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BIRATIONAL UNBOUNDEDNESS OF LOG TERMINAL $\mathbb{Q}$-FANO VARIETIES AND RATIONALLY CONNECTED STRICT MORI FIBER SPACES

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ABSTRACT. In this paper, we show that ($\mathbb{Q}$-factorial and log terminal) $\mathbb{Q}$-Fano varieties with Picard number one are birationally unbounded in each dimension ≥ 3. This result has been settled for 3-folds by J. Lin and n-folds with $n \geq 6$ by the author. We also prove that rationally connected Mori fiber spaces are birationally unbounded even if we fix dimensions of both total and base spaces.

1. INTRODUCTION

In this paper, a normal projective variety defined over the filed of complex numbers is said to be a (resp. terminal, resp. canonical) $\mathbb{Q}$-*Fano variety* if it is $\mathbb{Q}$-factorial, log terminal (resp. terminal, resp. canonical) and its anticanonical divisor is ample.

It is known that suitably restricted classes of $\mathbb{Q}$-Fano varieties are bounded. For examples, smooth Fano manifolds of arbitrary dimension are bounded (Kollár-Miyaoka-Mori [11]) and canonical $\mathbb{Q}$-Fano threefolds are bounded (Kollár-Miyaoka-Mori-Takagi [12]). There is a famous conjecture on the boundedness of $\mathbb{Q}$-Fano varieties.

Conjecture 1.1 (Borisov-Alexeev-Borisov)**.** *Fix a number $\varepsilon > 0$. Then $\mathbb{Q}$-Fano varieties with log discrepancies $> \varepsilon$ are bounded.*

This conjecture is proved for surfaces by Alexeev [1] and Nikulin [15], and for toric case by Borisov-Borisov [3].

If we consider every $\mathbb{Q}$-Fano varieties then they are unbounded even in the two dimensional case. We consider the generalized birational version of boundedness.

Definition 1.2. A class $\mathfrak{V}$ of varieties is *birationally bounded* if there is a morphism $\varphi\colon \mathcal{X} \to \mathcal{S}$ between algebraic schemes such that every member of $\mathfrak{V}$ is birational to one of the geometric fibers of φ. We say that $\mathfrak{V}$ is *birationally unbounded* if it is not birationally bounded.

In dimension two, $\mathbb{Q}$-Fano varieties, which are usually called log Del Pezzo surfaces, are rational and hence they are birationally bounded in a trivial sense. This cannot hold anymore in higher dimensional cases. Lin [13] proved that $\mathbb{Q}$-Fano threefolds with Picard number one are birationally unbounded and the author [16] proved the same result in each dimension at least six. Following is one of the main theorems of this paper, which completes the study of birational unboundedness of $\mathbb{Q}$-Fano varieties in arbitrary dimension ≥ 3.

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Theorem 1.3. *Fix $n \geq 3$. Then $\mathbb{Q}$-Fano n-folds with Picard number one are birationally unbounded.*

This implies that we cannot drop the assumption on ε in Conjecture 1.1 even if we replace the boundedness by the birational boundedness. In dimension three, this provides an alternate proof of Lin's result. He constructed an infinite sequence of conic bundles over $\mathbb{P}^2$, which are birational to $\mathbb{Q}$-Fano threefolds, and showed that they are birationally unbounded. As an immediate corollary to Theorem 1.3, smooth rationally connected n-folds are birationally unbounded if $n \geq 3$ since every $\mathbb{Q}$-Fano variety is rationally connected (Hacon-McKernan [7] and Zhang [17]). We can prove a finer result as we will explain below.

Definition 1.4. A normal projective variety X together with a morphism $\phi\colon X \to S$ onto a normal projective variety S is said to be a *Mori fiber space* if

- X has only terminal singularities,
- ϕ has connected fibers,
- $-K_X$ is relatively ample over S, and
- $\dim S < \dim X$.

We say that a Mori fiber space $\phi\colon X \to S$ is *strict* if $\dim S > 0$, that is, X is not a terminal $\mathbb{Q}$-Fano variety with Picard number one. For positive integers n and m with $0 \leq m \leq n-1$, a (n,m)-*Mori fiber space* is a Mori fiber space whose total space has dimension n and whose base space has dimension m.

The minimal model program reduces the birational classification of rationally connected varieties to that of Mori fiber spaces over rationally connected bases. Then the study of n-dimensional Mori fiber spaces can be divided into n cases, namely, (n,m)-Mori fiber spaces for $0 \leq m \leq n-1$. In dimension three, there are three classes: terminal $\mathbb{Q}$-Fano threefolds with Picard number one, conic bundles over rational surfaces and Del Pezzo fiber spaces over $\mathbb{P}^1$. We will construct an infinite sequence of families of (n,m)-Mori fiber spaces for $1 \leq m \leq n-1$ and consider the birational unboundedness of those families.

Theorem 1.5. *Fix $n \geq 3$ and m such that $1 \leq m \leq n-1$. Then n-dimensional smooth Mori fiber spaces over m-dimensional smooth rational varieties are birationally unbounded. In particular, rationally connected (n,m)-Mori fiber spaces are birationally unbounded.*

It follows that n-dimensional rationally connected Mori fiber spaces over m-dimensional bases are birationally unbounded for $m > 0$, while terminal $\mathbb{Q}$-Fano n-folds are conjectured to be bounded. In dimension three, neither conic bundles over rational surfaces nor Del Pezzo fiber spaces over $\mathbb{P}^1$ are birationally bounded while terminal $\mathbb{Q}$-Fano threefolds are bounded.

This paper is organized as follows. In Section 2, we briefly recall Kollár's reduction modulo p method to construct non-ruled varieties as a covering spaces. An important part in this section is to construct a specific invertible sheaf on those covering spaces as a subsheaf of the sheaf of differential $(n-1)$-forms, where n is the dimension of the covering space. In Section 3, we give a criterion for such a specific invertible sheaf to be birationally invariant. In Section 4, we construct an infinite sequence of families of (n,m)-Mori fiber spaces for $n \geq 3$ and $2 \leq m \leq n-1$, and study their properties especially when the ground field $\Bbbk$ has characteristic 2. Those (n,m)-Mori fiber spaces are obtained by blowing the singular loci of suitable $\mathbb{Q}$-Fano weighted hypersurfaces. In Section 5, we construct an infinite sequence of families of $(n,1)$-Mori fiber spaces (i.e. conic bundles) as double covers of suitable toric varieties for $n \geq 3$. The birational invariance of the specific invertible sheaf enables us to bound the dimensions of birationally trivial subfamilies of the families constructed in Sections 4 and 5, which is a key to the proof of main theorems. We prove main theorems in Section 6. On the one hand, we show that if (n,m)-Mori fiber spaces defined over $\mathbb{C}$ are birationally bounded then there are "large" birationally trivial families of families of (n,m)-Mori fiber spaces defined over $\Bbbk$ which are constructed in Sections 4 or 5. On the other hand, explicit computations of the dimensions of the birationally trivial subfamilies show that they are not so "large", which completes the proof of main theorems.

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2. Preliminaries

In this section, we recall results of Kollár from [8], [9] and [10] on the construction of a specific line bundle on suitable cyclic covering spaces, and then we partially generalize the argument. In this section, we work over an algebraically closed field $\Bbbk$ of characteristic $p > 0$.

For an invertible sheaf $\mathcal{N}$ on a scheme and a positive integer k, we write $\mathcal{N}^k$ and $\mathcal{N}^{-k}$ instead of $\mathcal{N}^{\otimes k}$ and $(\mathcal{N}^{-1})^{\otimes k}$, respectively.

Let us fix notation which we assume throughout the present section. Let X be a smooth variety of dimension $n \geq 3$ over $\Bbbk$, $\mathcal{L}$ a line bundle on X, $m > 0$ an integer divisible by p and s a global section of $\mathcal{L}^m$. We denote by $\pi\colon W \to X$ the total space of $\mathcal{L}$. We have

$$\pi_*\pi^*\mathcal{L} = \mathcal{L} \otimes \pi_*\mathcal{O}_W = \mathcal{L} \oplus \mathcal{O}_X \oplus \mathcal{L}^{-1} \oplus \mathcal{L}^{-2} \oplus \cdots .$$

Let w be the global section of $\pi^*\mathcal{L}$ which corresponds to $1 \in \mathcal{O}_X$ and we define $Y = X[\sqrt[m]{s}]$ to be the subscheme of W which is the zero locus of the global section $w^m - \pi^*s$ of $\pi^*\mathcal{L}^m$. With a slight abuse of notation, we also denote by $\pi\colon X[\sqrt[m]{s}] \to X$ the restriction of $W \to X$. We call $X[\sqrt[m]{s}]$ the *covering of X obtained by taking m-th root of s*.

2.1. Cyclic covering method.

For reader's convenience, we collect here some definitions and results which are due to Kollár without proofs.

Definition-Lemma 2.1 (Definition-Lemma V.5.4, [9])**.** There is a natural differential

$$d\colon \mathcal{L}^m \to \mathcal{L}^m \otimes \Omega^1_X,$$

constructed as follows. Let τ be a local generator of $\mathcal{L}$, $t = f\tau^m$ a local section of $\mathcal{L}^m$, and the x_i local coordinates. Set

$$d(t) := \sum \frac{\partial f}{\partial x_i} \tau^m dx_i.$$

This is independent of the choices made and thus defines d.

For the global section s of $\mathcal{L}^m$, we can view $d(s)$ as a sheaf homomorphism $\mathcal{O}_X \to \mathcal{L}^m \otimes \Omega^1_X$. Taking a Tensor product with $\mathcal{L}^{-m}$, we obtain $ds \colon \mathcal{L}^{-m} \to \Omega^1_X$.

Lemma 2.2 (Lemma V.5.3, [9]). (1) *There is an exact sequence*

$$0 \to \pi^*\Omega^1_X \to \Omega^1_W|_Y \to \pi^*\mathcal{L}^{-1} \to 0.$$

(2) *We have $\mathcal{O}_Y(-Y) \cong \pi^*\mathcal{L}^{-m}$ and there is an exact sequence*

$$\pi_Y^*\mathcal{L}^{-m} \xrightarrow{d_Y} \Omega^1_W|_Y \to \Omega^1_Y \to 0.$$

(3) *The image of d_Y is contained in $\pi^*\Omega^1_X$ and $d_Y \colon \pi^*\mathcal{L}^{-m} \to \pi^*\Omega^1_X$ coincides with $-\pi^*ds$.*

Definition 2.3. We define $\mathcal{F} = \mathcal{F}(\mathcal{L}, s) := \operatorname{Coker}(ds)$. We denote by $\mathcal{M} = \mathcal{M}(\mathcal{L}, s)$ the double dual of the sheaf $\bigwedge^{n-1}\mathcal{F}$ and by $q \colon \Omega^{n-1}_X \to \mathcal{M}$ the natural map.

Lemma 2.4. *We have an isomorphism $\mathcal{M} \cong \omega_X \otimes \mathcal{L}^m$ and an injection $\pi^*\mathcal{M} \hookrightarrow (\Omega^{n-1}_X)^{\vee\vee}$.*

Proof. By Lemma 2.2, we have an exact sequence

$$0 \to \operatorname{Coker}\left[\pi^*\mathcal{L}^{-m} \xrightarrow{d_Y} \pi^*\Omega^1_X\right] \to \Omega^1_Y \to \pi^*\mathcal{L}^{-1} \to 0$$

and the sheaf on the left is isomorphic to $\pi^*\mathcal{F}$. This gives rise to an injection $\mathcal{M} = (\bigwedge^{n-1}\mathcal{F})^{\vee\vee} \hookrightarrow (\Omega^{n-1}_X)^{\vee\vee}$. $\square$

Lemma 2.5 (Lemma V.5.9, [9]). *Let $x_1, \ldots, x_n$ be local coordinates of X at a closed point x and write $s = f\tau^m$, where $f \in \mathcal{O}_{X,x}$ and τ is a local generator of $\mathcal{L}$. Let*

$$\eta_i = \frac{dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n}{\partial f/\partial x_i}$$

for $i = 1, \ldots, n$ (η_i is undefined if $\partial f/\partial x_i$ is identically zero). Then $q(\eta_i) = \pm q(\eta_j)$ and they give local generators of $\mathcal{M}$.

Let us recall definitions and basic properties of critical points, which are necessary to analyze the singularity of Y.

Definition-Lemma 2.6 (cf. V.5.4, [9]). Let x be a closed point of X and $x_1, \ldots, x_n$ be local coordinates of X at x. We say that s has a *critical point* at x if $d(s) \in \Gamma(\mathcal{L}^m \otimes \Omega^1_X)$ vanishes at x. Assume that s has a critical point at x. Pick a local generator τ of $\mathcal{L}$ at x and write $s = f\tau^m$.

(1) The matrix

$$H(s) := \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$$

is called the *Hessian* of s. The rank of $H(s)$ at a point x is independent of the choices of the local coordinates and the local generator of $\mathcal{L}$.

(2) We say that s has a *nondegenerate critical point* at $x \in X$ if the rank of the Hessian $H(s)(x)$ is n.
(3) If n is even or $p \neq 2$ and n is odd then s has a nondegenerate critical point at x if and only if in suitable local coordinates f can be written as

$$f = c + x_1x_2 + x_3x_4 + \cdots + x_{n-1}x_n + f_3,$$

where $c \in \Bbbk$ and $f_3 \in \mathfrak{m}_x^3$.
(4) If $p = 2$ and n is odd then every critical point is degenerate.
(5) Assume that $p = 2$ and n is odd. A critical point of s is called *almost nondegenerate* if $\operatorname{length} \mathcal{O}_{X,x}/(\partial f/\partial x_1, \ldots, \partial f/\partial x_n) = 2$. Equivalently, in suitable local coordinates f can be written as

$$f = c + ax_1^2 + x_2x_3 + x_4x_5 + \cdots + x_{n-1}x_n + bx_1^3 + f_3,$$

where $a, b, c \in k$, $b \neq 0$, $f_3 \in \mathfrak{m}_x^3$ and the coefficients of x_1^3 in f_3 is 0.

We need to treat the case where critical points are not isolated. Hence we introduce the notion "admissible critical points".

Definition 2.7. Let (X, x) be a germ of a smooth variety and a closed point x. We say that s has an *admissible critical point* at $x \in X$ if we may choose local coordinates $x_1, \ldots, x_n$ of X at x such that, for some $k \geq 3$, s can be written as

$$s = c + \begin{cases} ax_1^2 + x_2x_3 + x_4x_5 + \cdots + x_{k-1}x_k + g, & \text{if } p = 2 \text{ and } k \text{ is odd,} \\ x_1x_2 + x_3x_4 + \cdots + x_{k-1}x_k + g, & \text{if } k \text{ is even,} \end{cases}$$

where $a, c \in \Bbbk$, $g = g(x_1, \ldots, x_n) \in (x_1, \ldots, x_k)^3$ and that the set of critical points of s is precisely the set $(x_1 = \cdots = x_k = 0)$ around x. If $p = 2$ and k is odd then we further require that the coefficient of x_1^3 in g is nonzero.

Note that if s has an isolated critical point at x then it is an admissible critical point if and only if it is an (almost) nondegenerate critical point.

Lemma 2.8. *Let (X, x) be a germ of a smooth variety and a closed point x. Assume that $p = m = 2$, that is, the ground field $\Bbbk$ has characteristic 2 and s is a global section of $\mathcal{L}^2$. If s has an admissible critical point at $x \in X$ then the morphism $r_x \colon Y' \to Y = X[\sqrt{s}]$ obtained by blowing up along the singular locus gives a resolution of singularities of (X, x). Moreover, the injection $\pi^*\mathcal{M} \hookrightarrow (\Omega_Y^{n-1})^{\vee\vee}$ lifts to an injection $r^*\pi^*\mathcal{M} \hookrightarrow \Omega_{Y'}^{n-1}$.*

Proof. Let $x_1, \ldots, x_n$ be local coordinates of X at x and $k \geq 3$ a positive integer for which s can be written as in Definition 2.7. We shall prove the assertion only when k is even. The case where k is odd can be proved similarly. Since Y is defined by the equation $w^2 - s = 0$, we see that the singular locus of Y is exactly $(w = x_1 = \cdots = x_k = 0)$ after replacing w by $w - \sqrt{c}$. Thus Y is defined on a smooth variety W with local coordinates $w, x_1, \ldots, x_n$ by the equation

$$w^2 - (x_1x_2 + x_3x_4 + \cdots + x_{k-1}x_k + g) = 0,$$

where $g \in (x_1, \ldots, x_k)^3$. Let $W' \to W$ be the blow up along $(w = x_1 = \cdots x_k = 0)$ and Y' the strict transform of Y on W'. Then the exceptional divisor of $W' \to W$ is covered by open subsets $U'_1, \dots, U'_k$ and U'_w, where $(x_i = 0)$ (resp. $(w = 0)$) defines the exceptional divisor on U'_i (resp. U'_w).

On U'_1, we may choose coordinates $w', x'_1, \dots, x'_k, x_{k+1}, \dots, x_n$ of W', where $x'_1 = x_1$, $x'_i = x_i/x_1$ for $2 \le i \le k$ and $w' = w/x_1$. Y' is defined on U'_1 by the equation

$$w'^2 - (x'_2 + x'_3 x'_4 + \cdots + x'_{k-1} x'_k + g') = 0,$$

where $g' = g(x'_1, x'_2 x'_1, \dots, x'_k x'_1, x_{k+1}, \dots, x_n)/x'^2_1$ vanishes along the exceptional divisor. It follows that Y' is smooth along U'_1. We can prove that Y' is smooth along U'_i for $i = 1, \dots, k$ similarly.

On U'_w, we may choose coordinates $w', x'_1, \dots, x'_k, x_{k+1}, \dots, x_n$, where $w' = w$ and $x'_i = x_i/w$ for $i = 1, \dots, k$ and Y' is defined by the equation

$$1 - (x'_1 x'_2 + x'_3 x'_4 + \cdots + x'_{k-1} x'_k + g') = 0,$$

where $g' = g(w' x'_1, \dots, w' x'_k, x_{k+1}, \dots, x_n)/w'^2$ vanishes along the exceptional divisor. This shows that Y' is smooth.

By Lemma 2.5, we can explicitly write down local generator $\pi^* \eta_i$ of $\pi^* \mathcal{M}$ using local coordinates $x_1, \dots, x_n$ and it is easy to see that $r_x^* \pi^* \eta_i$ does not have a pole along each exceptional divisor. Thus, we have an injection $r_x^* \pi^* \mathcal{M} \hookrightarrow \Omega^{n-1}_{Y'}$. $\square$

Remark 2.9. Let X° denote the open subset of X which is obtained by removing the set of critical points of s and Y° be the inverse image of X°. Then Y° is smooth and there is an injection $\pi^* \mathcal{L}|_{Y^\circ} \hookrightarrow \mathcal{T}_{Y^\circ}$. This injection can be seen as a foliation and the corresponding quotient is $\pi|_{Y^\circ} \colon Y^\circ \to X^\circ$. We refer the readers to [14, Part I, Lecture III] for a detailed account of foliations in positive characteristics.

2.2. **Non-ruledness criteria.** We collect non-ruledness criteria which are due to Kollár.

Lemma 2.10 (Lemma 7, [8])**.** *Let X be a smooth proper variety and $\mathcal{M}$ a big line bundle on X. Assume that there is an injection $\mathcal{M} \hookrightarrow \Omega^i_X$ for some $i > 0$.*

Then X is not separably uniruled.

Theorem 2.11 (Theorem 3.1.2, [10])**.** *Let $f \colon Y \to X$ be a surjective morphism between smooth proper varieties. Let $\mathcal{M}$ be a big line bundle on X and assume that for some $i > 0$ there is a nonzero map*

$$h \colon f^* \mathcal{M} \to \Omega^i_Y.$$

Let $F = k(X)$ be the field of rational functions on X and Y_F the generic fiber of f. Then there is a one-to-one correspondence between degree d separable unrulings of Y and degree d separable unirulings of Y_F. In particular, Y is ruled if and only if Y_F is ruled over F.

3. Birational invariance of $\pi^* \mathcal{M}$

We will keep notation in the previous section. In this section, we shall give a criterion for $r^* \pi^* \mathcal{M}$ to be birationally invariant assuming that X is a toric variety.

Let N be a lattice, $\mathsf{M} = \mathrm{Hom}_{\mathbb{Z}}(\mathsf{N}, \mathbb{Z})$ its dual and Σ a fan in $\mathsf{N}_{\mathbb{R}} = \mathsf{N} \otimes_{\mathbb{Z}} \mathbb{R}$. Let $X = X_\Sigma$ be the toric variety defined by Σ. In this section, we assume that the ground field $\Bbbk$ is an algebraically closed field of characteristic $p > 0$ and that X is smooth and projective. We denote by $\Sigma(1)$ the one dimensional cones in Σ and, for a cone σ in Σ, we set $\sigma(1) = \{\rho \mid \rho \in \Sigma(1) \text{ and } \rho \subset \sigma\}$. Let r be the Picard number of X and $d = |\Sigma(1)|$. We see that $n = \dim X = d - r$ since X is assumed to be smooth. We define $S = S_\Sigma := \Bbbk[x_\rho \mid \rho \in \Sigma(1)]$ which is a polynomial ring in d variables. For a torus invariant divisor $D = \sum_\rho a_\rho D_\rho$, we can associate the monomial $\prod_\rho x_\rho^{a_\rho}$, which we denote by x^D. We grade S by $\deg(x^D) = [D] \in \mathrm{Div}(X)$, where $\mathrm{Div}(X)$ is the divisor class group. For a divisor class $\alpha \in \mathrm{Div}(X)$, let $S_\alpha = \bigoplus_{\deg(x^D)=\alpha} \Bbbk \cdot x^D$ so that we have $S = \bigoplus_{\alpha \in \mathrm{Div}(X)} S_\alpha$. We call S the *homogeneous coordinate ring* of X.

Let F be a graded S-module, that is, F is a S-module and there is a direct sum decomposition $F = \bigoplus_{\alpha \in \mathrm{Div}(X)} F_\alpha$ such that $S_\alpha \cdot F_\beta \subset F_{\alpha+\beta}$ for all $\alpha, \beta \in \mathrm{Div}(X)$. We can define a sheaf of $\mathcal{O}_X$-modules $\tilde{F}$ as follows. Let $\sigma \in \Sigma$ be a cone and $\sigma^\vee \subset \mathsf{M}_\mathbb{R}$ be its dual cone. Set $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho$. Then there is a natural isomorphism $\Bbbk[\sigma^\vee \cap \mathsf{M}] \cong (S_\sigma)_0$, where S_σ is the localization of S at $x^{\hat{\sigma}}$. It follows that $X_\sigma = \mathrm{Spec}(S_\sigma)_0$ is an affine open subset of X. Put $F_\sigma = F \otimes_S S_\sigma$. Taking degree 0 part, we get a $(S_\sigma)_0$-module $(F_\sigma)_0$, which determines a quasi-coherent sheaf $\widetilde{(F_\sigma)_0}$ on X_σ. It can be checked that these X_σ cover X and these sheaves patch together to give a quasi-coherent sheaf $\tilde{F}$ on X. We refer the readers to [4] for a detailed account of this subject.

Proposition 3.1 (Proposition 1.1, [4])**.** *If $\alpha = [D] \in \mathrm{Div}(X)$ then there is an isomorphism*

$$\phi_D \colon S_\alpha \to H^0(X, \mathcal{O}_X(D)).$$

Proposition 3.2 (Proposition 3.1, [4])**.** *The map sending F to $\tilde{F}$ is an exact functor from graded S-modules to quasi-coherent $\mathcal{O}_X$-modules.*

Definition 3.3. Let E be the graded S-module $\bigoplus_{\rho \in \Sigma(1)} S(\alpha_\rho)$ with basis e_ρ in degree $-\alpha_\rho$. We define a (degree 0) homomorphism

$$\Psi \colon S^{\oplus r} = S \otimes_\mathbb{Z} \mathrm{Hom}_\mathbb{Z}(\mathrm{Div}(X), \mathbb{Z}) \to E$$

of graded S-modules by

$$\Psi(f \otimes \psi) = f \sum\nolimits_{\rho \in \Sigma(1)} \psi(\alpha_\rho) x_\rho e_\rho,$$

for a homogeneous element $f \in S$ and $\psi \in \mathrm{Hom}(\mathrm{Div}(X), \mathbb{Z})$.

Lemma 3.4. *There is an isomorphism $\widetilde{\mathrm{Coker}\,\Psi} \cong \mathcal{T}_X$ and the associated exact sequence*

$$0 \to \tilde{S}^{\oplus r} \xrightarrow{\tilde{\Psi}} \tilde{E} \to \widetilde{\mathrm{Coker}\,\Psi} \to 0$$

is the generalized Euler sequence

$$0 \to \mathcal{O}_X^{\oplus r} \to \bigoplus\nolimits_{\rho \in \Sigma(1)} \mathcal{O}_X(D_\rho) \to \mathcal{T}_X \to 0.$$

Proof. The homomorphism $\mathcal{O}_X^{\oplus r} \to \bigoplus_\rho \mathcal{O}_X(D_\rho)$ in the generalized Euler sequence is equal to the homomorphism

$$\mathcal{O}_X \otimes_\mathbb{Z} \mathrm{Hom}(\mathrm{Div}(X), \mathbb{Z}) \to \bigoplus\nolimits_{\rho \in \Sigma(1)} \mathcal{O}_X(D_\rho)$$

defined by sending $1 \otimes \psi$ to $(\psi(\alpha_\rho)x_\rho)_\rho$, which obviously coincides with $\tilde{\Psi}\colon \tilde{S}^{\oplus r} \to \tilde{E}$. $\square$

Let $\mathcal{L}$ be a line bundle on X and s a global section of $\mathcal{L}^m$, where m is a positive integer divisible by p. Let $\beta = [\mathcal{L}] \in \operatorname{Pic}(X) \cong \operatorname{Div}(X)$. We identify s with an element of $S_{m\beta}$ via the isomorphism $H^0(X, \mathcal{L}^m) \cong S_{m\beta}$. As in Definition-Lemma 2.1, we have a homomorphism $ds\colon \mathcal{L}^{-m} \to \Omega^1_X$. Let $ds^\vee\colon \mathcal{T}_X \to \mathcal{L}^m$ be the dual of ds. We shall reconstruct $ds^\vee$ in the toric case.

Definition 3.5. We define a (degree 0) homomorphism $\Theta'_s\colon E \to S(m\beta)$ of graded S-modules by

$$\Theta'_s\left(\sum\nolimits_\rho f_\rho e_\rho\right) = \sum\nolimits_\rho f_\rho \frac{\partial s}{\partial x_\rho}.$$

Definition-Lemma 3.6. The composite $\Theta'_s \circ \Psi\colon S^{\oplus r} \to S_{m\beta}$ is a zero map so that there is induced a homomorphism $\operatorname{Coker}\Psi \to S(m\beta)$, which we denote by Θ_s. Moreover, the induced homomorphism $\tilde{\Theta}_s\colon \widetilde{\operatorname{Coker}\Psi} \to \widetilde{S(m\beta)}$ coincides with $ds^\vee\colon \mathcal{T}_X \to \mathcal{L}^m$.

Proof. We have

$$(\Theta'_s \circ \Psi)(1 \otimes \phi) = \sum \phi(\alpha_\rho)x_\rho \frac{\partial s}{\partial x_\rho} = m\beta s,$$

where the last equality is so called generalized Euler relations. This shows that $\Theta'_s \circ \Psi = 0$ since the ground field has characteristic p and m is divisible by p. The last assertion follows from the construction and Lemma 3.4. $\square$

Definition 3.7. We denote by $V = V_\Sigma$ the $\Bbbk$-vector space

$$V := E_0 = \sum\nolimits_{\rho \in \Delta(1)} S_{\alpha_\rho} e_\rho.$$

For $\psi \in \operatorname{Hom}(\operatorname{Div}(X), \mathbb{Z})$, we define

$$v_\psi := \sum\nolimits_{\rho \in \Delta(1)} \psi([D_\rho]) x_\rho e_\rho \in V.$$

We denote by V' the subspace of V spanned by $\{v_\psi \mid \psi \in \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Div}(X), \mathbb{Z})\}$.

We define $\theta_s\colon V \to S_{m\beta}$ to be the map

$$\theta_s\colon V = E_0 \xrightarrow{(\Theta'_s)_0} S(m\beta)_0 = S_{m\beta}.$$

We may identify V with $H^0(X, \bigoplus_\rho \mathcal{O}_X(D_\rho))$. Then V' is considered as the image of $H^0(X, \mathcal{O}_X^{\oplus r})$ under the map Φ.

Definition 3.8. Let D be a torus invariant divisor on X with a class $\alpha = [D] \in \operatorname{Div}(X)$. We define $V_D := E(\alpha)_0 = \sum_{\rho \in \Delta(1)} S_{\alpha_\rho + \alpha} e_\rho$. Let V'_D be the degree 0 part of the image of $\Psi(\alpha)\colon S(\alpha)^{\oplus r} \to E(\alpha)$, which is a subspace of V_D. We define $\theta_{s,D}\colon V_D \to S_{m\beta+\alpha}$ to be the degree 0 part of the map $\Theta'_s(\alpha)\colon E(\alpha) \to S(m\beta + \alpha)$.

We have an exact sequence

$$0 \to S(\alpha)^{\oplus r} \xrightarrow{\Psi(\alpha)} E(\alpha) \to \operatorname{Coker}(\Psi(\alpha)) \to 0$$

and the corresponding exact sequence is

$$0 \to \mathcal{O}_X(D)^{\oplus r} \to \bigoplus\nolimits_\rho \mathcal{O}_X(D_\rho + D) \to \mathcal{T}_X(D) \to 0$$

which is the generalized Euler sequence tensored with $\mathcal{O}_X(D)$.

Lemma 3.9. *Let D be a torus invariant divisor on X with a class $\alpha = [D] \in \mathrm{Div}(X)$. Assume that the set of critical points of s has codimension at least two and $H^1(X, \mathcal{O}_X(D)) = 0$. Then $H^0(X, \mathcal{F}^\vee(D)) = 0$ if and only if the kernel of the map $\theta_{s,D}\colon V_D \to S_{m\beta+\alpha}$ is V'_D.*

Proof. We have a sequence

$$0 \to \mathcal{F}^\vee(D) \to \mathcal{T}_X(D) \to \mathcal{L}^m(D) \to 0$$

which is exact outside the closed subset on which s has a critical points. It follows that $H^0(X, \mathcal{F}^\vee(D)) = 0$ if and only if the map $\varphi\colon H^0(X, \mathcal{T}_X(D)) \to H^0(X, \mathcal{L}^m(D))$ is injective. We have the following exact sequence

$$0 \to \mathcal{O}_X(D)^{\oplus r} \to \bigoplus_{\rho \in \Sigma(1)} \mathcal{O}_X(D_\rho + D) \to \mathcal{T}_X(D) \to 0.$$

The assumption $H^1(X, \mathcal{O}_X(D)) = 0$ shows that the sequence

$$0 \to H^0(X, \mathcal{O}_X(D))^{\oplus r} \to \bigoplus_\rho H^0(X, \mathcal{O}_X(D_\rho + D)) \to H^0(X, \mathcal{T}_X(D)) \to 0$$

is exact. It follows that $H^0(X, \mathcal{T}_X(D))$ is naturally isomorphic to V_D/V'_D. Hence we have the following commutative diagram:

$$\begin{tikzcd} V_D \arrow[r, "\theta_{s,D}"] \arrow[d] & S_{m\beta} \arrow[d, "\phi_{m\beta}"] \\ H^0(X, \mathcal{T}_X(-D)) \arrow[r, "\varphi"] & H^0(X, \mathcal{L}^m(D)), \end{tikzcd}$$

and the kernel of the map $V_D \to H^0(X, \mathcal{T}_X(D))$ is V'_D. From this, we see that φ is injective if and only if the kernel of $\theta_{s,D}$ is V'_D. $\square$

Proposition 3.10. *Let $Y = X[\sqrt[m]{s}]$, $\mathcal{M} = \mathcal{M}(\mathcal{L}, s)$ and D be a torus invariant divisor on X such that $\mathcal{O}_X(D) \cong \mathcal{L}^{-1} \otimes \mathcal{M}$. Assume that the set of critical points of s has codimension at least two and $H^1(X, \mathcal{L}^{-1} \otimes \mathcal{M}) = 0$. Then, if the kernel of $\theta_{s,D}$ is V'_D then we have $H^0(Y, \pi^*\mathcal{M}) \cong H^0(Y, (\Omega_Y^{n-1})^{\vee\vee})$.*

In particular, for any resolution $r\colon Y' \to Y$ of singularities of Y such that the injection $\pi^*\mathcal{M} \hookrightarrow (\Omega_Y^{n-1})^{\vee\vee}$ lifts to an injection $r^*\pi^*\mathcal{M} \hookrightarrow \Omega_{Y'}^{n-1}$, we have $H^0(Y', r^*\pi^*\mathcal{M}) \cong H^0(Y', \Omega_{Y'}^{n-1})$.

Proof. Let X° be the open subset of X obtained by removing the set of critical points of s and $Y^\circ := \pi^{-1}(X^\circ)$ the smooth locus of Y. For a sheaf $\mathcal{H}$ on X, we denote by $\mathcal{H}^\circ$ the restriction on $\mathcal{H}$ to X°. By Lemma 2.2, there is an exact sequence

$$0 \to \pi^*\mathcal{F}^\circ \to \Omega^1_{Y^\circ} \to \pi^*\mathcal{L}^{\circ -1} \to 0$$

of locally free sheaves on Y°, where $\mathcal{F} = \mathcal{F}(\mathcal{L}, s)$ is the cokernel of $ds\colon \mathcal{L}^{-m} \to \Omega^1_X$. Taking a $(n-1)$-th wedge product, we obtain an exact sequence

$$0 \to \pi^*\mathcal{M}^\circ \to \Omega^{n-1}_{Y^\circ} \to \pi^*\left(\mathcal{L}^{\circ -1} \otimes \bigwedge^{n-2} \mathcal{F}^\circ\right) \to 0.$$

Since $\mathcal{F}^\circ$ is a locally free sheaf of rank $n-1$, we have an isomorphism

$$\bigwedge^{n-2} \mathcal{F}^\circ \cong \left(\bigwedge^{n-1} \mathcal{F}^\circ\right) \otimes \mathcal{F}^\circ = \mathcal{M}^\circ \otimes \mathcal{F}^{\circ\vee}.$$

Set $\mathcal{G} = \mathcal{L}^{-1} \otimes \mathcal{M} \otimes \mathcal{F}^\vee$. It follows that the assertion $H^0(Y, \pi^*\mathcal{M}) \cong H^0(Y, (\Omega_Y^{n-1})^{\vee\vee})$ follows from the assertion $H^0(Y, \pi^*\mathcal{G}) = 0$ since $X \setminus X^\circ$ has codimension at least two in X. The latter follows from $H^0(X, \mathcal{G}) = 0$ since we have $\pi_*\pi^*\mathcal{G} = \pi_*\mathcal{O}_Y \otimes \mathcal{G} \cong \mathcal{G} \oplus (\mathcal{G} \oplus \mathcal{L}^{-1})$ and $\mathcal{L}$ is effective.

We have an exact sequence

$$0 \to \mathcal{L}^{-m} \xrightarrow{ds} \Omega^1_{X^\circ} \to \mathcal{F}^\circ \to 0$$

of locally free sheaves on X°. Taking a dual and then taking a tensor product with $\mathcal{L}^{\circ -1} \otimes \mathcal{M}^\circ \cong \mathcal{O}_{X^\circ}(D)$, we get an exact sequence

$$0 \to \mathcal{G}^\circ \to \mathcal{T}_{X^\circ}(D) \to \mathcal{L}^{\circ m}(D) \to 0.$$

Thus, we see that $H^0(X, \mathcal{G}) = 0$ if and only if the map $H^0(X, \mathcal{T}_X(D)) \to H^0(X, \mathcal{L}^m(D))$ is injective. By Lemma 3.9, the latter is equivalent to $\operatorname{Ker} \theta_{s,D} = V'_D$. □

If $\mathcal{L}^{-1} \otimes \mathcal{M}$ is nef and big, which is the case in our applications, then we can show that the assumption $H^1(X, \mathcal{L}^{-1} \otimes \mathcal{M}) = 0$ is automatically satisfied using the following vanishing theorem.

Theorem 3.11 (Batyrev-Borisov vanishing, Theorem 2.5, [2])**.** *Let D be a nef $\mathbb{Q}$-Cartier divisor on a complete toric variety X. Then*

$$H^i(X, \mathcal{O}_X(-D)) = 0,$$

for all $i \neq \kappa(D)$.

Lemma 3.12. *Assume that $\mathcal{L}$ is nef and big. Then, $H^1(X, \mathcal{L}^{-1} \otimes \mathcal{M}) = 0$.*

Proof. We have $\mathcal{L}^{-1} \otimes \mathcal{M} \cong \mathcal{L}^{-1} \otimes (\omega_X \otimes \mathcal{L}^m) \cong \omega_X \otimes \mathcal{L}^{m-1}$. By the Serre duality, it suffices to show that $H^{n-1}(X, \mathcal{L}^{-(m-1)}) = 0$. This follows from Theorem 3.11. □

4. Non-ruled Mori fiber spaces

In this section, we construct a sequence of families of $\mathbb{Q}$-Fano weighted hypersurfaces and study their properties. We refer the readers to [5] for definitions and basic properties of weighted projective spaces.

For a (resp. homogeneous) ring A and (resp. homogeneous) elements $f_1, \ldots, f_m$ of A, we denote by

$$(f_1 = \cdots = f_m = 0)$$

the subscheme of $\operatorname{Spec} A$ (resp. $\operatorname{Proj} A$) defined by the (resp. homogeneous) ideal generated by $f_1, \ldots, f_m$. For positive integers $a_0, \ldots, a_k$ and $m_0, \ldots, m_k$, we denote by $\mathbb{P}(a_0^{m_0}, a_1^{m_1}, \ldots, a_k^{m_k})$ the weighted hypersurface

$$\mathbb{P}(\overbrace{a_0, \ldots, a_0}^{m_0}, \overbrace{a_1, \ldots, a_1}^{m_1}, \ldots, \overbrace{a_k, \ldots, a_k}^{m_k}).$$

In the following, we assume that positive integers l, m and n satisfy the following condition.

Condition 4.1.

$$\frac{n-m}{2} + 1 \leq l \leq n - m.$$

We note that l, m and n necessarily satisfy $l \geq 2$, $n - m \geq 2$ and $n \geq 3$ since $n - m \geq (n - m)/2 + 1$.

Definition 4.2. Let k be a field and a a positive integer. We denote by $k[x_0,\dots,x_n]$ and $k[x_0,\dots,x_n,w]$ the graded rings with $\deg x_i = 1$ for $0 \le i \le m$, $\deg x_i = a$ for $m+1 \le i \le n$ and $\deg w = la$. For an integer d, we denote by $k[x_0,\dots,x_n]_d$ the degree d part of the graded ring $k[x_0,\dots,x_n]$ with the above grading. For $f = f(x_0,\dots,x_n) \in k[x_0,\dots,x_n]_{2la}$, we define a weighted hypersurface

$$X_f := (w^2 - f(x_0,\dots,x_n) = 0) \subset \mathbb{P}(1^{m+1}, a^{n-m}, la) = \operatorname{Proj}(k[x_0,\dots,x_n,y])$$

of degree $2la$.

In this section, we fix positive integers l, m, n, a and weighted homogeneous polynomial $f = f(x_0,\dots,x_n)$ of degree $2la$ and put $X = X_f$. We assume that f is general.

Definition 4.3. We denote by P and V the weighted projective spaces

$$P = \mathbb{P}(1^{m+1}, a^{n-m}) \quad \text{and} \quad V = \mathbb{P}(1^{m+1}, a^{n-m}, la),$$

respectively. For $i = 0, 1, \dots, n+1$, we denote by p_i the vertex $(0 : \cdots : 1 : \cdots : 0)$, where the 1 is in the i-th position. We denote by $\pi_X \colon X \to P$ the projection from the point $\mathsf{p}_{n+1} \in V$. Let $\rho \colon W \to V$ be the blow up of V along the closed subscheme $(x_0 = \cdots = x_m = 0)$ and $\sigma \colon Q \to P$ the blow up of P along the closed subscheme $(x_0 = \cdots = x_m = 0)$. Let Y be the strict transform of X in W and denote by $\rho \colon Y \to X$ the induced birational morphism. We denote by $\pi_Y \colon Y \to Q$ the natural projection which sits in the following commutative diagram:

$$\begin{tikzcd} Y \arrow[r, "\rho"] \arrow[d, "\pi_Y"'] & X \arrow[d, "\pi_X"] \\ Q \arrow[r, "\sigma"'] & P. \end{tikzcd}$$

We often drop the subscript Y and write π instead of π_Y.

4.1. **Non-ruledness.** In this subsection, we work over an algebraically closed field $\Bbbk$ of characteristic 2 unless otherwise specified.

Let $\mathbb{P} = \mathbb{P}(a_0,\dots,a_n)$ be a weighted projective space and Z a subscheme of $\mathbb{P}$. For an integer k, we denote by $\mathcal{O}_{\mathbb{P}}(k)$ the tautological sheaf and by $\mathcal{O}_Z(k)$ the restriction of $\mathcal{O}_{\mathbb{P}}(k)$ to Z.

Definition 4.4. We denote by $\mathcal{L}$ the invertible sheaf $\sigma^*\mathcal{O}_P(la)$ on Q.

We identify f with a global section of $\mathcal{O}_P(2la)$ and let s be the pullback $\sigma^* f$ of f, which is a global section of $\mathcal{L}^2 = \sigma^*\mathcal{O}_Q(2la)$. Let V° be the open subset of V obtained by removing the point p_{n+1}, and W° be the open subset of W which is the inverse image of V° via the map $W \to V$.

Lemma 4.5. *Y is the covering $Q[\sqrt{s}]$ of Q taking a root of $s \in H^0(Q, \mathcal{L}^2)$ and the morphism $\pi \colon Y \to Q$ coincides with the covering map $Q[\sqrt{s}] \to Q$.*

Proof. The natural projection $V^\circ \to P$ can be seen as the total space of the line bundle $\mathcal{O}_P(la)$. Hence the morphism $W^\circ \to Q$ can be seen as the total space of the line bundle $\mathcal{L} = \sigma^*\mathcal{O}_P(la)$. The assertion follows easily. $\square$

Lemma 4.6. *Assume that n is even (resp. odd). Then f has only (resp. almost) nondegenerate critical points on the smooth locus of P.*

Proof. Let U be the smooth locus of P, that is, $U = P \setminus (x_0 = \cdots = x_m = 0)$. Since Condition 4.1 in particular implies that $l \geq 2$, it is easy to see that, for every closed point $\mathsf{p} \in U$, the restriction map

$$H^0(P, \mathcal{O}_P(2la)) \to \mathcal{O}_P(2la) \otimes (\mathcal{O}_P/\mathfrak{m}_{\mathsf{p}}^4)$$

is surjective, where $\mathfrak{m}_{\mathsf{p}}$ is the maximal ideal of the local ring $\mathcal{O}_{P,\mathsf{p}}$. Hence a general f has only (almost) nondegenerate critical points on U (cf. [9, Chapter V, Exercise 5.7]). $\square$

Lemma 4.7. *The closed subset*

$$\mathrm{Cr}(f) := \left(\frac{\partial f}{\partial x_0} = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0\right)$$

of P consists of closed points.

Proof. Lemma 4.6 implies that $\mathrm{Cr}(f)$ consists of closed points if it is restricted on the smooth locus of P. It is sufficient to show that $\mathrm{Cr}(f) \cap (x_0 = \cdots = x_m = 0)$ consists of closed point assuming that $a \geq 2$. We may write $f = g(x_{m+1}, \ldots, x_n) + h(x_0, \ldots, x_n)$, where g consists of the monomials in $x_{m+1}, \ldots, x_n$. We have

$$\mathrm{Cr}(f) \cap (x_0 = \cdots = x_m = 0) = \left(\frac{\partial g}{\partial x_{m+1}} = \cdots = \frac{\partial g}{\partial x_n} = 0\right) \cap (x_0 = \cdots = x_m = 0).$$

The right hand side consists of closed points if a general homogeneous polynomial of degree $2l$ has only isolated critical points on $\mathbb{P}^{n-m-1}$, which can be easily verified. Thus, $\mathrm{Cr}(f)$ is a finite set of closed points. $\square$

We denote by P_i the open subset $(x_i \neq 0)$ of P and by $\tau_i \colon \tilde{P}_i \to P_i$ the orbifold chart. By a slight abuse of notation, we think of $x_0, \ldots, \hat{x}_i, \ldots, x_n$ as affine coordinates of $\tilde{P}_i \cong \mathbb{A}^n$ under the identification. Note that $\tau_i \colon \tilde{P}_i \to P_i$ is an isomorphism for $i = 0, 1, \ldots, m$. For $i = m+1, \ldots, n$, the finite group scheme $\boldsymbol{\mu}_a = \operatorname{Spec} \Bbbk[t]/(t^a)$ acts on $\tilde{P}_i$ by

$$x_j \mapsto \begin{cases} x_j \otimes \bar{t}, & \text{for } j = 0, 1, \\ x_j \mapsto x_j \otimes 1, & \text{otherwise,} \end{cases}$$

and P_i is the geometric quotient $\tilde{P}_i/\boldsymbol{\mu}_a$. The blow up σ can be described as follows. For simplicity, we work over P_n. Let $\tilde{Q}_n \to \tilde{P}_n$ be the blow up along $(x_0 = \cdots = x_m = 0)$. Then the action of $\boldsymbol{\mu}_a$ on $\tilde{P}_n$ naturally extend to an action on $\tilde{Q}_n$ and the geometric quotient $Q_i = \tilde{Q}_i/\boldsymbol{\mu}_a$ is the inverse image of P_i by σ. We see that $\tilde{Q}_n$ is covered by open subsets $\tilde{U}_0, \ldots, \tilde{U}_{n-1}$ where the exceptional divisor is defined by x_i on $\tilde{U}_i$. For example, on $\tilde{U}_0$, we can choose affine coordinates $x_0, x'_1 \cdots, x'_m, x_{m+1}, \ldots, x_n$, where $x'_i = x_i/x_0$ for $1 \leq i \leq m$. Then the action of $\boldsymbol{\mu}_a$ on $\tilde{U}_0$ is given by $x_0 \mapsto x_0 \otimes \bar{t}$, $x'_i \mapsto x'_i \otimes 1$ for $1 \leq i \leq m$, and $x_j \mapsto x_j \otimes 1$ for $m+1 \leq j \leq n-1$. It follows that the geometric quotient U_i is the affine space with affine coordinates $x'_0 = x_0^a, x'_1, \ldots, x'_m, x_{m+1}, \ldots, x_{n-1}$ and the exceptional divisor of σ is defined by x'_0.

Lemma 4.8. *The global section $s = \sigma^* f$ has only admissible critical points on Q and the set* $\mathrm{Cr}(s)$ *of critical points of s endowed with the reduced induced scheme structure is smooth. Moreover,* $\mathrm{Cr}(s)$ *consists of isolated closed points and closed sets of codimension $n - m + 1$.*

Proof. By Lemma 4.6, s has only (almost) nondegenerate critical points outside the exceptional divisor E and hence it is sufficient to consider critical points contained in E. Let $\mathrm{Cr}(s)$ denote the set of critical points of s.

We write $f = f_0 + f_1 + \cdots + f_{2l}$, where $f_i = f_i(x_0, \ldots, x_n)$ has $(x_0, \ldots, x_m)$-degree ia, and let D be the strict transform of the divisor $(f_1 + \cdots + f_{2l} = 0) \subset P$. By $(x_0, \ldots, x_m)$-degree, we mean the degree of a polynomial in the polynomial ring $\Bbbk(x_{m+1}, \ldots, x_n)[x_0, \ldots, x_m]$. Let Z be the closed subset $(x_0 = \cdots = x_m = 0)$ of P. We shall show that the set $\mathrm{Cr}(s) \cap E$ is the disjoint union of $D \cap \sigma^{-1}(\mathsf{p})$, where p runs over the set $\mathrm{Cr}(f) \cap Z$. We work over the open subset $P_n = (x_n \neq 0)$. Put $x_n = 1$ and we identify $x_0, \ldots, x_{n-1}$ as affine coordinates of $\tilde{P}_n$. We see that $\sigma^{-1}(P_n)$ can be covered by open subsets $U_0, U_1, \ldots, U_m$, where the exceptional divisor is defined by x_i^a on U_i.

Put $x'_0 = x_0^a$ and $x'_i = x_i/x_0$ for $1 \le i \le m$. Then $x'_0, \ldots, x'_m$, $x_{m+1}, \ldots, x_{n-1}$ form affine coordinates of U_0. Put $g_i = f_i(1, x'_1, \ldots, x'_m, x_{m+1}, \ldots, x_{n-1}, 1)$. Then, we have $\sigma^* f_i = {x'_0}^i g_i$ on U_0 and hence

$$s_0 := s|_{U_0} = (\sigma^* f)|_{U_0} = g_0 + x'_0 g_1 + {x'_0}^2 h = g_0 + x'_0 z,$$

where $h = \sum_{i \ge 2} {x'_0}^{i-2} g_i$ and $z = g_1 + x'_0 h$ is the defining equation for D. An explicit calculation shows that

$$\mathrm{Cr}(s_0) \cap E = \left(\frac{\partial g_0}{\partial x_{m+1}} = \cdots = \frac{\partial g_0}{\partial x_{n-1}} = z = 0 \right).$$

Note that we have

$$\mathrm{Cr}(f) \cap Z \cap P_n = \left(\frac{\partial g_0}{\partial x_{m+1}} = \cdots = \frac{\partial g_0}{\partial x_{n-1}} = 0 \right) \cap P_n.$$

This shows that $\mathrm{Cr}(s) \cap E$ coincides with $\cup_{\mathsf{p} \in \mathrm{Cr}(f) \cap Z} \sigma^{-1}(\mathsf{p}) \cap D$ and is smooth on U_0. Let q be any closed point of $\mathrm{Cr}(s) \cap E \cap U_0$ and let $\mathsf{p} = \sigma(\mathsf{q})$. We shall show that s has an admissible critical point at q. By replacing homogeneous coordinates $x_{m+1}, \ldots, x_n$, we may assume that $\mathsf{p} = \mathsf{p}_n$. Then $\sigma^{-1}(\mathsf{p}) = (x'_0 = x_{m+1} = \cdots = x_{n-1} = 0)$ on U_0. Put $k = n - m - 1$. We may write

$$g_0 = \begin{cases} x_{m+1} x_{m+2} \cdots + x_{n-2} x_{n-1} + g'_0, & \text{if } k \text{ is even}, \\ a x_{m+1}^2 + x_{m+2} x_{m+3} + \cdots + x_{n-2} x_{n-1} + g'_0, & \text{if } k \text{ is odd}, \end{cases}$$

where g'_0 consists of monomials of degree ≥ 3 in $x_{m+1}, \ldots, x_{n-1}$ and if k is even then the coefficient of x_{m+1}^3 in g'_0 is nonzero. Therefore, we see that s has an admissible critical point at q since the hypersurface $D = (z = 0)$ is smooth and passes through q.

Arguing in the same way for other open subsets $U_1, \ldots, U_m$ and then for other P_i, $m < i < n$, we see that $\mathrm{Cr}(s) \cap E$ is the union of smooth subvarieties $\sigma^{-1}(\mathsf{p}) \cap D$ and s has only admissible critical points on Q. It is easy to see that each components $\sigma^{-1}(\mathsf{p}) \cap D$ has codimension $n - m + 1$, which completes the proof. $\square$

Remark 4.9. Over a field of characteristic 0, the same argument shows that $s_0 = g_0 + x_0'z$ on U_0, where we use the same notation as in the proof of Lemma 4.8. But the difference is that the set $\mathrm{Cr}(f) \cap Z \cap P_n$ is empty. It follows that we may write

$$\begin{aligned} g_0 = c + &\sum\nolimits_{i=m+1}^{n} a_i x_i \\ &+ \begin{cases} x_{m+1}x_{m+2} \cdots + x_{n-2}x_{n-1} + g_0', & \text{if } k \text{ is even,} \\ a x_{m+1}^2 + x_{m+2}x_{m+3} + \cdots + x_{n-2}x_{n-1} + g_0', & \text{if } k \text{ is odd,} \end{cases} \end{aligned}$$

where $(a_{m+1}, \dots, a_n) \neq (0, \dots, 0)$. This observation will be used in the proof of Lemma 6.2.

Lemma 4.10. *There is a resolution $r \colon Y' \to Y$ of singularities of Y such that the injection $\pi^*\mathcal{M}(\mathcal{L}, s) \hookrightarrow (\Omega_Y^{n-1})^{\vee\vee}$ lifts to an injection $r^*\pi^*\mathcal{M}(\mathcal{L}, s) \hookrightarrow \Omega_{Y'}^{n-1}$.*

Proof. This follows from Lemmas 2.8 and 4.8. □

Set $\mathcal{F} = \mathcal{F}(\mathcal{L}, s)$ and $\mathcal{M} = \mathcal{M}(\mathcal{L}, s)$, which are defined in Section 2.

For each $i = 0, 1, \dots, n$, let D_i be the Cartier divisor on Q which is the strict transform of the Weil divisor $(x_i = 0) \subset P$, and let E be the exceptional divisor of the blow up $\sigma \colon Q \to P$.

Lemma 4.11. *$\mathcal{M}$ is ample (resp. nef and big) if and only if $a > m + 1$ (resp. $a \geq m + 1$).*

Proof. By Lemma 2.4, $\mathcal{M}$ is isomorphic to $\mathcal{L}^2 \otimes \omega_Q$. It is easy to see that a $\mathbb{Q}$-divisor $\alpha D_0 + \beta E$ is ample (resp. nef and big) if and only if $0 < a\beta < \alpha$ (resp. $0 < a\beta \leq \alpha$). Note that $D_1 \sim D_0$ for $0 \leq i \leq m$ and $D_i \sim aD_0 + E$ for $m + 1 \leq i \leq n$. We have

$$\begin{aligned} \mathcal{L}^2 \otimes \omega_Q &\cong \sigma^*\mathcal{O}_P(2la) \otimes \mathcal{O}_Q\left(-\sum\nolimits_{i=0}^{n} D_i - E\right) \\ &\cong \mathcal{O}_Q(((2l - n + m)a - m - 1)D_0 + (2l - n + m - 1)E). \end{aligned}$$

Thus, our claim can be proved easily. □

Definition 4.12. A variety X is said to be *ruled* (resp. *uniruled*, resp. *separably uniruled*) if there is a variety Y with $\dim Y = \dim X - 1$ and a map $\mathbb{P}^1 \times Y \dashrightarrow X$ which is birational (resp. dominant, resp. dominant and separable).

Theorem 4.13. *We assume that $1 + (n - m)/2 \leq l \leq n - m$. Then the following assertions hold for any $a \geq m + 1$.*

(1) *If X_f is defined over $\mathbb{C}$ and f is very general then X_f is a non-ruled $\mathbb{Q}$-Fano variety with Picard number one and has at most log terminal singularities. Moreover, Y_f is smooth and it has a structure of (n, m)-Mori fiber space over $\mathbb{P}^m$.*
(2) *If X_f is defined over an algebraically closed field of characteristic 2 and f is general then X_f is not separably uniruled.*

Proof. We shall prove (2). Put $X = X_f$. By Lemmas 4.10 and 4.11, the invertible sheaf $r^*\pi^*\mathcal{M}$ is big and there is an injection $r^*\pi^*\mathcal{M} \hookrightarrow \Omega_{Y'}^{n-1}$. It follows from Theorem 2.10 that Y' is not separably uniruled.

Assume that $X = X_f$ is defined over $\mathbb{C}$. Then the non-ruledness of X follows from (2) by the degeneration method [9, Theorems 1.6 and 1.8]. We shall show that

X is quasismooth, that is, the affine cone C_X of X is smooth outside the origin. We see that

$$\begin{aligned}\mathrm{Sing}(C_X) &= \left(\frac{\partial f}{\partial x_0} = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = w = 0\right) \cap (w^2 - f = 0) \\ &= \left(f = \frac{\partial f}{\partial x_0} = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0\right) \cap (w = 0).\end{aligned}$$

By the criterion [6, Theorem 8.1], we see that the weighted hypersurface in P defined by f is quasismooth, which implies that $\mathrm{Sing}(C_X) = \{0\}$. Thus X is quasismooth. It follows that X is $\mathbb{Q}$-factorial and has only cyclic quotient singularities. By the adjunction formula, we have $\mathcal{O}_X(K_X) \cong \mathcal{O}_X(c)$, where

$$c = 2la - (m + 1 + (n - m)a + la) = (l - n + m)a - (m + 1).$$

By Condition 4.1, we have $c \le -(m+1)$, which shows that X is a $\mathbb{Q}$-Fano variety. We see from [5, Theorem 3.2.4] that the Picard number of X is one. Finally, let $X \dashrightarrow \mathbb{P}^m$ be the projection to the first $m+1$ coordinates $x_0, \dots, x_m$. Then the induced map $Y_f \dashrightarrow \mathbb{P}^m$ is a morphism and it is a Mori fiber space whose general fiber is isomorphic to a (smooth) Fano weighted hypersurface of degree $2l$ in $\mathbb{P}(1^{n-m+1}, l)$. $\square$

4.2. **Bounding birationally trivial subfamilies.**

Definition 4.14. We denote by $\mathcal{X}_a^{(n,m)} \to \mathcal{S}_a^{(n,m)}$ (resp. $\mathcal{X}'^{(n,m)}_a \to \mathcal{S}'^{(n,m)}_a$) the family of X_f defined over $\mathbb{C}$ (resp. $\Bbbk$), and by $\mathcal{Y}_a^{(n,m)}$ (resp. $\mathcal{Y}'^{(n,m)}_a$) the family of Y_f defined over $\mathbb{C}$ (resp. $\Bbbk$).

In the following, we put $\mathcal{X}'_a = \mathcal{X}'^{(n,m)}_a$ and $\mathcal{S}'_a = \mathcal{S}'^{(n,m)}_a$. We shall bound the dimensions of birationally trivial subfamilies of $\mathcal{X}'_a/\mathcal{S}'_a$ and then prove main theorems. Throughout the subsection we assume that (l, m, n) satisfies Condition 4.1 and $a > m + 1$.

We see that $\mathcal{L}^{-1} \otimes \mathcal{M} \cong \mathcal{O}_Q(D)$, where

$$D := -((n - m - l)a + m + 1)D_0 - (n - m - l + 1)E.$$

Note that $-D$ is effective and hence $H^0(Q, \mathcal{O}_X(D)) = 0$. Let S be the homogeneous coordinate ring of Q and V the $\Bbbk$-vector space defined in Definition 3.7.

Lemma 4.15. *If $l < n - m$ then $V_D = 0$.*

Proof. Let

$$\begin{aligned}\mathcal{E} &= \mathcal{O}_Q(D_0) \oplus \cdots \mathcal{O}_Q(D_m) \oplus \mathcal{O}_Q(D_{m+1}) \oplus \cdots \oplus \mathcal{O}_Q(D_n) \oplus \mathcal{O}_Q(E) \\ &\cong \mathcal{O}_Q(D_0)^{\oplus m+1} \oplus \mathcal{O}_Q(aD_0 + E)^{\oplus n-m} \oplus \mathcal{O}_Q(E).\end{aligned}$$

be the locally free sheaf which sits in the middle of the generalized Euler sequence. We have $V_D \cong H^0(Q, \mathcal{E}(D))$. It is easily verified that none of $D_0 + D$, $aD_0 + E + D$ and $E + D$ is effective if $l < n - m$, which shows that $H^0(Q, \mathcal{E}(D)) = 0$. $\square$

Proposition 4.16. *We have $H^0(Y', r^*\pi^*\mathcal{M}) \cong H^0(Y', \Omega^{n-1}_{Y'})$.*

Proof. By Lemma 3.12, we have $H^1(Q, \mathcal{O}_X(D)) = 0$. Hence by Proposition 3.10, it suffices to show that the kernel of $\theta_{s,D}$ is V'_D.

If $l < n - m$ then $V_D = 0$ by Lemma 4.15, and there is nothing to prove. We may assume that $l = n - m$. It is easy to see that $V'_D = 0$. Let y_i be the homogeneous coordinate corresponds to D_i for $0 \le i \le n$ and y' be the one corresponds to E. Then $S = \Bbbk[y_1, \dots, y_n, y']$. The section s can be expressed as

$$s = s(y_1, \dots, y_n, y') = f(y_0 {y'}^{1/a}, \dots, y_m {y'}^{1/a}, y_{m+1}, \dots, y_n) \in S_{2\beta},$$

where $\beta = [\mathcal{L}] \in \mathrm{Pic}(Q) \cong \mathrm{Div}(Q)$, and we may write

$$s = {y'}^{2l} s_{2l} + {y'}^{2l-1} s_{2l-1} + \cdots + y' s_1 + s_0,$$

where $s_i = s_i(y_1, \dots . y_n)$ is homogeneous of degree $2\beta - [iE]$. Note that $s_0 = s_0(y_{m+1}, \dots, y_n)$ is a polynomial in $y_{m+1}, \dots, y_n$. We have $V_D = (\bigoplus S_{[D_i + D]} e_i) \oplus S_{[E+D]} e'$, where e_i and e' are the generators. It is easy to see that $S_{[D_i+D]} = 0$ for $i = 0, \dots, m$, and $S_{[E+D]} = 0$. Hence an element $v \in V_D$ can be written as $v = g_{m+1} e_{m+1} + \cdots g_n e_n$, where $g_i \in S_{[D_i + D]} = S_{[(a-m-1)D_0]}$. It follows that g_i is a polynomial in $y_0, \dots, y_m$. Suppose that $\theta_s(v) = 0$. Then we have

$$\theta_s(v) = \sum_{j=0}^{2l} \sum_{i=m+1}^{n} g_i \frac{\partial ({y'}^j s_j)}{\partial y_j} = \sum_{j=0}^{2l} {y'}^j \sum_{i=m+1}^{n} g_i \frac{\partial s_j}{\partial y_i} = 0.$$

In particular, we must have

$$\sum_{i=m+1}^{n} g_i \frac{\partial s_0}{\partial y_i} = 0.$$

This shows that $g_i = 0$ for every i since g_i is a polynomial in $y_0, \dots, y_m$ and s_0 is a general polynomial in $y_{m+1}, \dots, y_n$ (cf. Remark 4.17 below). This shows that $\theta_{s,D}$ is injective. $\square$

Remark 4.17. Let $g = g(z_1, \dots, z_n)$ be a (usual) homogeneous polynomial of degree d with coefficients in an algebraically closed field and put $g_i = \partial g / \partial z_i$. Assume that g is general. We claim that $S_i \ne S$, where $S_i = \cap_{j \ne i} (g_j = 0)$ and $S = \cap (g_j = 0)$. We may write $g = x_1^d + x_1^{d-1} h_1 + \cdots + x_1 h_{d-1} + h_d$, where $h_j = h_j(x_2, \dots, x_n)$ is a homogeneous polynomial of degree j in $x_2, \dots, x_n$. We see that

$$S_1|_{(x_1 = 0)} = \left(\frac{\partial h_d}{\partial x_2} = \cdots = \frac{\partial h_d}{\partial x_n} = 0 \right)$$

and

$$S|_{(x_1=0)} = \left(h_{d-1} = \frac{\partial h_d}{\partial x_2} = \cdots = \frac{\partial h_d}{\partial x_n} = 0 \right).$$

Since g is general, we see that both h_{d-1} and h_d are general. It follows that S is a proper subset of S_1. Similarly, we have $S_i \ne S$ for every i.

In the proof of Lemma 4.16, s_0 is a homogeneous polynomial in $y_{m+1}, \dots, y_n$. From the claim above, for each i, we may find $\alpha = (\alpha_{m+1}, \dots, \alpha_n) \in \Bbbk^{n-m}$ such that $(\partial s_0 / \partial y_j)(\alpha) = 0$ for $j \ne i$ and $(\partial s_0 / \partial y_i)(\alpha) \ne 0$. It then follows that $g_i = 0$.

Lemma 4.18. *The rational map defined by the complete linear system of $r^* \pi^* \mathcal{M}$ is the composite $\pi_Y \circ r \colon Y' \to Q$ of r and π_Y.*

Proof. We have $\mathcal{M} \cong \mathcal{L}^2 \otimes \omega_Q$. As in the proof of Lemma 4.11, we have

$$\mathcal{M} \cong \mathcal{O}_Y(((2l - n + m)a - m - 1) H_0 + (2l - n + m - 1) E_\rho).$$

We deduce from $(2l - n + m)a - m - 1 < la$ that $\pi_Y^* \mathcal{M}$ is not very ample. Since $\mathcal{M}$ is ample on Q by Lemma 4.11 and Q is a smooth complete toric variety, it is indeed very ample on Q. It follows that the image of the rational map defined by $r^*\pi_Y^*\mathcal{M}$ coincides with $\pi_Y \circ r$. $\square$

Let X_f and X_g be general members of the family $\mathcal{X}'_a/\mathcal{S}'_a$. We denote by $\rho_f\colon Y_f \to X_f$ and $\mathcal{M}_f$, etc, for the corresponding blow up and invertible sheaf, etc.

Lemma 4.19. *Assume that there is a birational map $\varphi\colon X_f \dashrightarrow X_g$. Then, φ is an isomorphism. Moreover, there is an automorphism $\mu\colon P \to P$ such that the diagram*

$$\begin{tikzcd} X_f \arrow[r, "\varphi"] \arrow[d, "\pi_{X_f}"'] & X_g \arrow[d, "\pi_{X_g}"] \\ P \arrow[r, "\mu"'] & P \end{tikzcd}$$

commutes.

Proof. Let $\psi\colon Y_f \dashrightarrow Y_g$ and $\psi'\colon Y'_f \dashrightarrow Y'_g$ be the induced birational maps. By Proposition 4.16 and Lemma 4.18, there is an automorphism λ of Q which sits in the commutative diagram

$$\begin{tikzcd} Y_f \arrow[r, dotted, "\psi"] \arrow[d, "\pi_{Y_f}"'] & Y_g \arrow[d, "\pi_{Y_g}"] \\ Q \arrow[r, "\lambda"'] & Q. \end{tikzcd}$$

We see that Y_f and Y_g are normalizations of Q in the function field of Y'_f and Y'_g, respectively. In other words, they appear as Stein factorizations of $Y'_f \to Q$ and $Y'_g \to Q$, respectively. It follows that the birational map ψ is in fact an isomorphism. The isomorphism $\psi\colon Y_f \to Y_g$ descend to an isomorphism $X_f \to X_g$, which coincides with φ. The automorphism λ of Q also descends to an automorphism μ of P and we have the desired commutative diagram. $\square$

Lemma 4.20. *The following are equivalent.*

(1) *X_f and X_g are birational.*
(2) *X_f and X_g are isomorphic.*
(3) *There is an automorphism μ of P and a weighted homogeneous polynomial $h \in \Bbbk[x_0, \dots, x_n]_{la}$ such that $f = \mu^*g + h^2$.*

Proof. The equivalence of (1) and (2) in proved in Lemma 4.19. Assume that (3) holds. Let ψ be the automorphism of V defined by $\psi^*x_i = \mu^*x_i$ for $i = 0, 1 \dots, n$ and $\mu^*w = w - h$. Then the restriction of ψ on X_f defines an isomorphism $X_f \to X_g$. Conversely, assume that there is an isomorphism $\varphi\colon X_f \to X_g$. By Lemma 4.19, there is an automorphism μ of P such that $\mu \circ \pi_{X_f} = \pi_{X_g} \circ \varphi$. We may write $\varphi^*w = \alpha w + h$ for some $\alpha \in \Bbbk$ and $h \in \Bbbk[x_0, \dots, x_n]_{la}$. We see that $0 = \varphi^*(w^2 - g) = \alpha^2 w^2 + h^2 + \varphi^* g$ in the coordinate ring $\Bbbk[x_0, \dots, x_n, w]/(w^2 - f)$ of the affine cone of X_f. It follows that there is an element $\beta \in \Bbbk$ such that $\alpha^2 w^2 + h^2 + \varphi^* g = \beta(w^2 - f)$ in $\Bbbk[x_0, \dots, x_n, w]$. If $\beta = 0$ then $\alpha = 0$ and $\varphi^* g = h^2$. This is impossible since g is general. Thus, we have $\alpha^2 = \beta$ and $\varphi^* g = h^2 - \beta f$. Since $g \in \Bbbk[x_0, \dots, x_n]$ we see that $\varphi^* g = \mu^* g$. This is (3). $\square$

Remark 4.21. Let X_a and X_b be sufficiently general members of the family $\mathcal{X}_a/\mathcal{S}_a$ and $\mathcal{X}_b/\mathcal{S}_b$ defined over $\mathbb{C}$, respectively. Then X_a is never birational to X_b unless $a = b$. This can be seen as follows. If there is a birational map $\varphi \colon X_a \to X_b$ then Lemma 6.3 shows that a reduction modulo 2 model φ' of φ is a birational map. But, by Lemma 4.19, we must have an isomorphism between $\mathbb{P}(1^{m+1}, a^{n-m})$ and $\mathbb{P}(1^{m+1}, b^{n-m})$, which is impossible unless $a = b$.

Lemma 4.22. *The dimension of* $\operatorname{Aut} P$ *is at most* $(n-m)\dim \Bbbk[x_0,\dots,x_m]_a + (n-m)^2 + (m+1)^2$.

Proof. Let μ be an automorphism of P. We may write $\mu^* x_i = \alpha_0^{(i)} x_0 + \cdots + \alpha_m^{(i)} x_m$ for $i = 0,\dots,m$ and $\mu^* x_i = h_i + \beta_{m+1}^{(i)} x_{m+1} + \cdots + \beta_n^{(i)} x_n$ for $i = m+1,\dots,n$, where $\alpha_j^{(i)}, \beta_j^{(i)} \in \Bbbk$ and $h_i = h_i(x_0,\dots,x_m)$ is a homogeneous polynomial of degree a. Therefore, the dimension of $\operatorname{Aut} P$ is at most

$$\begin{aligned}(m+1)^2 &+ (n-m)(\dim \Bbbk[x_0,\dots,x_m]_a + (n-m)) \\ &= (n-m)\dim \Bbbk[x_0,\dots,x_m]_a + (n-m)^2 + (m+1)^2,\end{aligned}$$

which completes the proof. □

Lemma 4.23. *Let s'_a be a general point of S'_a. Then there is a closed subvariety $\mathcal{C}'_a$ of $\mathcal{S}'_a$ with the following properties.*

1. *$\mathcal{C}'_a$ parametrizes the members of $\mathcal{X}'_a/\mathcal{S}'_a$ which are birational to the member corresponds to s'_a.*
2. $\dim \mathcal{S}'_a - \dim \mathcal{C}'_a \to \infty$ *as* $a \to \infty$.

Proof. We see that $\operatorname{Aut} P$ naturally acts on $\mathcal{S}'_a$, that is, there is a morphism

$$\operatorname{Aut} P \times \mathcal{S}'_a \to \mathcal{S}'_a,$$

which sends $(\mu, [g])$ to $[\mu^* g]$. Let $\overline{\operatorname{Aut} P \cdot s'_a}$ be the closure of the image of $\operatorname{Aut} P \times \{s'_a\}$. Let $\mathcal{T}'_a$ be the linear subspace of $\mathcal{S}'_a$ which corresponds to the vector subspace $\{h^2 \mid h \in \Bbbk[x_0,\dots,x_n]_{la}\}$ of $\Bbbk[x_0,\dots,x_n]_{2la}$. Let $\mathcal{C}'_a$ be the cone over the image of $\overline{\operatorname{Aut} P \cdot s'_a}$ under the projection $\mathcal{S}'_a \dashrightarrow \mathbb{P}^N$ from the linear subspace $\mathcal{T}'_a$. By the construction and Lemma 4.20, we can verify (1).

We have

$$\begin{aligned}\dim \mathcal{C}'_a &\le \dim \overline{\operatorname{Aut} P \cdot s'_a} + (\dim \mathcal{T}'_a + 1) \\ &\le \dim \operatorname{Aut} P + h^0(P, \mathcal{O}_P(la)).\end{aligned}$$

For integers $k > 0$ and $i \ge 0$, let $d(k,i)$ be the dimension of the degree i part of a (usual) polynomial ring in k variables. Then we have

$$\dim \mathcal{S}'_a = h^0(P, \mathcal{O}_P(2la)) - 1 = \sum_{i=0}^{2l} d(n-m, 2l-i) d(m+1, ia) - 1$$

and

$$h^0(P, \mathcal{O}_P(la)) = \sum_{i=0}^{l} d(n-m, l-i) d(m+1, ia).$$

By Lemma 4.22, we have $\dim \operatorname{Aut} P \le (n-m) d(m+1, a) + c$, where $c = (n-m)^2 + (m+1)^2$. Note that $d(k,0) = 1$, $d(k,1) = k$ for every k, and $d(k,i) \le d(k,j)$ if $i < j$.

Combining altogether, we see that

$$\begin{aligned}
&\dim \mathcal{S}'_a - \dim \mathcal{C}'_a \\
&\quad \geq \sum_{i=l+1}^{2l} d(n-m, 2l-i) d(m+1, ia) - (n-m) d(m+1, a) - c - 1 \\
&\quad \geq d(m+1, 2la) + d(n-m, 1) d(m+1, (2l-1)a) - (n-m) d(m+1, a) - c - 1 \\
&\quad \geq d(m+1, 2la) - c - 1.
\end{aligned}$$

Therefore, we have (2) since $d(m+1, 2la) \to \infty$ as $a \to \infty$. $\square$

5. Non-ruled conic bundles

In this section, we shall construct a sequence of families of non-ruled conic bundles and study their properties.

5.1. **Non-ruledness.**

Definition 5.1. Let a be a positive integer. Let $k[x_0, \dots, x_{n-1}]$ be a graded ring whose grading is given by $\deg x_i = 1$ for $0 \leq i \leq n-2$ and $\deg x_{n-1} = a$, and $k[y_0, y_1]$ the usual polynomial ring. We define $P = \mathbb{P}(1^{n-1}, a)$ and $\mathbb{P}^1$ with homogeneous coordinates $x_0, \dots, x_{n-1}$ and y_0, y_1, respectively. Let $p_1 \colon P \to \mathbb{P}(1^{n-1}, a)$ and $p_2 \colon P \to \mathbb{P}^1$ be the projections. Let $\sigma_P \colon Q \to P$ be the blow up of P at the singular point $(0 : \cdots : 0 : 1)$ and set $\sigma = \sigma_P \times \mathrm{id} \colon Q \times \mathbb{P}^1 \to P \times \mathbb{P}^1$. We denote by $\mathcal{L}$ the pullback $\sigma^* \mathcal{O}_{P \times \mathbb{P}^1}(la, 1)$, where $\mathcal{O}_{P \times \mathbb{P}^1}(la, 1)$ is the invertible sheaf $p_1^* \mathcal{O}_{\mathbb{P}(1^{n-1}, a)}(la) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(1)$ on $P \times \mathbb{P}^1$. Let $f = f(x_0, \dots, x_{n-1}, y_0, y_1)$ be a general homogeneous polynomial of bidegree $(2la, 2)$, that is, a general global section of $\mathcal{O}_{P \times \mathbb{P}^1}(2la, 2)$, and $s = \sigma^* f$ be the pullback of f which is a global section of $\mathcal{L}^2$. We set $X = P \times \mathbb{P}^1[\sqrt{f}]$ and $Y = Q \times \mathbb{P}^1[\sqrt{s}]$.

Note that $\sigma \colon Q \times \mathbb{P}^1 \to P \times \mathbb{P}^1$ is the blow up of $P \times \mathbb{P}^1$ along the singular locus $(x_0 = \cdots = x_{n-2} = 0)$.

Lemma 5.2. *s has only (almost) nondegenerate critical points on $Q \times \mathbb{P}^1$.*

Proof. It follows from [10, Lemma 2.2.3] that f has only (almost) nondegenerate critical points on the smooth locus of $P \times \mathbb{P}^1$. We shall show that s does not have a critical point on E. We write $f = f_0 y_0^2 + f_1 y_0 y_1 + f_2 y_1^2$, where f_i is a homogeneous polynomial of degree $2la$ in $x_0, \dots, x_{n-1}$. Then we have $s = s_0 y_0^2 + s_1 y_0 y_1 + s_2 y_1^2$, where $s_i = \sigma^* f_i$.

Let U be an open subset of $Q \times \mathbb{P}^1$ on which y_0 does not vanish. Then, by setting $y_0 = 1$, we have $s = s_0 + s_1 y_1 + s_2 y_1^2$ on U and s_i does not involve y_1. It follows that $\partial s / \partial y_1 = s_1$. Thus the set of critical points of s which lie on E is contained in $(s_1 = 0) \cap E$. Note that f_1 is general and hence f_1 does not pass through the singular point of P. This shows that $(s_1 = 0) \cap E = \emptyset$ and s does not have a critical point on E. $\square$

Let $r \colon Y' \to Y$ be the resolution of singularities of Y which is given in Lemma 2.8. Put $\mathcal{M} := \mathcal{M}(\mathcal{L}, s)$. We see that $\mathcal{M} \cong \omega_{Q \times \mathbb{P}^1} \otimes \mathcal{L}^2$ and we have an injection $r^* \pi^* \mathcal{M} \hookrightarrow \Omega_{Y'}^{n-1}$.

Let D_i be the strict transform of the Weil divisor $(x_i = 0) \subset P \times \mathbb{P}^1$ for $0 \leq i \leq n-1$, E the exceptional divisor of σ and H_i the divisor $(y_i = 0)$ on $Q \times \mathbb{P}^1$ for

$i = 0, 1$. Similarly, let D'_i be the strict transform of the Weil divisor $(x_i = 0) \subset P$ and E' the exceptional divisor of σ_P. We have $q_1^* D'_i = D_i$ and $q_1^* E' = E$. Note that $D_0, \dots, D_{n-1}, H_0.H_1$ and E are the torus invariant prime divisors on $Q \times \mathbb{P}^1$. We see that $D_i \sim D_0$ for $1 \le i \le n-2$, $D_{n-1} \sim aD_0 + E$ and $H_0 \sim H_1$ so that the divisor class group $\mathrm{Div}(Q \times \mathbb{P}^1) \cong \mathrm{Pic}(Q \times \mathbb{P}^1)$ is a free abelian group generated by D_0, H_0 and E. Similarly, $\mathrm{Div}(Q) \cong \mathrm{Pic}(Q)$ is a free abelian group generated by D'_0 and E'.

We have

$$\mathcal{L} \cong \sigma^* \mathcal{O}_{P \times \mathbb{P}^1}(la, 1) \cong \mathcal{O}_{Q \times \mathbb{P}^1}(laD_0 + lE + H_0)$$

and

$$\begin{aligned} \mathcal{M} &\cong \omega_{Q \times \mathbb{P}^1} \otimes \mathcal{L}^2 \\ &\cong \mathcal{O}_{Q \times \mathbb{P}^1}\left(-\sum\nolimits_{i=0}^{n-1} D_i - \sum\nolimits_{i=0}^{1} H_i - E \right) \otimes \mathcal{L}^2 \\ &\cong \mathcal{O}_{Q \times \mathbb{P}^1}(((2l-1)a - (n-1))D_0 + (2l-2)E). \end{aligned}$$

Set $M := ((2l-1)a - (n-1))D_0 + (2l-2)E$.

Lemma 5.3. *Assume that $a > n-1$. Then $\mathcal{M}$ is the pullback of a very ample invertible sheaf on Q.*

Proof. We see that $\mathcal{M}$ is the pullback of $\mathcal{O}_Q(M')$, where $M' = ((2l-1)a - (n-1))D'_0 + E'$. The toric variety Q has Picard number two and two contractions $\sigma_P \colon Q \to P$ and $Q \to \mathbb{P}^{n-2}$, where the latter is the composite of σ_P and the projection $P \dashrightarrow \mathbb{P}^{n-2}$ to the first $n-1$ coordinates $x_0, \dots, x_{n-2}$. From this we deduce that a divisor $\alpha D'_0 + \beta E'$ is ample if and only if $0 < a\beta < \alpha$. Moreover, an ample Cartier divisor on Q is very ample since Q is smooth. It follows that M' is very ample if and only if $a > n-1$, which completes the proof. $\square$

Theorem 5.4. *Assume that $n \ge 3$ and $a > n-1$. Then the following hold.*

(1) *If $Y = Q \times \mathbb{P}^1[\sqrt{s}]$ is defined over $\mathbb{C}$ then it is a smooth non-ruled variety which has a conic bundle structure $q_1 \circ \pi \colon Y \to Q$.*
(2) *If $Y = Q \times \mathbb{P}^1[\sqrt{s}]$ is defined over an algebraically closed field $\Bbbk$ of characteristic 2 then it is not ruled.*

Proof. (1) follows from (2). We shall prove (2). By Theorem 2.11 and Lemma 5.3, it suffices to show that the generic fiber Y_F of $q_1 \circ \pi \colon Y \to Q$ is non-ruled over F, where F is the fields of rational functions on Q. Working on a suitable open subset of Q, we can identify F with the field $\Bbbk(x_1, \dots, x_{n-1})$. We see that Y_F is the conic defined in $\mathbb{P}^2_F$ with homogeneous coordinates y_0, y_1, w by the equation $w^2 = ay_0^2 + by_0y_1 + cy_1^2$, where $a, b, c \in \Bbbk[x_1, \dots, x_{n-1}]$. Thus Y_F is not ruled over F by Theorem 2.11. $\square$

Remark 5.5. Let Δ be the discriminant divisor of the conic bundle $Y \to Q$. Then we have $\mathcal{O}_Q(\Delta) \cong \mathcal{L}^4$. If $l = 1$ then $4K_Q + \Delta \sim -4(n-1)D_0 - 4E$ is not effective for any a. Hence, in this case, nonrationality of Y cannot follow from the Sarkisov's criterion.

5.2. **Bounding birationally trivial subfamilies.** In this subsection, we assume that $n \ge 3$, $l \ge 1$ and $a > n-1$.

Definition 5.6. Let $\mathcal{X}_a^{(n,n-1)} \to \mathcal{S}_a^{(n,n-1)}$ (resp. $\mathcal{X}'^{(n,n-1)}_a \to \mathcal{S}'^{(n,n-1)}_a$) be the family of degree 2 covers $P \times \mathbb{P}^1[\sqrt{f}]$ of $P \times \mathbb{P}^1$ ramified in a divisor of bidegree $(2la, 2)$ defined over $\mathbb{C}$ (resp. $\Bbbk$), and by $\mathcal{Y}_a^{(n,n-1)} \to \mathcal{S}_a^{(n,n-1)}$ (resp. $\mathcal{Y}'^{(n,n-1)}_a \to \mathcal{S}'^{(n,n-1)}_a$) be the family of $Q \times \mathbb{P}^1[\sqrt{s}]$ defined over $\mathbb{C}$ (resp. $\Bbbk$).

In the following, we put $\mathcal{X}'_a = \mathcal{X}'^{(n,n-1)}$, $\mathcal{S}'_a = \mathcal{S}'^{(n,n-1)}$, and similarly for $\mathcal{Y}'_a$ and $\mathcal{S}'_a$. We shall bound the dimensions of birationally trivial subfamilies of $\mathcal{X}'_a/\mathcal{S}'_a$. The argument is basically the same as in Section 4.2.

Lemma 5.7. *We have $H^0(Y', r^*\pi^*\mathcal{M}) \cong H^0(Y', \Omega^{n-1}_{Y'})$.*

Proof. We see that $\mathcal{L}^{-1} \otimes \mathcal{M} \cong \mathcal{O}_{Q\times\mathbb{P}^1}(D)$, where

$$D = ((l-1)a - (n-1))D_0 + (l-2)E - H_0$$

is a torus invariant divisor. It is easy to see that $H^0(Q \times \mathbb{P}^1, \mathcal{L}^{-1} \otimes \mathcal{M}) = 0$. Hence, by Proposition 3.10 and Lemma 3.12, we need to show that $\theta_{s,D} \colon V_D \to S_{2\beta+\gamma}$ is injective, where $\beta = [\mathcal{L}]$ and $\gamma = [D]$ are the classes in $\mathrm{Div}(Q)$. We have $H^0(Q\times\mathbb{P}^1, \mathcal{O}_{Q\times\mathbb{P}^1}(D_i+D)) = H^0(Q\times\mathbb{P}^1, \mathcal{O}_{Q\times\mathbb{P}^1}(E+D)) = 0$ since the coefficients of H_0 in the divisors $D_i + D$ and $E + D$ are negative. It follows that we have an isomorphism

$$V_D \cong S_{[H_0+D]} \oplus S_{[H_1+D]} \cong S_\delta^{\oplus 2},$$

where $\delta = [H_0 + D] = [H_1 + D] \in \mathrm{Div}(Q \times \mathbb{P}^1)$. Via the isomorphism above, $\theta_{s,D}$ maps $(t_0, t_1) \in S_\delta^{\oplus 2}$ to $t_0(\partial s/\partial y_0) + t_1(\partial s/\partial y_1)$. Assume that $\theta_{s,D}(t_0, t_1) = 0$. We may write $s = s_0 y_0^2 + s_1 y_1^2 + s_2 y_0 y_1$ for some $s_i \in S_{2\beta_x}$. It follows that $\theta_{s,D}(t_0, t_1) = t_0 s_2 y_1 + t_1 s_2 y_0 = s_2(t_0 y_1 + t_1 y_1) = 0$. Note that $s_2 \neq 0$ since s is general. This shows that $t_0 = t_1 = 0$ since t_i is a polynomial in $z_0, \ldots, z_{n-1}$ and z_e for $i = 0, 1$. $\square$

Let f_1 and f_2 be general global sections of $\mathcal{O}_{P\times\mathbb{P}^1}(2la, 2)$ and put $s_i = \sigma^* f_i$. We put $X_i = P \times \mathbb{P}^1[\sqrt{s_i}]$ and $Y_i = Q \times \mathbb{P}^1[\sqrt{s_i}]$. We denote by $\pi_i \colon Y_i \to Q \times \mathbb{P}^1$ the covering map.

Lemma 5.8. *Assume that there is a birational map $\psi \colon Y_1 \dashrightarrow Y_2$. Then φ is an isomorphism and there is an isomorphism $\nu \colon Q \times \mathbb{P}^1 \to Q \times \mathbb{P}^1$ such that $\pi_2 \circ \psi = \nu \circ \pi_1$. Moreover, there is a nonzero $\alpha \in \Bbbk$ and a global section t of $\mathcal{L}$ such that $\nu^* s_2 = \alpha s_1 + t^2$.*

Proof. Let $r_i \colon Y'_i \to Y_i$ be a resolution of singularities of Y_i such that $r_i^*\pi_i^*\mathcal{M} \hookrightarrow \Omega^{n-1}_{Y'_i}$, where $\mathcal{M}_i = \mathcal{M}(\mathcal{L}, s_i)$, for $i = 1, 2$. Let $\psi' \colon Y'_1 \dashrightarrow Y'_2$ be the birational map induced by ψ. By Lemma 5.7, we have $H^0(Y'_i, r_i^*\pi_i^*\mathcal{M}_i) \cong H^0(Y'_i, \Omega^{n-1}_{Y'_i})$. It follows that the image of the map Φ_i defined by the complete linear system of $r_i^*\pi_i^*\mathcal{M}_i$ is a birational invariant of Y'_i. Lemma 5.3 shows that Φ_i is the morphism $Y'_i \xrightarrow{r_i} Y_i \xrightarrow{\pi_i} Q \times \mathbb{P}^1 \xrightarrow{q_1} Q$. Thus, we have an isomorphism $\nu_Q \colon Q \to Q$ such that the diagram

$$\begin{tikzcd} Y_1 \arrow[r, dashed, "\psi"] \arrow[d] & Y_2 \arrow[d] \\ Q \arrow[r, "\nu_Z"'] & Q \end{tikzcd}$$

commutes.

Let U_1' be the open subset of Y_1' on which ψ' is defined. By Lemma 5.3, $r_i^*\pi^*\mathcal{M}_i$ is generated by global sections for $i = 1, 2$ and ψ'^* induces an isomorphism between $H^0(Y_2', r_2^*\pi_2^*\mathcal{M}_2)$ and $H^0(Y_1', r_1^*\pi_1^*\mathcal{M}_1)$. It follows that we have an isomorphism $(\psi'|_{U_1'})^* r_2^*\pi_2^*\mathcal{M}_2 \cong r_1^*\pi_1^*\mathcal{M}_1|_{U_1'}$. Let U_1 and U_2 be open subsets of Y_1 and Y_2, respectively, such that ψ induces an isomorphism $\psi\colon U_1 \to U_2$. We have an isomorphism $(\psi|_{U_1})^*\pi_2^*\mathcal{M}_2 \cong \pi_1^*\mathcal{M}_1|_{U_1}$. Since $\mathcal{M}_i \cong \pi_i^*(\omega_{Q\times\mathbb{P}^1} \otimes \mathcal{L}^2)$ and $\omega_{Y_i} \cong \pi_i^*(\omega_{Q\times\mathbb{P}^1} \otimes \mathcal{L})$, we have $\pi_i^*\mathcal{L} \cong \pi_i^*\mathcal{M} \otimes \omega_{Y_i}$. Thus, we have an isomorphism $\pi_1^*\mathcal{L}|_{U_1} \cong (\psi|_{U_1})^*\pi_2^*\mathcal{L}$ which sits in the following commutative diagram

$$\begin{tikzcd} \mathcal{T}_{U_1} \arrow[r, "\cong"] & (\psi|_{U_1})^*\mathcal{T}_{U_2} \\ \pi^*\mathcal{L}|_{U_1} \arrow[u, hook] \arrow[r, "\cong"] & (\psi|_{U_1})^*\pi_2^*\mathcal{L} \arrow[u, hook] \end{tikzcd}$$

This shows that we have a birational map $\nu\colon Q \times \mathbb{P}^1 \dashrightarrow Q \times \mathbb{P}^1$ which restricts to an isomorphism $\pi_1(U_1) \to \pi_2(U_2)$ such that $\pi_2 \circ \psi = \nu \circ \pi_1$ since the injection $\pi_i^*\mathcal{L} \hookrightarrow \mathcal{T}_{Y_i}$ is a foliation and the quotient is the morphism $\pi_i\colon Y_i \to Q \times \mathbb{P}^1$ for $i = 1, 2$ (cf. Remark 2.9).

We see that the following diagram

$$\begin{tikzcd} Q \times \mathbb{P}^1 \arrow[r, dotted, "\nu"] \arrow[d, "q_1"'] & Q \times \mathbb{P}^1 \arrow[d, "q_1"] \\ Q \arrow[r, "\nu_Q"'] & Q \end{tikzcd}$$

commutes. This implies that ν is an isomorphism. Then the map ψ is also an isomorphism since Y_1 and Y_2 are the normalizations of $Q \times \mathbb{P}^1$ in the function field of Y_1 and Y_2, respectively.

We have an isomorphism $\nu^*\mathcal{L} \cong \mathcal{L}$ which induces an isomorphism Ψ of the total spaces of $\mathcal{L}$ such that the restriction of Ψ to Y_1 coincides with ψ. Y_i is the zero locus of $w_i^2 - \pi_i^* s_i \in H^0(W, \pi_i^*\mathcal{L}^2)$, where $w_i \in H^0(W, \pi_i^*\mathcal{L})$. We must have $\Psi^* w_2 = \alpha w_1 + \pi_1^* t$ for some $\alpha \in \Bbbk^\times$ and $t \in H^0(Q, \mathcal{L})$. Therefore, we have

$$\pi_1^*\nu^* s_2 = \psi^*\pi_2^* s_2 = \psi^* w_2^2 = \alpha^2 w_1^2 + \pi_1^* t^2 = \pi_1^*(\alpha^2 s_1 + t^2),$$

which completes the proof. □

Definition 5.9. We define $G = \mathrm{Aut}(P) \times \mathrm{Aut}(\mathbb{P}^1)$ which can be seen as a subgroup of the automorphism group of $P \times \mathbb{P}^1$.

Lemma 5.10. *Notation as above. The following are equivalent.*

1. *X_1 and X_2 are isomorphic.*
2. *X_1 and X_2 are birational.*
3. *There is an automorphism $\mu \in G$ of $P \times \mathbb{P}^1$, a nonzero $\alpha \in \Bbbk$ and a homogeneous polynomial g of bidegree $(la, 1)$ such that $\mu^* f_2 = \alpha f_1 + g^2$.*

Proof. The implication (1) ⇒ (2) is obvious. Assume that (3) holds. Then, using μ, we can construct an automorphism the total space of $\mathcal{O}_{P\times\mathbb{P}^1}(la, 1)$ which restricts to an isomorphism between X_1 and X_2, which proves (1).

We assume that (2) holds and let $\varphi\colon X_1 \dashrightarrow X_2$ be a birational map. Then φ induces a birational map $\psi\colon Y_1 \dashrightarrow Y_2$. By Lemma 5.8, ψ is an isomorphism.Moreover, there are isomorphisms $\nu\colon Q\times\mathbb{P}^1 \to Q\times\mathbb{P}^1$ and $\nu_Q\colon Q \to Q$ such that $q_1\circ\nu = \nu_Q\circ q_1$ and $\nu^* s_2 = \alpha s_1 + t^2$ for some $\alpha \in \Bbbk^\times$ and $t \in H^0(Q\times\mathbb{P}^1, \mathcal{L})$. The isomorphisms ν and ν_Q descend to isomorphisms $\mu\colon P\times\mathbb{P}^1 \to P\times\mathbb{P}^1$ and $\mu_P\colon P \to P$, respectively, such that $p_1\circ\mu = \mu_P\circ p_1$ and that the diagram

$$\begin{tikzcd} X_1 \arrow[r, dashed, "\varphi"] \arrow[d] & X_2 \arrow[d] \\ P\times\mathbb{P}^1 \arrow[r, "\mu"'] & P\times\mathbb{P}^1 \end{tikzcd}$$

commutes since the contractions $Q\times\mathbb{P}^1 \to P\times\mathbb{P}^1$ and $Q \to P$ are unique. By arguing in the same way as in the proof of Lemma 5.8, we see that φ is an isomorphism and $\mu^* f_2 = \alpha f_1 + g^2$, where g is the element of $H^0(P, \mathcal{O}_P(la,1))$ such that $\sigma^* g = t$.

It remains to show that $\mu \in G$. Let $\mu' = (\mu_P \times \mathrm{id})^{-1}\circ\mu$ be the automorphism of $P\times\mathbb{P}^1$. Then μ' is an automorphism of $P\times\mathbb{P}^1$ over P and hence we have $\mu' = \mathrm{id}\times\mu_{\mathbb{P}^1}$ for some $\mu_{\mathbb{P}^1} \in \mathrm{Aut}(\mathbb{P}^1)$. It follows that $\mu = \mu_P \times \mu_{\mathbb{P}^1} \in G$. □

Lemma 5.11. *We have* $\dim G \le 3(\dim \Bbbk[x_0,\dots,x_{n-2}]_a + (n-1)^2)$.

Proof. □

Lemma 5.12. *Let s'_a be a general point of S'_a. Then there is a closed subvariety $\mathcal{C}'_a$ of $\mathcal{S}'_a$ with the following properties.*

(1) *$\mathcal{C}'_a$ parametrizes the members of $\mathcal{X}'_a/\mathcal{S}'_a$ which are birational to the member corresponds to s'_a.*
(2) $\dim\mathcal{S}'_a - \dim\mathcal{C}'_a \to \infty$ *as* $a \to \infty$.

Proof. By Lemma 5.10, we can construct $\mathcal{C}'_a$ with the property (1) in the same way as in the proof of Lemma 4.23 and we have

$$\dim\mathcal{C}'_a \le \dim G + h^0(P\times\mathbb{P}^1, \mathcal{O}_{P\times\mathbb{P}^1}(la,1)).$$

Let $d(m)$ be the dimension of the $\Bbbk$-vector space $\Bbbk[x_0,\dots,x_{n-2}]_m$. We have

$$\begin{aligned} \dim\mathcal{S}'_a - \dim\mathcal{C}'_a &\ge 3\sum\nolimits_{i=0}^{2l} d(ia) - 3\left(d(a) + (n-1)^2\right) - 2\sum\nolimits_{i=0}^{l} d(ia) \\ &\ge 3\sum\nolimits_{i=l+1}^{2l} d(ia) - 2d(a) - 3(n-1)^2 + 1 \\ &\ge d(2la) - 3(n-1)^2 + 1. \end{aligned}$$

This shows (2) since $d(2la) \to \infty$ as $a \to \infty$. □

6. Birational unboundedness

In this section, we shall prove Theorems 1.3 and 1.5.

6.1. Reduction modulo two.

Definition 6.1. We say that a family of varieties is *birationally trivial* if every two members of the family are birational.

Let $\mathcal{X}_a/\mathcal{S}_a = \mathcal{X}^{(n,m)}/\mathcal{S}_a^{(n,m)}$ be either the family of $\mathbb{Q}$-Fano weighted hypersurfaces defined in Section 4 or the family of degree 2 covers of $\mathbb{P}(1^{n-1}, a) \times \mathbb{P}^1$ defined in Section 5 and let $\mathcal{Y}_a/\mathcal{S}_a = \mathcal{X}_a^{(n,m)}/\mathcal{S}_a^{(n,m)}$ be the corresponding family obtained by blowing up $\mathcal{X}_a$. Let X_1 and X_2 be sufficiently general members of the family $\mathcal{X}_a/\mathcal{S}_a$. Let Y_i be the variety obtained by blowing up X_i so that it is a general member of $\mathcal{Y}_a/\mathcal{S}_a$. We assume that there is a birational map $\varphi\colon X_1 \dashrightarrow X_2$. Then, we may find a discrete valuation ring A with the following properties.

(1) A is a subring of $\mathbb{C}$ and its residue field is of characteristic 2.
(2) X_1 and X_2 descend to projective schemes $\mathfrak{X}_1$ and $\mathfrak{X}_2$ over $\operatorname{Spec} A$.
(3) φ descends to a birational map $\Phi\colon \mathfrak{X}_1 \dashrightarrow \mathfrak{X}_2$ defined over $\operatorname{Spec} A$.

Note that Y_i descends to projective scheme $\mathfrak{Y}_i$ over $\operatorname{Spec} A$ for $i = 1, 2$. Let $\varphi'\colon X_1' \dashrightarrow X_2'$ be the induced rational map between geometric special fibers of $\mathfrak{X}_1 \to \operatorname{Spec} A$ and $\mathfrak{X}_2 \to \operatorname{Spec} A$. We call X_1', X_2' and φ' a *reduction modulo* 2 *model* of X_1, X_2 and φ, respectively.

Lemma 6.2. *Let $Y = Y_i$ and A a discrete valuation ring as above. Then there is a resolution $\mathfrak{Y}' \to \mathfrak{Y}$ of singularities of $\mathfrak{Y}$ such that each exceptional divisor is contained in the special fiber and is geometrically ruled.*

Proof. The singular locus of $\mathfrak{Y}$ coincides with that of the special fiber of $\mathfrak{Y} \to \operatorname{Spec} A$. Let C be an irreducible component of the singular locus of $\mathfrak{Y}$ and t be a uniformizing parameter of A. If C is a point then it is proved in [16, Proof of Lemma 3.1] that the blow up at the point resolves the singularity and the exceptional divisor is ruled. Hence we may assume that $\dim C > 0$. By Lemma 4.8 and Remark 4.9, we may choose coordinates $x_1, \dots, x_n$ such that

$$s = c + t(a_1 x_1 + \cdots a_k x_k) + b_1 x_1 x_2 + \cdots + b_{k-1} x_{k-1} x_k + g,$$

where c, b_i are unit elements of A, $a_i \in A$ with $(a_1, \dots, a_k) \neq (0, \dots, 0)$ and g is contained in the ideal $(x_1, \dots, x_k)^3$. Here we assume that k is even for simplicity of the proof. It follows that $\mathfrak{Y}$ is defined by the equation

$$y^2 = t(a_1 x_1 + \cdots + a_k x_k) + b_1 x_1 x_2 + \cdots + b_{k-1} x_{k-1} x_k + g$$

and the singular locus C is defined by the ideal $(t, x_1, \dots, x_k)$. Let $\mathfrak{Y}' \to \mathfrak{Y}$ be the blow up along C. Then an explicit calculation shows that it resolves the singularity and that the exceptional divisor is isomorphic to $Z \times C$, where Z is the quadric hypersurface defined by

$$Y^2 = T(a_1 X_1 + \cdots + a_k X_k) + b_1 x_1 X_2 + \cdots + b_{k-1} X_{k-1} x_k$$

in $\mathbb{P}^{k+1}$ with homogeneous coordinates $X_1, \dots, X_k, Y, T$ over the residue field of A. Possibly taking a field extension, we see that Z is a cone over a quadric. Hence $E \cong Z \times C$ is geometrically ruled. $\square$

Lemma 6.3. *Let X_1 and X_2 be sufficiently general members of the family $\mathcal{X}_a/\mathcal{S}_a$ and assume that there is a birational map $\varphi\colon X_1 \dashrightarrow X_2$. Then, a reduction modulo 2 model $\varphi'\colon X_1' \dashrightarrow X_2'$ of φ is a birational map.*

Proof. Let A be a suitable discrete valuation ring as above so that X_1, X_2 and φ descend to $\mathfrak{X}_1$, $\mathfrak{X}_2$ and Φ. Let $\psi\colon Y_1 \dashrightarrow Y_2$ be the induced birational map. We see that ψ descends to a birational map $\Psi\colon \mathfrak{Y}_1 \dashrightarrow \mathfrak{Y}_2$ defined over $\operatorname{Spec} A$. By Theorems 4.13 and 5.4, the reduction modulo 2 model Y_i' of Y_i is not geometrically ruled. Therefore, by Lemma 6.2, Ψ does not contract Y_1 and induces a birational map $\psi'\colon Y_1 \dashrightarrow Y_2$. Thus a reduction modulo 2 model $\psi'\colon Y_1' \dashrightarrow Y_2'$ of ψ, and hence φ', is a birational map. $\square$

The following result reduces the proof of Theorem 1.3 to bound the birationally trivial subfamilies of $\mathcal{X}_a'/\mathcal{S}_a' = \mathcal{X}'^{(n,m)}_a/\mathcal{S}'^{(n,m)}_a$ in characteristic 2.

Proposition 6.4. *Suppose that the varieties in the infinite sequences of families $\mathcal{X}_a/\mathcal{S}_a$ are birationally bounded. Then, there exists a constant R' such that, for every positive integer a and a general point $s_a' \in \mathcal{S}_a'$, there is a closed subvariety $\mathcal{B}_a'$ of $\mathcal{S}_a'$ with the following properties.*

(1) *$\mathcal{B}_a'$ parametrizes a birationally trivial family.*
(2) *$\mathcal{B}_a'$ passes through s_a'.*
(3) *$\dim \mathcal{S}_a' - \dim \mathcal{B}_a' \le R'$.*

Proof. This is the combination of [16, Proposition 3.1 and Proposition 3.2], where the sequence $\mathcal{X}_a'/\mathcal{S}_a'$ of families are different from ours. Let us explain how to modify proofs.

The proof of [16, Proposition 3.1] applies to any sequence of families of varieties. It follows that we obtain a closed subvariety $\mathcal{B}_a$ of $\mathcal{S}_a$ which parametrizes a birationally trivial family and which contains a given general point $s_a \in \mathcal{S}_a$ with $\dim \mathcal{S}_a - \dim \mathcal{B}_a \le R$. Let $\mathcal{B}_a'$ be a reduction modulo 2 model of $\mathcal{B}_a$. It is enough to show that $\mathcal{B}_a'$ parametrizes a birationally trivial family (cf. [16, Proof of Proposition 3.2]). To this end, we need to show that a reduction modulo 2 model of a birational map $\varphi\colon X_f \dashrightarrow X_g$ between two general members parametrized by $\mathcal{B}_a$ is birational. Thus, by Lemma 6.3, we have the result. $\square$

6.2. **Proof of main theorems.**

Proof of Theorem 1.5. Let $n \ge 3$ and m be integers with $1 \le m \le n-1$. If $m \le n-2$ then we may pick (l,m,n) which satisfies Condition 4.1. In this case, let $\{\mathcal{X}_a/\mathcal{S}_a = \mathcal{X}_a^{(n,m)}/\mathcal{S}_a^{(n,m)}\}$ be the sequence of families of $\mathbb{Q}$-Fano weighted hypersurfaces with the fixed (l,m,n) defined in Section 4. If $m = n-1$ then let $\{\mathcal{X}_a/\mathcal{S}_a = \mathcal{X}^{(n,n-1)}/\mathcal{S}_a^{(n,n-1)}\}$ be the sequence of degree 2 covers of $\mathbb{P}(1^{n-1},a) \times \mathbb{P}^1$ defined in Section 5.

Suppose that n-dimensional smooth Mori fiber spaces over m-dimensional smooth rational varieties are birationally bounded. By Theorems 4.13 and 5.4, the corresponding family $\mathcal{Y}_a/\mathcal{S}_a$ is a family of smooth (n,m)-Mori fiber spaces over a rational base. It follows that the varieties in the sequence of families $\{\mathcal{X}_a/\mathcal{S}_a\}$ are birationally bounded. By Proposition 6.4, there are closed subvarieties $\mathcal{B}_a'$ of $\mathcal{S}_a'$ through a given general point s_a' which parametrize birationally trivial families such that $\dim \mathcal{S}_a' - \dim \mathcal{B}_a'$ is bounded from above by a constant R' which does not depend on a. Let $\mathcal{C}_a'$ be the subvariety obtained in Lemma 4.23 or 5.12. By the property (1) of Lemma 4.23 or 5.12, we may assume that $\mathcal{B}_a' \subset \mathcal{C}_a'$. It follows that

$$\dim \mathcal{S}_a' - \dim \mathcal{C}_a' \le \dim \mathcal{S}_a' - \dim \mathcal{B}_a' \le R'.$$

This contradicts to (2) of Lemma 4.23 or 5.12 and the proof is completed. $\square$

Proof of Theorem 1.3. Let (l, m, n) be a triplet which satisfies Condition 4.1 and let $\{\mathcal{X}_a/\mathcal{S}'_a\}$ be the family of $\mathbb{Q}$-Fano weighted hypersurfaces with the given (l, m, n) defined in Section 4. The proof of Theorem 1.5 shows that varieties in the sequence $\{\mathcal{X}_a/\mathcal{S}_a\}$ are birationally unbounded, which immediately implies the birational unboundedness of $\mathbb{Q}$-Fano n-folds with Picard number one. $\square$

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