RIMS-1707

Some topological aspects of 4-fold symmetric quandle invariants of 3-manifolds

By

Eri HATAKENAKA and Takefumi NOSAKA

October 2010

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

Some topological aspects of 4-fold symmetric quandle invariants of 3-manifolds

Eri Hatakenaka and Takefumi Nosaka

Abstract

The paper relates the 4-fold symmetric quandle homotopy (cocycle) invariants with topological objects. We show that the 4-fold symmetric quandle homotopy invariants are at least as powerful as the Dijkgraaf-Witten invariants. As an application, for an odd prime p, we show that the quandle cocycle invariant of a link in S^3 using the Mochizuki 3-cocycle is equivalent to the Dijkgraaf-Witten invariant with respect to $\mathbb{Z}/p\mathbb{Z}$ of the double covering of S^3 branched along the link. We also reconstruct the Chern-Simons invariant of closed 3-manifolds as a quandle cocycle invariant via the extended Bloch group, in analogy to [IK].

Keywords Quandle, symmetric quandle, quandle cocycle invariant, link, 3-manifold, branched covering, Dijkgraaf-Witten invariant, bordism group, Chern-Simons invariant, the extended Bloch group.

Contents

1 Introduction

Our motivation stems from a topic of Dijkgraaf-Witten invariant described in (1) below. Let M be an oriented compact 3-manifold. Dijkgraaf and Witten [DW] discussed the relation between Chern-Simons action and Wess-Zumino-Witten term on M through the cohomology of the Eilenberg-MacLane space $H^3(K(G,1);\mathbb{Z})$, where G is a compact Lie group. If M has no boundary, then the quantum field theory of the Chern-Simons functional in $H^3(K(G,1);\mathbb{C}/\mathbb{Z})$ can be interpreted as an obstruction class in the oriented bordism group $\Omega_3(K(G,1))$. Furthermore, one of their noteworthy results is that, when G is finite, the path integral of the distribution function of a 3-cocycle $\psi \in H^3(K(G,1);A)$ reduces to a finite sum

$$\mathrm{DW}_\psi(M) = \sum_{f\in \mathrm{Hom}_{\mathrm{grp}}(\pi_1(M),G)} \langle f^*(\psi), [M]\rangle \in \mathbb{Z}[A], \tag{1}$$

where $[M] \in H_3(M;A)$ is the fundamental class of M. From a mathematical viewpoint, Wakui [W] rigorously formulated $\mathrm{DW}_\psi(M)$ of certain 3-cocycles $\psi \in H^3(G;A)$ in terms of a triangulation of M. Recently, the first author [H1] reformulated the Wakui formula as a "quandle cocycle invariant" of links, using the fact that any closed 3-manifold is a 4-fold branched covering of S^3.

Inspired by her work, the second author [N2] introduced a 4-fold symmetric quandle homotopy invariant of 3-manifolds using a quandle $\widetilde{G}_c$. Here, G is a finite group and $c \in G$ is its central element satisfying $c^2 = e$, and $\widetilde{G}_c$ is a certain quandle defined with respect to the pair of (G,c). The invariant of M is defined to be a set of "$\widetilde{G}_c$-colorings" with a grading by an abelian group $\Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c)$. Here, if M is a 4-fold branched covering of S^3 along a link L, a $\widetilde{G}_c$-coloring is roughly a homomorphism $\pi_1(S^3\setminus L) \to G^4 \rtimes \mathfrak{S}_4$ compatible with the monodromy $\pi_1(S^3 \setminus L) \to \mathfrak{S}_4$. On the other hand, the group $\Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c)$ is defined to be a certain link bordism group of $\widetilde{G}_c$-colorings. The previous paper [N2] studied the quandle structure of $\widetilde{G}_c$ and estimated $\Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c)$ in an algebraic viewpoint. It is shown [N2, §7] that the invariant produces the above quandle cocycle invariants considered in [H1].

In this paper, we topologically study the 4-fold symmetric quandle homotopy invariant. For this, we give a topological interpretation of the $\widetilde{G}_c$-colorings, and relate the group $\Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c)$ with some topological objects. We first show a natural bijection between $G^3 \times \mathrm{Hom}_{\mathrm{grp}}(\pi_1(M),G)$ and the set of $\widetilde{G}_c$-colorings (Theorem 3.3). Note that the set of $\widetilde{G}_c$-colorings is a classical invariant and is independent of the central element $c \in G$.

We next give our invariants some functoriality (§3.4). We introduce a fundamental symmetric quandle $SQ(M)$ of M (Definition 3.5). $SQ(M)$ is roughly defined to be a universal quandle representing the all $\widetilde{G}_c$-colorings. Using the bijection of Theorem 3.3 and the universality, we show that the quandle $SQ(M)$ is quandle isomorphic to $\widetilde{G(M)}_{c(M)}$, where $G(M) := \pi_1(M) \times \mathbb{Z}/2\mathbb{Z}$ and $c(M) := (e,1) \in G(M)$ (Corollary 3.6). Further, we define the fundamental class to be a canonical class of a link bordism using $SQ(M)$

(Definition 3.7). Consequently, the study of the 4-fold homotopy invariant of M is a research of the fundamental class using relativity to other 4-fold symmetric quandles $\widetilde{G}_c$ (see §3.4 for detail).

Next, in order to study the group $\Pi_{2,\rho}^{4f}(\widetilde{G}_c)$ mentioned above, we compare the4-fold symmetric quandle homotopy invariants with the Dijkgraaf-Witten invariants given in (1). For this, we take a perspective of a bordism group rather than the state sum formula. We then work with the bordism Dijkgraaf-Witten invariant defined by using an oriented bordism group $\Omega_3(G,c)$ of the pair of (G,c) (see §5.1). From the viewpoint regarding $\Pi_{2,\rho}^{4f}(\widetilde{G}_c)$ as a certain link bordism, we canonically obtain an epimorphism $\Pi_{2,\rho}^{4f}(\widetilde{G}_c) \to \Omega_3(G,c)$, and show that the bordism invariant is derived from the 4-fold symmetric quandle homotopy invariant (Theorem 5.4). We remark that, when $c = e$, the bordism Dijkgraaf-Witten invariant produces $\mathrm{DW}_\psi(M)$ in (1) for *any* 3-cocycle $\psi \in H^3(K(G,1);A)$ (see Remark 5.1); hence, so does the 4-fold symmetric quandle homotopy invariant. For the future, it is a problem whether the epimorphism is isomorphic or not (Problem 5.7). If this is isomorphic, the two invariants in Theorem 5.4 are equivalent.

As an application of bordism groups, we succeed in giving a topological interpretation of two combinatorial invariants of links. For an odd $m \in \mathbb{Z}$, the dihedral quandle R_m is defined to be $\mathbb{Z}/m\mathbb{Z}$ with a binary operation $x * y = 2y - x$. Then the quandle homotopy invariant of oriented links $L \subset S^3$ with respect to R_m is defined by a combinatorial method (see, e.g., [FRS, N1]). Let M_L be the double branched cover of the link L. Then we show that the quandle homotopy invariant of L is equivalent to the Dijkgraaf-Witten invariant of M_L with respect to $G = \mathbb{Z}/m\mathbb{Z}$ (Corollary 5.9). As a special case, if m is a prime p, then it is known [N1] that "the quandle cocycle invariant $\Phi_{\theta_p}(L) \in \mathbb{Z}[t]/(t^p - 1)$ using Mochizuki 3-cocycle" is equivalent to the quandle homotopy invariant of links, leading to an equality $\Phi_{\theta_p}(L) = at^n \,\mathrm{DW}_\psi(M_L)$ for some $a, n \in \mathbb{Z}$ (Corollary 5.11). As a corollary, we compute the Dijkgraaf-Witten invariants of some 3-manifolds (Example 5.12, 5.13, 5.14), using known values of the quandle cocycle invariants $\Phi_{\theta_p}(L)$ in [AS, Iwa1, Iwa2].

In another direction, when $G = SL(2;\mathbb{C})$ or $PSL(2;\mathbb{C})$, we discuss the Cheeger-Chern-Simons class $\widehat{C}_2 \in H^3(G;\mathbb{C}/4\pi^2\mathbb{Z})$. Given a homomorphism $f : \pi_1(M) \to G$, the Chern-Simons invariant of M is defined to be the pairing $\langle f^*(\widehat{C}_2), [M]\rangle \in \mathbb{C}/4\pi^2\mathbb{Z}$. It had been a long-standing problem to provide a computation of the Cheeger-Chern-Simons class and the Chern-Simons invariant. Dupont [Dup] gave an answer modulo $\pi^2\mathbb{Q}$. Lifting his formula, Neumann [Neu] has obtained an explicit formula for $\widehat{C}_2$ with $G = PSL(2;\mathbb{C})$ via the extended Bloch group $\widehat{\mathcal{B}}(\mathbb{C})$, and a computation of the Chern-Simons invariant in term of a triangulation of M. Further, Dupont, Goette and Zickert succeeded in an extension of the formula suitable for $G = SL(2;\mathbb{C})$ [DG, DZ]. From quandle theory, Inoue-Kabaya [IK] in '09 reconstructed the Chern-Simons invariant of knot complements $S^3 \setminus K$ as a quandle cocycle invariant, using $\widehat{\mathcal{B}}(\mathbb{C})$. In this paper, as an analogy, for $G = SL(2;\mathbb{C})$, we reconstruct the Chern-Simons invariant of closed 3-manifolds as a quandle cocycle invariant through $\widehat{\mathcal{B}}(\mathbb{C})$, using the Dupont formula [Dup] and a result in [H1] (Theorem 6.5 and §6.4). Similar to Inoue-Kabaya's result [IK], a benefit of our reformulation is that we can combinatorially compute the Chern-Simons invariant only from the homomorphism $f : \pi_1(M) \to SL(2;\mathbb{C})$ and the monodromy $\phi : \pi_1(S^3 \setminus L) \to \mathfrak{S}_4$ of a 4-fold branched covering $M \to S^3$, without using triangulation of M.

Lastly, we outline the reconstruction. The point follows from the Dupont formula [Dup] rather than $\widehat{\mathcal{B}}(\mathbb{C})$. He reformulated $6\widehat{C}_2$ as a function on a configuration space $\mathrm{Conf}_4(SL(2;\mathbb{R}))$. The reformulation is adequate for a result of the first author [H1]; hence, we succeed in reconstructing 6 multiple of the Chern-Simons invariant (Theorem 6.5). Finally, the Chern-Simons invariant makes a recovery from the multiplication by 6, using the Dijkgraaf-Witten invariants with respect to cyclic groups (§6.4).

This paper is organized as follows. In §2, we review some notation of 4-fold symmetric quandle homotopy invariants. In §3, we give a topological interpretation of the $\widetilde{G}_c$-colorings. In §4, we show the connected sum formula. In §5, we compare the 4-fold symmetric quandle homotopy invariant with the bordism Dijkgraaf-Witten invariant. In §6, we reformulate the Chern-Simons invariant as a quandle cocycle invariant.

2 Review: 4-fold symmetric quandle homotopy invariant

We briefly review some notation of 4-fold symmetric quandle homotopy invariants in [N2, §2 and 3]. Throughout this paper, manifolds are assumed to be C^∞-smooth, oriented, connected and compact. Unless §5, we assume that manifolds have no boundary.

We first review symmetric quandles introduced by Kamada [Kam]. A *symmetric quandle* is a triple of a set X, a binary operation $*$ on X and an involution $\rho : X \to X$ satisfying that, for any $x, y, z \in X$, $x * x = x$, $(x * y) * z = (x * z) * (y * z)$, $\rho(x * y) = \rho(x) * y$, $(x * y) * \rho(y) = x$. For example, $\mathcal{S} := \{(ij) \in \mathfrak{S}_4\}$ with $x * y := yxy$ and $\rho(x) = x$ is a symmetric quandle. We give another example introduced in [N2, Example 4.1] as follows. We consider a pair of a group G and its central element $c \in G$ such that $c^2 = e$, where $e \in G$ is the identity element. Such a pair of (G, c) is called a *cored group*. Putting $T_{12} := \{(i, j) \in \mathbb{Z}^2 | 1 \le i, j \le 4, \ \ i \neq j\}$, we define $\widetilde{G}_c$ to be a quotient set $G \times T_{12}/{\sim}$, where the equivalence relation $\sim$ on $G \times T_{12}$ is defined by $(g, i, j) \sim (g^{-1}c, j, i)$, for $(i, j) \in T_{12}$ and $g \in G$. We equip $\widetilde{G}_c$ with an operation $* : \widetilde{G}_c \times \widetilde{G}_c \to \widetilde{G}_c$ defined by

$$(g, i, j) * (g', i, j) = (g'g^{-1}g', i, j), \qquad (g, i, j) * (g', j, k) = (gg', i, k),$$
$$(g, i, j) * (g', k, l) = (g, i, j),$$

where i, j, k, l are distinct indices. Further, define $\rho : \widetilde{G}_c \to \widetilde{G}_c$ by $\rho(g, i, j) = (gc, i, j)$. Then $(\widetilde{G}_c, \rho)$ is a symmetric quandle. Moreover, putting a projection $p_{\widetilde{G}_c} : \widetilde{G}_c \to \mathcal{S}$ which sends (g, i, j) to $(ij) \in \mathcal{S}$, the triple of $(\widetilde{G}_c, \rho, p_{\widetilde{G}_c})$ satisfies the axioms of the *4-fold symmetric quandle* (see [N2, Definition 3.1] for detail). Remark that if $G = \{e\}$, then $\widetilde{G}_e \cong \mathcal{S}$.

For simplicity, in this paper we denote three elements $(e,1,2)$, $(e,2,3)$, $(e,3,4) \in \widetilde{G}_c$ by e_{12}, e_{23}, e_{34}, respectively.

We review X_ρ-colorings. Let D be an unoriented link diagram on $\mathbb{R}^2$. For a symmetric quandle (X,ρ), an X_ρ-*coloring* of D is a map $C : \{$the two normal orientations on arcs of $D\}$ $\to X$ satisfying the following two conditions:

(X1) For the two orientations α_1, α_2 of the same arc as shown in Figure 1, the colors satisfy $C(\alpha_1) = \rho\big(C(\alpha_2)\big)$. (Hence, we will later draw the only one color of the two).
(X2) At each crossing shown in Figure 1, the three orientations satisfy $C(\gamma) = C(\alpha)*C(\beta)$.

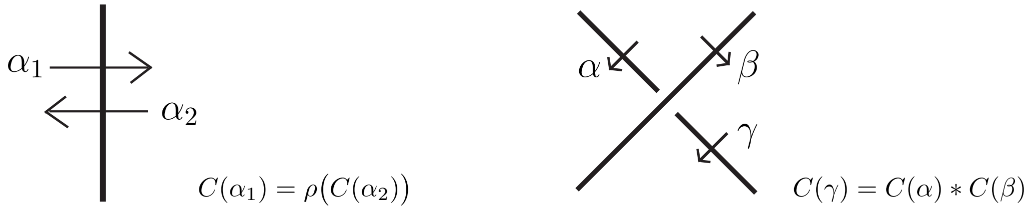
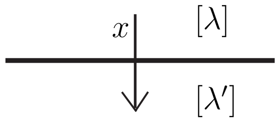


Figure 1: The conditions of symmetric colorings on orientations.

Note that the conditions (X1)(X2) are well-defined by the axioms of symmetric quandles. Denote by $\mathrm{Col}_{X,\rho}(D)$ the set of all X_ρ-colorings of D. It is known [KO, Proposition 6.2] that, if two diagrams D_1 and D_2 are related by Reidemeister moves, there exists a bijection between $\mathrm{Col}_{X,\rho}(D_1)$ and $\mathrm{Col}_{X,\rho}(D_2)$.

For a symmetric quandle (X,ρ), an (X,ρ)-*set* is a set Λ equipped with a map $* : \Lambda \times X \longrightarrow \Lambda$ satisfying $(\lambda * x) * x' = (\lambda * x') * (x * x')$ and $(\lambda * x) * \rho(x) = \lambda$ for any $\lambda \in \Lambda$ and $x, x' \in X$. For example, X is an (X,ρ)-*set* itself by the quandle operation. An X_Λ-*coloring of* D is defined to be an X_ρ-coloring of D with an assignment of elements of Λ to each complementary regions of D such that, for each regions separated by the arc with a color $x \in X$, the colors and assignments satisfy the following picture.



$\lambda * x = \lambda'.\ (\lambda, \lambda' \in \Lambda)$

We will interpret 3-manifolds as $\mathcal{S}_{\mathrm{id}}$-colorings, where $\mathcal{S} = \{(ij) \in \mathfrak{S}_4\}$ as above. It is well-known that any 3-manifold M is a 4-fold simple covering of S^3 branched over a link L with its monodromy $\phi : \pi_1(S^3 \setminus L) \to \mathfrak{S}_4$. Remark that ϕ is surjective and sends each meridian of L to a transposition in $\mathfrak{S}_4$ (see, e.g., [N2, §2.2]). A link diagram D of L with such a monodromy ϕ is called a *labeled diagram* and denoted by D_ϕ. Notice that a labeled diagram can be regarded as an $\mathcal{S}_{\mathrm{id}}$-coloring of D by Wirtinger presentation (see [PS, §24] for detail). A labeled diagram D_ϕ is said to be *3-fold*, if its subdiagram of D_ϕ labeled by (34) is a single unknot as shown in Figure 2. It is known that M can be regarded as a 3-fold labeled diagram D_ϕ (see, e.g., [R, §10.D]). We fix three orientations $\alpha_{12}, \alpha_{23}, \alpha_{34}$ of three distinct arcs in D_ϕ labeled by $(12), (23), (34) \in \mathfrak{S}_4$ as shown in Figure 2, respectively.

We denote by $\mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_\phi)$ the subset of $\widetilde{G}_c$-colorings C of D such that $p_{\widetilde{G}_c}(C) = D_\phi$ and these orientations α_{ij} are colored by $e_{ij} = (e, i, j) \in \widetilde{G}_c$, for $(ij) \in \{(12), (23), (34)\}$.

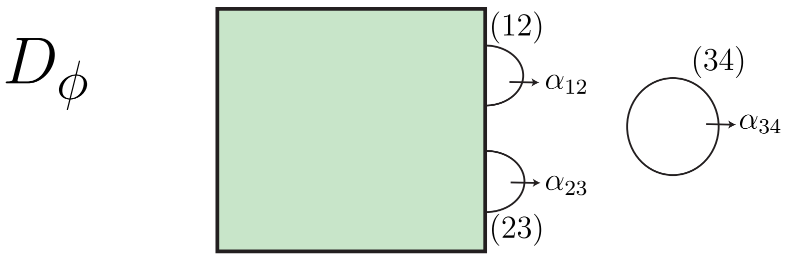


Figure 2: A 3-fold labeled diagram with three orientations $\alpha_{12}, \alpha_{23}, \alpha_{34}$.

For a cored group (G, c), we review an abelian group $\Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_c)$ defined in [N2, §3.1]. $\Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_c)$ is defined by a quotient set of all $\widetilde{G}_c$-colorings of all link diagrams D modulo Reidemeister moves, symmetric concordance relations and MI, II moves. Here, *symmetric concordance relations* and *MI, II moves* are local moves as shown in Figure 3 and 4, respectively. Then we impose an abelian structure on $\Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_c)$ by letting disjoint union define our multiplication. For a labeled diagram D_ϕ, we put a map $\Xi^{4\mathrm{f}}_{\widetilde{G}_c}(D_\phi; \bullet) : \mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_\phi) \to \Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_c)$ which sends a $\widetilde{G}_c$-coloring of D_ϕ to the canonical class (see [N2, (3)]).

Let M be a 3-manifold presented by a 3-fold labeled diagram D_ϕ. When G is finite, the second author defined a *4-fold symmetric quandle homotopy invariant* of M by the following formula (see [N2, Definition 3.3 and Lemma 4.6]).

$$\Xi^{4\mathrm{f}}_{\widetilde{G}_c}(M) = |G|^3 \sum_{C \in \mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_\phi)} \Xi^{4\mathrm{f}}_{\widetilde{G}_c}(D_\phi; C) \in \mathbb{Z}[\Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_c)]. \tag{2}$$

The definition does not depend on the choice of labeled diagrams and the three arcs (see [N2, Theorem 3.4 and Lemma 4.6]).

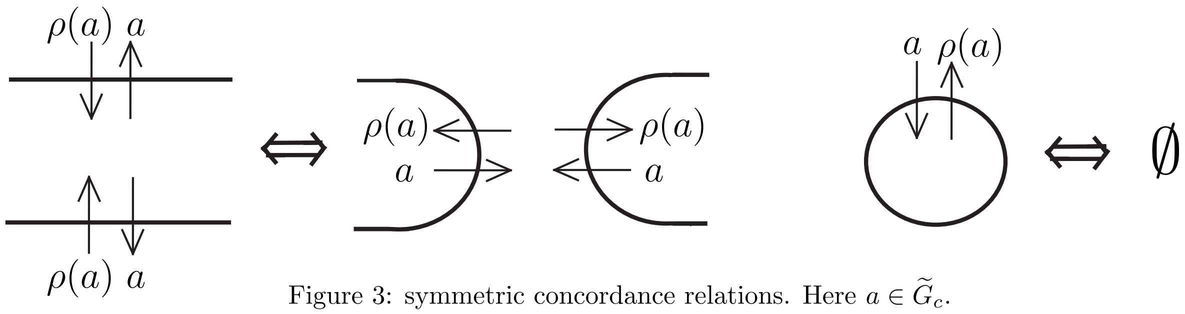


Figure 3: symmetric concordance relations. Here $a \in \widetilde{G}_c$.

3 4-fold symmetric quandle homotopy invariants as natural transformations

In §3.2, we give a topological meaning of $\widetilde{G}_c$-colorings. In §3.3, we introduce a fundamental symmetric quandle $SQ(M)$ of a 3-manifold M, and give an interpretation of $\widetilde{G}_c$-colorings

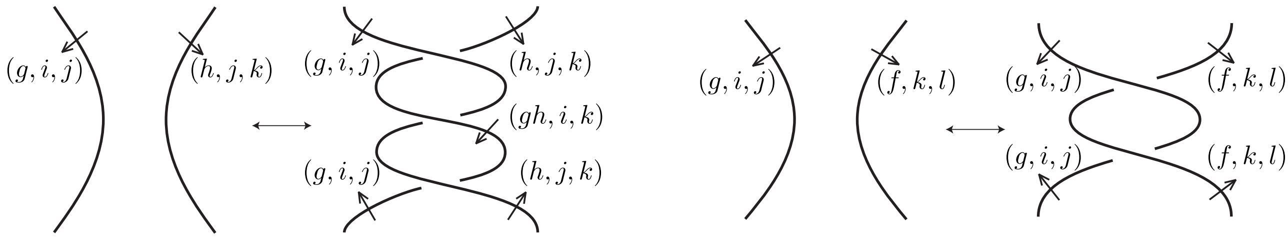


Figure 4: MI, II moves of $\widetilde{G}_c$-colorings. Here, $(g,i,j),\ (h,j,k),\ (f,k,l) \in \widetilde{G}_c$, and i,j,k,l are distinct.

as a representable functor using $SQ(M)$. Further, we interpret a 4-fold symmetric quandle homotopy invariant as a natural transformation.

3.1 Preliminaries: the symmetric link quandle and the associated group

Let L be an unoriented link in S^3. We briefly review the symmetric link quandle of L introduced by Kamada [Kam]. Let $SQ(L)$ be the set of homotopy classes of all pairs of $(\mathcal{D},\gamma)$, where $\mathcal{D}$ means an oriented meridian disk of L and γ means a path in $S^3 \setminus L$ starting from a point of the boundary $\partial\mathcal{D}$ and ending at a fixed base point in $S^3 \setminus L$. We equip $SQ(L)$ with a binary operation given by

$$[(\mathcal{D}_1,\gamma_1)] * [(\mathcal{D}_2,\gamma_2)] := [(\mathcal{D}_1, \gamma_1 \cdot \gamma_2^{-1} \cdot \partial\mathcal{D}_2 \cdot \gamma_2)],$$

and an involution ρ of $SQ(L)$ given by $\rho([(\mathcal{D},\gamma)]) = [(-\mathcal{D},\gamma)]$, where $-\mathcal{D}$ stands for the disk $\mathcal{D}$ with the opposite orientation. Then $SQ(L)$ turns out to be a symmetric quandle (see also [KO, Example 2.4]).

We will explain the correspondence (3) below. Let (X,ρ) be a symmetric quandle and D an unoriented link diagram of L. Let us denote by $\mathrm{Hom}_{\mathrm{sQnd}}(SQ(L),X)$ the set of maps $SQ(L) \to X$ preserving the operations $*$ and ρ, which are called (*symmetric quandle*) *homomorphisms*. Kamada [Kam] gave a canonical bijection

$$\mathscr{Q}(\bullet) : \mathrm{Col}_{X,\rho}(D) \longrightarrow \mathrm{Hom}_{\mathrm{sQnd}}(SQ(L),X), \tag{3}$$

where, for an X_ρ-coloring C, $\mathscr{Q}(C)$ is defined to be a homomorphism sending the meridian disk of an arc α to the color of C on α. This bijection is analogous to [Joy, §16].

We study labeled diagrams from a view of $SQ(L)$. Recall that any labeled diagram D_ϕ can be regarded as an $\mathcal{S}_{\mathrm{id}}$-coloring. By substituting the bijection (3) to $X = \mathcal{S}$, we see that the map $\phi \in \mathrm{Hom}_{\mathrm{sQnd}}(SQ(L),\mathcal{S})$ associated to D_ϕ through (3) is surjective. We denote three meridian disks obtained from the previous arcs $\alpha_{12},\ \alpha_{23},\ \alpha_{34}$, by $\mathcal{D}_{12},\ \mathcal{D}_{23},\ \mathcal{D}_{34} \in SQ(L)$, respectively. Then, as a restriction of the bijection (3), we obtain a bijection

$$\mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_\phi) \simeq \mathrm{Hom}^{(\mathcal{D}_{12},\mathcal{D}_{23},\mathcal{D}_{34})(e_{12},e_{23},e_{34})}_{4\mathrm{sQnd}}(SQ(L),\widetilde{G}_c), \tag{4}$$

where $\mathrm{Hom}^{(\mathcal{D}_{12},\mathcal{D}_{23},\mathcal{D}_{34})(e_{12},e_{23},e_{34})}_{4\mathrm{sQnd}}(SQ(L),\widetilde{G}_c)$ is defined to be the set of symmetric homomorphisms $f : SQ(L) \to \widetilde{G}_c$ satisfying $f(\mathcal{D}_{ij}) = e_{ij} = (e,i,j) \in \widetilde{G}_c$ and $p_{\widetilde{G}_c} \circ f = \phi \in \mathrm{Hom}_{\mathrm{sQnd}}(SQ(L),\mathcal{S})$.

We review the *associated group* [KO] of a symmetric quandle (X,ρ) defined by the following presentation:

$$\mathrm{As}(X)_\rho = \langle x \in X \mid y\cdot(x*y) = x\cdot y,\ \rho(x) = x^{-1}\ (x,y\in X)\rangle.$$

Notice that a symmetric quandle homomorphism $f:(X,\rho)\to(X',\rho')$ induces a group homomorphism $\mathrm{As}(f):\mathrm{As}(X)_\rho \to \mathrm{As}(X')_{\rho'}$.

Lastly, we discuss a canonical map $i_X : X \to \mathrm{As}(X)_\rho$ defined by $i_X(x) = x$. Note that if $X = \widetilde{G}_c$, then $i_{\widetilde{G}_c}$ is injective (cf. [N2, Lemma 3.8]). Further, we consider the case where X is the link quandle $SQ(L)$. Put a map $i_L : SQ(L) \to \pi_1(S^3\setminus L)$ given by that, for a meridian disk $\mathcal{D}$, $i_L(\mathcal{D}) \in \pi_1(S^3\setminus L)$ corresponds with a loop of the boundary $\partial\mathcal{D}$. Then i_L passes to a group homomorphism $\mathrm{As}(SQ(L))_\rho \to \pi_1(S^3\setminus L)$. It can be verified that this is an isomorphism by Wirtinger presentation (cf. [Joy, §14 and 15]). Moreover, we can check that i_L is entirely the above map $i_X : X \to \mathrm{As}(X)_\rho$ with $X = SQ(L)$, and that the image $\mathrm{Im}(i_L)$ consists of the conjugacy classes of the meridians of L.

3.2 $\widetilde{G}_c$-colorings of labeled diagrams

Our objective is to show Theorem 3.3.

Given a labeled diagram D_ϕ which presents a 3-manifold M, we first introduce the associated cored group denoted by $(\mathcal{G}_\phi, c_\phi)$ as follows. Recall the associated monodromy representation $\phi:\pi_1(S^3\setminus L)\longrightarrow \mathfrak{S}_4$ in §3.1. Let $(\mathfrak{S}_4)_1$ denote a subgroup $\{\sigma\in\mathfrak{S}_4 \mid \sigma(1)=1\}$ ($\cong \mathfrak{S}_3$). Let $\widetilde{S^3\setminus L}$ be the 4-fold (unbranched) covering associated to ϕ. Then an isomorphism $\pi_1(\widetilde{S^3\setminus L})\cong \phi^{-1}((\mathfrak{S}_4)_1)$ is known (see, e.g., [H1, §3.1]) and the boundary of $\widetilde{S^3\setminus L}$ consists of $3\sharp L$-tori. Let D be a link diagram of L, and l the number of the arcs of D. For $1\le t\le l$, we let $\widetilde{m}_{t,j}\in\pi_1(\widetilde{S^3\setminus L})$ be the meridian associated obtained from each lifted arcs, where $j\in\{1,2,3\}$. We may assume that, for $1\le t\le l$, $\widetilde{m}_{t,1}\in\pi_1(\widetilde{S^3\setminus L})$ is obtained from the branched locus of index 2. Then, we define a normal subgroup of $\pi_1(\widetilde{S^3\setminus L})$ by

$$N_\phi = \langle [\widetilde{m}_{t,1}, \pi_1(\widetilde{S^3\setminus L})],\ \widetilde{m}_{t,1}(\widetilde{m}_{t',1})^{-1},\ (\widetilde{m}_{t,1})^2,\quad \widetilde{m}_{t,2},\quad \widetilde{m}_{t,3}\quad (1\le t,t'\le l)\rangle, \qquad (5)$$

where $[\widetilde{m}_{t,1}, \pi_1(\widetilde{S^3\setminus L})]$ are their commutator subgroups and the symbol '$\langle\ \rangle$' temporarily stands for the normal closure in $\pi_1(\widetilde{S^3\setminus L})$. Define a group $\mathcal{G}_\phi = \pi_1(\widetilde{S^3\setminus L})/N_\phi$. Notice $\pi_1(M)\cong \pi_1(\widetilde{S^3\setminus L})/\langle \widetilde{m}_{t,j}\ \ (1\le t\le l,\ 1\le j\le 3)\rangle$ by Van Kampen theorem, leading a canonical epimorphism $\mathcal{G}_\phi \longrightarrow \pi_1(M)$. We can verify the epimorphism is a central extension, and the kernel is either $\mathbb{Z}/2\mathbb{Z}$ or 0. Let us denote by c_ϕ a generator of the kernel. We define a cored group $(\mathcal{G}_\phi, c_\phi)$ as required.

Proposition 3.1. *Let (G,c) be a cored group, and D_ϕ a labeled diagram which presents a 3-manifold M. Then, there is a canonical bijection*

$$\mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_\phi) \simeq \mathrm{Hom}_{\boldsymbol{Grp}_c}(\mathcal{G}_\phi, G), \qquad (6)$$

where $\mathrm{Hom}_{\boldsymbol{Grp}_c}(\mathcal{G}_\phi, G)$ *is defined to be the set of group homomorphisms* $f : \mathcal{G}_\phi \to G$ *satisfying* $f(c_\phi) = c \in G$.

Proof. In the case of $c = e$, the first author proved this theorem (see [H1, Proposition 3.5]). In general of c, our proof is analogous to her proof. Then, we sketch a proof of Proposition 3.1. We often regard $C \in \mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_\phi)$ as a homomorphism $\psi : SQ(L) \to \widetilde{G}_c$ by (4).

First, given a symmetric quandle homomorphism $\psi : SQ(L) \to \widetilde{G}_c$, we construct $\Psi \in \mathrm{Hom}_{\boldsymbol{Grp}_c}(\mathcal{G}_\phi, G)$ as follows. We consider an injection $\chi : \widetilde{G}_c \to G^4 \rtimes \mathfrak{S}_4$ defined by $\chi(g, i, j) = (a_1, a_2, a_3, a_4, (ij))$, where $g = a_i = c a_j^{-1}$ and $a_k = e$ $(k \neq i, j)$. Here $\mathfrak{S}_4$ acts on G^4 by the transformations of components of G^4. The map χ induces a group homomorphism $\bar{\chi} : \mathrm{As}(\widetilde{G}_c) \to G^4 \rtimes \mathfrak{S}_4$. Further, by the previous subsection, we have a commutative diagram:

$$\begin{array}{ccccc} SQ(L) & \xrightarrow{\psi} & \widetilde{G}_c & \xrightarrow{\chi} & G^4 \rtimes \mathfrak{S}_4 \\ \downarrow{\scriptstyle i_L} & & \downarrow{\scriptstyle i_{\widetilde{G}_c}} & \nearrow{\scriptstyle \bar{\chi}} & \\ \pi_1(S^3 \setminus L) & \xrightarrow{\mathrm{As}(\psi)} & \mathrm{As}((\widetilde{G}_c)_\rho) & & \end{array}$$

Denote by $\bar{\Psi}$ a composite group homomorphism $\bar{\chi} \circ \mathrm{As}(\psi)$. Hence, for each meridian $\mathfrak{m} \in \pi_1(S^3 \setminus L)$, if the associated arc of D_ϕ is colored by $(g, i, j) \in \widetilde{G}_c$, then

$$\bar{\Psi}(\mathfrak{m}) = (a_1, a_2, a_3, a_4, (ij)), \text{ where } g = a_i = c a_j^{-1} \text{ and } a_k = e \ (k \neq i, j). \tag{7}$$

Let us recall that $\pi_1(\widetilde{S^3 \setminus L}) \cong \phi^{-1}((\mathfrak{S}_4)_1)$ is a subgroup of $\pi_1(S^3 \setminus L)$. Denote by $\widetilde{\Psi}$ the restriction of $\bar{\Psi}$ on $\pi_1(\widetilde{S^3 \setminus L})$. Then, the image of $\widetilde{\Psi}$ is contained in the subgroup $G^4 \rtimes (\mathfrak{S}_4)_1 \subset G^4 \rtimes \mathfrak{S}_4$. Let $\pi_G^1 : G^4 \rtimes (\mathfrak{S}_4)_1 \longrightarrow G$ be the projection on the first component in G^4. Then by (7) we can check that the composite homomorphism $\pi_G^1 \circ \widetilde{\Psi}$ sends each meridians $\widetilde{m}_{t,1} \in \pi_1(\widetilde{S^3 \setminus L})$ to $c \in G$. Therefore $\pi_G^1 \circ \widetilde{\Psi}$ induces a homomorphism $\Psi : (\mathcal{G}_\phi, c_\phi) \longrightarrow (G, c)$ as desired.

Conversely, given $\Psi \in \mathrm{Hom}_{\boldsymbol{Grp}_c}(\mathcal{G}_\phi, G)$, we will construct a $\widetilde{G}_c$-coloring. Put the canonical section $s_G^1 : G \longrightarrow (\mathfrak{S}_4)_1 \ltimes G^4$ of π_G^1. We define $\widetilde{\Psi}$ to be a composite $s_G^1 \circ \Psi \circ \pi_{\mathcal{G}_\phi}$, where $\pi_{\mathcal{G}_\phi}$ is the canonical projection $\pi_1(\widetilde{S^3 \setminus L}) \longrightarrow \mathcal{G}_\phi$. According to [H1], we have known the group presentations of $\pi_1(S^3 \setminus L)$ and $\pi_1(\widetilde{S^3 \setminus L})$, although we do not write them (see [H1, §3.1] for more details). Then we can verify that $\pi_1(S^3 \setminus L)$ is generated by elements of $\pi_1(\widetilde{S^3 \setminus L})$ and three elements $i_L(\mathcal{D}_{12}),\ i_L(\mathcal{D}_{23}),\ i_L(\mathcal{D}_{34}) \in \pi_1(S^3 \setminus L)$, that is,

$$\pi_1(S^3 \setminus L) = \langle \pi_1(\widetilde{S^3 \setminus L}),\ i_L(\mathcal{D}_{12}),\ i_L(\mathcal{D}_{23}),\ i_L(\mathcal{D}_{34}) \rangle.$$

Moreover, we can verify that $\widetilde{\Psi}$ uniquely extends to a homomorphism $\bar{\Psi} : \pi_1(S^3 \setminus L) \longrightarrow G^4 \rtimes \mathfrak{S}_4$ given by $\bar{\Psi}(i_L(\mathcal{D}_{ij})) = \chi(e_{ij}) \in G^4 \rtimes \mathfrak{S}_4$ (see [H1, Page 278]). Then $\bar{\Psi}$ satisfies the condition (7). Hence, $\mathrm{Im}(\bar{\Psi})$ is contained in $\mathrm{Im}(\chi)$. We thus have a map $\psi : SQ(L) \longrightarrow \widetilde{G}_c$ given by $(\chi)^{-1} \circ \overline{\Psi} \circ i_L$ (see the diagram below). Further, since $\chi(\widetilde{G}_c) \subset G^4 \rtimes \mathfrak{S}_4$ is the conjugacy class of $(c, e, e, e, (12))$, we can check that the map ψ is a symmetric quandle homomorphism $SQ(L) \to \widetilde{G}_c$. By the bijection (4), we obtain the required $\widetilde{G}_c$-coloring of D_ϕ whose arcs α_{ij} are colored by $(e, i, j) \in \widetilde{G}_c$.

The two constructions give the required $1:1$ correspondence. For an understanding of the proof, we put the following commutative diagram:

$$
\begin{array}{ccccccc}
\mathcal{G}_\phi & \xleftarrow{\pi_{\mathcal{G}_\phi}} & \pi_1(\widetilde{S^3 \setminus L}) & \hookrightarrow & \pi_1(S^3 \setminus L) & \xleftarrow{i_L} & SQ(L) \\
\downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \widetilde{\Psi}} & & \downarrow{\scriptstyle \bar{\Psi}} & & \downarrow{\scriptstyle \psi} \\
G & \xleftarrow{\pi_G^1} & G^4 \rtimes (\mathfrak{S}_4)_1 & \hookrightarrow & G^4 \rtimes \mathfrak{S}_4 & \xhookleftarrow{\chi} & \widetilde{G}_c
\end{array}
$$

□

We next discuss the projection $\mathcal{G}_\phi \to \pi_1(M)$ mentioned above.

Lemma 3.2. *Let M be a 3-manifold. There exists a 3-fold labeled diagram D_ϕ which presents M such that the projection has a splitting: $\mathcal{G}_\phi \cong \pi_1(M) \times \mathbb{Z}/2\mathbb{Z}$.*

Proof. It is shown [HMT] that there exists a 3-fold irregular branched covering $p : M \to S^3$ such that the set of points at which p fails to be a local homeomorphism bounds a disk in M. We put the associated 3-fold labeled diagram D_ϕ.

Let us construct a homomorphism $f : \mathcal{G}_\phi \to \mathbb{Z}/2\mathbb{Z}$ as follows. Let $\phi : \pi_1(S^3 \setminus K) \to \mathfrak{S}_3$ be the monodromy, where K is a knot in S^3. Let $\tilde{l}$ (resp. $\tilde{m}$) $\in \pi_1(\widetilde{S^3 \setminus K})$ be the longitude (resp. meridian) of a torus in $\widetilde{S^3 \setminus K}$ of local index 2. Put a map $\iota_M : H_1(\widetilde{S^3 \setminus K}; \mathbb{Z}) \to H_1(M; \mathbb{Z})$ induced by the inclusion $\widetilde{S^3 \setminus K} \hookrightarrow M$. Since $\tilde{l}$ bounds a disk in M, we have $\iota_M(\tilde{l}) = 0 \in H_1(M; \mathbb{Z})$. Therefore, by a Mayer-Vietoris argument, we conclude that $H_1(\widetilde{S^3 \setminus K}; \mathbb{Z}) \cong H_1(M; \mathbb{Z}) \oplus \mathbb{Z}$, where the direct summand $\mathbb{Z}$ is generated by $\tilde{m} \in H_1(\widetilde{S^3 \setminus K}; \mathbb{Z})$. Putting the projection $H_1(\widetilde{S^3 \setminus K}; \mathbb{Z}) \to \mathbb{Z}$, we define a composite homomorphism by

$$
\pi_1(\widetilde{S^3 \setminus K}) \longrightarrow H_1(\widetilde{S^3 \setminus K}; \mathbb{Z}) \xrightarrow{\text{proj.}} \mathbb{Z} \xrightarrow{\text{proj.}} \mathbb{Z}/2\mathbb{Z},
$$

where the first map is the abelinization. Therefore, from the definition of $\mathcal{G}_\phi$, the composite induces a required homomorphism $f : \mathcal{G}_\phi \to \mathbb{Z}/2\mathbb{Z}$ satisfying $f(c_\phi) = 1$.

On the other hand, recall that the projection $\mathcal{G}_\phi \to \pi_1(M)$ is a central extension, and that the kernel is either 0 or $\mathbb{Z}/2\mathbb{Z}$. Since f gives a crossed section of $\mathcal{G}_\phi \to \pi_1(M)$, we easily see $\mathcal{G}_\phi \cong \pi_1(M) \times \mathbb{Z}/2\mathbb{Z}$ as desired. □

In conclusion, we give a topological interpretation of $\widetilde{G}_c$-colorings:

Theorem 3.3. *For a cored group (G, c) and a labeled diagram D_ϕ which presents a 3-manifold M, we thus have a bijection*

$$
\mathrm{Col}_{\widetilde{G}_c, \rho}(D_\phi) \simeq G^3 \times \mathrm{Hom}_{\mathrm{grp}}(\pi_1(M), G), \tag{8}
$$

where $\mathrm{Col}_{\widetilde{G}_c, \rho}(D_\phi)$ is the set of $\widetilde{G}_c$-colorings C satisfying $p_{\widetilde{G}_c}(C) = D_\phi \in \mathrm{Col}_{\mathcal{S}, \mathrm{id}}(D)$.

Proof. Since the set $\mathrm{Col}_{\widetilde{G}_c,\rho}(D_\phi)$ depends on only M (see [N2, Proposition 3.2]), we may choose a 3-fold labeled diagram D_ϕ in Lemma 3.2. Since $\mathcal{G}_\phi = \pi_1(M) \times \mathbb{Z}/2\mathbb{Z}$, we notice a bijection $\mathrm{Hom}_{\boldsymbol{Grp_C}}(\mathcal{G}_\phi, G) \simeq \mathrm{Hom}_{\mathrm{grp}}(\pi_1(M), G)$. Further, a bijection $\mathrm{Col}_{\widetilde{G}_c,\rho}(D_\phi) \simeq G^3 \times \mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_\phi)$ is shown [N2, Lemma 4.6]. Hence, the required bijection is obtained from Proposition 3.1. □

As a result, for a finite cored group (G, c), the cardinally of $\widetilde{G}_c$-colorings is a classical invariant, and does not depend on the central element $c \in G$. Hence, our next step in §4 is to study the group $\Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_c)$.

Incidentally, as a corollary, we give a topological interpretation of colorings of core quandles. Given a group G, we equip G with a symmetric quandle operation of $g * h = hg^{-1}h$ and $\rho = \mathrm{id}_G$, called a *core quandle.*

Corollary 3.4. *Let D be a link diagram of a link L, and G a group. Denote by Q_G the core quandle on G. Let M_L be the double branched covering of S^3 branched over the link. Then the set of Q_G-colorings $\mathrm{Col}_{Q_G,\mathrm{id}}(D)$ is in 1:1 correspondence with $G \times \mathrm{Hom}_{\mathrm{grp}}(\pi_1(M_L), G)$.*

Proof. By Figure 5, we obtain a labeled diagram D_ϕ from D, where we equip all arcs of D with labels $(12) \in \mathcal{S}$ and add two unknots labeled by (23) and (34). Then D_ϕ presents M_L. Note that the core quandle Q_G is isomorphic to the subquandle composed of $\{(g, 1, 2) \in \widetilde{G}_e\}$ by definitions. Hence, a Q_G-coloring of D is regarded as a $\widetilde{G}_e$-coloring of the labeled diagram D_ϕ, i.e., a homomorphism $\pi_1(M_L) \to G$ by Theorem 3.3. □

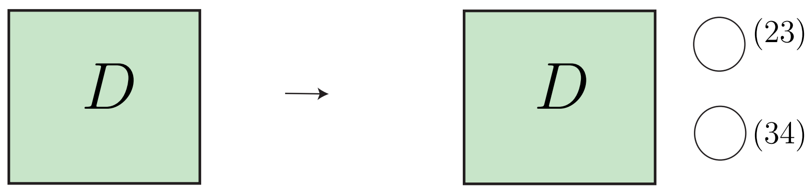


Figure 5: A labeled diagram D_ϕ obtained from a link diagram D.

3.3 A fundamental symmetric quandle of a 3-manifold

For a 3-manifold M, we will define a fundamental symmetric quandle of M and investigate its property.

Let D_ϕ be a labeled diagram which presents M. Recall the associated symmetric quandle epimorphism $\phi : SQ(L) \to \mathcal{S}$ in §3.1. We consider the following equivalent relations on $SQ(L)$:

$$R^{3,\phi}_L := \langle x_{ij} * y_{jk} = \rho(y_{jk}) * x_{ij} \quad (x_{ij} \in \phi^{-1}(ij),\ y_{jk} \in \phi^{-1}(jk))\rangle$$

$$R^{4,\phi}_L := \langle z_{ij} * w_{kl} = z_{ij} \quad (z_{ij} \in \phi^{-1}(ij),\ w_{kl} \in \phi^{-1}(kl))\rangle$$

Then, we define the quotient symmetric quandle $SQ(L)/\langle R_L^{3,\phi}, R_L^{4,\phi}\rangle$. It goes without saying that the quotient quandle satisfies the axioms of the 4-fold symmetric quandle by definition (see [N2, Definition 4.1]). By a discussion similar to [N2, Proposition 3.2], if two labeled diagrams D_ϕ and $D'_{\phi'}$ are related by some finite sequences of Reidemeister moves and MI, MII moves with $G = \{e\}$, then we can obtain a symmetric quandle isomorphism $SQ(L)/\langle R_L^{3,\phi}, R_L^{4,\phi}\rangle \cong SQ(L')/\langle R_{L'}^{3,\phi'}, R_{L'}^{4,\phi'}\rangle$. Thus, by the result in [Apo], $SQ(L)/\langle R_L^{3,\phi}, R_L^{4,\phi}\rangle$ does depend on only the 3-manifold M (see also [N2, Theorem 2.1]).

Definition 3.5. For a labeled diagram D_ϕ of a 3-manifold M, we define a *fundamental symmetric quandle* of M by the quandle $SQ(L)/\langle R_L^{3,\phi}, R_L^{4,\phi}\rangle$. We denote it by $SQ(M)$.

Assume that D_ϕ is 3-fold. We use notation $\mathcal{D}_{12}$, $\mathcal{D}_{23}$, $\mathcal{D}_{34} \in SQ(L)$ in §3.1. Recall the category of 4-fold symmetric quandles denoted by $\boldsymbol{Qnd_{4s}}$ (see [N2, Corollary 4.3]). The objects of $\boldsymbol{Qnd_{4s}}$ consist of $\widetilde{G}_c$ with respect to cored groups (G, c). Let us denote by $\mathrm{Hom}_{\boldsymbol{Qnd_{4s}}}(SQ(M), \widetilde{G}_c)$ the set of morphisms in $\boldsymbol{Qnd_{4s}}$ from $SQ(M)$ to $\widetilde{G}_c$ (see [N2, §4.1] for detail). Remark a natural bijection $\mathrm{Hom}_{\boldsymbol{Qnd_{4s}}}(SQ(M), \widetilde{G}_c) \simeq \mathrm{Hom}_{\mathrm{4sQnd}}^{(\mathcal{D}_{12},\mathcal{D}_{23},\mathcal{D}_{34})(e_{12},e_{23},e_{34})}(SQ(M), \widetilde{G}_c)$ described in [N2, Remark 4.4]. By the correspondence (4) and Proposition 3.1, we thus have a bijection

$$\mathrm{Hom}_{\boldsymbol{Qnd_{4s}}}(SQ(M), \widetilde{G}_c) \simeq \mathrm{Col}_{\widetilde{G}_c}^{e_{12},e_{23},e_{34}}(D_\phi). \tag{9}$$

Although the definition of $SQ(M)$ seems ad hoc, we give its concrete presentation as follows:

Corollary 3.6. *For a 3-manifold M, there exists a symmetric quandle isomorphism $SQ(M) \cong \widetilde{G(M)}_{c(M)}$, where $\big(G(M), c(M)\big)$ is a cored group $(\pi_1(M) \times \mathbb{Z}/2\mathbb{Z}, (e, 1))$.*

Proof. Let D_ϕ be a 3-fold labeled diagram which presents M in Lemma 3.2. Recall an equivalence of categories between $\boldsymbol{Qnd}_{4s}$ and the category of cored groups (see [N2, Corollary 4.3]). Hence, there exists a bijection

$$\mathrm{Hom}_{\boldsymbol{Grp_c}}(G(M), G) \simeq \mathrm{Hom}_{\boldsymbol{Qnd_{4s}}}(\widetilde{G(M)}_{c(M)}, \widetilde{G}_c),$$

for any cored group (G, c). By the canonical bijections (6) and (9), we have a natural equivalence of the following functors from $\boldsymbol{Qnd}_{4s}$ to the category of sets:

$$\mathrm{Hom}_{\boldsymbol{Qnd_{4s}}}(SQ(M), \widetilde{\bullet}_c) \simeq \mathrm{Hom}_{\boldsymbol{Qnd_{4s}}}(\widetilde{G(M)}_{c(M)}, \widetilde{\bullet}_c).$$

Hence, by Yoneda embedding, we conclude $SQ(M) \cong \widetilde{G(M)}_{c(M)}$. □

3.4 4-fold symmetric quandle homotopy invariants as natural transformations

Furthermore, we define a fundamental class of M, and give an interpretation of the 4-fold symmetric quandle homotopy invariant as a natural transformation.

We fix a 3-fold labeled diagram D_ϕ which presents a 3-manifold M. Let us regard $\mathrm{Hom}_{\boldsymbol{Qnd_{4s}}}(SQ(M), \widetilde{\bullet})$ as a functor from the category of 4-fold symmetric quandles. Further, we interpret the group $\Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{\bullet})$ described in §2 as such a functor. We now identify $\mathrm{Hom}_{\boldsymbol{Qnd_{4s}}}(SQ(M), \widetilde{G}_c)$ with $\mathrm{Col}_{\widetilde{G}_c,\rho}^{e_{12},e_{23},e_{34}}(D_\phi)$ by (9). Thus the map $\Xi_{\widetilde{G}_c}^{4\mathrm{f}}(D_\phi, \dagger) : \mathrm{Col}_{\widetilde{G}_c,\rho}^{e_{12},e_{23},e_{34}}(D_\phi) \to \Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c)$ can be regarded as a natural transformation:

$$\Xi_{\widetilde{\bullet}}^{4\mathrm{f}}(D_\phi; \dagger) : \mathrm{Hom}_{\boldsymbol{Qnd_{4s}}}(SQ(M), \widetilde{\bullet}) \longrightarrow \Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{\bullet}). \tag{10}$$

Let us consider a set of such natural transformations: by Yoneda lemma, we have a bijection

$$\mathrm{Nat}\big(\mathrm{Hom}_{\boldsymbol{Qnd_{4s}}}(SQ(M), \widetilde{\bullet}), \Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{\bullet})\big) \simeq \Pi_{2,\rho}^{4\mathrm{f}}(SQ(M)),$$

which sends $\Xi_{\widetilde{\bullet}}^{4\mathrm{f}}(D_\phi; \dagger)$ to $\Xi_{SQ(M)}^{4\mathrm{f}}(D_\phi; \mathrm{id}_{SQ(M)})$, where $\mathrm{id}_{SQ(M)}$ is the identity map of $SQ(M)$.

Definition 3.7. Let M be a 3-manifold, and $SQ(M)$ the fundamental symmetric quandle of M. A *fundamental class* of M is defined to be $\Xi_{SQ(M)}^{4\mathrm{f}}(D_\phi; \mathrm{id}_{SQ(M)}) \in \Pi_{2,\rho}^{4\mathrm{f}}(SQ(M))$.

By the naturality, we can reformulate the formula (2) of the 4-fold homotopy invariant as

$$\Xi_{\widetilde{G}_c}^{4\mathrm{f}}(M) = |G|^3 \cdot \sum_{F \in \mathrm{Hom}_{\boldsymbol{Qnd_{4s}}}(SQ(M), \widetilde{G}_c)} F_*\big(\, \Xi_{SQ(M)}^{4\mathrm{f}}(D_\phi; \mathrm{id}_{SQ(M)})\big) \in \mathbb{Z}[\Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c)]. \tag{11}$$

In conclusion, the study of the 4-fold symmetric quandle homotopy invariant of M is roughly the research of $\Pi_{2,\rho}^{4\mathrm{f}}(SQ(M))$ and of the fundamental class using relativity to other 4-fold symmetric quandles $\widetilde{G}_c$.

Remark 3.8. We compare the fundamental classes of knots with those of 3-manifolds. In the theory of quandle homotopy invariants valued in $\pi_2(BX)$, the second author showed that, for any non-trivial knots K, the homotopy group of the "knot quandle" is isomorphic to $\mathbb{Z}$ generated by the "fundamental class" (see [N1, Corollary 4.17]). On the other hand, on the 4-fold homotopy invariant of 3-manifolds M, $\Pi_{2,\rho}^{4\mathrm{f}}(SQ(M))$ is always neither $\mathbb{Z}$ nor generated by the fundamental class, but $\Pi_{2,\rho}^{4\mathrm{f}}(SQ(M))$ does depend on M.

4 Formulas for the connected sum and the opposite orientation

In this section, we show the formulas of the 4-fold quandle homotopy invariant for the connected sum and the opposite orientation.

Proposition 4.1. *Let M_1 and M_2 be 3-manifolds. Let $M_1\#M_2$ denote the connected sum of M_1 and M_2. For a finite cored group (G, c),*

$$\Xi_{\widetilde{G}_c}^{4\mathrm{f}}(M_1) \cdot \Xi_{\widetilde{G}_c}^{4\mathrm{f}}(M_2) = |G|^3 \cdot \Xi_{\widetilde{G}_c}^{4\mathrm{f}}(M_1\#M_2) \ \in \mathbb{Z}[\Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c)]. \tag{12}$$

Proof. The proof is analogous to [N1, Proposition 5.1]. Let D_1 and D_2 be 3-fold labeled diagrams which present 3-manifolds M_1 and M_2, respectively. Then, by Proposition 4.3 below, $M_1\#M_2$ is presented by $D_1\#D_2$ as shown in Figure 6. Let α^1_{ij}, α^2_{ij}, β_l be arcs of $D_1, D_2, D_1\#D_2$ shown in Figure 6, respectively. Also, define $\mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_1\#D_2)$ to be the subset of $\mathrm{Col}_{\widetilde{G}_c,\rho}(D_1\#D_2)$ such that $\beta_1, \beta_3, \beta_5$ are colored by $e_{12}, e_{23}, e_{34} \in \widetilde{G}_c$, respectively. Remark that, for a $\widetilde{G}_c$-coloring $C \in \mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_1\#D_2)$, it follows from Lemma 4.2 below that the arcs β_2 and β_4 are colored by e_{12} and e_{23}, respectively.

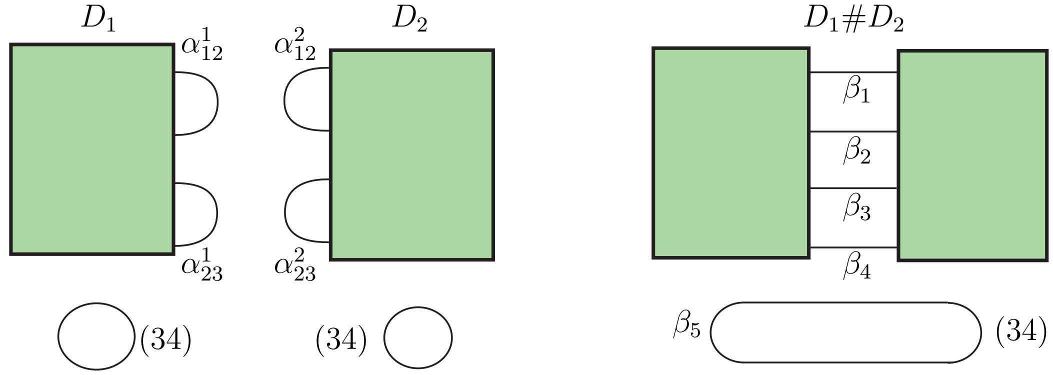


Figure 6: Labeled diagrams of D_1, D_2 and $D_1\#D_2$ with some arcs.

By the formula (2), the left hand side of (12) is

$$\Xi^{4\mathrm{f}}_{\widetilde{G}_c}(M_1)\cdot\Xi^{4\mathrm{f}}_{\widetilde{G}_c}(M_2) \;=\; |G|^6 \sum_{(C_1,C_2)\in \mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_1)\times \mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_2)} \Xi^{4\mathrm{f}}_{\widetilde{G}_c}(M_1;C_1)\cdot\Xi^{4\mathrm{f}}_{\widetilde{G}_c}(M_2;C_2).$$

Note that there is a natural 1-1 correspondence between $\mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_1)\times\mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_2)$ and $\mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_1\#D_2)$, which preserves the multiplication in $\Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_c)$. Therefore,

$$\Xi^{4\mathrm{f}}_{\widetilde{G}_c}(M_1)\cdot\Xi^{4\mathrm{f}}_{\widetilde{G}_c}(M_2) = |G|^6 \sum_{C\in \mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_1\#D_2)} \Xi^{4\mathrm{f}}_{\widetilde{G}_c}(D_1\#D_2;C) = |G|^3\cdot\Xi^{4\mathrm{f}}_{\widetilde{G}_c}(M_1\#M_2),$$

where we use the formula (2) again in the second equality. □

We show Lemma 4.2 and Proposition 4.3, which are used in the proof of Proposition 4.1.

Lemma 4.2. *Let $D_1\#D_2$ be a labeled diagram as shown in Figure 6. For any $C \in \mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_1\#D_2)$, the orientations β_2 and β_4 are colored by e_{12} and e_{23}, respectively.*

Proof. Let us regard $\widetilde{G}_c$ as a $(\widetilde{G}_c,\rho)$-set with the canonical action (see §2). For $\kappa \in \widetilde{G}_c$, put the associated $(\widetilde{G}_c)_\Lambda$-coloring as illustrated in Figure 7 whose region with the infty is assigned by κ. Then, by the coloring condition of the middle region, we have

$$(\kappa * e_{12}) * \rho(g,1,2) = (\kappa * e_{23}) * \rho(h,2,3). \tag{13}$$

Since $\kappa \in \widetilde{G}_c$ is arbitrary, by applying $\kappa = e_{34} \in \widetilde{G}_c$ to the equality (13), we have

$$e_{34} = (e_{34} * e_{12}) * \rho(g,1,2) = (e_{34} * e_{23}) * \rho(h,2,3) = (e,4,2) * \rho(h,2,3) = (h^{-1},3,4).$$

Hence, $h = e$. Similarly, by applying $\kappa = (e,1,4)$ to (13), we obtain $g = e$ as required. □

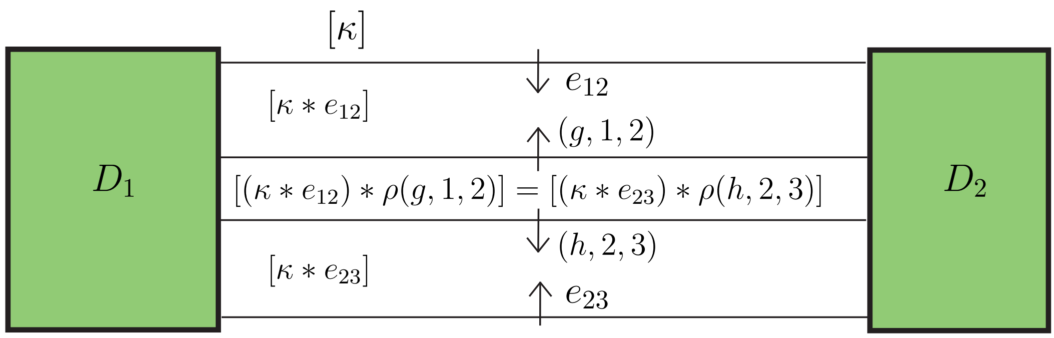


Figure 7: $C \in \mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_1 \# D_2)$ with arcs β_1, β_2, β_3 and β_4

We will show a presentation of a labeled diagram of the connected sum.

Proposition 4.3. *We let D_1 and D_2 be labeled diagrams presenting 3-manifolds M_1 and M_2, respectively. Then the connected sum of M_1 and M_2 is presented by the labeled diagram obtained from a transformation of D_1 and D_2 as illustrated in Figure 8.*

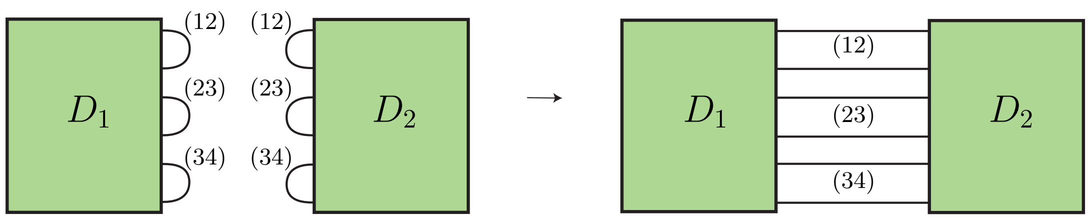


Figure 8: A transformation for a covering presentation of the connected sum of two 3-manifolds

We remark that this transformation does not depend on the choice of arcs of labeled diagrams D_1 and D_2, with the labels (12), (23) and (34).

Proof. We first explain a branched covering of a 3-ball B^3. We put three trivial tangles $\mathcal{T}$ in B^3 shown in Figure 9. Since $\pi_1(B^3 \setminus \mathcal{T})$ is a free group of rank 3, we put a homomorphism $\pi_1(B^3 \setminus \mathcal{T}) \to \mathfrak{S}_4$ which sends the three meridians to $(12), (23), (34) \in \mathfrak{S}_4$, respectively. Notice that a 4-fold branched covering associated to the homomorphism is also a 3-ball (cf. [PS, Example 23.5]).

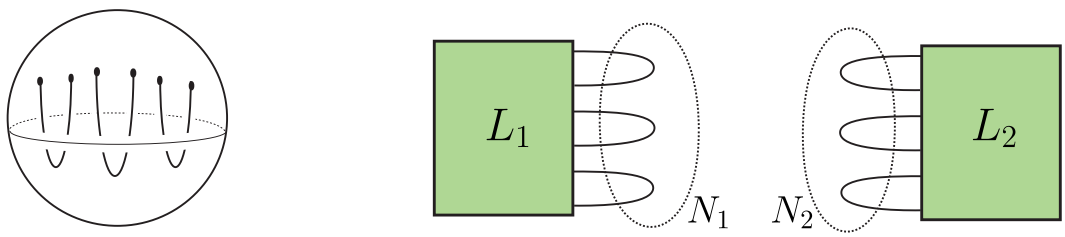


Figure 9: : The tangle $\mathcal{T}$ in B^3 and two neighborhoods $N_1 \subset S^3$ and $N_2 \subset S^3$.

Next, let $L_1 \subset S^3$ and $L_2 \subset S^3$ be links corresponding to the labeled diagrams D_1 and D_2, respectively. We remove from S^3 an open 3-ball N_1 (resp. N_2) which intersects L_1

(resp. L_2) transversally at 6-points as shown in Figure 9. Notice that, for $i \in \{1,2\}$, the 4-fold covering branched over the tangles $L_i \setminus (N_i \cap L_i) \subset S^3 \setminus N_i$ is precisely $M_i \setminus B^3$ by the previous discussion. Note that the boundaries of the pairs of $\big(S^3 \setminus N_1, L_1 \setminus (N_1 \cap L_1)\big)$ and of $\big(S^3 \setminus N_2, L_2 \setminus (N_2 \cap L_2)\big)$ are homeomorphic. We attach $S^3 \setminus N_1$ to $S^3 \setminus N_2$ by the homeomorphism, similar to the right in Figure 8. In sequel, the 4-fold covering branched over the resulting link turns out to be the connected sum of M_1 and M_2 by definition. $\square$

Lastly, we will deal with the quandle homotopy invariant of the inverse orientation.

Proposition 4.4. *Let $-M$ denote the 3-manifold M with the opposite orientation. For a finite cored group (G,c),*

$$\Xi^{4\mathrm{f}}_{\widetilde{G}_c}(-M) = \iota\big(\Xi^{4\mathrm{f}}_{\widetilde{G}_c}(M)\big) \in \mathbb{Z}[\Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_c)],$$

where ι is a map of $\mathbb{Z}[\Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_c)]$ induced by $\iota(x) = x^{-1}$ for any $x \in \Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_c)$.

Proof. We choose a labeled diagram D_ϕ of M. Then the mirror image $-D_\phi$ presents $-M$. It follows from the definition of $\widetilde{G}_c$-colorings that there exists a natural bijection $\varphi : \mathrm{Col}_{\widetilde{G}_c,\rho}(D_\phi) \to \mathrm{Col}_{\widetilde{G}_c,\rho}(-D_\phi)$. From the definition of $\Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_c)$, we have $\Xi^{4\mathrm{f}}_{\widetilde{G}_c}(-D_\phi;\varphi(C)) = \iota\big(\Xi^{4\mathrm{f}}_{\widetilde{G}_c,\rho}(D_\phi;C)\big) \in \Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_c)$. Hence, we obtain the required formula. $\square$

5 Dijkgraaf-Witten invariant

In this section, we compare the quandle homotopy invariants with bordism Dijkgraaf-Witten invariants. In §5.1, we prepare the bordism Dijkgraaf-Witten invariant. In §5.2, we show that the 4-fold symmetric quandle homotopy invariant is at least as strong as the bordism Dijkgraaf-Witten invariant (Theorem 5.4). We sometimes consider manifolds with some boundaries.

5.1 Preliminary: bordism Dijkgraaf-Witten invariant

Let (G,c) be a cored group and let $n \in \mathbb{Z}$ be ≥ 3. In this subsection, we make a modification of Dijkgraaf-Witten invariant in the view of an oriented bordism group of (G,c). To do this, we begin constructing the oriented bordism group. We consider a pair of a closed n-manifold M without boundary and a homomorphism $\pi_1(M) \to G$. Then a set $\Omega_n(G,c)$ is defined to be the quotient of such pairs of $(M, \pi_1(M) \to G)$ subject to the following (G,c)-bordant equivalence. Such a pair $(M,\ f : \pi_1(M) \to G)$ is (G,c)-*bordant*, if there exist an $(n+1)$-manifold W, two homomorphisms $\bar{f} : \pi_1(W) \times \mathbb{Z}/2\mathbb{Z} \to G$ and $\tilde{f} : \pi_1(M) \to \mathbb{Z}/2\mathbb{Z}$ such that $\bar{f}(e,1) = c \in G$, the boundary is $\partial W = M$, and $f = \bar{f} \circ \big((i_M)_* \times \tilde{f}\big)$, where $i_M : M \to W$ is the natural inclusion. Further, $\Omega_n(G,c)$ has an abelian group structure by the connected sum. More precisely, for such two pairs of $(M_i,\ f_i : \pi_1(M_i) \to G)$ with $i \in \{1,2\}$, the multiplication is defined by

$$(M_1,\ f_1 : \pi_1(M_1) \to G) \cdot (M_2,\ f_2 : \pi_1(M_2) \to G) := (M_1 \# M_2,\ f_1 * f_2 : \pi_1(M_1 \# M_2) \to G),$$

where $f_1 * f_2$ is the free product of f_1 and f_2. The inverse element of $(M, f : \pi_1(M) \to G)$ is $(-M, f : \pi_1(M) \to G)$. We call $\Omega_n(G, c)$ the *oriented bordism group* of (G, c). By using the obstruction theory, we notice that, when $c = e$, the group $\Omega_n(G, e)$ coincides with the usual oriented (SO-)bordism group of the Eilenberg-MacLane space $K(G, 1)$ (cf. [DW, §6.4]). It is known [Con, Theorem 15.2] that there is an isomorphism $\Omega_n(G, e) \cong \sum_{n=p+q} H_p(K(G, 1); \Omega_q(\text{pt.}))$ modulo the class of torsion groups of odd order. In the case where G is a finite subgroup of $SU(2)$, Katsube has obtained the complete list of $\Omega_n(G, e)$ [Kat].

Next, we construct bordism Dijkgraaf-Witten invariants. We fix an n-manifold M with no boundary. We put a natural map $\Omega_n^{G_c}(M; \bullet)$ from $\mathrm{Hom}_{\mathrm{grp}}(\pi_1(M), G)$ to $\Omega_n(G, c)$ which sends $f : \pi_1(M) \to G$ to the canonical class $[(M, f)]$. If G is finite, then the *bordism Dijkgraaf-Witten invariant* of M is defined by

$$\mathrm{DW}_\Omega^{G_c}(M) := \sum_{f \in \mathrm{Hom}_{\mathrm{grp}}(\pi_1(M), G)} \Omega_n^{G_c}(M; f) \in \mathbb{Z}[\Omega_n(G, c)].$$

Although the definition seems naive, however, this formulation plays a key role to study a relation to 4-fold symmetric quandle homotopy invariants in §5.2.

We now discuss the usual *Dijkgraaf-Witten invariant* of M given in (1) with using a group cocycle $\psi \in H^n(K(G, 1); A)$, where A is a trivial coefficient group. Here we assume $c = e$. For this, recall Thom homomorphism $\tau_{G,A} : \Omega_n(G, e) \to H_n(K(G, 1); A)$ obtained by assigning to every pair of (M, f) the image of $[M]$ under $f_* : H_n(M; A) \to H_n(K(G, 1); A)$, where $[M] \in H_n(M; A)$ is the fundamental class of M. Then, in the case of $c = e$, the usual Dijkgraaf-Witten invariant of M with $\psi \in H^n(K(G, 1); A)$ can be formulated as

$$\mathrm{DW}_\psi(M) = \sum_{f \in \mathrm{Hom}_{\mathrm{grp}}(\pi_1(M), G)} \langle \psi, f_*([M]) \rangle = \sum_{f \in \mathrm{Hom}_{\mathrm{grp}}(\pi_1(M), G)} \langle \psi, \tau_{G,A}(\Omega_n^{G_e}(M; f)) \rangle \in \mathbb{Z}[A], \quad (14)$$

where $\langle \, , \, \rangle$ means the canonical pairing. Furthermore, in a special case of $G = SL(2; \mathbb{C})$, let $\widehat{C}_2 \in H^3(K(G, 1); \mathbb{C}/4\pi^2\mathbb{Z})$ be the Cheeger-Chern-Simons class introduced in [CS]. The pairing $\langle \widehat{C}_2, \tau_{G, \mathbb{C}/4\pi^2\mathbb{Z}}(\Omega_3^{G_e}(M; f)) \rangle \in \mathbb{C}/4\pi^2\mathbb{Z}$ of $f : \pi_1(M) \to G$ is called the *Chern-Simons invariant* of f, which we will discuss in §6.

Remark 5.1. We see that, in the case $n = 3$ and $A = \mathbb{Z}$, the Thom homomorphism $\tau_{G,\mathbb{Z}}$ gives rise to an isomorphism $\Omega_3(G, e) \cong H_3(K(G, 1); \mathbb{Z})$ by Atiyah-Hirzebruch spectral sequence. In conclusion, this implies an equivalence between 3-dimensional Dijkgraaf-Witten invariants coming from the oriented bordism group and from the group homology.

Remark 5.2. We remark a relation between $\Omega_n(G, c)$ and $\Omega_n(G, e)$. By definition, (G, c)-bordance is stronger than (G, e)-bordance, unless $c = e$. We thus have a natural epimorphism $\Omega_n(G, e) \to \Omega_n(G, c)$, which implies bordism Dijkgraaf-Witten invariants of (G, c) are derived from that of (G, e).

5.2 From $\Pi_{2,\rho}^{4f}(\widetilde{G}_c)$ to the oriented bordism group $\Omega_3(G,c)$

Returning into the 4-fold symmetric quandle homotopy invariant, we mainly deal with 3-manifolds without boundary. Our goal is to obtain an epimorphism $\Phi_{\Pi\Omega} : \Pi_{2,\rho}^{4f}(\widetilde{G}_c) \to \Omega_3(G,c)$ (Theorem 5.4). For this, the following is a key lemma:

Lemma 5.3. *Assume that two $\widetilde{G}_c$-colorings $C_1 \in \mathrm{Col}_{\widetilde{G}_c,\rho}(D_\phi)$ and $C_2 \in \mathrm{Col}_{\widetilde{G}_c,\rho}(D'_{\phi'})$ are related by Reidemeister moves, MI, MII moves or symmetric concordance relations. For $i \in \{1,2\}$, let C_i correspond to a 3-manifold M_i with $\pi_1(M_i) \to G$ by Theorem 3.3. Then their connected sum $\big((-M_1)\# M_2,\ \pi_1(M_1\# M_2) \to G\big)$ is (G,c)-bordant.*

We defer the proof later, and will state Theorem 5.4. Put a composite map $\mathrm{Col}_{\widetilde{G}_c,\rho}(D_\phi) \to G^3 \times \mathrm{Hom}_{\mathrm{grp}}(\pi_1(M), G) \xrightarrow{\mathrm{proj}} \mathrm{Hom}_{\mathrm{grp}}(\pi_1(M), G)$, where the first map is the bijection obtained from Theorem 3.3. Moreover, recall the definition of $\Pi_{2,\rho}^{4f}(\widetilde{G}_c)$ in §2. By running over all homomorphisms $f : \pi_1(M) \to G$ of all 3-manifolds and over all $\widetilde{G}_c$-coloring of all link diagrams, the maps $\mathrm{Col}_{\widetilde{G}_c,\rho}(D_\phi) \to \mathrm{Hom}_{\mathrm{grp}}(\pi_1(M), G)$ give rise to a map

$$\Phi_{\Pi\Omega} : \Pi_{2,\rho}^{4f}(\widetilde{G}_c) \longrightarrow \Omega_3(G,c). \tag{15}$$

By the presentation of the connected sum in Proposition 4.3, the map is multiplicative. Moreover, the homomorphism is surjective by construction. In conclusion, when G is finite, we easily see

Theorem 5.4. *Let (G,c) be a finite cored group. Then the bordism Dijkgraaf-Witten invariant is derived from the 4-fold symmetric quandle homotopy invariant by the formula*

$$|G|^3 \cdot \mathrm{DW}_\Omega^{G_c}(M) = \Phi_{\Pi\Omega}\big(\Xi_{\widetilde{G}_c}^{4f}(M)\big) \in \mathbb{Z}[\Omega_3(G,c)].$$

In particular, when $c = e$, the Dijkgraaf-Witten invariant $\mathrm{DW}_\psi(M)$ of $\psi \in H^3(K(G,1);A)$ is derived from $\Xi_{\widetilde{G}_e}^{4f}(M)$ through the formula (14).

Remark 5.5. Let $\Xi_{SQ(M)}^{4f}(D_\phi; \mathrm{id}_{SQ(M)})$ be the fundamental class of M defined in §3.4. Then we have $\Phi_{\Pi\Omega}\big(\Xi_{SQ(M)}^{4f}(D_\phi; \mathrm{id}_{SQ(M)})\big) = (M, \mathrm{id}_{\pi_1(M)}) \in \Omega_3(G(M), c(M))$.

Remark 5.6. Wakui [W] formulated the Dijkgraaf-Witten invariant $\mathrm{DW}_\psi(M)$, and rigorously proved the topological invariance under "normalized conditions" of 3-cocycles ψ (see (17) for the definition). However, some 3-cocycles can not be related to any non-trivial normalized 3-cocycle up to coboundary. For instance, $G = \mathbb{Z}/6\mathbb{Z}$ has no normalized 3-cocycle, while $H^3(K(G,1))$ does not vanish. Namely, for some 3-cocycles ψ, the formulas in [W] are not permitted to manage $\mathrm{DW}_\psi(M)$. However, Theorem 5.4 enables us to deal with $\mathrm{DW}_\psi(M)$ of *any* 3-cocycle ψ from the perspective of 4-fold symmetric quandle homotopy invariants $\Xi_{\widetilde{G}_c}^{4f}(M)$.

Proof of Lemma 5.3. First, note that if C_1 and C_2 are related by some sequences of Reidemeister moves, MI and MII moves, then D_ϕ and $D'_{\phi'}$ present the same 3-manifold M_1 (see, e.g., [N2, §2.2] for details). Further, C_1 and C_2 present the same pair of $(M_1, \pi_1(M_1) \to G)$. Therefore, $\big((-M_1)\#M_2,\ \pi_1(M_1\#M_2) \to G\big)$ is (G,c)-bordant.

Next, assume that C_1 and C_2 are related by one of concordance relations shown as in the left of Figure 10. It is known that $S^2 \times S^1$ is presented by the labeled diagram consisting of four unknots illustrated in Figure 10 (see [R, §10.C.3]). Hence the labeled

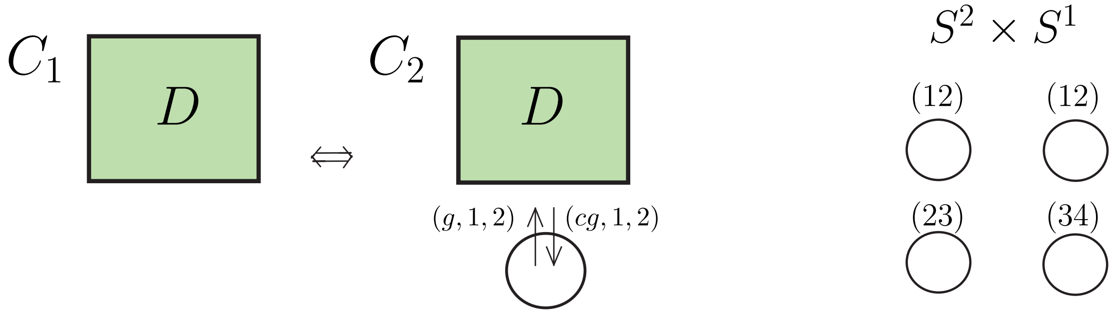


Figure 10: A symmetric concordance relation and the labeled diagram of $S^2 \times S^1$

diagram $D'_{\phi'}$ presents $M_2 = M_1\#(S^2 \times S^1)$ by Proposition 4.3. We therefore have

$$\big((-M_1)\#M_2, \pi_1(M_1\#M_2) \to G\big) = \big((-M_1)\#M_1\#(S^2\times S^1), \pi_1(M_1\#M_1\#(S^2\times S^1)) \to G\big).$$

To show its (G,c)-bordance, we have to check that, for any $f : \pi_1(S^2 \times S^1) \to G$, the pair of $\big(S^2 \times S^1, f : \pi_1(S^2 \times S^1) \to G\big)$ is (G,c)-bordant. In fact, $\big(B^3 \times S^1, \bar{f} = f : \pi_1(B^3 \times S^1) \to G\big)$ gives the required (G,c)-bordance, where B^3 is a 3-ball.

Next, we assume that C_1 and C_2 are related by another symmetric concordance relation shown in Figure 11. Let us define a surface F_L embeded in $S^3 \times [0,1]$ as follows. Let L_1 and $L_2 \subset S^3$ be links corresponding to C_1 and C_2, respectively. Let $N_{C_1} \subset S^3$ and $N_{C_2} \subset S^3$

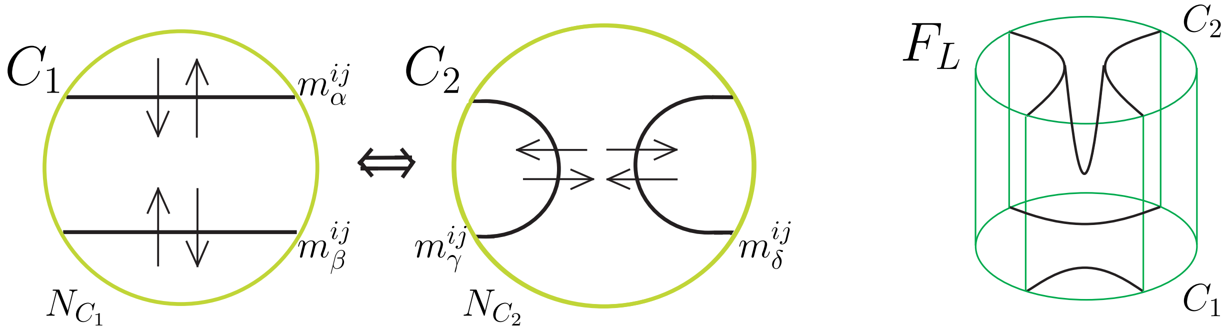


Figure 11: Another symmetric concordance relation and a suddle of $F_L \cap (B_C \times [0,1])$

be neighborhoods illustrated as Figure 11. Then we define $F_L \cap \big((S^3 \setminus N_{C_1}) \times [0,1]\big)$ to be $\big(L_1 \setminus (L_1 \cap N_{C_1})\big) \times [0,1]$, and define $F_L \cap (N_{C_1} \times [0,1])$ to be a saddle shown in Figure 11.

We construct a branched covering of $S^3 \times [0,1]$ branched over F_L as follows. Denote two arcs by m^{ij}_α and m^{ij}_β (resp. m^{ij}_γ and m^{ij}_δ) on C_1 (resp. C_2) as shown in Figure 11. Note that $\pi_1(S^3 \times [0,1] \setminus F_L)$ is presented by

$$\pi_1(S^3 \times [0,1] \setminus F_L) = \pi_1(S^3 \setminus L_1)/\langle m^{ij}_\alpha = m^{ij}_\beta\rangle = \pi_1(S^3 \setminus L_2)/\langle m^{ij}_\gamma = m^{ij}_\delta\rangle.$$

We put the monodromies $\phi : \pi_1(S^3 \setminus L_1) \to \mathfrak{S}_4$ and $\phi' : \pi_1(S^3 \setminus L_2) \to \mathfrak{S}_4$ associated to C_1 and C_2, respectively. Then, they can be uniquely extended to a homomorphism $\tilde{\phi} : \pi_1(S^3 \times [0,1] \setminus F_L) \to \mathfrak{S}_4$. We denote by W the associated 4-fold branched covering of $S^3 \times [0,1]$. Remark that the boundaries of W are precisely $(-M_1) \sqcup M_2$.

Next, let us construct a homomorphism $f_W : \pi_1(W) \times \mathbb{Z}/2\mathbb{Z} \to G$ as follows. We put the covering $W \setminus \widetilde{F_L} \to S^3 \times [0,1] \setminus F_L$ associated to $\tilde{\phi}$, where $\widetilde{F_L}$ is a surface composed of the preimage of the branched locus in W. Note that the fundamental group can be formulated as

$$\pi_1(W \setminus \widetilde{F_L}) \cong \pi_1(\widetilde{S^3 \setminus L_1})/\langle \widetilde{m}^{ij}_\alpha = \widetilde{m}^{ij}_\beta \rangle \cong \pi_1(\widetilde{S^3 \setminus L_2})/\langle \widetilde{m}^{ij}_\gamma = \widetilde{m}^{ij}_\delta \rangle.$$

We define the quotient group $G(W)$ by

$$G(W) := \pi_1(\widetilde{S^3 \setminus L_1})/\langle N_\phi,\ \widetilde{m}^{ij}_\alpha = \widetilde{m}^{ij}_\beta \rangle \ \cong \ \pi_1(\widetilde{S^3 \setminus L_2})/\langle N_\phi,\ \widetilde{m}^{ij}_\gamma = \widetilde{m}^{ij}_\delta \rangle, \tag{16}$$

where N_ϕ is the normal subgroup given in (5). Similar to Lemma 3.2, we can see $G(W) \cong \pi_1(W) \times \mathbb{Z}/2\mathbb{Z}$. On the other hand, by the proof of Proposition 3.1, the $\widetilde{G}_c$-colorings C_1 and C_2 give rise to homomorphisms $\widetilde{\Psi}_{f_1} : \pi_1(\widetilde{S^3 \setminus L_1}) \to G^4 \rtimes (\mathfrak{S}_4)_1$ and $\widetilde{\Psi}_{f_2} : \pi_1(\widetilde{S^3 \setminus L_2}) \to G^4 \rtimes (\mathfrak{S}_4)_1$, respectively. By projecting on the first component $\pi^G_1 : G^4 \rtimes (\mathfrak{S}_4)_1 \to G$, we notice

$$c = \pi^G_1(\widetilde{\Psi}_{f_1}(\widetilde{m}^{ij}_\alpha)) = \pi^G_1(\widetilde{\Psi}_{f_1}(\widetilde{m}^{ij}_\beta)) = \pi^G_1(\widetilde{\Psi}_{f_2}(\widetilde{m}^{ij}_\gamma)) = \pi^G_1(\widetilde{\Psi}_{f_2}(\widetilde{m}^{ij}_\delta)) \in G.$$

Hence, it follows from the presentation (16) that $\widetilde{\Psi}_{f_1}$ and $\widetilde{\Psi}_{f_2}$ passes to a homomorphism $f_W : G(W) = \pi_1(W) \times \mathbb{Z}/2\mathbb{Z} \to G$. For the natural inclusions $i_1 : M_1 \to W$ and $i_2 : M_2 \to W$, we see $f_1 = f_W \circ (i_1)_*$ and $f_2 = f_W \circ (i_2)_*$ by construction. Therefore, f_W gives a (G, c)-bordance of $(-M_1)\#M_2$. □

About Theorem 5.4, we pose a problem:

Problem 5.7. Is the epimorphism $\Phi_{\Pi\Omega}$ isomorphic? For what kinds of cored groups (G, c) are the 4-fold symmetric quandle homotopy invariants stronger than Dijkgraaf-Witten invariants?

Note that if the epimorphism $\Phi_{\Pi\Omega}$ is isomorphic, then the two invariants are equivalent by Theorem 5.4. If we expect that $\Phi_{\Pi\Omega}$ is isomorphic, it would come down to a problem whether any 4-manifold with boundaries is a 4-fold simple covering branched over a locally flat surface in a 4-ball or not. For reference, we remark the result of Iori and Piergallini [IP], which says that any *PL closed* 4-manifold is a *5-fold* simple covering of S^4 branched over a locally flat surface in S^4.

On the other hand, in order to show that the epimorphism $\Phi_{\Pi\Omega}$ would not be isomorphic, *4-fold symmetric quandle cocycle invariants* [N2, §7] might be useful, since they are computable and are derived from 4-fold symmetric quandle homotopy invariants.

5.3 Applications

We give two corollaries of Theorem 5.4. First, we conclude the 4-fold symmetric quandle homotopy invariant of $(G, c) = (\mathbb{Z}/2\mathbb{Z}, 0)$.

Corollary 5.8. *For $(G, c) = (\mathbb{Z}/2\mathbb{Z}, 0)$, $\Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_c) \cong \mathbb{Z}/2\mathbb{Z}$. Further, the 4-fold symmetric quandle homotopy invariant equals the Dijkgraaf-Witten invariant of G.*

Proof. By [N2, Proposition 6.6], $\Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_e)$ is either 0 or $\mathbb{Z}/2\mathbb{Z}$. However, the group homology $H_3(G;\mathbb{Z})$ is $\mathbb{Z}/2\mathbb{Z}$. Since $H_3(G;\mathbb{Z}) \cong \Omega_3(G, e)$, we have $\Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_c) \cong \mathbb{Z}/2\mathbb{Z}$ by the epimorphism (15). Further, Theorem 5.4 immediately results the latter statement. □

Next, we give another corollary (Corollary 5.9). For this, we briefly review the quandle homotopy invariant of a dihedral quandle (see [N2, §9] for detail). The *dihedral quandle* R_m of odd order is defined by $\mathbb{Z}/m\mathbb{Z}$ with a binary operation $x * y = 2y - x$. Let L be an oriented link in S^3, and $D_{\mathfrak{o}}$ an oriented link diagram of L. Then an R_m-coloring of $D_{\mathfrak{o}}$ is defined by a map from arcs on $D_{\mathfrak{o}}$ to R_m satisfying the right of Figure 1 at each crossings. Let us denote by $\mathrm{Col}_{R_m}(D_{\mathfrak{o}})$ the set of R_m-colorings of $D_{\mathfrak{o}}$. Further, $\Pi_2(R_m)$ is defined by the quotient of all R_m-colorings of all oriented link diagrams subject to Reidemeister moves and concordance relations. $\Pi_2(R_m)$ has an abelian group structure by disjoint union. *The quandle homotopy invariant* of L is defined by the formula

$$\Xi^{o}_{R_m}(L) = \sum_{C \in \mathrm{Col}_{R_m}(D_{\mathfrak{o}})} \Xi^{o}_{R_m}(D_{\mathfrak{o}}; C) \in \mathbb{Z}[\Pi_2(R_m)],$$

where $\Xi^{o}_{R_m}(D_{\mathfrak{o}}; \bullet)$ is a map $\mathrm{Col}_{R_m}(D_{\mathfrak{o}}) \to \Pi_2(R_m)$ sending an R_m-coloring of $D_{\mathfrak{o}}$ to the canonical class. The second author showed [N1, Remark 4.4] that $\Pi_2(R_m)$ is a quotient of $\mathbb{Z}/m\mathbb{Z}$, and [N1, Proposition 4.1] that if m is odd prime, then $\Pi_2(R_m) \cong \mathbb{Z}/m\mathbb{Z}$ [1].

Corollary 5.9. *Let m be an odd number, and M_L the double branched covering of a link $L \subset S^3$. Then, $\Pi_2(R_m) \cong \mathbb{Z}/m\mathbb{Z}$. Moreover, the quandle homotopy invariant $\Xi^{o}_{R_m}(L)$ is equal to a scalar multiple of the bordism Dijkgraaf-Witten invariant $\mathrm{DW}^{G_e}_{\Omega}(M_L) \in \mathbb{Z}[\Omega_3(G, e)]$, where $G = \mathbb{Z}/m\mathbb{Z}$ and we identify $\Omega_3(G, e)$ with $\mathbb{Z}/m\mathbb{Z}$.*

Proof. We first give a commutative diagram below. Let D be a link diagram of L without orientation. We consider the subquandle composed of $\{(g, 1, 2) \in \widetilde{G}_e \mid g \in G\}$, which is isomorphic to R_m by definition. Then, regarding D as a labeled diagram D_ϕ similar to Figure 5, we obtain $\mathrm{Col}_{R_m,\mathrm{id}}(D) \simeq G \times \mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_e,\rho}(D_\phi)$. Further, there is a canonical bijection $\mathfrak{P}_{R_m} : \mathrm{Col}_{R_m}(D_{\mathfrak{o}}) \to \mathrm{Col}_{R_m,\mathrm{id}}(D)$ (see [N2, (1)]). Then, from the definitions of the quandle homotopy invariant and of the bordism Dijkgraaf-Witten invariant, we obtain the following commutative diagram:

[1] In [N1], the second author discussed a homotopy group $\pi_2(BR_m)$. $\Pi_2(R_m) \cong \pi_2(BR_m)$ is known (see also [N2, §6.1]).

$$\begin{array}{ccccccc}
\mathrm{Col}_{R_m}(D_{\mathfrak{o}}) & \xrightarrow{\mathfrak{P}_{R_m}} \mathrm{Col}_{R_m,\mathrm{id}}(D) \xrightarrow{1:1} & G \times \mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_e,\rho}(D_\phi) & \xrightarrow{\text{Corollary 3.4}} & G \times \mathrm{Hom}(\pi_1(M_L), G) \\
\downarrow {\scriptstyle \Xi^o_{R_m}(D_{\mathfrak{o}};\bullet)} & & \downarrow {\scriptstyle \Xi^{4\mathrm{f}}_{\widetilde{G}_e}(D_\phi;\bullet)} & & \downarrow {\scriptstyle \Omega^{G_e}_3(M_L;\ \bullet)} \\
\Pi_2(R_m) & \xrightarrow{\qquad \mathcal{P}^{4\mathrm{f}} \qquad} & \Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_e) & \xrightarrow{\qquad \Phi_{\Pi\Omega} \qquad} & \Omega_3(G,e)
\end{array}$$

Here, $\Phi_{\Pi\Omega} : \Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_e) \to \Omega_3(G,e)$ is the homomorphism given in (15), and $\mathcal{P}^{4\mathrm{f}} : \Pi_2(R_m) \to \Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_e)$ is the homomorphism defined in [N2, Remark 6.4]. Notice that the top maps are bijective by Corollary 3.4. Hence, to complete the proof, it suffices to show that the composite $\mathcal{P}^{4\mathrm{f}} \circ \Phi_{\Pi\Omega}$ is isomorphic, since $\Omega_3(G,e) \cong H_3(K(G,1);\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}$.

To show the isomorphism, it is enough to prove the surjection, since $\Pi_2(R_m)$ is a quotient of $\mathbb{Z}/m\mathbb{Z}$. Notice that $\Omega_3(G,e)$ is generated by the lens space $L(m,1)$ with the identity map $\mathrm{id}_{\mathbb{Z}/m\mathbb{Z}} : \pi_1(L(m,1)) \to \mathbb{Z}/m\mathbb{Z}$. Remark that the double branched covering of the $(m,2)$-torus knot $T(m,2)$ is precisely $L(m,1)$. By Theorem 3.3 we have a non-trivial R_m-coloring $C_{T(m,2)}$ of $T(m,2)$ corresponding with $\mathrm{id}_{\mathbb{Z}/m\mathbb{Z}} : \pi_1(L(m,1)) \to \mathbb{Z}/m\mathbb{Z}$. Hence, $\mathcal{P}^{4\mathrm{f}} \circ \Phi_{\Pi\Omega}$ is surjective by the above commutative diagram. □

Remark 5.10. The first author [H2] shows the similar equivalence between the "quandle 3-cocycle invariant" of L and the Dijkgraaf-Witten invariant of M_L. However, her result is shown in a certain condition of group 3-cocycles: for instance, if m is divisible by 3, then her equivalence holds only for trivial 3-cocycles. Note that in Corollary 5.9 we drop the condition. Further, since it is known (see [N1]) that the quandle homotopy invariant is the universal among quandle cocycle invariants, Corollary 5.9 is a generalization of [H2].

However, it is not easy to directly compute the quandle homotopy invariant. In the case where $m = p$ is an odd prime, let us compute the invariant by using a shadow cocycle invariant. For this, we recall that Mochizuki [Moc] calculated the third "quandle cohomology" $H^3_Q(R_p;\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$ and obtained a presentation of the generator θ_p, called *Mochizuki* 3-*cocycle*. The cocycle θ_p gives an invariant $\Phi_{\theta_p}(L) \in \mathbb{Z}[\mathbb{Z}/p\mathbb{Z}]$ of links L, called a *shadow cocycle invariant* (see, e.g., [Iwa2, §2] for detail). The second author showed [N1, Corollary 4.2] that, under the isomorphism $\Pi_2(R_p) \cong \mathbb{Z}/p\mathbb{Z}$, the shadow cocycle invariant $\Phi_{\theta_p}(L)$ equals a scalar multiple of the quandle homotopy invariant $\Xi^o_{R_p}(L) \in \mathbb{Z}[\Pi_2(R_p)]$. In conclusion, by Corollary 5.9, we immediately give the shadow cocycle invariant a topological meaning as follows:

Corollary 5.11. *With notation $L \subset S^3$ and M_L in Corollary 5.9, if p is an odd prime, then the shadow cocycle invariant $\Phi_{\theta_p}(L)$ equals a scalar multiple of the Dijkgraaf-Witten invariant $\mathrm{DW}^{G_e}_\Omega(M_L) \in \mathbb{Z}[\Omega_3(G,e)]$, where $(G,e) = (\mathbb{Z}/p\mathbb{Z}, 0)$ and $\Omega_3(G,e) \cong \mathbb{Z}/p\mathbb{Z}$.*

We note that $\mathrm{DW}_\psi(M_L)$ in (1) equals $\mathrm{DW}^{G_e}_\Omega(M_L)$ up to scalar multiples by Remark 5.1.

The shadow cocycle invariants of several links have been computed. Using the computations, we give some values of the Dijkgraaf-Witten invariants as follows.

Example 5.12. For integers l and n, we consider the 2-bridge link of type (l, n). Putting an odd integer $s < l$ such that $sn = 1 \pmod l$, the 2-fold branched covering of S^3 branched along the link is the lens space $L(l, s)$ (see [R, §10.C. Exercise 6]). According to [Iwa1, Theorem 1.1], if l is divisible by p, then

$$\mathrm{DW}_\Omega^{G_e}\big(L(l,s)\big) \doteq \Phi_{\theta_p}(L) = p^2 \sum_{0 \le i \le p-1} t^{\frac{-ls}{p} i^2} \in \mathbb{Z}[t]/(t^p - 1) \cong \mathbb{Z}[\mathbb{Z}/p\mathbb{Z}];$$

otherwise, the Dijkgraaf-Witten invariant is trivial. The results of the lens spaces are shown in various ways (see, e.g., [DW, MOO, W]).

Example 5.13. For an integer n and an odd number l, we consider the (l, n)-torus link $T(l, n)$. It is known (see [R, §10.E. Exercise 5]) that the double covering branched over the link is the Brieskorn manifold $M(2, l, n)$ defined by

$$M(2,l,n) := \{(u,v,w) \in \mathbb{C}^3 \mid u^2 + v^l + w^n = 0,\ |u|^2 + |v|^2 + |w|^2 = 1\}.$$

According to [AS, Theorem 6.3], if l is divisible by p and n is even, then

$$\mathrm{DW}_\Omega^{G_e}\big(M(2,l,n)\big) \doteq \Phi_{\theta_p}(T(l,n)) = p \sum_{0 \le i \le p-1} t^{\frac{-ln}{2p} i^2} \in \mathbb{Z}[t]/(t^p - 1);$$

otherwise, the invariant is trivial.

Example 5.14. We consider the pretzel link of type $(m_1, \dots, m_n)$ (see [Iwa2, Figure 1]). The double branched cover of the link is the Seifert manifold denoted by $(S^2; m_1, \dots, m_n)$. Hence, the Dijkgraaf-Witten invariant of the Seifert manifold equals a scalar multiple of the corresponding computation described in [Iwa2, Theorem 1.1], although we omit writing it.

6 Chern-Simons invariant as a quandle cocycle invariant

Our goal of this section is to reconstruct the Chern-Simons invariant of closed 3-manifolds as a certain quandle cocycle invariant (Theorem 6.5). §6.1 and 6.2 are the preparation. In this section, we follow two notation: "4-fold symmetric quandle 2-cocycles" and "4-fold symmetric quandle cocycle invariants" (see [N2, §7.1 and 7.2] for the definitions).

6.1 Review: 4-fold symmetric quandle cocycle from normalized group cocycle

We review a 4-fold symmetric quandle cocycle introduced in [H1]. For a cored group (G, c), we recall a map $* : G^4 \times \widetilde{G}_c \to G^4$ defined by

$$(s_1,s_2,s_3,s_4)*(g,1,2) = (cgs_2, g^{-1}s_1, s_3, s_4), \quad (s_1,s_2,s_3,s_4)*(g,1,3) = (cgs_3, s_2, g^{-1}s_1, s_4),$$

$$(s_1,s_2,s_3,s_4)*(g,1,4) = (cgs_4, s_2, s_3, g^{-1}s_1), \quad (s_1,s_2,s_3,s_4)*(g,2,3) = (s_1, cgs_3, g^{-1}s_2, s_4),$$

$$(s_1,s_2,s_3,s_4)*(g,2,4) = (s_1, cgs_4, s_3, g^{-1}s_2), \quad (s_1,s_2,s_3,s_4)*(g,3,4) = (s_1, s_2, cgs_4, g^{-1}s_3),$$

where $g \in G$ and $(s_1, s_2, s_3, s_4) \in G^4$. Then it is not hard to see that G^4 is a $(\widetilde{G}_c, \rho)$-set via the operation $*$. We fix the case of $c = e$ later.

We briefly review the group cohomology. Let $B_n(G)$ be the free $\mathbb{Z}$-module generated by symbols $(g_1, \ldots, g_n) \in G^n$. The differential map $\partial_n : B_n(G) \to B_{n-1}(G)$ is defined by

$$\partial_n(g_1, \ldots, g_n) := (g_2, \ldots, g_n) + \sum_{t=1}^{n-1} (-1)^t (g_1, \ldots, g_t g_{t+1}, \ldots, g_n) + (-1)^n (g_1, \ldots, g_{n-1}).$$

The pair of $(B_n(G), \partial_n)$ is a complex called *the inhomogeneous complex* of G. The homology is denoted by $H_n(G; \mathbb{Z})$. Dually, for an abelian group A, we can define the cochain complex $(\mathrm{Hom}(B_n(G), A), (\partial_n)^*)$. A 3-cocycle θ of the cochain complex is said to be *(strong) normalized*, if it satisfies the following: for any $x, y \in G$,

$$\theta(e, x, y) = \theta(x, e, y) = \theta(x, y, e) = \theta(x, x^{-1}, y) = \theta(x, y^{-1}, y) = 1_A. \tag{17}$$

This definition is stronger than the usual one by adding the last two equalities.

For a normalized 3-cocycle θ, we give a function $\mathcal{X}_\theta : G^4 \times \widetilde{G}_e \times \widetilde{G}_e \to A$ as follows:

$$\begin{aligned}
&\mathcal{X}_\theta\big((s_1, s_2, s_3, s_4), (g, i, j), (g', i, j)\big) \\
&= \theta(g, g^{-1}g', g'^{-1}g s_j) \cdot \theta(g', g'^{-1}g, g^{-1}s_i) \cdot \theta(g'g^{-1}g', g'^{-1}g, s_j) \cdot \theta(g', g^{-1}g', g'^{-1}s_i), \\
&\mathcal{X}_\theta\big((s_1, s_2, s_3, s_4), (g, i, j), (g', j, k)\big) = \theta(g'^{-1}, g^{-1}, s_i)^{-1} \cdot \theta(g'^{-1}, g^{-1}, g s_j), \\
&\mathcal{X}_\theta\big((s_1, s_2, s_3, s_4), (g, i, j), (g', k, l)\big) = 1_A.
\end{aligned}$$

The function $\mathcal{X}_\theta$ is introduced in [H1, §4.2], and the first author showed [H1, Proposition 4.1] that the resulting map $\mathcal{X}_\theta : G^4 \times \widetilde{G}_e \times \widetilde{G}_e \to A$ is a 4-fold symmetric quandle 2-cocycle in the sense of [N2, §7.2].

Recall that if given 4-fold symmetric quandle 2-cocycle ψ and a $\widetilde{G}_e$-coloring $C \in \mathrm{Col}_{\widetilde{G}_e, \rho}(D_\phi)$, we obtain an invariant $\Phi_\psi(D_\phi; C) \in A$ called a 4-fold symmetric quandle cocycle invariant (see [N2, §7.2]). Moreover, the first author showed

Theorem 6.1. *([H1, Theorem 4.2.]) Let θ be a normalized 3-cocycle of G. Under the correspondence between $\mathrm{Col}^{e_{12}, e_{23}, e_{34}}_{\widetilde{G}_e, \rho}(D_\phi)$ and $\mathrm{Hom}(\pi_1(M), G)$ by Theorem 3.3, for any $C_f \in \mathrm{Col}^{e_{12}, e_{23}, e_{34}}_{\widetilde{G}_e, \rho}(D_\phi)$ corresponding with $f \in \mathrm{Hom}(\pi_1(M), G)$, the 4-fold symmetric quandle cocycle invariant $\Phi_{\mathcal{X}_\theta}(D_\phi; C_f) = \langle f^*(\theta), [M] \rangle \in A$.*

Remark 6.2. The first author [H1] stated this statement with respect to finite groups. However, her proof holds for any group G as well. We further remark that a $(\widetilde{G}_c, \rho)$-set was called a "switching map" and the 4-fold symmetric quandle cocycle invariant was called "the state sum" in [H1]. It is shown [N2, Proposition 7.4] that the value $\Phi_{\mathcal{X}_\theta}(D_\phi; C_f)$ is derived from the 4-fold symmetric quandle homotopy invariant with $\widetilde{G}_c$.

On the other hand, we prepare the homogenous cochain of G. From now on, we fix $G = SL(2;\mathbb{C})$. Let $C_n(G)$ denote the free $\mathbb{Z}$-module on all $(n+1)$-tuples $\langle g_0, \dots, g_n\rangle \in G^{n+1}$. Define the differential map by

$$\delta_n : C_n(G) \to C_{n-1}(G), \qquad \langle g_0, \dots, g_n\rangle \mapsto \sum_{t=0}^{n} (-1)^t \langle g_0, \dots, g_{t-1}, g_{t+1}, \dots, g_n\rangle.$$

The complex $(C_*(G), \delta_*)$ is called *the homogenous complex* of G. Further, put a submodule $C_n^{\neq}(G)$ generated by all $(n+1)$-tuples $\langle g_0, \dots, g_n\rangle$ of distinct elements of G. Then $(C_*^{\neq}(G), \delta_*)$ is a subcomplex. Since G is infinite, the complexes $(C_*(G), \delta_*)$ and $(C_*^{\neq}(G), \delta_*)$ are acyclic (see, e.g., [DZ, Lemma 1.3]). Let us regard $C_n(G)$ and $C_n^{\neq}(G)$ as projective right $\mathbb{Z}[G]$-modules by the diagonal action of G. Hence, the inclusion $C_n^{\neq}(G) \hookrightarrow C_n(G)$ induces an isomorphism between the homologies of $(C_n(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z},\ \delta_n \otimes_{\mathbb{Z}[G]} \mathrm{id})$ and of $(C_n^{\neq}(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z},\ \delta_n \otimes_{\mathbb{Z}[G]} \mathrm{id})$, where we regard $\mathbb{Z}$ as a trivial left $\mathbb{Z}[G]$-module. Also, it is well-known that the complex $B_n(G)$ above is chain isomorphic to $C_n(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ via a map

$$\Delta_n : B_n(G) \longrightarrow C_n(G), \qquad (g_1, \dots, g_n) \longmapsto \langle e, g_1, g_1 g_2, \dots, g_1 g_2 \cdots g_n\rangle. \tag{18}$$

6.2 Review: Cheeger-Chern-Simons class from the extended Bloch group.

In this section, we recall a description of the Cheeger-Chern-Simons class. Our brief description follows from Dupont, Goette and Zickert [DG, §2 and §4], [DZ, §2 and §3].

We will construct a map $\widehat{C}_2 : Z_3^{\neq}(G) \to \mathbb{C}/4\pi^2\mathbb{Z}$ as follows, where $Z_3^{\neq}(G)$ is a $\mathbb{Z}$-submodule $\{\sigma \in C_3^{\neq}(G) \mid \delta_3(\sigma) = 0\}$. Recall the natural representation of $G = SL(2;\mathbb{C})$ on $\mathbb{C}^2$. For any basis $\sigma = \sum_m a_m \langle g_0^{(m)}, g_1^{(m)}, g_2^{(m)}, g_3^{(m)}\rangle \in Z_3^{\neq}(G)$ with $a_m \in \mathbb{Z}$, we choose an element $v_\sigma \in \mathbb{C}^2 \setminus \{0\}$ such that $\det(g_k^{(m)} \cdot v_\sigma, g_l^{(m)} \cdot v_\sigma) \neq 0$ for each $0 \leq k < l \leq 3$ and m. Let us denote $g_k^{(m)} \cdot v_\sigma$ by v_k^m for short. We prepare a homomorphism given by

$$\mathcal{Z} : C_3^{\neq}(G) \longrightarrow \mathbb{Z}\langle(\mathbb{C} \setminus \{0,1\}) \times 2\mathbb{Z} \times 2\mathbb{Z}\rangle, \qquad \sigma \longmapsto \sum_m a_m (z_m, p_m, q_m),$$

where the triple of (z_m, p_m, q_m) is defined by the following formulae:

$$z_m := \frac{\det(v_0^m, v_3^m)\det(v_1^m, v_2^m)}{\det(v_0^m, v_2^m)\det(v_1^m, v_3^m)},$$

$$\pi\sqrt{-1}p_m = \mathrm{Log}\det(v_0^m, v_3^m) + \mathrm{Log}\det(v_1^m, v_2^m) - \mathrm{Log}\det(v_0^m, v_2^m) - \mathrm{Log}\det(v_1^m, v_3^m) - \mathrm{Log}(z_m),$$

$$\pi\sqrt{-1}q_m = \mathrm{Log}\det(v_0^m, v_2^m) + \mathrm{Log}\det(v_1^m, v_3^m) - \mathrm{Log}\det(v_0^m, v_1^m) - \mathrm{Log}\det(v_2^m, v_3^m) - \mathrm{Log}\Big(\frac{1}{1 - z_m}\Big).$$

Remark that $\mathcal{Z}$ depends on the choice of elements $v_\sigma \in \mathbb{C}^2 \setminus \{0\}$, and $\mathcal{Z}$ is written by $\Phi \circ \lambda$ in [DZ]. Further, put a map

$$\overline{L} : (\mathbb{C} \setminus \{0,1\}) \times 2\mathbb{Z} \times 2\mathbb{Z} \longrightarrow \mathbb{C}/4\pi^2\mathbb{Z},$$

$$(z; p, q) \longmapsto -\int_0^z \frac{\mathrm{Log}(1-t)}{t}\mathrm{dt} + \frac{1}{2}\big(\mathrm{Log}(z) + 2\pi\sqrt{-1}p\big)\big(\mathrm{Log}(1-z) + 2\pi\sqrt{-1}q\big) - \frac{\pi^2}{6}.$$

In summary, a map $\widehat{C}_2 : Z_3^{\neq}(G) \to \mathbb{C}/4\pi^2\mathbb{Z}$ is defined by the composite $\overline{L} \circ \mathcal{Z}$.

In addition, since $Z_3^{\neq}(G)$ is a direct summand of $C_3^{\neq}(G)$ as a free $\mathbb{Z}$-module by definition, we extend $\widehat{C}_2$ to a homomorphism from $C_3^{\neq}(G)$. It is shown [DG, Theorem 4.1] that $\widehat{C}_2$ is a 3-cocycle, and that $\widehat{C}_2$ coincides with the Cheeger-Chern-Simons class $H_3(G;\mathbb{Z}) \to \mathbb{C}/4\pi^2\mathbb{Z}$. Further, it is known that the map $\widehat{C}_2$ does not depend on the choice of elements $v_\sigma \in \mathbb{C}^2 \setminus \{0\}$ above (see, e.g., [DZ, Proposition 3.4]).

Remark 6.3. The (more) extended pre-Bloch group $\widehat{\mathcal{P}}(\mathbb{C})$ described in [DG] can be defined as a quotient group of the free module $\mathbb{Z}\langle(\mathbb{C} \setminus \{0,1\}) \times 2\mathbb{Z} \times 2\mathbb{Z}\rangle$ (see also [Neu, §2]). The extended Bloch group $\widehat{\mathcal{B}}(\mathbb{C})$ is defined by a subgroup of $\widehat{\mathcal{P}}(\mathbb{C})$. The map $\widehat{C}_2$ factors through $\widehat{\mathcal{B}}(\mathbb{C})$ (see [DG]).

6.3 Chern-Simons invariant as a quandle cocycle invariant

We put the projection [2] $P_3^{\neq} : C_3(G) \to C_3^{\neq}(G)$ and a multiplication by 6: $\mathbb{C}/4\pi^2\mathbb{Z} \xrightarrow{6} \mathbb{C}/4\pi^2\mathbb{Z}$. Using the map Δ_3 in (18) and the Cheeger-Chern-Simons class $\widehat{C}_2$, we denote by $\overline{C}_2^B$ a composite

$$B_3(G) \xrightarrow{\Delta_3} C_3(G) \xrightarrow{P_3^{\neq}} C_3^{\neq}(G) \xrightarrow{\widehat{C}_2} \mathbb{C}/4\pi^2\mathbb{Z} \xrightarrow{6} \mathbb{C}/4\pi^2\mathbb{Z}.$$

Lemma 6.4. *The composite $\overline{C}_2^B : B_3(G) \longrightarrow \mathbb{C}/4\pi^2\mathbb{Z}$ is a normalized 3-cocycle.*

We defer the proof later. Combing this with Theorem 6.1, we immediately conclude

Theorem 6.5. *Let $G = SL(2;\mathbb{C})$. Let $\overline{C}_2^B \in B^3(G;\mathbb{C}/4\pi^2\mathbb{Z})$ be as above. Let $\mathcal{X}_{\overline{C}_2^B} : G^4 \times \widetilde{G}_e \times \widetilde{G}_e \to \mathbb{C}/4\pi^2\mathbb{Z}$ be the map given in §6.1. For $f \in \mathrm{Hom}(\pi_1(M), G)$, we put the associated $\widetilde{G}_e$-coloring $C_f \in \mathrm{Col}_{\widetilde{G}_e,\rho}^{e_{12},e_{23},e_{34}}(D_\phi)$ by Proposition 3.1. Then, the 4-fold symmetric quandle cocycle invariant coincides with 6 multiple of the Chern-Simons invariant: $\Phi_{\mathcal{X}_{\overline{C}_2^B}}(D;C_f) = 6\langle f^*(\widehat{C}_2), [M]\rangle \in \mathbb{C}/4\pi^2\mathbb{Z}$.*

We now explain some benefits of Theorem 6.5. Following the description of [Neu, Z], for the computation of the Chern-Simons invariant we have to choose a (flattened) triangulation of M. However, in general, a triangulation of M is composed of many simplicies, which makes the computation complicated.

On the other hand, we recall that 4-fold symmetric quandle cocycle invariants are computable by a presentation of the cocycle. Thereby, Theorem 6.5 says that if we know a labeled diagram presenting M and a $\widetilde{G}_e$-coloring corresponding with $\pi_1(M) \to G$, the formulation in Theorem 6.5 makes the Chern-Simons invariant computable without triangulation of M. Also, in [Neu], Neumann dealt with only hyperbolic 3-manifolds M with the holonomy presentation $\pi_1(M) \to G$. Notice that Theorem 6.5 are applied to any closed 3-manifolds M with any presentations $\pi_1(M) \to G$, similar to [Z].

[2] Caution: the projection $P_3^{\neq}$ never commutes with the boundary map δ_n.

In general, for any 3-manifold M, it is not easy to find a labeled diagram of M. However, if we find a labeled diagram of M, for $f : \pi_1(M) \to G$, it is easy to find a $\widetilde{G}_c$-coloring C_f corresponding with f by Proposition 3.1. We expect a good computer program for the calculation of the Chern-Simons invariant of f from labeled diagrams. It goes without saying that a double branched covering of a link L is precisely presented by a diagram of L with a label (12), similar to Figure 5. So, the Chern-Simon invariants of the double branched coverings of S^3 would be easily computable.

Proof of Lemma 6.4. . The proof follows the arguments of Neumann [Neu] and Dupont-Zickert [DZ]. Let us decompose $\mathbb{C}/4\pi^2\mathbb{Z} \cong \mathbb{R}/4\pi^2\mathbb{Z} \oplus \mathbb{R}\sqrt{-1}$. We shall refer to the real and imaginary parts of $\overline{C}_2^B$ separately.

First, notice that the imaginary part $\mathrm{Im}(\overline{C}_2^B) : B_3(SL(2;\mathbb{C})) \to \mathbb{R}$ is equal to the map $\mathrm{Im}(c)$ defined by Dupont (see [Dup, §1]). Further, Dupont showed [Dup, Proposition 3.1] that $\mathrm{Im}(c)$ equals the volume form of the de Rham cohomology $H^3_{dR}(SL(2;\mathbb{C});\mathbb{R})$. Since the imaginary part of Cheeger-Chern-Simons class $\widehat{C}_2$ is the volume form, $\mathrm{Im}(\overline{C}_2^B) = \mathrm{Im}(\widehat{C}_2)$.

Next, we consider the real part $\mathrm{Re}(\overline{C}_2^B)$. Recall the fact by Dupont, Parry and Sah [DPS]: the inclusion $SL(2;\mathbb{R}) \hookrightarrow SL(2;\mathbb{C})$ induces an isomorphism

$$H_3(SL(2;\mathbb{R});\mathbb{Z}) \cong \{\sigma \in H_3(SL(2;\mathbb{C});\mathbb{Z}) \mid \tau\sigma = \sigma\},$$

where τ is the involution induced by complex conjugation. This means that it suffices to study the restriction of $\mathrm{Re}(\overline{C}_2^B)$ on $SL(2;\mathbb{R})^4$. Remark that the restriction coincides with the homomorphism l defined in [Dup, §1]. Further, it is shown [Dup, Theorem 1.11] that $\frac{1}{4\pi^2}\mathrm{Re}(\overline{C}_2^B)$ is equal to the imaginary part of Cheeger-Chern-Simons class in $\mathbb{R}/\frac{1}{6}\mathbb{Z}$.

In summary, the function $\overline{C}_2^B$ is a 3-cocycle of $SL(2;\mathbb{C})$. Further, from the definition of the map Δ_3 given in (18), we immediately see that $\overline{C}_2^B$ is normalized. □

6.4 Recovery of Chern-Simons invariant from the multiplication by 6.

We also discuss a recovery of the Chern-Simons invariant in Theorem 6.5 from the multiplication by 6. To begin, we recall the decomposition of the divisible $\mathbb{Z}$-module $\mathbb{C}/4\pi^2\mathbb{Z}$, that is, $\mathbb{C}/4\pi^2\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z} \oplus \big(\bigoplus_{\lambda\in\Lambda}\mathbb{Q}\big)$, where Λ is a uncoutable set of formal indices. Note that we may regard the canonical inclusion $\mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}/4\pi^2\mathbb{Z}$ as the direct summand. Hence, for the required recovery, it sufficies to assume that the Chern-Simons invariant lies in $\mathbb{Q}/\mathbb{Z}$.

Using isomorphisms $\mathbb{Q}/\mathbb{Z} \cong \varinjlim \mathbb{Z}/n\mathbb{Z} \cong \varinjlim H_3(\mathbb{Z}/n\mathbb{Z};\mathbb{Z})$, we recall an inclusion $\iota : \mathbb{Q}/\mathbb{Z} \hookrightarrow H_3(SL(2;\mathbb{C});\mathbb{Z})$ induced by the map $\mathbb{Z}/n\mathbb{Z} \hookrightarrow SL(2;\mathbb{C})$ sending 1 to the matrix of a rotation of $2\pi/n$ (see [DZ, (1-5)]). Moreover, it is known [DG, Corollary 3.5 and Theorem 4.1] that for any $\alpha \in \mathbb{Q}/\mathbb{Z}$ we have $\widehat{C}_2\big(\iota(\alpha)\big)/4\pi^2 = \alpha$. In particular, a composite $H_3(\mathbb{Z}/m\mathbb{Z};\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z} \overset{\iota}{\hookrightarrow} H_3(SL(2;\mathbb{C});\mathbb{Z})$ induces a surjection

$$H^3(SL(2;\mathbb{C});\mathbb{C}/4\pi^2\mathbb{Z}) \longrightarrow H^3(\mathbb{Z}/m\mathbb{Z};\mathbb{C}/4\pi^2\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}, \quad (19)$$

which sends the Cheeger-Chern-Simons class $\widehat{C}_2$ to a generator of $\mathbb{Z}/m\mathbb{Z}$. Hence, if the Chern-Simons invariant is contained in $\mathbb{Q}/\mathbb{Z}$, the invariant is equivalent to the Dijkgraaf-Witten invariants of $\mathbb{Z}/m\mathbb{Z}$ for all $m \in \mathbb{Z}$. In particular, if we know the values of the Dijkgraaf-Witten invariants of $\mathbb{Z}/6^a\mathbb{Z}$ for all $a \in \mathbb{N}$, the Chern-Simons invariant makes a recovery from the multiplication by 6 in Theorem 6.5, as required.

It is not so difficult to compute the Dijkgraaf-Witten invariant of $\mathbb{Z}/6^a\mathbb{Z}$. For example, it is known [MOO] that this invariant of M is derived from $U(1)$-quantum invariant, and can be computable from the linking matrix of the Kirby diagram of M or from the cohomology ring $H^*(M;\mathbb{Z}/6\mathbb{Z})$. In conclusion, the recovery can be obtained by an easy calculation.

Lastly, we discuss normalized 3-cocycles of cyclic groups, and a non-recovery of the Cheeger-Chern-Simon class $\widehat{C}_2$. The following corollary is useful for the discussion in [H2].

Corollary 6.6. *Let $m \in \mathbb{Z}$. For $\phi \in H^3(\mathbb{Z}/m\mathbb{Z}; A)$, there exists a representative normalized 3-cocycle which is cohomologous to 6ϕ.*

Proof. By Lemma 6.4, the induced 3-cocycle of $\overline{C}_2^B$ via the surjection (19) is normalized. □

Remark 6.7. We discussed the recovery of the Chern-Simons invariant. On the other hand, we will explain why the Cheeger-Chern-Simon class $\widehat{C}_2$ can not be recovered from the multiplication by 6 as follows. From the definition of $B^3(G;A)$, we can verify that $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ have no non-trivial normalized 3-cocycle (see also Remark 5.6). Hence, if we assumed that $\widehat{C}_2 \in H^3(G;\mathbb{C}/4\pi^2\mathbb{Z})$ could be normalized, then some 3-cocycle of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ would be normalized via the inclusion $\mathbb{Z}/m\mathbb{Z} \hookrightarrow G$, which implies a contradiction. Therefore, for the application of Theorem 6.1 using a normalized 3-cocycle, we have to choose $\overline{C}_2^B$ rather than the original Cheeger-Chern-Simons class $\widehat{C}_2$.

Acknowledgment

We are grateful to Yuichi Kabaya, Makoto Sakuma, Takao Satoh and Michihisa Wakui for several discussions. We express our gratitude to Tomotada Ohtsuki for invaluable advice and his suggestions. Also, parts of this work were carried out, while the first author stayed at Kyoto University and the second author was visiting Tokyo University of Agriculture and Technology. We sincerely thank both institutions for their hospitality.

References

[Apo] N. Apostolakis, *On 4-fold covering moves*, Algebr. Geom. Topol. **3** (2003), 117–145.

[AS] S. Asami, S. Satoh, *An infinite family of non-invertible surfaces in 4-space*, Bull. London Math. Soc. **37** (2005), 285–296.

[Ati] M. F. Atiyah, *Bordism and cobordism*, Proc. Cambridge Philos. Soc. **57** (1961) 200–208.

[Con] P. E. Conner, *Differentiable Periodic Maps*, Lecture Notes in Mathematics 46. Springer-Verlag, 1967.

[CS] S. S. Chern, J. Simons, *Characteristic forms and geometric invariants*, Ann. of Math. (2) **99** (1974) 48–69.

[DW] R. Dijkgraaf, E. Witten, *Topological gauge theories and group cohomology*, Comm. Math. Phys. **129** (1990), 393–429.

[Dup] J. L. Dupont, *The dilogarithm as a characteristic class for flat bundles*, J. Pure and App. Algebra **44** (1987) 137–164.

[DG] J. L. Dupont, S. Goette, *The extended Bloch group and the Cheeger-Chern-Simons class*, Geom. Topol. **11** (2007), 1623–1635.

[DPS] J. L. Dupont, W. Parry, C. H. Sah, *Homology of classical Lie groups made discrete. II.* H_2, H_3*, and relations with scissors congruences*, J. Algebra **113** (1988), no. 1, 215–260.

[DZ] J. L. Dupont, C. K. Zickert, *A dilogarithmic formula for the Cheeger-Chern-Simons class*, Geom. Topol. **10** (2006), 1347–1372.

[FRS] R. Fenn, C. Rourke, B. Sanderson, *The rack space*, Trans. Amer. Math. Soc. **359** (2007) 701–740.

[H1] E. Hatakenaka, *Invariants of 3-manifolds derived from covering presentations*, in Math. Proc. Camb. Phil. Soc. **149** (2010) 263–296.

[H2] ———, *On the Dijkgraaf-Witten invariant and the quandle cocycle invariant*, in preparation.

[HMT] H.M. Hilden, J.M. Montesinos, T. Thickstun, *Closed oriented 3-manifolds as 3-fold branched coverings of* S^3 *of special type*, Pacific J. Math. **65** (1976), 65–76.

[IK] A. Inoue, Y. Kabaya, *Quandle homology and complex volume*, in preparation.

[IP] M. Iori, R. Piergallini, *4-manifolds as covers of the 4-sphere branched over non-singular surfaces*, Geom. Topol. **6** (2002), 393–401 (electronic).

[Iwa1] M. Iwakiri, *Calculation of dihedral quandle cocycle invariants of twist spun 2-bridge knots*, J. Knot Theory Ramifications **14** (2005) 217–229.

[Iwa2] ———, *Quandle cocycle invariants of prentzel links*, Hiroshima Math. J. **36** (2006) 353–363.

[Joy] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra **23** (1982) 37–65.

[Kam] S. Kamada, *Quandles with good involutions, their homologies and knot invariants*, in "Intelligence of Low Dimensional Topology 2006, Eds. J. S. Carter et. al". 101–108 World Scientific Publishing Co. (2007).

[KO] S. Kamada, K. Oshiro, *Homology groups of symmetric quandles and cocycle invariants of links and surface-links*, Trans. Amer. Math. Soc. **362**, (2010) 5501–5527.

[Kat] Y. Katsube, *Principal oriented bordism modules of finite subgroups of* S^3, Hiroshima Math. J. **7** (1977), no. 1, 71–91.

[Moc] T. Mochizuki, *Some calculations of cohomology groups of finite Alexander quandles*, J.Pure Appl. Algebra **179** (2003) 287–330.

[MOO] H. Murakami, T. Ohtsuki, M. Okada, *Invariants of three-manifolds derived from linking matrices of framed links,* Osaka J. Math. **29** (1992), no. 3, 545–572.

[Neu] W. D. Neumann, *Extended Bloch group and the Cheeger-Chern-Simons class*, Geom. Topol. **8** (2004), 413–474 (electronic).

[N1] T. Nosaka, *On homotopy groups of quandle spaces and the quandle homotopy invariant of links* (revised version), preprint.

[N2] ———, *4-fold symmetric quandle invariants of 3-manifolds*, preprint available at **http://www.kurims.kyoto-u.ac.jp/˜nosaka/indexEnglish.html**

[R] D. Rolfsen, *Knots and links*, Math. Lecture Series, 7, Publish or Perish, Inc., Houston, Texas, 1990, Second printing, with corrections.

[PS] V. V. Prasolov, A. B. Sossinsky, *Knots, links, braids and 3-manifolds – an introduction to the new invariants in low-dimensional topology*, Trans. of Math. Monographs **154**, Amer. Math. Soc., 1997.

[W] M. Wakui, *On Dijkgraaf-Witten invariant for 3-manifolds*, Osaka J. Math. **29** (1992), 675–696.

[Z] C. K. Zickert, *The volume and Chern-Simons invariant of a representation*, Duke Math. J. **150** (2009), no. 3, 489–532.

Department of Mathematics, Tokyo University of Agriculture and Technology, Naka, Koganei, Tokyo 184-8588, Japan
E-mail address: `hataken@cc.tuat.ac.jp`

Research Institute for Mathematical Sciences, Kyoto University, Sakyo-ku, Kyoto, 606-8502, Japan
E-mail address: `nosaka@kurims.kyoto-u.ac.jp`