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On monodromically full points of configuration spaces of hyperbolic curves

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Abstract. In the present paper, we introduce and discuss the notion of *monodromically full points* of configuration spaces of hyperbolic curves. This notion leads to complements to M. Matsumoto's result concerning the difference between the kernels of the natural homomorphisms associated to a hyperbolic curve and its point from the Galois group to the *automorphism* and *outer automorphism* groups of the geometric fundamental group of the hyperbolic curve. More concretely, we prove that any hyperbolic curve over a number field has *many* "*nonexceptional*" closed points, i.e., closed points which do *not* satisfy a condition considered by Matsumoto, but that there exist infinitely many hyperbolic curves which admit *many* "*exceptional*" closed points, i.e., closed points which do satisfy the condition considered by Matsumoto. Moreover, we prove a Galois-theoretic characterization of equivalence classes of monodromically full points of configuration spaces, as well as a Galois-theoretic characterization of equivalence classes of quasi-monodromically full points of cores. In a similar vein, we also prove a necessary and sufficient condition for quasi-monodromically full Galois sections of hyperbolic curves to be geometric.

Contents

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Introduction

In the present §, let l be a prime number, k a field *of characteristic* 0, $\overline{k}$ an algebraic closure of k, and X a hyperbolic curve of type (g, r) over k. Moreover, for an algebraic extension $k' \subseteq \overline{k}$ of k, write $G_{k'} \stackrel{\text{def}}{=} \text{Gal}(\overline{k}/k')$ for the absolute Galois group of k' determined by the given algebraic closure $\overline{k}$. In the present paper, we introduce and discuss the notion of *monodromically full points* of configuration spaces of hyperbolic curves. The term "monodromically full" is a term introduced by the author in [9], but the corresponding notion was studied by M. Matsumoto and A. Tamagawa in [12]. If, for a positive integer n, we write X_n for the n-th configuration space of the hyperbolic curve X/k, then the natural projection $X_{n+1} \to X_n$ to the first n factors may be regarded as a *family of hyperbolic curves* of type $(g, r+n)$. In the present paper, we shall say that a closed point $x \in X_n$ of the n-th configuration space X_n is *l-monodromically full* if the $k(x)$-rational point — where $k(x)$ is the residue field at x — of $X_n \otimes_k k(x)$ determined by x is an l-monodromically full point with respect to the family of hyperbolic curves $X_{n+1} \otimes_k k(x)$ over $X_n \otimes_k k(x)$ in the sense of [9], Definition 2.1, i.e., roughly speaking, the image of the pro-l outer monodromy representation of $\pi_1(X_n \otimes_k \overline{k})$ with respect to the family of hyperbolic curves X_{n+1} over X_n is *contained in* the image of the pro-l outer Galois representation of $G_{k(x)}$ with respect to the hyperbolic curve $X_{n+1} \times_{X_n} \operatorname{Spec} k(x)$ over $k(x)$. (See Definition 3 for the precise definition of the notion of l-monodromically full points — cf. also Remark 4.)

By considering the notion of monodromically full points, one can give some complements to Matsumoto's result obtained in [13] concerning the difference between the kernels of the natural homomorphisms associated to a hyperbolic curve and its point from the Galois group to the *automorphism* and *outer automorphism* groups of the geometric fundamental group of the hyperbolic curve. To state these complements, let us review the result given in [13]: Write $\Delta^{\{l\}}_{X/k}$ for the geometric pro-l fundamental group of X — i.e., the maximal pro-l quotient of the étale fundamental group $\pi_1(X \otimes_k \overline{k})$ of $X \otimes_k \overline{k}$ — and $\Pi^{\{l\}}_{X/k}$ for the geometrically pro-l fundamental group of X — i.e., the quotient of the étale fundamental group $\pi_1(X)$ of X by the kernel of the natural surjection $\pi_1(X \otimes_k \overline{k}) \twoheadrightarrow \Delta^{\{l\}}_{X/k}$. Then since the closed subgroup $\Delta^{\{l\}}_{X/k} \subseteq \Pi^{\{l\}}_{X/k}$ is *normal* in $\Pi^{\{l\}}_{X/k}$, conjugation by elements of $\Pi^{\{l\}}_{X/k}$ determines a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta^{\{l\}}_{X/k} & \longrightarrow & \Pi^{\{l\}}_{X/k} & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \widetilde{\rho}^{\{l\}}_{X/k}} & & \downarrow{\scriptstyle \rho^{\{l\}}_{X/k}} & & \\ 1 & \longrightarrow & \mathrm{Inn}(\Delta^{\{l\}}_{X/k}) & \longrightarrow & \mathrm{Aut}(\Delta^{\{l\}}_{X/k}) & \longrightarrow & \mathrm{Out}(\Delta^{\{l\}}_{X/k}) & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are *exact*, and the left-hand vertical arrow is, in fact, an *isomorphism*. On the other hand, if $x \in X$ is a closed point of X, then we have a homomorphism $\pi_1(x)\colon G_{k(x)} \to \Pi^{\{l\}}_{X/k}$ induced by $x \in X$ (which is well-defined up to $\Pi^{\{l\}}_{X/k}$-conjugation). In [13], Matsumoto studied the difference between the kernels of the following two homomorphisms:

$$\rho^{\{l\}}_{X/k}|_{G_{k(x)}}\colon G_{k(x)} \longrightarrow \mathrm{Out}(\Delta^{\{l\}}_{X/k}) \ ;$$

$$G_{k(x)} \overset{\pi_1(x)}{\longrightarrow} \Pi^{\{l\}}_{X/k} \overset{\widetilde{\rho}^{\{l\}}_{X/k}}{\longrightarrow} \mathrm{Aut}(\Delta^{\{l\}}_{X/k}) \, .$$

Now we shall say that $E(X, x, l)$ *holds* if the kernels of the above two homomorphisms coincide and write

$$X^{E_l} \subseteq X^{\mathrm{cl}}$$

for the subset of the set X^{cl} of closed points of X consisting of "*exceptional*" $x \in X^{\mathrm{cl}}$ such that $E(X, x, l)$ holds (cf. [13], §1, §3, as well as Definition 4 in the present paper). Then the main result of [13] may be stated as follows:

> *Let $g \geq 3$ be an integer. Suppose that l* **divides** $2g-2$*; write l^ν for the highest power of l that divides* $2g-2$*. Then there are* **infinitely many** *isomorphism classes of pairs (K, C) of* **number fields** *K and* **proper hyperbolic curves** *C of genus g over K which satisfy the following condition: For any closed point $x \in C$ of C with residue field $k(x)$, if l^ν does* **not divide** $[k(x) : k]$*, then $E(C, x, l)$ does* **not hold**.

In the present paper, we prove that if a closed point $x \in X$ of the hyperbolic curve X is *l-monodromically full*, then $E(X, x, l)$ does *not hold* (cf. Proposition 11, (ii)). On the other hand, as a consequence of Hilbert's irreducibility theorem, any hyperbolic curve over a number field has *many* l-monodromically full points (cf. Proposition 2, as well as, [12], Theorem 1.2, or [9], Theorem 2.3). By applying these observations, one can prove the following result, which may be regarded as a *partial generalization* of the above theorem due to Matsumoto (cf. Theorem 1):

Theorem A (Existence of many nonexceptional closed points). *Suppose that k is a* **number field**. *If we regard the set X^{cl} of closed points of X as a subset of $X(\mathbb{C})$, then the complement*

$$X^{\mathrm{cl}} \setminus X^{E_l} \subseteq X(\mathbb{C})$$

is **dense with respect to the complex topology** *of $X(\mathbb{C})$. Moreover, the intersection*

$$X(k) \cap X^{E_l} \subseteq X(k)$$

is **finite**.

On the other hand, in [13], §2, Matsumoto proved that for any prime number l, the triple

$$(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}, \overrightarrow{01}, l)$$

— where $\overrightarrow{01}$ is a $\mathbb{Q}$-rational tangential base point — is a triple for which "$E(X, x, l)$" holds. As mentioned in [13], §2, the fact that "$E(X, x, l)$" holds for this triple was observed by P. Deligne and Y. Ihara. However, by definition, in fact, a *tangential base point is not a point*. In this sense, *no example of a triple* "(X, x, l)" for which $E(X, x, l)$ holds appears in [13]. The following result is a result concerning the *existence of triples* "(X, x, l)" for which $E(X, x, l)$ holds (cf. Theorem 2):

Theorem B (Existence of many exceptional closed points for certain hyperbolic curves). *Suppose that X is either of* **type $(0, 3)$** *or of* **type $(1, 1)$**. *Let $Y \to X$ be a finite étale covering over k which* **arises from** *an open subgroup of the geometrically pro-l fundamental group $\Pi^{\{l\}}_{X/k}$ of X and is* **geometrically connected** *over k. (Thus, Y is a* **hyperbolic curve over k**.*) Then the subset $Y^{E_l} \subseteq Y^{\mathrm{cl}}$ is* **infinite**. *In particular, the subset $X^{E_l} \subseteq X^{\mathrm{cl}}$ is* **infinite**.

Note that in Remark 13, we also give an example of a triple "(X, x, l)" such that X is a *proper* hyperbolic curve, and, moreover, $E(X, x, l)$ holds.

If $x \in X_n(k)$ is a k-rational point of the n-th configuration space X_n of the hyperbolic curve X/k, then it follows from the various definitions involved that the k-rational point $x \in X_n(k)$ determines n *distinct* k-rational points of X. Write

$$X[x] \subseteq X$$

for the hyperbolic curve of type $(g, r + n)$ over k obtained by taking the complement in X of the images of n distinct k-rational points of X determined by x, i.e., $X[x]$ may be regarded as the fiber product of the diagram of schemes

$$\begin{array}{ccc} & & X_{n+1} \\ & & \downarrow \\ \operatorname{Spec} k & \xrightarrow[x]{} & X_n\,. \end{array}$$

Here, for two k-rational points x and y of X_n, we shall say that x is *equivalent to* y if $X[x] \simeq X[y]$ over k. In [9], the author proved that the isomorphism class of a certain (e.g., *split* — cf. [9], Definition 1.5, (i)) l-monodromically full hyperbolic curve of genus 0 over a finitely generated extension of $\mathbb{Q}$ is completely determined by the kernel of the natural pro-l outer Galois representation associated to the hyperbolic curve (cf. [9], Theorem A). By a similar argument to the argument used in the proof of [9], Theorem A, one can prove the following Galois-theoretic characterization of equivalence classes of l-monodromically full points of configuration spaces (cf. Theorem 3):

Theorem C (Galois-theoretic characterization of equivalence classes of monodromically full points of configuration spaces). *Let n be a positive integer. Suppose that k is a* **finitely generated extension of** $\mathbb{Q}$. *Then for two k-rational points x and y of X_n which are* ***l*-monodromically full** (*cf. Definition 3*)*, the following three conditions are equivalent:*

(i) *x is* **equivalent to** *y.*
(ii) $\operatorname{Ker}(\rho^{\{l\}}_{X[x]/k}) = \operatorname{Ker}(\rho^{\{l\}}_{X[y]/k})$.
(iii) *If we write ϕ_x (respectively, ϕ_y) for the composite*

$$G_k \xrightarrow{\pi_1(x)} \pi_1(X_n) \xrightarrow{\widetilde{\rho}^{\{l\}}_{X_n/k}} \operatorname{Aut}(\Delta^{\{l\}}_{X_n/k})$$

$$(\textit{respectively,}\quad G_k \xrightarrow{\pi_1(y)} \pi_1(X_n) \xrightarrow{\widetilde{\rho}^{\{l\}}_{X_n/k}} \operatorname{Aut}(\Delta^{\{l\}}_{X_n/k}))$$

(*cf. Definition 1,* (ii), (iii))*, then* $\operatorname{Ker}(\phi_x) = \operatorname{Ker}(\phi_y)$.

In [17], S. Mochizuki introduced and studied the notion of a *k-core* (cf. [17], Definition 2.1, as well as [17], Remark 2.1.1). It follows from [14], Theorem 5.3, together with [17], Proposition 2.3, that if $2g-2+r > 2$, then *a general hyperbolic curve of type (g, r) over k is a k-core* (cf. also [17], Remark 2.5.1). For a hyperbolic curve over k which is a *k-core*, the following stronger Galois-theoretic characterization can be proven (cf. Theorem 4):

Theorem D (Galois-theoretic characterization of equivalence classes of quasi-monodromically full points of cores). *Suppose that k is a* **finitely generated extension of** $\mathbb{Q}$ *and that X is a* ***k*-core** (*cf.* [17], *Remark 2.1.1*)*. Then for two k-rational points x and y*

of X which are **quasi-l-monodromically full** (*cf. Definition 3*)*, the following four conditions are equivalent:*

(i) $x = y$.

(ii) x *is* **equivalent to** y.

(iii) *If we write*

$$U_x \stackrel{\text{def}}{=} X \setminus \mathrm{Im}(x) \ ; \ \ U_y \stackrel{\text{def}}{=} X \setminus \mathrm{Im}(y)\,,$$

then the intersection $\mathrm{Ker}(\rho^{\{l\}}_{U_x/k}) \cap \mathrm{Ker}(\rho^{\{l\}}_{U_y/k})$ *is* **open in** $\mathrm{Ker}(\rho^{\{l\}}_{U_x/k})$ *and* $\mathrm{Ker}(\rho^{\{l\}}_{U_y/k})$.

(iv) *If we write* ϕ_x (*respectively,* ϕ_y) *for the composite*

$$G_k \xrightarrow{\pi_1(x)} \pi_1(X) \xrightarrow{\widetilde{\rho}^{\{l\}}_{X/k}} \mathrm{Aut}(\Delta^{\{l\}}_{X/k})$$

$$(\textit{respectively,} \quad G_k \xrightarrow{\pi_1(y)} \pi_1(X) \xrightarrow{\widetilde{\rho}^{\{l\}}_{X/k}} \mathrm{Aut}(\Delta^{\{l\}}_{X/k}))\,,$$

then the intersection $\mathrm{Ker}(\phi_x) \cap \mathrm{Ker}(\phi_y)$ *is* **open in** $\mathrm{Ker}(\phi_x)$ *and* $\mathrm{Ker}(\phi_y)$.

Finally, in a similar vein, we prove a necessary and sufficient condition for a *quasi-l-monodromically full* Galois section (cf. Definition 5) of a hyperbolic curve to be *geometric* (cf. Theorem 5):

Theorem E (A necessary and sufficient condition for a quasi-monodromically full Galois section of a hyperbolic curve to be geometric). *Suppose that k is a* **finitely generated extension of** $\mathbb{Q}$. *Let* $s\colon G_k \to \Pi^{\{l\}}_{X/k}$ *be a pro-l Galois section of X (i.e., a continuous section of the natural surjection* $\Pi^{\{l\}}_{X/k} \twoheadrightarrow G_k$ *— cf.* [10], *Definition 1.1,* (i)) *which is* **quasi-l-monodromically full** (*cf. Definition 5*)*. Write* ϕ_s *for the composite*

$$G_k \xrightarrow{s} \Pi^{\{l\}}_{X/k} \xrightarrow{\widetilde{\rho}^{\{l\}}_{X/k}} \mathrm{Aut}(\Delta^{\{l\}}_{X/k})\,.$$

Then the following four conditions are equivalent:

(i) *The pro-l Galois section s is* **geometric** (*cf.* [10], *Definition 1.1,* (iii)).

(ii) *The pro-l Galois section s* **arises from** *a k-rational point of X* (*cf.* [10], *Definition 1.1,* (ii)).

(iii) *There exists a* **quasi-l-monodromically full** *k-rational point* (*cf. Definition 3*) $x \in X(k)$ *of X such that if we write ϕ_x for the composite*

$$G_k \xrightarrow{\pi_1(x)} \Pi^{\{l\}}_{X/k} \xrightarrow{\widetilde{\rho}^{\{l\}}_{X/k}} \mathrm{Aut}(\Delta^{\{l\}}_{X/k})\,,$$

then the intersection $\mathrm{Ker}(\phi_s) \cap \mathrm{Ker}(\phi_x)$ *is* **open in** $\mathrm{Ker}(\phi_s)$ *and* $\mathrm{Ker}(\phi_x)$.

(iv) *There exists a* **quasi-l-monodromically full** *k-rational point* (*cf. Definition 3*) $x \in X(k)$ *of* X *such that if we write*

$$U \stackrel{\mathrm{def}}{=} X \setminus \mathrm{Im}(x)\,,$$

then the intersection $\mathrm{Ker}(\phi_s) \cap \mathrm{Ker}(\rho^{\{l\}}_{U/k})$ *is* **open in** $\mathrm{Ker}(\phi_s)$ *and* $\mathrm{Ker}(\rho^{\{l\}}_{U/k})$.

The present paper is organized as follows: In §1, we introduce and discuss the notion of monodromically full points of configuration spaces of hyperbolic curves. In §2, we consider the fundamental groups of configuration spaces of hyperbolic curves. In §3, we consider the kernels of the outer representations associated to configuration spaces of hyperbolic curves. In §4, we prove Theorems A and B. In §5, we prove Theorems C, D, and E.

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0. Notations and Conventions

Numbers: The notation $\mathfrak{Primes}$ will be used to denote the set of all prime numbers. The notation $\mathbb{Z}$ will be used to denote the set, group, or ring of rational integers. The notation $\mathbb{Q}$ will be used to denote the set, group, or field of rational numbers. The notation $\mathbb{C}$ will be used to denote the set, group, or field of complex numbers.

Profinite groups: If G is a profinite group, and $H \subseteq G$ is a closed subgroup of G, then we shall write $N_G(H)$ for the *normalizer* of H in G, i.e.,

$$N_G(H) \stackrel{\mathrm{def}}{=} \{\, g \in G \mid gHg^{-1} = H \,\} \subseteq G\,,$$

$Z_G(H)$ for the *centralizer* of H in G, i.e.,

$$Z_G(H) \stackrel{\mathrm{def}}{=} \{\, g \in G \mid ghg^{-1} = h \text{ for any } h \in H \,\} \subseteq G\,,$$

$Z^{\mathrm{loc}}_G(H)$ for the *local centralizer* of H in G, i.e.,

$$Z^{\mathrm{loc}}_G(H) \stackrel{\mathrm{def}}{=} \varinjlim_{H' \subseteq H} Z_G(H') \subseteq G$$

— where $H' \subseteq H$ ranges over the open subgroups of H — $Z(G) \stackrel{\text{def}}{=} Z_G(G)$ for the *center* of G, and $Z^{\text{loc}}(G) \stackrel{\text{def}}{=} Z_G^{\text{loc}}(G)$ for the *local center* of G. It is immediate from the various definitions involved that $H \subseteq N_G(H) \supseteq Z_G(H) \subseteq Z_G^{\text{loc}}(H)$ and that if H_1, $H_2 \subseteq G$ are closed subgroups of G such that $H_1 \subseteq H_2$ (respectively, $H_1 \subseteq H_2$; $H_1 \cap H_2$ is *open* in H_1 and H_2), then $Z_G(H_2) \subseteq Z_G(H_1)$ (respectively, $Z_G^{\text{loc}}(H_2) \subseteq Z_G^{\text{loc}}(H_1)$; $Z_G^{\text{loc}}(H_1) = Z_G^{\text{loc}}(H_2)$).

We shall say that a profinite group G is *center-free* (respectively, *slim*) if $Z(G) = \{1\}$ (respectively, $Z^{\text{loc}}(G) = \{1\}$). Note that it follows from [16], Remark 0.1.3, that a profinite group G is *slim* if and only if every open subgroup of G is *center-free*.

If G is a profinite group, then we shall denote the group of continuous automorphisms of G by $\text{Aut}(G)$ and the group of inner automorphisms of G by $\text{Inn}(G) \subseteq \text{Aut}(G)$. Conjugation by elements of G determines a surjection $G \twoheadrightarrow \text{Inn}(G)$. Thus, we have a homomorphism $G \to \text{Aut}(G)$ whose image is $\text{Inn}(G) \subseteq \text{Aut}(G)$. We shall denote by $\text{Out}(G)$ the quotient of $\text{Aut}(G)$ by the normal subgroup $\text{Inn}(G) \subseteq \text{Aut}(G)$. If, moreover, G is *topologically finitely generated*, then one verifies easily that the topology of G admits a basis of *characteristic open subgroups*, which thus induces a *profinite topology* on the group $\text{Aut}(G)$, hence also a *profinite topology* on the group $\text{Out}(G)$.

Curves: Let S be a scheme and C a scheme over S. Then for a pair (g, r) of nonnegative integers, we shall say that C is a *smooth curve of type* (g, r) over S if there exist a scheme C^{cpt} which is *smooth*, *proper*, *geometrically connected*, and *of relative dimension* 1 over S and a closed subscheme $D \subseteq C^{\text{cpt}}$ of C^{cpt} which is *finite* and *étale* over S such that the complement of D in C^{cpt} is isomorphic to C over S, any geometric fiber of $C^{\text{cpt}} \to S$ is (a necessarily smooth, proper, and connected curve) of genus g, and, moreover, the degree of the finite étale covering $D \hookrightarrow C^{\text{cpt}} \to S$ is r. Moreover, we shall say that C is a *hyperbolic curve* (respectively, *tripod*) over S if there exists a pair (g, r) of nonnegative integers such that C is a smooth curve of type (g, r) over S, and, moreover, $2g - 2 + r > 0$ (respectively, $(g, r) = (0, 3)$).

For a pair (g, r) of nonnegative integers such that $2g-2+r > 0$, write $\mathcal{M}_{g,r}$ for the moduli stack of r-pointed smooth curves of genus g over $\mathbb{Z}$ whose marked points are equipped with orderings (cf. [4], [11]) and $\mathcal{M}_{g,[r]}$ for the moduli stack of hyperbolic curves of type (g, r) over $\mathbb{Z}$. Then we have a natural finite étale Galois $\mathfrak{S}_r$-covering $\mathcal{M}_{g,r} \to \mathcal{M}_{g,[r]}$ — where $\mathfrak{S}_r$ is the symmetric group on r letters.

1. Monodromically full points

In the present §, we introduce and discuss the notion of *monodromically full points* of configuration spaces of hyperbolic curves. Let $\Sigma \subseteq$

$\mathfrak{Primes}$ be a nonempty subset of $\mathfrak{Primes}$ (cf. the discussion entitled "Numbers" in §0), and X and S regular and connected schemes. Suppose, moreover, that X is a *scheme over* S.

Definition 1.

(i) Let $1 \to \Delta \to \Pi \to G \to 1$ be an exact sequence of profinite groups. Suppose that Δ is *topologically finitely generated.* Then conjugation by elements of Π determines a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta & \longrightarrow & \Pi & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}(\Delta) & \longrightarrow & \mathrm{Aut}(\Delta) & \longrightarrow & \mathrm{Out}(\Delta) & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are *exact*, and we refer to the discussion entitled "Profinite Groups" in §0 concerning the topology of $\mathrm{Aut}(\Delta)$ (respectively, $\mathrm{Out}(\Delta)$). We shall refer to the continuous homomorphism

$$\Pi \longrightarrow \mathrm{Aut}(\Delta) \ \ (\text{respectively, } G \longrightarrow \mathrm{Out}(\Delta))$$

obtained as the middle (respectively, right-hand) vertical arrow in the above diagram as the *representation associated to* $1 \to \Delta \to \Pi \to G \to 1$ (respectively, *outer representation associated to* $1 \to \Delta \to \Pi \to G \to 1$).

(ii) We shall write

$$\Delta^{\Sigma}_{X/S}$$

for the maximal pro-Σ quotient of the kernel of the natural homomorphism

$$\pi_1(X) \longrightarrow \pi_1(S)$$

and

$$\Pi^{\Sigma}_{X/S}$$

for the quotient of $\pi_1(X)$ by the kernel of the natural surjection from the kernel of $\pi_1(X) \to \pi_1(S)$ to $\Delta^{\Sigma}_{X/S}$. (Note that since $\mathrm{Ker}(\pi_1(X) \to \pi_1(S))$ is a *normal* closed subgroup of $\pi_1(X)$, and the kernel of the natural surjection $\mathrm{Ker}(\pi_1(X) \to \pi_1(S)) \twoheadrightarrow \Delta^{\Sigma}_{X/S}$ is a *characteristic* closed subgroup of $\mathrm{Ker}(\pi_1(X) \to \pi_1(S))$, it holds that the kernel of $\mathrm{Ker}(\pi_1(X) \to \pi_1(S)) \twoheadrightarrow \Delta^{\Sigma}_{X/S}$ is a *normal* closed subgroup of $\pi_1(X)$.) Thus, we have a commutative diagram of profinite groups

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathrm{Ker}(\pi_1(X) \to \pi_1(S)) & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(S) \\ & & \downarrow & & \downarrow & & \| \\ 1 & \longrightarrow & \Delta^{\Sigma}_{X/S} & \longrightarrow & \Pi^{\Sigma}_{X/S} & \longrightarrow & \pi_1(S) \end{array}$$

— where the horizontal sequences are *exact*, and the vertical arrows are *surjective*. If S is the spectrum of a ring R, then we shall write $\Delta^{\Sigma}_{X/R} \stackrel{\mathrm{def}}{=} \Delta^{\Sigma}_{X/S}$ and $\Pi^{\Sigma}_{X/R} \stackrel{\mathrm{def}}{=} \Pi^{\Sigma}_{X/S}$.

(iii) Suppose that the natural homomorphism $\pi_1(X) \to \pi_1(S)$ is *surjective* — or, equivalently, $\Pi^{\Sigma}_{X/S} \to \pi_1(S)$ is *surjective* — and that the profinite group $\Delta^{\Sigma}_{X/S}$ is *topologically finitely generated.* Then we have an exact sequence of profinite groups

$$1 \longrightarrow \Delta^{\Sigma}_{X/S} \longrightarrow \Pi^{\Sigma}_{X/S} \longrightarrow \pi_1(S) \longrightarrow 1\,.$$

We shall write

$$\widetilde{\rho}^{\Sigma}_{X/S} \colon \Pi^{\Sigma}_{X/S} \longrightarrow \mathrm{Aut}(\Delta^{\Sigma}_{X/S})$$

for the representation associated to the above exact sequence (cf. (i)) and refer to as the *pro-Σ representation associated to* X/S. Moreover, we shall write

$$\rho^{\Sigma}_{X/S} \colon \pi_1(S) \longrightarrow \mathrm{Out}(\Delta^{\Sigma}_{X/S})$$

for the outer representation associated to the above exact sequence (cf. (i)) and refer to as the *pro-Σ outer representation associated to* X/S. If S is the spectrum of a ring R, then we shall write $\widetilde{\rho}^{\Sigma}_{X/R} \stackrel{\mathrm{def}}{=} \widetilde{\rho}^{\Sigma}_{X/S}$ and $\rho^{\Sigma}_{X/R} \stackrel{\mathrm{def}}{=} \rho^{\Sigma}_{X/S}$. Moreover, if l is a prime number, then for simplicity, we write "pro-l representation associated to X/S" (respectively, "pro-l outer representation associated to X/S") instead of "pro-$\{l\}$ representation associated to X/S" (respectively, "pro-$\{l\}$ outer representation associated to X/S").

(iv) Suppose that the natural homomorphism $\pi_1(X) \to \pi_1(S)$ is *surjective* — or, equivalently, $\Pi^{\Sigma}_{X/S} \to \pi_1(S)$ is *surjective* — and that the profinite group $\Delta^{\Sigma}_{X/S}$ is *topologically finitely generated.* Then we shall write

$$\Pi^{\Sigma}_{X/S} \twoheadrightarrow \Phi^{\Sigma}_{X/S} \stackrel{\mathrm{def}}{=} \mathrm{Im}(\widetilde{\rho}^{\Sigma}_{X/S})$$

for the quotient of $\Pi^{\Sigma}_{X/S}$ determined by the pro-Σ representation $\widetilde{\rho}^{\Sigma}_{X/S}$ associated to X/S. Moreover, we shall write

$$\pi_1(S) \twoheadrightarrow \Gamma^{\Sigma}_{X/S} \stackrel{\mathrm{def}}{=} \mathrm{Im}(\rho^{\Sigma}_{X/S})$$

for the quotient of $\pi_1(S)$ determined by the pro-Σ outer representation $\rho^{\Sigma}_{X/S}$ associated to X/S. If S is the spectrum of a ring R, then we shall write $\Phi^{\Sigma}_{X/R} \stackrel{\mathrm{def}}{=} \Phi^{\Sigma}_{X/S}$ and $\Gamma^{\Sigma}_{X/R} \stackrel{\mathrm{def}}{=} \Gamma^{\Sigma}_{X/S}$.

(v) Let $\pi_1(X) \twoheadrightarrow Q$ be a quotient of $\pi_1(X)$. Then we shall say that a finite étale covering $Y \to X$ is a *finite étale Q-covering* if Y is connected, and the finite étale covering $Y \to X$ arises from an open subgroup of Q, i.e., the open subgroup of $\pi_1(X)$

corresponding to the connected finite étale covering $Y \to X$ contains the kernel of the surjection $\pi_1(X) \twoheadrightarrow Q$.

Remark 1. If S is the spectrum of a field k, then it follows from [5], Exposé V, Proposition 6.9, that $\pi_1(X) \to \pi_1(S)$ is *surjective* if and only if X is *geometrically connected*, i.e., $X \otimes_k \overline{k}$, where $\overline{k}$ is an algebraic closure of k, is *connected* — or, equivalently, $X \otimes_k k^{\mathrm{sep}}$, where k^{sep} is a separable closure of k, is *connected*. Suppose, moreover, that X is *geometrically connected* and *of finite type* over S. Then it follows from [5], Exposé IX, Théorème 6.1, that the natural sequence of profinite groups

$$1 \longrightarrow \pi_1(X \otimes_k k^{\mathrm{sep}}) \longrightarrow \pi_1(X) \longrightarrow \pi_1(S) \longrightarrow 1$$

is *exact*. Thus, it follows from the various definitions involved that $\Delta^{\Sigma}_{X/k}$ is *naturally isomorphic to the maximal pro-*Σ *quotient* of the étale fundamental group $\pi_1(X \otimes_k k^{\mathrm{sep}})$ of $X \otimes_k k^{\mathrm{sep}}$. In particular, if, moreover, k is *of characteristic* 0, then it follows from [6], Exposé II, Théorème 2.3.1, that $\Delta^{\Sigma}_{X/k}$ is *topologically finitely generated*.

Remark 2. Suppose that X is a *hyperbolic curve* over S (cf. the discussion entitled “Curves” in §0). Then since S is *regular*, it follows immediately from [5], Exposé X, Théorème 3.1, that the natural homomorphism $\pi_1(\eta_S) \to \pi_1(S)$ — where η_S is the generic point of S — is *surjective*. Thus, in light of the *surjectivity* of the natural homomorphism $\pi_1(X \times_S \eta_S) \to \pi_1(\eta_S)$ (cf. Remark 1), we conclude that the natural homomorphism $\pi_1(X) \to \pi_1(S)$ is *surjective*. In particular, we have an exact sequence of profinite groups

$$1 \longrightarrow \Delta^{\Sigma}_{X/S} \longrightarrow \Pi^{\Sigma}_{X/S} \longrightarrow \pi_1(S) \longrightarrow 1\,.$$

If, moreover, every element of Σ is *invertible on* S, then it follows from a similar argument to the argument used in the proof of [7], Lemma 1.1, that $\Delta^{\Sigma}_{X/S}$ is *naturally isomorphic to the maximal pro-*Σ *quotient* of the étale fundamental group $\pi_1(X \times_S \overline{s})$ — where $\overline{s} \to S$ is a geometric point of X — of $X \times_S \overline{s}$. In particular, it follows immediately from the well-known structure of the maximal pro-Σ quotient of the fundamental group of a smooth curve over an algebraically closed field of characteristic $\notin \Sigma$ that $\Delta^{\Sigma}_{X/S}$ is *topologically finitely generated* and *slim* — where we refer to the discussion entitled “Profinite Groups” in §0 concerning the term “slim”. Thus, we have continuous homomorphisms

$$\widetilde{\rho}^{\Sigma}_{X/S} \colon \Pi^{\Sigma}_{X/S} \longrightarrow \mathrm{Aut}(\Delta^{\Sigma}_{X/S})\,;$$

$$\rho^{\Sigma}_{X/S} \colon \pi_1(S) \longrightarrow \mathrm{Out}(\Delta^{\Sigma}_{X/S})\,.$$

Moreover, there exists a *natural bijection* between the set of the cusps of X/S and the set of the conjugacy classes of the cuspidal inertia subgroups of $\Delta^{\Sigma}_{X/S}$.

Lemma 1 (**Outer representations arising from certain extensions**). *Let*

$$1 \longrightarrow \Delta \longrightarrow \Pi \longrightarrow G \longrightarrow 1$$

be an exact sequence of profinite groups. Suppose that Δ *is* **topologically finitely generated** *and* **center-free**. *Write*

$$\widetilde{\rho}\colon \Pi \longrightarrow \mathrm{Aut}(\Delta) \quad ; \quad \rho\colon G \longrightarrow \mathrm{Out}(\Delta)$$

for the continuous homomorphisms arising from the above exact sequence of profinite groups (*cf. Definition 1,* (i)). *Then the following hold:*

(i) $\mathrm{Ker}(\widetilde{\rho}) = Z_{\Pi}(\Delta)$. *Moreover, the natural surjection* $\Pi \twoheadrightarrow G$ *induces an isomorphism*

$$\mathrm{Ker}(\widetilde{\rho})\,(= Z_{\Pi}(\Delta)) \xrightarrow{\sim} \mathrm{Ker}(\rho)\,.$$

In particular, $\Delta \cap \mathrm{Ker}(\widetilde{\rho}) = \{1\}$.

(ii) *The normal closed subgroup* $\mathrm{Ker}(\widetilde{\rho}) \subseteq \Pi$ *is the* **maximal** *normal closed subgroup* N *of* Π *such that* $N \cap \Delta = \{1\}$.

(iii) *Write*

$$\mathrm{Aut}(\Delta \subseteq \Pi) \subseteq \mathrm{Aut}(\Pi)$$

for the subgroup of the group $\mathrm{Aut}(\Pi)$ *of automorphisms of* Π *consisting of automorphisms which* **preserve** *the closed subgroup* $\Delta \subseteq \Pi$. *Suppose that* $Z_{\Pi}(\Delta) = \{1\}$. *Then the natural homomorphism* $\mathrm{Aut}(\Delta \subseteq \Pi) \to \mathrm{Aut}(\Delta)$ *is* **injective**, *and its image* **coincides with** $N_{\mathrm{Aut}(\Delta)}(\mathrm{Im}(\widetilde{\rho})) \subseteq \mathrm{Aut}(\Delta)$, *i.e.,*

$$\mathrm{Aut}(\Delta \subseteq \Pi) \xrightarrow{\sim} N_{\mathrm{Aut}(\Delta)}(\mathrm{Im}(\widetilde{\rho})) \subseteq \mathrm{Aut}(\Delta)\,.$$

Proof. Assertion (i) follows immediately from the various definitions involved. Next, we verify assertion (ii). Let $N \subseteq \Pi$ be a normal closed subgroup of Π such that $\mathrm{Ker}(\widetilde{\rho}) \subseteq N$, and, moreover, $N \cap \Delta = \{1\}$. Write $\overline{N} \subseteq G$ for the image of N via the natural surjection $\Pi \twoheadrightarrow G$. Then since the image of $\mathrm{Ker}(\widetilde{\rho}) \subseteq \Pi$ via the natural surjection $\Pi \twoheadrightarrow G$ is $\mathrm{Ker}(\rho)$ (cf. assertion (i)), we obtain a commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta & \longrightarrow & \Pi/\mathrm{Ker}(\widetilde{\rho}) & \longrightarrow & G/\mathrm{Ker}(\rho) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta & \longrightarrow & \Pi/N & \longrightarrow & G/\overline{N} & \longrightarrow & 1
\end{array}$$

— where the horizontal sequences are *exact*, and the vertical arrows are *surjective*. Thus, it follows immediately from the *exactness* of the lower horizontal sequence of the above diagram that the homomorphism ρ *factors through* $G/\overline{N}$. Therefore, it holds that $\overline{N} = \mathrm{Ker}(\rho)$. In particular, the right-hand vertical arrow, hence also the middle vertical arrow, is an *isomorphism*. This completes the proof of assertion (ii). Finally, we verify assertion (iii). It follows from the various definitions involved that the natural homomorphism $\mathrm{Aut}(\Delta \subseteq \Pi) \to \mathrm{Aut}(\Delta)$ *factors through* $N_{\mathrm{Aut}(\Delta)}(\mathrm{Im}(\widetilde{\rho})) \subseteq \mathrm{Aut}(\Delta)$. On the other hand, since the natural surjection $\Pi \twoheadrightarrow \mathrm{Im}(\widetilde{\rho})$ is an *isomorphism* (cf. assertion (i)), conjugation by elements of $N_{\mathrm{Aut}(\Delta)}(\mathrm{Im}(\widetilde{\rho}))$ determines a homomorphism $N_{\mathrm{Aut}(\Delta)}(\mathrm{Im}(\widetilde{\rho})) \to \mathrm{Aut}(\mathrm{Im}(\widetilde{\rho})) \simeq \mathrm{Aut}(\Pi)$, which *factors through* $\mathrm{Aut}(\Delta \subseteq \Pi) \to \mathrm{Aut}(\Pi)$. Now it may be easily verified that this homomorphism is the *inverse* of the homomorphism in question $\mathrm{Aut}(\Delta \subseteq \Pi) \to N_{\mathrm{Aut}(\Delta)}(\mathrm{Im}(\widetilde{\rho}))$. This completes the proof of assertion (iii). ☐

Lemma 2 (**Certain automorphisms of slim profinite groups).** *Let G be a* **slim** (*cf. the discussion entitled "Profinite Groups" in §0*) *profinite group and α an automorphism of G. If α induces the* **identity automorphism** *on an open subgroup, then α is the* **identity automorphism** *of G.*

Proof. Let $H \subseteq G$ be an open subgroup of G such that α induces the *identity automorphism* of H. To verify Lemma 2, by replacing H by the intersection of all G-conjugates of H, we may assume without loss of generality that H is *normal* in G. Then since $Z_G(H) = \{1\}$, it follows immediately from Lemma 1, (iii), that α is the *identity automorphism* of G. This completes the proof of Lemma 2. ☐

Proposition 1 (**Fundamental exact sequences associated to certain schemes).** *Suppose that the natural homomorphism $\pi_1(X) \to \pi_1(S)$ is* **surjective** *and that the profinite group $\Delta^{\Sigma}_{X/S}$ is* **topologically finitely generated** *and* **center-free**. *Then the following hold:*

(i) *We have a commutative diagram of profinite groups*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta^{\Sigma}_{X/S} & \longrightarrow & \Pi^{\Sigma}_{X/S} & \longrightarrow & \pi_1(S) & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \widetilde{\rho}^{\Sigma}_{X/S}} & & \downarrow{\scriptstyle \rho^{\Sigma}_{X/S}} & & \\ 1 & \longrightarrow & \Delta^{\Sigma}_{X/S} & \longrightarrow & \Phi^{\Sigma}_{X/S} & \longrightarrow & \Gamma^{\Sigma}_{X/S} & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are **exact**, *and the vertical arrows are* **surjective**.

(ii) *The quotient $\Pi^{\Sigma}_{X/S} \twoheadrightarrow \Phi^{\Sigma}_{X/S}$ determined by $\widetilde{\rho}^{\Sigma}_{X/S}$ is the* **minimal** *quotient $\Pi^{\Sigma}_{X/S} \twoheadrightarrow Q$ of $\Pi^{\Sigma}_{X/S}$ such that $\mathrm{Ker}(\Pi^{\Sigma}_{X/S} \twoheadrightarrow Q) \cap \Delta^{\Sigma}_{X/S} = \{1\}$.*

Proof. Assertion (i) (respectively, (ii)) follows immediately from Lemma 1, (i) (respectively, (ii)). ☐

Definition 2. Let n be a nonnegative integer and (g, r) a pair of nonnegative integers such that $2g-2+r > 0$. Suppose that X is a *hyperbolic curve* of type (g, r) over S (cf. the discussion entitled "Curves" in §0).

(i) We shall write

$$X_0 \overset{\text{def}}{=} S$$

and

$$X_n$$

for the *n-th configuration space of X/S*, i.e., the open subscheme of the fiber product of n copies of X over S which represents the functor from the category of schemes over S to the category of sets given by

$$T \rightsquigarrow \{ (x_1, \cdots, x_n) \in X(T)^{\times n} \mid x_i \neq x_j \text{ if } i \neq j \} .$$

For a nonnegative integer $m \leq n$, we always regard X_n as a scheme over X_m by the natural projection $X_n \to X_m$ to the first m factors. Then it follows immediately from the various definitions involved that X_{n+1} is a *hyperbolic curve* of type $(g, r+n)$ over X_n. In particular, if every element of Σ is *invertible on S*, then we have continuous homomorphisms

$$\widetilde{\rho}^{\Sigma}_{X_{n+1}/X_n} \colon \Pi^{\Sigma}_{X_{n+1}/X_n} \longrightarrow \mathrm{Aut}(\Delta^{\Sigma}_{X_{n+1}/X_n}) ;$$

$$\rho^{\Sigma}_{X_{n+1}/X_n} \colon \pi_1(X_n) \longrightarrow \mathrm{Out}(\Delta^{\Sigma}_{X_{n+1}/X_n})$$

(cf. Remark 2). Moreover, it follows immediately from the various definitions involved that X_n is *naturally isomorphic to* the $(n-m)$-th configuration space of the hyperbolic curve X_{m+1}/X_m.

(ii) Let $m \leq n$ be a nonnegative integer, T a regular and connected scheme over S, and $x \in X_m(T)$ a T-valued point of X_m. Then we shall write

$$X[x] \subseteq X \times_S T$$

for the open subscheme of $X \times_S T$ obtained by taking the complement in $X \times_S T$ of the images of the m *distinct* T-valued points of $X \times_S T$ determined by the T-valued point x. Then it follows immediately from the various definitions involved that $X[x]$ is equipped with a *natural structure of hyperbolic curve of type $(g, r+m)$ over T* and that the base-change of $X_n \to X_m$ via x is *naturally isomorphic to* the $(n-m)$-th configuration space $X[x]_{n-m}$ of the hyperbolic curve $X[x]/T$, i.e., we have a *cartesian* diagram of schemes

$$\begin{array}{ccc} X[x]_{n-m} & \longrightarrow & X_n \\ \big\downarrow & & \big\downarrow \\ T & \underset{x}{\longrightarrow} & X_m \, . \end{array}$$

(iii) Let T be a regular and connected scheme over S and x, $y \in X_n(T)$ two T-valued points of X_n. Then we shall say that x *is* *equivalent to* y if there exists an isomorphism $X[x] \xrightarrow{\sim} X[y]$ over T.

Definition 3. Suppose that every element of Σ is *invertible on* S and that X is a *hyperbolic curve* over S. Let n be a positive integer and T a regular and connected scheme over S. Then we shall say that a T-valued point $x \in X_n(T)$ of the n-th configuration space X_n of X/S is Σ-*monodromically full* (respectively, *quasi*-Σ-*monodromically full*) if the following condition is satisfied: For any $l \in \Sigma$, if we write $\Gamma_T \subseteq \mathrm{Out}(\Delta^{\{l\}}_{X_{n+1}/X_n})$ (respectively, $\Gamma_{\mathrm{geom}} \subseteq \mathrm{Out}(\Delta^{\{l\}}_{X_{n+1}/X_n})$) for the image of the composite

$$\pi_1(T) \stackrel{\pi_1(x)}{\to} \pi_1(X_n) \stackrel{\rho^{\{l\}}_{X_{n+1}/X_n}}{\to} \mathrm{Out}(\Delta^{\{l\}}_{X_{n+1}/X_n})$$

$$\text{(respectively, } \mathrm{Ker}\Big(\pi_1(X_n) \to \pi_1(S)\Big) \hookrightarrow \pi_1(X_n) \stackrel{\rho^{\{l\}}_{X_{n+1}/X_n}}{\to} \mathrm{Out}(\Delta^{\{l\}}_{X_{n+1}/X_n})\text{)}$$

— cf. Definition 2, (i) — then Γ_T contains Γ_{geom} (respectively, $\Gamma_T \cap \Gamma_{\mathrm{geom}}$ is an open subgroup of Γ_{geom}). Note that since the closed subgroup $\Gamma_{\mathrm{geom}} \subseteq \Gamma^{\{l\}}_{X_{n+1}/X_n}$ ($\subseteq \mathrm{Out}(\Delta^{\{l\}}_{X_{n+1}/X_n})$) is *normal* in $\Gamma^{\{l\}}_{X_{n+1}/X_n}$, one may easily verify that whether or not Γ_T contains Γ_{geom} (respectively, $\Gamma_T \cap \Gamma_{\mathrm{geom}}$ is an open subgroup of Γ_{geom}) does *not depend on* the choice of the homomorphism "$\pi_1(T) \stackrel{\pi_1(x)}{\to} \pi_1(X_n)$" induced by $x \in X_n(T)$ among the various $\pi_1(X_n)$-conjugates.

Moreover, we shall say that a point $x \in X_n$ of X_n is Σ-*monodromically full* (respectively, *quasi*-Σ-*monodromically full*) if for any $l \in \Sigma$, the $k(x)$-valued point of X_n — where $k(x)$ is the residue field at x — naturally determined by x is Σ-monodromically full (respectively, quasi-Σ-monodromically full).

If l is a prime number, then for simplicity, we write "l-monodromically full" (respectively, "quasi-l-monodromically full") instead of "$\{l\}$-monodromically full" (respectively, "quasi-$\{l\}$-monodromically full").

Remark 3. In the notation of Definition 3, as the terminologies suggest, it follows immediately from the various definitions involved that the Σ-*monodromic fullness* of $x \in X_n(T)$ implies the *quasi*-Σ-*monodromic fullness* of $x \in X_n(T)$.

Remark 4. In the notation of Definition 3, if S is the spectrum of a field k *of characteristic* 0, then it follows immediately from the various definitions involved that for a closed point $x \in X_n$ of X_n with residue field $k(x)$, the following two conditions are equivalent:

- The closed point $x \in X_n$ is a Σ-*monodromically full* (respectively, *quasi*-Σ-*monodromically full*) point in the sense of Definition 3.
- The $k(x)$-rational point of $X_n \otimes_k k(x)$ determined by x is a Σ-*monodromically full* (respectively, *quasi*-Σ-*monodromically full*)

point with respect to the hyperbolic curves $X_{n+1} \otimes_k k(x)/X_n \otimes_k k(x)$ in the sense of [9], Definition 2.1.

If, moreover, X is the complement $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$ of $\{0, 1, \infty\}$ in the projective line $\mathbb{P}^1_k$ over k, then since the n-th configuration space X_n of X/k is *naturally isomorphic to* the moduli stack $\mathcal{M}_{0,n+3} \otimes_{\mathbb{Z}} k$ of $(n+3)$-pointed smooth curves of genus 0 over k-schemes whose marked points are equipped with orderings, for a closed point $x \in X_n$ with residue field $k(x)$, the following two conditions are equivalent:

- The closed point $x \in X_n$ is a Σ-*monodromically full* (respectively, *quasi*-Σ-*monodromically full*) point in the sense of Definition 3.
- The hyperbolic curve $X[x]$ over $k(x)$ (cf. Definition 2, (ii)) is a Σ-*monodromically full* (respectively, *quasi*-Σ-*monodromically full*) hyperbolic curve over $k(x)$ in the sense of [9], Definition 2.2.

Remark 5. In the notation of Definition 3, suppose that $S = T$. Then it follows from the various definitions involved that the following two conditions are equivalent:

(i) The S-valued point $x \in X_n(S)$ is a Σ-*monodromically full* (respectively, *quasi*-Σ-*monodromically full*) point.
(ii) For any $l \in \Sigma$, the composite

$$\pi_1(S) \stackrel{\pi_1(x)}{\to} \pi_1(X_n) \stackrel{\rho^{\{l\}}_{X_{n+1}/X_n}}{\twoheadrightarrow} \Gamma^{\{l\}}_{X_{n+1}/X_n}$$

is *surjective* (respectively, has *open image*).

Proposition 2 (**Existence of many monodromically full points**). *Let Σ be a nonempty* **finite** *set of prime numbers, k a* **finitely generated extension of** $\mathbb{Q}$*, X a hyperbolic curve over k (cf. the discussion entitled "Curves" in §0), n a positive integer, X_n the n-th configuration space of X/k (cf. Definition 2,* (i)*), X_n^{cl} the set of closed points of X_n, and $X_n^{\Sigma\text{-MF}} \subseteq X_n^{\mathrm{cl}}$ the subset of X_n^{cl} consisting of closed points of X_n which are* **Σ-monodromically full** *(cf. Definition 3). If we regard X_n^{cl} as a subset of $X_n(\mathbb{C})$, then the subset*

$$X_n^{\Sigma\text{-MF}} \subseteq X_n(\mathbb{C})$$

is **dense with respect to the complex topology** *of $X_n(\mathbb{C})$. If, moreover, X is* **of genus 0***, then the complement*

$$X_n(k) \setminus (X_n(k) \cap X_n^{\Sigma\text{-MF}}) \subseteq X_n(k)$$

forms a **thin set** *in $X_n(k)$ in the sense of Hilbert's irreducibility theorem.*

Proof. This follows from [9], Theorem 2.3, together with Remark 4. □

2. Fundamental groups of configuration spaces

In the present §, we consider the fundamental groups of configuration spaces of hyperbolic curves. We maintain the notation of the preceding §. Suppose, moreover, that

- X is a *hyperbolic curve* over S (cf. the discussion entitled "Curves" in §0),
- Σ is either $\mathfrak{Primes}$ *itself* (cf. the discussion entitled "Numbers" in §0), or *of cardinality 1*, and
- every element of Σ is *invertible on* S.

Lemma 3 (**Fundamental groups of configuration spaces**). *Let $m < n$ be nonnegative integers. Then the following hold:*

(i) *The natural homomorphism $\pi_1(X_n) \to \pi_1(X_m)$ is* **surjective**. *Thus, we have an exact sequence of profinite groups*

$$1 \longrightarrow \Delta^{\Sigma}_{X_n/X_m} \longrightarrow \Pi^{\Sigma}_{X_n/X_m} \longrightarrow \pi_1(X_m) \longrightarrow 1\,.$$

(ii) *If $\overline{x} \to X_m$ is a geometric point of X_m, then $\Delta^{\Sigma}_{X_n/X_m}$ is* **naturally isomorphic to the maximal pro-Σ quotient** *of the étale fundamental group $\pi_1(X_n \times_{X_m} \overline{x})$ of $X_n \times_{X_m} \overline{x}$.*

(iii) *Let T be a regular and connected scheme over S and $x \in X_m(T)$ a T-valued point of X_m. Then the homomorphism*

$$\Delta^{\Sigma}_{X[x]_{n-m}/T} \longrightarrow \Delta^{\Sigma}_{X_n/X_m}$$

determined by the **cartesian** *square of schemes*

$$\begin{array}{ccc} X[x]_{n-m} & \longrightarrow & X_n \\ \big\downarrow & & \big\downarrow \\ T & \xrightarrow[x]{} & X_m \end{array}$$

(*cf. Definition 2,* (ii)) *is an* **isomorphism**. *In particular, the right-hand square of the commutative diagram of profinite groups*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta^{\Sigma}_{X[x]_{n-m}/T} & \longrightarrow & \Pi^{\Sigma}_{X[x]_{n-m}/T} & \longrightarrow & \pi_1(T) & \longrightarrow & 1 \\ & & \wr\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \pi_1(x)} & & \\ 1 & \longrightarrow & \Delta^{\Sigma}_{X_n/X_m} & \longrightarrow & \Pi^{\Sigma}_{X_n/X_m} & \longrightarrow & \pi_1(X_m) & \longrightarrow & 1 \end{array}$$

— *where the horizontal sequences are* **exact** (*cf. assertion* (i)) — *is* **cartesian**.

(iv) *The natural sequence of profinite groups*

$$1 \longrightarrow \Delta^{\Sigma}_{X_n/X_m} \longrightarrow \Delta^{\Sigma}_{X_n/S} \longrightarrow \Delta^{\Sigma}_{X_m/S} \longrightarrow 1$$

is **exact**.

(v) *The profinite group* $\Delta^{\Sigma}_{X_n/X_m}$ *is* **topologically finitely generated** *and* **slim** (*cf. the discussion entitled "Profinite Groups" in §0*). *Thus, we have*

$$\widetilde{\rho}^{\Sigma}_{X_n/X_m} : \Pi^{\Sigma}_{X_n/X_m} \longrightarrow \mathrm{Aut}(\Delta^{\Sigma}_{X_n/X_m})\,;$$
$$\rho^{\Sigma}_{X_n/X_m} : \pi_1(X_m) \longrightarrow \mathrm{Out}(\Delta^{\Sigma}_{X_n/X_m})$$

(*cf. assertion* (i)).

(vi) *Let T be a regular and connected scheme over S and $x \in X_m(T)$ a T-valued point of X_m. Then the diagram of profinite groups*

$$\begin{array}{ccc} \pi_1(T) & \xrightarrow{\rho^{\Sigma}_{X[x]_{n-m}/T}} & \mathrm{Out}(\Delta^{\Sigma}_{X[x]_{n-m}/T}) \\ {\scriptstyle \pi_1(x)}\big\downarrow & & \big\downarrow\wr \\ \pi_1(X_m) & \xrightarrow[\rho^{\Sigma}_{X_n/X_m}]{} & \mathrm{Out}(\Delta^{\Sigma}_{X_n/X_m}) \end{array}$$

(*cf. assertion* (v)) — *where the right-hand vertical arrow is the* **isomorphism** *determined by the isomorphism obtained in assertion* (iii) — **commutes**.

(vii) *The centralizer* $Z_{\Delta^{\Sigma}_{X_n/S}}(\Delta^{\Sigma}_{X_n/X_m})$ *of* $\Delta^{\Sigma}_{X_n/X_m}$ *in* $\Delta^{\Sigma}_{X_n/S}$ (*cf. assertion* (iv)) *is* **trivial**.

(viii) *The pro-Σ outer representation associated to X_n/X_m*

$$\rho^{\Sigma}_{X_n/X_m} : \pi_1(X_m) \longrightarrow \mathrm{Out}(\Delta^{\Sigma}_{X_n/X_m})$$

factors through *the natural surjection* $\pi_1(X_m) \twoheadrightarrow \Pi^{\Sigma}_{X_m/S}$, *and, moreover, the composite of the natural inclusion* $\Delta^{\Sigma}_{X_m/S} \hookrightarrow \Pi^{\Sigma}_{X_m/S}$ *and the resulting homomorphism* $\Pi^{\Sigma}_{X_m/S} \to \mathrm{Out}(\Delta^{\Sigma}_{X_n/X_m})$ *is* **injective**.

Proof. First, we verify assertion (i). By induction on $n - m$, we may assume without loss of generality that $n = m + 1$. On the other hand, if $n = m + 1$, then $X_n \to X_m$ is a *hyperbolic curve* over X_m (cf. Definition 2, (i)). Thus, the desired surjectivity follows from Remark 2. This completes the proof of assertion (i). Next, we verify assertion (ii). It is immediate that there exists a connected finite étale covering $Y \to X_m$ of X_m which satisfies the condition (c) in the statement of [18], Proposition 2.2, hence also the three conditions (a), (b), and (c) in the statement of [18], Proposition 2.2. Now it follows from [18], Proposition 2.2, (iii), that if $\overline{y} \to Y$ is a geometric point, then $\Delta^{\Sigma}_{X_n \times_{X_m} Y/Y}$ is *naturally isomorphic to the maximal pro-Σ quotient* of $\pi_1(X_n \times_{X_m} \overline{y})$. On the other hand, it follows from the various definitions involved that $\Delta^{\Sigma}_{X_n \times_{X_m} Y/Y}$ is *naturally isomorphic to* $\Delta^{\Sigma}_{X_n/X_m}$. Thus, assertion (ii) follows from the fact that any geometric point of X_m *arises from* a geometric point of Y. This completes the proof of assertion (ii). Assertion (iii) follows immediately from assertion (ii). Assertion (iv) (respectively, (v)) follows immediately from [18], Proposition 2.2, (iii) (respectively, (ii)), together with assertion (ii). Assertion (vi) follows immediately from the various definitions involved. Next, we verify assertion (vii). Since $\Delta^{\Sigma}_{X_n/X_m}$ is *center-free* (cf. assertion (v)), it holds that $Z_{\Delta^{\Sigma}_{X_n/S}}(\Delta^{\Sigma}_{X_n/X_m}) \cap \Delta^{\Sigma}_{X_n/X_m} = \{1\}$. Thus, to verify assertion (vii), by replacing $\Delta^{\Sigma}_{X_n/S}$ by the quotient

$$\Delta^{\Sigma}_{X_{m+1}/S} \simeq \Delta^{\Sigma}_{X_n/S}/\Delta^{\Sigma}_{X_n/X_{m+1}}$$

(cf. assertion (iv)) of $\Delta^{\Sigma}_{X_n/S}$ by $\Delta^{\Sigma}_{X_n/X_{m+1}} \subseteq (\Delta^{\Sigma}_{X_n/X_m} \subseteq)\ \Delta_{X_n/S}$, we may assume without loss of generality that $n = m+1$. Then it follows from Lemma 1, (i), that, to verify assertion (vii), it suffices to show that the outer representation $\Delta^{\Sigma}_{X_m/S} \to \mathrm{Out}(\Delta^{\Sigma}_{X_{m+1}/X_m})$ associated to the exact sequence of profinite groups

$$1 \longrightarrow \Delta^{\Sigma}_{X_{m+1}/X_m} \longrightarrow \Delta^{\Sigma}_{X_{m+1}/S} \longrightarrow \Delta^{\Sigma}_{X_m/S} \longrightarrow 1$$

(cf. assertion (iv)) is *injective*. On the other hand, this *injectivity* follows immediately from [2], Theorem 1, together with [2], Remark following the proof of Theorem 1. This completes the proof of assertion (vii). Finally, we verify assertion (viii). The fact that the pro-Σ outer representation $\rho^{\Sigma}_{X_n/X_m}$ *factors through* the natural surjection $\pi_1(X_m) \twoheadrightarrow \Pi^{\Sigma}_{X_m/S}$ follows immediately from assertion (iv). The fact that the composite in question is *injective* follows immediately from assertion (vii), together with Lemma 1, (i). This completes the proof of assertion (viii). □

Proposition 3 (**Base-changing and monodromic fullness**). *Let n be a positive integer, T a regular and connected scheme over S, and $x \in X_n(T)$ a T-valued point of X_n. Then the following hold:*

(i) *The T-valued point $x \in X_n(T)$ is* **Σ-monodromically full** (*respectively,* **quasi-Σ-monodromically full**) *if and only if the T-valued point of $X_n \times_S T$ determined by x is* **Σ-monodromically full** (*respectively,* **quasi-Σ-monodromically full**).

(ii) *Let T' be a regular and connected scheme over S and $T' \to T$ a morphism over S such that the natural outer homomorphism $\pi_1(T') \to \pi_1(T)$ is* **surjective** (*respectively, has* **open image**, *e.g., $T' \to T$ is a connected finite étale covering of T). Then the T-valued point $x \in X_n(T)$ is* **Σ-monodromically full** (*respectively,* **quasi-Σ-monodromically full**) *if and only if the T'-valued point of X_n determined by x is* **Σ-monodromically full** (*respectively,* **quasi-Σ-monodromically full**).

Proof. This follows immediately from Lemma 3, (vi), together with Remark 5. □

Lemma 4 (**Extensions arising from FC-admissible outer automorphisms**). *Let $m < n$ be positive integers, G a profinite group,* *and*

$$1 \longrightarrow \Delta^{\Sigma}_{X_n/S} \longrightarrow E_n \longrightarrow G \longrightarrow 1$$

an exact sequence of profinite groups. Write

$$\phi \colon G \longrightarrow \mathrm{Out}(\Delta^{\Sigma}_{X_n/S})$$

for the outer representation associated to the above exact sequence of profinite groups (*cf. Definition 1,* (i))*. Suppose that* ϕ **factors through** *the closed subgroup*

$$\mathrm{Out}^{\mathrm{FC}}(\Delta^{\Sigma}_{X_n/S}) \subseteq \mathrm{Out}(\Delta^{\Sigma}_{X_n/S})$$

— where we refer to [19]*, Definition 1.1,* (ii)*, concerning "*$\mathrm{Out}^{\mathrm{FC}}$*". Write, moreover,* B_m *for the quotient of* E_n *by* $\Delta^{\Sigma}_{X_n/X_m} \subseteq (\Delta^{\Sigma}_{X_n/S} \subseteq)$ E_n (*cf. Lemma 3,* (iv))*, i.e.,*

$$B_m \stackrel{\mathrm{def}}{=} E_n/\Delta^{\Sigma}_{X_n/X_m} .$$

Thus, relative to the natural isomorphism

$$\Delta^{\Sigma}_{X_m/S} \simeq \Delta^{\Sigma}_{X_n/S}/\Delta^{\Sigma}_{X_n/X_m}$$

(*cf. Lemma 3,* (iv))*, we have exact sequences of profinite groups*

$$1 \longrightarrow \Delta^{\Sigma}_{X_n/X_m} \longrightarrow E_n \longrightarrow B_m \longrightarrow 1 ;$$

$$1 \longrightarrow \Delta^{\Sigma}_{X_m/S} \longrightarrow B_m \longrightarrow G \longrightarrow 1 .$$

In particular, we obtain continuous homomorphisms

$$\rho \colon B_m \longrightarrow \mathrm{Out}(\Delta^{\Sigma}_{X_n/X_m}) ;$$

$$\widetilde{\rho} \colon B_m \longrightarrow \mathrm{Aut}(\Delta^{\Sigma}_{X_m/S})$$

(*cf. Definition 1,* (i))*. Then the following hold:*

(i) *The natural surjection* $B_m \twoheadrightarrow G$ *induces an isomorphism* $\mathrm{Ker}(\widetilde{\rho}) \xrightarrow{\sim} \mathrm{Ker}(\phi)$.

(ii) $\mathrm{Ker}(\rho) = \mathrm{Ker}(\widetilde{\rho}) = Z_{B_m}(\Delta^{\Sigma}_{X_m/S})$.

(iii) $Z_{E_n}(\Delta^{\Sigma}_{X_n/X_m}) = Z_{E_n}(\Delta^{\Sigma}_{X_n/S})$.

(iv) *The natural surjections* $E_n \twoheadrightarrow B_m \twoheadrightarrow G$ *induce isomorphisms*

$$Z_{E_n}(\Delta^{\Sigma}_{X_n/X_m}) \, (= Z_{E_n}(\Delta^{\Sigma}_{X_n/S})) \xrightarrow{\sim} \mathrm{Ker}(\rho)$$

$$(= \mathrm{Ker}(\widetilde{\rho}) = Z_{B_m}(\Delta^{\Sigma}_{X_m/S})) \xrightarrow{\sim} \mathrm{Ker}(\phi) .$$

Proof. First, we verify assertion (i). Write

$$Z \subseteq B_m$$

for the image of the centralizer $Z_{E_n}(\Delta^{\Sigma}_{X_n/S})$ of $\Delta^{\Sigma}_{X_n/S}$ in E_n via the natural surjection $E_n \twoheadrightarrow B_m$. Then it follows immediately from the definition of the closed subgroup $Z \subseteq B_m$ that

$$Z \subseteq Z_{B_m}(\Delta^{\Sigma}_{X_m/S}) .$$

Now I *claim* that

$(*_1)$ the surjection $B_m \twoheadrightarrow G$ induces an isomorphism $Z \xrightarrow{\sim} \mathrm{Ker}(\phi)$.

Indeed, since $\Delta^{\Sigma}_{X_n/S}$ is *center-free* (cf. Lemma 3, (v)), it holds that $\Delta^{\Sigma}_{X_n/S} \cap Z_{E_n}(\Delta^{\Sigma}_{X_n/S}) = \{1\}$. In particular, the natural surjection $Z_{E_n}(\Delta^{\Sigma}_{X_n/S}) \twoheadrightarrow Z$ is an *isomorphism*. Thus, it follows from the definition of $Z \subseteq B_m$ that the *claim* $(*_1)$ is equivalent to the fact that the surjection $E_n \twoheadrightarrow G$ induces an isomorphism $Z_{E_n}(\Delta^{\Sigma}_{X_n/S}) \xrightarrow{\sim} \mathrm{Ker}(\phi)$. On the other hand, this follows immediately from the fact that $\Delta^{\Sigma}_{X_n/S}$ is *center-free* (cf. Lemma 3, (v)), together with Lemma 1, (i). This completes the proof of the *claim* $(*_1)$. Next, I *claim* that

$(*_2)$ $Z = Z_{B_m}(\Delta^{\Sigma}_{X_m/S})$.

Indeed, it follows immediately from Lemma 1, (i), together with the various definitions involved, that the image of $Z_{B_m}(\Delta^{\Sigma}_{X_m/S}) \subseteq B_m$ via the natural surjection $B_m \twoheadrightarrow G$ *coincides with* the kernel of the composite

$$G \xrightarrow{\phi} \mathrm{Out}^{\mathrm{FC}}(\Delta^{\Sigma}_{X_n/S}) \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\Delta^{\Sigma}_{X_m/S})$$

— where the second arrow is the homomorphism induced by the natural surjection $\Delta^{\Sigma}_{X_n/S} \twoheadrightarrow \Delta^{\Sigma}_{X_m/S}$ (cf. Lemma 3, (iv)). Thus, it follows immediately from [8], Theorem B, that the image of $Z_{B_m}(\Delta^{\Sigma}_{X_m/S}) \subseteq B_m$ via the surjection $B_m \twoheadrightarrow G$ *coincide with* $\mathrm{Ker}(\phi)$. On the other hand, since $\Delta^{\Sigma}_{X_m/S}$ is *center-free* (cf. Lemma 3, (v)), it holds that $\Delta^{\Sigma}_{X_m/S} \cap Z_{B_m}(\Delta^{\Sigma}_{X_m/S}) = \{1\}$. Thus, since $Z \subseteq Z_{B_m}(\Delta^{\Sigma}_{X_m/S})$, it follows immediately from the *claim* $(*_1)$ that $Z = Z_{B_m}(\Delta^{\Sigma}_{X_m/S})$. This completes the proof of the *claim* $(*_2)$. Now it follows from Lemma 1, (i), that $\mathrm{Ker}(\widetilde{\rho}) = Z_{B_m}(\Delta^{\Sigma}_{X_m/S})$. Thus, assertion (i) follows from the *claims* $(*_1)$, $(*_2)$. This completes the proof of assertion (i).

Next, we verify assertion (ii). Now I *claim* that

$(*_3)$ $\mathrm{Ker}(\widetilde{\rho}) \subseteq \mathrm{Ker}(\rho)$.

Indeed, it follows from the *claim* $(*_2)$, together with Lemma 1, (i), that $\mathrm{Ker}(\widetilde{\rho}) = Z_{B_m}(\Delta^{\Sigma}_{X_m/S}) = Z$. On the other hand, since $Z_{E_n}(\Delta^{\Sigma}_{X_n/S}) \subseteq Z_{E_n}(\Delta^{\Sigma}_{X_n/X_m})$ (cf. Lemma 3, (iv)), it follows from Lemma 1, (i), together with the definition of $Z \subseteq B_m$, that $Z \subseteq \mathrm{Ker}(\rho)$. This completes the proof of the *claim* $(*_3)$. Now it follows immediately from Lemma 3, (viii), that $\mathrm{Ker}(\rho) \cap \Delta^{\Sigma}_{X_m/S} = \{1\}$. Thus, assertion (ii) follows immediately from the *claim* $(*_3)$, together with Lemma 1, (ii). This completes the proof of assertion (ii).

Next, we verify assertion (iii). Observe that $Z_{E_n}(\Delta^{\Sigma}_{X_n/S}) \subseteq Z_{E_n}(\Delta^{\Sigma}_{X_n/X_m})$ (cf. Lemma 3, (iv)). Moreover, it follows immediately from Lemma 1, (i), (respectively, the *claim* $(*_2)$, together with Lemma 1, (i)) that the image of $Z_{E_n}(\Delta^{\Sigma}_{X_n/X_m})$ (respectively, $Z_{E_n}(\Delta^{\Sigma}_{X_n/S})$) via the natural surjection $E_n \twoheadrightarrow B_m$ *coincides with* $\mathrm{Ker}(\rho)$ (respectively, $\mathrm{Ker}(\widetilde{\rho})$). On the other hand, it follows from the fact that $\Delta^{\Sigma}_{X_n/X_m}$ is *center-free* (cf.

Lemma 3, (v)) that $\Delta^{\Sigma}_{X_n/X_m} \cap Z_{E_n}(\Delta^{\Sigma}_{X_n/X_m}) = \{1\}$. Therefore, assertion (iii) follows immediately from assertion (ii). This completes the proof of assertion (iii).

Assertion (iv) follows immediately from assertions (i), (ii), and (iii). □

Remark 6. A similar result to Lemma 4, (ii), can be found in [3], Theorem 2.5.

Proposition 4 (Two quotients of the fundamental group of a configuration space). *Let $m < n$ be positive integers, T a regular and connected scheme over S, and $x \in X_m(T)$ a T-valued point of X_m. Then the following hold:*

(i) *The kernel of the pro-Σ representation associated to X_m/S*

$$\pi_1(X_m) \twoheadrightarrow \Pi^{\Sigma}_{X_m/S} \stackrel{\widetilde{\rho}^{\Sigma}_{X_m/S}}{\to} \mathrm{Aut}(\Delta^{\Sigma}_{X_m/S})$$

coincides with *the kernel of the pro-Σ outer representation associated to X_n/X_m*

$$\rho^{\Sigma}_{X_n/X_m} \colon \pi_1(X_m) \to \mathrm{Out}(\Delta^{\Sigma}_{X_n/X_m})$$

— i.e., the two quotients $\Phi^{\Sigma}_{X_m/S}$ and $\Gamma^{\Sigma}_{X_n/X_m}$ of $\Pi^{\Sigma}_{X_m/S}$ **coincide**. *In particular, we obtain a commutative diagram of profinite groups*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta^{\Sigma}_{X_m/S} & \longrightarrow & \Pi^{\Sigma}_{X_m/S} & \longrightarrow & \pi_1(S) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta^{\Sigma}_{X_m/S} & \longrightarrow & \Gamma^{\Sigma}_{X_n/X_m} & \longrightarrow & \Gamma^{\Sigma}_{X_m/S} & \longrightarrow & 1
\end{array}$$

— where the horizontal sequences are **exact**, *and the vertical arrows are* **surjective**.

(ii) *The kernel of the pro-Σ outer representation associated to $X[x]_{n-m}/T$*

$$\rho^{\Sigma}_{X[x]_{n-m}/T} \colon \pi_1(T) \to \mathrm{Out}(\Delta^{\Sigma}_{X[x]_{n-m}/T})$$

and the kernel of the composite

$$\pi_1(T) \stackrel{\pi_1(x)}{\to} \pi_1(X_m) \twoheadrightarrow \Pi^{\Sigma}_{X_m/S} \stackrel{\widetilde{\rho}^{\Sigma}_{X_m/S}}{\to} \mathrm{Aut}(\Delta^{\Sigma}_{X_m/S})$$

coincide. *In particular, if $x \in X(T)$ is a T-valued point of X, and we write*

$$U \stackrel{\mathrm{def}}{=} (X \times_S T) \setminus \mathrm{Im}(x)\,,$$

then the kernel of the pro-Σ outer representation associated to U/T

$$\rho^{\Sigma}_{U/T} \colon \pi_1(T) \to \mathrm{Out}(\Delta^{\Sigma}_{U/T})$$

and the kernel of the composite

$$\pi_1(T) \stackrel{\pi_1(x)}{\to} \pi_1(X) \twoheadrightarrow \Pi^{\Sigma}_{X/S} \stackrel{\widetilde{\rho}^{\Sigma}_{X/S}}{\to} \mathrm{Aut}(\Delta^{\Sigma}_{X/S})$$

coincide.

(iii) *The following two conditions are equivalent:*

(iii-1) *The T-valued point $x \in X_n(T)$ is* **Σ-monodromically full** (*respectively,* **quasi-Σ-monodromically full**).

(iii-2) *For any $l \in \Sigma$, if we write $\Phi_T \subseteq \mathrm{Aut}(\Delta_{X_m/S})$ (respectively, $\Phi_{\mathrm{geom}} \subseteq \mathrm{Aut}(\Delta_{X_m/S})$) for the image of the composite*

$$\pi_1(T) \stackrel{\pi_1(x)}{\to} \pi_1(X_m) \twoheadrightarrow \Pi^{\{l\}}_{X_m/S} \stackrel{\widetilde{\rho}^{\{l\}}_{X_m/S}}{\to} \mathrm{Aut}(\Delta_{X_m/S})$$

$$(\textit{respectively},\ \mathrm{Ker}\Big(\pi_1(X_m) \to \pi_1(S)\Big) \hookrightarrow \pi_1(X_m)$$

$$\twoheadrightarrow \Pi^{\{l\}}_{X_m/S} \stackrel{\widetilde{\rho}^{\{l\}}_{X_m/S}}{\to} \mathrm{Aut}(\Delta_{X_m/S})\,),$$

then Φ_T **contains** *Φ_{geom} (respectively, $\Phi_T \cap \Phi_{\mathrm{geom}}$ is an* **open** *subgroup of Φ_{geom}).*

(iv) *If $S = T$, then the following two conditions are equivalent:*

(iv-1) *The S-valued point $x \in X_n(S)$ is* **Σ-monodromically full** (*respectively,* **quasi-Σ-monodromically full**).

(iv-2) *For any $l \in \Sigma$, the composite*

$$\pi_1(S) \stackrel{\pi_1(x)}{\to} \pi_1(X_m) \twoheadrightarrow \Pi^{\{l\}}_{X_m/S} \stackrel{\widetilde{\rho}^{\{l\}}_{X_m/S}}{\twoheadrightarrow} \Phi^{\{l\}}_{X_m/S}$$

is **surjective** (*respectively, has* **open image**).

Proof. Assertion (i) follows immediately from Lemma 4, (ii), together with Proposition 1, (i). Assertion (ii) follows immediately from assertion (i), together with Lemma 3, (vi). Assertion (iii) follows immediately from assertion (i). Assertion (iv) follows immediately from assertion (iii). □

Lemma 5 (**Extension via the outer universal monodromy representations).** *Let (g, r) be a pair of nonnegative integers such that $2g - 2 + r > 0$, n a positive integer, k a field of characteristic 0, $s\colon \mathrm{Spec}\, k \to \mathcal{M}_{g,r}$ a morphism of stacks (where we refer to the discussion entitled "Curves" in §0 concerning "$\mathcal{M}_{g,r}$"), and X an r-pointed smooth curve of genus g over k corresponding to s. By the morphism of stacks $\mathcal{M}_{g,r+n} \to \mathcal{M}_{g,r}$ obtained by forgetting the last n sections, regard the stack $\mathcal{M}_{g,r+n}$ as a stack over $\mathcal{M}_{g,r}$. Then the morphism of stacks s naturally induces a morphism $X_n \to \mathcal{M}_{g,r+n}$ — where X_n is the n-th configuration space of X/k (cf. Definition 2,* (i)*). Moreover, this morphism and the pro-Σ outer universal monodromy representations*

$$\rho^{\Sigma}_{g,r}\colon \pi_1(\mathcal{M}_{g,r} \otimes_{\mathbb{Z}} k) \longrightarrow \mathrm{Out}(\Delta^{\Sigma}_{g,r})\,;$$

$$\rho^{\Sigma}_{g,r+n}\colon \pi_1(\mathcal{M}_{g,r+n} \otimes_{\mathbb{Z}} k) \longrightarrow \mathrm{Out}(\Delta^{\Sigma}_{g,r+n})$$

(*cf.* [9], *Definition 1.3,* (ii)) *determine a commutative diagram of profinite groups*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(X_n \otimes_k \overline{k}) & \longrightarrow & \pi_1(\mathcal{M}_{g,r+n} \otimes_{\mathbb{Z}} k) & \longrightarrow & \pi_1(\mathcal{M}_{g,n} \otimes_{\mathbb{Z}} k) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow {\scriptstyle \rho^{\Sigma}_{g,r+n}} & & \downarrow {\scriptstyle \rho^{\Sigma}_{g,r}} & & \\ 1 & \longrightarrow & \Delta^{\Sigma}_{X_n/k} & \longrightarrow & \mathrm{Im}(\rho^{\Sigma}_{g,r+n}) & \longrightarrow & \mathrm{Im}(\rho^{\Sigma}_{g,r}) & \longrightarrow & 1 \end{array}$$

— *where the horizontal sequences are* **exact**, *and the vertical arrows are the* **natural surjections**.

Proof. It follows immediately from the various definitions involved that one may naturally regard the fiber product of $\mathcal{M}_{g,r+n} \to \mathcal{M}_{g,r}$ and $s\colon \mathrm{Spec}\, k \to \mathcal{M}_{g,r}$ as X_n. Thus, we obtain a morphism $X_n \to \mathcal{M}_{g,r+n}$, hence also a sequence of stacks $X_n \otimes_k \overline{k} \to \mathcal{M}_{g,r+n} \otimes_{\mathbb{Z}} k \to \mathcal{M}_{g,r} \otimes_{\mathbb{Z}} k$. In particular, we obtain the top sequence in the commutative diagram of the statement of Lemma 5, hence also the commutative diagram of the statement of Lemma 5. Now the exactness of the top sequence in the commutative diagram of the statement of Lemma 5 follows immediately from [12], Lemma 2.1.

To verify the exactness of the lower sequence in the commutative diagram of the statement of Lemma 5, write Π for the quotient of $\pi_1(\mathcal{M}_{g,r+n} \otimes_{\mathbb{Z}} k)$ by the kernel of the natural surjection $\pi_1(X_n \otimes_k \overline{k}) \twoheadrightarrow \Delta^{\Sigma}_{X_n/k}$ — i.e., $\Pi \overset{\mathrm{def}}{=}$ "$\Pi^{\Sigma}_{\mathcal{M}_{g,r+n}\otimes_{\mathbb{Z}}k/\mathcal{M}_{g,r}\otimes_{\mathbb{Z}}k}$" — and $\widetilde{\rho}\colon \Pi \to \mathrm{Aut}(\Delta^{\Sigma}_{X_n/k})$ (respectively, $\rho\colon \pi_1(\mathcal{M}_{g,r} \otimes_{\mathbb{Z}} k) \to \mathrm{Out}(\Delta^{\Sigma}_{X_n/k})$) for the representation (respectively, outer representation) associated to the natural exact sequence of profinite groups

$$1 \longrightarrow \Delta^{\Sigma}_{X_n/k} \longrightarrow \Pi \longrightarrow \pi_1(\mathcal{M}_{g,r} \otimes_{\mathbb{Z}} k) \longrightarrow 1\,,$$

i.e., $\widetilde{\rho} \overset{\mathrm{def}}{=}$ "$\widetilde{\rho}^{\Sigma}_{\mathcal{M}_{g,r+n}\otimes_{\mathbb{Z}}k/\mathcal{M}_{g,r}\otimes_{\mathbb{Z}}k}$" (respectively, $\rho \overset{\mathrm{def}}{=}$ "$\rho^{\Sigma}_{\mathcal{M}_{g,r+n}\otimes_{\mathbb{Z}}k/\mathcal{M}_{g,r}\otimes_{\mathbb{Z}}k}$"). Then it follows immediately from [8], Theorem B, that $\mathrm{Ker}(\rho) = \mathrm{Ker}(\rho^{\Sigma}_{g,r})$. On the other hand, it follows immediately from Lemma 3, (viii), together with the various definitions involved, that $\rho^{\Sigma}_{g,r+n}\colon \pi_1(\mathcal{M}_{g,r+n} \otimes_{\mathbb{Z}} k) \to \mathrm{Out}(\Delta^{\Sigma}_{g,r+n})$ *factors through* the natural surjection $\pi_1(\mathcal{M}_{g,r+n} \otimes_{\mathbb{Z}} k) \twoheadrightarrow \Pi$; moreover, it follows immediately from Lemma 4, (ii), that the kernel of the homomorphism $\Pi \to \mathrm{Out}(\Delta^{\Sigma}_{g,r+n})$ determined by $\rho^{\Sigma}_{g,r+n}$ *coincides with* $\mathrm{Ker}(\widetilde{\rho})$. Therefore, the exactness of the lower sequence in the commutative diagram of the statement of Lemma 5 follows immediately from Lemma 3, (v), together with Lemma 1, (i). This completes the proof of Lemma 5. ☐

Proposition 5 (Monodromically full curves and monodromically full points). *Suppose that S is the spectrum of a field k of characteristic* 0. *Let n be a positive integer and $x \in X_n(k)$ a k-rational point of X_n. Then the hyperbolic curve $X[x]$ over k is* **Σ-monodromically**

full (*respectively,* **quasi-Σ-monodromically full**) — *cf.* [9]*, Definition 2.2 — if and only if the following two conditions are satisfied:*

(i) *The hyperbolic curve X over k is* **Σ-monodromically full** (*respectively,* **quasi-Σ-monodromically full**) — *cf.* [9]*, Definition 2.2.*

(ii) *The k-rational point $x \in X_n(k)$ of X_n is* **Σ-monodromically full** (*respectively,* **quasi-Σ-monodromically full**)*.*

Proof. Suppose that X is *of type* (g, r). Now it follows immediately from the definitions of the terms "monodromically full", "quasi-monodromically full" that, by replacing Σ by $\{l\}$ for $l \in \Sigma$, we may assume without loss of generality that Σ is *of cardinality* 1. Moreover, again by the definitions the terms "monodromically full", "quasi-monodromically full", by replacing k by the (necessarily finite) minimal Galois extension of k over which X is *split* (where we refer to [9], Definition 1.5, (i), concerning the term "split"), we may assume without loss of generality that X is *split over* k. Then since X is *split*, the classifying morphism $\mathrm{Spec}\, k \to \mathcal{M}_{g,[r]}$ of the hyperbolic curve X/k *factors through* the natural finite étale Galois covering $\mathcal{M}_{g,r} \to \mathcal{M}_{g,[r]}$ (cf. the discussion entitled "Curves" in §0). Let $s_X \colon \mathrm{Spec}\, k \to \mathcal{M}_{g,r}$ be a lift of the classifying morphism $\mathrm{Spec}\, k \to \mathcal{M}_{g,[r]}$ of X/k. Then by Lemma 5, we have a commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
 & & & & \pi_1(\mathrm{Spec}\, k) & & & & \\
 & & & & \downarrow{\scriptstyle \pi_1(x)} & & & & \\
1 & \longrightarrow & \pi_1(X_n \otimes_k \overline{k}) & \longrightarrow & \pi_1(X_n) & \longrightarrow & \pi_1(\mathrm{Spec}\, k) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \pi_1(s_X)} & & \\
1 & \longrightarrow & \pi_1(X_n \otimes_k \overline{k}) & \longrightarrow & \pi_1(\mathcal{M}_{g,r+n} \otimes_{\mathbb{Z}} k) & \longrightarrow & \pi_1(\mathcal{M}_{g,r} \otimes_{\mathbb{Z}} k) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \rho^{\Sigma}_{g,r+n}} & & \downarrow{\scriptstyle \rho^{\Sigma}_{g,r}} & & \\
1 & \longrightarrow & \Delta^{\Sigma}_{X_n/k} & \longrightarrow & \mathrm{Im}(\rho^{\Sigma}_{g,r+n}) & \longrightarrow & \mathrm{Im}(\rho^{\Sigma}_{g,r}) & \longrightarrow & 1
\end{array}$$

— where the horizontal sequences are *exact*, and "$\pi_1(-)$" is the outer homomorphism induced by "$(-)$".

Now it follows from the various definitions involved that the composite of the three middle vertical arrows $\pi_1(\mathrm{Spec}\, k) \to \mathrm{Im}(\rho^{\Sigma}_{g,r+n})$ *coincides with* the outer pro-Σ representation $\rho^{\Sigma}_{X[x]/k}$ associated to $X[x]/k$, and the composite of the two right-hand vertical arrows $\pi_1(\mathrm{Spec}\, k) \to \mathrm{Im}(\rho^{\Sigma}_{g,r})$ *coincides with* the outer pro-Σ representation $\rho^{\Sigma}_{X/k}$ associated to X/k. Therefore, again by the various definitions involved, if we write $\rho \colon \pi_1(X_n) \to \mathrm{Im}(\rho^{\Sigma}_{g,r+n})$ for the composite of the middle vertical arrow $\pi_1(X_n) \to \pi_1(\mathcal{M}_{g,r+n} \otimes_{\mathbb{Z}} k)$ and the lower middle vertical arrow $\rho^{\Sigma}_{g,r+n}$, then

- the hyperbolic curve $X[x]$ is *Σ-monodromically full* (respectively, *quasi-Σ-monodromically full*) if and only if the composite of the three middle vertical arrows — i.e., $\rho^{\Sigma}_{X[x]/k}$ — is *surjective* (respectively, *has open image in* $\mathrm{Im}(\rho^{\Sigma}_{g,r+n})$),
- the hyperbolic curve X is *Σ-monodromically full* (respectively, *quasi-Σ-monodromically full*) if and only if the composite of the two right-hand vertical arrows — i.e., $\rho^{\Sigma}_{X/k}$ — is *surjective* (respectively, *has open image in* $\mathrm{Im}(\rho^{\Sigma}_{g,r+n})$),
- and the k-rational point $x \in X_n(k)$ of X_n is *Σ-monodromically full* (respectively, *quasi-Σ-monodromically full*) if and only if the image of the composite of the three middle vertical arrows — i.e., $\rho^{\Sigma}_{X[x]/k}$ — *coincides with* the image of ρ (respectively, is an *open* subgroup of the image of ρ).

Thus, one may easily verify that Proposition 5 holds. This completes the proof of Proposition 5. □

Remark 7. In the notation of Proposition 5, since the complement $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$ of $\{0, 1, \infty\}$ in the projective line $\mathbb{P}^1_k$ over k is a $\mathfrak{Primes}$-*monodromically full* hyperbolic curve (cf. [9], Definition 2.2, (i)), one may regard Proposition 5 as a generalization of the second equivalence in Remark 4.

3. Kernels of the outer representations associated to configuration spaces

In the present §, we consider the kernels of the outer representations associated to configuration spaces of hyperbolic curves. We maintain the notation and assumption of the preceding §. Let n be a positive integer and $Y \to X_n$ a finite étale $\Pi^{\Sigma}_{X_n/S}$-covering (cf. Definition 1, (v)) over S. Suppose, moreover, that

- the natural homomorphism $\pi_1(Y) \to \pi_1(S)$ induced by the structure morphism $Y \to S$ of Y is *surjective.*

Then we have a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta^{\Sigma}_{Y/S} & \longrightarrow & \Pi^{\Sigma}_{Y/S} & \longrightarrow & \pi_1(S) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Delta^{\Sigma}_{X_n/S} & \longrightarrow & \Pi^{\Sigma}_{X_n/S} & \longrightarrow & \pi_1(S) & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are *exact*, and the vertical arrows are *open injections*. In particular, $\Delta^{\Sigma}_{Y/S}$ is *topologically finitely generated* (cf. Lemma 3, (v)).

Lemma 6 (**Difference between kernels of outer representations arising from extensions**)**.** *Let*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta' & \longrightarrow & \Pi' & \longrightarrow & G & \longrightarrow & 1 \\ & & \alpha\downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Delta & \longrightarrow & \Pi & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

be a commutative diagram of profinite groups, where the horizontal sequences are **exact***, and the right-hand vertical arrow is the* **identity automorphism** *of G. Suppose that Δ and Δ' are* **topologically finitely generated***. Write*

$$\rho\colon G \longrightarrow \mathrm{Out}(\Delta) \ \ (\textit{respectively},\ \rho'\colon G \longrightarrow \mathrm{Out}(\Delta')\,)$$

for the outer representation associated to the lower (*respectively, top*) *horizontal sequence in the above commutative diagram of profinite groups* (*cf. Definition 1,* (i))*. Then the following hold:*

(i) *If α is* **injective***, then we have a natural* **exact** *sequence of profinite groups*

$$1 \longrightarrow \mathrm{Ker}(\rho) \cap \mathrm{Ker}(\rho') \longrightarrow \mathrm{Ker}(\rho) \xrightarrow{\rho'} \mathrm{Im}(\phi)$$

— where ϕ is the outer representation

$$N_{\Delta}(\Delta')/\Delta' \longrightarrow \mathrm{Out}(\Delta')$$

associated to the exact sequence of profinite groups

$$1 \longrightarrow \Delta' \longrightarrow N_{\Delta}(\Delta') \longrightarrow N_{\Delta}(\Delta')/\Delta' \longrightarrow 1$$

(*cf. Definition 1,* (i))*.*

(ii) *If α is* **surjective***, then we have an inclusion*

$$\mathrm{Ker}(\rho') \subseteq \mathrm{Ker}(\rho)\,.$$

(iii) *If α is an* **open injection***, and Δ is* **slim** (*cf. the discussion entitled "Profinite Groups" in* §*0*)*, then*

$$Z_{\Pi'}(\Delta') = Z_{\Pi}(\Delta) \cap \Pi' \ \ ; \ \ \mathrm{Ker}(\rho') \subseteq \mathrm{Ker}(\rho)\,.$$

Moreover, $Z_{\Pi}(\Delta) \subseteq \Pi'$ if and only if $\mathrm{Ker}(\rho') = \mathrm{Ker}(\rho)$.

Proof. Assertions (i) and (ii) follow immediately from the various definitions involved. Finally, we verify assertion (iii). Since Δ and Δ' are *center-free*, it follows from Lemma 1, (i), that, to verify assertion (iii), it suffices to verify that $Z_{\Pi'}(\Delta') = Z_{\Pi}(\Delta) \cap \Pi'$. On the other hand, since Δ is *slim*, this follows immediately from Lemma 2. This completes the proof of assertion (iii). □

Proposition 6 (**Hyperbolic partial compactifications and monodromic fullness**)**.** *Let T be a regular and connected scheme over S and $x \in X_n(T)$ a T-valued point of X_n. Suppose that the T-valued point $x \in X_n(T)$ is* **Σ-monodromically full** (*respectively,* **quasi-Σ-monodromically full**)*. Then the following hold:*

(i) *Let Z be a hyperbolic partial compactification of X over S — i.e., a hyperbolic curve over S which contains X as an open subscheme over S. (Note that the natural open immersion $X \hookrightarrow Z$ induces an open immersion $X_n \hookrightarrow Z_n$). Then the T-valued point of Z_n determined by x is* **Σ-monodromically full** (*respectively,* **quasi-Σ-monodromically full**).

(ii) *Let $m < n$ be a positive integer. Then the T-valued point of X_m determined by x is* **Σ-monodromically full** (*respectively,* **quasi-Σ-monodromically full**).

Proof. It follows from Proposition 3, (i), that, to verify Proposition 6, by replacing X by $X \times_S T$, we may assume without loss of generality that $S = T$. Let $l \in \Sigma$ be an element of Σ. First, we verify assertion (i). The natural open immersion $X \hookrightarrow Z$ induces a commutative diagram of schemes

$$\begin{array}{ccc} X_{n+1} & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ Z_{n+1} & \longrightarrow & Z_n\,; \end{array}$$

thus, we obtain a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta^{\{l\}}_{X_{n+1}/X_n} & \longrightarrow & \Pi^{\{l\}}_{X_{n+1}/S} & \longrightarrow & \Pi^{\{l\}}_{X_n/S} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta^{\{l\}}_{Z_{n+1}/Z_n} & \longrightarrow & \Pi^{\{l\}}_{Z_{n+1}/S} & \longrightarrow & \Pi^{\{l\}}_{Z_n/S} & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are *exact* (cf. Lemma 3, (iv)), and the vertical arrows are *surjective* (cf. Lemma 3, (ii)). In particular, it follows immediately from Lemma 6, (ii), that we obtain a natural *surjection* $\Gamma^{\{l\}}_{X_{n+1}/X_n} \twoheadrightarrow \Gamma^{\{l\}}_{Z_{n+1}/Z_n}$. Thus, assertion (i) follows immediately from Remark 5. This completes the proof of assertion (i). Next, we verify assertion (ii). Now we have a commutative diagram of schemes

$$\begin{array}{ccc} X_{n+1} & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ X_{m+1} & \longrightarrow & X_m \end{array}$$

— where the left-hand vertical arrow is the projection obtained as "$(x_1, \cdots, x_{n+1}) \mapsto (x_1, \cdots, x_m, x_{n+1})$" and other arrows are the natural projections. Thus, we obtain a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta^{\{l\}}_{X_{n+1}/X_n} & \longrightarrow & \Pi^{\{l\}}_{X_{n+1}/S} & \longrightarrow & \Pi^{\{l\}}_{X_n/S} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta^{\{l\}}_{X_{m+1}/X_m} & \longrightarrow & \Pi^{\{l\}}_{X_{m+1}/S} & \longrightarrow & \Pi^{\{l\}}_{X_m/S} & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are *exact* (cf. Lemma 3, (iv)), and the vertical arrows are *surjective* (cf. Lemma 3, (i), (ii)). In particular, it follows immediately from Lemma 6, (ii), that we obtain a natural *surjection* $\Gamma^{\{l\}}_{X_{n+1}/X_n} \twoheadrightarrow \Gamma^{\{l\}}_{X_{m+1}/X_m}$. Thus, assertion (ii) follows immediately from Remark 5. This completes the proof of assertion (ii). $\square$

Proposition 7 (**Kernels of the outer representations associated to the fundamental groups of configuration spaces**). *Let T be a regular and connected scheme over S and $y \in Y(T)$ a T-valued point of Y. Write $x \in X_n(T)$ for the T-valued point of X_n determined by y. Then the following hold:*

(i) $\mathrm{Ker}(\rho^{\Sigma}_{Y/S})$ *is an* **open** *subgroup of* $\mathrm{Ker}(\rho^{\Sigma}_{X_n/S}) \subseteq \pi_1(S)$. *Moreover,* $\mathrm{Ker}(\rho^{\Sigma}_{X_n/S}) = \mathrm{Ker}(\rho^{\Sigma}_{Y/S})$ *if and only if the covering $Y \to X_n$ is a finite étale $\Phi^{\Sigma}_{X_n/S}$-covering* (*cf. Definition 1,* (v)).

(ii) *The natural inclusion* $\Pi^{\Sigma}_{Y/S} \hookrightarrow \Pi^{\Sigma}_{X_n/S}$ *induces a commutative diagram of profinite groups*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta^{\Sigma}_{Y/S} & \longrightarrow & \Phi^{\Sigma}_{Y/S} & \longrightarrow & \Gamma^{\Sigma}_{Y/S} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta^{\Sigma}_{X_n/S} & \longrightarrow & \Phi^{\Sigma}_{X_n/S} & \longrightarrow & \Gamma^{\Sigma}_{X_n/S} & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are **exact**, *the left-hand and middle vertical arrows are* **open injection**, *and the right-hand vertical arrow is a* **surjection with finite kernel**.

(iii) *The kernels of the two composites*

$$\pi_1(T) \overset{\pi_1(y)}{\to} \pi_1(Y) \overset{\widetilde{\rho}^{\Sigma}_{Y/S}}{\to} \mathrm{Aut}(\Delta^{\Sigma}_{Y/S}) \ ; \ \ \pi_1(T) \overset{\pi_1(x)}{\to} \pi_1(X_n) \overset{\widetilde{\rho}^{\Sigma}_{X_n/S}}{\to} \mathrm{Aut}(\Delta^{\Sigma}_{X_n/S})$$

coincide.

(iv) *If Y is a* **hyperbolic curve** *over S* (*thus, $n = 1$*), *and we write*

$$U_Y \overset{\mathrm{def}}{=} (Y \times_S T) \setminus \mathrm{Im}(y) \ ; \ \ U_X \overset{\mathrm{def}}{=} (X \times_S T) \setminus \mathrm{Im}(x) \, ,$$

then $\mathrm{Ker}(\rho^{\Sigma}_{U_Y/T}) = \mathrm{Ker}(\rho^{\Sigma}_{U_X/T})$.

(v) *Suppose that $S = T$, $\Sigma = \{l\}$ for some $l \in \mathfrak{Primes}$, and that $x \in X_n(T)$ is* ***l*-monodromically full**. *Then the covering $Y \to X_n$ is an* **isomorphism** *if and only if* $\mathrm{Ker}(\rho^{\Sigma}_{Y/S}) = \mathrm{Ker}(\rho^{\Sigma}_{X_n/S})$, *i.e., if the covering $Y \to X_n$ is* **not** *an isomorphism, then the cokernel of the natural inclusion* $\mathrm{Ker}(\rho^{\Sigma}_{Y/S}) \hookrightarrow \mathrm{Ker}(\rho^{\Sigma}_{X_n/S})$ (*cf. assertion* (i)) *is* **nontrivial**. *Moreover, if* $\Delta^{\Sigma}_{Y/S} \subseteq \Delta^{\Sigma}_{X_n/S}$ *is* **normal**, *then we have an exact sequence of profinite groups*

$$1 \longrightarrow \mathrm{Ker}(\rho^{\Sigma}_{Y/S}) \longrightarrow \mathrm{Ker}(\rho^{\Sigma}_{X_n/S}) \longrightarrow \Delta^{\Sigma}_{X_n/S}/\Delta^{\Sigma}_{Y/S} \longrightarrow 1 \, .$$

Proof. Assertion (i) follows immediately from Lemmas 1, (i); 6, (i), (iii). Assertion (ii) follows immediately from assertion (i), together with Proposition 1, (i). Assertion (iii) follows immediately from assertion (ii). Assertion (iv) follows immediately from assertion (iii), together with Proposition 4, (ii). Finally, we verify assertion (v). Since $S = T$, and $x \in X_n(T)$ is *l-monodromically full*, it follows immediately from Proposition 4, (iv), that the composite

$$\pi_1(S) \overset{\pi_1(x)}{\to} \pi_1(X_n) \overset{\widetilde{\rho}^{\Sigma}_{X_n/S}}{\twoheadrightarrow} \Phi^{\Sigma}_{X_n/S}$$

is *surjective*. Thus, it follows immediately from assertion (ii) that we obtain a commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta^{\Sigma}_{Y/S} & \longrightarrow & \Phi^{\Sigma}_{Y/S} & \longrightarrow & \Gamma^{\Sigma}_{Y/S} & \longrightarrow & 1 \\
 & & \downarrow & & \wr\downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta^{\Sigma}_{X_n/S} & \longrightarrow & \Phi^{\Sigma}_{X_n/S} & \longrightarrow & \Gamma^{\Sigma}_{X_n/S} & \longrightarrow & 1
\end{array}$$

— where the horizontal sequences are *exact*, and the middle vertical arrow is an *isomorphism*. Therefore, if the covering $Y \to X_n$ is *not* an isomorphism, then it holds that the kernel of the natural surjection $\Gamma^{\Sigma}_{Y/S} \twoheadrightarrow \Gamma^{\Sigma}_{X_n/S}$ is *nontrivial*. This completes the proof of assertion (v). □

Remark 8. Let k be a number field (i.e., a finite extension of the field of rational numbers), (g_0, r_0) a pair of nonnegative integers such that $2g_0 - 2 + r_0 > 0$, and $N \subseteq \pi_1(\operatorname{Spec} k)$ a normal closed subgroup of $\pi_1(\operatorname{Spec} k)$. Write

$$\mathcal{I}^{\mathrm{Gal}}(l, k, g_0, r_0, N)$$

for the set of the isomorphism classes over k of hyperbolic curves C of type (g_0, r_0) over k such that the kernels of the pro-l outer representations associated to C/k

$$\rho^{\{l\}}_{C/k} \colon \pi_1(\operatorname{Spec} k) \longrightarrow \operatorname{Out}(\Delta^{\{l\}}_{C/k})$$

coincide with $N \subseteq \pi_1(\operatorname{Spec} k)$ (cf. [9], Definition A. 1). Then it follows from [9], Theorem C, that $\mathcal{I}^{\mathrm{Gal}}(l, k, g_0, r_0, N)$ is *finite*. On the other hand, it follows from Proposition 7, (i), that in general,

$$\mathcal{I}^{\mathrm{Gal}}(l, k, N) \overset{\mathrm{def}}{=} \bigcup_{2g-2+r>0} \mathcal{I}^{\mathrm{Gal}}(l, k, g, r, N)$$

is *not finite*. Indeed, let C be a hyperbolic curve over k such that there exists a k-rational point $x \in C(k)$ which is *not l-monodromically full* (e.g., C is *of type* $(0, 3)$ — cf. Proposition 11, (ii) below; Theorem 2 below). Then it follows from Remark 5 that the image of the composite

$$\phi \colon \pi_1(\operatorname{Spec} k) \overset{\pi_1(x)}{\to} \pi_1(C) \twoheadrightarrow \Phi^{\{l\}}_{C/k}$$

is *not open*. Thus, there exists an infinite sequence of open subgroups of $\Phi^{\{l\}}_{C/k}$ which contain $\operatorname{Im}(\phi)$

$$\operatorname{Im}(\phi) \subseteq \cdots \subsetneq \Phi_n \subsetneq \cdots \subsetneq \Phi_2 \subsetneq \Phi_1 \subsetneq \Phi_0 = \Phi^{\{l\}}_{C/k},$$

hence also an infinite sequence of connected finite étale coverings of C

$$\cdots \longrightarrow C^n \longrightarrow \cdots \longrightarrow C^2 \longrightarrow C^1 \longrightarrow C^0 = C$$

— where C^n is the finite étale covering of C corresponding to the open subgroup $\Phi_n \subseteq \Phi^{\{l\}}_{C/k}$. Then for any n, since $\mathrm{Im}(\phi) \subseteq \Phi_n$, C^n is a *hyperbolic curve over* k; moreover, since C^n is a finite étale $\Phi^{\{l\}}_{C/k}$-covering (cf. Definition 1, (v)), it follows from Proposition 7, (i), that $\mathrm{Ker}(\rho^{\{l\}}_{C/k}) = \mathrm{Ker}(\rho^{\{l\}}_{C^n/k})$. In particular,

$$\mathcal{I}^{\mathrm{Gal}}(l, k, \mathrm{Ker}(\rho^{\{l\}}_{C/k})) = \bigcup_{2g-2+r>0} \mathcal{I}^{\mathrm{Gal}}(l, k, g, r, \mathrm{Ker}(\rho^{\{l\}}_{C/k}))$$

is *not finite*.

Proposition 8 (**Monodromic fullness and finite étale coverings**)**.** *Suppose that* Y *is a* **hyperbolic curve** *over* S (*thus,* $n = 1$) *and that* $\Sigma = \{l\}$ *for some* $l \in \mathfrak{Primes}$. *Let* T *be a regular and connected scheme over* S *and* $y \in Y(T)$ *a* T*-valued point of* Y. *Write* $x \in X(T)$ *for the* T*-valued point of* X *determined by* y. *Then the following hold:*

(i) *If* $x \in X(T)$ *is* ***l*-monodromically full**, *then* $y \in Y(T)$ *is* ***l*-monodromically full**.

(ii) $x \in X(T)$ *is* **quasi-*l*-monodromically full** *if and only if* $y \in Y(T)$ *is* **quasi-*l*-monodromically full**.

Proof. It follows from Proposition 3, (i), that, to verify Proposition 8, by replacing X by $X \times_S T$, we may assume without loss of generality that $S = T$. Then Proposition 8 follows immediately from Propositions 4, (iv); 7, (ii). □

Lemma 7 (**Kernels of outer representations associated to certain finite étale coverings**)**.** *Suppose that the following five conditions are satisfied:*

(i) Y *is a* **hyperbolic curve** *over* S. (*Thus,* $n = 1$.)

(ii) $\Sigma = \{l\}$ *for some* $l \in \mathfrak{Primes}$.

(iii) *The subgroup* $\Delta^{\Sigma}_{Y/S} \subseteq \Delta^{\Sigma}_{X/S}$ *is* **normal**, *i.e., there exists a geometric point* $\overline{s} \to S$ *such that the connected finite étale covering* $Y \times_S \overline{s} \to X \times_S \overline{s}$ *is* **Galois**.

(iv) *The action of the Galois group* $\mathrm{Gal}(Y \times_S \overline{s}/X \times_S \overline{s})$ (*cf. condition* (iii)) *on the set of the cusps of* $Y \times_S \overline{s}$ *is* **faithful**. (*In particular, if the covering* $Y \to X$ *is* **not** *an isomorphism, then* Y*, hence also* X*, is* **not proper** *over* S.)

(v) *Every cusp of* Y/S *is* **defined over** *the connected* (*possibly infinite*) *étale covering of* S *corresponding to the kernel* $\mathrm{Ker}(\rho^{\Sigma}_{X/S}) \subseteq \pi_1(S)$ *of* $\rho^{\Sigma}_{X/S}$.

Then $\mathrm{Ker}(\rho^{\Sigma}_{X/S}) = \mathrm{Ker}(\rho^{\Sigma}_{Y/S})$.

Proof. It follows from Proposition 7, (i), that $\mathrm{Ker}(\rho^{\Sigma}_{Y/S}) \subseteq \mathrm{Ker}(\rho^{\Sigma}_{X/S})$. Thus, to verify Lemma 7, it suffices to verify that $\mathrm{Ker}(\rho^{\Sigma}_{X/S}) \subseteq \mathrm{Ker}(\rho^{\Sigma}_{Y/S})$. Now it follows from condition (iii), together with Lemma 6, (i), that if we write

$$\phi\colon \Delta^{\Sigma}_{X/S}/\Delta^{\Sigma}_{Y/S} \longrightarrow \mathrm{Out}(\Delta^{\Sigma}_{Y/S})$$

for the outer representation associated to the natural exact sequence of profinite groups

$$1 \longrightarrow \Delta^{\Sigma}_{Y/S} \longrightarrow \Delta^{\Sigma}_{X/S} \longrightarrow \Delta^{\Sigma}_{X/S}/\Delta^{\Sigma}_{Y/S} \longrightarrow 1$$

(cf. Definition 1, (i)), then the image of $\mathrm{Ker}(\rho^{\Sigma}_{X/S}) \subseteq \pi_1(S)$ via $\rho^{\Sigma}_{Y/S}$ *contained in* $\mathrm{Im}(\phi) \subseteq \mathrm{Out}(\Delta^{\Sigma}_{Y/S})$, i.e.,

$$\rho^{\Sigma}_{Y/S}(\mathrm{Ker}(\rho^{\Sigma}_{X/S})) \subseteq \mathrm{Im}(\phi) \subseteq \mathrm{Out}(\Delta^{\Sigma}_{Y/S})\,.$$

Now it follows from condition (iv) that the action of $\mathrm{Im}(\phi)$ on the set of the $\Delta^{\Sigma}_{Y/S}$-conjugacy classes of cuspidal inertia subgroups of $\Delta^{\Sigma}_{Y/S}$ is *faithful* (cf. Remark 2). On the other hand, it follows from condition (v) that the action of $\rho^{\Sigma}_{Y/S}(\mathrm{Ker}(\rho^{\Sigma}_{X/S}))$ on the set of the $\Delta^{\Sigma}_{Y/S}$-conjugacy classes of cuspidal inertia subgroups of $\Delta^{\Sigma}_{Y/S}$ is *trivial*. Therefore, it follows that $\rho^{\Sigma}_{Y/S}(\mathrm{Ker}(\rho^{\Sigma}_{X/S})) = \{1\}$, i.e., $\mathrm{Ker}(\rho^{\Sigma}_{X/S}) \subseteq \mathrm{Ker}(\rho^{\Sigma}_{Y/S})$. This completes the proof of Lemma 7. □

Proposition 9 (**Kernels of outer representations associated to certain coverings of tripods**)**.** *Let l be a prime number which is* **invertible on** *S. Write $P \stackrel{\mathrm{def}}{=} \mathrm{Spec}\,\mathbb{Z}[t^{\pm 1}, 1/(t-1)]$ — where t is an indeterminate — and $P_S \stackrel{\mathrm{def}}{=} P \times_{\mathrm{Spec}\,\mathbb{Z}} S$. Let $U \to P_S$ be a finite étale $\Pi^{\{l\}}_{P_S/S}$-covering* (*cf. Definition 1,* (v)) *over S such that U is a* **hyperbolic curve** *over S. Suppose that the following three conditions are satisfied:*

(i) *The subgroup $\Delta^{\{l\}}_{U/S} \subseteq \Delta^{\{l\}}_{P_S/S}$ is* **normal**, *i.e., there exists a geometric point $\overline{s} \to S$ such that the connected finite étale covering $U \times_S \overline{s} \to P_S \times_S \overline{s}$ is* **Galois**.

(ii) *The action of the Galois group* $\mathrm{Gal}(U \times_S \overline{s}/P_S \times_S \overline{s})$ (*cf. condition* (i)) *on the set of the cusps of $U \times_S \overline{s}$ is* **faithful**.

(iii) *Every cusp of U/S is* **defined over** *the connected* (*possibly infinite*) *étale covering of S corresponding to the kernel* $\mathrm{Ker}(\rho^{\{l\}}_{P_S/S}) \subseteq \pi_1(S)$ *of $\rho^{\{l\}}_{P_S/S}$.*

Let V be a hyperbolic partial compactification of U over S — i.e., a hyperbolic curve over S which contains U as an open subscheme over S. Then $\mathrm{Ker}(\rho^{\{l\}}_{V/S}) = \mathrm{Ker}(\rho^{\{l\}}_{P_S/S})$. *In particular,* $\mathrm{Ker}(\rho^{\{l\}}_{U/S}) = \mathrm{Ker}(\rho^{\{l\}}_{P_S/S})$.

Proof. It follows immediately from [8], Theorem C, together with Lemma 6, (ii), that we obtain natural inclusions

$$\mathrm{Ker}(\rho^{\{l\}}_{U/S}) \subseteq \mathrm{Ker}(\rho^{\{l\}}_{V/S}) \subseteq \mathrm{Ker}(\rho^{\{l\}}_{P_S/S})\,.$$

Thus, to verify Proposition 9, it suffices to verify that $\mathrm{Ker}(\rho^{\{l\}}_{P_S/S}) \subseteq \mathrm{Ker}(\rho^{\{l\}}_{U/S})$. On the other hand, this follows immediately from Lemma 7. □

Remark 9. A similar result to Proposition 9 can be found in [1], Corollary 3.8.1.

Proposition 10 (Kernels of outer representations associated to certain hyperbolic curves arising from elliptic curves). *Let l be a prime number which is* **invertible on** *S, N a positive integer, and E an elliptic curve over S. Write $o \in E(S)$ for the identity section of E, $E[l^N] \subseteq E$ for the kernel of the endomorphism of E given by multiplication by l^N, and*

$$Z \stackrel{\mathrm{def}}{=} E \setminus \mathrm{Im}(o) \ ; \ \ U \stackrel{\mathrm{def}}{=} E \setminus E[l^N] \, .$$

[Thus, Z (respectively, U) is a hyperbolic curve of type $(1,1)$ (respectively, $(1,l^{2N})$) over S.] Let V be an open subscheme of Z such that $U \subseteq V \subseteq Z$. Then $\mathrm{Ker}(\rho^{\{l\}}_{V/S}) = \mathrm{Ker}(\rho^{\{l\}}_{Z/S})$. In particular, $\mathrm{Ker}(\rho^{\{l\}}_{U/S}) = \mathrm{Ker}(\rho^{\{l\}}_{Z/S})$.

Proof. Observe that the endomorphism of E given by multiplication by l^N determines an *open injection* $\Pi^{\{l\}}_{U/S} \hookrightarrow \Pi^{\{l\}}_{Z/S}$ over $\pi_1(S)$. First, I *claim* that the natural action of the kernel $\mathrm{Ker}(\rho^{\{l\}}_{Z/S}) \subseteq \pi_1(S)$ of $\rho^{\{l\}}_{Z/S}$ on the finite group $E[l^N] \times_S \overline{s}$ — where $\overline{s} \to S$ is a geometric point of S — is *trivial*. Indeed, this follows immediately from the existence of a natural $\pi_1(S)$-*equivariant* isomorphism of $E[l^N] \times_S \overline{s}$ with $\Delta^{\{l\}}_{Z/S}/\Delta^{\{l\}}_{U/S}$. This completes the proof of the above *claim*. Now it follows from Lemma 6, (ii), that the natural open immersions $U \hookrightarrow V \hookrightarrow Z$ induce inclusions

$$\mathrm{Ker}(\rho^{\{l\}}_{U/S}) \subseteq \mathrm{Ker}(\rho^{\{l\}}_{V/S}) \subseteq \mathrm{Ker}(\rho^{\{l\}}_{Z/S}) \, .$$

Thus, to verify Proposition 10, it suffices to verify that $\mathrm{Ker}(\rho^{\{l\}}_{Z/S}) \subseteq \mathrm{Ker}(\rho^{\{l\}}_{U/S})$. On the other hand, by applying Lemma 7 to the *open injection* $\Pi^{\{l\}}_{U/S} \hookrightarrow \Pi^{\{l\}}_{Z/S}$ over $\pi_1(S)$, it follows immediately from the above *claim* that $\mathrm{Ker}(\rho^{\{l\}}_{Z/S}) = \mathrm{Ker}(\rho^{\{l\}}_{U/S})$. This completes the proof of Proposition 10. □

4. Some complements to Matsumoto's result concerning the representations arising from hyperbolic curves

In the present §, we give some complements to Matsumoto's result obtained in [13] concerning the difference between the kernels of the natural homomorphisms associated to a hyperbolic curve and its point from the Galois group to the *automorphism* and *outer automorphism* groups of the geometric fundamental group of the hyperbolic curve. Let

l be a prime number, k a field of characteristic $\neq l$, and X a *hyperbolic curve* over k. Write

$$X^{\mathrm{cl}}$$

for the set of closed points of X.

Definition 4. Let $x \in X^{\mathrm{cl}}$ be a closed point of X. Then we shall say that $E(X, x, l)$ *holds* if the kernel of the composite

$$\pi_1(\operatorname{Spec} k(x)) \stackrel{\pi_1(x)}{\longrightarrow} \pi_1(X) \stackrel{\widetilde{\rho}^{\{l\}}_{X/k}}{\longrightarrow} \operatorname{Aut}(\Delta^{\{l\}}_{X/k})$$

— where $k(x)$ is the residue field at x — coincides with the kernel of the composite

$$\pi_1(\operatorname{Spec} k(x)) \longrightarrow \pi_1(\operatorname{Spec} k) \stackrel{\rho^{\{l\}}_{X/k}}{\longrightarrow} \operatorname{Out}(\Delta^{\{l\}}_{X/k}),$$

i.e., the intersection of the closed subgroup

$$\operatorname{Inn}(\Delta^{\{l\}}_{X/k}) \subseteq \operatorname{Aut}(\Delta^{\{l\}}_{X/k})$$

and the image of the composite

$$\pi_1(\operatorname{Spec} k(x)) \stackrel{\pi_1(x)}{\longrightarrow} \pi_1(X) \stackrel{\widetilde{\rho}^{\{l\}}_{X/k}}{\longrightarrow} \operatorname{Aut}(\Delta^{\{l\}}_{X/k})$$

is trivial (cf. [13], §1, as well as §3). Note that since the closed subgroup $\operatorname{Inn}(\Delta^{\{l\}}_{X/k}) \subseteq \operatorname{Aut}(\Delta^{\{l\}}_{X/k})$ is *normal*, one may easily verify that whether or not the intersection in question is trivial does *not depend on* the choice of the homomorphism "$\pi_1(\operatorname{Spec} k(x)) \stackrel{\pi_1(x)}{\to} \pi_1(X)$" induced by $x \in X$ among the various $\pi_1(X)$-conjugates. Moreover, we shall write

$$X^{E_l} \subseteq X^{\mathrm{cl}}$$

for the set of closed points x of X such that $E(X, x, l)$ holds.

Proposition 11 (**Properties of exceptional points**)**.** *Let $x \in X(k)$ be a k-rational point of X. Then the following hold:*

(i) *Write $U \stackrel{\mathrm{def}}{=} X \setminus \operatorname{Im}(x)$. Then the following four conditions are equivalent:*
  (1) *$E(X, x, l)$ holds.*
  (2) *The section of the natural surjection $\pi_1(X) \twoheadrightarrow \pi_1(\operatorname{Spec} k)$ induced by x determines a section of the natural surjection $\Phi^{\{l\}}_{X/k} \twoheadrightarrow \Gamma^{\{l\}}_{X/k}$ (cf. Proposition 1,* (i)*).*
  (3) $\operatorname{Ker}(\rho^{\{l\}}_{X/k}) = \operatorname{Ker}(\rho^{\{l\}}_{U/k})$.
  (4) *The cokernel of the natural inclusion* $\operatorname{Ker}(\rho^{\{l\}}_{U/k}) \subseteq \operatorname{Ker}(\rho^{\{l\}}_{X/k})$ (*cf. Lemma 6,* (ii)) *is* **finite**.

(ii) *The following implications hold:*

$$x \text{ is a } l\text{-monodromically full point.}$$

$$\Longrightarrow x \text{ is a quasi-}l\text{-monodromically full point.}$$

$$\Longrightarrow E(X, x, l) \text{ does not hold.}$$

Proof. First, we verify assertion (i). The equivalence (1) $\Leftrightarrow$ (2) follows immediately from the various definitions involved. The equivalence (1) $\Leftrightarrow$ (3) follows from Proposition 4, (ii). The implication (3) $\Rightarrow$ (4) is immediate. Finally, we verify the implication (4) $\Rightarrow$ (3). Now it follows immediately from Proposition 4, (ii), together with the various definitions involved, that the natural surjection $\Gamma^{\{l\}}_{U/k} \twoheadrightarrow \Gamma^{\{l\}}_{X/k}$ *factors through* the natural injection $\Gamma^{\{l\}}_{U/k} \hookrightarrow \Phi^{\{l\}}_{X/k}$. In particular, the cokernel of the natural inclusion $\mathrm{Ker}(\rho^{\{l\}}_{U/k}) \subseteq \mathrm{Ker}(\rho^{\{l\}}_{X/k})$ may be naturally regarded as a closed subgroup of $\Delta^{\{l\}}_{X/k}$. Therefore, since $\Delta^{\{l\}}_{X/k}$ is *torsion-free*, if the cokernel of $\mathrm{Ker}(\rho^{\{l\}}_{U/k}) \subseteq \mathrm{Ker}(\rho^{\{l\}}_{X/k})$ is finite, then it is trivial. This completes the proof of the implication (4) $\Rightarrow$ (3).

Assertion (ii) follows immediately from the equivalence (1) $\Leftrightarrow$ (3) in assertion (i), together with the various definitions involved. $\square$

Remark 10. In the notation of Proposition 11, (ii), in general, the implication

$$E(X, x, l) \text{ does not hold.}$$
$$\Longrightarrow x \text{ is a quasi-}l\text{-monodromically full point.}$$

does *not hold*. Indeed, if we write $X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ for the complement of $\{0, 1, \infty\}$ in the projective line $\mathbb{P}_{\mathbb{Q}}$ over $\mathbb{Q}$ and $x \in X(\mathbb{Q})$ for the $\mathbb{Q}$-rational point of X corresponding to $2 \in \mathbb{Q} \setminus \{0, 1\}$ via the natural identification $\mathbb{Q} \setminus \{0, 1\} \simeq X(\mathbb{Q})$, then the $\mathbb{Q}$-rational point $x \in X(\mathbb{Q})$ is *not quasi-l-monodromically full*, i.e., the hyperbolic curve $U \stackrel{\mathrm{def}}{=} X \setminus \mathrm{Im}(x) = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, 2, \infty\}$ is *not quasi-l-monodromically full* (cf. Remark 4, together with the equivalence (ii) $\Leftrightarrow$ (iv) in [9], Corollary 7.12). On the other hand, since $U = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, 2, \infty\}$ does *not have good reduction* at the nonarchimedean prime $2\mathbb{Z} \subseteq \mathbb{Z}$, if $l \neq 2$, then it follows from [21], Theorem 0.8, that the algebraic extension of $\mathbb{Q}$ corresponding to $\mathrm{Ker}(\rho^{\{l\}}_{U/\mathbb{Q}})$ is *not unramified over* the prime $2\mathbb{Z}$; in particular, the natural surjection $\Gamma^{\{l\}}_{U/\mathbb{Q}} \twoheadrightarrow \Gamma^{\{l\}}_{X/\mathbb{Q}}$ is *not an isomorphism*. Thus, it follows from the equivalence (1) $\Leftrightarrow$ (3) in Proposition 11, (i), that $E(X, x, l)$ does *not hold*.

Proposition 12 (**Exceptional points and finite étale coverings**). *Let $Y \to X$ be a finite étale $\Pi^{\{l\}}_{X/k}$-covering (cf. Definition 1,* (v)*) over k and $y \in Y(k)$ a k-rational point of Y.* (*Thus, Y is a* **hyperbolic curve over $\boldsymbol{k}$**.) *Write $x \in X(k)$ for the k-rational point of X determined by y. Then $E(X, x, l)$ holds if and only if $E(Y, y, l)$ holds. If, moreover, $E(X, x, l)$, hence also $E(Y, y, l)$, holds, then* $\mathrm{Ker}(\rho^{\{l\}}_{X/k}) = \mathrm{Ker}(\rho^{\{l\}}_{Y/k})$.

Proof. If we write

$$U_X \stackrel{\mathrm{def}}{=} X \setminus \mathrm{Im}(x) \ ; \quad U_Y \stackrel{\mathrm{def}}{=} Y \setminus \mathrm{Im}(y)\,,$$

then the diagram of schemes

$$\begin{array}{ccc} U_Y & \longrightarrow & Y \\ & & \downarrow \\ U_X & \longrightarrow & X \end{array}$$

— where the horizontal arrows are *open immersion* — induces a commutative diagram of profinite groups

$$\begin{array}{ccc} \Gamma^{\{l\}}_{U_Y/k} & \longrightarrow & \Gamma^{\{l\}}_{Y/k} \\ \wr\downarrow & & \downarrow \\ \Gamma^{\{l\}}_{U_X/k} & \longrightarrow & \Gamma^{\{l\}}_{X/k} \end{array}$$

— where the left-hand vertical arrow is the *isomorphism* obtained by Proposition 7, (iv). Now observe that

- the horizontal arrows are *surjective* (cf. Lemma 6, (ii)), and
- the right-hand vertical arrow is *surjective* and has *finite kernel* (Proposition 7, (i)).

If $E(X, x, l)$ holds, then it follows from the equivalence (1) $\Leftrightarrow$ (3) in Proposition 11, (i), that the lower horizontal arrow is an *isomorphism*. Thus, it follows that the top horizontal arrow and the right-hand vertical arrow are *isomorphisms*. In particular, again by the equivalence (1) $\Leftrightarrow$ (3) in Proposition 11, (i), $E(Y, y, l)$ holds, and $\mathrm{Ker}(\rho^{\{l\}}_{X/k}) = \mathrm{Ker}(\rho^{\{l\}}_{Y/k})$.

On the other hand, if $E(X, x, l)$ does *not* holds, then it follows from the equivalence (1) $\Leftrightarrow$ (4) in Proposition 11, (i), that the kernel of the lower horizontal arrow is *infinite*. Thus, it follows from the fact that the kernel of the right-hand vertical arrow in the above diagram is *finite* that the kernel of the top horizontal arrow is *infinite*. In particular, again by the equivalence (1) $\Leftrightarrow$ (4) in Proposition 11, (i), $E(Y, y, l)$ does *not* hold. This completes the proof of Proposition 12. $\square$

Theorem 1 (**Existence of many nonexceptional points**). *Let l be a prime number, k a* **number field** (*i.e., a finite extension of the field of rational numbers*)*, and X a hyperbolic curve over k* (*cf. the discussion entitled "Curves" in §0*)*. Then if we regard the set X^{cl} of closed points of X as a subset of $X(\mathbb{C})$, then the complement*

$$X^{\mathrm{cl}} \setminus X^{E_l} \subseteq X(\mathbb{C})$$

(*cf. Definition 4*) *is* **dense with respect to the complex topology** *of $X(\mathbb{C})$. Moreover, the intersection*

$$X(k) \cap X^{E_l} \subseteq X(k)$$

is **finite**.

Proof. The fact that the complement $X^{\mathrm{cl}} \setminus X^{E_l}$ is *dense with respect to the complex topology* of $X(\mathbb{C})$ follows immediately from Propositions 2; 11, (ii). The fact that $X(k) \cap X^{E_l}$ is *finite* follows immediately from the equivalence (1) $\Leftrightarrow$ (3) in Proposition 11, (i); [9], Theorem C; together with the Lemma 8 below. □

Lemma 8 (**Finiteness of the set consisting of equivalent points**). *Let n be a positive integer and $x \in X_n(k)$ a k-rational point of X_n. Then the set of k-rational points of X_n which are* **equivalent to** x (*cf. Definition 2,* (iii)) *is* **finite***.*

Proof. Write X^{cpt} for the smooth compactification of X over k and $x_1, \cdots, x_n \in X(k)$ the n distinct k-rational points of X determined by $x \in X_n(k)$. Now it follows from a similar argument to the argument used in the proof of [9], Lemma A.6, that by replacing k by a finite separable extension of k, we may assume without loss of generality that every cusp of X is *defined over* k, i.e., $X^{\mathrm{cpt}}(k^{\mathrm{sep}}) \setminus X(k^{\mathrm{sep}}) \subseteq X^{\mathrm{cpt}}(k)$ — where k^{sep} is a separable closure of k. Write g for the genus of X and

$$n_g \stackrel{\mathrm{def}}{=} \begin{cases} 3 & (\text{if } g = 0) \\ 1 & (\text{if } g = 1) \\ 0 & (\text{if } g \geq 2) \end{cases}$$

and fix n_g *distinct* elements $a_1, \cdots, a_{n_g}$ of the set $S \stackrel{\mathrm{def}}{=} X^{\mathrm{cpt}}(k^{\mathrm{sep}}) \setminus X(k^{\mathrm{sep}})$ $(\subseteq X^{\mathrm{cpt}}(k))$ of cusps of X. Then it is immediate that, to verify the *finiteness* of the set of k-rational points of X_n which are *equivalent to* $x \in X_n(k)$, it suffices to verify the *finiteness* of the set

$$\{ f \in \mathrm{Aut}_k(X^{\mathrm{cpt}}) \mid \{a_i\}_{i=1}^{n_g} \subseteq f(S) \cup \{f(x_1), \cdots, f(x_n)\} \} .$$

On the other hand, this *finiteness* follows from the fact that for any n_g *distinct* elements $b_1, \cdots, b_{n_g}$ of $S \cup \{x_1, \cdots, x_n\}$, the set

$$\{ f \in \mathrm{Aut}_k(X^{\mathrm{cpt}}) \mid f(b_i) = a_i \text{ for any } 1 \leq i \leq n_g \}$$

is *finite*. This completes the proof of Lemma 8. □

Remark 11. In [13], Matsumoto proved the following theorem (cf. [13], Theorem 1):

> *Let l be a prime number and $g \geq 3$ an integer. Suppose that l* **divides** $2g - 2$*; write l^{ν} for the highest power of l that divides* $2g - 2$*. Then there are* **infinitely many** *isomorphism classes of pairs (k, X) of* **number fields** *k and* **hyperbolic curves** *X of type $(g, 0)$ over k which satisfy the following condition: For any closed point $x \in X$ with residue field $k(x)$, if l^{ν} does* **not divide** $[k(x) : k]$*, then $E(X, x, l)$ does* **not hold***.*

Theorem 1 may be regarded as a *partial generalization* of this theorem.

Proposition 13 (**Exceptional points via tripods**)**.** *Write* $P_k \stackrel{\mathrm{def}}{=} \mathrm{Spec}\, k[t^{\pm 1}, 1/(t-1)]$ *— where* t *is an indeterminate. Then the following hold:*

(i) *Let* $U \to P_k$ *be a finite étale* $\Pi^{\{l\}}_{P_k/k}$*-covering* (*cf. Definition 1,* (v)) *over* k*. Suppose that the following three conditions are satisfied:*

(1) *If* k^{sep} *is a separable closure of* k*, then the covering* $U \otimes_k k^{\mathrm{sep}} \to P_k \otimes_k k^{\mathrm{sep}}$ *is* **Galois**.

(2) *The action of the Galois group* $\mathrm{Gal}(U \otimes_k k^{\mathrm{sep}}/P_k \otimes_k k^{\mathrm{sep}})$ (*cf. condition* (1)) *on the set of cusps of* $U \otimes_k k^{\mathrm{sep}}$ *is* **faithful**.

(3) *Every cusp of* U *is* **defined over** *the* (*possibly infinite*) *Galois extension of* k *corresponding to the kernel* $\mathrm{Ker}(\rho^{\{l\}}_{P_k/k}) \subseteq \pi_1(\mathrm{Spec}\, k)$ *of* $\rho^{\{l\}}_{P_k/k}$.

Let V *be a hyperbolic partial compactification of* U *over* k *— i.e., a hyperbolic curve over* k *which contains* U *as an open subscheme over* k *— and* $x \in V^{\mathrm{cl}}$ *a closed point of* V *such that the complement* $V \setminus \{x\}$ **contains** U*. Then* $E(V, x, l)$ *holds.*

(ii) *Let* N *be a positive integer. If a closed point* $x \in P_k^{\mathrm{cl}}$ *of* P_k *is* **contained in** *the closed subscheme of* P_k *determined by the principal ideal*

$$(t^{l^N} - 1) \subseteq k[t^{\pm 1}, 1/(t-1)]\,,$$

then $E(P_k, x, l)$ *holds.*

Proof. First, we verify assertion (i). By replacing k by the residue field $k(x)$ at x, we may assume without loss of generality that $x \in X(k)$. Then it follows immediately from Proposition 9 that

$$\mathrm{Ker}(\rho^{\{l\}}_{V/k}) = \mathrm{Ker}(\rho^{\{l\}}_{(V \setminus \{x\})/k})\,.$$

Thus, assertion (i) follows immediately from the equivalence (1) $\Leftrightarrow$ (3) in Proposition 11, (i). Assertion (ii) may be verified by applying assertion (i) to the finite étale covering

$$U \stackrel{\mathrm{def}}{=} \mathrm{Spec}\, k[s^{\pm 1}, 1/(s^{l^N} - 1)] \longrightarrow P_k$$

— where s is an indeterminate — given by "$t \mapsto s^{l^N}$". This completes the proof of Proposition 13. ☐

Proposition 14 (**Exceptional points via elliptic curves**)**.** *Let* N *be a positive integer and* E *an elliptic curve over* k*. Write* $o \in E(k)$ *for the identity section of* E*,* $E[l^N] \subseteq E$ *for the kernel of the endomorphism of* E *given by multiplication by* l^N*, and*

$$Z \stackrel{\mathrm{def}}{=} E \setminus \mathrm{Im}(o) \ ; \ \ U \stackrel{\mathrm{def}}{=} E \setminus E[l^N]\,.$$

[*Thus,* Z (*respectively,* U) *is a hyperbolic curve of type* $(1,1)$ (*respectively,* $(1, l^{2N})$) *over* k*.*] *Let* V *be an open subscheme of* Z *such that* $U \subseteq V \subseteq$

Z and $x \in V^{\mathrm{cl}}$ a closed point of V such that the complement $V \setminus \{x\}$ **contains** *U. Then $E(V, x, l)$ holds. In particular, for any closed point $z \in Z^{\mathrm{cl}}$ of Z* **contained in** *$E[l^N]$, $E(Z, z, l)$ holds.*

Proof. By replacing k by the residue field $k(x)$ at x, we may assume without loss of generality that $x \in X(k)$. Then it follows immediately from Proposition 10 that

$$\mathrm{Ker}(\rho^{\{l\}}_{V/k}) = \mathrm{Ker}(\rho^{\{l\}}_{(V \setminus \{x\})/k}) \, .$$

Thus, Proposition 14 follows immediately from the equivalence (1) $\Leftrightarrow$ (3) in Proposition 11, (i). $\square$

Theorem 2 (**Existence of many exceptional points for certain hyperbolic curves).** *Let l be a prime number, k a field* **of characteristic 0**, *X a hyperbolic curve over k (cf. the discussion entitled "Curves" in §0) which is either of* **type (0, 3)** *or* **type (1, 1)**, *and $Y \to X$ a finite étale $\Pi^{\{l\}}_{X/k}$-covering (cf. Definition 1,* (v)*) which is* **geometrically connected** *over k. (Thus, Y is a* **hyperbolic curve over *k***.*) Then the subset $Y^{E_l} \subseteq Y^{\mathrm{cl}}$ (cf. Definition 4) is* **infinite**. *In particular, the subset $X^{E_l} \subseteq X^{\mathrm{cl}}$ is* **infinite**.

Proof. It follows immediately from the definition of the set "$(-)^{E_l}$" — to verify Theorem 2 — by replacing k by the finite Galois extension of k corresponding to the kernel of the natural action of the absolute Galois group of k on the set of cusps of X, we may assume without loss of generality that every cusp of X is *defined over* k. Then it follows immediately from Propositions 13, (ii); 14, that the set X^{E_l} is *infinite*. Therefore, it follows immediately from Proposition 12 that the set Y^{E_l} is *infinite*. This completes the proof of Theorem 2. $\square$

Remark 12. By a similar argument to the argument used in the proof of Theorem 2, one may also prove the following *assertion*:

> *Let l be a prime number, k a field* **of characteristic 0**, *and $r \geq 3$ (respectively, $r \geq 1$) an integer. Then there exist a finite extension k' of k and a hyperbolic curve X over k' of* **type** *$(0, r)$ (respectively, $(1, r)$) such that the subset $X^{E_l} \subseteq X^{\mathrm{cl}}$ is* **infinite**.

Indeed, write $C_0 \stackrel{\mathrm{def}}{=} \mathrm{Spec}\, k[t^{\pm 1}, 1/(t-1)]$ — where t is an indeterminate — and C_1 for the complement in an elliptic curve E over k of the origin of E. Let N be a positive integer such that $r \leq l^N$. Moreover, write $F^N_0 \subseteq C_0$ for the closed subscheme of C_0 defined by the principal ideal

$$(t^{l^N} - 1) \subseteq k[t^{\pm 1}, 1/(t-1)]$$

and $F^N_1 \subseteq C_1$ for the closed subscheme of C_1 obtained as the kernel of the endomorphism of E given by multiplication by l^N. Now, by replacing k by a finite extension of k, we may assume without loss generality that every geometric points of F^N_0, F^N_1, respectively, can be *defined over* k. Then it is immediate that for $g = 0$ or 1, there exists an open subscheme $X_g \subseteq C_g$ of C_g such that X_g is *of type* (g, r), and, moreover, X_g contains the complement of F_g^N in C_g. Now it follows immediately from Propositions 9, 10, that $\mathrm{Ker}(\rho^{\{l\}}_{C_g/k}) = \mathrm{Ker}(\rho^{\{l\}}_{X_g/k})$. Therefore, by Propositions 13, (ii); 14, together with the equivalence (1) ⇔ (3) in Proposition 11, (i), one may easily verify that the set $X_g^{E_l}$ is *infinite*. This completes the proof of the above *assertion*.

Remark 13. An example of a triple "(X, x, l)" such that X is a *proper* hyperbolic curve over a number field k, and, moreover, $E(X, x, l)$ *holds* is as follows: Suppose that $l > 3$. Let k be a number field and

$$U \stackrel{\mathrm{def}}{=} \mathrm{Spec}\, k[t_1^{\pm 1}, t_2^{\pm 1}]/(t_1^l + t_2^l + 1)$$

— where t_1 and t_2 are indeterminates. Then one may easily verify that the connected finite étale covering

$$U \longrightarrow \mathrm{Spec}\, k[t^{\pm 1}, 1/(t-1)]$$

— where t is an indeterminate — given by "$t \mapsto t_1^l$" satisfies the three conditions appearing in the statement of Proposition 13, (i). In particular, if we write

$$X \stackrel{\mathrm{def}}{=} \mathrm{Proj}\, k[t_1, t_2, t_3]/(t_1^l + t_2^l + t_3^l)$$

— where t_1, t_2, and t_3 are indeterminates — and $x \stackrel{\mathrm{def}}{=}$ "$[1, -1, 0]$" $\in X(k)$, then it follows immediately from Proposition 13, (i), that $E(X, x, l)$ *holds*.

Remark 14. In [13], §2, Matsumoto proved that for any prime number l, the triple

$$(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}, \overrightarrow{01}, l)$$

— where $\overrightarrow{01}$ is a $\mathbb{Q}$-rational tangential base point — is a triple for which "$E(X, x, l)$" holds. As mentioned in [13], §2, the fact that "$E(X, x, l)$" holds for this triple was observed by P. Deligne and Y. Ihara. However, by definition, in fact, a *tangential base point is not a point*. In this sense, *no example of a triple* "(X, x, l)" for which $E(X, x, l)$ holds appears in [13].

5. Galois-theoretic characterization of equivalence classes of monodromically full points

In the present §, we prove that the equivalence class (cf. Definition 2, (iii)) of a monodromically full point of a configuration space of a hyperbolic curve is completely determined by the kernel of the representation associated to the point (cf. Theorems 3, 4 below). Moreover, we also give a necessary and sufficient condition for a quasi-monodromically full Galois section (cf. Definition 5 below) of a hyperbolic curve to be geometric (cf. Theorem 5 below). We maintain the notation of the preceding §. Suppose, moreover, that k is *of characteristic* 0. Let n be a positive integer, $\overline{k}$ an algebraic closure of k, and $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$ the absolute Galois group of k determined by the given algebraic closure $\overline{k}$ of k.

Lemma 9 (**Equivalence and automorphisms**)**.** *Let $x,\ y \in X_n(k)$ be k-rational points of X_n. If there exists an automorphism α of X_n over k such that $y = \alpha \circ x$, then x is* **equivalent to** y (*cf. Definition 2,* (iii))*.*

Proof. If the hyperbolic curve X is of type $(0,3)$ (respectively, *neither* of type $(0,3)$ nor of type $(1,1)$), then Lemma 9 follows immediately from [9], Lemma 4.1, (i), (ii) (respectively, [20], Theorem A, Corollary B; [15], Theorem A). Thus, to verify Lemma 9, it suffices to verify Lemma 9 in the case where the hyperbolic curve X is *of type* $(1,1)$. Suppose that X is *of type* $(1,1)$. Write E for the smooth compactification of X over k and $o \in E(k)$ for the k-rational point whose image is the marked divisor of the hyperbolic curve X. Now since X is *of genus* 1, by regarding the k-rational point $o \in E(k)$ as the *origin*, one may regard E as an *abelian group scheme* over k whose identity section is the k-rational point o. Then it follows immediately from [20], Theorem A, Corollary B; [15], Theorem A, that the group $\mathrm{Aut}_k(X_n)$ of automorphisms over k of the n-th configuration space X_n of X/k is generated by the images of the natural inclusions

$$\mathrm{Aut}_k(X)\ ,\ \ \mathfrak{S}_n \hookrightarrow \mathrm{Aut}_k(X_n)$$

— where $\mathfrak{S}_n$ is the symmetric group on n letters — together with the automorphism of X_n induced by

$$\begin{array}{ccc} \overbrace{E \times_k \cdots \times_k E}^{n} & \longrightarrow & \overbrace{E \times_k \cdots \times_k E}^{n} \\ (x_1, \cdots, x_n) & \mapsto & (x_1, x_1 - x_2, x_1 - x_3, \cdots, x_1 - x_n)\,. \end{array}$$

Therefore, to verify Lemma 9 in the case where X is *of type* $(1,1)$, it suffices to verify that for any n distinct k-rational points $x_1, \cdots, x_n \in X(k)$ of X, the hyperbolic curve of type $(1, n+1)$

$$E \setminus \{o, x_1, \cdots x_n\}$$

is *isomorphic to* the hyperbolic curve of type $(1, n+1)$

$$E \setminus \{o, x_1, x_1 - x_2, x_1 - x_3, \cdots, x_1 - x_n\}$$

over k. On the other hand, one may easily verify that the composite of the automorphism of E given by multiplication by -1 and the automorphism of E given by "$a \mapsto a + x_1$" determines the desired isomorphism. This completes the proof of Lemma 9. □

Definition 5. Let $\Sigma' \subseteq \Sigma \subseteq \mathfrak{Primes}$ be nonempty subsets of $\mathfrak{Primes}$ and s a pro-Σ Galois section of X_n/k, i.e., a continuous section of the natural surjection $\Pi^{\Sigma}_{X_n/k} \twoheadrightarrow G_k$ (cf. [10], Definition 1.1, (i)). Then we shall say that s is Σ'*-monodromically full* (respectively, *quasi-*Σ'*-monodromically full*) if for any $l' \in \Sigma'$, the composite

$$G_k \xrightarrow{s} \Pi^{\Sigma}_{X_n/k} \twoheadrightarrow \Pi^{\{l'\}}_{X_n/k} \overset{\rho^{\{l'\}}_{X_{n+1}/X_n}}{\twoheadrightarrow} \Gamma^{\{l'\}}_{X_{n+1}/X_n}$$

(cf. Lemma 3, (viii)) is surjective (respectively, has open image).

If l' is a prime number, then for simplicity, we write "l'-monodromically full" (respectively, "quasi-l'-monodromically full") instead of "$\{l'\}$-monodromically full" (respectively, "quasi-$\{l'\}$-monodromically full").

Remark 15. Let $\Sigma \subseteq \mathfrak{Primes}$ be a nonempty subset of $\mathfrak{Primes}$ and $x \in X_n(k)$ a k-rational point of X_n. Then it follows from Remark 5 that the following two conditions are equivalent:

(i) The k-rational point $x \in X_n(k)$ is Σ*-monodromically full* (respectively, *quasi-*Σ*-monodromically full*).

(ii) A pro-$\mathfrak{Primes}$ Galois section of X_n arising from $x \in X_n(k)$ (cf. [10], Definition 1.1, (ii)) is Σ*-monodromically full* (respectively, *quasi-*Σ*-monodromically full*).

Lemma 10 (**Certain two monodromically full Galois sections).** *Let s, t be pro-l Galois sections of X_n/k. For $\mathfrak{a} \in \{s, t\}$, write $\phi_{\mathfrak{a}}$ for the composite*

$$G_k \xrightarrow{\mathfrak{a}} \Pi^{\{l\}}_{X_n/k} \xrightarrow{\widetilde{\rho}^{\{l\}}_{X_{n+1}/X_n}} \mathrm{Aut}(\Delta^{\{l\}}_{X_n/k})\,.$$

Then the following hold:

(i) *If the pro-l Galois sections s and t are* ***l*-monodromically full**, *and* $\mathrm{Ker}(\phi_s) = \mathrm{Ker}(\phi_t)$, *then there exists an automorphism α of $\Pi^{\{l\}}_{X_n/k}$ over G_k such that $\alpha \circ s = t$.*

(ii) *If the pro-l Galois sections s and t are* **quasi-*l*-monodromically full**, *and the intersection* $\mathrm{Ker}(\phi_s) \cap \mathrm{Ker}(\phi_t)$ *is* **open in** $\mathrm{Ker}(\phi_s)$ *and* $\mathrm{Ker}(\phi_t)$, *then — after replacing G_k by a suitable open subgroup of G_k — there exist*
- *for $\mathfrak{a} \in \{s, t\}$, a finite étale $\Phi^{\{l\}}_{X_n/k}$-covering $C_{\mathfrak{a}} \to X_n$ over k (cf. Definition 1,* (v)) *which is* **geometrically connected** *over k, and*
- *an isomorphism $\alpha \colon \Pi^{\{l\}}_{C_s/k} \xrightarrow{\sim} \Pi^{\{l\}}_{C_t/k}$ over G_k*

such that
- *for $\mathfrak{a} \in \{s, t\}$, the pro-$l$ Galois section $\mathfrak{a} \colon G_k \to \Pi^{\{l\}}_{X_n/k}$* **factors through** $\Pi^{\{l\}}_{C_{\mathfrak{a}}/k} \subseteq \Pi^{\{l\}}_{X_n/k}$, *and*
- *the composite*

$$G_k \xrightarrow{s} \Pi^{\{l\}}_{C_s/k} \xrightarrow[\sim]{\alpha} \Pi^{\{l\}}_{C_t/k}$$

coincides with t.

Proof. First, we verify assertion (i). Since the pro-l Galois sections s and t are *l-monodromically full*, it follows immediately from Proposition 4, (i), that $\mathrm{Im}(\phi_s) = \mathrm{Im}(\phi_t) = \Phi^{\{l\}}_{X_n/k}$. Write β for the automorphism of $\Phi^{\{l\}}_{X_n/k}$ obtained as the composite

$$\Phi^{\{l\}}_{X_n/k} = \mathrm{Im}(\phi_s) \overset{\sim}{\longleftarrow} G_k/N \overset{\sim}{\longrightarrow} \mathrm{Im}(\phi_t) = \Phi^{\{l\}}_{X_n/k}$$

— where $N \overset{\mathrm{def}}{=} \mathrm{Ker}(\phi_s) = \mathrm{Ker}(\phi_t) \subseteq G_k$. Then it follows immediately from the various definitions involved that this automorphism β is an *automorphism over* $\Gamma^{\{l\}}_{X_n/k}$. Thus, since the right-hand square in the commutative diagram of profinite groups appearing in the statement of Proposition 1, (i), is *cartesian*, by base-changing β via the natural surjection $G_k \twoheadrightarrow \Gamma^{\{l\}}_{X_n/k}$, we obtain an automorphism α of $\Pi^{\{l\}}_{X_n/k}$ over G_k. Now it follows immediately from the various definitions involved that this automorphism α satisfies the condition in the statement of assertion (i). This completes the proof of assertion (i).

Next, we verify assertion (ii). To verify assertion (ii), by replacing G_k by an open subgroup of G_k, we may assume without loss of generality that $\mathrm{Ker}(\phi_s) = \mathrm{Ker}(\phi_t)$. (Note that if $k' \subseteq \overline{k}$ is a finite extension of k, then it follows immediately from the various definitions involved that the pro-l Galois sections of $X_n \otimes_k k'/k'$ determined by s, t, respectively, are *quasi-l-monodromically full*.) Now since the pro-l Galois sections s and t are *quasi-l-monodromically full*, it follows immediately from Proposition 4, (i), that the images of the homomorphisms ϕ_s and ϕ_t are *open* in $\Phi^{\{l\}}_{X_n/k}$. For $\mathfrak{a} \in \{s, t\}$, write

$$C_{\mathfrak{a}} \longrightarrow X_n$$

for the connected finite étale coverings of X_n corresponding to the *open* subgroups $\mathrm{Im}(\phi_{\mathfrak{a}}) \subseteq \Phi^{\{l\}}_{X_n/k}$ of $\Phi^{\{l\}}_{X_n/k}$. Then it follows from Proposition 7, (ii), that the natural open injection $\Pi^{\{l\}}_{C_{\mathfrak{a}}} \hookrightarrow \Pi^{\{l\}}_{X_n/k}$ determines a diagram of profinite groups

$$\Pi^{\{l\}}_{C_{\mathfrak{a}}/k} \twoheadrightarrow \Phi^{\{l\}}_{C_{\mathfrak{a}}/k} = \mathrm{Im}(\phi_{\mathfrak{a}}) \subseteq \Phi^{\{l\}}_{X_n/k}\,.$$

Now it follows from Proposition 7, (i), that the natural surjection $\Gamma^{\{l\}}_{C_{\mathfrak{a}}/k} \twoheadrightarrow \Gamma^{\{l\}}_{X_n/k}$ is an *isomorphism*. Write β for the isomorphism obtained as the composite

$$\Phi^{\{l\}}_{C_s/k} = \mathrm{Im}(\phi_s) \overset{\sim}{\longleftarrow} G_k/N \overset{\sim}{\longrightarrow} \mathrm{Im}(\phi_t) = \Phi^{\{l\}}_{C_t/k}$$

— where $N = \mathrm{Ker}(\phi_s) = \mathrm{Ker}(\phi_t)$. Then it follows immediately from the various definitions involved that we obtain a commutative diagram of profinite groups

$$\begin{array}{ccc}\Phi^{\{l\}}_{C_s/k} = \mathrm{Im}(\phi_s) & \xrightarrow{\beta} & \mathrm{Im}(\phi_t) = \Phi^{\{l\}}_{C_t/k} \\ \downarrow & & \downarrow \\ \Phi^{\{l\}}_{X_n/k} & & \Phi^{\{l\}}_{X_n/k} \\ \downarrow & & \downarrow \\ \Gamma^{\{l\}}_{C_s/k} = \Gamma^{\{l\}}_{X_n/k} & = & \Gamma^{\{l\}}_{X_n/k} = \Gamma^{\{l\}}_{C_t/k}\, . \end{array}$$

Thus, since the right-hand square in the commutative diagram of profinite groups appearing in the statement of Proposition 1, (i), is *cartesian*, by base-changing β via the natural surjection $G_k \twoheadrightarrow \Gamma^{\{l\}}_{X_n/k} = \Gamma^{\{l\}}_{C_s/k} = \Gamma^{\{l\}}_{C_t/k}$, we obtain an isomorphism $\alpha\colon \Pi^{\{l\}}_{C_s/k} \xrightarrow{\sim} \Pi^{\{l\}}_{C_t/k}$ over G_k. Now it follows immediately from the various definitions involved that this isomorphism α satisfies the condition in the statement of assertion (ii). This completes the proof of assertion (ii). □

Theorem 3 (**Galois-theoretic characterization of equivalence classes of monodromically full points of configuration spaces**). *Let l be a prime number, n a positive integer, k a* **finitely generated extension of** $\mathbb{Q}$*, $\overline{k}$ an algebraic closure of k, and X a hyperbolic curve over k* (*cf. the discussion entitled "Curves" in* §*0*)*. Write $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$ for the absolute Galois group of k determined by the fixed algebraic closure $\overline{k}$ and X_n for the n-th configuration space of the hyperbolic curve X/k* (*cf. Definition 2,* (i))*. Then for two k-rational points x and y of X_n which are* ***l*-monodromically full** (*cf. Definition 3*)*, the following three conditions are equivalent:*

(i) *x is* **equivalent to** *y* (*cf. Definition 2,* (iii))*.*

(ii) $\mathrm{Ker}(\rho^{\{l\}}_{X[x]/k}) = \mathrm{Ker}(\rho^{\{l\}}_{X[y]/k})$ (*cf. Definitions 1,* (iii)*; 2,* (ii))*.*

(iii) *If we write ϕ_x* (*respectively, ϕ_y*) *for the composite*

$$G_k \xrightarrow{\pi_1(x)} \pi_1(X_n) \xrightarrow{\widetilde{\rho}^{\{l\}}_{X_n/k}} \mathrm{Aut}(\Delta^{\{l\}}_{X_n/k})$$

$$(\textit{respectively,}\quad G_k \xrightarrow{\pi_1(y)} \pi_1(X_n) \xrightarrow{\widetilde{\rho}^{\{l\}}_{X_n/k}} \mathrm{Aut}(\Delta^{\{l\}}_{X_n/k}))$$

(*cf. Definition 1,* (ii), (iii))*, then* $\mathrm{Ker}(\phi_x) = \mathrm{Ker}(\phi_y)$*.*

Proof. It is immediate that the implication (i) ⇒ (ii) holds. On the other hand, it follows Proposition 4, (ii), that the equivalence (ii) ⇔ (iii) holds. Thus, to verify Theorem 3, it suffices to show the implication (iii) ⇒ (i). Suppose that condition (iii) is satisfied. Then it follows from Lemma 10, (i), that there exists an automorphism α of $\Pi^{\{l\}}_{X_n/k}$ over G_k

such that two homomorphisms

$$G_k \xrightarrow{\pi_1(x)} \Pi^{\{l\}}_{X_n/k} \overset{\alpha}{\underset{\sim}{\longrightarrow}} \Pi^{\{l\}}_{X_n/k} \;\; ; \;\; G_k \xrightarrow{\pi_1(y)} \Pi^{\{l\}}_{X_n/k}$$

coincide. Now it follows from [20], Corollary B; [15], Theorem A, together with [9], Lemmas 4.1, (i); 4.3, (iii), that the automorphism α of $\Pi^{\{l\}}_{X_n/k}$ *arises from* an automorphism f_α of X_n over k. Thus, it follows from [15], Theorem C, that $f_\alpha \circ x = y$; in particular, it follows from Lemma 9 that condition (i) is satisfied. This completes the proof of Theorem 3. □

Remark 16. If, in Theorem 3, one drops the assumption that x and y are *l-monodromically full*, then the conclusion no longer holds in general. Such a counter-example is as follows: Suppose that $l \neq 2$. Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$ and $\zeta_l \in \overline{\mathbb{Q}}$ a primitive l-th root of unity. Write $k \overset{\text{def}}{=} \mathbb{Q}(\zeta_l)$, $X \overset{\text{def}}{=} \mathbb{P}^1_k \setminus \{0, 1, \infty\}$ for the complement of $\{0, 1, \infty\}$ in the projective line $\mathbb{P}^1_k$ over k, $x \in X(k)$ (respectively, $y \in X(k)$) for the k-rational point of X corresponding to $\zeta_l \in k \setminus \{0, 1\}$ (respectively, $\zeta_l^2 \in k \setminus \{0, 1\}$) via the natural identification $k \setminus \{0, 1\} \simeq X(k)$, and

$$U_x \overset{\text{def}}{=} U \setminus \text{Im}(x) \;\; ; \;\; U_y \overset{\text{def}}{=} U \setminus \text{Im}(y) \, .$$

Then it follows from Proposition 13, (ii), that $E(X, x, l)$ and $E(X, y, l)$ hold. Thus, it follows from the equivalence (1) ⇔ (3) in Proposition 11, (i), that

$$\text{Ker}(\rho^{\{l\}}_{X/k}) = \text{Ker}(\rho^{\{l\}}_{U_x/k}) = \text{Ker}(\rho^{\{l\}}_{U_y/k}) \, ;$$

in particular, x and y satisfy condition (ii) in the statement of Theorem 3. On the other hand, if $l \neq 3$, then U_x is *not isomorphic to* U_y over k, i.e., x and y do *not satisfy* condition (i) in the statement of Theorem 3.

Theorem 4 (**Galois-theoretic characterization of equivalence classes of quasi-monodromically full points of cores**). *Let l be a prime number, k a* **finitely generated extension of** $\mathbb{Q}$*, $\overline{k}$ an algebraic closure of k, and X a hyperbolic curve over k (cf. the discussion entitled "Curves" in §0) which is a* **k-core** *(cf. [17], Remark 2.1.1). Write $G_k \overset{\text{def}}{=} \text{Gal}(\overline{k}/k)$ for the absolute Galois group of k determined by the fixed algebraic closure $\overline{k}$. Then for two k-rational points x and y of X which are* **quasi-l-monodromically full** *(cf. Definition 3), the following four conditions are equivalent:*

(i) *$x = y$.*
(ii) *x is* **equivalent to** *y.*
(iii) *If we write*

$$U_x \overset{\text{def}}{=} X \setminus \text{Im}(x) \;\; ; \;\; U_y \overset{\text{def}}{=} X \setminus \text{Im}(y) \, ,$$

then the intersection $\mathrm{Ker}(\rho^{\{l\}}_{U_x/k}) \cap \mathrm{Ker}(\rho^{\{l\}}_{U_x/k})$ (*Definition 1,* (iii)) *is* **open in** $\mathrm{Ker}(\rho^{\{l\}}_{U_x/k})$ *and* $\mathrm{Ker}(\rho^{\{l\}}_{U_x/k})$.

(iv) *If we write* ϕ_x (*respectively,* ϕ_y) *for the composite*

$$G_k \xrightarrow{\pi_1(x)} \pi_1(X) \xrightarrow{\widetilde{\rho}^{\{l\}}_{X/k}} \mathrm{Aut}(\Delta^{\{l\}}_{X/k})$$

$$(\textit{respectively,} \quad G_k \xrightarrow{\pi_1(y)} \pi_1(X) \xrightarrow{\widetilde{\rho}^{\{l\}}_{X/k}} \mathrm{Aut}(\Delta^{\{l\}}_{X/k}))$$

(*cf. Definition 1,* (ii), (iii)), *then the intersection* $\mathrm{Ker}(\phi_x) \cap \mathrm{Ker}(\phi_y)$ *is* **open in** $\mathrm{Ker}(\phi_x)$ *and* $\mathrm{Ker}(\phi_y)$.

Proof. It is immediate that the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) hold. On the other hand, it follows Proposition 4, (ii), that the equivalence (iii) $\Leftrightarrow$ (iv) holds. Thus, to verify Theorem 4, it suffices to show the implication (iv) $\Rightarrow$ (i). Suppose that condition (iv) is satisfied. Then it follows from Lemma 10, (ii), that — after replacing G_k by a suitable open subgroup of G_k — there exist

- for $\mathfrak{a} \in \{x, y\}$, a finite étale $\Phi^{\{l\}}_{X/k}$-covering $C_{\mathfrak{a}} \to X$ over k (cf. Definition 1, (v)) which is *geometrically connected* over k, and
- an isomorphism $\alpha \colon \Pi^{\{l\}}_{C_x/k} \xrightarrow{\sim} \Pi^{\{l\}}_{C_y/k}$ over G_k

such that

- for $\mathfrak{a} \in \{x, y\}$, the pro-$l$ Galois section $\pi_1(\mathfrak{a}) \colon G_k \to \Pi^{\{l\}}_{X/k}$ determined by $\mathfrak{a} \in X(k)$ *factors through* $\Pi^{\{l\}}_{C_{\mathfrak{a}}/k} \subseteq \Pi^{\{l\}}_{X/k}$, and
- the two homomorphisms

$$G_k \xrightarrow{\pi_1(x)} \Pi^{\{l\}}_{C_x/k} \xrightarrow[\sim]{\alpha} \Pi^{\{l\}}_{C_y/k} \ ; \ G_k \xrightarrow{\pi_1(y)} \Pi^{\{l\}}_{C_y/k}$$

coincide.

(Note that if $k' \subseteq \overline{k}$ is a finite extension of k, then it follows from [17], Proposition 2.3, (i), that $X \otimes_k k'$ is a k'-*core*.) Now it follows from [15], Theorem A, that the isomorphism $\alpha \colon \Pi^{\{l\}}_{C_x/k} \xrightarrow{\sim} \Pi^{\{l\}}_{C_y/k}$ *arises from* an isomorphism $f_\alpha \colon C_x \xrightarrow{\sim} C_y$ over k; moreover, since X is a k-*core*, it follows that the isomorphism f_α is an *isomorphism over* X. Therefore, it follows immediately from [15], Theorem C, together with the various definitions involved, that condition (i) is satisfied. This completes the proof of Theorem 4. □

Theorem 5 (**A necessary and sufficient condition for a quasi–monodromically full Galois section of a hyperbolic curve to be geometric**)**.** *Let* l *be a prime number,* k *a* **finitely generated extension of** $\mathbb{Q}$*,* $\overline{k}$ *an algebraic closure of* k*,* $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$ *for the absolute Galois group of* k *determined by the fixed algebraic closure* $\overline{k}$*,* X *a hyperbolic curve over* k (*cf. the discussion entitled "Curves" in* §*0*)*, and* $s \colon G_k \to \Pi^{\{l\}}_{X/k}$ *a pro-*l *Galois section of* X (*cf.* [10], *Definition 1.1,* (i)) *which is* **quasi-l-monodromically full** *(cf. Definition 5). Write ϕ_s for the composite*

$$G_k \xrightarrow{s} \Pi^{\{l\}}_{X/k} \xrightarrow{\widetilde{\rho}^{\{l\}}_{X/k}} \mathrm{Aut}(\Delta^{\{l\}}_{X/k})$$

(cf. Definition 1, (ii), (iii)*). Then the following four conditions are equivalent:*

(i) *The pro-l Galois section s is* **geometric** *(cf.* [10]*, Definition 1.1,* (iii)*).*

(ii) *The pro-l Galois section s* **arises from** *a k-rational point of X (cf.* [10]*, Definition 1.1,* (ii)*).*

(iii) *There exists a* **quasi-l-monodromically full** *k-rational point (cf. Definition 3) $x \in X(k)$ of X such that if we write ϕ_x for the composite*

$$G_k \xrightarrow{\pi_1(x)} \Pi^{\{l\}}_{X/k} \xrightarrow{\widetilde{\rho}^{\{l\}}_{X/k}} \mathrm{Aut}(\Delta^{\{l\}}_{X/k})$$

(cf. Definition 1, (ii), (iii)*), then the intersection* $\mathrm{Ker}(\phi_s) \cap \mathrm{Ker}(\phi_x)$ *is* **open in** $\mathrm{Ker}(\phi_s)$ *and* $\mathrm{Ker}(\phi_x)$*.*

(iv) *There exists a* **quasi-l-monodromically full** *k-rational point (cf. Definition 3) $x \in X(k)$ of X such that if we write*

$$U \stackrel{\mathrm{def}}{=} X \setminus \mathrm{Im}(x)\,,$$

then the intersection $\mathrm{Ker}(\phi_s) \cap \mathrm{Ker}(\rho^{\{l\}}_{U/k})$ *(cf. Definition 1,* (iii)*) is* **open in** $\mathrm{Ker}(\phi_s)$ *and* $\mathrm{Ker}(\rho^{\{l\}}_{U/k})$*.*

Proof. It follows from Remark 15 that the implication (ii) $\Rightarrow$ (iii) holds. On the other hand, it follows Proposition 4, (ii), that the equivalence (iii) $\Leftrightarrow$ (iv) holds. Thus, to verify Theorem 5, it suffices to show the implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i).

To verify the implication (i) $\Rightarrow$ (ii), suppose that condition (i) is satisfied. Since the pro-l Galois section s is *geometric*, there exists a k-rational point $x \in X^{\mathrm{cpt}}(k)$ of the smooth compactification X^{cpt} of X such that the image of s is *contained in* a decomposition subgroup $D \subseteq \Pi^{\{l\}}_{X/k}$ of $\Pi^{\{l\}}_{X/k}$ associated to x. Suppose that x is a *cusp of* X, i.e., an element of $X^{\mathrm{cpt}}(k) \setminus X(k)$. Write $I \subseteq D$ for the inertia subgroup of D. Then since x is a *cusp* of X, it follows immediately from [16], Lemma 1.3.7, that $D = N_{\Pi^{\{l\}}_{X/k}}(I)$ and $I = N_{\Pi^{\{l\}}_{X/k}}(I) \cap \Delta_{X/k}$. Therefore, since the composite $\Delta^{\{l\}}_{X/k} \hookrightarrow \Pi^{\{l\}}_{X/k} \twoheadrightarrow \Phi^{\{l\}}_{X/k}$ is *injective* (cf. Lemma 3, (v)), if we write $\overline{D}, \overline{I} \subseteq \Phi^{\{l\}}_{X/k}$ for the images of the composites $D \hookrightarrow \Pi^{\{l\}}_{X/k} \twoheadrightarrow \Phi^{\{l\}}_{X/k}$, $I \hookrightarrow \Pi^{\{l\}}_{X/k} \twoheadrightarrow \Phi^{\{l\}}_{X/k}$, respectively, then it holds that $\overline{D} \subseteq N_{\Phi^{\{l\}}_{X/k}}(\overline{I})$ and $\overline{I} = N_{\Phi^{\{l\}}_{X/k}}(\overline{I}) \cap \Delta^{\{l\}}_{X/k}$; in particular, $\overline{D} \subseteq \Phi^{\{l\}}_{X/k}$ is *not open in* $\Phi^{\{l\}}_{X/k}$. On the other hand, since s is *quasi-l-monodromically full*, it follows immediately from Proposition 4, (ii), that the image of ϕ_s, hence also $\overline{D} \subseteq \Phi^{\{l\}}_{X/k}$, is *open in* $\Phi^{\{l\}}_{X/k}$. Thus, we obtain a contradiction. Therefore, x is *not a cusp of* X. This completes the proof of the implication (i) $\Rightarrow$ (ii).

Next, to verify the implication (iii) $\Rightarrow$ (i), suppose that condition (iii) is satisfied. Then it follows from Lemma 10, (ii), that — after replacing G_k by a suitable open subgroup of G_k — there exist

- for $\mathfrak{a} \in \{s, x\}$, a finite étale $\Phi^{\{l\}}_{X/k}$-covering $C_{\mathfrak{a}} \to X$ over k (cf. Definition 1, (v)) which is *geometrically connected* over k, and
- an isomorphism $\alpha \colon \Pi^{\{l\}}_{C_x/k} \xrightarrow{\sim} \Pi^{\{l\}}_{C_s/k}$ over G_k

such that

- the pro-l Galois section $\pi_1(x) \colon G_k \to \Pi^{\{l\}}_{X/k}$ *factors through* $\Pi^{\{l\}}_{C_x/k} \subseteq \Pi^{\{l\}}_{X/k}$, and
- the composite
$$G_k \overset{\pi_1(x)}{\to} \Pi^{\{l\}}_{C_x/k} \overset{\alpha}{\xrightarrow{\sim}} \Pi^{\{l\}}_{C_s/k} \hookrightarrow \Pi^{\{l\}}_{X/k}$$
coincides with s.

Now it follows from [15], Theorem A, that the isomorphism $\alpha \colon \Pi^{\{l\}}_{C_x/k} \xrightarrow{\sim} \Pi^{\{l\}}_{C_s/k}$ *arises from* an isomorphism $C_x \xrightarrow{\sim} C_s$ over k. Therefore, it follows immediately from Lemma 11 below, together with the various definitions involved, that condition (i) is satisfied. This completes the proof of Theorem 5. $\square$

Lemma 11 (Geometricity and base-changing). *Suppose that k is a* **finitely generated extension of** $\mathbb{Q}$. *Let $\Sigma \subseteq \mathfrak{Primes}$ be a nonempty subset of $\mathfrak{Primes}$, $s \colon G_k \to \Pi^{\Sigma}_{X/k}$ a pro-Σ Galois section of X/k, and $k' \subseteq \overline{k}$ a* **finite** *extension of k. Write $G_{k'} \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k') \subseteq G_k$. Then s is* **geometric** *if and only if the restriction $s|_{G_{k'}}$ of s to $G_{k'} \subseteq G_k$ is* **geometric**.

Proof. The necessity of the condition is immediate; thus, to verify Lemma 11, it suffices to verify the sufficiency of the condition. Suppose that the restriction $s|_{G_{k'}}$ of s to $G_{k'} \subseteq G_k$ is *geometric*. Now by replacing k' by a finite *Galois* extension of k, we may assume without loss of generality that k' is *Galois* over k. Moreover, by replacing $\Pi^{\Sigma}_{X/k}$ by an open subgroup of $\Pi^{\Sigma}_{X/k}$ which contains the image $\mathrm{Im}(s)$ of s, we may assume without loss of generality that X is *of genus* ≥ 2. Let
$$\cdots \subseteq \Delta_n \subseteq \cdots \subseteq \Delta_2 \subseteq \Delta_1 \subseteq \Delta_0 = \Delta^{\Sigma}_{X/k}$$
be a sequence of characteristic closed subgroups of $\Delta^{\Sigma}_{X/k}$ such that
$$\bigcap_i \Delta_i = \{1\}.$$

Write $\Pi_n \stackrel{\text{def}}{=} \Delta_n \cdot \mathrm{Im}(s) \subseteq \Pi^{\Sigma}_{X/k}$, $X^n \to X$ for the connected finite étale covering of X corresponding to the open subgroup $\Pi_n \subseteq \Pi^{\Sigma}_{X/k}$ (thus, $\Pi^{\Sigma}_{X^n/k} = \Pi_n$), and $s_n \colon G_k \to \Pi_n = \Pi^{\Sigma}_{X^n/k}$ for the pro-Σ Galois section of X^n/k determined by s. Note that it holds that

$$\bigcap_i \Pi_i = \mathrm{Im}(s)\,.$$

Now I *claim* that $(X^n)^{\mathrm{cpt}}(k) \neq \emptyset$ — where $(X^n)^{\mathrm{cpt}}$ is the smooth compactification of X^n. Indeed, since the restriction $s|_{G_{k'}}$ is *geometric*, it holds that the image of the composite

$$G_{k'} \stackrel{s_n|_{G_{k'}}}{\hookrightarrow} \Pi^{\Sigma}_{X^n/k} \twoheadrightarrow \Pi^{\Sigma}_{(X^n)^{\mathrm{cpt}}/k}$$

is a decomposition subgroup D associated to a k'-rational point x_n of $(X^n)^{\mathrm{cpt}}$. On the other hand, since this decomposition subgroup D associated to x_n is *contained in* the image of the homomorphism s_n from G_k, and k' is *Galois* over k, by considering the $\mathrm{Im}(s_n)$-conjugates of D, it follows immediately from [15], Theorem C, that the k'-rational point x_n is *defined over* k. In particular, it holds that $(X^n)^{\mathrm{cpt}}(k) \neq \emptyset$. This completes the proof of the above *claim*.

Now since the set $(X^n)^{\mathrm{cpt}}(k)$ is *finite* by Mordell-Faltings' theorem, it follows from the above *claim* that the projective limit $(X^\infty)^{\mathrm{cpt}}(k)$ of the sequence of sets

$$\cdots \longrightarrow (X^n)^{\mathrm{cpt}}(k) \longrightarrow \cdots \longrightarrow (X^2)^{\mathrm{cpt}}(k) \longrightarrow (X^1)^{\mathrm{cpt}}(k) \longrightarrow (X^0)^{\mathrm{cpt}}(k)$$

is *nonempty*. Let $x_\infty \in (X^\infty)^{\mathrm{cpt}}(k)$ be an element of $(X^\infty)^{\mathrm{cpt}}(k)$. Then it follows immediately from the fact that $\bigcap_i \Pi_i = \mathrm{Im}(s)$ that the pro-Σ Galois section of X/k arising from x_∞ *coincides with* s. In particular, s is *geometric*. This completes the proof of Lemma 11. □

Remark 17. In [10], the author proved that there exist a prime number l, a number field k, a hyperbolic curve X over k, and a pro-l Galois section s of X/k such that the pro-l Galois section s is *not geometric* (cf. [10], Theorem A). On the other hand, it seems to the author that the *nongeometric* pro-l Galois sections appearing in [10] are *not quasi-l-monodromically full*. It is not clear to the author at the time of writing whether or not there exists a pro-l Galois section of a hyperbolic curve over a number field which is *nongeometric* and *quasi-l-monodromically full*.

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