^n(B_7(p_{n(i)}))}\int_{B_7(p_{n(i)})} (\mathrm{Lip} u_1^{n(i)})^2 dH^n.$$

On the other hand, by Claim 5.42, we have

$$\lambda_j(X_{n(i)}) \le C(n)\left(\frac{j}{H^{n-1}(X)}\right)^{\frac{2}{n-1}}$$

for every j. Thus, we have

$$\mathbf{Lip}(u_1^{n(i)}|_{B_2(p_{n(i)})}) \leq C(n, V_M).$$

By Proposition 2.11, there exist a subsequence of $\{m(i)\}_i$ of $\{n(i)\}_i$, a Lipschitz harmonic function u_1^∞ on $B_2(p_\infty)$, a Lipschitz function f_1^∞ on X_∞ and a nonnegative real number α_1^∞ such that $u_1^{m(i)} \to u_1^\infty$ on $B_2(p_\infty)$, $f_1^{m(i)} \to f_1^\infty$ on X_∞ and that $\alpha_1^{m(i)} \to \alpha_1^\infty$. Thus, we have $u_1^\infty(r,x) = r^{\alpha_1^\infty} f_1^\infty(x)$ on $B_2(p_\infty)$,

$$\lim_{i\to\infty} \int_{X_{m(i)}} (f_1^{m(i)})^2 dH^{n-1} = \int_{X_\infty} (f_1^\infty)^2 dH^{n-1}.$$

On the other hand, by Proposition 5.6 and Theorem 5.9, we have

$$\begin{aligned}
\lim_{i\to\infty} \int_{1-\epsilon}^1 t^{3-n} \int_{\partial B_t(p_{m(i)})} |d_{\partial B_t(p_{m(i)})} u_1^{m(i)}|^2 dH^{n-1} dt &= \lim_{i\to\infty} \left(\int_{1-\epsilon}^1 t D_{u_1^{m(i)}}(t) dt - \int_{1-\epsilon}^1 F_{u_1^{m(i)}}(t) dt \right) \\
&= \int_{1-\epsilon}^1 t D_{u_1^\infty}(t) dt - \int_{1-\epsilon}^1 F_{u_1^\infty}(t) dt \\
&= \int_{1-\epsilon}^1 t^{3-n} \int_{\partial B_t(p_\infty)} |d_{\partial B_t(p_\infty)} u_1^\infty|^2 dH^{n-1} dt
\end{aligned}$$

for every $0 < \epsilon < 1$. Since $|d_{\partial B_t(p_{m(i)})} u_1^{m(i)}|^2 = t^{2\alpha_1^{m(i)}-2} |d_{X(m(i))} u_1^{m(i)}|^2$, we have

$$\begin{aligned}
\int_{1-\epsilon}^1 t^{3-n} \int_{\partial B_t(p_{m(i)})} |d_{\partial B_t(p_{m(i)})} u_1^{m(i)}|^2 dH^{n-1} dt &= \int_{1-\epsilon}^1 t^{3-n} t^{2\alpha_1^{m(i)}-2} t^{n-1} \int_{X_{m(i)}} |d_{X_{m(i)}} u_1^{m(i)}|^2 dH^{n-1} dt \\
&= \int_{1-\epsilon}^1 t^{2\alpha_1^{m(i)}} \lambda_1(X_{m(i)}) H^{n-1}(X_{m(i)}) dt \\
&= \frac{1-(1-\epsilon)^{2\alpha_1^{m(i)}+1}}{2\alpha_1^{m(i)}+1} \lambda_1(X_{m(i)}) H^{n-1}(X_{m(i)}).
\end{aligned}$$

Similarly, we have

$$\int_{1-\epsilon}^1 t^{3-n} \int_{\partial B_t(p_\infty)} |d_{\partial B_t(p_\infty)} u_1^\infty|^2 dH^{n-1} dt = \frac{1-(1-\epsilon)^{2\alpha_1^\infty+1}}{2\alpha_1^\infty+1} \int_{X_\infty} |df_1^\infty|^2 dH^{n-1}.$$

Therefore, we have

$$\lim_{i\to\infty} \frac{1}{H^{n-1}(X_{m(i)})} \int_{X_{m(i)}} |df_1^{m(i)}|^2 dH^{n-1} = \lim_{i\to\infty} \lambda_1(X_{m(i)}) = \frac{1}{H^{n-1}(X_\infty)} \int_{X_\infty} |df_1^\infty|^2 dH^{n-1}.$$

Therefore, since $\{n(i)\}_i$ is arbitrary, we have

$$\liminf_{i\to\infty} \lambda_1(X_i) \geq \lambda_1(X_\infty).$$

On the other hand, by [9, Theorem 7.1], we have

$$\limsup_{i\to\infty} \lambda_1(X_i) \le \lambda_1(X_\infty).$$

Therefore we have

$$\lim_{i\to\infty} \lambda_1(X_i) = \lambda_1(X_\infty),$$

f_1^∞ is $\lambda_1(X_\infty)$-eigenfunction.

Next, we fix an integer $k \ge 2$. We assume that

$$\lim_{i\to\infty} \lambda_j(X_i) = \lambda_j(X_\infty)$$

for every $1 \le j \le k-1$ and that for every subsequence $\{n(i)\}_i$ of $\mathbf{N}$, there exist a subsequence $\{m(i)\}_i$ of $\{n(i)\}_i$, $\lambda_j(X_{m(i)})$-eigenfunction $f_j^{m(i)}$ on $X_{m(i)}$ and $\lambda_j(X_\infty)$-eigenfunction f_j^∞ on X_∞ such that $f_j^{m(i)} \to f_j^\infty$ on X_∞, $\mathbf{Lip}(f_j^{m(i)}|_{B_2(p_{m(i)})}) \le C(n, j, V_M)$ for every $1 \le j \le k-1$ and that

$$\frac{1}{H^{n-1}(X_{m(i)})} \int_{X_{m(i)}} f_l^{m(i)} f_j^{m(i)} dH^{n-1} = \delta_{jl}$$

for every $1 \le j \le l \le k-1$. Especially, $\{f_j^\infty\}_{1\le j\le k-1}$ are linearly independent in $L^2(X_\infty)$. We fix a subsequence $\{n(i)\}_i$ of $\mathbf{N}$ and take a subsequece $\{m(i)\}_i$ of $\{n(i)\}_i$ as above. We also take a $\lambda_k(X_{m(i)})$-eigenfunction $f_k^{m(i)}$ such that

$$\frac{1}{H^{n-1}(X_{m(i)})} \int_{X_{m(i)}} (f_k^{m(i)})^2 dH^{n-1} = 1.$$

We define a harmonic function $u_k^{m(i)}$ on $C(X_{m(i)})$ by $u_k^{m(i)}(r, x) = r^{\alpha_k^{m(i)}} f_k^{m(i)}(x)$. Here $\alpha_k^{m(i)}$ is the positive number satisfying $\alpha_k^{m(i)}(\alpha_k^{m(i)} + n - 2) = \lambda_k(X_{m(i)})$.

By Proposition 2.11 and an argument similar to one of the case $k = 1$, we can assume that there exist a locally Lipschitz harmonic function u_k^∞ on $C(X_\infty)$, a Lipschitz function f_k^∞ on X_∞ and a nonnegative number α_k^∞ such that $\mathbf{Lip}(u_k^{m(i)}|_{B_2(p_{m(i)})}) \le C(n, k, V_M)$, $\mathbf{Lip} f_k^{m(i)} \le C(n, k, V_M)$, $u_k^{m(i)} \to u_k^\infty$ on $C(X_\infty)$, $f_k^{m(i)} \to f_k^\infty$ on X_∞ and $\alpha_k^{m(i)} \to \alpha_k^\infty$. Thus, we have $u_k^\infty(r, x) = r^{\alpha_k^\infty} f_k^\infty(x)$. By an argument similar to one of the case $k = 1$, we have

$$\lim_{i\to\infty} \int_{X_{m(i)}} |df_k^{m(i)}|^2 dH^{n-1} = \int_{X_\infty} |df_k^\infty|^2 dH^{n-1}.$$

On the other hand, by Proposition 4.13,

$$\lim_{i\to\infty} \int_{X_{m(i)}} f_j^{m(i)} f_l^{m(i)} dH^{n-1} = \int_{X_\infty} f_j^\infty f_l^\infty dH^{n-1}.$$

for every $1 \leq j \leq l \leq k$. Thus, we have $f_k^\infty \in (\mathrm{span}\{f_1^\infty, \dots, f_{k-1}^\infty\})^\perp$ and $f_k^\infty \neq 0$. Therefore, by min-max principle, we have

$$\lambda_k(X_\infty) \leq \frac{\int_{X_\infty} |df_k^\infty|^2 dH^{n-1}}{\int_{X_\infty} (f_k^\infty)^2 dH^{n-1}}.$$

Since $\{n(i)\}_i$ is arbitrary, we have

$$\lambda_k(X_\infty) \leq \liminf_{i\to\infty} \lambda_k(X_i).$$

On the other hand, by [9, Theorem 7.1], we have

$$\limsup_{i\to\infty} \lambda_k(X_i) \leq \lambda_k(X_\infty).$$

Therefore, we have

$$\lim_{i\to\infty} \lambda_k(X_i) = \lambda_k(X_\infty),$$

f_k^∞ is a $\lambda_k(X_\infty)$-eigenfunction. Thus, by induction, we have the assertion. □

Remark 5.36. By the proof of Theorem 5.35, with same assumption as in Theorem 5.35, if a sequence of $\lambda_k(X_i)$-eigenfunction f_k^i on X_i converges to some Lipschitz function f_k^∞ on X_∞, then f_k^∞ is also a $\lambda_k(X_\infty)$-eigenfunction.

5.3 Asymptotic behavior of spaces of harmonic functions on asymptotic cones

In this subsection, we shall give a *Weyl type asymptotic formula* for harmonic functions on asymptotic cones of a fixed nonnegatively Ricci curved manifold M with Euclidean volume growth, as in [22] by Colding-Minicozzi. See [22, Theorem 0.26], [22, Proposition 6.1] and Corollary 5.47. On asymptotic cones of such manifolds, we can give a Weyl type *two-sided bound* asymptotic formula. See Theorem 5.43.

Proposition 5.37. *For every n-dimensional complete Riemannian manifold M with $\mathrm{Ric}_M \geq 0$ and $V_M > 0$, $(M_\infty, m_\infty) \in \hat{\mathcal{M}}_M$ and $d > 0$, we have $\dim H^d(M_\infty) \leq C(n) d^{n-1}$. Moreover, for every $V > 0$, there exists $d(V, n) > 1$ such that for every n-dimensional complete Riemannian manifold M with $\mathrm{Ric}_M \geq 0$ and $V_M \geq V$, $d > d(V, n)$ and $(M_\infty, m_\infty) \in \hat{\mathcal{M}}_M$, we have*

$$\dim H^d(M_\infty) \leq C(n) V_M d^{n-1}.$$

Proof. This follows from proofs of [22, Proposition 3.1], [22, Proposition 6.1] and Theorem 5.34. We shall introduce important ideas used in proofs of their propositions

and give an outline of a proof of our assertion only. We fix $V > 0$, an n-dimensional complete Riemannian manifold M with $\mathrm{Ric}_M \geq 0$ and $V_M \geq V$ and $(M_\infty, m_\infty) \in \hat{\mathcal{M}}_M$. There exists a compact geodesic space X such that $(M_\infty, m_\infty) = (C(X), p)$. We take $d_1 = d_1(n) \geq 1$ satisfying that $d(d+n-2) \leq 2d^2$ for every $d \geq d_1$. We take an i-th eigenfunction u_i of Δ_X and the i-th eigenvalue $\lambda_i(X)$ of Δ_X satisfying

$$\int_X u_i u_j dH^{n-1} = \delta_{ij}.$$

We put $N_d = \max\{l \in \mathbf{N}; \lambda_l(X) \leq d(d+n-2)\}$. Then, we have

$$\int_X |du_i|^2 dH^{n-1} = \lambda_i(X) \leq d(d+n-2)$$

for every $1 \leq i \leq N_d$. On the other hand, by the proof of [22, Proposition 6.1] (and Proposition 7.25), there exists $d_2 = d_2(n, V_M) \geq d_1$ such that for every $d \geq d_2$ and $\{x_i\}_{1 \leq i \leq l}$ which is a maximal $1/d$-separated subset of X, we have

$$l \leq C(n) V_M d^{n-1}.$$

We fix $C > 1$ and $d \geq d_2$. (We will decide C depending only on n later.) Let $\{x_j\}_{1 \leq j \leq l}$ be a maximal $1/(Cd)$-separated subset of X. We put $\mathcal{V} = \mathrm{span}\{u_i; 1 \leq i \leq N_d\}$. We define a linear map $\mathcal{M}$ from $\mathcal{V}$ to $\mathbf{R}^l$ by

$$\mathcal{M}(v) = \left(\int_{B_{2/Cd}(x_1)} v dH^{n-1}, \dots, \int_{B_{2/Cd}(x_l)} v dH^{n-1} \right).$$

We put $\mathcal{K} = \mathrm{Ker}\mathcal{M}$. Let $w_1, \dots, w_k$ be an $L^2(X)$-orthonormal basis of $\mathcal{K}$. We take $w_{k+1}, \dots, w_{N_d} \in \mathcal{V}$ satisfying that $\{w_i\}_{1 \leq i \leq N_d}$ are an $L^2(X)$-orthonormal basis of $\mathcal{V}$. By Poincaré inequality on X (see [24, Lemma 4.3]), we have

$$\int_{B_{2/Cd}(x_i)} w_j^2 dH^{n-1} \leq \frac{C(n)}{(Cd)^2} \int_{B_{2/Cd}(x_i)} |dw_j|^2 dH^{n-1}$$

for every $1 \leq j \leq k$ and $1 \leq i \leq l$. Therefore, we have

$$1 \leq \sum_{i=1}^{l} \int_{B_{2/Cd}(x_i)} w_j^2 dH^{n-1} \leq \frac{C(n)}{(Cd)^2} \sum_{i=1}^{l} \int_{B_{2/Cd}(x_i)} |dw_j|^2 dH^{n-1} \leq \frac{C(n)}{(Cd)^2} \int_X |dw_j|^2 dH^{n-1}$$

for $1 \leq j \leq k$. Thus we have

$$k \leq \frac{C(n)}{(Cd)^2} \sum_{j=1}^{k} \int_X |dw_j|^2 dH^{n-1} \leq \frac{C(n)}{(Cd)^2} \sum_{j=1}^{N_d} \int_X |dw_j|^2 dH^{n-1} \leq \frac{C(n)}{(Cd)^2} 2d^2 N_d \leq \frac{C(n)}{C^2} N_d.$$

We put $C = \sqrt{2C(n)}$ for $C(n)$ as above. Then we have $k \leq N_d/2$. Since $N_d = k + \dim(\mathrm{Image}\,\mathcal{M})$, we have $N_d \leq 2l \leq C(n) V_M d^{n-1}$. On the other hand, by Theorem 5.34, we have $\dim H^d(M_\infty) \leq N_d$. Therefore, we have the assertion. □

PROPOSITION 5.38. *For every $V > 0$, there exists $d(V,n) > 1$ such that*

$$\dim H^d(M_\infty) \geq C(n) V_M d^{n-1}$$

holds for every n-dimensional complete Riemannian manifold M with $\mathrm{Ric}_M \geq 0$ and $V_M \geq V$, $d > d(V,n)$ and $(M_\infty, m_\infty) \in \hat{\mathcal{M}}_M$.

PROOF. We fix $V > 0$, an n-dimensional complete Riemannian manifold M with $\mathrm{Ric}_M \geq 0$ and $V_M \geq V$, and $X \in \mathcal{M}_M$. First, we remark the following. This follows from Proposition 7.25, directly.

CLAIM 5.39. *Let ϵ be a positive number, k a positive integer and $\{x_i\}_{1 \leq i \leq k}$ points in X. We assume that $\{x_i\}_{1 \leq i \leq k}$ are an ϵ-separated subset of X. Then we have $k \leq C(n)/\epsilon^{n-1}$.*

We shall give an upper bound of the first eigenvalue for Dirichlet problem on each balls:

CLAIM 5.40. *We have*

$$\inf_{k \in \mathcal{K}(B_r(x)), k \neq 0} \frac{\int_{B_r(x)} |d_X k|^2 dH^{n-1}}{\int_{B_r(x)} k^2 dH^{n-1}} \leq \frac{C(n)}{r^2}$$

for every $x \in X$ and $0 < r \leq \pi$.

The proof is as follows. We define a Lipschitz function k on X by $k(w) = \max\{r/2 - \overline{x,w}, 0\}$. By the definition, we have $k \in \mathcal{K}(B_r(x))$,

$$\int_{B_r(x)} |d_X k|^2 dH^{n-1} = H^{n-1}(B_{\frac{r}{2}}(x))$$

and

$$\int_{B_r(x)} k^2 dH^{n-1} \geq \int_{B_{\frac{r}{4}(x)}(x)} k^2 dH^{n-1} \geq \int_{B_{\frac{r}{4}}(x)} \frac{r^2}{16} dH^{n-1} \geq \frac{r^2}{16} H^{n-1}(B_{\frac{r}{4}}(x)).$$

By Proposition 7.25, we have

$$\frac{\int_{B_r(x)} |dk|^2 dH^{n-1}}{\int_{B_r(x)} k^2 dH^{n-1}} \leq \frac{16}{r^2} \frac{H^{n-1}(B_r(x))}{H^{n-1}(B_{\frac{r}{4}}(x))} \leq \frac{C(n)}{r^2}.$$

Thus, we have Claim 5.40.

CLAIM 5.41. *We have*

$$\limsup_{r \to 0} \frac{H^{n-1}(B_r(x))}{r^{n-1}} \leq C(n)$$

for every $x \in X$.

The proof is as follows. For every sufficiently small $r > 0$, we put $A = \{(s, w) \in C(X); 1 - r \leq s \leq 1 + r, w \in B_r(x)\}$. By Proposition 7.22, we have

$$H^n(B_{5r}(1, x)) = \int_{1-r}^{1+r} H^{n-1}(\partial B_t(p) \cap B_{5r}(1, x))dt \tag{226}$$

$$\geq \int_{1-r}^{1+r} H^{n-1}(\partial B_t(p) \cap A)dt \tag{227}$$

$$\geq C(n)rH^{n-1}(B_r(x)). \tag{228}$$

Therefore, by Bishop-Gromov volume comparison theorem on limit spaces, we have Claim 5.41.

Claim 5.42. *We have*

$$\lambda_d(X) \leq C(n) \left(\frac{d}{H^{n-1}(X)} \right)^{\frac{2}{n-1}}$$

for every $d \geq 1$.

The proof is as follows. We fix $0 < C < 1$. (We will decide C depending only on n later.) We put

$$\epsilon = C \left(\frac{H^{n-1}(X)}{d} \right)^{\frac{1}{n-1}}$$

and take maximum ϵ-separated subset $\{x_i\}_{1 \leq i \leq k}$ of X. By Claim 5.39, we have $k \leq C(n)/\epsilon^{n-1} \leq C(n)d^{n-1}/(C^{n-1}H^{n-1}(X))$. On the other hand, we have

$$\sum_{i=1}^{k} H^{n-1}(B_{2\epsilon}(x_i)) \geq H^{n-1}(X).$$

By Claim 5.41 and Proposition 7.25, we have $H^{n-1}(B_{5\epsilon}(x_i)) \leq C(n)\epsilon^{n-1}$. Thus, we have

$$H^{n-1}(X) \leq \sum_{i=1}^{k} H^{n-1}(B_{2\epsilon}(x_i)) \leq kC(n)\epsilon^{n-1}.$$

Therefore, we have

$$k \geq \frac{C_1(n)H^{n-1}(X)}{\epsilon^{n-1}} = \frac{C_1(n)}{C^{n-1}} \frac{H^{n-1}(X)d}{H^{n-1}(X)} \geq \frac{C_1(n)}{C^{n-1}}d.$$

Here $C_1(n)$ is a sufficiently small positive constant depending only on n. We define C by $C = C_1(n)^{1/(n-1)}$. Then, we have $k \geq d$. By Claim 5.40, for every $1 \leq i \leq k$, there exists $\phi_i \in \mathcal{K}(B_{\epsilon/10}(x_i))$ such that $\phi_i \neq 0$ and

$$\frac{\int_{B_{\epsilon/10}(x_i)} |d\phi_i|^2 dH^{n-1}}{\int_{B_{\epsilon/10}(x_i)} (\phi_i)^2 dH^{n-1}} \leq \frac{C(n)}{\epsilon^2}.$$

Since $\{B_{\epsilon/10}(x_i)\}_i$ are pairwise disjoint, $\{\phi_i\}_i$ are linearly independent in $L^2(X)$. Then, for every $a_1,\dots,a_k \in \mathbf{R}$ satisfying $\sum_{i=1}^k (a_i)^2 \neq 0$, we have

$$\int_X \left| d\left(\sum_{i=1}^k a_i\phi_i\right)\right|^2 dH^{n-1} = \sum_{i=1}^k \int_X |d(a_i\phi_i)|^2 dH^{n-1} \tag{229}$$

$$\leq \sum_{i=1}^k \frac{C(n)}{\epsilon^2}\int_X (a_i\phi_i)^2 dH^{n-1} \tag{230}$$

$$= \frac{C(n)}{\epsilon^2}\int_X \left|\sum_{i=1}^k a_i\phi_i\right|^2 dH^{n-1}. \tag{231}$$

Thus, by min-max principle, we have $\lambda_k(X) \leq C(n)/\epsilon^2$. Therefore, we have

$$\lambda_d(X) \leq \lambda_k(X) \leq \frac{C(n)}{\epsilon^2} \leq C(n)\left(\frac{d}{H^{n-1}(X)}\right)^{\frac{2}{n-1}}.$$

Thus, we have Claim 5.42.

The assertion follows from Claim 5.42 and Theorem 5.34. □

The main result in this subsection is the following:

THEOREM 5.43 (Weyl type asymptotic formula on asymptotic cones). *For every $V > 0$, there exists $d(n,V) \geq 1$ such that*

$$C(n)^{-1}V_M d^{n-1} \leq \dim H^d(M_\infty) \leq C(n)V_M d^{n-1}$$

holds for every n-dimensional complete Riemannian manifold M with $\mathrm{Ric}_M \geq 0$ and $V_M \geq V$, $d \geq d(n,V)$ and $(M_\infty, m_\infty) \in \hat{\mathcal{M}}_M$.

PROOF. It follows from Proposition 5.37 and 5.38 directly. □

5.4 A dimension comparison theorem and Liouville type theorem

In this subsection, we shall give a comparison theorem for dimensions between a space of harmonic functions on a fixed nonnegatively Ricci curved manifold with Euclidean volume growth, and one on an asymptotic cone of the manifold (Theorem 5.45 below). Essential tools to prove it are [18, Lemma 3.1] (or [19, Lemma 7.1]) and several properties of frequency functions on asymptotic cones given in section 5. As a corollary, we will give a Liouville type theorem on the manifold. See Corollary 5.48. First, we shall introduce an important mean value inequality for subharmonic functions on nonnegatively Ricci curved manifolds by Li-Schoen:

THEOREM 5.44 (Li-Schoen, [61]). *Let M be a complete n-dimensional Riemannian manifold with $\mathrm{Ric}_M \geq 0$, m a point in M and R a positive number. Then for every nonnegative subharmonic function f on $B_{3R/2}(m)$, we have*

$$\sup_{B_R(m)} f \leq \frac{C(n)}{\mathrm{vol}\, B_{\frac{3R}{2}}(m)} \int_{B_{\frac{3R}{2}}(m)} f d\mathrm{vol}.$$

We remark that if $\mathrm{Ric}_M \geq 0$, then, by Bochner's formula, for every harmonic function h on $B_R(m)$, we have, $|\nabla h|^2$ is a subharmonic function. We fix an n-dimensional complete Riemannian manifold M with $\mathrm{Ric}_M \geq 0$ and $V_M > 0$ below.

THEOREM 5.45. *For every $d \geq 0$, $\epsilon > 0$ and nonnegative integers $k \leq \dim H^d(M) - 1$ and $0 \leq l \leq k$, , there exists $(M_\infty, m_\infty) \in \hat{\mathcal{M}}_M$ such that*

$$l \leq \dim H^{\frac{k}{k-l+1}\left(d-1+\frac{n}{2}\right)+1-\frac{n}{2}+\epsilon}(M_\infty) - 1.$$

PROOF. Without loss of generality, we can assume that $k \geq 1$ and $l \geq 1$. We take linearly independent harmonic functions $u_1, u_2, \ldots, u_k \in H^d(M)$ satisfying $u_i(m) = 0$. We put

$$J_r(u_i, u_j) = \int_{b^{g_M} \leq r} \langle du_i, du_j \rangle d\mathrm{vol}^{g_M}$$

for every $r > 0$. We define $u_i = \sum_{j=1}^{i-1} \lambda_{ji}(r) u_j + w_{i,r}$ by satisfying $J_r(w_{i,r}, w_{j,r}) = 0$ for $i \neq j$ and put

$$f_i(r) = \int_{b^{g_M} \leq r} |dw_{i,r}|^2 d\mathrm{vol}^{g_M}.$$

CLAIM 5.46. *We have the following:*

1. *There exists $K > 0$ such that $f_i(r) \leq K(r^{2d-2+n} + 1)$ for every $i = 1, \ldots, k$ and $r > 0$.*
2. *$f_i > 0$.*
3. *$f_i(r) \leq f_i(s)$ for $r \leq s$.*
4. *f_i is a barrier for $t^{n-2} D^{g_M}_{w_{i,s}}(t)$ at every $s > 0$. Here, for functions g, h on $\mathbf{R}$ and a real number $r \in \mathbf{R}$, we say that f is a barrier for g at r if $f(r) = g(r)$ and $f(s) \leq g(s)$ for $s < r$. (see also [18, Definition 4.6]).*

By the trivial monotonicity of $t^{n-2} D^{g_M}_u(t)$ and an argument similar to the proof of [19, Proposition 8.6] (or [18, Proposition 4.7]), we have Claim 5.46.

We put $\lambda = \frac{k}{k-l+1}$. By [18, Lemma 3.1], for every $N \in \mathbf{N}_{\geq 2}$, there exist a subsequence $\{m(N,i)\}_{i \in \mathbf{N}}$ of $\mathbf{N}$ and a pairwise distinct integers $\alpha_1^N, \ldots, \alpha_l^N \in \{1, \ldots, k\}$ such that

$f_j(N^{m(N,i)+1}) \le 2N^{\lambda(2d-2+n)} f_j(N^{m(N,i)})$ for every $j \in \{\alpha_1^N, \dots, \alpha_k^N\}$ and $i \in \mathbf{N}$. By Claim 5.46, we have

$$\frac{f_j(N^{m(N,i)+1})}{f_j(N^{m(N,i)})} \ge \frac{(N^{m(N,i)+1})^{n-2} D^{g_M}_{w_{j,N^{m(N,i)+1}}}(N^{m(N,i)+1})}{(N^{m(N,i)})^{n-2} D^{g_M}_{w_{j,N^{m(N,i)+1}}}(N^{m(N,i)})}.$$

Thus, we have

$$\frac{D^{g_M}_{w_{j,N^{m(N,i)+1}}}(N^{m(N,i)+1})}{D^{g_M}_{w_{j,N^{m(N,i)+1}}}(N^{m(N,i)})} \le 2N^{\lambda(2d-2+n)+2-n}.$$

We define a harmonic function $w_j^{N,i}$ on $B^{(N^{m(N,i)})^{-2}g_M}_{N/10}(m)$ by

(232)

$$w_j^{N,i}(w) = w_{j,N^{m(N,i)+1}}$$

(233)
$$\times \left(N^{m(N,i)} \sqrt{\frac{1}{\mathrm{vol}^{g_M}(\{b^{g_M} \le N^{m(N,i)}\})} \int_{b^{g_M} \le N^{m(N,i)}} |dw_{j,N^{m(N,i)+1}}|^2 d\mathrm{vol}^{g_M}} \right)^{-1}.$$

We assume that N is sufficiently large below. Then, for $x_1, x_2 \in B^{(N^{m(N,i)})^{-2}g_M}_{N/10}(m)$, by Li-Schoen's gradient estimate, we have

(234)

$$|w_j^{N,i}(x_1) - w_j^{N,i}(x_2)|$$

(235)
$$\le \sup_{B_{N^{m(N,i)}\frac{N}{5}}(m)} |\nabla w_{j,N^{m(n,i)+1}}| \overline{x_1, x_2}^{g_M}$$

(236)
$$\times \left(N^{m(N,i)} \sqrt{\frac{1}{\mathrm{vol}^{g_M}(\{b^{g_M} \le N^{m(N,i)}\})} \int_{b^{g_M} \le N^{m(N,i)}} |dw_{j,N^{m(N,i)+1}}|^2 d\mathrm{vol}^{g_M}} \right)^{-1}$$

(237)
$$\le C(n) \sqrt{\frac{1}{\mathrm{vol}^{g_M}(\{b^{g_M} \le N^{m(N,i)}\frac{2N}{3}\})} \int_{b^{g_M} \le N^{m(N,i)}\frac{2N}{3}} |dw_{j,N^{m(N,i)+1}}|^2 d\mathrm{vol}^{g_M}}$$

(238)
$$\times \left(\sqrt{\frac{1}{\mathrm{vol}^{g_M}(\{b^{g_M} \le N^{m(N,i)}\})} \int_{b^{g_M} \le N^{m(N,i)}} |dw_{j,N^{m(N,i)+1}}|^2 d\mathrm{vol}^{g_M}} \right)^{-1} \times \overline{x_1, x_2}^{(N^{m(N,i)})^{-2}g_M}$$

(239)
$$\le C(n) N^{\lambda(d-1+n/2)+1-n/2} \overline{x_1, x_2}^{(N^{m(N,i)})^{-2}g_M}.$$

By Proposition 2.11 and compactness of $\mathcal{M}_M$, without loss of generality, we can assume that there exist $X_N \in \mathcal{M}_M$ and Lipschitz functions $w_j^{N,\infty}$ on $B_{N/10}(p_N)$ such that $(M, m, (N^{m(N,i)})^{-1}d_M, w_j^{N,i}) \to (C(X_N), p_N, w_j^{N,\infty})$ $(j \in \{\alpha_1, \cdots, \alpha_l\})$. On the other hand,

$$\frac{1}{\mathrm{vol}^{(N^{m(N,i)})^{-2}g_M} B_1^{(N^{m(N,i)})^{-2}g_M}(m)} \tag{240}$$

$$\times \int_{B_1^{(N^{m(N,i)})^{-2}g_M}(m)} |d^{(N^{m(N,i)})^{-2}g_M} w_j^{N,i}|^2 d\mathrm{vol}^{(N^{m(N,i)})^{-2}g_M} \tag{241}$$

$$= \frac{1}{\mathrm{vol}^{g_M} B_{N^{m(N,i)}}(m)} \int_{B_{N^{m(N,i)}}(m)} |dw_{j,N^{m(N,i)+1}}|^2 (N^{m(N,i)})^2 d\mathrm{vol}^{g_M} \tag{242}$$

$$\times \left(N^{2m(N,i)} \frac{1}{\mathrm{vol}^{g_M}(\{b^{g_M} \le N^{m(N,i)}\})} \int_{b^{g_M} \le N^{m(N,i)}} |dw_{j,N^{m(N,i)+1}}|^2 d\mathrm{vol}^{g_M} \right)^{-1} \tag{243}$$

$$= 1 \pm \Psi(i^{-1}; n, N). \tag{244}$$

By Corollary 4.37 and Theorem 5.1, we have

$$\frac{1}{H^n(B_1(p_N))} \int_{B_1(p_N)} |dw_j^{N,\infty}|^2 dH^n = 1.$$

Similarly, we have

$$\int_{B_1(p_N)} \langle dw_i^{N,\infty}, dw_j^{N,\infty} \rangle dH^n = 0$$

for $i \neq j$. Therefore, $\{w_j^{N,\infty}\}_j$ are linearly independent harmonic functions. For convenience, we shall change the notation: $\{\alpha_1^N, \cdots, \alpha_l^N\} = \{1, \cdots, l\}$. By Proposition 5.14, we have

$$\frac{I_{w_j^{N,\infty}}(\frac{N}{100})}{I_{w_j^{N,\infty}}(1)} = \frac{U_{w_j^{N,\infty}}(1)}{U_{w_j^{N,\infty}}(\frac{N}{100})} \frac{D_{w_j^{N,\infty}}(\frac{N}{100})}{D_{w_j^{N,\infty}}(1)} \le \frac{D_{w_j^{N,\infty}}(\frac{N}{100})}{D_{w_j^{N,\infty}}(1)} \le 2N^{\lambda(2d-2+n)+2-n}.$$

Thus, by Proposition 5.11, we have

$$\exp \int_1^{\frac{N}{100}} 2 \frac{U_{w_j^{N,\infty}}(t)}{t} dt \le 2N^{\lambda(2d-2+n)+2-n}.$$

We take $1 \le \hat{l} < N/100$. Since

$$\exp \int_{\hat{l}}^{\frac{N}{100}} 2 \frac{U_{w_j^{N,\infty}}(t)}{t} dt \le 2N^{\lambda(2d-2+n)+2-n},$$

by Proposition 5.14, we have

$$\left(\frac{N}{100\hat{l}} \right)^{2U_{w_j^{N,\infty}}(\hat{l})} \le 2N^{\lambda(2d-2+n)+2-n}$$

i.e.

$$2U_{w_j^{N,\infty}}(\hat{l}) \le \frac{\log N}{\log N - \log(100\hat{l})} + \frac{\log N}{\log N - \log(100\hat{l})}(\lambda(2d-2+n)+2-n).$$

Therefore, for every $\hat{l} \ge 1$, there exists $N_{\hat{l}}$ such that $U_{w_j^{N,\infty}}(a) \le \lambda(d-1+n/2)+1-n/2+\epsilon$ for every $N \ge N_{\hat{l}}$ and $1 \le a \le \hat{l}$. We take $x_1 \in B_{\frac{\hat{l}}{10}}(p_N)$. By Li-Schoen's gradient estimate and Theorem 5.9, we have

$$\mathrm{Lip} w_j^{N,\infty}(x_1) \le C(n)\sqrt{\frac{1}{H^n(B_{\hat{l}}(p_N))}\int_{B_{\hat{l}}(p_N)} (\mathrm{Lip} w_j^{N,\infty})^2 dH^n} \tag{245}$$

$$\le C(n, V_M)\sqrt{\hat{l}^{-n}\int_{B_{\hat{l}}(p_N)} |dw_j^{N,\infty}|^2 dH^n} \tag{246}$$

$$\le C(n, V_M, \lambda, d)\sqrt{\hat{l}^{-1-n}\int_{\partial B_{\hat{l}}(p_N)} |w_j^{N,\infty}|^2 dH^n} \tag{247}$$

$$\le C(n, V_M, \lambda, d)\hat{l}^{-1}\sqrt{\frac{1}{H^{n-1}(\partial B_{\hat{l}}(p_N))}\int_{\partial B_{\hat{l}}(p_N)} |w_j^{N,\infty}|^2 dH^n}. \tag{248}$$

On the other hand, by Proposition 5.14, we have

$$I_{w_j^{N,\infty}}(\hat{l}) = \exp\left(\int_1^{\hat{l}} \frac{2U_{w_j^{N,\infty}}(t)}{t} dt\right) I_{w_j^{N,\infty}}(1) \tag{249}$$

$$\le \exp\left(\int_1^{\hat{l}} \frac{\lambda(2d-2+n)+2-n+2\epsilon}{t} dt\right) I_{w_j^{N,\infty}}(1) \tag{250}$$

$$\le \hat{l}^{\lambda(2d-2+n)+2-n+2\epsilon} I_{w_j^{N,\infty}}(1) \tag{251}$$

for $N \ge N_l$. By Proposition 5.19, we have

$$0 \le I_{w_j^{N,\infty}}(1) \le I_{w_j^{N,\infty}}(1) U_{w_j^{N,\infty}}(1) \le D_{w_j^{N,\infty}}(1) = 1.$$

Thus, we have $I_{w_j^{N,\infty}}(\hat{l}) \le \hat{l}^{\lambda(2d-2+n)+2-n+2\epsilon}$. Therefore, we have

$$\mathbf{Lip}\left(w_j^{N,\infty}|_{B_{\frac{\hat{l}}{10}}(p_N)}\right) \le C(n, V_M, \lambda, d)\hat{l}^{\lambda(d-1+n/2)-n/2+\epsilon}.$$

By Proposition 2.11 and compactness of $\mathcal{M}_M$, we can assume that there exist $X_\infty \in \mathcal{M}_M$ and locally Lipschitz harmonic functions $w_j^\infty \in H^{\lambda(d-1+n/2)+1-n/2+\epsilon}(C(X_\infty))$ such that $X_N \to X_\infty$ and that $w_j^{N,\infty} \to w_j^\infty$ on $B_R(p_\infty)$ for every $R > 0$. By Corollary 4.37, we have

$$\frac{1}{H^n(B_1(p_\infty))}\int_{B_1(p_\infty)} \langle dw_j^\infty, dw_i^\infty\rangle dH^n = \delta_{ij}.$$

Especially, $\{w_j^\infty\}_j$ are linearly independent nonconstant harmonic functions. Therefore we have the assertion. □

As a corollary of Theorem 5.45, we have the following result by Colding-Minicozzi:

COROLLARY 5.47 (Colding-Minicozzi, [22]). *For every $V > 0$, there exists $d(V,n) > 1$ such that*

$$\dim H^d(M) \leq C(n)V_M d^{n-1}$$

for every $d > d(V,n)$ and n-dimensional complete Riemannian manifold M with $\mathrm{Ric}_M \geq 0$ and $V_M \geq V$.

PROOF. By taking $k = [(\dim H^d(M) - 1)/2]$ as in Theorem 5.45, the assertion follows from Theorem 5.43 and Theorem 5.45 directly. Here $[a] = \inf\{l \in \mathbf{Z}; a \leq l\}$ for every $a \in \mathbf{R}$. □

We put $\lambda_1 = \inf\{\lambda_1(X); X \in \mathcal{M}_M\}$ and define $d_1 \geq 1$ by

$$d_1 = \frac{-(n-1) + \sqrt{(n-2)^2 + 4\lambda_1}}{2}.$$

By Theorem 5.35, we have the following:

1. $H^d(M_\infty) = \{\text{Constant functions}\}$ for every $(M_\infty, m_\infty) \in \hat{\mathcal{M}}_M$ and $0 < d < d_1$.
2. $H^{d_1}(\hat{M}_\infty) \neq \{\text{Constant functions}\}$ for some $(\hat{M}_\infty, \hat{m}_\infty) \in \hat{\mathcal{M}}_M$.

COROLLARY 5.48 (Liouville type theorem). *We have $H^d(M) = \{$Constant functions$\}$ for every $0 < d < d_1$.*

PROOF. We assume that the assertion is false. We take $\epsilon > 0$ satisfying $\epsilon < d_1 - d$. By taking $k = l = 1$ as in Theorem 5.45, there exists $(M_\infty, m_\infty) \in \hat{\mathcal{M}}_M$ such that $2 \leq \dim H^{d+\epsilon}(M_\infty)$. This is a contradiction. □

Finally, we end this subsection by showing the following. See also [20, Conjecture 0.9].

COROLLARY 5.49. *Let d be a positive number and $u \in H^d(M)$. Then we have*

$$\liminf_{t\to\infty} \left(\sup_{s\in K} U_u^{g_M}(ts) \right) \leq d$$

for every compact set $K \subset (0,\infty)$.

PROOF. Assume that u is not a constant. By the proof of Theorem 5.45, for every $\epsilon > 0$, there exist sequences of positive numbers $\{R_i\}_i$, $\{\hat{R}_i\}_i$, an asymptotic cone $(M_\infty, m_\infty) \in \hat{\mathcal{M}}_M$ and a nonconstant harmonic function $u_\infty \in H^{d+\epsilon}(M_\infty)$ such that $R_i \to \infty$, $\hat{R}_i \to \infty$, $(M, m, R_i^{-1} d_M) \to (M_\infty, m_\infty)$, $\sup_i \mathbf{Lip}^{R_i^{-1}d_M}\left((u)_{\hat{R}_i}|_{B_R^{R_i^{-1}d_M}(m_i)} \right) < \infty$ for every $R > 0$

and that $(u)_{\hat{R}_i}(x_i) \to u_\infty(x_\infty)$ for every sequence $x_i \to x_\infty$ with respect to the convergence $(M, m, R_i^{-1}d_M) \to (M_\infty, m_\infty)$. By the definition of $U_u^{g_M}(t)$, we have $U_{(u)_{\hat{R}_i}}^{R_i^{-2}g_M}(s) = U_u^{R_i^{-2}g_M}(s) = U_u^{g_M}(R_i s)$ for every $s > 0$. Thus, since $\lim_{i\to\infty}\left(\sup_{s\in K}|U_{(u)_{\hat{R}_i}}^{R_i^{-2}g_M}(s) - U_{u_\infty}(s)|\right) = 0$ and $U_{u_\infty} \le d + \epsilon$, we have $\liminf_{t\to\infty}(\sup_{s\in K} U_u^{g_M}(ts)) \le d + \epsilon$. Therefore, we have the assertion.

□

6 Stability of lower bounds on Ricci curvature via Laplacian comparison theorem

In this section, as an application of Theorem 4.27, we shall establish Laplacian comparison theorem on Ricci limit spaces. For $H \in \mathbf{R}$, we define a smooth function $\underline{k}_H$ on $\mathbf{R}$ by

$$\underline{k}_H''(r) + H\underline{k}_H''(r) = 0, \quad \underline{k}(0) = 0, \quad \underline{k}_H'(0) = 1.$$

Here $f' = df/dr$ for every differentiable function f on $\mathbf{R}$. We remark the following:

1. (Laplacian comparison theorem on manifolds). For every n-dimensional complete Riemannian manifold M with $\mathrm{Ric}_M \ge H(n-1)$ and point $p \in M$, we have

$$\Delta r_p(x) \ge -(n-1)\frac{\underline{k}_H'(\overline{p,x})}{\underline{k}_H(\overline{p,x})}$$

for every $x \in M \setminus (C_p \cup \{p\})$.

2. For the n-dimensional space form $\underline{M}_H^n$ whose sectional curvature is equal to H and every point $\underline{p} \in \underline{M}_H$, we have

$$\Delta r_{\underline{p}}(x) = -(n-1)\frac{\underline{k}_H'(\overline{p,x})}{\underline{k}_H(\overline{p,x})}$$

for every $x \in \underline{M}_H \setminus (C_{\underline{p}} \cup \{\underline{p}\})$.

3. If an n-dimensional complete Riemannian manifold M satisfies that

$$\Delta r_p(x) \ge -(n-1)\frac{\underline{k}_H'(\overline{p,x})}{\underline{k}_H(\overline{p,x})}$$

for every $p \in M$ and $x \in M \setminus (C_p \cup \{p\})$, then we have $\mathrm{Ric}_M \ge H(n-1)$.

See for instance [4], [7], [53], [72] and [93]. The following theorem is the main result in this subsection. This formulation is given in [53] by Kuwae-Shioya on weighted Alexandrov spaces.

THEOREM 6.1 (Laplacian comparison theorem). *Let H be a real number, (Y, y, υ) a (n, H)-Ricci limit space $(n \geq 2)$, x a point in Y and R a positive number and f a nonnegative valued Lipschitz function on $B_R(x)$. Then, we have*

$$\int_{B_R(x)} \langle df, dr_x \rangle d\upsilon \geq -(n-1) \int_{B_R(x)} \frac{\underline{k}'_H(\overline{x,w})}{\underline{k}_H(\overline{x,w})} f(w) d\upsilon.$$

PROOF. Let $(M_i, m_i, \underline{\mathrm{vol}}) \to (Y, y, \upsilon)$ with $\mathrm{Ric}_{M_i} \geq H_i(n-1)$ satisfying $H_i \to H$. We take $L \geq 1$ and $x(j) \in M_j$ satisfying $|f|_{L^\infty(B_R(x))} + \mathbf{Lip} f + \upsilon(B_R(x)) \leq L$ and $x(j) \to x$. First, we assume that $\mathrm{supp} f \cap (\{x\} \cup \partial B_{\pi/\sqrt{H}}(x)) = \emptyset$. Here, if $H \leq 0$, then $\partial B_{\pi/\sqrt{H}}(x)) = \emptyset$. Then there exists $\tau > 0$ such that $\mathrm{supp} f \cap \overline{B}_\tau \left(\{x\} \cup \partial B_{\pi/\sqrt{H}}(x) \right) = \emptyset$. By Theorem 4.27, for every $\epsilon > 0$, there exist an open set $\Omega^\epsilon \subset B_R(x) \setminus \overline{B}_\tau \left(\{x\} \cup \partial B_{\pi/\sqrt{H}}(x) \right)$, $2L$-Lipschitz function f^ϵ on $B_R(x)$ and a sequence of $2L$-Lipschitz function f_i^ϵ on $B_R(x_i)$ such that $\mathrm{supp} f^\epsilon \cap \overline{B}_\tau \left(\{x\} \cup \partial B_{\pi/\sqrt{H}}(x) \right) = \emptyset$, $\mathrm{supp} f_i^\epsilon \cap \overline{B}_\tau \left(\{x(i)\} \cup \partial B_{\pi/\sqrt{H}}(x(i)) \right) = \emptyset$, $(f_i^\epsilon, df_i^\epsilon) \to (f^\epsilon, df^\epsilon)$ on Ω^ϵ and

$$\frac{\upsilon \left(\Omega^\epsilon \cup \overline{B}_\tau \left(\{x\} \cup \partial B_{\pi/\sqrt{H}}(x) \right) \right)}{\upsilon(B_R(x))} + |f - f^\epsilon|_{L^\infty(B_R(x))} + \frac{1}{\upsilon(B_R(x))} \int_{B_R(x)} |df - df^\epsilon|^2 d\upsilon < \epsilon.$$

By Proposition 2.12, we can assume that there exists a finite pairwise disjoint collection $\{\overline{B}_{r_i}(x_i)\}_{1 \leq i \leq N}$ such that $\Omega^\epsilon = \bigcup_{i=1}^N B_{r_i}(x_i)$. We take $x_i(j) \in M_j$ satisfying $x_i(j) \to x_i$. Then, by Proposition 4.13, we have

$$\begin{aligned}
(252) \qquad \int_{B_R(x(j))} \langle df_j^\epsilon, dr_{x(j)} \rangle d\underline{\mathrm{vol}} &= \int_{B_R(x(j)) \setminus \overline{B}_\tau \left(\{x(j)\} \cup \partial B_{\pi/\sqrt{H}}(x(j)) \right)} \langle df_j^\epsilon, dr_{x(j)} \rangle d\underline{\mathrm{vol}} \\
(253) \qquad &= \sum_{i=1}^N \int_{B_{r_i}(x_i(j))} \langle df_j^\epsilon, dr_{x(j)} \rangle d\underline{\mathrm{vol}} \pm \Psi(\epsilon; n, L, H) \\
(254) \qquad &= \sum_{i=1}^N \int_{B_{r_i}(x_i(j))} \langle df^\epsilon, dr_x \rangle d\upsilon \pm \Psi(\epsilon; n, L, H) \\
(255) \qquad &= \int_{B_R(x) \setminus \overline{B}_\tau \left(\{x\} \cup \partial B_{\pi/\sqrt{H}}(x) \right)} \langle df^\epsilon, dr_x \rangle d\upsilon \pm \Psi(\epsilon; n, L, H) \\
(256) \qquad &= \int_{B_R(x)} \langle df^\epsilon, dr_x \rangle d\upsilon \pm \Psi(\epsilon; n, L, H) \\
(257) \qquad &= \int_{B_R(x)} \langle df, dr_s \rangle d\upsilon \pm \Psi(\epsilon; n, L, H)
\end{aligned}$$

for every sufficiently large j. On the other hand, for every i, there exists a Lipschitz function ψ_i on M_i such that $0 \leq \psi_i \leq 1$, $\psi_i|_{\overline{B}_{\tau/2} \left(\{x\} \cup \partial B_{\pi/\sqrt{H}}(x) \right)} = 0$, $\psi_i|_{M_i \setminus \overline{B}_\tau \left(\{x\} \cup \partial B_{\pi/\sqrt{H}}(x) \right)} = 1$ and $\mathbf{Lip} \psi_i \leq C(n, \tau)$. Since $f_i^\epsilon + \Psi(\epsilon; n, L, H) \geq 0$ on $B_R(x(i))$ for every sufficiently

large i, we have $f_i^\epsilon + \Psi(\epsilon; n, L, H)\psi_i \geq 0$ on $B_R(x(i))$. Therefore by Proposition 4.13 and Corollary 7.18, we have

(258)
$$\int_{B_R(x(i))} \langle d\left(f_i^\epsilon + \Psi(\epsilon; n, L, H)\psi_i\right), dr_{x(i)}\rangle d\underline{\mathrm{vol}}$$

(259)
$$\geq -(n-1)\int_{B_R(x(i))} \frac{\underline{k}'_{H_i}(\overline{x(i), w})}{\underline{k}_{H_i}(\overline{x(i), w})}(f_i^\epsilon + \Psi(\epsilon; n, L, H)\psi_i)d\underline{\mathrm{vol}}$$

(260)
$$\geq -(n-1)\int_{B_R(x(i))} \frac{\underline{k}'_{H_i}(\overline{x(i), w})}{\underline{k}_{H_i}(\overline{x(i), w})} f_i^\epsilon d\underline{\mathrm{vol}} - \Psi(\epsilon; n, L, H)\int_{B_R(x(i))} \left|\frac{\underline{k}'_{H_i}(\overline{x(i), w})}{\underline{k}_{H_i}(\overline{x(i), w})}\psi_i\right| d\underline{\mathrm{vol}}$$

(261)
$$\geq -(n-1)\int_{B_R(x(i))} \frac{\underline{k}'_{H_i}(\overline{x(i), w})}{\underline{k}_{H_i}(\overline{x(i), w})} f_i^\epsilon d\underline{\mathrm{vol}} - \Psi(\epsilon; n, L, H, \tau, R)$$

(262)
$$= -(n-1)\int_{B_R(x)} \frac{\underline{k}'_{H}(\overline{x, w})}{\underline{k}_{H}(\overline{x, w})} f^\epsilon d\upsilon - \Psi(\epsilon; n, L, H, \tau, R)$$

(263)
$$= -(n-1)\int_{B_R(x)} \frac{\underline{k}'_{H}(\overline{x, w})}{\underline{k}_{H}(\overline{x, w})} f d\upsilon - \Psi(\epsilon; n, L, H, \tau, R)$$

for every sufficiently large i. Since

$$\int_{B_R(x(i))} |df_i^\epsilon - d\left(f_i^\epsilon + \Psi(\epsilon; n, L, H)\psi_i\right)| d\underline{\mathrm{vol}} \leq \Psi(\epsilon; n, L, H, \tau),$$

we have

$$\int_{B_R(x)} \langle df, dr_x\rangle d\upsilon \geq -(n-1)\int_{B_R(x)} \frac{\underline{k}'_{H}(\overline{x, w})}{\underline{k}_{H}(\overline{x, w})} f(w) d\upsilon - \Psi(\epsilon; n, L, H, \tau, R).$$

By letting $\epsilon \to 0$, we have the assertion of the case $\mathrm{supp} f \cap (\{x\} \cup \partial B_{\pi/\sqrt{H}}(x)) = \emptyset$.

Next, we shall discuss the assertion of the case $\mathrm{supp} f \cap (\{x\} \cup \partial B_{\pi/\sqrt{H}}(x)) \neq \emptyset$. We assume that $H \leq 0$ and $\liminf_{r\to 0} \upsilon(B_r(x))/r = 0$. We take a sequence of positive numbers $\{r_i\}_i$ satisfying $r_i \to 0$ and $\lim_{i\to\infty} \upsilon(B_{r_i}(x))/r_i = 0$. We also take a Lipschitz function ϕ_i on Y satisfying $\phi_i|_{B_{r_i/2}(x)} = 1$, $0 \leq \phi_i \leq 1$, $\mathrm{supp}\phi_i \subset B_{r_i}(x)$ and $\mathbf{Lip}\phi_i \leq C(n)/r_i$. We fix

$\epsilon > 0$. Then we have

$$\left|\int_Y \langle df, dr_x\rangle d\upsilon - \int_Y \langle d(1-\phi_i)f, dr_x\rangle d\upsilon\right| \leq \int_Y |d(\phi_i f)| d\upsilon \tag{264}$$

$$= \int_{B_{r_i}(x)} |d(\phi_i f)| d\upsilon \tag{265}$$

$$\leq \frac{C(n,L)}{r_i}\upsilon(B_{r_i}(x)). \tag{266}$$

On the other hand, since $\underline{k}'_H/\underline{k}_H \geq 0$, we have

$$\int_Y \langle d(1-\phi_i)f, dr_x\rangle d\upsilon \geq -(n-1)\int_Y \frac{\underline{k}'_H(\overline{x,w})}{\underline{k}_H(\overline{x,w})}(1-\phi_i)f d\upsilon \tag{267}$$

$$\geq -(n-1)\int_Y \frac{\underline{k}'_H(\overline{x,w})}{\underline{k}_H(\overline{x,w})} f(w) d\upsilon. \tag{268}$$

Thus, by letting $i \to \infty$, we have the assertion of the case $H \leq 0$ and $\liminf_{r\to 0}\upsilon(B_r(x))/r = 0$.

Next, we shall discuss the assertion the case $H \leq 0$, $\liminf_{r\to 0}\upsilon(B_r(x))/r > 0$ and $f(x) = 0$. We take a sequence of positive numbers $\{r_i\}_i$ satisfying $r_i \to 0$. We also take ϕ_i as above. Then we have

$$\left|\int_Y \langle df, dr_x\rangle d\upsilon - \int_Y \langle d(1-\phi_i)f, dr_x\rangle d\upsilon\right| \leq \int_Y |d(\phi_i f)| d\upsilon \tag{269}$$

$$= \int_{B_{r_i}(x)} |d(\phi_i f)| d\upsilon \tag{270}$$

$$= \int_{B_{r_i}(x)} |f d\phi_i + \phi_i df| d\upsilon \tag{271}$$

$$\leq \int_{B_{r_i}(x)} |f||d\phi_i| d\upsilon + \mathbf{Lip} f \upsilon(B_{r_i}(x)) \tag{272}$$

$$\leq r_i \mathbf{Lip} f \frac{\upsilon(B_{r_i}(x))}{r_i} + L\upsilon(B_{r_i}(x)) \tag{273}$$

$$= 2L\upsilon(B_{r_i}(x)). \tag{274}$$

Therefore, we have the assertion of the case $H \leq 0$, $\liminf_{r\to 0}\upsilon(B_r(x))/r > 0$ and $f(x) = 0$.

We shall discuss the case $H \leq 0$, $\liminf_{r\to 0}\upsilon(B_r(x))/r > 0$ and $f(x) > 0$. Then we remark the following:

Claim 6.2. *We have*

$$\liminf_{r\to 0} \upsilon_{-1}(\partial B_r(x) \setminus C_x) > 0.$$

The proof is as follows. For every sufficiently small $r > 0$, there exists an isometric embedding γ from $[0, 3r]$ to Y satisfying $\gamma(0) = x$. We put $x_r = \gamma(5r/2)$. Then we have

$$\upsilon(B_{3r}(x) \setminus B_{2r}(x)) \geq \upsilon(B_{\frac{r}{100}}(x_r)) \geq C(n, H)\upsilon(B_r(x)).$$

By [42, Theorem 4.6], we have

$$\upsilon_{-1}(\partial B_r(x) \setminus C_x) \geq \upsilon_{-1}\left(\partial B_r(x) \cap C_x(B_{3r}(x) \setminus B_{2r}(x))\right) \tag{275}$$

$$\geq \frac{C(n, H)\upsilon(B_{3r}(x) \setminus B_{2r}(x))}{\operatorname{vol} B_{3r}(\underline{p}) - \operatorname{vol} B_{2r}(\underline{p})} \operatorname{vol}_{n-1} \partial B_r(\underline{p}) \tag{276}$$

$$\geq C(n, H)\frac{\upsilon(B_r(x))}{r^n} r^{n-1} \geq C(n)\frac{\upsilon(B_r(x))}{r}. \tag{277}$$

Therefore, we have Claim 6.2.

By the assumption, there exist $r_0 > 0$ and $\tau_0 > 0$ such that $f(w) > \tau_0$ for every $w \in B_{r_0}(x)$. Thus, by [42, Theorem 5.2], we have

$$\int_Y \frac{k'_H(\overline{x, w})}{k_H(\overline{x, w})} f(w) d\upsilon \geq C(n, r_0, H, \tau_0) \int_{B_{r_0}(x)} \frac{1}{r_x(w)} d\upsilon \tag{278}$$

$$\geq C(n, r_0, H, \tau_0) \int_0^{r_0} \int_{\partial B_r(x) \setminus C_x} \frac{1}{r} d\upsilon_{-1} dr \tag{279}$$

$$= C(n, r_0, H, \tau_0) \int_0^{r_0} \frac{\upsilon_{-1}(\partial B_r(x) \setminus C_x)}{r} dr = \infty \tag{280}$$

Therefore, we have the assertion of the case $H \leq 0$, $\liminf_{r \to 0} \upsilon(B_r(x))/r > 0$ and $f(x) > 0$.

Finally, we shall discuss the assertion of the case $H > 0$. By rescaling, without loss of generality, we can assume that $H = 1$. If $R < \pi$, then we can prove the assertion by an argument similar to one above. Therefore, we assume that $R = \pi$ and $\partial B_\pi(x) \neq \emptyset$ below. Then, by [6] (or [44]), we have $Y = \mathbf{S}^0 * \partial B_{\pi/2}(x)$. Here, for every metric space X, we define a distance on $[0, \pi] \times X / \{0, \pi\} \times X$ by

$$\overline{(t_1, x_1), (t_2, x_2)} = \arccos(\cos t_1 \cos t_2 + \sin t_1 \sin t_2 \cos \min\{\overline{x_1, x_2}, \pi\}),$$

$\mathbf{S}^0 * X$ denote this metric space. We take $z \in \partial B_\pi(x)$. By Bishop-Gromov volume comparison theorem for υ, we have

$$\frac{\upsilon(B_r(x))}{\upsilon(Y)} = \frac{\upsilon(Y \setminus B_{\pi - r}(z))}{\upsilon(Y)} = 1 - \frac{\upsilon(B_{\pi - r}(x))}{\upsilon(Y)} \leq 1 - \frac{\operatorname{vol} B_{\pi - r}(\underline{p})}{\operatorname{vol} B_\pi(\underline{p})} = \frac{\operatorname{vol} B_r(\underline{p})}{\operatorname{vol} \mathbf{S}^n}$$

for every $0 < r \leq \pi/2$. On the other hand, by Bishop-Gromov volume comparison theorem, since $\upsilon(B_r(x))/\upsilon(Y) \geq \operatorname{vol} B_r(\underline{p})/\operatorname{vol} \mathbf{S}^n$, we have

$$\frac{\upsilon(B_r(x))}{\upsilon(Y)} = \frac{\operatorname{vol} B_r(\underline{p})}{\operatorname{vol} \mathbf{S}^n}.$$

Similarly, we have $\upsilon(B_r(z))/\upsilon(Y) = \operatorname{vol} B_r(\underline{p})/\operatorname{vol}\mathbf{S}^n$. Especially, we have

$$\lim_{r\to 0}\frac{\upsilon(B_r(x))}{\omega_n r^n} = \lim_{r\to 0}\frac{\upsilon(B_r(z))}{\omega_n r^n} = \frac{\upsilon(Y)}{\operatorname{vol}\mathbf{S}^n}.$$

Since $\underline{k}_1'(r)/\underline{k}_1(r) \geq 0$ for every $0 < r \leq \pi/2$, by [42, Theorem 4.2] and [42, Theorem 5.2], we have

$$\int_{B_{\frac{\pi}{2}}(x)} \frac{\underline{k}_1'(\overline{x,w})}{\underline{k}_1(\overline{x,w})} d\upsilon \leq \int_0^{\frac{\pi}{2}} \int_{\partial B_t(x)\setminus C_x} \frac{C(n)}{r_x} d\upsilon_{-1} dt \tag{281}$$

$$= C(n)\int_0^{\frac{\pi}{2}} \frac{\upsilon_{-1}(\partial B_t(x)\setminus C_x)}{t} dt \tag{282}$$

$$\leq C(n)\int_0^{\frac{\pi}{2}} \frac{\upsilon(B_t(x))}{t}\frac{\operatorname{vol}_{n-1}\partial B_t(\underline{p})}{\operatorname{vol} B_t(\underline{p})} dt \tag{283}$$

$$\leq C(n)\int_0^{\frac{\pi}{2}} \frac{\upsilon(B_t(x))}{t^2} dt \leq C(n). \tag{284}$$

We remark that $C_z = \{x\}$ and $C_x = \{z\}$. Similarly, we have

$$\int_{M\setminus B_{\frac{\pi}{2}}(x)} \left|\frac{\underline{k}_1'(\overline{x,w})}{\underline{k}_1(\overline{x,w})}\right| d\upsilon = \int_{B_{\frac{\pi}{2}}(z)} \left|\frac{\underline{k}_1'(\overline{x,w})}{\underline{k}_1(\overline{x,w})}\right| d\upsilon \tag{285}$$

$$\leq C(n)\int_0^{\frac{\pi}{2}} \int_{\partial B_t(z)} \frac{1}{t} d\upsilon_{-1} dt \tag{286}$$

$$\leq C(n)\int_0^{\frac{\pi}{2}} \frac{\upsilon_{-1}(\partial B_t(z)\setminus C_z)}{t} dt \leq C(n). \tag{287}$$

We take $r_i > 0$ satisfying $r_i \to 0$ and ϕ_i as above. We also take Lipschitz functions $\hat{\phi}_i$ on Y satisfying $0 \leq \hat{\phi}_i \leq 1$, $\hat{\phi}_i|_{B_{r_i/2}(z)} = 1$, $\operatorname{supp}\hat{\phi}_i \subset B_{r_i}(z)$ and $\mathbf{Lip}\hat{\phi}_i \leq C(n)/r_i$. Then we have

(288)
$$\left|\int_Y \langle df, dr_x\rangle d\upsilon - \int_Y \langle d(1-\phi_i)(1-\hat{\phi}_i)f, dr_x\rangle d\upsilon\right| \leq \int_Y |d(f-(1-\phi_i)(1-\hat{\phi}_i)f)| d\upsilon$$

$$= \int_{B_{r_i}(x)} |d(\phi_i f)| d\upsilon + \int_{B_{r_i}(z)} |d(\hat{\phi}_i f)| d\upsilon \tag{289}$$

$$\leq \mathbf{Lip} f \frac{\upsilon(B_{r_i}(x))}{r_i} + \mathbf{Lip} f \frac{\upsilon(B_{r_i}(z))}{r_i} \tag{290}$$

$$\overset{i\to\infty}{\to} 0. \tag{291}$$

On the other hand, by dominated convergence theorem, we have

(292)
$$\int_Y \langle d(1-\phi_i)(1-\hat{\phi}_i)f, dr_x\rangle d\upsilon$$

(293)
$$\geq -(n-1)\int_Y \frac{\underline{k}_1'(\overline{x,w})}{\underline{k}_1(\overline{x,w})}(1-\phi_i)(1-\hat{\phi}_i)f(w)d\upsilon$$

(294)
$$\geq -(n-1)\int_Y \frac{\underline{k}_1'(\overline{x,w})}{\underline{k}_1(\overline{x,w})}f(w)d\upsilon - (n-1)\int_Y \left|\frac{\underline{k}_1'(\overline{x,w})}{\underline{k}_1(\overline{x,w})}\right| |(1-\phi_i)(1-\hat{\phi}_i)f(w) - f(w)|d\upsilon$$

(295)
$$\overset{i\to\infty}{\to} -(n-1)\int_Y \frac{\underline{k}_1'(r_x(w))}{\underline{k}_1(r_x(w))}f(w)d\upsilon.$$

Therefore we have the assertion. □

We end this section by giving a corollary of Theorem 6.1. The corollary is well known in the setting of metric measure spaces. See for instance [72, 88, 89, 65, 66, 92, 93]. We will give a new proof via Laplacian comparison theorem on Ricci limit spaces:

Corollary 6.3. *Let $\{H_i\}_{i=1,2,\dots,\infty}$ be a sequence of real numbers, $\{(M_i, m_i)\}_{i\in\mathbf{N}}$ a sequence of pointed n-dimensional complete Riemannian manifolds with $\mathrm{Ric}_{M_i} \geq H_i(n-1)$ and (M_∞, m_∞) a pointed n-dimensional complete Riemannian manifold ($n \geq 2$). We assume that $H_i \to H_\infty$ and $(M_i, m_i) \to (M_\infty, m_\infty)$. Then we have $\mathrm{Ric}_{M_\infty} \geq H_\infty(n-1)$.*

Proof. By [6, Theorem 5.9], we have $(M_i, m_i, H^n) \to (M_\infty, m_\infty, H^n)$. Then, by Theorem 6.1, we have, $\Delta r_x(w) \geq -(n-1)\underline{k}'_{H_\infty}(\overline{x,w})/\underline{k}_{H_\infty}(\overline{x,w})$ for every $x \in M_\infty$ and $w \in M_\infty \setminus (C_x \cup \{x\})$. Therefore, we have the assertion. □

7 Appendix

7.1 Infinitesimal doubling condition and Lebesgue set

In this subsection, we shall study metric measure spaces satisfying a good property (Definition 7.1). On such metric measure spaces, we can construct an outer measure associated to the measure and give several properties about it. Especially, we will define Lebesgue set and give several properties of the set (see Corollary 7.6 and Proposition 7.7).

Definition 7.1. Let (Z, υ) be a metric measure space, A a Borel subset of Z and $C \geq 1$. We say that (Z, υ) *satisfies infinitesimal doubling condition on A with doubling constant C* if the following properties hold:

1. $v(K) < \infty$ for every bounded Borel subset K of A.

2. For every $z \in A$, there exists $r > 0$ such that

$$v(\overline{B}_{2s}(z)) \leq Cv(\overline{B}_s(z))$$

for every $0 < s < r$.

We shall give an example:

EXAMPLE 7.2. Let (Y, y, v) be a Ricci limit space, x a point in Y, R a positive number satisfying $\partial B_R(x) \setminus C_x \neq \emptyset$. Then, the metric measure space $(\partial B_R(x), v_{-1})$ satisfies infinitesimal doubling condition on $\partial B_R(x) \setminus C_x$. In fact, we have

$$\limsup_{r \to 0} \frac{v_{-1}(\partial B_R(x) \cap \overline{B}_{2r}(z))}{v_{-1}(\partial B_R(x) \cap B_r(z) \setminus C_x)} \leq C(n)$$

for every $z \in \partial B_R(x) \setminus C_x$. This follows from [42, Corollary 4.7] and [42, Theorem 5.2].

We fix a metric measure space (Z, v) and a Borel subset A of Z satisfying that (Z, v) satisfies infinitesimal doubling condition on A with doubling constant $C \geq 1$ below. For every $\delta > 0$ and $\hat{A} \subset Z$, we put

$$v^*_\delta(\hat{A}) = \inf \left\{ \sum_{i=1}^{\infty} v(B_{r_\lambda}(x_\lambda));\ 0 \leq r_i < \delta,\ \hat{A} \subset \bigcup_{i=1}^{\infty} B_{r_i}(x_i) \right\}$$

and define an outer measure v^* on Z by

$$v^*(\hat{A}) = \lim_{\delta \to 0} v^*_\delta(\hat{A}).$$

We also put $\mathcal{M} = \{\hat{A} \in 2^Z; v^*(B \cap \hat{A}) + v^*(B \setminus \hat{A}) \leq v^*(B) \text{ for every } B \in 2^Z\}$. We shall recall that $(Z, \mathcal{M}, v^*)$ is a complete measure space and that $\mathcal{B}(Z) = \{B \in 2^Z; B \text{ is a Borel subset of } Z\ \} \subset \mathcal{M}$. See for instance chapter 1 in [81]. By the definition, we have $v(\hat{A}) \leq v^*(\hat{A})$ for every Borel subset $\hat{A}$ of Z.

PROPOSITION 7.3. *We have $v^*(\hat{A}) = v(\hat{A})$ for every Borel subset $\hat{A} \subset A$.*

PROOF. Without loss of generality, we can assume that $v(\hat{A}) < \infty$. We fix $\epsilon, \delta > 0$. There exists an open set $O \subset Z$ such that $\hat{A} \subset O$ and $v(O \setminus \hat{A}) < \epsilon$. For every $a \in \hat{A}$, there exists $r_a > 0$ such that $\overline{B}_{r_a}(a) \subset O$ and that $v(B_{2r}(a)) \leq Cv(B_r(a))$ for every $0 < r < r_a$. By Proposition 2.12, there exists a pairwise disjoint collection $\{\overline{B}_{r_i}(a_i)\}$ such

that $a_i \in \hat{A}$, $r_i < \min\{\delta, r_{a_i}\}/100$ and $\hat{A} \setminus \bigcup_{i=1}^{N} \overline{B}_{r_i}(a_i) \subset \bigcup_{i=N+1}^{\infty} \overline{B}_{5r_i}(a_i)$ for every N. Since $\upsilon(O) < \infty$, there exists N such that $\sum_{i=N+1}^{\infty} \upsilon(\overline{B}_{r_i}(a_i)) < \epsilon$. Then we have

$$\upsilon^*_\delta(\hat{A}) \leq \sum_{i=1}^{N} \upsilon(\overline{B}_{r_i}(a_i)) + \sum_{i=N+1}^{\infty} \upsilon(\overline{B}_{5r_i}(a_i)) \tag{296}$$

$$\leq \sum_{i=1}^{N} \upsilon(\overline{B}_{r_i}(a_i)) + \sum_{i=N+1}^{\infty} C^3 \upsilon(\overline{B}_{r_i}(a_i)) \tag{297}$$

$$\leq \upsilon(O) + C^3\epsilon \leq \upsilon(\hat{A}) + (1 + C^3)\epsilon. \tag{298}$$

By letting $\delta \to 0$ and $\epsilon \to 0$, we have the assertion. □

The following corollary is a fundamental property for a relation to Hausdorff measure on metric measure spaces satisfying infinitesimal doubling condition.

Corollary 7.4. *Assume that there exists $\alpha \geq 0$ such that υ is Ahlfors α-regular at every $z \in A$. Then, υ and H^α are mutually absolutely continuous on A.*

Proof. For every $i \in \mathbf{N}$, we put $A_i = \{a \in A; i^{-1}r^\alpha \leq \upsilon(B_r(a)) \leq ir^\alpha$ for every $0 < r < i^{-1}\}$. Let D be a Borel subset of A. First, we assume that $H^\alpha(D) = 0$. Then, we have $H^\alpha(D \cap A_i) \leq H^\alpha(D) = 0$ for every i. We fix i. Then, for every positive numbers ϵ, δ satisfying $\epsilon, \delta << i^{-1}$, there exists a countable collection $\{B_{r_j}(x_j)\}_j$ such that $r_j < \delta$, $x_j \in D \cap A_i$ and $\sum_j r_j^\alpha < \epsilon$. Thus, we have $\sum_j \upsilon(B_{r_j}(x_j)) < \Psi(\epsilon; i)$. Therefore, we have $\upsilon^*(D \cap A_i) = 0$. Since $(Z, \upsilon^*, \mathcal{M})$ is a complete measure space, we have $\upsilon^*(D) = 0$. Especially, we have $\upsilon(D) = 0$. Next, we assume that $\upsilon(D) = 0$. By Proposition 7.3, we have $\upsilon^*(D \cap A_i) \leq \upsilon^*(D) = \upsilon(D) = 0$ for every i. Then, by an argument similar to that above, we have $H^\alpha(D \cap A_i) = 0$. Thus, we have $H^\alpha(D) = 0$. □

For subset $\hat{A} \subset Z$, let $\operatorname{Leb} \hat{A}$, denote the set of points, $a \in A$, such that for every $\epsilon > 0$, there exists $r > 0$ such that $\upsilon^*(\overline{B}_s(a) \cap \hat{A}) \geq (1 - \epsilon)\upsilon(\overline{B}_s(a))$ for every $0 < s < r$. We call $\operatorname{Leb} \hat{A}$ *Lebesgue set of* $\hat{A}$.

Proposition 7.5. *We have*

$$\upsilon^*(\hat{A} \setminus \operatorname{Leb} \hat{A}) = 0$$

for every Borel subset $\hat{A}$ of A.

Proof. We fix $z \in Z$ and $\epsilon > 0$. For $\tau > 0$ and $N \in \mathbf{N}$, let $\hat{A}_{\tau,N}$, denote the set of points, $a \in \hat{A} \cap B_N(z)$, such that there exists a sequence of positive numbers $r_i > 0$ such that $r_i \to 0$ and that $\upsilon^*(\overline{B}_{r_i}(a) \cap \hat{A}) \leq (1 - \tau)\upsilon(\overline{B}_{r_i}(a))$ holds for every i. We remark that $\upsilon^*(\hat{A}_{\tau,N}) \leq \upsilon^*(\hat{A} \cap B_N(z)) = \upsilon(\hat{A} \cap B_N(z)) < \infty$. Thus, by the definition

of υ^*, there exists a countable collection $\{B_{s_i}(x_i)\}_i$ such that $\hat{A}_{\tau,N} \subset \bigcup_{i=1}^{\infty} B_{s_i}(x_i)$ and $|\upsilon^*(A_{\tau,N}) - \sum_{i=1}^{\infty} \upsilon(B_{s_i}(x_i))| < \epsilon$. We put $O = B_N(z) \cap \bigcup_{i=1}^{\infty} B_{s_i}(x_i)$. By the definition of $\hat{A}_{\tau,N}$ and Proposition 2.12, there exists a pairwise disjoint collection $\{\overline{B}_{r_i}(a_i)\}_i$ such that $a_i \in \hat{A}_{\tau,N}$, $\upsilon(\overline{B}_{2r_i}(a_i)) \leq C\upsilon(\overline{B}_{r_i}(a_i))$, $\overline{B}_{100r_i}(a_i) \subset O$, $\upsilon(\overline{B}_{r_i}(a_i) \cap \hat{A}) \leq (1-\tau)\upsilon(\overline{B}_{r_i}(a_i))$ for every i, and $\hat{A}_{\tau,N} \setminus \bigcup_{i=1}^{\hat{N}} \overline{B}_{r_i}(a_i) \subset \bigcup_{i=\hat{N}+1}^{\infty} \overline{B}_{5r_i}(a_i)$ for every $\hat{N}$. We take $\hat{N}$ satisfying $\sum_{i=\hat{N}+1}^{\infty} \upsilon(\overline{B}_{r_i}(a_i)) < \epsilon$. Then we have

$$\upsilon^*(\hat{A}_{\tau,N}) \leq \sum_{i=1}^{\hat{N}} \upsilon^*(\hat{A}_{\tau,N} \cap \overline{B}_{r_i}(a_i)) + \sum_{i=\hat{N}+1}^{\infty} \upsilon(\overline{B}_{5r_i}(a_i)) \tag{299}$$

$$\leq \sum_{i=1}^{\hat{N}} \upsilon(\hat{A} \cap \overline{B}_{r_i}(a_i)) + C^3 \sum_{i=\hat{N}+1}^{\infty} \upsilon(\overline{B}_{r_i}(a_i)) \tag{300}$$

$$\leq (1-\tau) \sum_{i=1}^{\hat{N}} \upsilon(\overline{B}_{r_i}(a_i)) + \epsilon C^3 \tag{301}$$

$$\leq (1-\tau)\upsilon(O) + \epsilon C^3 \tag{302}$$

$$\leq (1-\tau) \sum_{i=1}^{\infty} \upsilon(B_{s_i}(x_i)) + \epsilon C^3 \tag{303}$$

$$\leq (1-\tau)(\upsilon^*(\hat{A}_{\tau,N}) + \epsilon) + \epsilon C^3. \tag{304}$$

By letting $\epsilon \to 0$, we have $\upsilon^*(\hat{A}_{\tau,N}) = 0$. Thus, we have $\hat{A}_{\tau,N} \in \mathcal{M}$ and $\upsilon^*(\hat{A} \setminus \operatorname{Leb} \hat{A}) = \upsilon^*(\bigcup_{\tau>0, N\in\mathbf{N}} \hat{A}_{\tau,N}) = 0$. □

By Proposition 7.5, we remark $\operatorname{Leb}(\operatorname{Leb}(\hat{A})) = \operatorname{Leb}(\hat{A})$ for every Borel subset $\hat{A} \subset A$.

Corollary 7.6 (Lebesgue differentiation theorem for locally bounded functions). *Let f be a Borel function f on Z satisfying that f is locally bounded at every $a \in A$. Then, there exists a Borel subset $\hat{A}$ of A such that $\upsilon(A \setminus \hat{A}) = 0$ and that for every $a \in \hat{A}$ and $\epsilon > 0$, there exists $r > 0$ such that*

$$-\epsilon\upsilon(\overline{B}_s(x)) \leq \int_{\overline{B}_s(x)} |f - f(x)| d\upsilon \leq \epsilon\upsilon(\overline{B}_s(x))$$

for every $0 < s < r$.

Proof. We fix $\epsilon > 0$ and $z \in A$. For every $N \in \mathbf{N}$, by Lusin's theorem, there exists a compact set $K_{\epsilon,N} \subset A \cap B_N(z)$ such that $\upsilon(A \cap B_N(z) \setminus K_{\epsilon,N}) < \epsilon$ and that f is continuous on $K_{\epsilon,N}$. We put $\hat{K}_{\epsilon,N} = \operatorname{Leb} K_{\epsilon,N}$. Then, it is easy to check that for every $x \in \hat{K}_{\epsilon,N}$ and $\epsilon > 0$, there exists $r > 0$ such that

$$-\epsilon\upsilon(\overline{B}_s(x)) \leq \int_{\overline{B}_s(x)} |f - f(x)| d\upsilon \leq \epsilon\upsilon(\overline{B}_s(x))$$

for every $0 < s < r$. Therefore, we have the assertion. □

We end this subsection by giving a fundamental property of Lebesgue sets for Lipschitz functions on metric measure spaces satisfying *doubling condition*:

PROPOSITION 7.7. *Assume that the following properties hold:*

1. $0 < \upsilon(B_r(z))$ *for every* $z \in Z$ *and* $r > 0$
2. *There exist* $r_0 > 0$ *and* $C > 1$ *such that*

$$\upsilon(B_{2r}(z)) \leq C\upsilon(B_r(z))$$

for every $z \in Z$ *and* $0 < r < r_0$.

Then, for every Lipschitz function f *on* Z *and Borel subset* A *of* Z*, we have* $\mathrm{Lip}f(a) = \mathrm{Lip}(f|_A)(a)$ *and* $lipf(a) = lip(f|_A)(a)$ *for every* $a \in \mathrm{Leb}(A)$.

PROOF. Without loss of generality, we can assume that a is not isolated point. There exists a sequence $a_i \in Z \setminus \{a\}$ such that $a_i \to a$ and that $|f(a_i) - f(a)|/\overline{a_i, a} \to \mathrm{Lip}f(a)$. By the assumption, for every sufficiently large i, there exists $\hat{a}_i \in A$ such that $\overline{a_i\hat{a}_i} \leq \Psi(\overline{a, a_i}; C)\overline{a, a_i}$. Especially we have $\hat{a}_i \neq a$, i.e. a is not an isolated point in A. It is easy to check

$$\lim_{i\to\infty} \frac{|f(a) - f(a_i)|}{\overline{a, a_i}} = \lim_{i\to\infty} \frac{|f(a) - f(\hat{a}_i)|}{\overline{a, \hat{a}_i}}.$$

Therefore, we have $\mathrm{Lip}f(a) \leq \mathrm{Lip}(f|_A)(a)$. Thus we have the first assertion. Similarly, we have the second assertion. □

7.2 A proof of Claim 3.25

In this subsection, we shall give a proof of Claim 3.25. We define a function π_1 on $\mathbf{R}^k$ by $\pi_1((x_1, \cdots, x_k)) = x_1$. Then, by the definition, we have

$$sl_1 - \mathrm{Leb}A = \left\{ a = (a_1, \cdots, a_k) \in A; \liminf_{r\to 0} \frac{H^{k-1}(\overline{B}_r(a) \cap A \cap \pi_1^{-1}(\pi_1(a)))}{\omega_{k-1}r^{k-1}} = 1 \right\}.$$

We define a function f_r^A on $\mathbf{R}^k$ by $f_r^A(x) = H^{k-1}\left(\overline{B}_r(x) \cap A \cap \pi_1^{-1}(\pi_1(x))\right) 1_A(x)$. First, we assume that A is compact.

CLAIM 7.8. *The function* f_r^A *is upper semi-continuous. Especially,* f_r^A *is* H^k*-measurable function.*

PROOF. Let x_∞ be a point in A and $\{x_i\}_i$ a sequence of points in $\mathbf{R}^k$ satisfying $x_i \to x_\infty$. It suffices to check that $\limsup_{i\to\infty} f_r^A(x_i) \leq f_r^A(x_\infty)$. Without loss of generality, we

can assume that $x_j \in A$ for every sufficiently large j. We fix $\delta > 0$. We take a subsequence $\{n(i)\}_{i\in\mathbf{N}}$ of $\mathbf{N}$ such that

$$\lim_{j\to\infty} H^{k-1}\left(\overline{B}_r(x_{n(j)}) \cap A \cap \pi_1^{-1}(\pi_1(x_{n(j)}))\right) = \limsup_{i\to\infty} H^{k-1}\left(\overline{B}_r(x_i) \cap A \cap \pi_1^{-1}(\pi_1(x_i))\right).$$

On the other hand, since a sequence of compact set $\{\overline{B}_r(x_{n(j)}) \cap A \cap \pi_1^{-1}(\pi_1(x_{n(j)}))\}$ is precompact with respect to the Hausdroff distance on $\mathbf{R}^k$. Thus, without loss of generality, we can assume that there exists a compact subset K_∞ of $\mathbf{R}^k$ such that $\overline{B}_r(x_{n(j)}) \cap A \cap \pi_1^{-1}(\pi_1(x_{n(j)}))$ converges to K_∞ in the sense of Hausdorff distance on $\mathbf{R}^k$. Then, it is easy to check $K_\infty \subset \overline{B}_r(x_\infty) \cap A \cap \pi_1^{-1}(\pi_1(x_\infty))$. There exists a finite collection $\{B_{r_i}(y_i)\}_{i=1,\dots,N}$ such that $r_i << \delta$, $\overline{B}_r(x_\infty) \cap A \cap \pi_1^{-1}(\pi_1(x_\infty)) \subset \bigcup_{i=1}^N B_{r_i}(y_i)$ and

$$\left| H^{k-1}(\overline{B}_r(x_\infty) \cap A \cap \pi_1^{-1}(\pi_1(x_\infty)) - \sum_{i=1}^N \omega_{k-1} r_i^{k-1} \right| < \delta.$$

Since $\overline{B}_r(x_\infty) \cap A \cap \pi_1^{-1}(\pi_1(x_\infty))$ is compact, there exists $\tau_0 > 0$ such that $B_{\tau_0}(\overline{B}_r(x_\infty) \cap A \cap \pi_1^{-1}(\pi_1(x_\infty))) \subset \bigcup_{i=1}^N B_{r_i}(y_i)$. Since $\overline{B}_r(x_{n(j)}) \cap A \cap \pi_1^{-1}(\pi_1(x_{n(j)})) \subset B_{\tau_0}(K_\infty)$ for every sufficiently large j, we have $\overline{B}_r(x_{n(j)}) \cap A \cap \pi_1^{-1}(\pi_1(x_{n(j)})) \subset \bigcup_{i=1}^N B_{r_i}(y_i)$. Thus, we have

$$H^{k-1}(\overline{B}_r(x_{n(j)}) \cap A \cap \pi_1^{-1}(\pi_1(x_{n(j)}))) \leq \sum_{i=1}^N H^{k-1}(\overline{B}_r(y_i) \cap \pi_1^{-1}(\pi_1(x_{n(j)}))) \tag{305}$$

$$\leq \sum_{i=1}^N \omega_{k-1} r^{k-1} \tag{306}$$

$$\leq H^{k-1}(\overline{B}_r(x_\infty) \cap A \cap \pi_1^{-1}(\pi_1(x_\infty))) + \delta \tag{307}$$

for every sufficiently large j. Therefore, we have Claim 7.8. □

By Claim 7.8, we have the statement 1 in Claim 3.25. The statement 2 follows from Lebesgue differentiation theorem on Euclidean spaces. Finally, by Fubini's theorem, we have

$$H^k(A \setminus sl_1 - \mathrm{Leb}A) = \int_{\mathbf{R}} H^{k-1}(A \cap \{t\} \times \mathbf{R}^{k-1} \setminus sl_1 - \mathrm{Leb}A)dt = 0.$$

Thus, we have the statement 3. Therefore, we have Claim 3.25 if A is compact.

We shall give a proof of Claim 3.25 in general case. We fix $R > 0$. There exists a sequence of compact sets $K_i \subset B_R(0_k) \cap A$ such that $H^k(B_R(0_k) \cap A \setminus K_i) \to 0 (i \to \infty)$. By the definition, we have $sl_1 - \mathrm{Leb}K_i \subset sl_1 - \mathrm{Leb}(B_R(0_k) \cap A)$. As an outer measure,

we have

(308)

$$H^k(B_R(0_k)\cap A\setminus sl_1-\mathrm{Leb}(B_R(0_k)\cap A))\leq H^k(B_R(0_k)\cap A\setminus sl_1-\mathrm{Leb}K_i)$$

(309)
$$\leq H^k(B_R(0_k)\cap A\setminus K_i)+H^k(K_i\setminus sl_1-\mathrm{leb}K_i)$$

(310)
$$\overset{i\to\infty}{\to} 0$$

Thus, $sl_1-\mathrm{leb}(B_R(0)\cap A)$ is a H^k-measurable set. Since $sl_1-\mathrm{Leb}A=\bigcup_{R>0}sl_1-\mathrm{Leb}(A\cap B_R(0))$, we have the statement 1 in Claim 3.25. By Lebesgue differentiation theorem and Fubini's theorem, we have statements 2 and 3. Thus, we have Claim 3.25.

7.3 Distributional Laplacian comparison theorem on manifolds

Our aim in this subsection is to state distributional Laplacian comparison theorem on manifolds we want to use in section 6. It is Corollary 7.18. Throughout this subsection, we fix a positive number $R>0$ and (M,m) be a pointed n-dimensional complete Riemannian manifold ($n\geq 2$). We put $C^\infty(\overline{B}_R(m))=\{f\in C^0(\overline{B}_R(m))$; there exist an open subset U of M and a smooth function g on U such that $\overline{B}_R(m)\subset U$ and $g|_{\overline{B}_R(m)}=f\}$. We define a linear functional $\Delta_R^{\mathrm{dist}}r_m$ on $C^\infty(\overline{B}_R(m))$ by

$$\Delta_R^{\mathrm{dist}}r_m(f)=\int_{B_R(m)}\langle dr_m,df\rangle d\mathrm{vol}.$$

Proposition 7.9. *There exists unique Radon measure $\upsilon_{R,m}^{\mathrm{sing}}$ on $\overline{B}_R(m)$ satisfying the following properties:*

1. *A smooth function Δr_m on $B_R(m)\setminus(C_m\cup\{m\})$ is in $L^1(B_R(m))$.*
2. $\mathrm{supp}(\upsilon_{R,m}^{\mathrm{sing}})\subset C_m\cap\overline{B}_R(m)$
3. *For every $f\in C^\infty(\overline{B}_R(m))$, we have*
$$\Delta_R^{\mathrm{dist}}r_m(f)=\int_{B_R(m)}f\Delta r_m d\mathrm{vol}+\int_{\partial B_R(m)\setminus C_m}f d\mathrm{vol}_{n-1}+\int_{\overline{B}_R(m)}f d\upsilon_{R,x}^{\mathrm{sing}}.$$
4. *We have*
$$\int_{B_R(x)}|\Delta r_x|d\mathrm{vol}+\upsilon_{R,m}^{\mathrm{sing}}(B_R(x))+\mathrm{vol}_{n-1}(\partial B_R(x)\setminus C_x)=-2\int_{B_R(x)\cap\{\Delta r_x<0\}}\Delta r_x.$$

Proof. We put $S_mM=\{u\in T_mM;|u|=1\}$ and define $t(u)>0$ as the supremum of $t\in(0,\infty)$ such that $\exp_m su|_{[0,t]}$ is a minimal geodesic segment from m to $\exp_m tu$ for $u\in S_mM$. We also define a continuous function ϕ_R on S_mM by $\phi_R(u)=\min\{t(u),R\}$. We take a sequence of C^∞-functions $\{\phi_R^j\}_j$ on S_mM and a sequence of open sets $O_i\subset S_mM$ satisfying the following properties:

1. $\overline{O}_i \subset O_{i+1}$, $\bigcup_{i=1}^{\infty} O_i = \{u \in S_m M; t(u) > R\}$, $\phi_R - j^{-1} \leq \phi_R^j \leq \phi_R$ and $\phi_R^j(u) < t(u)$.

2. For every i, there exists l such that $\phi_R^j|_{O_i} = R$ for every $j \geq l$.

REMARK 7.10. There exists $\{\phi_R^j\}_j$ and $\{O_i\}_i$ as above. We shall explain it below. We take a sequence of C^∞-functions ψ_R^j on $S_m M$ satisfying $|\psi_R^j - \phi_R|_{L^\infty(S_m M)} \to 0$. Without loss of generality, we can assume that $\psi_R^j < \phi_R$. We take a sequence of open subsets O_i of $S_m M$ satisfying $\overline{O}_i \subset O_{i+1}$ and $\bigcup_{i=1}^{\infty} O_i = \{u \in S_m M; t(u) > R\}$. We put $O = \{u \in S_m M; t(u) > R\}$. We take a C^∞-function ϕ_i on $S_m M$ satisfying $0 \leq \phi_i \leq 1$, $\phi_i|_{O_i} = 1$ and $\mathrm{supp}\phi_i \subset O_{i+1}$. We define a C^∞-function $\phi_R^{i,j}$ on $S_x M$ by $\phi_R^{i,j}(u) = (1 - \phi_i(u))\psi_R^j(u) + \phi_i(u)R$. Then, we have $\phi_R^{i,j}(u) = \psi_R^j(u) < \phi_R(u) = t(u)$ for every $u \in S_m M \setminus O_{i+1}$. and $\phi_R^{i,j}(u) = R$ for every $u \in O_i$. For every i, there exists $j_0(i)$ such that $|\psi_R^j - \phi_R|_{L^\infty(S_m M)} < i^{-1}$ for every $j \geq j_0(i)$. We put $\phi_R^i = \phi_R^{i,j_0(i)}$. Then, we have $\phi_R^i|_{O_i} = R$ and $\phi_R^i(u) = (1 - \phi_i(u))\psi_R^{j_0(i)}(u) + \phi_i(u)R \leq (1 - \phi_i(u))\phi_R(u) + \phi_i(u)R \leq (1 - \phi_i(u))\phi_R(u) + \phi_i(u)\phi_R(u) = \phi_R(u) < t(u)$ for every $u \in O$ and $\phi_R^i(u) = (1 - \phi_i(u))\psi_R^{j_0(i)}(u) + \phi_i(u)R = (1 - \phi_i(u))\psi_R^{j_0(i)}(u) \leq \psi_R^{j_0(i)}(u) < \phi_R(u) = t(u)$ for every $u \in S_x M \setminus O$. Therefore, we have $\phi_R^i(u) \leq \phi_R(u)$ and $\phi_R^i(u) < t(u)$ for every $u \in S_m M$. Since $\phi_R|_{O_{i+1}} = R$, we have $|\phi_R^i(u) - \phi_R(u)| = |(1 - \phi_i(u))\psi_R^{j_0(i)}(u) + \phi_i(u)R - \phi_R(u)| \leq (1-\phi_i(u))|\phi_R^{j_0(i)}(u) - \phi_R(u)| + \phi_i(u)|R - \phi_R(u)| \leq i^{-1}$ for every $u \in O_{i+1}$. On the other hand, since $\phi_i|_{S_m M \setminus O_{i+1}} = 0$, by an argument similar to one above, we have $|\phi_R^i(u) - \phi_R(u)| \leq i^{-1}$ for every $u \in S_m M \setminus O_{i+1}$. Thus, we get an existence of sequences $\{\phi_R^j\}_j$ and $\{O_i\}_i$ as above.

We define an open subset V_R^j of M by $V_R^j = \{\exp_m tu \in M; u \in S_m M, 0 \leq t < \phi_R^j(u)\}$.

CLAIM 7.11. *We have* $\partial V_R^j = \{\exp_m tu \in M; u \in S_m M, t = \phi_R^j(u)\}$ *for every* j *satisfying* $j^{-1} < \overline{m, C_m}$.

The proof is as follows. We take $w \in \partial V_R^j$. By the definition, there exist $u_i \in S_x M$ and $0 \leq t_i < \phi_R^j$ such that $w_i = \exp_x t_i u_i \to w$. By the compactness of $S_m M$, we can assume that there exist $t \in [0, R]$ and $u \in S_m M$ such that $t_i \to t$ and $u_i \to u$. Thus, we have $w = \lim_{i\to\infty} \exp_m t_i u_i = \exp_m tu$. Since $t_i < \phi_R^j(u_i)$, we have $t \leq \phi_R^j(u) < t(u)$. Thus, we have $w \in M \setminus C_x$. If $t < \phi_R^j(u)$, then by the continuity of ϕ_R^j, there exists $\tau > 0$ such that $\hat{t} < \phi_R^j(\hat{u})$ for every $\hat{t}$ and $\hat{u} \in S_m M$ satisfying $|\hat{t} - t| < \tau$ and $\overline{\hat{u}, u} < \tau$. Thus, we have $\exp_x tu \in M \setminus \partial V_R^j$. This is a contradiction. Therefore, we have $\phi_R^j(u) \geq t$. Similarly, if $t > \phi_R^j(u)$, by the continuity of ϕ_R^j, there exists $\tau > 0$ such that $\phi_R^j(\hat{u}) < \hat{t} < t(\hat{u})$ for every $\hat{t}$ and $\hat{u} \in S_m M$ satisfying $|\hat{t} - t| < \tau$ and $\overline{\hat{u}, u} < \tau$. Thus, we have $\exp_m tu \in M \setminus \partial V_R^j$. This is a contradiction. Therefore, we have $\partial V_R^j \subset \{\exp_m tu \in M; u \in S_m M, t = \phi_R^j(u)\}$. On the other hand, for every $u \in S_m M$, we take a increasing sequence $0 < t_i < \phi_R^j(u)$ such that $t_i \to \phi_R^j(u)$. Since $\exp_m \phi_R^j(u)u = \lim_{i\to\infty} \exp_m t_i u \in \overline{V}_R^j$ and $\exp_m^{-1}|_{M\setminus C_m}$

gives diffeomorphism to the image, we have $\exp_m \phi_R^j(u)u \in M \setminus V_R^j$. Especially, we have $\exp_m \phi_R^j(u)u \in \partial V_R^j$. Therefore, we have Claim 7.11.

It is easy to check that ∂V_R^j is a compact $(n-1)$-dimensional C^∞-Riemannian submanifold of M and is diffeomorphic to $\mathbf{S}^{n-1}$ for every j satisfying $j^{-1} < \overline{m, C_m}$. Especially, $\overline{V}_R^j$ is a compact n-dimensional C^∞-Riemannian submanifold with C^∞-boundary.

Claim 7.12. *We have $\langle \nabla r_m, N_w \rangle(w) \geq 0$ for every j satisfying $j^{-1} < \overline{m, C_m}$, and $w \in \partial V_R^j$. Here N_w is the unit outer normal vector of $\overline{V}_R^j$ at w.,*

Because, since N_w is outer vector, we have $\langle N_w, \gamma'(0) \rangle \leq 0$ for the minimal geodesic γ from w to m. Thus, we have Claim 7.12.

For every j satisfying $j^{-1} < \overline{m, C_m}$, we define open subsets $A_+^{R,j}, A_-^{R,j}$ of V_R^j by $A_+^{R,j} = \{w \in V_R^j \setminus \{m\}; \Delta r_m(w) > 0\}$ and $A_-^{R,j} = \{w \in V_R^j \setminus \{m\}; \Delta r_m(w) < 0\}$.

Claim 7.13. *We have*

$$\int_{A_+^{R,j}} \Delta r_m d\mathrm{vol} \leq -\int_{A_-^{R,j}} \Delta r_m d\mathrm{vol} < \infty.$$

The proof is as follows. We put $\theta(s,u) = s^{n-1}\sqrt{\det(g_{ij}|_{\exp_m su})}$ for $u \in S_m M$ and $0 < s < t(u)$. Here, $g_{ij} = g(\partial/\partial x_i, \partial/\partial x_j)$ for a normal coordinate $(x_1, x_2, \dots, x_n)$ around m. By rescaling, without loss of generality, we can assume that $\mathrm{Ric}_M \geq -(n-1)$ on $B_{100R}(x)$. Then, we have

$$-\int_{A_-^{R,j}} \Delta r_x d\mathrm{vol} \leq \int_{B_R(x)} (n-1)\frac{\cosh \overline{x,w}}{\sinh \overline{x,w}} d\mathrm{vol} \tag{311}$$

$$= \int_{S_m M} \int_0^{\min\{t(u),R\}} (n-1)\frac{\cosh t}{\sinh t}\theta(t,u) dt du \tag{312}$$

$$\leq \int_{S_m M} \int_0^{R} (n-1)\frac{\cosh t}{\sinh t}\sinh^{n-1} t dt du \tag{313}$$

$$\leq \int_{S_m M} \int_0^{R} (n-1)\cosh t \sinh^{n-2} t dt du < \infty. \tag{314}$$

Since

$$\Delta r_m(w) = -\frac{n-1}{\overline{m,w}} + O(\overline{m,w})$$

for every w satisfying that $\overline{m,w}$ is sufficiently small, we have

$$\int_{B_\tau(x)} |\Delta r_x| d\mathrm{vol} \leq \int_{B_\tau(x)} \left(\frac{n-1}{\overline{x,w}} + 1\right) d\mathrm{vol} \tag{315}$$

$$= \int_{S_x M} \int_0^{\tau} \frac{n-1}{t}\theta(t,u) dt du + \mathrm{vol}\, B_\tau(x) \tag{316}$$

$$\leq C(n) \int_{S_x M} \int_0^{\tau} \frac{n-1}{t} t^{n-1} dt du + \mathrm{vol}\, B_\tau(x) \tag{317}$$

$$\overset{\tau \to 0}{\to} 0. \tag{318}$$

Therefore, we have

$$\int_{V_R^j} \Delta r_x d\mathrm{vol} = \lim_{\tau \to 0} \int_{V_R^j \setminus B_\tau(x)} \Delta r_x d\mathrm{vol}.$$

Thus, by divergence formula and Claim 7.12, we have

$$\int_{V_R^j} \Delta r_x d\mathrm{vol} = -\int_{\partial V_R^j} \langle \nabla r_x, N_w \rangle d\mathrm{vol}_{n-1} - \lim_{\tau \to 0} \int_{\partial B_\tau(x)} \langle -\nabla r_x, \nabla r_x \rangle d\mathrm{vol}_{n-1} \tag{319}$$

$$= -\int_{\partial V_R^j} \langle \nabla r_x, N_w \rangle d\mathrm{vol}_{n-1} \leq 0. \tag{320}$$

Thus, we have

$$\int_{A_+^{R,j}} \Delta r_x d\mathrm{vol} + \int_{A_-^{R,j}} \Delta r_x d\mathrm{vol} \leq 0.$$

Therefore we have Claim 7.13.

Next claim follows from Claim 7.13 directly:

Claim 7.14. *We have*

$$\int_{V_R^j} |\Delta r_x| d\mathrm{vol} \leq -2 \int_{A_-^{R,j}} \Delta r_x d\mathrm{vol} < \infty.$$

Especially, $\Delta r_x \in L^1(B_R(x))$.

Therefore, for $f \in C^\infty(\overline{B}_R(x))$, we have

$$\Delta_R^{\mathrm{dist}} r_m(f) = \lim_{j \to \infty, \tau \to 0} \int_{V_R^j \setminus B_\tau(m)} \langle df, dr_m \rangle d\mathrm{vol} \tag{321}$$

$$= \lim_{j \to \infty, \tau \to 0} \left(\int_{V_R^j \setminus B_\tau(m)} f \Delta r_x d\mathrm{vol} + \int_{\partial V_R^j} \langle dr_m, N_w \rangle f d\mathrm{vol}_{n-1} \right. \tag{322}$$

$$\left. - \int_{\partial B_\tau(m)} \langle dr_m, dr_m \rangle d\mathrm{vol}_{n-1} \right) \tag{323}$$

$$= \int_{B_R(m)} f \Delta r_m d\mathrm{vol} + \lim_{j \to \infty} \int_{\partial V_R^j} \langle dr_m, N_w \rangle f d\mathrm{vol}_{n-1}. \tag{324}$$

Claim 7.15. *For every* $w \in M$*, we have*

$$\lim_{j \to \infty} 1_{\partial V_R^j \cap \partial B_R(x)}(w) = 1_{\partial B_R(x) \setminus C_x}(w).$$

The proof is as follows. We take $w \in M$.

1. The case $w \in \partial B_R(m) \setminus C_x$. Then, there exists $u \in S_m M$ such that $R < t(u)$ and $w = \exp_m Ru$. By the definition of ϕ_R^j, we have $\phi_R^j(u) = R$ for every sufficiently large j. Thus, by Claim 7.11, we have $w = \exp_x \phi_R^j(u) u \in \partial V_R^j$. Therefore, we have $\lim_{j \to \infty} 1_{\partial V_R^j \cap \partial B_R(m)}(w) = 1_{\partial B_R(m) \setminus C_m}(w)$.

2. The case $w \in (M \setminus (\partial B_R(m) \setminus C_m)) \cap \partial B_R(m)$. Then $w \in C_x$. By Claim 7.11, since $\overline{V}_R^j \cap C_x = \emptyset$, we have $\lim_{j\to\infty} 1_{\partial V_R^j \cap \partial B_R(m)}(w) = 0 = 1_{\partial B_R(m)\setminus C_m}(w)$.

3. The case $w \in (M\setminus(\partial B_R(m)\setminus C_m))\setminus\partial B_R(m)$. Then, we have $w \in M\setminus(\partial V_R^j \cap \partial B_R(m))$ for every j. Especially, $\lim_{j\to\infty} 1_{\partial V_R^j \cap \partial B_R(m)}(w) = 0 = 1_{\partial B_R(m)\setminus C_m}(w)$.

Thus, we have Claim 7.14.

Then, since $\langle \nabla r_m, N_w\rangle f(w) 1_{\partial V_R^j \cap \partial B_R(m)}(w) \to f(w)$ for every $w \in \partial B_R(m) \setminus C_m$, by dominated convergence theorem, we have

(325)
$$\int_{\partial V_R^j \cap \partial B_R(m)} \langle \nabla r_m, N_w\rangle f d\mathrm{vol}_{n-1} = \int_{\partial B_R(m)\setminus C_m} \langle \nabla r_m, N_w\rangle f(w) 1_{\partial V_R^j \cap \partial B_R(m)}(w) d\mathrm{vol}_{n-1}$$

(326)
$$\overset{j\to\infty}{\to} \int_{\partial B_R(m)\setminus C_m} f d\mathrm{vol}_{n-1}.$$

Therefore, we have

(327)
$$\lim_{j\to\infty} \int_{\partial V_R^j} \langle \nabla r_m, N_w\rangle f(w) d\mathrm{vol}_{n-1}$$

(328)
$$= \lim_{j\to\infty} \left(\int_{\partial V_R^j \cap \partial B_R(m)} \langle \nabla r_m, N_w\rangle f(w) d\mathrm{vol}_{n-1} + \int_{\partial V_R^j \setminus \partial B_R(m)} \langle \nabla r_m, N_w\rangle f(w) d\mathrm{vol}_{n-1} \right)$$

(329)
$$= \int_{\partial B_R(m)\setminus C_x} f d\mathrm{vol}_{n-1} + \lim_{j\to\infty} \int_{\partial V_R^j \setminus \partial B_R(m)} \langle \nabla r_x, N_w\rangle f(w) d\mathrm{vol}_{n-1}.$$

We define a linear functional Φ on $C_c^\infty(B_{2R}(m))$ by

$$\Phi(f) = \lim_{j\to\infty} \int_{\partial V_R^j \setminus \partial B_R(m)} \langle \nabla r_m, N_w\rangle f(w) d\mathrm{vol}_{n-1}.$$

By Claim 7.12, if $f \geq 0$, then $\Phi(f) \geq 0$. Therefore, by Riesz's theorem, there exists a Radon measure $v_{R,m}^{\mathrm{sing}}$ on $B_{2R}(m)$ such that

$$\Phi(f) = \int_{B_{2R}(m)} f dv_{R,m}^{\mathrm{sing}}.$$

for every $f \in C_c^\infty(B_{2R}(m))$.

Claim 7.16. *We have* $\mathrm{supp}(v_{R,m}^{\mathrm{sing}}) \subset \overline{B}_R(m)$, *i.e. for every Borel set* $A \subset B_{2R}(m) \setminus \overline{B}_R(m)$, *we have* $v_{R,m}^{\mathrm{sing}}(A) = 0$.

The proof is as follows. Since $v_{R,x}^{\mathrm{sing}}$ is a Radon measure, without loss of generality, we can assume that A is compact. We take $\tau > 0$ satisfying $\tau << \min\{\overline{A, B_R(x)}, \overline{A, \partial B_{2R}(x)}\}$. We also take $\phi \in C_c^\infty(B_{2R}(x))$ satisfying $\phi|_A = 1$, $0 \leq \phi \leq 1$, $\mathrm{supp}\phi \subset B_\tau(A)$. Since $\phi|_{\overline{B}_R(x)} = 0$, by the definition of Φ, we have $\Phi(\phi) = 0$. On the other hand,

$$v_{R,x}^{\mathrm{sing}}(A) \leq \int_{B_{2R}(x)} \phi dv_{R,x}^{\mathrm{sing}} = \Phi(\phi) = 0.$$

Thus, we have Claim 7.16.

Since $V_R^j \subset \overline{B}_R(m)$, if $f_1, f_2 \in C_c^\infty(B_{2R}(m))$ satisfies $f_1|_{\overline{B}_R(m)} = f_2|_{\overline{B}_R(m)}$, then we have

$$\int_{\partial V_R^j \setminus \partial B_R(m)} \langle \nabla r_m, N_w \rangle f_1(w) d\mathrm{vol}_{n-1} = \int_{\partial V_R^j \setminus \partial B_R(m)} \langle \nabla r_m, N_w \rangle f_2(w) d\mathrm{vol}_{n-1}$$

for every j. Especially, we have $\Phi(f_1) = \Phi(f_2)$. By the definition, for every $f \in C^\infty(\overline{B}_R(m))$, there exists $F \in C_c^\infty(B_{2R}(m))$ such that $F|_{\overline{B}_R(m)} = f$. If we put $\Phi(f) = \Phi(F)$, then, $\Phi(f)$ does not depend on the choice of F. Thus for $f \in C^\infty(\overline{B}_R(m))$, $\Phi(f)$ is well defined, we have,

$$\Phi(f) = \Phi(F) = \lim_{j \to \infty} \int_{\partial V_R^j \setminus \partial B_R(m)} \langle \nabla r_m, N_w \rangle F(w) d\mathrm{vol}_{n-1} \tag{330}$$

$$= \lim_{j \to \infty} \int_{\partial V_R^j \setminus \partial B_R(m)} \langle \nabla r_m, N_w \rangle f(w) d\mathrm{vol}_{n-1} \tag{331}$$

and

$$\Phi(f) = \Phi(F) = \int_{B_{2R}(m)} F dv_{R,m}^{\mathrm{sing}} = \int_{\overline{B}_R(m)} f dv_{R,m}^{\mathrm{sing}}.$$

Therefore, we have

$$\Delta_R^{\mathrm{dist}} r_m(f) = \int_{B_R(m)} f \Delta r_m d\mathrm{vol} + \int_{\partial B_R(m) \setminus C_m} f d\mathrm{vol}_{n-1} + \int_{\overline{B}_R(m)} f dv_{R,m}^{\mathrm{sing}}$$

for every $f \in C^\infty(\overline{B}_R(x))$. By taking $f = 1$ and the definition of $\Delta_R^{\mathrm{dist}} r_m$, we have

$$0 = \Delta_R^{\mathrm{dist}} r_m(1) = \int_{B_R(m)} \Delta r_m d\mathrm{vol} + \mathrm{vol}_{n-1}(\partial B_R(m) \setminus C_m) + v_{R,m}^{\mathrm{sing}}(\overline{B}_R(m)).$$

Thus, we have

$$v_{R,x}^{\mathrm{sing}}(\overline{B}_r(x)) = -\int_{B_R(x)} \Delta r_x d\mathrm{vol} - \mathrm{vol}_{n-1}(\partial B_R(x) \setminus C_x).$$

Especially, we have

$$\int_{B_R(m)} |\Delta r_m| d\mathrm{vol} + \mathrm{vol}(\partial B_R(m) \setminus C_m) + v_{R,m}^{\mathrm{sing}}(\overline{B}_R(m)) = \int_{B_R(m)} |\Delta r_m| d\mathrm{vol} - \int_{B_R(m)} \Delta r_m d\mathrm{vol} \tag{332}$$

$$= -2 \int_{B_R(m) \cap \{\Delta r_m < 0\}} \Delta r_m d\mathrm{vol}. \tag{333}$$

Claim 7.17. *We have* $\mathrm{supp}(v_{R,m}^{\mathrm{sing}}) \subset C_m \cap \overline{B}_R(m)$.

The proos is as follows. First, we shall prove $\mathrm{supp}(v_{R,m}^{\mathrm{sing}}) \subset \partial B_R(m) \cup C_m$. It suffices to check that $v_{r,m}^{\mathrm{sing}}(A) = 0$ for every compact set $A \subset \overline{B}_R(m)$ satisfying $A \cap (\partial B_R(m) \cup C_m) = \emptyset$. We take $\tau > 0$ satisfying $\tau << \overline{A, C_m \cup \partial B_R(m)}$. We also take $\phi \in C_c^{\infty}(B_{2R}(m))$ satisfying $0 \leq \phi \leq 1$, $\phi|_A = 1$ and $\mathrm{supp}\phi \subset B_\tau(A)$. Then, we have

$$v_{R,m}^{\mathrm{sing}}(A) \leq \int_{B_{2R}(m)} \phi dv_{R,m}^{\mathrm{sing}} = \lim_{j \to \infty} \int_{\partial V_R^j \setminus \partial B_R(m)} \langle \nabla r_m, N_w \rangle \phi(w) d\mathrm{vol}_{n-1}.$$

We take j satisfying $j^{-1} < \frac{\tau}{100}$, and $w \in \partial V_R^j \setminus \partial B_R(m)$. By Claim 7.11, there exists $u \in S_x M$ such that $w = \exp_x \phi_R^j(u) u$. Since $\phi_R(u) - j^{-1} \leq \phi_R^j(u) \leq \phi_R(u)$, if $\phi_R(u) = t(u)$, then, since $\overline{w, C_m} \leq j^{-1} < \frac{\tau}{100}$, we have $\phi(w) = 0$. On the other hand, if $\phi_R(u) = R$, then, since $\overline{w, \partial B_R(m)} \leq j^{-1} < \frac{\tau}{100}$, we have $\phi(w) = 0$. Therefore, we have $\phi|_{\partial V_R^j \setminus \partial B_R(m)} = 0$. Thus, we have $v_{r,m}^{\mathrm{sing}}(A) = 0$. Finally, we shall prove $\mathrm{supp}(v_{R,m}^{\mathrm{sing}}) \subset C_m \cap \overline{B}_R(m)$. It suffices to check that $v_{r,m}^{\mathrm{sing}}(\partial B_R(m) \setminus C_m) = 0$. Since $\overline{O}_i$ is compact, there exists a sequence of nonincresing sequence $\tau_i > 0$ such that $\tau_i \to 0$ and $t(u) > R + \tau_i$ for every $u \in O_i$. We put $U_i = \{\exp_m tu; u \in O_i, R - \tau_i < t < R + \tau_i\}$ and $V_i = \{\exp_m tu; ; u \in O_i, R - \tau_{i+1}/2 < t < R + \tau_{i+1}/2\}$. Since $\overline{O}_i \subset O_{i+1}$, we have $\overline{V}_i \subset U_{i+1}$. We take $\phi_i \in C_c^{\infty}(B_{2R}(m))$ satisfying $\phi_i|_{\overline{V}_i} = 1$, $0 \leq \phi_i \leq 1$ and $\mathrm{supp}\phi_i \subset U_{i+1}$. We fix i. Then, since $U_i \cap \partial V_R^j \subset U_i \cap \partial B_R(m)$ for every sufficiently large j, we have $\mathrm{supp}\phi_i \cap (\partial V_R^j \setminus \partial B_R(m)) \subset U_{i+1} \cap (\partial V_R^j \setminus \partial B_R(m)) = \emptyset$ for every sufficiently large j. Thus, we have

$$v_{R,m}^{\mathrm{sing}}(\partial B_R(m) \cap V_i) \leq \int_{B_{2R}(m)} \phi_i dv_{R,m}^{\mathrm{sing}} \tag{334}$$

$$= \lim_{j \to \infty} \int_{\partial V_R^j \setminus \partial B_R(m)} \langle \nabla r_m, N_w \rangle \phi_i(w) d\mathrm{vol}_{n-1} \tag{335}$$

$$= 0. \tag{336}$$

By letting $i \to \infty$, we have $v_{R,x}^{\mathrm{sing}}(\partial B_R(x) \setminus C_x) = 0$. Therefore, we have Claim 7.17.

Thus, we have the assertion. □

The following corollary is used in the proof of Theorem 6.1. See also [4, Theorem 4.1].

Corollary 7.18. *Let H be a real number, (M, m) a pointed complete n-dimensional $(n \geq 2)$ Riemannian manifold with $\mathrm{Ric}_M \geq (n-1)H$, R a positive number and f a nonnegative valued Lipschitz function on $\overline{B}_R(m)$. Then, we have*

$$\int_{B_R(m)} \langle df, dr_m \rangle d\mathrm{vol} \geq -(n-1) \int_{B_R(m)} \frac{\underline{k}_H'(\overline{m, w})}{\underline{k}_H(\overline{m, w})} f(w) d\mathrm{vol}.$$

7.4 Co-area formula for distance functions

In this subsection, we shall give several measure theoretical properties on non-collapsing Euclidean cones. For example, we will prove co-area formula for distance functions (see Proposition 7.22). Throughout this subsection, we fix an $(n, -1)$-Ricci limit space $(n \geq 2)$ (Y, y, υ) and assume that the following properties hold:

1. There exists a compact geodesic space X such that $\mathrm{diam} X \leq \pi$ and $(Y, y) = (C(X), p)$.

2. $\dim_H X = n - 1$. Here, $\dim_H X$ is the Hausdorff dimension of X.

Then by [7, Theorem 5.9], there exists $C > 0$ such that $\upsilon = CH^n$. First, we shall recall definitions of lower dimensional Hausdorff measures associated to υ and standard (spherical) Hausdorff measures (see section 2 in [8]). For convenience, we will use the notaion below: $r^{-\alpha}\upsilon(B_r(x)) = 0$ for every $x \in Y$ and $\alpha \geq 0$ if $r = 0$. For $\alpha \in \mathbf{R}_{\geq 0}$, $\delta > 0$ and a set $A \subset Y$, we put

$$(\upsilon_{-\alpha})_\delta(A) = \inf\left\{\sum_{i=1}^{\infty} r_i^{-\alpha}\upsilon(B_{r_i}(x_i));\ x_i \in Y,\ 0 \leq r_i < \delta,\ A \subset \bigcup_{i=1}^{\infty} B_{r_i}(x_i)\right\},$$

$$(H^\alpha)_\delta(A) = \inf\left\{\sum_{i=1}^{\infty} \omega_\alpha r_i^\alpha;\ x_i \in Y,\ 0 \leq r_i < \delta,\ A \subset \bigcup_{i=1}^{\infty} B_{r_i}(x_i)\right\}$$

and

$$\upsilon_{-\alpha}(A) = \lim_{\delta \to 0}(\upsilon_{-\alpha})_\delta(A), \quad H^\alpha(A) = \lim_{\delta \to 0}(H^\alpha)_\delta(A).$$

For a subset $A \subset \{1\} \times X \subset C(X)$. we also put

$$(\upsilon_{-\alpha})_{X,\delta}(A) = \left\{\sum_{i=1}^{\infty} r_i^{-\alpha}\upsilon(B_{r_i}(x_i));\ x_i \in \{1\} \times X,\ 0 \leq r_i < \delta,\ A \subset \bigcup_{i=1}^{\infty} B_{r_i}(x_i)\right\},$$

$$(H^\alpha)_{X,\delta}(A) = \left\{\sum_{i=1}^{\infty} \omega_\alpha r_i^\alpha;\ x_i \in \{1\} \times X,\ 0 \leq r_i < \delta,\ A \subset \bigcup_{i=1}^{\infty} B_{r_i}(x_i)\right\}$$

and

$$(\upsilon_{-\alpha})_X(A) = \lim_{\delta \to 0}(\upsilon_{-\alpha})_\delta(A), \quad H^\alpha_X(A) = \lim_{\delta \to 0}(H^\alpha)_\delta(A).$$

We remark that $\upsilon_{-\alpha}(A) \leq (\upsilon_{-\alpha})_X(A)$, $H^\alpha(A) \leq H^\alpha_X(A)$ for every subset $A \subset \{1\} \times X$ and that if we define a map ϕ from $(X, d_X) \to (\{1\} \times X, d_{C(X)})$ by $\phi(x) = (1, x)$, then $H^{n-1}(A) = H^{n-1}_X(\phi(A))$ for every $A \subset X$.

Lemma 7.19. *We have $\upsilon_{-1}(A) = (\upsilon_{-1})_X(A)$ for every Borel set $A \subset X$.*

PROOF. We fix sufficiently small positive numbers $\delta, \epsilon > 0$. By definition, there exists $\{B_{r_i}(x_i)\}_i$ such that $0 \leq r_i < \delta$, $x_i = (t_i, w_i) \in C(X) = \mathbf{R}_{\geq 0} \times X/\{0\} \times X$ and $\left|(\upsilon_1)_\delta(A) - \sum_{i=1}^\infty r_i^{-1}\upsilon(B_{r_i}(x_i))\right| < \epsilon$. Without loss of generality, we can assume that $B_{r_i}(x_i) \cap A \neq \emptyset$ for every i. We put $y_i = (1, w_i) \in C(X)$ and $\hat{y}_i = (1, w_i) \in (\mathbf{R} \times X, \sqrt{d_{\mathbf{R}^2}^2 + d_X^2})$. It is easy to check that the map $\Phi_i(s, z) = (s, z)$ from $B_{5r_i}(x_i)$ to $\mathbf{R} \times X$ gives $(1 \pm \Psi(\delta))$-bi-Lipschitz equivalent to the image. Therefore, we have $B_{r_i}(x_i) \cap (\{1\} \times X) \subset B_{(1+\Psi(\delta))\sqrt{r_i^2 - \overline{x_i, y_i}^2}}(y_i)$. On the other hand, since $|t_i - 1| \leq \delta$, a map $\hat{\Phi}_i(t, w) = (t + t_i - 1, w)$ from $B_{(1+\Psi(\delta))r_i}(\hat{y}_i)$ to $C(X)$ gives $(1 \pm \Psi(\delta))$-bi-Lipschitz equivalent to the image. By $\hat{\Phi}_i(\hat{y}_i) = x_i$, we have $\operatorname{Image}\hat{\Phi} \subset B_{(1+\Psi(\delta))r_i}(x_i)$. Therefore, we have $H^n(B_{(1+\Psi(\delta))r_i}(y_i)) \leq (1 + \Psi(\delta))H^n(B_{(1+\Psi(\delta))r_i}(x_i)) \leq (1 + \Psi(\delta))H^n(B_{r_i}(x_i))$. Thus, since $\upsilon = CH^n$, we have

$$(\upsilon_{-1})_{X,(1+\Psi(\delta))\delta}(A) \leq \sum_{i=1}^\infty ((1 + \Psi(\delta))r_i)^{-1} CH^n(B_{(1+\Psi(\delta))r_i}(y_i)) \tag{337}$$

$$\leq (1 + \Psi(\delta)) \sum_{i=1}^\infty r_i^{-1} CH^n(B_{r_i}(x_i)) \tag{338}$$

$$\leq (1 + \Psi(\delta))((\upsilon_{-1})_{X,\delta}(A) + \epsilon). \tag{339}$$

By letting $\epsilon \to 0$ and $\delta \to 0$, we have the assertion. □

Similarly, we have the following lemma:

LEMMA 7.20. *We have $H_X^{n-1}(A) = H^{n-1}(A)$ for every Borel set $A \subset \{1\} \times X$.*

We shall remark the following: By Bishop-Gromov volume comparison theorem for υ, there exists $V > 1$ such that $V^{-1} \leq \lim_{r\to 0} \upsilon(B_r(x))/\omega_n r^n \leq V$ for every $x \in B_2(p)$. On the other hand, since $\upsilon = CH^n$, we have $\lim_{r\to 0} \upsilon(B_r((t, w)))/\omega_n r^n = \lim_{r\to 0} \upsilon(B_r((s, w)))/\omega_n r^n$ for every $0 < s < t < \infty$ and $w \in X$. By these facts and Corollary 7.4, it is easy to check that there exists $C_1 > 1$ such that $C_1^{-1}\upsilon_{-1}(A) \leq H^{n-1}(A) \leq C_1\upsilon_{-1}(A)$ for every Borel subset A of $C(X)$.

LEMMA 7.21. *The product measure $H^1 \times H^{n-1}$ on $\mathbf{R} \times X$ is equal to H^n.*

PROOF. It suffices to check that $H^n([0, a] \times A) = aH^{n-1}(A)$ for every Borel subset A os X and $a > 0$. By Corollary 3.58, there exists a Borel subset $\hat{X}$ of X such that the following properties hold:

1. $H^{n-1}(X \setminus \hat{X}) = 0$.
2. For every $x \in \hat{X}$ and $\epsilon > 0$, there exist $r_x^\epsilon > 0$ such that for every $0 < r < r_x^\epsilon$, there exist a compact set $C_r^x \subset \overline{B}_r(x)$ and a Lipschitz ϕ_r^x from C_r^x to $\mathbf{R}^{n-1}$ such that
$$\frac{H^{n-1}(\overline{B}_r(x) \setminus C_r^x)}{H^{n-1}(\overline{B}_r(x))} \leq \epsilon$$

and that ϕ_r^x gives $(1 \pm \epsilon)$-bi-Lipschitz equivalent to the image.

For every $x \in \hat{X}$ and $\epsilon > 0$, by Fubini's theorem, we have

$$H^n([0,a] \times C_r^x) = (1 \pm \epsilon) H^n([0,a] \times \phi_r^x(C_r^x)) \tag{340}$$

$$= (1 \pm \epsilon) a H^{n-1}(\phi_r^x(C_r^x)) \tag{341}$$

$$= (1 \pm \epsilon) a H^{n-1}(C_r^x) \tag{342}$$

$$= (1 \pm \epsilon) a H^{n-1}(\overline{B}_r(x)) \tag{343}$$

for every sufficiently small $r > 0$. On the other hand, by the proof of [44, Lemma 5.2], we have $H^n([0,a] \times \hat{A}) \leq C(n) a H^{n-1}(\hat{A})$ for every $\hat{A} \subset X$. Thus, we have

$$\lim_{r \to 0} \frac{H^n([0,a] \times \overline{B}_r(x))}{a H^{n-1}(\overline{B}_r(x))} = 1$$

for every $x \in \hat{X}$. Therefore, there exists a Borel set $\hat{A} \subset A$ such that $H^{n-1}(A \setminus \hat{A}) = 0$

$$\lim_{r \to 0} \frac{H^n([0,a] \times \overline{B}_r(x))}{a H^{n-1}(\overline{B}_r(x))} = 1$$

and

$$\lim_{r \to 0} \frac{H^{n-1}(A \cap \overline{B}_r(x))}{H^{n-1}(\overline{B}_r(x))} = 1$$

for every $x \in \hat{A}$. We remark that $H^n([0,a] \times (A \setminus \hat{A})) \leq C(n) a H^{n-1}(A \setminus \hat{A}) = 0$. We fix a sufficiently small $\epsilon > 0$. By Proposition 2.12, there exists a pairwise disjoint collection $\{\overline{B}_{r_i}(x_i)\}_{i \in \mathbf{N}}$ such that $x_i \in \hat{A}$, $r_i < \epsilon$, $\hat{A} \setminus \bigcup_{i=1}^N \overline{B}_{r_i}(x_i) \subset \bigcup_{i=N+1}^\infty \overline{B}_{5r_i}(x_i)$ for every $N \in \mathbf{N}$ and

$$\left| \frac{H^n([0,a] \times \overline{B}_r(x_i))}{a H^{n-1}(\overline{B}_r(x_i))} - 1 \right| + \left| \frac{H^{n-1}(A \cap \overline{B}_r(x_i))}{H^{n-1}(\overline{B}_r(x_i))} - 1 \right| < \epsilon$$

for every $0 < r < r_i$. We take N satisfying $\sum_{i=N+1}^\infty H^{n-1}(\overline{B}_{r_i}(x_i)) < \epsilon$. Then, we have

$$H^n([0,a] \times \hat{A}) \leq \sum_{i=1}^N H^n([0,a] \times \overline{B}_{r_i}(x_i)) + \sum_{i=N+1}^\infty H^n([0,a] \times \overline{B}_{5r_i}(x_i)) \tag{344}$$

$$\leq \sum_{i=1}^N H^n([0,a] \times \overline{B}_{r_i}(x_i)) + aC(n) \sum_{i=N+1}^\infty H^{n-1}(\overline{B}_{5r_i}(x_i)) \tag{345}$$

$$\leq \sum_{i=1}^N H^n([0,a] \times \overline{B}_{r_i}(x_i)) + \Psi(\epsilon; n, a, C_1) \tag{346}$$

$$\leq a \sum_{i=1}^N H^{n-1}(\overline{B}_{r_i}(x_i)) + \Psi(\epsilon; n, a, C_1) \tag{347}$$

$$\leq a(1+\epsilon)(H^{n-1}(\hat{A}) + \epsilon) + \Psi(\epsilon; n, a, C_1). \tag{348}$$

Therefore, we have

$$H^n([0,a]\times A)\le aH^{n-1}(A).$$

On the other hand, we have

$$aH^{n-1}(A)=a\left(\sum_{i=1}^N H^{n-1}(\overline{B}_{r_i}(x_i))+\Psi(\epsilon;n,C_1)\right)\le(1+\epsilon)\sum_{i=1}^N H^n([0,a]\times\overline{B}_{r_i}(x_i))+\Psi(\epsilon;n,a,C_1)$$

and

$$\frac{H^n([0,a]\times(\overline{B}_{r_i}(x_i)\setminus A))}{H^n([0,a]\times\overline{B}_{r_i}(x_i))}\le C(n)(1+\epsilon)\frac{aH^{n-1}(\overline{B}_{r_i}(x_i)\setminus A)}{aH^{n-1}(\overline{B}_{r_i}(x_i))}\le\Psi(\epsilon;n).$$

Therefore, we have

$$aH^{n-1}(A)\le(1+\epsilon)\sum_{i=1}^N H^n([0,a]\times\overline{B}_{r_i}(x_i))+\Psi(\epsilon;n,a,C_1) \tag{349}$$

$$\le(1+\Psi(\epsilon;n))\sum_{i=1}^N H^n\left(([0,a]\times\overline{B}_{r_i}(x_i))\cap A\right)+\Psi(\epsilon;n,a,C_1) \tag{350}$$

$$\le(1+\Psi(\epsilon;n))H^n([0,a]\times A)+\Psi(\epsilon;n,a,C_1). \tag{351}$$

Therefore, we have

$$aH^{n-1}(A)\le H^n([0,a]\times A).$$

Thus, we have the assertion. □

Proposition 7.22 (Co-area formula for distance functions on non-collapsing Euclidean cones). *We have*

$$\int_{C(X)}fdH^n=\int_0^\infty\int_{\partial B_t(p)}fdH^{n-1}dt$$

for every $f\in L^1(C(X))$.

Proof. By [42, Theorem 5.2] and $C_1\upsilon_{-1}\le H^{n-1}\le C_1\upsilon_{-1}$, it suffices to check that

$$\lim_{r\to0}\frac{1}{H^n(B_r(x))}\int_0^\infty H^{n-1}(\partial B_t(p)\cap\overline{B}_r(x))dt=1$$

for every $x\in C(X)\setminus\{p\}$. We put $R=\overline{p,x}>0$ and fix sufficiently small $r>0$. Then, since a map $\Phi(t,w)=(t,w)$ from $B_r(x)$ to $\mathbf{R}\times X$ gives $(1\pm\Psi(r))$-bi-Lipschitz equivalent to the image, we have

$$B_{(1-\Psi(r))r}(\Phi(x))\subset\Phi(B_r(x))\subset B_{(1+\Psi(r))r}(\Phi(x)).$$

On the other hand, by Lemma 7.21 and Fubini's Theorem, we have

$$H^n\left(B_{(1+\Psi(r))r}(\Phi(x))\right) = \int_{R-(1+\Psi(r))r}^{R+(1+\Psi(r))r} H^{n-1}\left((\{t\}\times X)\cap B_{(1+\Psi(r))r}(\Phi(x))\right)dt.$$

Since $\Phi(\partial B_t(p)\cap B_r(x)) \subset (\{t\}\times X)\cap B_{(1+\Psi(r))r}(\Phi(x))$, we have

$$H^n\left(B_{(1+\Psi(r))r}(\Phi(x))\right) \geq (1-\Psi(r;n))\int_{R-(1+\Psi(r))r}^{R+(1+\Psi(r))r} H^{n-1}(\partial B_t(p)\cap B_r(x))dt.$$

Therefore, we have

$$1 \geq \limsup_{r\to 0}\frac{1}{H^n(B_r(x))}\int_0^\infty H^{n-1}(\partial B_t(p)\cap B_r(x))dt.$$

Similarly, we have

$$1 \leq \liminf_{r\to 0}\frac{1}{H^n(B_r(x))}\int_0^\infty H^{n-1}(\partial B_t(p)\cap B_r(x))dt.$$

Therefore, we have the assertion. □

PROPOSITION 7.23. *We have $\upsilon_{-1}(A) = C(n)CH^{n-1}(A)$ for every Borel set $A\subset\{1\}\times X$.*

PROOF. By [16], we have

$$\lim_{r\to 0}\frac{H^n(B_r(z))}{\omega_n r^n} = 1$$

for every $z\in\mathcal{R}_n(Y)$. Since $\mathcal{R}_n(Y)\cap(\{1\}\times X) = \{1\}\times\mathcal{R}_{n-1}(X)$, by Proposition 7.22, we have $H^{n-1}(X\setminus\mathcal{R}_{n-1}(X)) = 0$. We fix $\epsilon,\delta,\tau>0$. We put

$$A_\tau = \left\{a\in A\cap\mathcal{R}_{n-1}(X);\ \left|\frac{H^n(B_r(a))}{\omega_n r^n}-1\right| < \epsilon \text{ for every } 0<r\leq\tau\right\}.$$

By the definition of υ_{-1}, there exists $\{B_{r_i}(x_i)\}_i$ such that $x_i\in A_\tau$, $r_i<\min\{\delta,\tau\}$ and $|\upsilon_{-1}(A_\tau)-\sum_{i=1}^\infty r_i^{-1}\upsilon(B_{r_i}(x_i))|<\epsilon$. Thus, we have

$$(H^{n-1})_\delta(A_\tau) \leq \sum_{i=1}^\infty \omega_{n-1}r_i^{n-1} \tag{352}$$

$$\leq \sum_{i=1}^\infty \frac{\omega_{n-1}}{\omega_n}r_i^{-1}(1+\epsilon)H^n(B_{r_i}(x_i)) \tag{353}$$

$$= \sum_{i=1}^\infty \frac{\omega_{n-1}}{\omega_n}(1+\epsilon)r_i^{-1}C^{-1}\upsilon(B_{r_i}(x_i)) \tag{354}$$

$$\leq \sum_{i=1}^\infty \frac{\omega_{n-1}}{\omega_n}(1+\epsilon)C^{-1}(\upsilon_{-1}(A_\tau)+\epsilon). \tag{355}$$

By letting $\delta\to 0$, $\tau\to 0$ and $\epsilon\to 0$, we have

$$CH^{n-1}(A) \leq \frac{\omega_{n-1}}{\omega_n}\upsilon_{-1}(A).$$

Claim 7.24. *There exists a Borel subset Z of $\{1\}\times X$ such that $H^{n-1}((\{1\}\times X)\setminus Z) = 0$,*

$$\lim_{r\to 0}\frac{H^{n-1}(\overline{B}_r(z)\cap(\{1\}\times X))}{\omega_{n-1}r^{n-1}} = 1$$

for every $z\in Z$.

The proof is as follows. Let x be a point in X and $\{r_i\}_i$ a sequence of positive numbers satisfying $r_i\to 0$. We assume that there exists a tangent cone $(T_xX, 0_x)$ of X at x such that $(X, x, r_i^{-1}d_X)\to(T_xX, 0_x)$. By [44, Claim 4.5] and [7, Theorem 5.9], we have $(C(X), r_i^{-1}d_{C(X)}, (1,x), H^n)\to(\mathbf{R}\times T_xX, (0,0_x), H^n)$. Moreover, By the H^{n-1}-rectifiability of T_xX (Corollary 3.58) and an argument similar to the proof of Lemma 7.21, we have $H^1\times H^{n-1} = H^n$ on $\mathbf{R}\times T_xX$. Since a sequence of compact sets $[-1,1]\times\overline{B}_1^{r_i^{-1}d_X}(x)\subset C(X)$ converges to $[-1,1]\times\overline{B}_1(0_x)$, by Proposition 2.14 and Proposition 4.13, we have

$$\lim_{i\to\infty}H^n([-1,1]\times\overline{B}_1^{r_i^{-1}d_X}(x)) = H^n([-1,1]\times\overline{B}_1(0_x)).$$

By Proposition 7.22, we have $H^n([-1,1]\times\overline{B}_1^{r_i^{-1}d_X}(x)) = 2H^{n-1}(\overline{B}_1^{r_i^{-1}d_X}(x))$. Especially, we have

$$\lim_{i\to\infty}H^{n-1}(\overline{B}_1^{r_i^{-1}d_X}(x)) = H^{n-1}(\overline{B}_1(0_x)).$$

Therefore, if we put $Z = \mathcal{R}_n(Y)\cap(\{1\}\times X)$, then we have Claim 7.24.

We put $W = \mathrm{Leb}(A\cap Z)$ with respect to the measure H^{n-1}. By Proposition 2.12, there exists a pairwise disjoint collection $\{\overline{B}_{r_i}(a_i)\}_i$ such that $a_i\in W$, $r_i<\delta/100$, $W\setminus\bigcup_{i=1}^N\overline{B}_{r_i}(a_i)\subset\bigcup_{i=N+1}^\infty\overline{B}_{5r_i}(a_i)$ for every N and

$$\left|\frac{H^n(B_{r_i}(a_i))}{\omega_n r_i^n}-1\right| + \left|\frac{H^{n-1}(\overline{B}_{r_i}(a_i)\cap W)}{\omega_{n-1}r_i^{n-1}}-1\right| < \epsilon$$

for every i. We take N satisfying $\sum_{i=N+1}^\infty H^{n-1}(\overline{B}_{r_i}(a_i)\cap W)<\epsilon$. Therefore, we have $\sum_{N+1}^\infty H^{n-1}(\overline{B}_{5r_i}(a_i)\cap W)<\Psi(\epsilon;n,C_1)$. Then, by the assumption, we have $\sum_{i=N+1}^\infty\omega_{n-1}r_i^{n-1}\le$

$\Psi(\epsilon; n, C_1)$. Therefore, we have

$$(\upsilon_{-1})_\delta(W) \le \sum_{i=1}^{N} r_i^{-1}\upsilon(\overline{B}_{r_i}(a_i)) + \sum_{i=N+1}^{\infty} (5r_i)^{-1}\upsilon(\overline{B}_{5r_i}(a_i)) \tag{356}$$

$$\le \sum_{i=1}^{N} r_i^{-1} C H^n(\overline{B}_{r_i}(a_i)) + \sum_{i=N+1}^{\infty} C(n) C r_i^{n-1} \tag{357}$$

$$\le \sum_{i=1}^{N} r_i^{-1} C H^n(\overline{B}_{r_i}(a_i)) + \Psi(\epsilon; n, C, C_1) \tag{358}$$

$$\le \sum_{i=1}^{N} C\omega_n r_i^{n-1}(1+\epsilon) + \Psi(\epsilon; n, C, C_1) \tag{359}$$

$$\le \frac{C\omega_n}{\omega_{n-1}}(1+\epsilon)\sum_{i=1}^{N} H^{n-1}(\overline{B}_{r_i}(a_i) \cap W) + \Psi(\epsilon; n, C, C_1) \tag{360}$$

$$\le \frac{C\omega_n}{\omega_{n-1}}(1+\epsilon) H^{n-1}(W) + \Psi(\epsilon; n, C, C_1). \tag{361}$$

By letting $\delta \to 0$ and $\epsilon \to 0$, we have

$$\upsilon_{-1}(A) \le \frac{C\omega_n}{\omega_{n-1}} H^{n-1}(A).$$

Thus, we have the assertion. □

We end this subsection by giving a proof of the following proposition:

PROPOSITION 7.25. *We have*

$$H^{n-1}(B_t(x)) \le C(n)\frac{t^{n-1}}{s^{n-1}} H^{n-1}(B_s(x))$$

for every $0 < s < t \le \pi$ *and* $x \in X$.

PROOF. We remark that there exists $C_2 > 1$ such that for every metric space $\hat{X}$, a bi-Lipschitz map $f_{\hat{X}}(\hat{x}) = (1, \hat{x})$ from $\hat{X}$ to $\{1\} \times \hat{X} \subset C(\hat{X})$ satisfies $\mathbf{Lip} f_{\hat{X}} + \mathbf{Lip} f_{\hat{X}}^{-1} \le C_2$.

Therefore, by [42, Theorem 5.7] and Proposition 7.22, we have

$$H^{n-1}(B_t(x)) \le C(n)H^{n-1}(B_{C_2t}(1,x)\cap(\{1\}\times X)) \tag{362}$$

$$= C(n)C^{-1}\upsilon_{-1}(B_{C_2t}(1,x)\cap(\{1\}\times X)) \tag{363}$$

$$\le \frac{C(n)\upsilon\left(C_p(B_{C_2t}(1,x)\cap(\{1\}\times X))\cap A_p(\max\{0,1-C_2t\},1)\right)}{C\mathrm{vol}\,A_p(\max\{0,1-C_2t\},1)} \tag{364}$$

$$\le \frac{C(n)}{Ct}\upsilon(B_{5C_2t}(1,x)) \tag{365}$$

$$\le \frac{C(n)}{Ct}\frac{t^n}{s^n}\upsilon(B_{C_2^{-1}s}(1,x)) \tag{366}$$

$$\le C(n)\frac{t^{n-1}}{s^n}\int_{\max\{0,1-C_2^{-1}s\}}^{1+C_2^{-1}s} H^{n-1}(\partial B_r(p)\cap B_{C_2^{-1}s}(1,x))dr \tag{367}$$

$$\le C(n)\frac{t^{n-1}}{s^n}\int_{\max\{0,1-C_2^{-1}s\}}^{1+C_2^{-1}s} r^{n-1}H^{n-1}(\partial B_1(p)\cap B_{C_2^{-1}s}(1,x))dr \tag{368}$$

$$\le C(n)\frac{t^{n-1}}{s^n}sH^{n-1}(\partial B_1(p)\cap B_{C_2^{-1}s}(1,x)) \tag{369}$$

$$\le C(n)\frac{t^{n-1}}{s^{n-1}}H^{n-1}(\partial B_1(p)\cap B_{C_2^{-1}s}(1,x)) \tag{370}$$

$$\le C(n)\frac{t^{n-1}}{s^{n-1}}H^{n-1}(B_s(x)). \tag{371}$$

□

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