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On a problem of Matsumoto and Tamagawa concerning monodromic fullness of hyperbolic curves: Genus zero case

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ON A PROBLEM OF MATSUMOTO AND TAMAGAWA CONCERNING MONODROMIC FULLNESS OF HYPERBOLIC CURVES: GENUS ZERO CASE

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ABSTRACT. In the present paper, we discuss a problem concerning monodromic fullness of hyperbolic curves over number fields posed by M. Matsumoto and A. Tamagawa in the case where a given hyperbolic curve is *of genus* 0.

CONTENTS

INTRODUCTION

Write $\mathfrak{Primes}$ for the set of all prime numbers. In [4], M. Matsumoto and A. Tamagawa posed the following problem concerning monodromic fullness of hyperbolic curves over number fields (cf. [4], Problem 4.1):

> Let X be a hyperbolic curve over a number field [where we refer to the discussion entitled "Curves" (respectively, "Numbers") in "Notations and Conventions" concerning the term "hyperbolic curve" (respectively, "number field")]. Then are the following three conditions equivalent?
>
> (MT_1) X is *quasi-$\mathfrak{Primes}$-monodromically full* (cf. [2], Definition 2.2, (iii)).
>
> (MT_2) There exists a prime number l such that X is *l-monodromically full* (cf. [2], Definition 2.2, (i)).
>
> (MT_3) There exists a *finite* subset $\Sigma \subseteq \mathfrak{Primes}$ of $\mathfrak{Primes}$ such that X is *$(\mathfrak{Primes} \setminus \Sigma)$-monodromically full*.

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Note that this is an *analogue for hyperbolic curves* of the equivalences "$(1) \Leftrightarrow (2) \Leftrightarrow (3)$" in the following result due to J. P. Serre (cf. [5]):

> Let E be an elliptic curve over a number field k, $\overline{k}$ an algebraic closure of k, and $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$. Moreover, for each prime number l, write $T_l(E)$ for the l-adic Tate module of E. Then the following four conditions are equivalent:
> (0) E does *not admit complex multiplication* over $\overline{k}$.
> (1) For any prime number l, the image of the pro-l Galois representation $G_k \to \mathrm{Aut}(T_l(E))$ is an *open* subgroup of $\mathrm{Aut}(T_l(E))$.
> (2) There exists a prime number l such that the pro-l Galois representation $G_k \to \mathrm{Aut}(T_l(E))$ is *surjective*.
> (3) There exists a *finite* subset $\Sigma \subseteq \mathfrak{Primes}$ of $\mathfrak{Primes}$ such that if $l \notin \Sigma$, then the pro-l Galois representation $G_k \to \mathrm{Aut}(T_l(E))$ is *surjective*.

In the present paper, we discuss the above problem due to M. Matsumoto and A. Tamagawa in the case where the given hyperbolic curve X is *of genus* 0. More concretely, we prove the following two results.

Theorem A. *Let k be a number field. Then there exists a split* (*where we refer to the discussion entitled "Curves" in "Notations and Conventions" concerning the term "split"*) *hyperbolic curve* **of type $(0, 4)$** *over k which* **satisfies** (MT_3), *hence also* (MT_2), *but does* **not satisfy** (MT_1). *Moreover, for any positive integer $r > 4$, there exists a split hyperbolic curve* **of type $(0, r)$** *over k which* **satisfies** (MT_2) *but does* **not satisfy** (MT_1).

Theorem B. *Let k be an imaginary quadratic field and X a hyperbolic curve* **of type $(0, 4)$** *over a subfield k_0 of k such that $X \otimes_{k_0} k$ is* **split**. *Then the following four conditions are equivalent:*

1. *There exists a prime number l such that X is* **quasi-l-monodromically full** (*cf.* [2], *Definition 2.2,* (iii)).
2. *There exists a prime number l such that X is* **l-monodromically full** (*cf.* [2], *Definition 2.2,* (i)).
3. *There exists a* **finite** *subset $\Sigma \subseteq \mathfrak{Primes}$ of $\mathfrak{Primes}$ such that X is* **$(\mathfrak{Primes} \setminus \Sigma)$-monodromically full**.
4. $\mathfrak{m}_X$ (*cf.* [2], *Definition 7.10*) *does* **not contain any unit** *of the ring of integers of k, i.e., if $X \otimes_{k_0} k$ is isomorphic to*

$$\mathbb{P}^1_k \setminus \{0, 1, \lambda, \infty\}$$

— *where $\lambda \in k \setminus \{0, 1\}$ — over k, then*

$$\{\lambda, 1-\lambda, \frac{\lambda}{\lambda - 1}\} \cap \mathfrak{o}_k^\times = \emptyset$$

— *where* $\mathfrak{o}_k$ *is the ring of integers of* k.

In particular, the equivalence "$(\mathrm{MT}_2) \Leftrightarrow (\mathrm{MT}_3)$*" for such an* X *holds.*

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NOTATIONS AND CONVENTIONS

Numbers: The notation $\mathfrak{Primes}$ will be used to denote the set of all prime numbers. The notation $\mathbb{Z}$ will be used to denote the ring of rational integers. If p is a prime number, then the notation $\mathbb{F}_p$ will be used to denote the finite field with p elements and the notation $\mathbb{Z}_p$ will be used to denote the p-adic completion of $\mathbb{Z}$. We shall refer to a finite extension of the field of rational numbers as a *number field*.

Profinite Groups: If G is a profinite group, then we shall write $\mathrm{Aut}(G)$ for the group of (continuous) automorphisms of G, $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ for the group of inner automorphisms of G, and

$$\mathrm{Out}(G) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(G)/\mathrm{Inn}(G)\,.$$

If, moreover, G is *topologically finitely generated*, then one verifies easily that the topology of G admits a basis of *characteristic open subgroups*, which thus induces a *profinite topology* on the group $\mathrm{Aut}(G)$, hence also a *profinite topology* on the group $\mathrm{Out}(G)$.

Curves: Let k be a field and X a scheme over k. For a pair (g, r) of nonnegative integers, we shall say that X is a *smooth curve of type* (g, r) over k if there exist a scheme X^{cpt} of dimension 1 which is smooth, proper, and geometrically connected over k and a closed subscheme $D \subseteq X^{\mathrm{cpt}}$ of X^{cpt} which is étale and of degree r over k such that the complement of D in X^{cpt} is isomorphic to X over k, and, moreover, a geometric fiber of $X^{\mathrm{cpt}} \to \mathrm{Spec}\, k$ is (a necessarily smooth, proper, and connected curve) of genus g. Note that it follows immediately that if X is a smooth curve of type (g, r) over k, then the pair "(X^{cpt}, D)" is *uniquely determined up to isomorphism*. We shall say that X is a *hyperbolic curve* over k if there exists a pair (g, r) of nonnegative integers such that $2g - 2 + r > 0$, and, moreover, X is a smooth curve of type (g, r) over k. We shall say that X is a *tripod* over k if X is a smooth curve of type $(0, 3)$ over k. (Thus, any tripod over k is a *hyperbolic curve* over k.) Suppose that there exists a pair (g, r) of nonnegative integers such that X is a smooth curve of type (g, r) over k. Then we shall say that X is *split* if "D" appearing in the definition of the term "smooth curve of type (g, r)" is isomorphic to the disjoint union of r copies of $\mathrm{Spec}\, k$ over k.

PROOFS OF MAIN RESULTS

Let k be a field *of characteristic* 0 and $\overline{k}$ an algebraic closure of k. Write $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$ for the absolute Galois group of k determined by the algebraic closure $\overline{k}$ and

$$\mathcal{M} \stackrel{\mathrm{def}}{=} \mathbb{P}^1_k \setminus \{0, 1, \infty\} = \mathrm{Spec}\, k[t^{\pm 1}, 1/(1-t)]$$

— where t is an indeterminate — for the *split tripod* over k (where we refer to the discussion entitled "Curves" in "Notations and Conventions" concerning the terms "split" and "tripod"). Now we have a natural identification

$$\mathcal{M}(k) \simeq k \setminus \{0, 1\}$$

and an exact sequence of profinite groups

$$1 \longrightarrow \pi_1(\mathcal{M} \otimes_k \overline{k}) \longrightarrow \pi_1(\mathcal{M}) \longrightarrow G_k \longrightarrow 1\,.$$

Moreover, for each prime number l, write

$$\mu_{l^\infty} \subseteq \overline{k}^{\times}$$

for the subgroup of $\overline{k}^{\times}$ of all l-powers roots of unity.

Definition 1. Let l be a prime number.

(i) We shall write

$$\Delta^{\{l\}}$$

for the maximal pro-l quotient of $\pi_1(\mathcal{M} \otimes_k \overline{k})$.

(ii) Since the closed subgroup $\pi_1(\mathcal{M} \otimes_k \overline{k}) \subseteq \pi_1(\mathcal{M})$ of $\pi_1(\mathcal{M})$ is *normal*, conjugation by elements of $\pi_1(\mathcal{M})$ naturally determines continuous homomorphisms

$$\pi_1(\mathcal{M}) \longrightarrow \mathrm{Aut}(\Delta^{\{l\}}) \ ; \ G_k \longrightarrow \mathrm{Out}(\Delta^{\{l\}})$$

— where we refer to the discussion entitled "Profinite Groups" in "Notations and Conventions" concerning the profinite topologies of $\mathrm{Aut}(\Delta^{\{l\}})$ and $\mathrm{Out}(\Delta^{\{l\}})$. We shall write

$$\widetilde{\rho}^{\{l\}} \ ; \ \rho^{\{l\}}$$

for the above continuous homomorphisms, respectively. It follows immediately from the various definitions involved that these homomorphisms fit into the following commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\mathcal{M} \otimes_k \overline{k}) & \longrightarrow & \pi_1(\mathcal{M}) & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\rho}^{\{l\}}} & & \big\downarrow{\scriptstyle \rho^{\{l\}}} & & \\
1 & \longrightarrow & \mathrm{Inn}(\Delta^{\{l\}}) & \longrightarrow & \mathrm{Aut}(\Delta^{\{l\}}) & \longrightarrow & \mathrm{Out}(\Delta^{\{l\}}) & \longrightarrow & 1
\end{array}$$

— where the horizontal sequences are *exact*; moreover, since $\Delta^{\{l\}}$ is *center-free*, the left-hand vertical arrow factors as the composite of the natural *surjection* $\pi_1(\mathcal{M} \otimes_k \overline{k}) \twoheadrightarrow \Delta^{\{l\}}$ and the natural *isomorphism* $\Delta^{\{l\}} \xrightarrow{\sim} \mathrm{Inn}(\Delta^{\{l\}})$.

(iii) We shall write

$$\pi_1(\mathcal{M}) \twoheadrightarrow \Phi^{\{l\}} \quad (\text{respectively, } G_k \twoheadrightarrow G_k^{\text{tpd-}l})$$

for the quotient of $\pi_1(\mathcal{M})$ (respectively, G_k) by the kernel of the homomorphism $\widetilde{\rho}^{\{l\}}$ (respectively, $\rho^{\{l\}}$). Thus, the commutative diagram in (ii) determines an exact sequence of profinite groups

$$1 \longrightarrow \Delta^{\{l\}} \longrightarrow \Phi^{\{l\}} \longrightarrow G_k^{\text{tpd-}l} \longrightarrow 1\,.$$

(iv) We shall write

$$k^{\text{tpd-}l}\ (\subseteq \overline{k})$$

for the algebraic extension of k corresponding to the quotient $G_k \twoheadrightarrow G_k^{\text{tpd-}l}$, i.e., $G_k^{\text{tpd-}l} = \mathrm{Gal}(k^{\text{tpd-}l}/k)$.

Remark 2. In [3], the notation $\Delta^{\{l\}}_{\mathcal{M}/k}$ (respectively, $\widetilde{\rho}^{\{l\}}_{\mathcal{M}/k}$; $\rho^{\{l\}}_{\mathcal{M}/k}$; $\Phi^{\{l\}}_{\mathcal{M}/k}$; $\Gamma^{\{l\}}_{\mathcal{M}/k}$) was used to denote the object $\Delta^{\{l\}}$ (respectively, $\widetilde{\rho}^{\{l\}}$; $\rho^{\{l\}}$; $\Phi^{\{l\}}$; $G_k^{\text{tpd-}l}$) defined in Definition 1 of the present paper (cf. [3], Definition 1).

Lemma 3. *Let l be a prime number and $\lambda \in k \setminus \{0,1\}$. Then the following three conditions are equivalent:*

(1) *The split hyperbolic curve of type $(0,4)$ over k*

$$\mathbb{P}^1_k \setminus \{0, 1, \lambda, \infty\}$$

*is **l-monodromically full** (respectively, **quasi-l-monodromically full**) [cf. [2], Definition 2.2].*

(2) *The k-rational point of $\mathcal{M}$ naturally corresponding to $\lambda \in k \setminus \{0,1\}$ is **l-monodromically full** (respectively, **quasi-l-monodromically full**) [cf. [3], Definition 3].*

(3) *The image of the composite*

$$G_k \longrightarrow \pi_1(\mathcal{M}) \longrightarrow \Phi^{\{l\}}$$

*— where the first arrow is the outer homomorphism induced by $\lambda \in k \setminus \{0,1\} \simeq \mathcal{M}(k)$ — is $\Phi^{\{l\}}$ (respectively, is an **open** subgroup of $\Phi^{\{l\}}$).*

Proof. The equivalence "(1) $\Leftrightarrow$ (2)" follows from the second equivalence in [3], Remark 4. The equivalence "(2) $\Leftrightarrow$ (3)" follows from [3], Proposition 4, (iv). $\square$

Definition 4 (cf. [1], §2.3; 2.5). Let l be an odd prime number.

(i) We shall write
$$\mathbb{S}_l$$
for the minimal set of finite subsets of $\mathbb{P}^1_{\overline{k}}(\overline{k}) \simeq \overline{k} \cup \{\infty\}$ which satisfies the following three conditions:
  (1) $\{0, 1, \infty\} \in \mathbb{S}_l$.
  (2) If $S \in \mathbb{S}_l$, then $\{\, a \in \overline{k} \mid a^l \in S \,\} \cup \{\infty\} \in \mathbb{S}_l$.
  (3) If $S \in \mathbb{S}_l$, and ϕ is an automorphism of $\mathbb{P}^1_{\overline{k}}$ over $\overline{k}$ such that $\{0, 1, \infty\} \subseteq \phi(S)$, then $\phi(S) \in \mathbb{S}_l$.

(ii) We shall write
$$\mathbb{E}_l \subseteq \overline{k}^{\times}$$
for the subgroup of $\overline{k}^{\times}$ generated by the elements of $S \setminus \{0, \infty\}$ for all $S \in \mathbb{S}_l$.

Some of main results of [1] are as follows.

Proposition 5. *Let l be an odd prime number. Then the following hold:*
  (i) $k^{\mathrm{tpd}\text{-}l} = k(\mathbb{E}_l)$.
  (ii) $(\mathbb{E}_l)^l = \mathbb{E}_l$.
  (iii) $\mu_{l^\infty} \subseteq \mathbb{E}_l$.

Proof. This follows from [1], Theorems A and B. □

Lemma 6. *Let l be an odd prime number. Then $l \in \mathbb{E}_l$.*

Proof. It follows from condition (1) of Definition 4, (i), that $\{0, 1, \infty\} \in \mathbb{S}_l$. Thus, it follows from condition (2) of Definition 4, (i), that
$$S \stackrel{\mathrm{def}}{=} \{0, 1, \zeta_l, \zeta_l^2, \cdots, \zeta_l^{l-1}, \infty\} \in \mathbb{S}_l$$
— where $\zeta_l \in \overline{k}$ is an l-th root of unity. Now since the automorphism ϕ of $\mathbb{P}^1_{\overline{k}}$ over $\overline{k}$ given by "$t \mapsto 1 - t$" satisfies that $\{0, 1, \infty\} \subseteq \phi(S)$, it follows from condition (3) of Definition 4, (i), that
$$\phi(S) = \{1, 0, 1 - \zeta_l, 1 - \zeta_l^2, \cdots, 1 - \zeta_l^{l-1}, \infty\} \in \mathbb{S}_l\,.$$
Therefore,
$$l = \prod_{i=1}^{l-1} (1 - \zeta_l^i) \in \mathbb{E}_l\,.$$
This completes the proof of Lemma 6. □

Lemma 7. *Let l be a prime number. Suppose that $\mu_{l^\infty} \subseteq k$. For each positive integer n, write*
$$C_{l^n} \stackrel{\mathrm{def}}{=} \mathrm{Spec}\, k[x^{\pm 1}, y^{\pm 1}]/(x^{l^n} + y^{l^n} + 1) \longrightarrow \mathcal{M}$$
— where x and y are indeterminates — for the finite étale Galois $(\mathbb{Z}/l^n\mathbb{Z})^{\oplus 2}$-covering of $\mathcal{M}$ given by "$t \mapsto x^{l^n}$" and
$$\pi_1(\mathcal{M}) \twoheadrightarrow Q_l \simeq \mathbb{Z}_l^{\oplus 2}$$
for the quotient of $\pi_1(\mathcal{M})$ determined by the C_{l^n}'s. Then the quotient $\pi_1(\mathcal{M}) \twoheadrightarrow Q_l$ **factors through** *the quotient $\pi_1(\mathcal{M}) \twoheadrightarrow \Phi^{\{l\}}$.*

Proof. To verify Lemma 7, it is immediate that it suffices to verify the *fact* that for any positive integer n, the quotient $\pi_1(\mathcal{M}) \twoheadrightarrow (\mathbb{Z}/l^n\mathbb{Z})^{\oplus 2}$ determined by the finite étale covering $C_{l^n} \to \mathcal{M}$ *factors through* the quotient $\pi_1(\mathcal{M}) \twoheadrightarrow \Phi^{\{l\}}$. Moreover, to verify this *fact*, it follows immediately from [3], Proposition 7, (i), that it suffices to verify that the kernel of the pro-l outer Galois representation associated to $\mathcal{M}/k$ (i.e., $\mathrm{Ker}(\rho^{\{l\}})$) *coincides with* the kernel of the pro-l outer Galois representation associated to C_{l^n}/k. On the other hand, this follows immediately from [3], Proposition 9. This completes the proof of Lemma 7. $\square$

Proposition 8. *Let l be an odd prime number and $\lambda \in k \setminus \{0, 1\}$. If either $\lambda \in \mathbb{E}_l \cap k$ (cf. Definition 4,* (ii)*) or λ is a* **root of unity***, then the split hyperbolic curve of type $(0, 4)$ over k*

$$\mathbb{P}^1_k \setminus \{0, 1, \lambda, \infty\}$$

is **not quasi-*l*-monodromically full**.

Proof. To verify Proposition 8, it follows immediately from Lemma 3, together with the exactness of the sequence appearing in Definition 1, (iii), that, by replacing k by $k^{\text{tpd-}l} \subseteq \overline{k}$, we may assume without loss of generality that $k = k^{\text{tpd-}l}$. Write ϕ for the composite

$$G_k \longrightarrow \pi_1(\mathcal{M}) \longrightarrow Q_l$$

— where the first arrow is the outer homomorphism induced by $\lambda \in k \setminus \{0, 1\} \simeq \mathcal{M}(k)$, and the second arrow is the natural surjection from $\pi_1(\mathcal{M})$ to the quotient Q_l defined in the statement of Lemma 7 (cf. Proposition 5, (i), (iii)). Moreover, for each positive integer n, write ϕ_n for the composite of ϕ and the natural surjection $Q_l \twoheadrightarrow Q_l/l^n Q_l$ ($\simeq (\mathbb{Z}/l^n\mathbb{Z})^{\oplus 2}$) and $k_n \subseteq \overline{k}$ for the finite Galois extension of k corresponding to the quotient of G_k determined by the homomorphism ϕ_n. Then it follows immediately from the definition of the finite étale covering $C_{l^n} \to \mathcal{M}$ (where we refer to the statement of Lemma 7 concerning "$C_{l^n} \to \mathcal{M}$") that

$$k_n = k(\lambda^{1/l^n}, (1-\lambda)^{1/l^n}).$$

Now I *claim* that for any positive integer n, it holds that $k_n = k((1-\lambda)^{1/l^n})$. Indeed, if $\lambda \in \mathbb{E}_l$, then it follows immediately from Proposition 5, (i), (ii), that $\lambda^{1/l^n} \in \mathbb{E}_l \subseteq k^{\text{tpd-}l} = k$; in particular, $k_n = k(\lambda^{1/l^n}, (1-\lambda)^{1/l^n}) = k((1-\lambda)^{1/l^n})$. On the other hand, if λ is a *root of unity*, then it follows immediately from Proposition 5, (i), (iii), that $\lambda^{1/l^n} \in k(\mu_{l^\infty}, \lambda) \subseteq k^{\text{tpd-}l}(\lambda) = k(\lambda)$; in particular, $k_n = k(\lambda^{1/l^n}, (1-\lambda)^{1/l^n}) = k((1-\lambda)^{1/l^n})$. This completes the proof of the above *claim*.

Now it follows immediately from Lemma 3 that the hyperbolic curve of type $(0,4)$ over k

$$\mathbb{P}^1_k \setminus \{0,1,\lambda,\infty\}$$

is *quasi-l-monodromically full* if and only if the image of the composite

$$G_k \longrightarrow \pi_1(\mathcal{M}) \longrightarrow \Phi^{\{l\}}$$

— where the first arrow is the outer homomorphism induced by $\lambda \in k \setminus \{0,1\} \simeq \mathcal{M}(k)$ — is an *open* subgroup of $\Phi^{\{l\}}$. In particular, it follows from Lemma 7 that if $\mathbb{P}^1_k \setminus \{0,1,\lambda,\infty\}$ is *quasi-l-monodromically full*, then the image of ϕ is an *open* subgroup of Q_l. On the other hand, it follows immediately from the above *claim* that for any positive integer n, the image of ϕ_n is a *cyclic group*. In particular, the image of ϕ is *not open* in Q_l. Therefore, $\mathbb{P}^1_k \setminus \{0,1,\lambda,\infty\}$ is *not quasi-l-monodromically full*. This completes the proof of Proposition 8. $\square$

Proof of Theorem A. Let l be an odd prime number. Then since $l \in \mathbb{E}_l$ (cf. Lemma 6), it follows immediately from Proposition 8 that the hyperbolic curve of type $(0,4)$ over k

$$X \overset{\text{def}}{=} \mathbb{P}^1_k \setminus \{0,1,l,\infty\}$$

is *not quasi-l-monodromically full*. On the other hand, since neither l, $1-l$, nor $l/(l-1)$ is a *unit* of the ring of integers of k, it follows from [2], Corollary 7.11, that there exists a *finite* subset $\Sigma \subseteq \mathfrak{Primes}$ of $\mathfrak{Primes}$ such that X is $(\mathfrak{Primes} \setminus \Sigma)$*-monodromically full*. In particular, X satisfies the condition (MT_3) but does not satisfy the condition (MT_1). This completes the proof of the fact that there exists a split hyperbolic curve *of type* $(0,4)$ over k which *satisfies* (MT_3), hence also (MT_2), but does *not satisfy* (MT_1).

Moreover, let $r > 4$ be a positive integer and $l' \notin \Sigma$ a prime number. Then it follows from [3], Proposition 2, that there exists an *l'-monodromically full* k-rational point x (cf. [3], Definition 3) of the $(r-4)$-th configuration space of the hyperbolic curve X/k. Since X is *l'-monodromically full* and x is *l'-monodromically full*, it follows from [3], Proposition 5, that the split hyperbolic curve Y of type $(0,r)$ determined by x — i.e., the hyperbolic curve obtained by taking the complement in X of the images of $r-4$ distinct k-rational points of X determined by x — is *l'-monodromically full*. In particular, Y satisfies the condition (MT_2). On the other hand, since $Y \subseteq X$, and X is *not quasi-l-monodromically full*, it follows from [2], Remark 2.2.5, that Y is *not quasi-l-monodromically full*. In particular, Y does not satisfy the condition (MT_1). This completes the proof of the fact that for any positive integer $r > 4$, there exists a split hyperbolic curve *of type* $(0,r)$ over k which *satisfies* (MT_2) but does *not satisfy* (MT_1). $\square$

Proof of Theorem B. The implication "$(4) \Rightarrow (3)$" follows from [2], Corollary 7.11. The implications "$(3) \Rightarrow (2) \Rightarrow (1)$" are immediate.

The implication "(1) $\Rightarrow$ (4)" follows from Proposition 8, together with the fact that every unit of the ring of integers of an imaginary quadratic field is a root of unity. $\square$

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