RIMS-1705

GROWTH PARTITION FUNCTIONS FOR CANCELLATIVE INFINITE MONOIDS

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September 2010

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ABSTRACT. We introduce the *growth partition function* $Z_{\Gamma,G}(t)$ associated with any cancellative infinite monoid Γ with a finite generator system G. It is a power series in t whose coefficients lie in integral Lie-like space $\mathcal{L}_{\mathbb{Z}}(\Gamma, G)$ in the configuration algebra associated with the Cayley graph (Γ, G). We determine them for homogeneous monoids admitting left greatest common divisor and right common multiple. Then, for braid monoids and Artin monoids of finite type, using that formula, we explicitley determine their limit partition functions $\omega_{\Gamma,G}$.

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1. INTRODUCTION

In a previous paper [S1, §11.1.3], we introduced the set $\Omega(\Gamma, G)$ of *partition functions* (which were called pre-partition functions there) associated with a cancellative infinite monoid Γ [1] with a fixed finite generator system G. In the present paper, using the same framework, we introduce the *growth partition function* $Z_{\Gamma,G}(t)$, which we already have studied without a name (see (1.1) and its following explanations in the next paragraph). We determine the growth partition functions for a class of homogeneous monoids which admit left greatest common divisors and right common multiples. Then, using that formula, we show that Artin monoids of finite type, in particular braid monoids, up to possible finite exceptions, are *simple accumulating* (i.e. $\Omega(\Gamma, G) =$

[1]We call a semigroup with an identity element a *monoid*. A monoid is called *cancellative* if the identity $aub = avb$ for a, b, u, v in the monoid implies $u = v$. If an equality $ab = c$ for $a, b, c \in \Gamma$ holds in a cancellative monoid Γ, then a (resp. b) is uniquely determined by b, c (resp. a, c), which we shall denote by cb^{-1} (resp. $a^{-1}c$).

$\{\omega_{\Gamma,G}\}$ for a single element $\omega_{\Gamma,G}$), and then we determine explicitly the limit partition function $\omega_{\Gamma,G}$ for them by solving an algebraic equation arising from the denominator of their growth functions.

In the following, we briefly recall the definition of the set $\Omega(\Gamma,G)$ of partition functions associated with an infinite Cayley graph (Γ,G) [2], and recall in (1.1) the main formula in [S1] for them. The main term of the formula is a proportion of two growth functions, which we will call the *growth partition function* and denote by $Z_{\Gamma,G}(t)$.

An isomorphism class of a finite subgraph of (Γ,G) is called a *configuration*. The set of all configurations, denoted by $Conf(\Gamma,G)$, form a partial ordered semi-group by taking the disjoint union as the product. Consider the algebra $\mathbb{A}[[Conf(\Gamma,G)]]$:=the adic completion of the group ring $\mathbb{A}{\cdot}Conf(\Gamma,G)$ with respect to the grading $\deg(S) := \#S$ for $S\in Conf(\Gamma,G)$, where $\mathbb{A}$ is a commutative coefficient ring. We can attach to it a topological Hopf algebra structure and call it the *configuration algebra*. It is also equipped with the classical topology if $\mathbb{A}$ is $\mathbb{R}$ or $\mathbb{C}$. For any configuration $S\in Conf(\Gamma,G)$, let $\mathcal{A}(S)$ be the sum of isomorphism classes of all subgraphs of S. Then, $\mathcal{M}(S):=\log(\mathcal{A}(S))$ becomes a Lie-like element of the Hopf algebra, where we shall denote by $\mathcal{L}_{\mathbb{A}}(\Gamma,G)$ the space of all Lie-like elements of $\mathbb{A}[[Conf(\Gamma,G)]]$.

Inspired by statistical mechanics, we call $\frac{\mathcal{M}(S)}{\#S}$ the *free energy* of S. We introduce the space $\Omega(\Gamma,G)$ of partition functions as the compact accumulation set (with respect to the classical topology) in $\mathcal{L}_{\mathbb{R}}(\Gamma,G)$ of the sequence of free energies $\frac{\mathcal{M}(\Gamma_n)}{\#\Gamma_n}$ for the balls Γ_n in (Γ,G) of radius $n\in\mathbb{Z}_{\geq 0}$ centered at the unit e. Parallely, we introduce the *space* $\Omega(P_{\Gamma,G})$ *of opposite series of the growth function* $P_{\Gamma,G}(t):=\sum_{n=0}^{\infty}\#(\Gamma_n)t^n$ ([S1, §11.2.3], see also §2). Then, we obtain a natural surjective map:

$$\pi_\Omega \ : \ \Omega(\Gamma,G) \ \longrightarrow \ \Omega(P_{\Gamma,G})$$

which is equivariant with certain actions $\tilde{\tau}_\Omega$ and τ_Ω on $\Omega(\Gamma,G)$ and $\Omega(P_{\Gamma,G})$, respectively. Both actions are transitive if $\Omega(\Gamma,G)$ is finite. Therefore, the fiber of π_Ω over a point in $\Omega(P_{\Gamma,G})$ is an orbit of a finite cyclic group $\mathbb{Z}/m_{\Gamma,G}\mathbb{Z}:=\ker(\langle\tilde{\tau}_\Omega\rangle\to\langle\tau_\Omega\rangle)$, called the *inertia group*.

The main formula of [S1, §11.5 Theorem] states that

$$(1.1)\qquad Trace^{[e]}\Omega(\Gamma,G) - E = \frac{m_{\Gamma,G}}{h_{\Gamma,G}} \sum_{x_i\in V(\Delta^{top}_{\Gamma,G})} A^{[e]}(x_i^{-1})\, \frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\Big|_{t=x_i}$$

where $Trace^{[e]}\Omega(\Gamma,G)$:=the sum of partition function in a fiber of π_Ω (=an orbit of the inertia group) over a point $[e]\in\Omega(P_{\Gamma,G})$, E =an error

[2] To be exact, we consider colored and oriented graph (see Footnote 5). Depending on the setting, we shall sometimes assume further three conditions H, I and S on (Γ,G) (even though they are unnecessary for the definitions of $\Omega(\Gamma,G)$ and $Z_{\Gamma,G}(t)$).

term (conjecturally zero), $h_{\Gamma,G} = ord(\tau_{\Gamma,G})$,[3] $V(\Delta^{top}_{\Gamma,G})$:=the set of zero loci of the top-denominator polynomial $\Delta^{top}_{\Gamma,G}(t)$ of $P_{\Gamma,G}(t)$ (see Footnote 4.A), $A^{[e]}(s)$ =the numerator polynomial in s of degree $h_{\Gamma,G}-1$ of the opposite series indexed by $[e] \in \Omega(P_{\Gamma,G})$ and, finally,

$$P_{\Gamma,G}\mathcal{M}(t) \ := \ \sum_{n0}^{\infty} \mathcal{M}(\Gamma_n) t^n$$

is a newly introduced growth function of Lie-like elements [S1, §11.2.7]. Due to formula (1.1), we are interested in the ratio $\frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}$, and give it a name: a *growth partition function* and denote it by $Z_{\Gamma,G}(t)$.

In the following 1)-4), we contrast the growth partition function $Z_{\Gamma,G}(t)$ with partition functions in $\Omega(\Gamma, G)$.

1) *Family over* $\Omega(\Gamma, G)$ *v.s. a single function with one variable* t*.*

The partition functions for (Γ, G) are parametrized by a compact set $\Omega(\Gamma, G)$, whereas there is only one growth partition function $Z_{\Gamma,G}(t)$ with one variable t, and data of $Z_{\Gamma,G}(t)$ can be disclosed by specializing the variable t to special values t_i at some zero loci of the *denominator* [4] of the growth function $P_{\Gamma,G}(t)$. We do not know whether $Z_{\Gamma,G}(t)$ recovers the whole functions of $\Omega(\Gamma, G)$ or not. On the other hand, we shall see in the following 4) that $Z_{\Gamma,G}(t)$ contains "new partition functions" which may not be covered by the functions in $\Omega(\Gamma, G)$.

2) *Completed coefficient field* $\mathbb{R}$ *v.s. small coefficient ring* $\mathbb{Z}$*.*

We use the real number field $\mathbb{R}$ as the coefficient ring $\mathbb{A}$ to describe the partition functions $\Omega(\Gamma, G)$, since they are defined by classical limits of sequences of free energies whose coefficients are in rational number field $\mathbb{Q}$, whereas the growth partition function $Z_{\Gamma,G}(t)$ is defined as

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[3] $h_{\Gamma,G} = ord(\tau_{\Gamma,G}) = \#\Omega(P_{\Gamma,G})$ is called the *period*, characterized as the smallest integer s.t. $\Delta^{top}_{P_{\Gamma,G}} | (t^{h_{\Gamma,G}} - r_{\Gamma,G}^{h_{\Gamma,G}})$, where $r_{\Gamma,G}$ is the radius of convergence of $P_{\Gamma,G}(t)$.

[4] Here, we are abusing the terminology *denominator* of $P_{\Gamma,G}(t)$ as follows.
A: In 1)-3), we consider the cases when the growth function $P_{\Gamma,G}(t)$ belongs to $\mathbb{C}\{t\}_{r_{\Gamma,G}}$, where we put $\mathbb{C}\{t\}_r := \{P(t) \in \mathbb{C}[[t]] \mid$ (i) $P(t)$ converges on the disc $D(r) := \{t \in \mathbb{C} \mid |t| < r\}$, and (ii) there exists a denominator polynomial $\Delta(t)$ in t such that $\Delta(t)P(t)$ is holomorphic on a neighbourhood of $\overline{D}(r)\}$ for $r \in \mathbb{R}_{>0}$ (see [S1, S11.4 Def.]). Let $\Delta_P(t) = \prod_i (t - x_i)^{d_i}$, where $x_i \in \mathbb{C}$ with $|x_i| = r$ and $d_i \in \mathbb{Z}_{>0}$, be such denominator polynomial of minimal degree. Then, by the top-denominator of $P(t)$, we mean $\Delta^{top}_P(t) := \prod_{i,d_i=d}(t - x_i)$ (where $d = \max\{d_i\}$). If $\Omega(P)$ is finite, $\Delta^{top}_P(t)$ is a factor of $t^h - r^h$ for $h \in \mathbb{Z}_{>0}$ called the period [S1, §11.3].
B: In 4), we consider the cases when the growth function is a rational function or a global meromorphic function in t (they are included in the above case **A**). Then the denominator of the growth function means the denominator in usual sense, up to unit factor. Obviously, $\Delta^{top}_P(t)$ is a factor of the denominator in this sense.

power series with coefficients in integral lattice points $\mathcal{L}_{\mathbb{Z}}(\Gamma, G)$ in the Lie-like space (see §2 for the lattice $\mathcal{L}_{\mathbb{Z}}(\Gamma, G)$).

3) *Coefficients of partition functions.*

For a reason in above 2), it is hard to determine explicit values of the coefficients of partition functions in $\Omega(\Gamma, G)$ with respect to the integral lattice basis. However, once it is expressed using growth partition function, then the coefficients appear explicitly by the substitution of the parameter t to some special values: zero-loci x_i of the denominator polynomial of the growth function, which are often "calculable".

4) *New partition functions.*

In the above 1), 2) and 3), t is specialized at the zero-loci of denominators of the growth function whose absolute values is the smallest (see Footnote 4.**A**). Let us call these partition functions tentatively "old". On the other hand, specialization of $Z_{\Gamma,G}(t)$ at other zero-loci of the denominator (see Footnote 4.**B**) give "new partition functions" in the sense that they satisfy the kabi condition (see [S1, §12, **2. Assertion**] and Footnote 6.). The Galois group of the splitting field of the denominator polynomial acts on and mixes up old and new partition functions.

The above 1), 2), 3) and 4) altogether seem to suggest that $Z_{\Gamma,G}(t)$ gives some structural insight on partition functions, even though we do not yet understand the *global phenomenon* described in 4) (see [S1, §12, **2.** and **3.**] and §4 Artin monoid of finite type).

Let us give an overview of the present paper.

In §2, we recall from [S1] basic concepts and notations on configuration algebras, introduce the space $\Omega(\Gamma, G)$ of partition functions, and define the growth partition function $Z_{\Gamma,G}(t)$. We loosen a technical assumption in [S1] that Γ is embeddable into a group to a weaker one, which we call **Assumption H**. In §3, we calculate the growth partition function for a class $\mathcal{C}$ of cancellative homogeneous monoids which admit left greatest common divisors and right common multiple. Finally in §4, we show that an Artin monoid of finite type is *simple accumulating.* Applying (1.1), we determine the unique partition function explicitly by a help of the denominator polynomial of the growth function.

2. Growth partition functions

We recall basic notation and concepts (as minimal as possible) on configuration algebra on a Cayley graph of a cancellative monoid (see [S1] for details). Then, we introduce the limit set $\Omega(\Gamma, G)$ of partition functions and the growth partition function $Z_{\Gamma,G}(t)$.

Let (Γ,G) be the colored Cayley graph[5] associated with a pair of an infinite cancellative monoid Γ and its finite generator system G. An isomorphism class denoted by $S=[\mathbb{S}]$ of a finite subgraph $\mathbb{S}$ [6] of (Γ,G) is called a *configuration*. The set of all configurations (resp. connected configurations) is denoted by $Conf(\Gamma,G)$ (resp. $Conf_0(\Gamma,G)$). The set $Conf(\Gamma,G)$ has a monoid structure generated by $Conf_0(\Gamma,G)$ by taking the disjoint union as the product and the empty graph class $[\emptyset]$ as the unit element. The completion $\mathbb{A}[[Conf(\Gamma,g)]]$ of the group ring $\mathbb{A}\cdot Conf(\Gamma,G)$ with respect to the adic topology defined by the grading $\deg(T):=$ number of vertices of T for $T\in Conf(\Gamma,G)$ is called the *canfiguration algebra*, where $\mathbb{A}$ is a commutative coefficient ring containing $\mathbb{Q}$. The configuration algebra is equipped with a topological Hopf algebra structure, induced from the (higher) co-multiplications:

$$\Phi_n:\ S\ \mapsto \sum_{S_1,\cdots,S_n\in Conf(\Gamma,G)} \binom{S_1,\cdots,S_n}{S} S_1\otimes\cdots\otimes S_n$$

for $n\in\mathbb{Z}_{\geq 0}$ and $S\in Conf(\Gamma,G)$, where the coefficient is a combinatorial invariant, called the *covering coefficient* (see [S1, §2.4 & §4.1]).

For $T\in Conf(\Gamma,G)$, let $\mathbb{T}$ be a representative graph of T. Put

$\mathcal{A}(T)\ :=\sum_{\mathbb{S}\subset\mathbb{T}}[\mathbb{S}]=$ the sum of isomorphism classes of all subgraphs of $\mathbb{T}$.

That is, $\mathcal{A}(T)=\sum_{S\in Conf} SA(S,T)$ for $A(S,T):=\#\mathbb{A}(S,\mathbb{T})$ where $\mathbb{A}(S,\mathbb{T}):=\{\mathbb{S}\mid\mathbb{S}\subset\mathbb{T}\ \&\ [\mathbb{S}]=S\}$. Then $\mathcal{A}(T)$ is a *group-like element* in the Hopf algebra, i.e. $\Phi_n(\mathcal{A}(T))=\overset{n}{\otimes}\mathcal{A}(T)$. In fact, this fact gives a characterization of the Hopf algebra structure. Thus, the logarithm $\mathcal{M}(T):=\log(\mathcal{A}(T))$ for $T\in Conf(\Gamma,G)$ generate over $\mathbb{A}$ a dense (w.r.t. the adic topology) submodule of the module $\mathcal{L}_{\mathbb{A}}(\Gamma,G)$ of all Lie-like elements of $\mathbb{A}[[Conf(\Gamma,G)]]$. However, they cannot form topological basis of $\mathcal{L}_{\mathbb{A}}(\Gamma,G)$, since $\mathcal{M}(T)=\#T\cdot[pt]+\cdots$ contains low degree terms. Thus, we are lead to introduce a new (topological) $\mathbb{A}$-basis $\{\varphi(S)\}_{S\in Conf_0(\Gamma,G)}$ of $\mathcal{L}_{\mathcal{A}}(\Gamma,G)$ by the base change:

$$\mathcal{M}(T)\ =\sum_{S\in Conf_0(\Gamma,g)}\varphi(S)\cdot A(S,T), \tag{2.2}$$

$$\varphi(S)\ =\sum_{T\in Conf_0(\Gamma,g)}\mathcal{M}(T)\cdot(-1)^{\#T-\#S}K(T,S), \tag{2.3}$$

[5]Cayley graph $(\Gamma,G):=$ a graph whose vertex set is Γ, and two vertices $u,v\in\Gamma$ are connected by an edge if and only if $u^{-1}v$ or $v^{-1}u\in G$. Each oriented edge $a - \beta$ is labelled by the element $\alpha^{-1}\beta$ in G (called color and orientation).

[6]By a subgraph we mean a *full-subgraph*, i.e. two vertices connected by an edge in the subgraph if and only if they are connected in the Cayley graph. Thus, an isomorphism of two subgraphs $\mathbb{S}$ and $\mathbb{T}$ is a bijection φ of vertices such that, for $\alpha\in G$ and $x,y\in\mathbb{S}$, $x\alpha=y$ holds if and only if $\varphi(x)\alpha=\varphi(y)$ holds (see [S1, §2.1])

where $K(T,S)$ is a combinatorial constant $\in \mathbb{Z}_{\geq 0}$, called kabi-coefficient, satisfying an inversion formula:$\sum_{U\in Conf_0}(-1)^{\#U-\#S}K(S,U)A(U,T) = \delta(S,T)$ ([S1, §7.3.1]). Further more, they satisfy $A(S,T)=0$ if $S \not\lesssim T$ and $K(T,S)=0$ if $T \not\lesssim S$ or $\deg(S) \not\leq \deg(T)(\#G-1)+2$. In particular, $\varphi(S)$ consists only of terms of degree greater or equal than $\deg(S)$ so that

$$\mathcal{L}_{\mathbb{A}}(\Gamma,G) \ = \prod_{S\in Conf_0(\Gamma,G)} \varphi(S)\cdot \mathbb{A}. \tag{2.4}$$

Regarding $\{\varphi(S)\}_{S\in Conf(\Gamma,G)}$ as integral basis of $\mathcal{L}_{\mathbb{A}}(\Gamma,G)$, we put $\mathcal{L}_{\mathbb{B}}(\Gamma,G) := \prod_{S\in Conf_0(\Gamma,G)} \varphi(S)\cdot \mathbb{B}$ for any subalgebra $\mathbb{B}$ of $\mathbb{A}$.

Recall that for any $g\in\Gamma$, its length is defined by

$$l(g) := \min\{n\in \mathbb{Z}_{\geq 0} \mid \exists g_1,\cdots,g_n \in G \text{ s.t. } g=g_1\cdots g_n\} \tag{2.5}$$

Note that $l(g) \geq d(g,e)$:= the distance in the Cayley graph between g and the unit element e, but the equality may not hod in general. If Γ is a group and $G=G^{-1}$, the equality holds.

Definition. We call a monoid Γ *homogeneous with respect to the generator system* G, if l (2.5) is additive, i.e. $l(gh)=l(g)+l(h)$, or equivalently, if Γ is presented by homogeneous relations in G.

Using $l(g)$, we define a "ball" of radius $n\in\mathbb{Z}_{\geq 0}$ centered at e by

$$\Gamma_n := \{g\in\Gamma \mid l(g)\leq n\}. \tag{2.6}$$

By an abuse of notation, we shall confuse the ball Γ_n with its isomorphism class in $Conf_0(\Gamma,G)$.

We recall definitions of the spaces of partition functions and of opposite sequences, and then state a Theorem on them. For the purpose, we specialize the coefficient ring $\mathbb{A}$ to the real number field $\mathbb{R}$. Here, we remark that the configuration algebra $\mathbb{R}[[Conf(\Gamma,G)]]$ and the Lie-like space $\mathcal{L}_{\mathbb{R}}(\Gamma,G)$ over $\mathbb{R}$ are also equipped with *classical topology.*

Definition. 1. The space of *partition function* of (Γ,G) is

$$\Omega(\Gamma,G) := \begin{cases} \text{the accumulating set of free energies } \{\frac{\mathcal{M}(\Gamma_n)}{\#\Gamma_n}\}_{n\in\mathbb{Z}_{\geq 0}} \\ \text{in } \mathbb{R}[[Conf(\Gamma,G)]] \text{ with respect to the classical topology.}\end{cases}$$

2. The space of opposite sequences for the growth sequence $\{\#\Gamma_n\}$ is

$$\Omega(P_{\Gamma,G}) := \begin{cases} \text{the accumulating set of polynomials } \{\sum_{k=0}^{n}\frac{\#\Gamma_{n-k}}{\#\Gamma_n}s^k\}_{n\in\mathbb{Z}_{\geq 0}} \\ \text{in } \mathbb{R}[[s]] \text{ with respect to the classical topology.}\end{cases}$$

Note. Using formula (2.2), we see that the convergence of $\frac{\mathcal{M}(\Gamma_n)}{\#\Gamma_n}$ on a subsequence of $\{n\}_{n\in\mathbb{Z}_{\geq 0}}$ is equivalent to the convergence of $\frac{A(S,\Gamma_n)}{\#\Gamma_n}$ for all $S\in Conf(\Gamma,G)$. Therefore, $\Omega(\Gamma,G)$ and $\Omega(P_{\Gamma,G})$ are closed subset of the

Hilbert cubes $\prod_{S\in Conf_0(\Gamma,G)}\varphi(S)\cdot[0,1]$ and $\prod_{n=0}^{\infty}s^n\cdot[0,1]$, respectively, so that $\Omega(\Gamma,G)$ *and* $\Omega(P_{\Gamma,G})$ *are non-empty compact sets.*[7]

Theorem ([S1, 11.2]). *Let* (Γ,G) *be the Cayley graph of an infinite cancellative monoid with a finite generator system* G *satisfying* **Assumptions H**, **I'** *and* **S** *stated in the following proof.*

1. The correspondence $\sum_S\varphi(S)\cdot a_S\mapsto\sum_k s^k\cdot a_{\Gamma_k}$ *defines a continuous surjective map:*

$$\pi_\Omega:\Omega(\Gamma,G)\to\Omega(P_{\Gamma,G})$$

2. Define maps:

$$\tilde\tau_\Omega:\quad \mathcal{L}_{\mathbb{R}}\to\mathcal{L}_{\mathbb{R}},\qquad \sum_{S\in Conf_0}\varphi(S)\cdot a_S\mapsto\frac{1}{a_{\Gamma_1}}\sum_{S\in Conf_0}\varphi(S)\cdot a_{S\Gamma_1}$$

$$\tau_\Omega:\quad \mathbb{R}[[s]]\to\mathbb{R}[[s]],\qquad \sum_{k=0}^{\infty}s^k\cdot a_k\mapsto\frac{1}{a_1}\sum_{k=0}^{\infty}s^k\cdot a_{k+1},$$

respectively, where i) their domains are restricted to the subspaces $\{a_{\Gamma_1}\neq 0\}$ *and* $\{a_1\neq 0\}$*, respectively, and ii)* $S\Gamma_1$ *for* $S\in Conf_0(\Gamma,G)$ *means the isomorphism class of the graph* $\mathbb{S}\Gamma_1=\cup_{\alpha\in\mathbb{S},\beta\in\Gamma_1}\alpha\beta$ *for a representative* $\mathbb{S}$ *of the configuration* S *(for the well-definedness of* $S\Gamma_1$*, see* **1.** *in the following proof). Then, they induce continuous self-maps:*

$$\tilde\tau_\Omega:\Omega(\Gamma,G)\to\Omega(\Gamma,G)\quad\textit{and}\quad\tau_\Omega:\Omega(P_{\Gamma,G})\to\Omega(P_{\Gamma,G}),$$

respectively, so that the map π_Ω *is equivariant with their actions. That is, we obtain a commutative diagram:*

$$\begin{array}{ccc}\Omega(\Gamma,G) & \xrightarrow{\pi_\Omega} & \Omega(P_{\Gamma,G})\\ \tilde\tau_\Omega\downarrow & & \tau_\Omega\downarrow\\ \Omega(\Gamma,G) & \xrightarrow{\pi_\Omega} & \Omega(P_{\Gamma,G})\end{array}\quad.$$

Proof. Theorem is already proven in [S1, §11.2] under slightly stronger assumptions [S1, §11.1 Assumption 1, §11.2 Assumption 2.], which shall be replaced by **H**, **I'** and **S** given below. Therefore, in the following **1.** and **2**, we only explain new assumptions and sketch how they are used.

1. In [S1, §11.1 Assumption 1.], we assumed that the monoid Γ is embeddable in a group, say $\hat\Gamma$. Assumption 1. was used only to define the right action $\Gamma_1: Conf\to Conf$, $S\mapsto S\Gamma_1$. We shall replace Assumption 1. by the following weaker **Assumption H.**, which is sufficient to define the action of Γ_1, as we shall see in following **Assertion A**. This weakening of the assumption shall be used when we study partition functions for a monoid of class $\mathcal{C}$ in §3.

Assumption H. Assume the condition b) in next **Assertion A**. holds.

[7]Further more, any element $\sum_{S\in Conf_0}\varphi(S)\cdot a_S\in\Omega(\Gamma,G)$ satisfies a constraint $\sum_{S\in Conf_0}(-1)^{\#T-\#S}K(T,S)a_S=0$ for any $T\in Conf_0$ (kabi condition [S1, §11.1]). However, we shall not discuss further on the condition in the present paper.

We shall refer to this as the *homogeneity assumption on* (Γ, G).

Assertion A. *Let Γ be a cancellative monoid generated by a finite set G. Then, in the following, a) implies b), and b) is equivalent to c).*

a) The monoid Γ is embedded into a group, say $\hat{\Gamma}$.

b) Let $U_0, U_1, \cdots, U_n$ and $V_0, V_1, \cdots, V_n$ $(n \in \mathbb{Z}_{\geq 0})$ be two sequences in Γ such that every successive points U_{i-1}, U_i and V_{i-1}, V_i for $i = 1, \cdots, n$ are connected by edges in (Γ, G) of the same label. If $U_0 = U_n$ then $V_0 = V_n$.

c) Any isomorphism $\varphi : \mathbb{S}_1 \simeq \mathbb{S}_2$ between connected subgraphs of (Γ, G) induces an isomorphism $\hat{\varphi} : \mathbb{S}_1\Gamma_1 \simeq \mathbb{S}_2\Gamma_1$ such that $\hat{\varphi}|_{\mathbb{S}_1} = \varphi$.

Proof. a) $\Rightarrow$ b): Regard U_i and V_i for $i = 0, \cdots, n$ as elements in $\hat{\Gamma}$. Then $U_0 = U_n$ implies that $e = U_0^{-1}U_n = (U_0^{-1}U_1)(U_1^{-1}U_2)\cdots(U_{n-1}^{-1}U_n)$ $= (V_0^{-1}V_1)(V_1^{-1}V_2)\cdots(V_{n-1}^{-1}V_n) = V_0^{-1}V_n$ and $V_0 = V_n$.

b) $\Rightarrow$ c): We need to show that i) a map $\hat{\varphi} : \mathbb{S}\Gamma_1 \to \mathbb{S}_2\Gamma_1$ is well-defined by putting $\hat{\varphi}(\alpha\beta) := \varphi(\alpha)\beta$ for $\alpha \in \mathbb{S}$ and $\beta \in G$, and ii) the map $\hat{\varphi}$ is an isomorphism of graphs.

i) By definition of φ, if $\alpha, \alpha\beta \in \mathbb{S}_1$ and $\beta \in G$, then $\varphi(\alpha)\beta = \varphi(\alpha\beta)$. If $\alpha_1\beta_1 = \alpha_2\beta_2 \notin \mathbb{S}_1$ for $\alpha_1, \alpha_2 \in \mathbb{S}_1$ and $\beta_1, \beta_2 \in G \cup G^{-1}$ where at least one of β_1 or β_2 belongs to G, then we need to show $\varphi(\alpha_1)\beta_1 = \varphi(\alpha_2)\beta_2$. Since $\mathbb{S}_1$ is a connected graph, there is a sequence of points $U_1 := \alpha_1, U_2, \cdots, U_{n-1} := \alpha_2$ which are successively adjacent to each other by an element of $G \cup G^{-1}$. Then, put $U_0 := U_1\beta_1$, $U_n := \alpha_2\beta_2$ and $V_0 := \varphi(\alpha_1)\beta_1, V_1 = \varphi(U_1), \cdots, V_{n-1} = \varphi(\alpha_2), V_n := \varphi(\alpha_2)\beta_2$, we obtain two sequences satisfying the assumption in b). Then b) says that $V_0 = V_n$ i.e. $\varphi(\alpha_1)\beta_1 = \varphi(\alpha_2)\beta_2$. Thus the map $\hat{\varphi}$ is well-defined. By applying the same argument for φ^{-1}, we see that $\hat{\varphi}$ is bijective.

ii) It remains only to show that two distinct points $\alpha_1\beta_1$ and $\alpha_2\beta_2$ in $\mathbb{S}_1\Gamma_1 \setminus \mathbb{S}_1$ is connected by an edge if and only if $\hat{\varphi}(\alpha_1\beta_1)$ and $\hat{\varphi}(\alpha_2\beta_2)$ are connected by the same labeled of edge. This can be verified by comparing the sequence $\alpha_1\beta_1, \alpha_1, \cdots, \alpha_2, \alpha_2\beta_2, \alpha_1\beta_2\gamma$ (here, $\alpha_1, \cdots, \alpha_2$ means a path in $\mathbb{S}_1$ connecting the two points α_1 and α_2, and $\gamma := (\alpha_2\beta_2)^{-1}\alpha_1\beta_1 \in G$, by changing of the role of α_1, β_1 and α_2, β_2 if necessary) and the sequence $\hat{\varphi}(\alpha_1\beta_1), \hat{\varphi}(\alpha_1), \cdots, \hat{\varphi}(\alpha_2\beta_2), \hat{\varphi}(\alpha_2\beta_2)\gamma$. The condition b) says $\hat{\varphi}(\alpha_1\beta_1) = \hat{\varphi}(\alpha_2\beta_2)\gamma$, as desired.

c) $\Rightarrow$ b): We show b) by induction on $n \in \mathbb{Z}_{\geq 0}$, where the case $n = 0$ is trivially true. Let two sequences as in b) are given. If $U_i = U_j$ (resp. $V_i = V_j$) for $0 \leq i < j \leq n$ and $(i, j) \neq (0, n)$, then by induction hypothesis, we have $V_i = V_j$ (resp. $U_i = U_j$). Then, applying the induction hypothesis to the shorter sequences $U_0, \cdots, U_i = U_j, \cdots, U_n$ and $V_0, \cdots, V_i = V_j, \cdots, V_n$, we obtain $V_0 = V_n$. Thus, we may assume $U_0, \cdots, U_n$ and $V_0, \cdots, V_n$ are mutually distinct except for $U_0 = U_n$ and possible $V_0 = V_n$.

Suppose we have $U_i = U_j\beta$ (resp. $V_i = V_j\beta$) for $\beta \in G$ and $0 \le i, j \le n$ such that $|i-j| \neq 1$, $\{i,j\} \neq \{0,n\}, \{0,n-1\}$ or $\{1,n\}$, then by applying the induction hypothesis to the sequences $U_i, \cdots, U_j, U_j\beta$ and $V_i, \cdots, V_j, V_j\beta$, we get $V_i = V_j\beta$ (resp. $U_i = U_j\beta$).

i) Case $\beta := U_{n-1}^{-1}U_n \in G$. We have the natural isomorphism $\varphi : \mathbb{S}_1 = \{U_1, \cdots, U_{n-1}\} \simeq \mathbb{S}_2 = \{V_1, \cdots, V_{n-1}\}, U_i \mapsto V_i$ $(i = 1, \cdots, n-1)$. The c) implies the existence of an isomorphism $\hat{\varphi} : \mathbb{S}_1\Gamma_1 \simeq \mathbb{S}_2\Gamma_1$. It then implies $\hat{\varphi}(U_0 = U_n) = \hat{\varphi}(U_{n-1}\beta) = \varphi(U_{n-1})\beta = V_{n-1}\beta = V_n$. Since the vertex U_0 is connected with U_1 by an edge, so is the vertex $\hat{\varphi}(U_0 = U_n) = V_n$ with $\varphi(U_1) = V_1$. That is, in $\mathbb{S}_2\Gamma_1$ two vertices V_n and V_0 are connected with V_1 by the same labeled edges, then the left cancellation implies $V_n = V_0$.

ii) Case $\beta := U_n^{-1}U_{n-1} \in G$. Applying c) to the isomorphic graphs $\mathbb{S}_1 = \{U_0, \cdots, U_{n-2}\}$ and $\mathbb{S}_2 = \{V_0, \cdots, V_{n-2}\}$, isomorphism $\hat{\varphi} : \mathbb{S}_1\Gamma_1 \simeq \mathbb{S}_2\Gamma_1$ implies $\hat{\varphi}(U_{n-1}) = \varphi(U_{n-2})U_{n-2}^{-1}U_{n-1} = V_{n-1}$. On the other hand $U_0 = U_n$ implies $U_{n-1} = U_0\beta$ and hence $\hat{\varphi}(U_{n-1}) = \varphi(U_0)\beta = V_0\beta$. That is, two vetices V_0 and V_n are connected with V_{n-1} by edges of the same type β. Then the left cancellation by β implies $V_0 = V_n$. □

2. In [S1, §11.2 Assumption 2.], the following two were assumed.

Assumption I. Let $\mathbb{S}$ be a connected finite subgraph of (Γ, G). If an equality $\mathbb{S}\Gamma_1 = g\mathbb{S}\Gamma_1$ for $g \in \hat{\Gamma}$ holds then $\mathbb{S} = g\mathbb{S}$ holds, where $\hat{\Gamma}$ is a group in which Γ is embedded by Assumption 1.

Assumption S. Define the set of dead elements[8] by

$$D(\Gamma, G) := \{g \in \Gamma \mid l(g\alpha) \le l(g) \quad \forall \alpha \in G\}. \tag{2.7}$$

Then the ratio $\frac{\#(\Gamma_n \cap D(\Gamma,G))}{\#\Gamma_n}$ tends to 0 as $n \to \infty$.

Note. If Γ is homogeneous with respect to G, then $D(\Gamma, G) = \emptyset$. Therefore, **Assumption S.** is automatically satisfied.

Since we removed Assumption 1. that Γ is embeddable into a group $\hat{\Gamma}$, we need to reformulate **I** in the following form **I'** without using $\hat{\Gamma}$.

Assumption I'. Let $\mathbb{S}$ and $\mathbb{S}'$ be isomorphic finite connected subgraphs of (Γ, G). Then, any isomorphism $\hat{\varphi} : \mathbb{S}\Gamma_1 \simeq \mathbb{S}'\Gamma_1$ induces $\hat{\varphi}|_{\mathbb{S}} : \mathbb{S} \simeq \mathbb{S}'$.

[8]Since Γ may not be a group and we do not assume $G = G^{-1}$, we should note that our definition of dead elements is different from the followings

$$D_0(\Gamma, G) := \{g \in \Gamma \mid l(h) \le l(g) \ \ \forall h \in \Gamma \text{ s.t. } h = g\alpha \text{ or } g = h\alpha \ \exists \alpha \in G\}.$$
$$D_1(\Gamma, G) := \{g \in \Gamma \mid d(h) \le d(g) \ \ \forall h \in \Gamma \text{ s.t. } h = g\alpha \text{ or } g = h\alpha \ \exists \alpha \in G\}.$$

Actually, Assumption **I.** was used in [S1] only once at the proof of the following formula (2.8), which we will prove now by assuming only **H** and **I'** but not **Assumption 1** and **I**.

Formula. For $S \in Conf_0(\Gamma, G)$ *and* $n \in \mathbb{Z}_{>0}$, *we have*

$$0 \le A(S\Gamma_1, \Gamma_n) - A(S, \Gamma_{n-1}) \le \#S \cdot \#(\dot{\Gamma}_n \cap D(\Gamma, G)). \tag{2.8}$$

Proof of (2.8). The proof is parallel to that of [S1, §11.2.10]. For a sake of completeness of the present paper, we sketch it.

Assumption I' implies that the map $\cdot\Gamma_1 : \mathbb{A}(S, \Gamma_{n-1}) \to \mathbb{A}(S\Gamma_1, \Gamma_n)$ is injective. This implies the first inequality.

On the other hand, any element $\mathbb{T} \in \mathbb{A}(S\Gamma_1, \Gamma_n)$ is of the form $\mathbb{S}\Gamma_1$ for a graph $\mathbb{S} \in \mathbb{A}(S, \Gamma_n)$ (*Proof.* Fix $\mathbb{S}_0$ with $[\mathbb{S}_0] = S$. Put $\mathbb{S} :=$ the image of $\mathbb{S}_0$ by an isomorphism $\mathbb{S}_0\Gamma_1 \simeq \mathbb{T}$. Then $\mathbb{S} \subset \mathbb{T} \subset \Gamma_n$).

If an element $\mathbb{S}\Gamma_1 \in \mathbb{A}(S\Gamma_1, \Gamma_n)$ with $[\mathbb{S}] = S$ is not in the Γ_1-image from $\mathbb{A}(S, \Gamma_{n-1})$, i.e. $\mathbb{S} \not\subset \Gamma_{n-1}$ then $\mathbb{S} \cap \dot{\Gamma}_n \cap D(\Gamma, G) \neq \emptyset$. Let $\varphi : \mathbb{S}_0 \simeq \mathbb{S}$ be an isomorphism. Choose points $d \in \mathbb{S} \cap \dot{\Gamma}_n \cap D(\Gamma, G)$ and $s := \varphi^{-1}(d) \in \mathbb{S}_0$. Then, due to **Assumption H.** and the connectedness of $\mathbb{S}_0$, a pointed graph $(\mathbb{S}, d)$ is uniquely determined (if it exists) as the isomorphic image of the pointed graph $(\mathbb{S}_0, s)$, where the choice depends only on $(s, d) \in \mathbb{S}_0 \times (\dot{\Gamma}_n \cap D(\Gamma, G))$. That is, the number of $\mathbb{S}\Gamma_1 \in \mathbb{A}(S, \Gamma_n)$ with $\mathbb{S} \not\subset \Gamma_{n-1}$ is at most $\#(S) \cdot \#(\dot{\Gamma}_n \cap D(\Gamma, G))$.

This proves the the second inequality of (2.8). □

Let us check how the inequality (2.8) together with **Assumption S.** imply the existence of the surjective map π_Ω. First, we see easily that (2.8) implies an inequality [S1, §11.2.11]:

$$0 \le A(\Gamma_k, \Gamma) - \#\Gamma_{n-k} \le \#(\Gamma_{k-1})\#(\Gamma_n \cap D(\Gamma, G))$$

Then, one has $0 \le \frac{A(\Gamma_k,\Gamma)}{\#\Gamma_n} - \frac{\#\Gamma_{n-k}}{\#\Gamma_n} \le \#\Gamma_{k-1}\frac{\#(\Gamma_n \cap D(\Gamma,G))}{\Gamma_n}$, where **Assumption S.** implies that the right hand side converges to 0 for any sub-sequence of $\{n\}_{n\in\mathbb{Z}_{\ge 0}}$ tending to infinity. Thus, the convergence of the first term to a_{Γ_k} implies the convergence of the second term to a_k such that $a_{\Gamma_k} = a_k$. This implies the map π_Ω is well defined. Surjectivity of the map π_Ω follows from the compactness of $\Omega(\Gamma, G)$: since for any sub-sequence $\{n\}_{n\in\mathbb{Z}_{\ge 0}}$ tending to infinity such that $\frac{\#\Gamma_{n-k}}{\#\Gamma_n}$ converges for all $k \in \mathbb{Z}_{\ge 0}$, we can choose sub-sequence of the subsequence such that $\frac{A(S,\Gamma_n)}{\#\Gamma_n}$ converges for all $S \in Conf(\Gamma, G)$.

In order to show $\tilde{\tau}_\Omega(\Omega(\Gamma,G))\subset\Omega(\Gamma,G)$ and $\tau_\Omega(\Omega(P_{\Gamma,G)})\subset\Omega(P_{\Gamma,G)})$, using again the formula (2.8) and **Assumption S.**, we show that

$$\begin{aligned}\tilde{\tau}_\Omega\big(\lim_{m\to\infty}\tfrac{\mathcal{M}(\Gamma_{n_m})}{\#\Gamma_{n_m}}\big) &= \lim_{m\to\infty}\tfrac{\mathcal{M}(\Gamma_{n_m-1})}{\#\Gamma_{n_m-1}}\\ \tau_\Omega\big(\lim_{m\to\infty}\textstyle\sum_{k=0}^{n_m}\tfrac{\#\Gamma_{n_m-k}}{\#\Gamma_{n_m}}s^k\big) &= \lim_{m\to\infty}\textstyle\sum_{k=0}^{n_m-1}\tfrac{\#\Gamma_{n_m-1-k}}{\#\Gamma_{n_m-1}}s^k.\end{aligned}$$

For the details of the proof, we refer to [S1, §11.2].

The equivariance of π_Ω with the actions $\tilde{\tau}_\Omega$ and τ_Ω is trivial since $\Gamma_k\Gamma_1=\Gamma_{k+1}$ for $k\in\mathbb{Z}_{\geq 0}$.

This completes the proof of the Theorem. □

Question. Do the conditions a), b) and c) in **Assertion A.** equivalent? Precisely, does b) imply a)? That is, do b) and c) give characterizations of the embeddability of a monoid Γ into a group?

Next in the remaining part of the present section, we introduce the *growth partition functions* and discuss some of its descriptions. For the definition, we do not need either **Assumptions H.**, **I'.** nor **S**. Therefore, until the end of this section 2, we assume only the cancellativity on Γ.

Let us consider two growth series in the variable t:

$$P_{\Gamma,G}(t) := \sum_{n=0}^{\infty}\#\Gamma_n\cdot t^n \tag{2.9}$$

$$P_{\Gamma,G}\mathcal{M}(t) := \sum_{n=0}^{\infty}\mathcal{M}(\Gamma_n)\cdot t^n \tag{2.10}$$

where the first one is the usual growth function introduced by Milnor [M] as an element of $\mathbb{Z}[[t]]$, and the second one is a growth series, introduced in [S1, (11.2.7)] as an element in $\mathcal{L}_{\mathbb{Z}[[t]]}(\Gamma,G)$.

Definition. The *growth partition function* of (Γ,G) is the series

$$Z_{\Gamma,G}(t) := \frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)} \tag{2.11}$$

Since the initial term of $P_{\Gamma,G}(t)$ is $\#\Gamma_0=1$, the growth function is invertible in $\mathbb{Z}[[t]]$ so that $Z_{\Gamma,G}(t)\in\mathcal{L}_{\mathbb{Z}[[t]]}(\Gamma,G)$.

The following is an elementary remark.

Assertion. The growth partition function has a development

$$Z_{\Gamma,G}(t) = \sum_{S\in Conf_0(\Gamma,G)}\varphi(S)\cdot Z_{\Gamma,G}(S,t). \tag{2.12}$$

with respect to integral basis $\{\varphi(S)\}_{S\in Conf_0(\Gamma,G)}$*, where*

$$Z_{\Gamma,G}(S,t) := \frac{P_{\Gamma,G}A(S,t)}{P_{\Gamma,G}(t)} \tag{2.13}$$

$$P_{\Gamma,G}A(S,t) := \sum_{n=0}^{\infty} A(S,\Gamma_n)\cdot t^n \tag{2.14}$$

(recall $A(S,\Gamma_n)$ *:=the number of subgraphs in* Γ_n *isomorphic to* S*).*

Proof. Apply (2.2) for $T=\Gamma_n$, and, (2.10) together, we get

$$P_{\Gamma,G}\mathcal{M}(t) = \sum_{S\in Conf_0(\Gamma,G)} \varphi(S)\cdot P_{\Gamma,G}A(S,t). \tag{2.15}$$

This together with (2.13) implies (2.12). □

It was shown [S1, 10.6] that $P_{\Gamma,G}A(S,t)$ and $P_{\Gamma,G}(t)$ have same radius , say $r_{\Gamma,G}$, of convergence, so that the growth partition function converges at least in the radius $r_{\Gamma,G}$.

Conjecture 1. The growth partition function $Z_{\Gamma,G}(t)$ has the radius of convergence larger than that $r_{\Gamma,G}$ of the growth function $P_{\Gamma,G}(t)$.

Let us observe that the conjecture is true if $P_{\Gamma,G}(t)$ belongs to $\mathbb{C}\{t\}_{r_{\Gamma,G}}$, where we recall that

$$\mathbb{C}\{t\}_r := \begin{cases} \text{the set of all power series in } t \text{ of radius of convergence} \\ \text{greater or equal than } r \text{ which analytically continued} \\ \text{as meromorphic functions on a neighborhood of the closed} \\ \text{disc of radius } r \text{ centered at the origin } 0. \end{cases}$$

Conjecture 2. If the growth function $P_{\Gamma,G}(t)$ is a rational function in t, then the partition function coefficient $Z_{\Gamma,G}(S,t)$ for any $S\in Conf_0(\Gamma,G)$ is a rational function in t, whose order at infinity is bounded by $L(S) := \min\{n\in\mathbb{Z}_{>0} \mid A(S,\Gamma_n)\neq 0\}$.

Example. Let F_f be a free group generated by $G_f=\{g_1^{\pm1},\cdots,g_f^{\pm1}\}$ for $f\in\mathbb{Z}_{\geq 0}$. The growth partition function for (F_f,G_f) for $f\geq 2$ is

$$Z_{F_f,G_f}(t) = \sum_{\substack{S\in Conf_0(F_f,G_f),\\ d(S)\text{ even.}}} \varphi(S)t^{[d(S)/2]} + \frac{2t}{1+t}\sum_{\substack{S\in Conf_0(F_f,G_f),\\ d(S)\text{ odd.}}} \varphi(S)t^{[d(S)/2]}. \tag{2.16}$$

where $d(S) := max\{d(x,y)\mid x,y\in S\}$ for $S\in Conf_0(\Gamma,G)$.

Proof. For $n \geq [d(S)/2]$, the following formula holds [S1, §11.1]:

$$(2.17) \qquad A(S,\Gamma_n) \;=\; \begin{cases} \frac{f(2f-1)^{n-[d(S)/2]}-1}{f-1} & \text{if } d(S) \text{ is even,} \\ \frac{(2f-1)^{n-[d(S)/2]}-1}{f-1} & \text{if } d(S) \text{ is odd.} \end{cases}$$

Thus, in view of the fact that $A(S,\Gamma_n) = 0$ for $n < [d(S)/2]$, we calculate the growth function for S as follows.

$$(2.18) \qquad P_{F_f,G_f}A(S,t) = \begin{cases} t^{[d(S)/2]}\frac{1+t}{(1-t)(1-(2f-1)t)} & \text{if } d(S) \text{ is even,} \\ t^{[d(S)/2]}\frac{2t}{(1-t)(1-(2f-1)t)} & \text{if } d(S) \text{ is odd.} \end{cases}$$

In particular, $\#\Gamma_n = \frac{f(2f-1)^n-1}{f-1}$ and $P_{F_f,G_f}(t) = \frac{1+t}{(1-t)(1-(2f-1)t)}$.

As a consequence, we obtain

$$(2.19) \qquad Z_{F_f,G_f}(S,t) = \begin{cases} t^{[d(S)/2]} & \text{if } d(S) \text{ is even,} \\ t^{[d(S)/2]}\frac{2t}{1+t} & \text{if } d(S) \text{ is odd.} \end{cases}$$

Combining this with the formula (2.12), we obtain Formula (2.16). □

Remark. Specializing the growth partition function at the two places $t = 1/(2f-1) = r_{F_f,G_f}$ and $t = 1$ of poles of $P_{F_f,G_f}(t)$, we obtain:

$$(2.20) \qquad \omega_{F_f,G_f} = \sum_{\substack{S\in Conf_0(F_f,G_f),\\ d(S) \text{ even.}}} \frac{\varphi(S)}{(2f-1)^{[d(S)/2]}} + \frac{1}{f}\sum_{\substack{S\in Conf_0(F_f,G_f),\\ d(S) \text{ odd.}}} \frac{\varphi(S)}{(2f-1)^{[d(S)/2]}}$$

$$(2.21) \qquad \omega_{F_f,1} = \sum_{S\in Conf_0(F_f,G_f)} \varphi(S).$$

where the first formula coincides with the partition function for (F_f, G_f) already directly (without using $Z_{F_f,G_f}(t)$) calculated in [S1, §11.1.9].

3. Monoids of class $\mathcal{C}$

We consider a class, which we call $\mathcal{C}$, of cancellative homogeneous monoids with respect to finite generator system, admitting conditions GCD_l and CM_r. Any configuration S of a monoid of class $\mathcal{C}$ admits a unique minimal representative, whose "radius" $L(S)$ is a numerical invariant of S. Then, the growth partition function for the monoid of class $\mathcal{C}$ is a sum of the main term calculated by the invariant L and the additional term coming from dead elements.

Let Γ be a cancellative monoid. Let us consider conditions on Γ:

GCD_l: For any two elements u, v of Γ, there exists a unique maximal common left divisor $\gcd_l(u,v) \in \Gamma$ of them. That is, $\gcd_l(u,v)|_l u$ and $\gcd_l(u,v)|_l v$, and if $w|_l u$ and $w|_l v$ for $w \in \Gamma$ then $w|\gcd_l(u,v)$.

CM$_r$: For any two elements u, v of Γ, there exists a common right multiple of them. That is, there exists $w \in \Gamma$ such that $u|_r w$ and $v|_r w$. In the other words, there exists $a, b \in \Gamma$ such that $au = bv$.

Note that uniqueness assumption in GCD$_l$, in particular, asks that no element in Γ except for the unit element e is invertible. In particular, Γ contains no non-trivial subgroups.

Definition. A cancellative infinite homogeneous monoid Γ is called of class $\mathcal{C}$ if it satisfies conditions GCD$_l$ and CM$_r$.

Let us state a general proporty of monoids satisfying condition CM$_r$.

Lemma 1. *Let Γ be a cancellative monoid satisfying condition CM$_r$. Then, for any finite generator system G of Γ, the Cayley graph (Γ, G) satisfies* **Assumption H.** *(recall §2 for a definition).*

Proof. Let $U_0, U_1, \cdots, U_n$ and $V_0, V_1, \cdots, V_n$ $(n \in \mathbb{Z}_{\geq 0})$ be two sequences in Γ as in b) of §2 Assertion A. Let us consider a common right multiple W_0 of U_0 and V_0. That is, there exists $A, B \in \Gamma$ such that $W_0 = AU_0 = BV_0$. We now compare two sequences $AU_0, AU_1, \cdots, AU_n$ and $BV_0, BV_1, \cdots, BV_n$ $(n \in \mathbb{Z}_{\geq 0})$ in Γ. Let us show that the two sequences are the same. The initial terms have already the equality $AU_0 = BV_0$. As induction hypothesis, assume $W_k := AU_k = BV_k$ for a k with $0 \leq k < n$. However, by the assumption on the sequences in b), two points AU_{k+1} and BV_{k+1} are connected with W_k by the same type edge. This implies that there exists $\alpha \in G$ such that either $AU_{k+1} = BV_{k+1} = W_k\alpha$ or $AU_{k+1}\alpha = BV_{k+1}\alpha = W_k$. In both cases, we get $AU_{k+1} = BV_{k+1}$. Thus we get finally $AU_n = BV_n$. If $U_0 = U_n$, then $BV_0 = AU_0 = AU_n = BV_n$. The left cancellation by B implies $V_0 = V_n$. □

Corollary. *Monoids of class $\mathcal{C}$ satisfies* **Assumption H**.

Remark. If Γ satisfies both CM$_r$ and CM$_l$ simultaneously, then, together with the cancellativity, we know that Γ is injectively embedded into its localization group $\hat{\Gamma}$ of Γ (Öre's criterion), implying **Assumption H.** We will observe in [S4] that CM$_r$ alone together with cancellativity is sufficient not only to get **Assumption H.**, but leads to an embedding of Γ into a "homogeneous set" $\hat{\Gamma}$ (which may no longer have a group structure), where we define the growth partition function for $\hat{\Gamma}$ (since for a definition of partition functions and growth partition functions, group structure is unnecessary [S1]).

Let us return to the study of monoids of class $\mathcal{C}$.

Lemma 2. *Let* (Γ, G) *be of class* $\mathcal{C}$. *Then, for any configuration* $S \in Conf_0(\Gamma, G)$, *there exists a unique subgraph* $\mathbb{S}_0$ *of* (Γ, G) *such that i)* $[\mathbb{S}_0] = S$ *and ii)* $\mathrm{gcd}_l(\mathbb{S}_0) = e$. *In particular, these imply iii)* $\mathrm{Aut}(\mathbb{S}_0) = 1$, *and iv) for any subgraph* $\mathbb{S}$ *with* $[\mathbb{S}] = S$ *we have* $\mathbb{S} = \mathrm{gcd}_l(\mathbb{S})\ \mathbb{S}_0$.

Proof. Let $\mathbb{S}$ and $\mathbb{T}$ be any two presentative of the class S. That is, there is an isomorphism $\varphi : \mathbb{S} \simeq \mathbb{T}$. Choose any element $u \in \mathbb{S}$. Let $w \in \Gamma$ be a common right multiple of u and $\varphi(u)$. That is, there are $a, b \in \Gamma$ such that $w = au = b\varphi(u)$. Then, let us show that $a\mathbb{S} = b\mathbb{T}$ (*Proof.* It is sufficient to show that we have $ax = b\varphi(x)$ for any $x \in \mathbb{S}$. But, this can be shown by induction on the distance inside the graph $\mathbb{S}$ of x from u by using the cancellativity and **Assumption H.** of Γ.)

The uniqueness of the gcd_l of elements of $a\mathbb{S} = b\mathbb{T}$ implies the equality: $a\ \mathrm{gcd}_l(\mathbb{S}) = b\ \mathrm{gcd}_l(\mathbb{T})$. This implies the relation: $\mathrm{gcd}_l(\mathbb{S})^{-1}\mathbb{S} = \mathrm{gcd}_l(\mathbb{T})^{-1}\mathbb{T}$. This means that $\mathbb{S}_0 := \mathrm{gcd}_l(\mathbb{S})^{-1}\mathbb{S}$ does not depend on the choice of a representative $\mathbb{S}$ of S. Thus, i), ii) and iv) are proven.

Suppose that there is an automorphism φ of $\mathbb{S}_0$. Then, applying the same argument above, consider a common right multiple $au = b\varphi(u)$ for an element $u \in \mathbb{S}_0$. Then the automorphism φ is realized by $\mathbb{S}_0 \overset{\cdot a}{\sim} a\mathbb{S}_0 = b\mathbb{S}_0 \overset{/b}{\sim} \mathbb{S}_0$. Again, the uniqueness of GCD_l implies $a = b$ and $\varphi = 1$. □

Corollary. *Let* (Γ, G) *be of type* $\mathcal{C}$. *Then, for any configuration* $S \in Conf_0(\Gamma, G)$, *the automorphism group* $\mathrm{Aut}(S)$ *is trivial. In particular,* (Γ, G) *satisfies* **Assumption I'**.

Let us call $\mathbb{S}_0$ in Lemma 2. the *minimal representative* of $S \in Conf(\Gamma, G)$. Using the minimal representative, we introduce a numerical invariant for S, which is to present the growth partition function.

Notation. For $S \in Conf_0(\Gamma, G)$, put

$$L(S) := \max\{l(u) \mid u \in \mathbb{S}_0\}. \tag{3.22}$$

Formula. The growth partition function for a pair of a monoid Γ of class $\mathcal{C}$ and a finite generator system G is given by

$$Z_{\Gamma,G}(t) = \sum_{S \in Conf_0(\Gamma,G)} \varphi(S)\ t^{L(S)}. \tag{3.23}$$

Proof. Let us denote by $\mathbb{A}(S, \Gamma_n)$ the set of subgraphs of Γ_n whose isomorphism class is equal to S (recall §2).

Lemma 3. *For* $S \in Conf_0(\Gamma, G)$, *let* $\mathbb{S}_0$ *be the minimal representative of* S. *Then, for* $n \in \mathbb{Z}_{\geq 0}$, *we have a natural bijection:*

$$\Gamma_n \simeq \mathbb{A}(S, \Gamma_{n+L(S)}), \quad g \mapsto g\mathbb{S}_0. \tag{3.24}$$

Proof. The correspondence is well-defined and is injective due to the uniqueness in Lemma 1. Surjectivity is also clear from homogeneity of (Γ, G), since if $\mathbb{S} \in \mathbb{A}(S, \Gamma_{n+L(S)})$ then, again by Lemma 1, we have $\mathbb{S} = \gcd_l(\mathbb{S}) \cdot \mathbb{S}_0$, where $n + L(S) \geq \max\{l(u) \mid u \in \mathbb{S}\} = l(\gcd_l(\mathbb{S})) + L(S)$ implies $n \geq l(\gcd_l(\mathbb{S}))$ and $\gcd_l(\mathbb{S}) \in \Gamma_n$. □

Corollary 1. *Under the same assumptions, we have the equality:*

$$P_{\Gamma,G}A(S,t) = t^{L(S)} \cdot P_{\Gamma,G}(t) \tag{3.25}$$

$$Z_{\Gamma,G}(S,t) = t^{L(S)}. \tag{3.26}$$

Proof. We have $A(S,\Gamma_n) = 0$ if $n < L(S)$ and $\#\Gamma_{n-L(S)}$ if $n \geq L(S)$. □

Corollary 1. together with (1.9) implies Formula (3.23). □

As an application of Lemma 3., let us state about the map π_Ω (§2).

Corollary 2. *Let* (Γ, G) *be of class* $\mathcal{C}$*. Then, the map* $\pi_\Omega : \Omega(\Gamma, G) \to \Omega(P_{\Gamma,G})$ *is a bijection.*

Proof. Generally, π_Ω is surjective under **Assumption S.**, which is automatically satisfied for homogeneous (Γ, G) (recall *Note.* after (2.7)).

For injectivity, we need to show that if the opposite polynomials $X_n(P_{\Gamma,G}) := \sum_{k=0}^{n} \frac{\#\Gamma_{n-k}}{\#\Gamma_n} s^k$ (recall §2 Definition for $\Omega(\Gamma, G)$, [S1, §11.2.2]) for a subsequence $\{n_m\}_{m \in \mathbb{Z}_{\geq 0}}$ converges, then the free energies $\mathcal{M}(\Gamma_n)/\#\Gamma_n$ should converge also for the same subsequence $\{n_m\}_{m \in \mathbb{Z}_{\geq 0}}$. Actually, using (2.1), for the convergence of the sequence $\mathcal{M}(\Gamma_n)/\#\Gamma_n$, it is sufficient to show the convergence of $A(S, \Gamma_n)/\#\Gamma_n$ for all $S \in Conf_0(\Gamma, G)$ for the same sequence $\{n_m\}_{m \in \mathbb{Z}_{\geq 0}}$.

But it was shown (see §2) that, under **Assumptions I'.** and **S**, the sequence $A(\Gamma_k, \Gamma_n)/\#\Gamma_n$ and the sequence $\#\Gamma_{n-k}/\#\Gamma_n$ converge simultaneously for the same sequence $\{n_m\}_{m \in \mathbb{Z}_{\geq 0}}$. Thus, $A(S, \Gamma_n)/\#\Gamma_n = A(\Gamma_{L(S)}, \Gamma_n)/\#\Gamma_n$ for $S \in Conf_0(\Gamma, G)$ (4.23) for $n \geq L(S)$ converges for the same sequence $\{n_m\}_{m \in \mathbb{Z}_{\geq 0}}$. □

Note that the proof of Corollary 2. is independent from whether $\Omega(\Gamma, G)$ and/or $\Omega(P_{\Gamma,G})$ is finite or not, and whether $P_{\Gamma,G}$ is a rational function or not.

4. Artin monoids of finite type

Let G be a finite set of letters and let $M = (m_{\alpha,\beta})_{\alpha,\beta \in G}$ be a Coxeter matrix (i.e. $m_{\alpha,\alpha} = 1$ for $\alpha \in G$ and $m_{\alpha,\beta} = m_{\beta,\alpha} \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$ for $\alpha \neq \beta \in G$). Then, an Artin monoid Γ_M ([B-S]) or a generalized braid monoid ([D]) associated with the Coxeter matrix M is a monoid defined by the positive homogeneous relations

$$\langle \alpha\beta \rangle^{m_{\alpha,\beta}} = \langle \beta\alpha \rangle^{m_{\alpha,\beta}} \quad \text{for} \quad \alpha, \beta \in G$$

on the free monoid generated by the letters in G. Here, we denote by $\langle \alpha\beta \rangle^m$ a word of alternating sequence of letters α and β starting from α of length $m \in \mathbb{Z}_{\geq 0}$. We shall refer to G as the *standard generator system* for the Artin monoid. An Artin monoid is called *of finite type*, if the associated Coxeter group (i.e. the quotient group of Γ_M divided by the relations $\alpha^2 = 1$ for $\alpha \in G$) is finite. Indecomposable Artin monoids of finite type are classified into types A_l $(l \geq 1)$, B_l $(l \geq 2)$, D_l $(l \geq 4)$, E_6, E_7, E_8, F_4, G_2, H_3, H_4 and $I_2(p)$ $(p \geq 3)$. A *braid monoid* $B(n)^+$ of n-strings is isomorphic to an Artin monoid of type A_{n-1}.

Assertion. An Artin monoid of finite type belongs to the class $\mathcal{C}$.

Proof. Clearly, any Artin monoid is homogeneous by the definition.

It is shown ([B-S],[D]) that an Artin monoid is an cancellative infinite monoid, satisfying conditions GCD_l and GCD_r. If, further, it is of finite type, it they satisfies LCM_l and LCM_r, and hence CM_l and CM_r. □

Note. An Artin monoid is known to be embeddable into its group (Paris [P]) so that it satisfies **Assumption H**. However, we shall not use this result in the present paper. See also **Remark 3.** at the end of the paper.

Recall ([S2],[S3]) that the growth function for an Artin monoid Γ_M with respect to the standard generator system G is given by

$$P_M(t) := P_{\Gamma,G}(t) = \frac{1}{N_M(t)}, \tag{4.27}$$

where

$$N_M(t) := \sum_{J \subset G} (-1)^{\#J} t^{\deg(\Delta_J)}. \tag{4.28}$$

Here the summation index J runs over all subsets of G such that the restriction $M|_J := (m_{ij})_{i,j \in J}$ is a Coxeter matirx of finite type, and Δ_J is the fundamental element in the monoid $\Gamma_{M|_J}$ so that $\deg(\Delta_J)$=length of the longest element in the associated Coxeter group ([B-S]).

For an indecomposable Artin monoid of finite type Γ_M, the following (1), (2) and (3) are conjectured [S2].

(1) $\tilde{N}_M(t) := N_M(t)/(1-t)$ is an irreducible polynomial over $\mathbb{Z}$,

(2) there are $\#G - 1$ distinct real roots on the interval $(0, 1)$, and

(3) the smallest real root on the interval (0,1), say $r_{\Gamma,G}$, of $N_M(t) = 0$ is strictly smaller than the absolute value of any other root.

Actually, conjectures are affirmatively solved for types $A_l, B_l = C_l$ and D_l for $l \leq 30$ and $E_6, E_7, E_8, F_4, G_2, H_3, H_4$ and $I_2(p)$ $(p \geq 3)$ by a help of computer. Conjecture (3) is affirmatively solved by Kobayashi-Tsuchioka-Yasuda [K-T-Y] for the types A_l, B_l and D_l for $l \geq M$ for some M, where the author still expect that $M = 1$.

Conjecture 3 implies $\Delta^{top}_{P_{\Gamma,G}}(t) = t - r_{\Gamma,G}$, where $r_{\Gamma,G}$ is the radius of convergence of the series $P_{\Gamma,G}(t)$. As a consequence, the period $h_{\Gamma,G}$ is equal to 1 and $\#\Omega(P_{\Gamma,G}) = 1$ (recall Footnote 3) except for possible finite exceptions in types A_l, B_l or D_l with $30 < l < M$. Together with §3 Corollary 2. to Lemma 3, this implies $\#\Omega(\Gamma, G) = 1$, that is, (Γ, G) is *simple accumulating* in the terminology of [S1, §11.1]. Let us denote by $\omega_{\Gamma,G}$ the single element of $\Omega(\Gamma, G)$. Then, since we have $\omega_{\Gamma,G} = Trace^{[e]}\Omega(\Gamma, G)$ where the class $[e]$ denotes the single element in $\Omega(P_{\Gamma,G})$, this limit element is now calculable from the growth partition function (3.23) by the use of a formula (1.1). Let us describe the other terms in (1.1).

$E =$ a sum of terms depending the root of $\delta := (t^{h_{\Gamma,G}} - r_{\Gamma,G}^{h_{\Gamma,g}})/\Delta^{top}_{\Gamma,G}$
$\quad = 0$ since $\deg(\delta) = 0$ (see [S1, (11.3.6), (11.5.6)]).
$h_{\Gamma,G} = \#(\Omega(P_{\Gamma,G})) = 1$ (Conjecture 3 in [S1], solved by [K-T-Y]).
$m_{\Gamma,G} =$ covering sheet number of $\Omega(\Gamma, G) \to \Omega(P_{\Gamma,G})$
$\quad = 1$ (see §3 Lemma 3, Corollary 2).
$A^{[e]}(s) =$ a polynomial in s of degree $h_{\Gamma,G} - 1$ with a constant term 1
$\quad = 1$, since $h_{\Gamma,G} - 1 = 0$ (see [S1, §11.3.2]).

Finally, substituting in $Z_{\Gamma_M,G}(t)$ (3.23) the single summation index $r_{\Gamma_M,G}$: the smallest real root of the equation $N_M(t) = 0$, we arrive at the goal formula of the present paper.

Theorem. *Artin monoid of finite type, except for finite possible exceptions in types A_l, B_l or D_l, is simple accumulating. That is, $\#\Omega(\Gamma_M, G) = 1$. The partition function is given by*

$$\omega_{\Gamma_M,G} = Z_{\Gamma_M,G}(r_{\Gamma_M,G}) = \sum_{S \in Conf_0(\Gamma_M,G)} \varphi(S)\ r_{\Gamma_M,G}^{L(S)}, \tag{4.29}$$

where $r_{\Gamma_M,G}$ is the smallest real root in the interval (0,1) of the denominator polynomial (4.29).

Remark 1. Let M be an indecomposable Coxeter matrix of finite type. Let us consider the set $\Delta := \{\alpha \in \mathbb{C} \mid \tilde{N}_M(\alpha) = 0\}$ of roots of $N_M(t) = 0$, and, for $\alpha \in \Delta$, put

$$\omega_{M,\alpha} := Z_{\Gamma_M,G}(t)|_{t=\alpha} = \sum_{S \in Conf_0(\Gamma_M,G)} \varphi(S)\ \alpha^{L(S)}. \tag{4.30}$$

It was shown ([S1, §11.4, **4.** Assertion.]) that each $\omega_{M,\alpha}$ belongs to the Lie-like space $\mathcal{L}_{\mathbb{C},\infty}$ at infinity. Then assuming Conjecture (1) in [S2],[S3], the Galois group of the splitting field of $N_M(t)$ acts transitively on the set Δ, inducing also a transitive action on the set $\{\omega_{M,\alpha}\}_{\alpha\in\Delta}$. In particular, the action mixes up the partition function $\omega_{\Gamma,G}$ with the other functions. We do not know the meaning of this action.

2. Above Theorem is valid not only for Artin monoids of finite type but for any monoid (Γ,G) of type $\mathcal{C}$ whose growth function belongs to $\mathbb{C}\{t\}_{r_{\Gamma,G}}$ and has period $h_{\Gamma,G}$ equal to 1 (i.e. $r_{\Gamma,G}$ is the unique pole of $P_{\Gamma,G}$ on the circle $|t| = r_{\Gamma,G}$).

3. Artin monoids of non-finite type do not belong to the class $\mathcal{C}$, since they do not satisfy CM_r. However, we conjectured in [S3, §3 Conjecture 3] that Artin monoids of affine type have the period $h_{\Gamma,G}$ equal to 1. Thus, it seems to be interesting to ask whether §3 **Lemma 2.** of the present paper holds for Artin monoids (in general or, in particular, of affine type) or not.

References

[B-S] E.Brieskorn & K.Saito: Artin-Gruppen und Coxeter-Gruppen, Invent. Math. **17** (1972), 241-271. Zbl 0243.20037 MR 0323910

[D] P.Deligne: Les immeubles des groupes de tresses généralisés, Invent. Math. **17** (1972), 273-302.

[K-T-Y] K.Kobayashi, S.Tsuchioka and S.Yasuda: Partial theta and growth series of Artin monoids of finite type, in preparation.

[M] J.Milnor: A note on curvature and fundamental group, J. Differential Geom. **2** (1968), 1-7. Zbl 0162.25401. MR 0232311

[P] L.Paris: Artin monoids inject in their groups, Comment. Math. Helv. **77** (2002),no.3, 609-637.

[S1] K.Saito: Limit elements in the Configuration Algebra for a Cancellative Monoid, Publ. RIMS Kyoto Univ. **46** (2010), 37-113. DOI 10.2977/PRIMS/2

[S2] K.Saito: Growth functions associated with Artin Monoids of finite type, Proc. Japan Acd. Ser. A **84** (2008), 179-183. Zbl 1159.20330 MR 2105710

[S3] K.Saito: Growth functions associated with Artin Monoids, Proc. Japan Acd. Ser. A **85** (2009), 84-88.

[S4] K.Saito: Embedding of cancellative monoids into groups, in preparation.

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