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Abstract. In a series of works [18, 21, 19, 20, 23, 22], Geiß-Leclerc-Schröer defined the cluster algebra structure on the coordinate ring $\mathbb{C}[N(w)]$ of the unipotent subgroup, associated with a Weyl group element w. And they proved cluster monomials are contained in Lusztig's *dual semicanonical basis* $\mathcal{S}^*$. We give a set up for the quantization of their results and propose a conjecture which relates the quantum cluster algebras in [4] to the *dual canonical basis* $\mathbf{B}^{\mathrm{up}}$. In particular, we prove that the quantum analogue $\mathcal{O}_q[N(w)]$ of $\mathbb{C}[N(w)]$ has the induced basis from $\mathbf{B}^{\mathrm{up}}$, which contains quantum flag minors and satisfies a factorization property with respect to the 'q-center' of $\mathcal{O}_q[N(w)]$. This generalizes Caldero's results [7, 8, 9] from ADE cases to an arbitary symmetrizable Kac-Moody Lie algebra.

Contents

1. Introduction

1.1. The canonical basis $\mathbf{B}$ and the dual canonical basis $\mathbf{B}^{\mathrm{up}}$. Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra, $\mathbf{U}_q(\mathfrak{g})$ its associated quantized enveloping algebra, and $\mathbf{U}_q^-(\mathfrak{g})$ its negative part. In [39], Lusztig constructed the canonical basis $\mathbf{B}$ of $\mathbf{U}_q^-(\mathfrak{g})$ by a geometric method when $\mathfrak{g}$ is symmetric. In [25], Kashiwara constructed the (lower) global basis $G^{\mathrm{low}}(\mathscr{B}(\infty))$ by a purely algebraic method. Grojnowski-Lusztig [24] showed that the two bases coincide when $\mathfrak{g}$ is symmetric. We call the basis the *canonical basis*. There are two remarkable properties of the canonical basis, one is the positivity of structure constants of multiplication and comultiplications, and another is Kashiwara's crystal structure $\mathscr{B}(\infty)$, which is a combinatorial machinery useful for applications to representation theory, such as tensor product decomposition.

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Since $\mathbf{U}_q^-(\mathfrak{g})$ has a natural pairing which makes it into a (twisted) self-dual bialgebra, we consider the dual basis $\mathbf{B}^{\mathrm{up}}$ of the canonical basis in $\mathbf{U}_q^-(\mathfrak{g})$. We call it the *dual canonical basis*.

1.2. **Cluster algebras.** Cluster algebras were introduced by Fomin and Zelevinsky [15] and intensively studied also with Berenstein [16, 2, 17] with an aim of providing a concrete and combinatorial setting for the study of Lusztig's (dual) canonical basis and total positivity. Quantum cluster algebras were also introduced by Berenstein and Zelevinsky [4], Fock and Goncharov [13, 14, 12] independently. The definition of (quantum) cluster algebra was motivated by Berenstein and Zelevinsky's earlier work [3] where combinatorial and multiplicative structures of the dual canonical basis were studied for $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{sl}_3$. Let us quote from [15]:

> We conjecture that the above examples can be extensively generalized: for any simply-connected connected semisimple group G, the coordinate rings $\mathbb{C}[G]$ and $\mathbb{C}[G/N]$, as well as coordinate rings of many other interesting varieties related to G, have a natural structure of a cluster algebra. This structure should serve as an algebraic framework for the study of "dual canonical basis" in these coordinate rings and their q-deformations. In particular, we conjecture that all monomials in the variables of any given cluster (the cluster monomials) belong to this dual canonical basis.

In [2], it was shown that the coordinate ring of the double Bruhat cell has a part of structures of a cluster algebra.

A cluster algebra $\mathscr{A}$ is a subalgebra of rational function field $\mathbb{Q}(x_1, x_2, \cdots, x_r)$ of r indeterminates which is equipped with a distinguished set of generators (*cluster variables*) which is grouped into overlapping subsets (*clusters*) consisting of precisely r elements. Each subset is defined inductively by a sequence of certain combinatorial operation (*seed mutations*) from the initial seed. The monomials in the variables of a given single cluster are called *cluster monomials*. However, it is not known that a cluster algebra have a basis, related to the dual canonical basis, which includes all cluster monomials in general.

1.3. **Cluster algebra and the dual semicanonical basis.** In a series of works [18, 21, 19, 20, 23, 22], Geiß, Leclerc and Schröer introduced a cluster algebra structure on the coordinate ring $\mathbb{C}[N(w)]$ of the unipotent subgroup associated with a Weyl group element w. Furthermore they show that the *dual semicanonical basis* $\mathcal{S}^*$ is compatible with the inclusion $\mathbb{C}[N(w)] \subset U(\mathfrak{n})^*_{\mathrm{gr}}$ and contains all cluster monomials. Here the dual semicanonical basis is the dual basis of the semicanonical basis of $U(\mathfrak{n})$, introduced by Lusztig [40, 44], and "compatible" means that $\mathcal{S}^* \cap \mathbb{C}[N(w)]$ forms a $\mathbb{C}$-basis of $\mathbb{C}[N(w)]$.

It is known that canonical and semicanonical bases share similar combinatorial properties (crystal structure), but they are different (examples can be found in [32] [1]).

1.4. **Cluster algebra and the dual canonical basis.** Our main result is to give a set up of a quantum analogue of Geiß-Leclerc-Schröer's results:

(1) The dual canonical basis is compatible with the quantum unipotent subgroup $\mathcal{O}_q[N(w)]$ which is a quantum analogue of $\mathbb{C}[N(w)]$, that is $\mathbf{B}^{\mathrm{up}}(w) := \mathbf{B}^{\mathrm{up}} \cap \mathcal{O}_q[N(w)]$ forms a $\mathbb{Q}(q)$-basis of $\mathcal{O}_q[N(w)]$. (See Theorem 4.22.)

[1] In [32], $\underline{\mathcal{S}}$ is the specialization of the dual canonical basis, while $\underline{\Sigma}$ is the dual semicanonical basis thanks to [22].

(2) Quantum flag minors are mutually q-commuting and their monomials are contained in the dual canonical basis up to some q-shifts. Here quantum flag minors are defined as certain matrix coefficients with respect to extremal vectors in integrable highest weight modules. (See Theorem 6.20.)

(3) The "q-center" of $\mathcal{O}_q[N(w)]$ is generated by some of the quantum flag minors. Moreover any dual canonical basis element in $\mathbf{B}^{\mathrm{up}}(w)$ can be factored into the product of an element in the "q-center" of $\mathcal{O}_q[N(w)]$ and an "interval-free" element. (See Theorem 6.21.)

When $\mathfrak{g}$ is of type ADE, Caldero proved the above results in a series of works [7, 8, 9] (see also [6, 6.3]). ($\mathcal{O}_q[N(w)]$ is denoted by $\mathbf{U}_q(\mathfrak{n}_w)$ in [9].) We generalize them to an arbitary symmetrizable Kac-Moody Lie algebra. Key tools are the Poincaré-Birkhoff-Witt basis of $\mathcal{O}_q[N(w)]$ and the crystal structures. They are already used by Caldero, but the author cannot follow several claims, and give a self-contained proof in this paper.

1.5. **Quantization conjectures and its consequences.** The above properties (1), (2), (3) can be thought as a part of structures of a quantum cluster algebra. The corresponding properties of the "classical limit" $\mathbb{C}[N(w)]$ were shown in [23] if the dual canonical basis is replaced by the dual semicanoncial basis. We conjecture that remaining structures of a quantum cluster algebra exist on $\mathcal{O}_q[N(w)]$ as in [23]. Let $\mathcal{O}_q[N(w)]_{\mathcal{A}}$ be the integral form defined by the dual canonical basis $\mathbf{B}^{\mathrm{up}}(w)$ where $\mathcal{A} = \mathbb{Q}[q^{\pm}]$.

Conjecture 1.1 (Quantization conjecture)**.** (1) We take a reduced expression $\widetilde{w} = (i_1, \cdots, i_l)$ of the Weyl group element w, then we have an isomorphism of algebras

$$\Phi_{\widetilde{w}} \colon \mathscr{A}^q(\Gamma_{\widetilde{w}}, \Lambda_{\widetilde{w}}) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Q}[q^{\pm}] \simeq \mathcal{O}_q[N(w)]_{\mathcal{A}},$$

which sends the initial seed to the quantum flag minors $\{\Delta^q_{s_{i_1}\cdots s_{i_k}\varpi_{i_k},\varpi_{i_k}}\}_{1\le k\le l}$, defined as matrix coefficients of certain extremal vectors associated with $\widetilde{w}$, where $\Gamma_{\widetilde{w}}$ is the frozen quiver in [2] and [23] and $\Lambda_{\widetilde{w}}$ is the compatible pair in [4, §10.3].

(2) Under this isomorphism, the quantum cluster monomials of $\mathscr{A}^q(\Gamma_{\widetilde{w}}, \Lambda_{\widetilde{w}})$ are contained in the dual canonical base $\mathbf{B}^{\mathrm{up}}(w)$ up to some q-shifts.

Let $\mathcal{A} \to \mathbb{C}$ be the algebra homomorphism defined by $q \mapsto 1$. If we specialize Conjecture 1.1 to $q = 1$, we obtain the following "weak" conjecture.

Conjecture 1.2 (Weak quantization conjecture)**.** (1) Let $\widetilde{w}$ be as above, We have an isomorphism of algebras

$$\Phi_{\widetilde{w}} \colon \mathscr{A}(\Gamma_{\widetilde{w}}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathbb{C}[N(w)],$$

which sends the initial seed to the specialized quantum flag minors $\{\Delta_{s_{i_1}\cdots s_{i_k}\varpi_{i_k},\varpi_{i_k}}\}_{1\le k\le l}$, where $\Gamma_{\widetilde{w}}$ is the frozen quiver as above.

(2) Under this isomorphism, the cluster monomials of $\mathbb{C}[N(w)]$ are contained in the specialized dual canonical base $\mathbf{B}^{\mathrm{up}}(w)$ at $q = 1$.

Some parts of Conjecture 1.1 were shown for A_2, A_3, A_4 cases with $w = w_0$ in [3] and [18, §12] and $A_1^{(1)}$ with $w = c^2$ in [32].

The definition of the quantum cluster algebra $\mathscr{A}^q(\Gamma_{\widetilde{w}}, \Lambda_{\widetilde{w}})$ will not be explained. So we explain the meaning of this conjecture as properties of the dual canonical basis without referring to the axiom of a quantum cluster algebra [4].

An element $x \in \mathbf{B}^{\mathrm{up}} \setminus \{1\}$ is called *prime* if it does not have a non-trivial factorization $x = q^N x_1 x_2$ with $x_1, x_2 \in \mathbf{B}^{\mathrm{up}}$ and $N \in \mathbb{Z}$. A subset $\mathbf{x} = \{x_1, \cdots, x_l\} \subset \mathbf{B}^{\mathrm{up}}$ is called *strongly* *compatible* if for any $m_1, \cdots, m_l \in \mathbb{Z}_{\geq 0}$, the monomial $x_1^{m_1} \cdots x_l^{m_l} \in q^{\mathbb{Z}}\mathbf{B}^{\mathrm{up}}$, that is $x_1^{m_1} \cdots x_l^{m_l}$ is contained in the dual canonical basis $\mathbf{B}^{\mathrm{up}}$ up to some q-shifts. In particular, x is contained in a compatible family, then it satisfies $x^m \in q^{\mathbb{Z}}\mathbf{B}^{\mathrm{up}}$ for any $m \geq 1$. A strongly compatible subset $\mathbf{x} = \{x_1, \cdots, x_l\}$ is called *maximal* in $\mathbf{B}^{\mathrm{up}}(w)$ if $y \in \mathbf{B}^{\mathrm{up}}(w)$ satisfies $yx_i \in q^{\mathbb{Z}}\mathbf{B}^{\mathrm{up}}(w)$ for any x_i, then there exists $m_1, \cdots, m_l$ and N such $y = q^N x_1^{m_1} \cdots x_l^{m_l}$.

Our quantization conjecture means that there are lots of maximal strongly compatible subsets of $\mathbf{B}^{\mathrm{up}}(w)$, constructed recursively from $\{\Delta^q_{s_{i_1}\cdots s_{i_k}\varpi_{i_k},\varpi_{i_k}}\}_{1\leq k\leq l}$. For example, for finite type $\mathfrak{g}$ with $w = c^2$ for a (bipartite) Coxeter element c, it is expected that the dual canonical basis $\mathbf{B}^{\mathrm{up}}(w)$ is covered by the (finite) union of the maximal compatible families. But the union is not the whole $\mathbf{B}^{\mathrm{up}}(w)$ in general.

Our quantization conjecture implies several conjectures on (quantum) cluster algebras. Let us spell out a few.

If $\mathfrak{g}$ is symmetric, we have the positivity result for the dual canonical base by the construction of [39]. This implies the positivity conjecture for the quantum cluster algebras $\mathscr{A}^q(\Gamma_{\widetilde{w}}, \Lambda_{\widetilde{w}})$, stating that cluster monomials are Laurent polynomials with positive coefficients in q and cluster variables of any seed. This conjecture is known only special cases:

- cluster algebras of finite type [16],
- cluster algebras with bipartite seeds [47],
- cluster algebras coming from triangulated surfaces [45],
- acyclic cluster algebras at the initial seed [49].

In fact, these results apply only to cluster algebras, not quantum ones except [49]. Thus we have much stronger positivity.

The quantization conjecture also provides us a *monoidal categorification* of $\mathbb{C}[N(w)]$ in the sense of Hernandez-Leclerc [35]. It roughly says that there is a monoidal abelian category $\mathfrak{N}(w)$ whose complexified Grothendieck ring $K_0(\mathfrak{N}(w)) \otimes_{\mathbb{Z}} \mathbb{C}$ has the cluster algebra structure of $\mathbb{C}[N(w)]$ so that the cluster monomials are classes of simple objects. If the weak quantizaton conjecture is true (and $\mathfrak{g}$ is symmetric), the category $\mathfrak{N}(w)$ is given as the category of finite dimensional modules of the (equivariant) Ext algebras of the simple (equivariant) perverse sheaves belonging to $\mathbf{B}^{\mathrm{up}}(w)$. Thanks to [53], $\mathfrak{N}(w)$ is also considered as the extension-closed subcategory of the module category of Khovanov-Lauda-Rouquier's algebra [30, 29, 51] consisting of finite dimensional modules whose composition factors are contained in $\mathbf{B}^{\mathrm{up}}(w)$.

When $\mathfrak{g}$ is symmetric, Geiß, Leclerc and Schröer conjecture that certain dual semicanonical basis elements are specialization of the corresponding dual canonical basis elements. This is called the *open orbit conjecture*. This class of the dual semicanonical basis element contains all the cluster monomials. (Conjecturally it exactly consists of the cluster monomials [5, Conjecture II 5.3].) The open orbit conjecture for the cluster monomials is equivalent to the weak quantization conjecture.

This paper is organized as follows. In §2, we give a review the quantized enveloping algebra and its canonical basis. In §3, we give a review the dual canonical basis $\mathbf{B}^{\mathrm{up}}$ and its multiplicative properties. In §4, we define the quantum unipotent subgroup and prove its compatibility with the dual canonical basis. In §5, we define the quantum closed unipotent cell and study its relationship with the quantum unipotent subgroup. In §6, we give quantum flag minors and prove their multiplicative properties.

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2. Preliminaries: Quantized enveloping algebras and the canonical bases

We briefly recall the definition of the quantized enveloping algebra and its canonical base in this section.

2.1. Definition of $\mathbf{U}_q(\mathfrak{g})$.

2.1.1. A *root datum* consists of

(1) $\mathfrak{h}$: a finite-dimensional $\mathbb{Q}$-vector space,
(2) a finite index set I,
(3) $P \subset \mathfrak{h}^*$: a lattice (weight lattice),
(4) $P^\vee = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$ with natural pairing $\langle \ , \ \rangle : P \otimes P^\vee \to \mathbb{Z}$,
(5) $\alpha_i \in P$ for $i \in I$ (simple roots),
(6) $h_i \in P^\vee$ for $i \in I$ (simple coroots),
(7) $(\cdot, \cdot)$ a $\mathbb{Q}$-valued symmetric bilinear form on $\mathfrak{h}^*$

satisfying following conditions:

(a) $\langle h_i, \lambda \rangle = 2(\alpha_i, \lambda)/(\alpha_i, \alpha_i)$ for $i \in I$ and $\lambda \in P$,
(b) $a_{ij} = \langle h_i, \alpha_j \rangle = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ gives a symmetrizable generalized Cartan matrix, i.e., $\langle h_i, \alpha_i \rangle = 2$, and $\langle h_i, \alpha_j \rangle \in \mathbb{Z}_{\leq 0}$ and $\langle h_i, \alpha_j \rangle = 0 \Leftrightarrow \langle h_j, \alpha_i \rangle = 0$ for $i \neq j$,
(c) $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$, i.e. $d_i := (\alpha_i, \alpha_i)/2 \in \mathbb{Z}_{>0}$,
(d) $\{\alpha_i\}_{i \in I}$ are linearly independent.

We call $(I, \mathfrak{h}, (\ , \ ))$ a *Cartan datum*. Let $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \subset P$ be the root lattice. Let $Q_\pm = \pm \sum_{i \in I} \mathbf{Z}_{\geq 0}\alpha_i$. Fot $\xi = \sum_{i \in I} \xi_i \alpha_i \in Q$, we define $\mathrm{tr}(\xi) = \sum_{i \in I} \xi_i$. And we assume that there exists $\varpi_i \in P$ such that $\langle h_i, \varpi_j \rangle = \delta_{i,j}$ for any $i, j \in I$. We call ϖ_i the *fundamental weight* corrsponding to $i \in I$. We say $\lambda \in P$ is *dominant* if $\langle h_i, \lambda \rangle \geq 0$ for any $i \in I$ and denote by P_+ the set of dominant integral weights. We denote by $\overline{P} := \bigoplus_{i \in I} \mathbb{Z}\varpi_i$ and $\overline{P}_+ := \overline{P} \cap P_+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\varpi_i$.

2.1.2. Let $(I, \mathfrak{h}, (\ , \ ))$ be a Cartan datum. Let $\mathfrak{g}$ be the symmetrizable Kac-Moody Lie algebra corresponding to the generalized Cartan matrix $A = (a_{ij})$ with the Cartan subalgebra $\mathfrak{h}$, i.e., $\mathfrak{g}$ is the Lie algebra generated by $\{h; h \in \mathfrak{h}\}$, e_i, and f_i $(i \in I)$ with the following relations:

(i) $[h, h'] = 0$ for $h, h' \in \mathfrak{h}$,
(ii) $[h, e_i] = \langle h, \alpha_i \rangle e_i$, $[h, f_i] = -\langle h, \alpha_i \rangle f_i$,
(iii) $[e_i, f_j] = \delta_{ij} h_i$, and
(iv) $(\mathrm{ad} e_i)^{1 - \langle h_i, \alpha_j \rangle} e_j = (\mathrm{ad} f_i)^{1 - \langle h_i, \alpha_j \rangle} f_j = 0$ for $i \neq j$.

We denote the Lie subalgebra generated by $\{f_i\}_{i \in I}$ by $\mathfrak{n}$.

2.1.3. Suppose a root datum is given. We introduce an indeterminate q. For $i \in I$, we set $q_i = q^{(\alpha_i, \alpha_i)/2}$. For $\xi = \sum_{i \in I} \xi_i \alpha_i \in Q$, we set $q_\xi := \prod_{i \in I} (q_i)^{\xi_i} = q^{(\xi, \rho)}$, where ρ is the sum of all fundamental weights. We define $\mathbb{Q}$-subalgebras $\mathcal{A}_0$, $\mathcal{A}_\infty$ and $\mathcal{A}$ of $\mathbb{Q}(q)$ by

$$\mathcal{A}_0 := \{f \in \mathbb{Q}(q); f \text{ is regular at } q = 0\},$$
$$\mathcal{A}_\infty := \{f \in \mathbb{Q}(q); f \text{ is regular at } q = \infty\},$$
$$\mathcal{A} := \mathbb{Q}[q^\pm].$$

2.1.4. The *quantized enveloping algebra* $\mathbf{U}_q(\mathfrak{g})$ associated with a root datum is the $\mathbb{Q}(q)$-algebra generated by e_i, f_i $(i \in I)$, q^h $(h \in d^{-1}P^*)$ with the following relations:

(i) $q^0 = 1, q^h q^{h'} = q^{h+h'}$,

(ii) $q^h e_i q^{-h} = q^{\langle h,\alpha_i \rangle} e_i, q^h f_i q^{-h} = q^{-\langle h,\alpha_i \rangle} f_i$,

(iii) $e_i f_j - f_j e_j = \delta_{ij}(t_i - t_i^{-1})/(q_i - q_i^{-1})$,

(iv) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(k)} e_j e_i^{(1-a_{ij}-k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(k)} f_j f_i^{(1-a_{ij}-k)} = 0$ (q-Serre relations),

where $t_i = q^{\frac{(\alpha_i,\alpha_i)}{2}h_i}$, $[n]_i = (q_i^n - q_i^{-n})/(q_i - q_i^{-1})$, $[n]_i! = [n]_i[n-1]_i \cdots [1]_i$ for $n > 0$ and $[0]! = 1$, $e_i^{(k)} = e_i^k/[k]_i!, f_i^{(k)} = f_i^k/[k]_i!$ for $i \in I$ and $k \in \mathbb{Z}_{\geq 0}$.

2.1.5. Let $\mathbf{U}_q^+(\mathfrak{g})$ (resp. $\mathbf{U}_q^-(\mathfrak{g})$) be the $\mathbb{Q}(q)$-subalgebra of $\mathbf{U}_q(\mathfrak{g})$ generated by e_i (resp. f_i) for $i \in I$. Then we have the triangular decomposition

$$\mathbf{U}_q(\mathfrak{g}) \simeq \mathbf{U}_q^-(\mathfrak{g}) \otimes_{\mathbb{Q}(q)} \mathbb{Q}(q)[P^\vee] \otimes_{\mathbb{Q}(q)} \mathbf{U}_q^+(\mathfrak{g}),$$

where $\mathbb{Q}(q)[P^\vee]$ is the group algebra over $\mathbb{Q}(q)$, i.e., $\bigoplus_{h \in P^\vee} \mathbb{Q}(q) q^h$.

2.1.6. For $\xi \in Q$, we define its *root space* $\mathbf{U}_q(\mathfrak{g})_\xi$ by

$$\mathbf{U}_q(\mathfrak{g})_\xi = \{x \in \mathbf{U}_q(\mathfrak{g}) | q^h x q^{-h} = q^{\langle h,\xi \rangle} x \text{ for any } h \in P^*\}.$$

Then we have the root space decomposition

$$\mathbf{U}_q^\pm(\mathfrak{g}) = \bigoplus_{\xi \in Q_\pm} \mathbf{U}_q(\mathfrak{g})_\xi.$$

An element $x \in \mathbf{U}_q(\mathfrak{g})$ is *homogenous* if $x \in \mathbf{U}_q(\mathfrak{g})_\xi$ for some $\xi \in Q$, and we set $\mathrm{wt}(x) = \xi$.

2.1.7. Let $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}$ be the $\mathcal{A}$-subalgebra of $\mathbf{U}_q^-(\mathfrak{g})$ generated by $f_i^{(k)}$ for $i \in I$ and $k \in \mathbb{Z}_{\geq 0}$. Let $(\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}})_\xi := \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}} \cap \mathbf{U}_q^-(\mathfrak{g})_\xi$. We have

$$\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}} = \bigoplus_{\xi \in Q_-} (\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}})_\xi.$$

2.1.8. We define a $\mathbb{Q}(q)$-algebra anti-involution $*\colon \mathbf{U}_q(\mathfrak{g}) \to \mathbf{U}_q(\mathfrak{g})$ by

$$*(e_i) = e_i, \qquad *(f_i) = f_i, \qquad *(q^h) = q^{-h}. \tag{2.1}$$

We call this the $*$-*involution.*

We define a $\mathbb{Q}$-algebra automorphism $^-\colon \mathbf{U}_q(\mathfrak{g}) \to \mathbf{U}_q(\mathfrak{g})$ by

$$\overline{e_i} = e_i, \qquad \overline{f_i} = f_i, \qquad \overline{q} = q^{-1}, \qquad \overline{q^h} = q^{-h}. \tag{2.2}$$

We call this the *bar involution.*

We remark that these two involutions preserve $\mathbf{U}_q^+(\mathfrak{g})$ and $\mathbf{U}_q^-(\mathfrak{g})$, and we have $^- \circ * = * \circ ^-$.

2.1.9. In this article, we choose the coproduct on $\mathbf{U}_q(\mathfrak{g})$ following [25]:

$$\Delta(q^h) = q^h \otimes q^h, \tag{2.3a}$$

$$\Delta(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i, \tag{2.3b}$$

$$\Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i. \tag{2.3c}$$

2.1.10. We introduce Lusztig's $\mathbb{Q}(q)$-valued symmetric nondegenerate bilinear form $(\ ,\ )_L$ on $\mathbf{U}_q^-(\mathfrak{g})$. We first define a $\mathbb{Q}(q)$-algebra structure on $\mathbf{U}_q^-(\mathfrak{g}) \otimes \mathbf{U}_q^-(\mathfrak{g})$ by

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = q^{-(\mathrm{wt}(x_2),\mathrm{wt}(y_1))} x_1 x_2 \otimes y_1 y_2,$$

where x_i, y_i $(i = 1, 2)$ are homogenous elements.

Let $r\colon \mathbf{U}_q^-(\mathfrak{g}) \to \mathbf{U}_q^-(\mathfrak{g}) \otimes \mathbf{U}_q^-(\mathfrak{g})$ be a $\mathbb{Q}(q)$-algebra homomorphism defined by

$$r(f_i) = f_i \otimes 1 + 1 \otimes f_i \ (i \in I).$$

We call this the *twisted coproduct.*

By [41, 1.2.5], the algebra $\mathbf{U}_q^-(\mathfrak{g})$ has a unique nondegenerate $\mathbb{Q}(q)$-valued symmetric bilinear form $(\ ,\ )_L\colon \mathbf{U}_q^-(\mathfrak{g}) \times \mathbf{U}_q^-(\mathfrak{g}) \to \mathbb{Q}(q)$ which satisfies

$$(1,1)_L = 1, \tag{2.4a}$$

$$(f_i, f_j)_L = \frac{\delta_{i,j}}{1-q_i^2}, \tag{2.4b}$$

$$(x, yy')_L = (r(x), y \otimes y')_L, \tag{2.4c}$$

$$(xx', y)_L = (x \otimes x', r(y))_L, \tag{2.4d}$$

where the form on $\mathbf{U}_q^-(\mathfrak{g}) \otimes \mathbf{U}_q^-(\mathfrak{g})$ is defined by $(x_1 \otimes y_1, x_2 \otimes y_2)_L = (x_1, x_2)_L (y_1, y_2)_L$.

2.1.11. The relation between the coproduct Δ and the twisted coproduct r is given as follows:

Lemma 2.5. For homogenous $x \in \mathbf{U}_q^-(\mathfrak{g})_\xi$, we have

$$\Delta(x) = \sum x_{(1)} t_{-\mathrm{wt}(x_{(2)})} \otimes x_{(2)}, \tag{2.6}$$

where $r(x) = \sum x_{(1)} \otimes x_{(2)}$, $t_\xi = q^{\nu(\xi)}$, and $\nu(\xi) = \sum_i \frac{(\alpha_i,\alpha_i)}{2} \xi_i h_i$ for $\xi = \sum \xi_i \alpha_i \in Q$.

2.1.12. For $i \in I$, we define the unique $\mathbb{Q}(q)$-linear map ${}_ir\colon \mathbf{U}_q^- \to \mathbf{U}_q^-$ (resp. $r_i\colon \mathbf{U}_q^- \to \mathbf{U}_q^-$) given by ${}_ir(1) = 0, {}_ir(f_j) = \delta_{i,j}$ (resp. $r_i(1) = 0, r_i(f_j) = \delta_{i,j}$) for any $i, j \in I$ and

$${}_ir(xy) = {}_ir(x)y + q^{-(\mathrm{wt}\, x, \alpha_i)} x\, {}_ir(y), \tag{2.7a}$$

$$r_i(xy) = q^{-(\mathrm{wt}\, y, \alpha_i)} r_i(x) y + x r_i(y) \tag{2.7b}$$

for homogenous $x, y \in \mathbf{U}_q^-$. From the definition, we have

$$(f_i x, y)_L = \frac{1}{1-q_i^2} (x, {}_ir y)_L, \tag{2.8a}$$

$$(x f_i, y)_L = \frac{1}{1-q_i^2} (x, r_i y)_L. \tag{2.8b}$$

2.2. **Canonical basis of $\mathbf{U}_q^-(\mathfrak{g})$.** In this subsection, we give a brief review of the theory of the canonical base following Kashiwara [25, 28]. Note that Kashiwara call it the *lower global base.*

2.2.1.

Lemma 2.9 ([25, Lemma 3.4.1], [48])**.** For $x \in \mathbf{U}_q^-(\mathfrak{g})$ and any $i \in I$, we have

$$[e_i, x] = \frac{r_i(x) t_i - t_i^{-1}\, {}_ir(x)}{q_i - q_i^{-1}}.$$

2.2.2. Kashiwara [25, §3.4] has proved that there is a unique non-degenerate symmetric bilinear form $(\cdot,\cdot)_K$ on $\mathbf{U}_q^-(\mathfrak{g})$ such that

$$(f_i x, y)_K = (x, {}_i r y)_K, \tag{2.10a}$$

$$(1,1)_K = 1. \tag{2.10b}$$

Lemma 2.11 ([25, Lemma 3.4.7], [41, Lemma 1.2.15])**.** For $x \in \mathbf{U}_q^-(\mathfrak{g})$ with ${}_i r(x) = 0$ for any $i \in I$ and $\mathrm{wt}(x) \neq 0$, then we have $x = 0$.

2.2.3. We have the following relation between Kashiwara's bilinear form $(\ ,\ )_K$ and Lusztig's one $(\ ,\ )_L$.

Lemma 2.12 ([34, 2.2])**.** For homogenous $x, y \in \mathbf{U}_q^-(\mathfrak{g})_\xi$ with $\xi = -\sum n_i \alpha_i \in Q_-$, we have

$$(x,y)_K = \prod_{i\in I}(1-q_i^2)^{n_i}(x,y)_L.$$

This can be proved by an induction on $\mathrm{wt}(x)$ by using Lemma 2.11, (2.10a) and (2.8a).

Lemma 2.13 ([41, Lemma 1.2.8(b)])**.** For any homogenous $x, y \in \mathbf{U}_q^-(\mathfrak{g})$, we have

$$(x,y)_K = (x^*, y^*)_K.$$

2.2.4. The *reduced q-analogue* $\mathscr{B}_q(\mathfrak{g})$ of a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$ is the $\mathbb{Q}(q)$-algebra generated by ${}_i r$ and f_i with the q-Boson relations ${}_i r f_j = q^{-(\alpha_i,\alpha_j)} f_j {}_i r + \delta_{i,j}$ for $i, j \in I$ and the q-Serre relations for ${}_i r$ and f_i for $i \in I$. Then $\mathbf{U}_q^-(\mathfrak{g})$ becomes a $\mathscr{B}_q(\mathfrak{g})$-modules by Lemma 2.9.

By the q-Boson relation, any element $x \in \mathbf{U}_q^-(\mathfrak{g})$ can be uniquely written as $x = \sum_{n\geq 0} f_i^{(n)} x_n$ with ${}_i r(x_n) = 0$ for any $n \geq 0$. So we define Kashiwara's *modified root operators* $\widetilde{f}_i$ and $\widetilde{e}_i$ by

$$\widetilde{e}_i x = \sum_{n\geq 1} f_i^{(n-1)} x_n,$$

$$\widetilde{f}_i x = \sum_{n\geq 0} f_i^{(n+1)} x_n.$$

By using these operators, Kashiwara introduced the crystal basis $(\mathscr{L}(\infty), \mathscr{B}(\infty))$ of $\mathbf{U}_q^-(\mathfrak{g})$:

Theorem 2.14 ([25])**.** Let

$$\mathscr{L}(\infty) := \sum_{l\geq 0, i_1, i_2, \cdots, i_l \in I} \mathcal{A}_0 \widetilde{f}_{i_1} \cdots \widetilde{f}_{i_l} 1 \subset \mathbf{U}_q^-(\mathfrak{g}),$$

$$\mathscr{B}(\infty) := \{\widetilde{f}_{i_1} \cdots \widetilde{f}_{i_l} 1 \bmod q\mathscr{L}(\infty); l \geq 0, i_1, i_2, \cdots, i_l \in I\} \subset \mathscr{L}(\infty)/q\mathscr{L}(\infty).$$

Then we have the followings:

(1) $\mathscr{L}(\infty)$ is a free $\mathcal{A}_0$-module with $\mathbb{Q}(q) \otimes_{\mathcal{A}_0} \mathscr{L}(\infty) = \mathbf{U}_q^-(\mathfrak{g})$.
(2) $\widetilde{e}_i \mathscr{L}(\infty) \subset \mathscr{L}(\infty)$ and $\widetilde{f}_i \mathscr{L}(\infty) \subset \mathscr{L}(\infty)$.
(3) $\mathscr{B}(\infty)$ is a $\mathbb{Q}$-basis of $\mathscr{L}(\infty)/q\mathscr{L}(\infty)$.
(4) $\widetilde{f}_i \colon \mathscr{B}(\infty) \to \mathscr{B}(\infty)$ and $\widetilde{e}_i \colon \mathscr{B}(\infty) \to \mathscr{B}(\infty) \cup \{0\}$.
(5) For $b \in \mathscr{B}(\infty)$ with $\widetilde{e}_i(b) \neq 0$, we have $\widetilde{f}_i \widetilde{e}_i b = b$.

We call $(\mathscr{L}(\infty), \mathscr{B}(\infty))$ the *(lower) crystal basis* of $\mathbf{U}_q^-(\mathfrak{g})$, and $\mathscr{L}(\infty)$ the *(lower) crystal lattice*. We denote $1 \bmod q\mathscr{L}(\infty) \in \mathscr{B}(\infty)$ by u_∞ hereafter. For $b \in \mathscr{B}(\infty)$, we set $\varepsilon_i(b) := \max\{n \in \mathbb{Z}_{\geq 0}; \widetilde{e}_i^n b \neq 0\} < \infty$, and $\widetilde{e}_i^{\max}(b) := \widetilde{e}_i^{\varepsilon_i(b)} b \in \mathscr{B}(\infty)$.

2.2.5. We have the following compatibility of the $*$-involution with the crystal lattice $\mathscr{L}(\infty)$.

Theorem 2.15 ([25, Proposition 5.2.4], [26, Theorem 2.1.1])**.** We have

$$*(\mathscr{L}(\infty)) = \mathscr{L}(\infty), \tag{2.16a}$$

$$*(\mathscr{B}(\infty)) = \mathscr{B}(\infty). \tag{2.16b}$$

For $i \in I$ and $b \in \mathscr{B}(\infty)$, we set

$$\widetilde{f}_i^*(b) := (* \circ \widetilde{f}_i \circ *)(b), \tag{2.17a}$$

$$\widetilde{e}_i^*(b) := (* \circ \widetilde{e}_i \circ *)(b). \tag{2.17b}$$

For $b \in \mathscr{B}(\infty)$, we set $\varepsilon_i^*(b) := \max\{n \in \mathbb{Z}_{\geq 0}; \widetilde{e}_i^{*n} b \neq 0\} < \infty$. We have $\varepsilon_i^*(b) = \varepsilon_i(*b)$.

2.2.6. We recall some results on relationship between the crystal lattice $\mathscr{L}(\infty)$ and Kashiwara's form $(\cdot, \cdot)_K$.

Proposition 2.18 ([25, Proposition 5.1.2])**.** We have

$$(\mathscr{L}(\infty), \mathscr{L}(\infty))_K \subset \mathcal{A}_0.$$

Therefore the $\mathbb{Q}$-valued inner product on $\mathscr{L}(\infty)/q\mathscr{L}(\infty)$ given by $(\cdot, \cdot)|_{q=0}$ is well-defined, which we denote by $(\cdot, \cdot)_0$. Then we have the following properties:

(1) $(\widetilde{e}_i u, u')_0 = (u, \widetilde{f}_i u')_0$ for $u, u' \in \mathscr{L}(\infty)/q\mathscr{L}(\infty)$,
(2) $\mathscr{B}(\infty) \subset \mathscr{L}(\infty)/q\mathscr{L}(\infty)$ is an orthonormal basis with respect to $(\ ,\ )_0$.

Moreover we have

$$\mathscr{L}(\infty) = \{x \in \mathbf{U}_q^-(\mathfrak{g}); (x, \mathscr{L}(\infty))_K \subset \mathcal{A}_0\}, \tag{2.19}$$

that is the crystal lattice $\mathscr{L}(\infty)$ is a self-dual lattice with respect to $(\cdot, \cdot)_K$.

2.2.7. Let $\overline{}\colon \mathbb{Q}(q) \to \mathbb{Q}(q)$ be the $\mathbb{Q}$-algebra involution sending q to q^{-1}. Let V be a vector space over $\mathbb{Q}(q)$, $\mathscr{L}_0$ be an $\mathcal{A}_0$-submodule of V, $\mathscr{L}_\infty$ be an $\mathcal{A}_\infty$-submodule of V, and $V_{\mathcal{A}}$ be an $\mathcal{A}$-submodule of V. We set $E := \mathscr{L}_0 \cap \mathscr{L}_\infty \cap V_{\mathcal{A}}$.

Definition 2.20. We say that a triple $(\mathscr{L}_0, \mathscr{L}_\infty, V_{\mathcal{A}})$ is *balanced* if each $\mathscr{L}_0, \mathscr{L}_\infty$, and $V_{\mathcal{A}}$ generates V as $\mathbb{Q}(q)$-vector space and if one of the following equivalent conditions is satisfied

(1) $E \to \mathscr{L}_0/q\mathscr{L}_0$ is an isomorphism,
(2) $E \to \mathscr{L}_\infty/q^{-1}\mathscr{L}_\infty$ is an isomorphism,
(3) $(\mathscr{L}_0 \cap V_{\mathcal{A}}) \oplus (q^{-1}\mathscr{L}_\infty \cap V_{\mathcal{A}}) \to V_{\mathcal{A}}$ is an isomorphism,
(4) $\mathcal{A}_0 \otimes_{\mathbb{Q}} E \to \mathscr{L}_0$, $\mathcal{A}_\infty \otimes_{\mathbb{Q}} E \to \mathscr{L}_\infty$, $\mathcal{A} \otimes_{\mathbb{Q}} E \to V_{\mathcal{A}}$, and $\mathbb{Q}(q) \otimes_{\mathbb{Q}} E \to V$ are isomorphisms.

Theorem 2.21 ([25, Theorem 6])**.** The triple $(\mathscr{L}(\infty), \overline{\mathscr{L}(\infty)}, \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}})$ is balanced.

Let $G^{\mathrm{low}}\colon \mathscr{L}(\infty)/q\mathscr{L}(\infty) \to E := \mathscr{L}(\infty) \cap \overline{\mathscr{L}(\infty)} \cap \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}$ be the inverse of $E \xrightarrow{\sim} \mathscr{L}(\infty)/q\mathscr{L}(\infty)$. Then $\{G^{\mathrm{low}}(b); b \in \mathscr{B}(\infty)\}$ forms an $\mathcal{A}$-basis of $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}$. This basis is called the *canonical basis* of $\mathbf{U}_q^-(\mathfrak{g})$. Using this characterization, we obtain the following compatibility of the canonical basis and the $*$-involution.

Proposition 2.22. We have

$$*G^{\mathrm{low}}(b) = G^{\mathrm{low}}(*b).$$

2.2.8. For integrable highest weight modules, we can define the (lower) crystal basis and the canonical basis of them as for $\mathbf{U}_q^-(\mathfrak{g})$, see [25, Theorem 2, Theorem 6] for more details. Let M be an integrable $\mathbf{U}_q(\mathfrak{g})$-module and $M = \bigoplus_{\lambda \in P} M_\lambda$ be its weight decomposition. For $u \in \mathrm{Ker}(e_i) \cap M_\lambda$ and $0 \leq n \leq \langle h_i, \lambda \rangle$, we define Kashiwara's modified operators or (lower) crystal operators $\widetilde{e}_i^{\mathrm{low}}$ and $\widetilde{f}_i^{\mathrm{low}}$ by

$$\widetilde{e}_i^{\mathrm{low}}(f_i^{(n)}u) = f_i^{(n-1)}u,$$
$$\widetilde{f}_i^{\mathrm{low}}(f_i^{(n)}u) = f_i^{(n+1)}u.$$

Here we understand $f_i^{(-1)}u$ and $f_i^{(\langle h_i, \lambda \rangle + 1)}u$ as 0. Note that we denote the operators $\widetilde{f}_i$ and $\widetilde{e}_i$ in [25, 2.2] by $\widetilde{f}_i^{\mathrm{low}}$ and $\widetilde{e}_i^{\mathrm{low}}$ following [27].

Let $\lambda \in P_+$ and $V(\lambda)$ be the integrable highest weight $\mathbf{U}_q(\mathfrak{g})$-module generated by a highest weight vector u_λ of weight λ. Let $\mathscr{L}^{\mathrm{low}}(\lambda)$ be the $\mathcal{A}_0$-submodule spanned by $\widetilde{f}_{i_1}^{\mathrm{low}} \cdots \widetilde{f}_{i_l}^{\mathrm{low}} u_\lambda$. Let $\mathscr{B}^{\mathrm{low}}(\lambda)$ be the subset of $\mathscr{L}^{\mathrm{low}}(\lambda)/q_s\mathscr{L}^{\mathrm{low}}(\lambda)$ consisting of the non-zero vectors of the form $\widetilde{f}_{i_1}^{\mathrm{low}} \cdots \widetilde{f}_{i_l}^{\mathrm{low}} u_\lambda$, that is

$$\mathscr{L}^{\mathrm{low}}(\lambda) := \sum \mathcal{A}_0 \widetilde{f}_{i_1}^{\mathrm{low}} \cdots \widetilde{f}_{i_l}^{\mathrm{low}} u_\lambda \subset V(\lambda),$$
$$\mathscr{B}^{\mathrm{low}}(\lambda) := \{\widetilde{f}_{i_1}^{\mathrm{low}} \cdots \widetilde{f}_{i_l}^{\mathrm{low}} u_\lambda \bmod q\mathscr{L}^{\mathrm{low}}(\lambda)\} \setminus \{0\} \subset \mathscr{L}^{\mathrm{low}}(\lambda)/q_s\mathscr{L}^{\mathrm{low}}(\lambda).$$

Theorem 2.23 ([25, Theorem 2])**.** (1) $\mathscr{L}^{\mathrm{low}}(\lambda)$ is a free $\mathcal{A}_0$-submodule with $\mathbb{Q}(q) \otimes_{\mathcal{A}_0} \mathscr{L}^{\mathrm{low}}(\lambda) \simeq V(\lambda)$ and $\mathscr{L}^{\mathrm{low}}(\lambda) = \bigoplus_{\mu \in P} \mathscr{L}^{\mathrm{low}}(\lambda)_\mu$ where $\mathscr{L}^{\mathrm{low}}(\lambda)_\mu = \mathscr{L}^{\mathrm{low}}(\lambda) \cap M_\mu$.

(2) $\widetilde{e}_i^{\mathrm{low}} \mathscr{L}^{\mathrm{low}}(\lambda) \subset \mathscr{L}^{\mathrm{low}}(\lambda)$ and $\widetilde{f}_i^{\mathrm{low}} \mathscr{L}^{\mathrm{low}}(\lambda) \subset \mathscr{L}^{\mathrm{low}}(\lambda)$.

(3) $\mathscr{B}^{\mathrm{low}}(\lambda)$ is a $\mathbb{Q}$-basis of $\mathscr{L}^{\mathrm{low}}(\lambda)/q\mathscr{L}^{\mathrm{low}}(\lambda)$ and $\mathscr{B}^{\mathrm{low}}(\lambda) = \bigsqcup_{\mu \in P} \mathscr{B}^{\mathrm{low}}(\lambda)_\mu$ where $\mathscr{B}^{\mathrm{low}}(\lambda)_\mu = \mathscr{B}^{\mathrm{low}}(\lambda) \cap \mathscr{L}^{\mathrm{low}}(\lambda)_\mu / q\mathscr{L}^{\mathrm{low}}(\lambda)_\mu$.

(4) For $i \in I$, we have $\widetilde{e}_i \mathscr{B}(\lambda) \subset \mathscr{B}(\lambda) \cup \{0\}$ and $\widetilde{f}_i \mathscr{B}(\lambda) \subset \mathscr{B}(\lambda) \cup \{0\}$.

(5) For $b, b' \in \mathscr{B}^{\mathrm{low}}(\lambda)$, $b' = \widetilde{f}_i^{\mathrm{low}} b$ is equivalent to $b = \widetilde{e}_i^{\mathrm{low}} b'$.

We call $(\mathscr{L}^{\mathrm{low}}(\lambda), \mathscr{B}^{\mathrm{low}}(\lambda))$ the *lower crystal basis* of $V(\lambda)$, and $\mathscr{L}^{\mathrm{low}}(\lambda)$ the *lower crystal lattice*.

Let $^-$ be the bar-involution defined by $\overline{Pu_\lambda} = \overline{P}u_\lambda$. Set $V(\lambda)_\mathcal{A} := \mathbf{U}_q^-(\mathfrak{g})_\mathcal{A} u_\lambda$.

Theorem 2.24 ([25, Theorem 6])**.** The triple $(\mathscr{L}^{\mathrm{low}}(\lambda), \overline{\mathscr{L}^{\mathrm{low}}(\lambda)}, V(\lambda)_\mathcal{A})$ is balanced.

Let G_λ^{low} be the inverse of $\mathscr{L}^{\mathrm{low}}(\lambda) \cap \overline{\mathscr{L}^{\mathrm{low}}(\lambda)} \cap V(\lambda)_\mathcal{A} \xrightarrow{\sim} \mathscr{L}^{\mathrm{low}}(\lambda)/q\mathscr{L}^{\mathrm{low}}(\lambda)$. We call $G_\lambda^{\mathrm{low}}(\mathscr{B}^{\mathrm{low}}(\lambda))$ the *canonical basis* of $V(\lambda)$.

2.2.9. We have a compatibility of the (lower) crystal basis of $\mathbf{U}_q^-(\mathfrak{g})$ and the integrable modules $V(\lambda)$. Let $\pi_\lambda \colon \mathbf{U}_q^-(\mathfrak{g}) \to V(\lambda)$ be the $\mathbf{U}_q^-(\mathfrak{g})$-module homomorphism defined by $x \mapsto xu_\lambda$.

Theorem 2.25 ([25, Theorem 5])**.** We have the following properties:

(1) $\pi_\lambda \mathscr{L}(\infty) = \mathscr{L}(\lambda)$, hence π_λ induces a surjection homomorphism $\pi_\lambda \colon \mathscr{L}(\infty)/q\mathscr{L}(\infty) \to \mathscr{L}^{\mathrm{low}}(\lambda)/q\mathscr{L}^{\mathrm{low}}(\lambda)$.
(2) π_λ induces a bijection $\{b \in \mathscr{B}(\infty); \pi_\lambda(b) \neq 0\} \simeq \mathscr{B}^{\mathrm{low}}(\lambda)$.
(3) $\widetilde{f}_i^{\mathrm{low}} \circ \pi_\lambda(b) = \pi_\lambda \circ \widetilde{f}_i(b)$ if $\pi_\lambda(b) \neq 0$.
(4) $\widetilde{e}_i^{\mathrm{low}} \circ \pi_\lambda(b) = \pi_\lambda \circ \widetilde{e}_i(b)$ if $\widetilde{e}_i \circ \pi_\lambda(b) \neq 0$.

We denote the inverse of the bijection π_λ by j_λ.

2.2.10. We also have a compatibility of the canonical basis of $\mathbf{U}_q^-(\mathfrak{g})$ and the integrable modules $V(\lambda)$ via π_λ.

Theorem 2.26 ([25, 7.3 Lemma 7.3.2])**.** For $\lambda \in P_+$ and $b \in \mathscr{B}(\infty)$ with $\pi_\lambda(b) \neq 0$, we have

$$G^{\mathrm{low}}(b)u_\lambda = G^{\mathrm{low}}_\lambda(\pi_\lambda(b)).$$

2.2.11. For the canonical basis, we have the following expansion of left and right multiplication with respect to $f_i^{(m)}$.

Theorem 2.27 ([26, (3.1.2)])**.** For $b \in \mathscr{B}(\infty)$, we have

$$f_i^{(m)}G^{\mathrm{low}}(b) = \begin{bmatrix}\varepsilon_i(b)+m \\ m\end{bmatrix} G^{\mathrm{low}}(\widetilde{f}_i^m b) + \sum_{\varepsilon_i(b')>\varepsilon_i(b)+m} f^{(m)}_{bb';i}(q)G^{\mathrm{low}}(b'), \tag{2.28a}$$

$$G^{\mathrm{low}}(b)f_i^{(m)} = \begin{bmatrix}\varepsilon_i^*(b)+m \\ m\end{bmatrix} G^{\mathrm{low}}(\widetilde{f}_i^{*m} b) + \sum_{\varepsilon_i^*(b')>\varepsilon_i^*(b)+m} f^{*(m)}_{bb';i}(q)G^{\mathrm{low}}(b'), \tag{2.28b}$$

where $f^{(m)}_{bb';i}(q) = \overline{f^{(m)}_{bb';i}(q)}, f^{*(m)}_{bb';i}(q) = \overline{f^{*(m)}_{bb';i}(q)} \in \mathcal{A}$.

As a corollary of the above theorem, we have the following compatibilities of the right and left ideals $f_i^n\mathbf{U}_q^-(\mathfrak{g})$ and $\mathbf{U}_q^-(\mathfrak{g})f_i^n$ with the canonical basis.

Theorem 2.29 ([25, Theorem 7])**.** For $i \in I$ and $n \geq 1$, we have

$$f_i^n\mathbf{U}_q^-(\mathfrak{g}) \cap \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}} = \bigoplus_{b\in\mathscr{B}(\infty),\varepsilon_i(b)\geq n} \mathcal{A}G^{\mathrm{low}}(b),$$

$$\mathbf{U}_q^-(\mathfrak{g})f_i^n \cap \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}} = \bigoplus_{b\in\mathscr{B}(\infty),\varepsilon_i^*(b)\geq n} \mathcal{A}G^{\mathrm{low}}(b).$$

2.3. **Abstract crystal.** The notion of a (abstract) crystal was introduced in [26] by abstracting the crystal basis of $\mathbf{U}_q^-(\mathfrak{g})$ and of irreducible highest weight representations which are constructed in [25]. We recall it briefly. For more detail, see [28].

2.3.1.

Definition 2.30. A *crystal* $\mathscr{B}$ associated with a root datum is a set $\mathscr{B}$ endowed with maps $\mathrm{wt}\colon \mathscr{B} \to P, \varepsilon_i, \varphi_i : \mathscr{B} \to \mathbb{Z} \sqcup \{-\infty\}$, $\widetilde{e}_i, \widetilde{f}_i\colon \mathscr{B} \to \mathscr{B} \sqcup \{0\}$ $(i \in I)$ satisfying following conditions:

(a) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$,
(b) $\mathrm{wt}(\widetilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$, $\varepsilon_i(\widetilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\widetilde{e}_i b) = \varphi_i(b) + 1$, if $\widetilde{e}_i b \in \mathscr{B}$,
(c) $\mathrm{wt}(\widetilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$, $\varepsilon_i(\widetilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\widetilde{f}_i b) = \varphi_i(b) - 1$, if $\widetilde{f}_i b \in \mathscr{B}$,
(d) $b' = \widetilde{f}_i b \Leftrightarrow b = \widetilde{e}_i b'$ for $b, b' \in \mathscr{B}$,
(e) if $\varphi_i(b) = -\infty$ for $b \in \mathscr{B}$, then $\widetilde{e}_i b = \widetilde{f}_i b = 0$.

Let $\mathrm{wt}_i(b) = \langle h_i, \mathrm{wt}(b)\rangle$.

Definition 2.31. For given two crystals $\mathscr{B}_1, \mathscr{B}_2$ and $h \in \mathbb{Z}_{\geq 1}$, a map $\psi\colon \mathscr{B}_1 \sqcup \{0\} \to \mathscr{B}_2 \sqcup \{0\}$ is called a *morphism of amplitude* h of crystals from $\mathscr{B}_1$ to $\mathscr{B}_2$ if it satisfies the following properties for $b \in \mathscr{B}_1$ and $i \in I$:

(a) $\psi(0) = 0$,
(b) $\mathrm{wt}(\psi(b)) = h\,\mathrm{wt}(b)$, $\varepsilon_i(\psi(b)) = h\varepsilon_i(b)$, $\varphi_i(\psi(b)) = h\varphi_i(b)$ if $\psi(b) \in \mathscr{B}_2$,

(c) $\widetilde{e}_i^h\psi(b) = \psi(\widetilde{e}_i b)$ if $\psi(b) \in \mathscr{B}_2$, $\widetilde{e}_i b \in \mathscr{B}_1$,
(d) $\widetilde{f}_i^h\psi(b) = \psi(\widetilde{f}_i b)$ if $\psi(b) \in \mathscr{B}_2$, $\widetilde{f}_i b \in \mathscr{B}_1$.

When $h = 1$, it is simply called a *morphism of crystal.* A morphism $\psi\colon \mathscr{B}_1 \to \mathscr{B}_2$ is *strict* if ψ commutes with $\widetilde{e}_i, \widetilde{f}_i$ for all $i \in I$ without any restriction. A strict morphism $\psi\colon \mathscr{B}_1 \to \mathscr{B}_2$ is called an *strict embedding* if ψ is an injective map from $\mathscr{B}_1 \sqcup \{0\}$ to $\mathscr{B}_2 \sqcup \{0\}$.

Definition 2.32. The *tensor product* $\mathscr{B}_1 \otimes \mathscr{B}_2$ of crystals $\mathscr{B}_1$ and $\mathscr{B}_2$ is defined to be the set $\mathscr{B}_1 \times \mathscr{B}_2$ with maps given by

$$\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2), \tag{2.33a}$$

$$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \mathrm{wt}_i(b_1)), \tag{2.33b}$$

$$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \mathrm{wt}_i(b_2)), \tag{2.33c}$$

$$\widetilde{e}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{e}_i b_2 & \text{otherwise,} \end{cases} \tag{2.33d}$$

$$\widetilde{f}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{f}_i b_2 & \text{otherwise.} \end{cases} \tag{2.33e}$$

Here (b_1, b_2) is denoted by $b_1 \otimes b_2$ and $0 \otimes b_2$, $b_1 \otimes 0$ are identified with 0.

Iterating (2.33d) and (2.33e), we obtain the followings:

$$\widetilde{e}_i^n(b_1 \otimes b_2) = \begin{cases} \widetilde{e}_i^n b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ \widetilde{e}_i^{n-\varepsilon_i(b_2)+\varphi_i(b_2)} b_1 \otimes \widetilde{e}_i^{\varepsilon_i(b_2)-\varphi_i(b_1)} & \text{if } \varepsilon_i(b_2) \geq \varphi_i(b_1) \geq \varepsilon_i(b_2) - n, \\ b_1 \otimes \widetilde{e}_i^n b_2 & \text{if } \varepsilon_i(b_2) - n \geq \varphi_i(b_1). \end{cases} \tag{2.34a}$$

$$\widetilde{f}_i^n(b_1 \otimes b_2) = \begin{cases} \widetilde{f}_i^n b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) + n, \\ \widetilde{f}_i^{\varphi_i(b_1)-\varepsilon_i(b_2)} b_1 \otimes \widetilde{f}_i^{n-\varphi_i(b_1)+\varepsilon_i(b_2)} b_2 & \text{if } \varepsilon_i(b_2) + n \geq \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{f}_i^n b_2 & \text{if } \varepsilon_i(b_2) \geq \varphi_i(b_1), \end{cases} \tag{2.34b}$$

2.3.2. The (lower) crystal basis $\mathscr{B}(\infty)$ of $\mathbf{U}_q^-(\mathfrak{g})$ is an example of an abstract crystal. The same is true for $\mathscr{B}^{\mathrm{low}}(\lambda)$ of $V(\lambda)$ for $\lambda \in P_+$. We may also write $\mathscr{B}(\lambda)$ instead of $\mathscr{B}^{\mathrm{low}}(\lambda)$, when it is considered as an abstract crystal.

Example 2.35. For $i \in I$, let $\mathscr{B}_i = \{b_i(n); n \in \mathbb{Z}\}$. We can endow it with a structure of the abstract crystal by $\mathrm{wt}(b_i(n)) = n\alpha_i$, $\varepsilon_i(b_i(n)) = -n$, $\varphi_i(b_i(n)) = n$, $\varepsilon_j(b_i(n)) = \varphi_j(b_i(n)) = -\infty$, for $j \neq i$, and

$$\widetilde{f}_j b_i(n) = \begin{cases} b_i(n-1) & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases}$$

$$\widetilde{e}_j b_i(n) = \begin{cases} b_i(n+1) & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

2.3.3. For the crystal $\mathscr{B}(\infty)$, we have the following strict embedding.

Theorem 2.36 ([26, Theorem 2.2.1])**.** (1) For each $i \in I$, there exits a strict embedding $\Psi_i\colon \mathscr{B}(\infty) \to \mathscr{B}(\infty) \otimes \mathscr{B}_i$ which satisfies $\Psi_i(u_\infty) = u_\infty \otimes b_i$.

(2) If $\Psi_i(b) = b' \otimes \widetilde{f}_i^n b_i$, we have

$$\Psi_i(\widetilde{e}_i^* b) = \begin{cases} b' \otimes \widetilde{f}_i^{n-1} b_i & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \end{cases}$$
$$\Psi_i(\widetilde{f}_i^* b) = b' \otimes \widetilde{f}_i^{n+1} b_i.$$

(3) $\operatorname{Im} \Psi_i = \{b' \otimes \widetilde{f}_i^n b_i; \varepsilon_i^*(b') = 0, n \geq 0\}$.

By the above theorem, we have $\Psi_i(b) = \widetilde{e}_i^{*\max} b \otimes \widetilde{f}_i^{\varepsilon_i^*(b)} b_i$. For a sequence $(i_1, i_2, \cdots, i_r) \in I^r$, we have a strict embedding

$$\Psi_{(i_1, i_2, \cdots, i_r)} := (\Psi_{i_r} \otimes \cdots \otimes 1) \cdots (\Psi_{i_2} \otimes 1)\Psi_{i_1} \colon \mathscr{B}(\infty) \hookrightarrow \mathscr{B}(\infty) \otimes \mathscr{B}_{i_r} \otimes \cdots \otimes \mathscr{B}_{i_1}.$$

2.3.4. For $m \geq 1$, we have the following crystal morphism of amplitude m which is called *inflation of order m* in [28, Definition 8.1.4].

Proposition 2.37 ([28, Proposition 8.1.3], [46, Proposition 3.2])**.** (1) For $m \in \mathbb{Z}_{\geq 1}$, there exists a unique crystal morphism $S_m \colon \mathscr{B}(\infty) \to \mathscr{B}(\infty)$ of amplitude m satisfying

$$\operatorname{wt}(S_m b) = m \operatorname{wt}(b), \quad \varepsilon_i(S_m b) = m\varepsilon_i(b), \quad \varphi_i(S_m b) = m\varphi_i(b),$$
$$S_m(\widetilde{e}_i b) = \widetilde{e}_i^m S_m(b), \quad S_m(\widetilde{f}_i b) = \widetilde{f}_i^m S_m(b),$$
$$S_m(u_\infty) = u_\infty.$$

(2) Let $b \in \mathscr{B}(\infty)$. Then we have $(* \circ S_m)(b) = (S_m \circ *)(b)$. In particular, for any $b \in \mathscr{B}(\infty)$, we have

$$\varepsilon_i^*(S_m b) = m\varepsilon_i^*(b), \quad \varphi_i^*(S_m b) = m\varphi_i^*(b),$$
$$S_m(\widetilde{e}_i^* b) = \widetilde{e}_i^{*m} S_m(b), \quad S_m(\widetilde{f}_i^* b) = \widetilde{f}_i^{*m} S_m(b).$$

3. The dual canonical basis

3.1. In this subsection, we recall the definition of the dual canonical basis and its charactrization in terms of the *dual bar involution* σ with a balanced triple. We define $\mathbf{B}^{\mathrm{up}} \subset \mathbf{U}_q^-(\mathfrak{g})$ by the dual basis of $\mathbf{B}$ under the Kashiwara's bilinear form $(\ ,\ )_K$. We define the *dual bar involution* $\sigma \colon \mathbf{U}_q^-(\mathfrak{g}) \to \mathbf{U}_q^-(\mathfrak{g})$ so that

$$(\sigma(x), y)_K = \overline{(x, \overline{y})_K}$$

holds for any y ([4, 10.2]). This is well-defined since $(\cdot, \cdot)_K$ is non-degenerate. By its definition, we have $\sigma(x) = x$ for $x \in \mathbf{B}^{\mathrm{up}}$ and this is a $\mathbb{Q}$-linear involutive automorphism of $\mathbf{U}_q^-(\mathfrak{g})$ which satisfies $\sigma(fx) = \overline{f}\sigma(x)$ for any $f \in \mathbb{Q}(q)$ and $x \in \mathbf{U}_q^-(\mathfrak{g})$.

3.1.1. For $\xi = \sum \xi_i \alpha_i \in Q$, we define

$$N(\xi) := \frac{1}{2}\left((\xi, \xi) + \sum \xi_i(\alpha_i, \alpha_i)\right) = \frac{1}{2}\left((\xi, \xi) + 2(\xi, \rho)\right). \tag{3.1}$$

We have $N(-\alpha_i) = 0$ for any $i \in I$ and $N(\xi + \eta) = N(\xi) + N(\eta) + (\xi, \eta)$ for any $\xi, \eta \in Q$.

Proposition 3.2. We assume $x, y \in \mathbf{U}_q^-(\mathfrak{g})$ are homogenous.

(1) If $r(x) = \sum x_{(1)} \otimes x_{(2)}$, we have

$$r(\overline{x}) = \sum q^{-(\mathrm{wt}\, x_{(1)}, \mathrm{wt}\, x_{(2)})} \overline{x_{(2)}} \otimes \overline{x_{(1)}}.$$

(2) We set $\{x, y\}_K := \overline{(\overline{x}, \overline{y})_K}$, then we have

$$\{x, y\}_K = q^{N(\mathrm{wt}\, x)} (x, *y)_K.$$

(3) We have

$$\sigma(x) = q^{N(\mathrm{wt}\, x)} (* \circ {}^-)(x).$$

Proof. For convenience of the reader, we give a proof.

(1) We follow the argument in [41, 1.2.10]. For generators of $\mathbf{U}_q^-(\mathfrak{g})$, we have $r(f_i) = f_i \otimes 1 + 1 \otimes f_i = r(\overline{f_i})$. We prove the assertion by the induction on wt, so we assume that (1) holds for homogenous x', x'' and show that it holds also for $x = x'x''$. First we write $r(x') = \sum x'_{(1)} \otimes x'_{(2)}$ and $r(x'') = \sum x''_{(1)} \otimes x''_{(2)}$. By assumption, we have $r(\overline{x'}) = \sum q^{-(\mathrm{wt}\, x'_{(1)}, \mathrm{wt}\, x'_{(2)})} \overline{x'_{(2)}} \otimes \overline{x'_{(1)}}$ and $r(\overline{x''}) = \sum q^{-(\mathrm{wt}\, x''_{(1)}, \mathrm{wt}\, x''_{(2)})} \overline{x''_{(2)}} \otimes \overline{x''_{(1)}}$. We have $r(x'x'') = r(x')r(x'') = \sum q^{-(\mathrm{wt}\, x'_{(2)}, \mathrm{wt}\, x''_{(1)})} x'_1 x''_1 \otimes x'_2 x''_2$ and

$$r(\overline{x'}) r(\overline{x''}) = \sum q^{-(\mathrm{wt}\, x''_{(1)}, \mathrm{wt}\, x''_{(2)}) - (\mathrm{wt}\, x'_{(1)}, \mathrm{wt}\, x'_{(2)}) - (\mathrm{wt}\, x'_{(1)}, \mathrm{wt}\, x''_{(2)})} \overline{x'_{(2)} x''_{(2)}} \otimes \overline{x'_{(1)} x''_{(1)}}.$$

Then the assertion follows.

(2) We follow the argument in [41, Lemma 1.2.11 (2)]. For the generators, we have $\{f_i, f_i\}_K = (f_i, f_i)_K = q^{N(\mathrm{wt}\, f_i)} (f_i, f_i)_K$.

We prove the assertion by the induction on $\mathrm{tr}(\mathrm{wt}\, x) = \mathrm{tr}(\mathrm{wt}\, y)$. We prove that (2) holds for $y = y'y''$ and for any x assuming it holds for y', y''. First we write $r(x) = \sum x_{(1)} \otimes x_{(2)}$ with $x_{(1)}$ and $x_{(2)}$ homogenous. We have

$$\begin{aligned}
&(\overline{x}, \overline{y})_K \\
&= (r(\overline{x}), \overline{y'} \otimes \overline{y''}) = \sum q^{-(\mathrm{wt}\, x_{(1)}, \mathrm{wt}\, x_{(2)})} (\overline{x_{(2)}} \otimes \overline{x_{(1)}}, \overline{y'} \otimes \overline{y''})_K \\
&= \sum q^{-(\mathrm{wt}\, x_{(1)}, \mathrm{wt}\, x_{(2)})} (\overline{x_{(2)}}, \overline{y'})_K (\overline{x_{(1)}}, \overline{y''})_K \\
&= \sum q^{-(\mathrm{wt}\, x_{(1)}, \mathrm{wt}\, x_{(2)}) - N(\mathrm{wt}\, x_{(1)}) - N(\mathrm{wt}\, x_{(2)})} \overline{(x_{(2)}, *y')_K (x_{(1)}, *y'')_K} \\
&= \sum q^{-N(\mathrm{wt}\, x)} \overline{(x_{(2)}, *y')_K (x_{(1)}, *y'')_K},
\end{aligned}$$

where we have used the induction hypothesis in the fourth equality. On the other hand, we have

$$\begin{aligned}
&q^{-N(\mathrm{wt}\, x)} \overline{(x, *y)_K} \\
&= q^{-N(\mathrm{wt}\, x)} \overline{(r(x), *y'' \otimes *y')_K} \\
&= q^{-N(\mathrm{wt}\, x)} \sum \overline{(x_{(1)} \otimes x_{(2)}, *y'' \otimes *y')_K}.
\end{aligned}$$

Hence we obtain the assertion.

(3) We have $(\sigma(x), y) = \overline{(x, \overline{y})} = q^{N(\mathrm{wt}(x))} (\overline{x}, *y) = q^{N(\mathrm{wt}(x))} ((* \circ {}^-)(x), y)$, where we used Lemma 2.13. Since this holds for any y, assertion follows. q.e.d

3.1.2. By its construction, we have a characterization of the dual canonical basis $\mathbf{B}^{\mathrm{up}}$ in terms of the dual bar involution σ and the crystal lattice $\mathscr{L}(\infty)$ of $\mathbf{U}_q^-(\mathfrak{g})$. We note that $\mathscr{L}(\infty)$ is a self-dual $\mathcal{A}_0$ lattice, see (2.19), and hence we do not need to the introduce the dual lattice of $\mathscr{L}(\infty)$.

Proposition 3.3. We set

$$\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} := \{x \in \mathbf{U}_q^-(\mathfrak{g}); (x, \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}})_K \subset \mathcal{A}\}.$$

Then $(\mathscr{L}(\infty), \sigma(\mathscr{L}(\infty)), \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}})$ is a balanced triple for the dual canonical basis $\mathbf{B}^{\mathrm{up}}$.

Here we have the following isomorphism of $\mathbb{Q}$-vector spaces:

$$\mathscr{L}(\infty) \cap \sigma(\mathscr{L}(\infty)) \cap \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} \xrightarrow{\sim} \mathscr{L}(\infty)/q\mathscr{L}(\infty).$$

We denote its inverse by G^{up}. Then we have $\mathbf{B}^{\mathrm{up}} = G^{\mathrm{up}}(\mathscr{B}(\infty))$.

3.1.3. The above proposition gives a characterization of the dual canonical basis elements.

Corollary 3.4 ([36, Proposition 16])**.** A homogenous $x \in \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} \cap \mathscr{L}(\infty) \cap \sigma(\mathscr{L}(\infty))$ is an element of the dual canonical basis if and only if there exists $b \in \mathscr{B}(\infty)$ such that

$$\sigma(x) = x,$$
$$x \equiv b \bmod q\mathscr{L}(\infty).$$

3.1.4. We have the following compatibility of the dual canonical basis and the $*$-involution from Proposition 2.22.

Lemma 3.5. For $b \in \mathscr{B}(\infty)$, we have

$$G^{\mathrm{up}}(*b) = *G^{\mathrm{up}}(b).$$

3.2. **Compatible subset.** In this subsection, we introduce the concept of *compatible subsets* of $\mathscr{B}(\infty)$. Roughly speaking, they are closed under the multiplication up to q-shifts, considered as subsets of the dual canonical basis $\mathbf{B}^{\mathrm{up}}$.

3.2.1. By Proposition 3.2 (3), we obtain the following.

Proposition 3.6. For homogenous $x_1, x_2 \in \mathbf{U}_q^-(\mathfrak{g})$, we have

$$\sigma(x_1 x_2) = q^{(\mathrm{wt}\, x_1, \mathrm{wt}\, x_2)} \sigma(x_2)\sigma(x_1). \tag{3.7}$$

Then we obtain the following property.

Corollary 3.8. Let $b_1, b_2 \in \mathscr{B}(\infty)$ and consider the following expansion

$$G^{\mathrm{up}}(b_1)G^{\mathrm{up}}(b_2) = \sum_{\mathrm{wt}(b) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)} d^b_{b_1,b_2}(q) G^{\mathrm{up}}(b).$$

Then we have $d^b_{b_1,b_2}(q^{-1}) = q^{(\mathrm{wt}\, b_1, \mathrm{wt}\, b_2)} d^b_{b_2,b_1}(q)$. In particular, if we have $G^{\mathrm{up}}(b_1)G^{\mathrm{up}}(b_2) = q^N G^{\mathrm{up}}(b_1 \circledast b_2)$ for $b_1 \circledast b_2 \in \mathscr{B}(\infty)$ and $N \in \mathbf{Z}$, then we have

$$G^{\mathrm{up}}(b_1)G^{\mathrm{up}}(b_2) = q^{-N-(\mathrm{wt}\, b_1, \mathrm{wt}\, b_2)} G^{\mathrm{up}}(b_2)G^{\mathrm{up}}(b_1).$$

Proof. The first statement is clear from (3.7). Suppose that $G^{\mathrm{up}}(b_1)G^{\mathrm{up}}(b_2) = q^N G^{\mathrm{up}}(b_1 \circledast b_2)$ for $b_1 \circledast b_2 \in \mathscr{B}(\infty)$ and $N \in \mathbf{Z}$, i.e., $d^b_{b_1,b_2}(q) = q^N \delta_{b,b_1\circledast b_2}$ for $b_1 \circledast b_2 \in \mathscr{B}(\infty)$. Then we have

$$d^b_{b_2,b_1}(q) = q^{-(\mathrm{wt}\, b_1, \mathrm{wt}\, b_2)} d^b_{b_1,b_2}(q^{-1}) = q^{-(\mathrm{wt}\, b_1,\mathrm{wt}\, b_2)} q^{-N} \delta_{b,b_1 * b_2}.$$

This implies that if $G^{\mathrm{up}}(b_1)$ and $G^{\mathrm{up}}(b_2)$ satisfies $d^b_{b_1,b_2}(q) = q^N \delta_{b,b_1\circledast b_2}$ for some $b_1 \circledast b_2 \in \mathscr{B}(\infty)$, then $G^{\mathrm{up}}(b_1)$ and $G^{\mathrm{up}}(b_2)$ q-commutes. q.e.d

Motivated by this corollary, we introduce the following definition.

Definition 3.9. (1) We denote $x \simeq y$ for $x, y \in \mathbf{U}^-_q(\mathfrak{g})$ if there exists $N \in \mathbb{Z}$ such that $x = q^N y$.

(2) For $b_1, b_2 \in \mathscr{B}(\infty)$, we call b_1 and b_2 are *multiplicative* or *compatible* if there exists a unique $b_1 \circledast b_2 \in \mathscr{B}(\infty)$ such that

$$G^{\mathrm{up}}(b_1 \circledast b_2) \simeq G^{\mathrm{up}}(b_1)G^{\mathrm{up}}(b_2).$$

By Corollary 3.8 this condition is independent of the order on b_1 and b_2. We write $b_1 \perp b_2$ when this holds.

(3) Elements $b_1, \cdots, b_l \in \mathscr{B}(\infty)$ are called *compatible* if the following holds

$$G^{\mathrm{up}}(b_1)\cdots G^{\mathrm{up}}(b_l) \simeq G^{\mathrm{up}}(b_1 \circledast \cdots \circledast b_l)$$

for a unique $b_1 \circledast \cdots \circledast b_l \in \mathscr{B}(\infty)$. This condition is also independent of the ordering on $b_1, \cdots, b_l$.

(4) An element $b \in \mathscr{B}(\infty)$ is called *real* if $G^{\mathrm{up}}(b)G^{\mathrm{up}}(b) \simeq G^{\mathrm{up}}(b^{[2]})$ for a unique $b^{[2]} \in \mathscr{B}(\infty)$, that is $b \perp b$.

(5) An element $b \in \mathscr{B}(\infty)$ is called *strongly real* if $G^{\mathrm{up}}(b)^m \simeq G^{\mathrm{up}}(b^{[m]})$ for a unique $b^{[m]} \in \mathscr{B}(\infty)$ for any m, that is $\underbrace{b, \cdots, b}_{m \text{ times}}$ is compatible for any m.

(6) Elements $b_1, \cdots, b_l$ is called *strongly compatible* if for any $m_1, \cdots, m_l \in \mathbb{Z}_{\geq 0}$, the product $G^{\mathrm{up}}(b_1)^{m_1} \cdots G^{\mathrm{up}}(b_l)^{m_l} \simeq G^{\mathrm{up}}(b_1^{[m_1]} \circledast \cdots \circledast b_l^{[m_l]})$ for a unique $b_1^{[m_1]} \circledast \cdots \circledast b_l^{[m_l]} \in \mathscr{B}(\infty)$.

Remark 3.10. For $b_1, b_2 \in \mathscr{B}(\infty)$, we say a pair (b_1, b_2) is *quasi-commutative* if we have $G^{\mathrm{up}}(b_1)G^{\mathrm{up}}(b_2) \simeq G^{\mathrm{up}}(b_2)G^{\mathrm{up}}(b_1)$ following [3] and [50]. In [3], Berenstein and Zelevinsky conjectured that the quasi-commutativity and compatibility is equivalent. The above corollary proves the Reineke's result that the compatibility for b_1 and b_2 implies the quasi-commutativity generalizes Reineke's result from when $\mathfrak{g}$ is symmetric to arbitrary symmetrizable $\mathfrak{g}$.

Remark 3.11. The relation $b_1 \perp b_2$ is *not* an equivalence relation, as there exists b which does not satisfies $b \perp b$. In particular, such elements are counter-examples for Berenstein-Zelevinsky's conjecture in [3]. In [33], Leclerc said that b is *real* if $b \perp b$ and *imaginary* otherwise. He constructed examples of imaginary elements in [33]. Other examples closely related to this paper are given in [32, Corollary 4.4].

Remark 3.12. Even if $b_1 \perp b_2$, we can not determine N in $d^b_{b_1,b_2} = q^N \delta_{b,b_1\circledast b_2}$ in terms of weight of b_1, b_2. In §4, we have its explicit form in terms of the Lusztig data of b and b' associated with a reduced expression $\widetilde{w}$.

Corollary 3.13. (1) If $b_1 \perp b_2$, then $*b_1 \perp *b_2$.

(2) If b is real, then $*b$ is also real.

3.2.2. Let ${}_ir^{(m)} := {}_ir^m/[m]!$ and ${r_i}^{(m)} := {r_i}^m/[m]!$. These operators are adjoint of the left and right multiplications of $f_i^{(m)}$ by (2.10a). From Theorem 2.27, we get the following expansions for the actions of ${}_ir^{(m)} := {}_ir^m/[m]!$ and ${r_i}^{(m)} := {r_i}^m/[m]!$.

Theorem 3.14. For $b \in \mathscr{B}(\infty)$, we have

$$ {}_ir^{(m)} G^{\mathrm{up}}(b) = \begin{bmatrix} \varepsilon_i(b) \\ m \end{bmatrix} G^{\mathrm{up}}(\widetilde{e}_i^m b) + \sum_{\varepsilon_i(b') < \varepsilon_i(b) - m} E^{(m)}_{bb';i}(q) G^{\mathrm{up}}(b'), \tag{3.15a} $$

$$ {r_i}^{(m)} G^{\mathrm{up}}(b) = \begin{bmatrix} \varepsilon_i^*(b) \\ m \end{bmatrix} G^{\mathrm{up}}(\widetilde{e}_i^{*m} b) + \sum_{\varepsilon_i^*(b') < \varepsilon_i^*(b) - m} E^{*(m)}_{bb';i}(q) G^{\mathrm{up}}(b'), \tag{3.15b} $$

where $E^{(m)}_{bb';i}(q) = \overline{E^{(m)}_{bb';i}(q)}, E^{*(m)}_{bb';i}(q) = \overline{E^{*(m)}_{bb';i}(q)} \in \mathcal{A}$.

As a special case, we have the following result.

Corollary 3.16 ([27, Lemma 5.1.1.])**.** Let $b \in \mathscr{B}(\infty)$ with $\varepsilon_i(b) = c$ (resp. $\varepsilon_i^*(b) = c$). Then we have ${}_ir^{(c)} G^{\mathrm{up}}(b) = G^{\mathrm{up}}(\widetilde{e}_i^{\max} b)$ (resp. ${r_i}^{(c)} G^{\mathrm{up}}(b) = G^{\mathrm{up}}(\widetilde{e}_i^{*\max} b)$).

By the above corollary and (2.7a), we obtain the following result.

Corollary 3.17 ([50, Lemma 2.1])**.** For $b_1, b_2 \in \mathscr{B}(\infty)$ with

$$ G^{\mathrm{up}}(b_1) G^{\mathrm{up}}(b_2) = \sum d^b_{b_1,b_2}(q) G^{\mathrm{up}}(b), $$

we have $\varepsilon_i(b) \le \varepsilon_i(b_1) + \varepsilon_i(b_2)$ for any $i \in I$ if $d^b_{b_1,b_2}(q) \neq 0$. An equality holds at least one b.

If fact, we can prove prove $d^b_{b_1,b_2}(q) = 0$ if $\varepsilon_i(b) > \varepsilon_i(b_1) + \varepsilon_i(b_2)$ by the descending induction on $\varepsilon_i(b)$. In particular, the positivity of $d^b_{b_1,b_2}$, assumed in [50], is not used in the proof. The second assertion follows from

$$ \begin{aligned} & {}_ir^{(\varepsilon_i(b_1)+\varepsilon_i(b_2))} (G^{\mathrm{up}}(b_1) G^{\mathrm{up}}(b_2)) \\ &= q^N G^{\mathrm{up}}(\widetilde{e}_i^{\max} b_1) G^{\mathrm{up}}(\widetilde{e}_i^{\max} b_2) \\ &= \sum_{\varepsilon_i(b_1)+\varepsilon_i(b_2)=\varepsilon_i(b)} q^N d^b_{b_1,b_2}(q) G^{\mathrm{up}}(\widetilde{e}_i^{\max} b) \end{aligned} \tag{3.18} $$

for some $N \in \mathbb{Z}$.

As a corollary of Corollary 3.16 and Corollary 3.17, we obtain the following criterion.

Corollary 3.19. (1) If $b_1 \perp b_2$, then $\widetilde{e}_i^{\max} b_1 \perp \widetilde{e}_i^{\max} b_2$ for any $i \in I$. In fact, we have $\varepsilon_i(b_1 \circledast b_2) = \varepsilon_i(b_1) + \varepsilon_i(b_2)$ and $\widetilde{e}_i^{\max}(b_1) \circledast \widetilde{e}_i^{\max}(b_2) = \widetilde{e}_i^{\max}(b_1 \circledast b_2)$. Similar statement holds for $\widetilde{e}_i^{*\max}$.

(2) If b is (resp. strongly) real, $\widetilde{e}_i^{\max}(b)$ is (resp. strongly) real for any $i \in I$. In fact, we have $\varepsilon_i(b^{[m]}) = m\varepsilon_i(b)$ and $(\widetilde{e}_i^{\max} b)^{[m]} = \widetilde{e}_i^{\max}(b^{[m]})$ for $m = 2$ (resp. any m). Similar statement holds for $\widetilde{e}_i^{*\max}$.

Lemma 3.20. If b is (resp. strongly) real, we have $b^{[2]} = S_2(b)$ (resp. $b^{[m]} = S_m(b)$).

Proof. For any b with $\mathrm{tr}(\mathrm{wt}(b)) > 0$, there exists $i \in I$ such that $\varepsilon_i(b) > 0$. Therefore we can connect b to u_∞ by a path consisting of (strongly) real elements by successive applications of $\widetilde{e}_i^{\max}$'s. From the formula in Corollary 3.19 (2), we get the assertion. q.e.d

3.3. **Compatibilities of the dual canonical basis.** In this subsection, we study the dual canonical basis of integrable highest weight modules and its compatibilities with tensor products.

3.3.1. We recall the definition of the dual canonical base of the integrable highest weight module $V(\lambda)$ following [27, 4.2]. Kashiwara call it the *upper global basis*. Let M be an integrable $\mathbf{U}_q(\mathfrak{g})$-module with a weight decomposition $M = \bigoplus_{\lambda\in P} M_\lambda$. For $u \in \operatorname{Ker}(e_i) \cap M_\lambda$ and $0 \leq n \leq \langle h_i, \lambda\rangle$, we define other modified root operators called the *upper crystal operators*:

$$\widetilde{e}_i^{\mathrm{up}}(f_i^{(n)}u) = \frac{[\langle h_i, \lambda\rangle - n + 1]_i}{[n]_i} f_i^{(n-1)}u,$$
$$\widetilde{f}_i^{\mathrm{up}}(f_i^{(n)}u) = \frac{[n+1]_i}{[\langle h_i, \lambda\rangle - n]_i} f_i^{(n+1)}u.$$

We have a $\mathbb{Q}(q)$-linear anti-automorphism φ on $\mathbf{U}_q(\mathfrak{g})$ defined by

$$\varphi(e_i) = f_i, \qquad \varphi(f_i) = e_i, \qquad \varphi(q^h) = q^h. \tag{3.21}$$

For $\lambda \in P_+$, we have a unique symmetric non-degenerate bilinear form $(\ ,\ )_\lambda \colon V(\lambda) \otimes V(\lambda) \to \mathbb{Q}(q)$ which satisfies

$$(\varphi(x)u, v)_\lambda = (u, xv)_\lambda, \qquad \text{for } u, v \in V(\lambda) \text{ and } x \in \mathbf{U}_q(\mathfrak{g}), \tag{3.22a}$$
$$(u_\lambda, u_\lambda)_\lambda = 1. \tag{3.22b}$$

Then we have

$$(\widetilde{e}_i^{\mathrm{up}}u, v)_\lambda = (u, \widetilde{f}_i^{\mathrm{low}}v)_\lambda, \tag{3.23a}$$
$$(\widetilde{f}_i^{\mathrm{up}}u, v)_\lambda = (u, \widetilde{e}_i^{\mathrm{low}}v)_\lambda. \tag{3.23b}$$

Using $(\cdot,\cdot)_\lambda$, we define the dual bar involution σ_λ by

$$(\sigma_\lambda u, v)_\lambda := \overline{(u, \overline{v})_\lambda}.$$

This is well-defined since $(\cdot,\cdot)_\lambda$ is a non-degenerate bilinear form. We set

$$V(\lambda)_{\mathcal{A}}^{\mathrm{up}} := \{u \in V(\lambda); (u, V(\lambda)_{\mathcal{A}})_\lambda \subset \mathcal{A}\}, \tag{3.24a}$$
$$\mathscr{L}^{\mathrm{up}}(\lambda) := \{u \in V(\lambda); (u, \mathscr{L}^{\mathrm{low}}(\lambda))_\lambda \subset \mathcal{A}_0\}. \tag{3.24b}$$

Then we have $\sigma_\lambda(\mathscr{L}^{\mathrm{up}}(\lambda)) = \{u \in V(\lambda); (u, \overline{\mathscr{L}^{\mathrm{low}}(\lambda)})_\lambda \subset \mathcal{A}_\infty\}$. Kashiwara denote $\sigma_\lambda(\mathscr{L}^{\mathrm{up}}(\lambda))$ by $\overline{\mathscr{L}}^{\mathrm{up}}(\lambda)$. The triple $(\mathscr{L}^{\mathrm{up}}(\lambda), \sigma_\lambda(\mathscr{L}^{\mathrm{up}}(\lambda)), V(\lambda)_{\mathcal{A}}^{\mathrm{up}})$ is balanced by [27, Lemma 2.2.3].

Proposition 3.25. Let $\mathscr{B}^{\mathrm{up}}(\lambda)$ be the dual basis of $\mathscr{B}^{\mathrm{low}}(\lambda)$ with respect to the induced pairing $(\cdot,\cdot)_\lambda \colon \mathscr{L}^{\mathrm{up}}(\lambda)/q\mathscr{L}^{\mathrm{up}}(\lambda) \times \mathscr{L}^{\mathrm{low}}(\lambda)/q\mathscr{L}^{\mathrm{low}}(\lambda) \to \mathbb{Q}$, then the pair $(\mathscr{L}^{\mathrm{up}}(\lambda), \mathscr{B}^{up}(\lambda))$ is an *upper crystal base*, that is

(1) $\mathscr{L}^{\mathrm{up}}(\lambda)$ is a free $\mathcal{A}_0$-module with $\mathbb{Q}(q) \otimes_{\mathcal{A}_0} \mathscr{L}^{\mathrm{up}}(\lambda) \simeq V(\lambda)$,
(2) $\widetilde{f}_i^{\mathrm{up}}\mathscr{L}^{\mathrm{up}}(\lambda) \subset \mathscr{L}^{\mathrm{up}}(\lambda)$ and $\widetilde{e}_i^{\mathrm{up}}\mathscr{L}^{\mathrm{up}}(\lambda) \subset \mathscr{L}^{\mathrm{up}}(\lambda)$,
(3) $\mathscr{B}^{\mathrm{up}}(\lambda) \subset \mathscr{L}^{\mathrm{up}}(\lambda)/q\mathscr{L}^{\mathrm{up}}(\lambda)$ is a $\mathbb{Q}$-basis,
(4) $\widetilde{e}_i^{\mathrm{up}}\mathscr{B}^{\mathrm{up}}(\lambda) \subset \mathscr{B}^{\mathrm{up}}(\lambda) \sqcup \{0\}$ and $\widetilde{f}_i^{\mathrm{up}}\mathscr{B}^{\mathrm{up}}(\lambda) \subset \mathscr{B}^{\mathrm{up}}(\lambda) \sqcup \{0\}$,
(5) For $b, b' \in \mathscr{B}^{\mathrm{up}}(\lambda)$, $b = \widetilde{f}_i^{\mathrm{up}}b'$ is equivalent to $\widetilde{e}_i^{\mathrm{up}}b = b'$.

Let G_λ^{up} be the inverse of $V(\lambda)_{\mathcal{A}}^{\mathrm{up}} \cap \mathscr{L}^{\mathrm{up}}(\lambda) \cap \sigma_\lambda(\mathscr{L}^{\mathrm{up}}(\lambda)) \xrightarrow{\sim} \mathscr{L}^{\mathrm{up}}(\lambda)/q\mathscr{L}^{\mathrm{up}}(\lambda)$. The set $G_\lambda^{\mathrm{up}}(\mathscr{B}^{\mathrm{up}}(\lambda))$ is called the *dual canonical basis* of $V(\lambda)$. By its construction, the dual canonical basis is the dual basis of the canonical basis with respect to $(\ ,\ )_\lambda$. We also have

$$\mathscr{L}^{\mathrm{up}}(\lambda)_\mu = q^{(\lambda,\lambda)/2-(\mu,\mu)/2}\mathscr{L}^{\mathrm{low}}(\lambda)_\mu \text{ for } \mu \in P, \tag{3.26}$$

see [27, (4.2.9)]. By (3.26), we obtain an isomorphism $\mathscr{L}^{\mathrm{up}}(\lambda)/q\mathscr{L}^{\mathrm{up}}(\lambda) \simeq \mathscr{L}^{\mathrm{low}}(\lambda)/q\mathscr{L}^{\mathrm{low}}(\lambda)$. Through this identification, we have a bijection $\mathscr{B}^{\mathrm{up}}(\lambda) \simeq \mathscr{B}^{\mathrm{low}}(\lambda)$, and this bijection is an isomorphism of abstract crystals associated with upper and lower crystal basis. Hence we can identify $\mathscr{B}^{\mathrm{up}}(\lambda)$ with $\mathscr{B}^{\mathrm{low}}(\lambda)$ and denote both by $\mathscr{B}(\lambda)$ hereafter. If $\mu \in W\lambda$, this identification is given by the identity as $(\lambda,\lambda) = (\mu,\mu)$. We can also prove that the canonical base elements and the dual canonical base elements coincide in this case.

Remark 3.27. For $\mathbf{U}_q^-(\mathfrak{g})$, we consider the $\mathbb{Q}(q)$-linear anti-automorphism a of the reduced q-analogue $\mathscr{B}_q(\mathfrak{g})$ defined by

$$a({}_ir) = f_i, \qquad a(f_i) = {}_ir. \tag{3.28}$$

Since the (lower) crystal lattice is self-dual by Kashiwara's bilinear form $(\cdot,\cdot)_K$, we do not need to consider the dual lattice of $\mathscr{L}(\infty)$.

3.3.2. Using the pairing $(\ ,\ )_\lambda$, we consider an $\mathbb{Q}(q)$-linear embedding $j_\lambda\colon V(\lambda) \to \mathbf{U}_q^-(\mathfrak{g})$ which is defined in the following commutative diagram:

$$\begin{array}{ccc} V(\lambda) & \xrightarrow{\sim} & V(\lambda)^* \\ \downarrow{\scriptstyle j_\lambda} & & \downarrow{\scriptstyle \pi_\lambda^*} \\ \mathbf{U}_q^-(\mathfrak{g}) & \xrightarrow{\sim} & \mathbf{U}_q^-(\mathfrak{g})^*, \end{array}$$

where the horizontal isomorphisms are induced by the non-degenerate inner products on $V(\lambda)$ and $\mathbf{U}_q^-(\mathfrak{g})$ and the right vertical homomorphism is the transpose of $\mathbf{U}_q^-(\mathfrak{g})$-module homomorphism $\pi_\lambda\colon \mathbf{U}_q^-(\mathfrak{g}) \to V(\lambda)$ given by $P \mapsto Pu_\lambda$. Then for $b \in \mathscr{B}(\lambda)$, we have $j_\lambda G_\lambda^{\mathrm{up}}(b) = G^{\mathrm{up}}(j_\lambda(b))$, where j_λ in the right hand side was defined just after Theorem 2.25. Thanks to this equality, there is no fear of confusion even though we use the same symbol j_λ for different maps.

3.3.3. We use the following result in [36, 7.3.2]. For $\lambda, \lambda_1, \lambda_2, \cdots, \lambda_r \in P_+$ with $\lambda = \sum_j \lambda_j$, let $\Phi(\lambda_1,\cdots,\lambda_r)\colon V(\lambda_1+\lambda_2+\cdots+\lambda_r) \to V(\lambda_1)\otimes\cdots\otimes V(\lambda_r)$ be the unique $\mathbf{U}_q(\mathfrak{g})$-module homomorphism with $\Phi(\lambda_1,\cdots,\lambda_r)(u_\lambda) = u_{\lambda_1}\otimes\cdots\otimes u_{\lambda_r}$. Then we have the corresponding embeddings

$$\begin{aligned} &\Phi(\lambda_1,\cdots,\lambda_r)\colon \mathscr{B}(\lambda) \hookrightarrow \mathscr{B}(\lambda_1)\otimes\cdots\otimes\mathscr{B}(\lambda_r), \\ &\Phi(\lambda_1,\cdots,\lambda_r)(\mathscr{L}^{\mathrm{low}}(\lambda)) \subset \mathscr{L}^{\mathrm{low}}(\lambda_1)\otimes_{\mathcal{A}_0}\cdots\otimes_{\mathcal{A}_0}\mathscr{L}^{\mathrm{low}}(\lambda_r). \end{aligned}$$

(See [25, §4.2].) Hence we obtain

$$\Phi(\lambda_1,\cdots,\lambda_r)(G_\lambda^{\mathrm{low}}(b)) \equiv G_{\lambda_1}^{\mathrm{low}}(b_1)\otimes\cdots\otimes G_{\lambda_r}^{\mathrm{low}}(b_r) \bmod q(\mathscr{L}^{\mathrm{low}}(\lambda_1)\otimes\cdots\otimes\mathscr{L}^{\mathrm{low}}(\lambda_r))$$

for $\Phi(\lambda_1,\cdots,\lambda_r)(b) = b_1\otimes\cdots\otimes b_r$ for some $b_j \in \mathscr{B}^{\mathrm{low}}(\lambda_j)$.

Let $q_{\lambda_1,\cdots,\lambda_r}\colon V(\lambda_1)\otimes\cdots\otimes V(\lambda_r)\to V(\lambda)$ be the homomorphism defined by the commutative diagram

$$\begin{CD} V(\lambda_1)\otimes\cdots\otimes V(\lambda_r) @>{\sim}>> V(\lambda_1)^*\otimes\cdots\otimes V(\lambda_r)^* \\ @V{q_{\lambda_1,\cdots,\lambda_r}}VV @VV{\Phi(\lambda_1,\cdots,\lambda_r)^*}V \\ V(\lambda) @>{\sim}>> V(\lambda)^*, \end{CD}$$

where the upper horizontal isomorphism is induced by the non-degenerate inner product $(\cdot,\cdot)_{\lambda_1,\cdots,\lambda_r} := (\cdot,\cdot)_{\lambda_1}\cdots(\cdot,\cdot)_{\lambda_r}$ on $V(\lambda_1)\otimes\cdots\otimes V(\lambda_r)$, the lower horizontal isomorphism is induced by the non-degenrate inner product $(\cdot,\cdot)_\lambda$ on $V(\lambda)$ and the right vertical homomorphism is the transpose of $\Phi(\lambda_1,\cdots,\lambda_r)$.

Proposition 3.29. Let $\lambda_1,\cdots,\lambda_r\in P_+$ and $b_j\in\mathscr{B}(\lambda_j)$ $(1\le j\le r)$. Assume that there exists $b_1\diamond\cdots\diamond b_r\in\mathscr{B}(\sum\lambda_j)$ with $\Phi(\lambda_1,\lambda_2,\cdots,\lambda_r)(b_1\diamond\cdots\diamond b_r)=b_1\otimes\cdots\otimes b_r\in\mathscr{B}(\lambda_1)\otimes\cdots\otimes\mathscr{B}(\lambda_r)$. Then we have the following equality

$$q_{\lambda_1,\cdots,\lambda_r}(G^{\mathrm{up}}_{\lambda_1}(b_1)\otimes\cdots\otimes G^{\mathrm{up}}_{\lambda_r}(b_r)) = G^{\mathrm{up}}_{\lambda}(b_1\diamond b_2\diamond\cdots\diamond b_r) \bmod q\mathscr{L}^{\mathrm{up}}(\lambda).$$

We give the proof for a completeness.

Proof. We have $q_{\lambda_1,\cdots,\lambda_r}(\mathscr{L}^{\mathrm{up}}(\lambda_1)\otimes_{\mathcal{A}_0}\cdots\otimes_{\mathcal{A}_0}\mathscr{L}^{\mathrm{up}}(\lambda_r))\subset\mathscr{L}^{\mathrm{up}}(\lambda)$, in particular we have $q_{\lambda_1,\cdots,\lambda_r}(G^{\mathrm{up}}_{\lambda_1}(b_1)\otimes\cdots\otimes G^{\mathrm{up}}_{\lambda_r}(b_r))\in\mathscr{L}^{\mathrm{up}}(\lambda)$.

Hence to show the statement, it suffices to compute the following inner product

$$(q_{\lambda_1,\cdots,\lambda_r}(G^{\mathrm{up}}_{\lambda_1}(b_1)\otimes\cdots\otimes G^{\mathrm{up}}_{\lambda_r}(b_r)), G^{\mathrm{low}}_{\lambda}(b))_\lambda|_{q=0}$$

for $b\in\mathscr{B}(\lambda)$. By its definition of $q_{\lambda_1,\cdots,\lambda_r}$, this is equal to $(b_1\otimes\cdots\otimes b_r,\Phi(\lambda_1,\cdots,\lambda_r)(b))_{\lambda_1,\cdots,\lambda_r}|_{q=0}$. Since the tensor product of the dual canonical basis is the dual of the tensor product of the canonical basis, this is equal to $\delta_{b_1\otimes\cdots\otimes b_r,\Phi(\lambda_1,\cdots,\lambda_r)(b)}=\delta_{b_1\diamond b_2\diamond\cdots\diamond b_r,b}$. Hence we obtained the assertion. q.e.d

3.3.4. To compute a product of dual canonical basis elements of integble highest weight modules, we need the following modification of the coproduct as in [36, 7.2.5, 7.2.6].

Lemma 3.30. For $\lambda,\mu\in P_+$, let $r_{\lambda,\mu}\colon\mathbf{U}_q^-(\mathfrak{g})\to\mathbf{U}_q^-(\mathfrak{g})\otimes\mathbf{U}_q^-(\mathfrak{g})$ be the $\mathbb{Q}(q)$-linear map defined by

$$r_{\lambda,\mu}G^{\mathrm{low}}(b)=\sum_{b_1,b_2}d^b_{b_1,b_2}(q)q^{-(\mathrm{wt}(b_2),\lambda)}G^{\mathrm{low}}(b_1)\otimes G^{\mathrm{low}}(b_2)$$

for $G^{\mathrm{low}}(b)\in\mathbf{U}_q^-(\mathfrak{g})$ with $r(G^{\mathrm{low}}(b))=\sum_{b_1,b_2}d^b_{b_1,b_2}(q)G^{\mathrm{low}}(b_1)\otimes G^{\mathrm{low}}(b_2)$. Then we have the commutative diagram of $\mathbb{Q}(q)$-vector spaces

$$\begin{CD} \mathbf{U}_q^-(\mathfrak{g}) @>{\pi_{\lambda+\mu}}>> V(\lambda+\mu) \\ @V{r_{\lambda,\mu}}VV @VV{\Phi(\lambda,\mu)}V \\ \mathbf{U}_q^-(\mathfrak{g})\otimes\mathbf{U}_q^-(\mathfrak{g}) @>>{\pi_\lambda\otimes\pi_\mu}> V(\lambda)\otimes V(\mu). \end{CD}$$

Using the above modification, we obtain the following formula.

Proposition 3.31. For $b_1\in\mathscr{B}(\lambda)$ and $b_2\in\mathscr{B}(\mu)$, we have

$$q^{(\mathrm{wt}\,b_2-\mu,\lambda)}G^{\mathrm{up}}(j_\lambda(b_1))G^{\mathrm{up}}(j_\mu(b_2))=j_{\lambda+\mu}q_{\lambda,\mu}(G^{\mathrm{up}}_\lambda(b_1)\otimes G^{\mathrm{up}}_\mu(b_2)).$$

3.3.5. Combining Proposition 3.31 with Proposition 3.29, we obtain the following proposition.

Proposition 3.32. Let $\lambda_1, \cdots, \lambda_r \in P_+$ and $b_j \in \mathscr{B}(\lambda_j)$ $(1 \le j \le r)$. Assume that there exists $b_1 \diamond \cdots \diamond b_r \in \mathscr{B}(\sum \lambda_j)$ with $\Phi(\lambda_1, \lambda_2, \cdots, \lambda_r)(b_1 \diamond \cdots \diamond b_r) = b_1 \otimes \cdots \otimes b_r \in \mathscr{B}(\lambda_1) \otimes \cdots \otimes \mathscr{B}(\lambda_r)$. Then there exists a unique $m \in \mathbb{Z}$ such that

$$q^m G^{\mathrm{up}}(j_{\lambda_1}(b_1)) \cdots G^{\mathrm{up}}(j_{\lambda_r}(b_r)) = G^{\mathrm{up}}(j_{\sum \lambda_i}(b_1 \diamond \cdots \diamond b_r)) \bmod q\mathscr{L}(\infty).$$

4. Quantum unipotent subgroup and the dual canonical basis

4.1. The Lie algebra $\mathfrak{n}(w)$.

4.1.1. Let $w \in W$ be an element of the Weyl group associated with $\mathfrak{g}$. Let $\Delta^+(w) := \Delta^+ \cap w\Delta^- = \{\alpha \in \Delta^+ | w^{-1}\alpha < 0\} \subset \Delta^+$. We have the following description of $\Delta^+(w)$ as follows ([31, Lemma 1.3.14]).

For a Weyl group element w, let $\widetilde{w} = (i_1, i_2, \cdots, i_l) \in R(w)$ be a reduced expression of w, where $R(w)$ is the set of reduced expression of w. For each $1 \le k \le l = l(w)$, we set

$$\beta_k := s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}).$$

Then $\Delta^+(w)$ has cardinality exactly equal to $l = l(w)$ and we have

$$\Delta^+(w) = \{\beta_k\}_{1 \le k \le l}.$$

Let

$$\mathfrak{n}(w) = \bigoplus_{\alpha \in \Delta^+(w)} \mathfrak{g}_{-\alpha}.$$

Let $N(w)$ be the corresponding (pro-)unipotent (pro-) group in [31, VI]. Then $N(w)$ is a unipotent algebraic group of dimension $l(w)$ and its Lie algebra is $\mathfrak{n}(w)$. We can identify the restricted dual $U(\mathfrak{n}(w))^*_{\mathrm{gr}}$ of $U(\mathfrak{n}(w))$ with the coordinate ring of $N(w)$, that is $U(\mathfrak{n}(w))^*_{\mathrm{gr}} \simeq \mathbb{C}[N(w)]$, see [23, 5.2] for more details.

4.2. Braid group symmetry on $\mathbf{U}_q(\mathfrak{g})$.

We define (quantum) root vectors, using Lusztig's braid group symmetry $\{T_i\}$ on $\mathbf{U}_q(\mathfrak{g})$. See [41, Chapter 32] for more details.

4.2.1. Following [41, 37.1.3], we define the $\mathbb{Q}(q)$-algebra automorphisms $T'_{i,e} \colon \mathbf{U}_q(\mathfrak{g}) \to \mathbf{U}_q(\mathfrak{g})$ for $i \in I$ and $e \in \{\pm 1\}$ by

$$T'_{i,e}(q^h) = q^{s_i(h)}, \tag{4.1a}$$

$$T'_{i,e}(e_i) = -t_i^e f_i, \tag{4.1b}$$

$$T'_{i,e}(f_i) = -e_i t_i^{-e}, \tag{4.1c}$$

$$T'_{i,e}(e_j) = \sum_{r+s=-\langle h_i, \alpha_j \rangle} (-1)^r q_i^{er} e_i^{(r)} e_j e_i^{(s)} \text{ for } j \neq i, \tag{4.1d}$$

$$T'_{i,e}(f_j) = \sum_{r+s=-\langle h_i, \alpha_j \rangle} (-1)^r q_i^{-er} f_i^{(s)} f_j f_i^{(r)} \text{ for } j \neq i. \tag{4.1e}$$

For $i \in I$ and $e \in \{\pm 1\}$, we also define the $\mathbb{Q}(q)$-algebra automorphisms $T''_{i,e} \colon \mathbf{U}_q(\mathfrak{g}) \to \mathbf{U}_q(\mathfrak{g})$ by

$$T''_{i,-e}(q^h) = q^{s_i(h)}, \tag{4.2a}$$

$$T''_{i,-e}(e_i) = -f_i t_i^{-e}, \tag{4.2b}$$

$$T''_{i,-e}(f_i) = -t_i^{e} e_i, \tag{4.2c}$$

$$T''_{i,-e}(e_j) = \sum_{r+s=-\langle h_i,\alpha_j\rangle} (-1)^r q_i^{er} e_i^{(s)} e_j e_i^{(r)} \text{ for } j \neq i, \tag{4.2d}$$

$$T''_{i,-e}(f_j) = \sum_{r+s=-\langle h_i,\alpha_j\rangle} (-1)^r q_i^{-er} f_i^{(r)} f_j f_i^{(s)} \text{ for } j \neq i. \tag{4.2e}$$

We have

$$T'_{i,e} T''_{i,-e} = T''_{i,-e} T'_{i,e} = \mathrm{id}. \tag{4.3}$$

In the following, we write $T_i = T''_{i,1}$ and $T_i^{-1} = T'_{i,-1}$ as in [52, Proposition 1.3.1].

4.2.2. We define a q-analogue of the action of the Weyl group on integrable module following [41, Chapter 5] and [52]. We use a q-analogue of exponential $\exp_q(x)$ defined by

$$\exp_q(x) := \sum_{n \geq 0} \frac{q^{n(n-1)/2}}{[n]_q!} x^n.$$

We have

$$\exp_q(x) \exp_{q^{-1}}(-x) = 1. \tag{4.4}$$

For $i \in I$, we define S_i ([52, (1.2.2), (1.2.13)]) by

$$S_i := \exp_{q_i^{-1}}(q_i^{-1} e_i t_i^{-1}) \exp_{q_i^{-1}}(-f_i) \exp_{q_i^{-1}}(q_i e_i t_i) q_i^{h_i(h_i+1)/2} \tag{4.5a}$$

$$= \exp_{q_i^{-1}}(-q_i^{-1} f_i t_i) \exp_{q_i^{-1}}(e_i) \exp_{q_i^{-1}}(-q_i f_i t_i^{-1}) q_i^{h_i(h_i+1)/2}. \tag{4.5b}$$

For integrable $\mathbf{U}_q(\mathfrak{g})$-modules, the action of S_i is well-defined. It is known that the action of $\{S_i\}_{i \in I}$ satisfies the braid group relations for the Weyl group W.

The braid group symmetry $\{T_i\}_{i \in I}$ defined above is described as

$$T_i(x) = S_i x S_i^{-1}, \tag{4.6}$$

where the elements are considered in the endomorphism ring of integrable modules, see [52, 1.3] for more details.

4.2.3. We have the following relationship between T_i, T_i^{-1} and the $*$-involution.

Proposition 4.7 ([41, 37.2.4])**.** We have

$$* \circ T_i \circ * = T_i^{-1}. \tag{4.8}$$

4.3. **Quantum nilpotent subalgebra $\mathbf{U}_q^-(w, e)$.**

4.3.1. We define root vectors associated with $\widetilde{w} = (i_1, \cdots, i_l) \in R(w)$ for $w \in W$. See [41, Proposition 40.1.3, Proposition 41.1.4] for more detail. For $w \in W$ and $\widetilde{w} \in R(w)$, we define β_k as above. We define the *root vectors* $F_e(\beta_k)$ associated with $\beta_k \in \Delta(w)$ and $e \in \{\pm 1\}$ by

$$F_e(\beta_k) := T_{i_1}^e \cdots T_{i_{k-1}}^e(f_{i_k}).$$

It is known that $F_e(\beta_k) \in \mathbf{U}_q^-(\mathfrak{g})$. We note that $F_e(\beta_k)$ *does* depend on the choice of $\widetilde{w} \in R(w)$. We define its divided power by

$$F_e(c\beta_k) := T_{i_1}^e \cdots T_{i_{k-1}}^e(f_{i_k}^{(c)})$$

for $c \geq 1$. It is known that $F_e(c\beta_k) \in \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}$.

4.3.2.

Theorem 4.9 ([41, Proposition 40.2.1, Proposition 41.1.3])**.** (1) For $w \in W$, $\widetilde{w} \in R(w)$, $e \in \{\pm 1\}$ and $\mathbf{c} \in \mathbb{Z}_{\geq 0}^l$, we set

$$F_e(\mathbf{c}, \widetilde{w}) := \begin{cases} F_e(c_1\beta_1) \cdots F_e(c_l\beta_l) & \text{if } e = +1, \\ F_e(c_l\beta_l) \cdots F_e(c_1\beta_1) & \text{if } e = -1. \end{cases}$$

Then $\{F_e(\mathbf{c}, \widetilde{w})\}_{\mathbf{c} \in \mathbb{Z}_{\geq 0}^l}$ forms a basis of a subspace $\mathbf{U}_q^-(w, e)$ of $\mathbf{U}_q^-(\mathfrak{g})$ which does not depend on $\widetilde{w}$.

(2) We have $F_e(\mathbf{c}, \widetilde{w}) \in \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}$ for any $\mathbf{c} \in \mathbb{Z}_{\geq 0}^l$.

4.3.3. We recall commutation relations for root vectors and its divied powers $\{F(c_k\beta_k)\}_{1 \leq k \leq l, c_k \geq 1}$, known as the *Levendorskii-Soibelman formula.* See [38], [1] or [48] for more details.

In this subsections, we give statements for the $e = +1$ case. We can obtain the corresponding results for the $e = -1$ case, applying the $*$-involution (4.8). So we denote $F_e(c\beta)$, $F_e(\mathbf{c}, \widetilde{w})$ by $F(c\beta)$, $F(\mathbf{c}, \widetilde{w})$ by omitting e.

Let $w \in W$, $\widetilde{w} = (i_1, i_2, \cdots, i_l) \in R(w)$ and fix a total order on $\Delta^+(w)$ given by

$$\beta_1 < \beta_2 < \cdots < \beta_l.$$

Theorem 4.10 ([48, Proposition 3.6], [38, 5.5.2 Proposition])**.** For $j < k$, let us write

$$F(c_k\beta_k)F(c_j\beta_j) - q^{-(c_j\beta_j, c_k\beta_k)}F(c_j\beta_j)F(c_k\beta_k) = \sum f_{\mathbf{c}'}F(\mathbf{c}', \widetilde{w})$$

with $f_{\mathbf{c}'} \in \mathbb{Q}(q)$. If $f_{\mathbf{c}'} \neq 0$, then $c'_j < c_j$ and $c'_k < c_k$ with $\sum_{j \leq m \leq k} c'_m\beta_m = c_j\beta_j + c_k\beta_k$.

4.3.4. The following proposition is a consequence of Theorem 4.10. (cf. [37, 2.4.2 Proposition Theorem b)] and [11, 2.2 Proposition].)

Proposition 4.11. Let $\widetilde{w} = (i_1, i_2, \cdots, i_l)$ be a reduced expression for $w \in W$ and $e \in \{\pm 1\}$. Then the subspace $\mathbf{U}_q^-(w, e)$ is a $\mathbb{Q}(q)$-subalgebra generated by $\{F_e(\beta_k)\}_{1 \leq k \leq l}$.

We call it the *quantum nilpotent subalgebra* associated with $w \in W$.

4.3.5. We define a lexicographic order $\leq$ on $\mathbb{Z}_{\geq 0}^l$ associated with $\widetilde{w} \in R(w)$ by

$$\begin{aligned} &\mathbf{c} = (c_1, c_2, \cdots, c_l) < \mathbf{c}' = (c'_1, c'_2, \cdots, c'_l) \\ \iff &\text{there exists } 1 \leq p \leq l \text{ such that } c_1 = c'_1, \cdots, c_{p-1} = c'_{p-1}, c_p < c'_p. \end{aligned}$$

The following theorem is obtained as a consequence of the Levendorskii-Soibelmann formula.

Theorem 4.12. Let $w \in W$ and $\widetilde{w} \in R(w)$ be its reduced expression. For $\mathbf{c} \in \mathbb{Z}_{\geq 0}^{l}$, we consider the following $\mathbb{Q}(q)$-subspace $\mathcal{F}_{\leq \mathbf{c}}^{\widetilde{w}} \mathbf{U}_q^-(w)$:

$$\mathcal{F}_{\leq \mathbf{c}}^{\widetilde{w}} \mathbf{U}_q^-(w) := \bigoplus_{\mathbf{c}' \leq \mathbf{c}} \mathbb{Q}(q) F(\mathbf{c}', \widetilde{w}).$$

Then

(1) $\{\mathcal{F}_{\leq \mathbf{c}}^{\widetilde{w}} \mathbf{U}_q^-(w)\}_{\mathbf{c} \in \mathbb{Z}_{\geq 0}^{l}}$ forms an increasing filtration on $\mathbf{U}_q^-(w)$.

(2) The associated graded algebra $\mathrm{gr}^{\widetilde{w}} \mathbf{U}_q^-(w)$ is generated by $\{\mathrm{gr}^{\widetilde{w}}(F(\beta_k)) | 1 \leq k \leq l\}$ with relations:

$$\mathrm{gr}^{\widetilde{w}}(F(\beta_k)) \mathrm{gr}^{\widetilde{w}}(F(\beta_j)) = q^{-(\beta_j, \beta_k)} \mathrm{gr}^{\widetilde{w}}(F(\beta_j)) \mathrm{gr}^{\widetilde{w}}(F(\beta_k)) \ (j < k).$$

We call this the *De Concini-Kac filtration*.

4.4. **PBW basis and the canonical base.** In this subsection, we recall compatibilities between Lusztig's braid symmetry $\{T_i\}_{i \in I}$ and the canonical base. For more details, see [41, Chapter 38], [43] and [52].

Lemma 4.13 ([41, Proposition 38.1.6, Lemma 38.1.5])**.** (1) For $i \in I$, we have

$$\begin{aligned} \mathbf{U}_q^-[i] :=& \{x \in \mathbf{U}_q^- ; {}_i r(x) = 0\}, \\ =& \{x \in \mathbf{U}_q^- ; T_i^{-1}(x) \in \mathbf{U}_q^-\}, \\ {}^*\mathbf{U}_q^-[i] :=& \{x \in \mathbf{U}_q^- ; r_i(x) = 0\}, \\ =& \{x \in \mathbf{U}_q^- ; T_i(x) \in \mathbf{U}_q^-\}. \end{aligned}$$

(2) For $i \in I$, we have the following orthogonal decompositions:

$$\mathbf{U}_q^- = \mathbf{U}_q^-[i] \oplus f_i \mathbf{U}_q^- = {}^*\mathbf{U}_q^-[i] \oplus \mathbf{U}_q^- f_i.$$

From Lemma 4.13 and Theorem 2.29, we obtain the following result.

Proposition 4.14. For $n \geq 0$ and $i \in I$, the subspaces $\bigoplus_{k=0}^{n} f_i^k \mathbf{U}_q^-[i]$ and $\bigoplus_{k=0}^{n} {}^*\mathbf{U}_q^-[i] f_i^k$ are compatible with the dual canonical base and we have

$$\begin{aligned} \bigoplus_{k=0}^{n} f_i^k \mathbf{U}_q^-[i] &= \bigoplus_{b \in \mathscr{B}(\infty), \varepsilon_i(b) \leq n} \mathbb{Q}(q) G^{\mathrm{up}}(b), \\ \bigoplus_{k=0}^{n} {}^*\mathbf{U}_q^-[i] f_i^k &= \bigoplus_{b \in \mathscr{B}(\infty), \varepsilon_i^*(b) \leq n} \mathbb{Q}(q) G^{\mathrm{up}}(b). \end{aligned}$$

Let ${}^i\pi \colon \mathbf{U}_q^- \to \mathbf{U}_q^-[i]$ (resp. $\pi^i \colon \mathbf{U}_q^- \to {}^*\mathbf{U}_q^-[i]$) be the orthogonal projection whose kernel is $f_i \mathbf{U}_q^-(\mathfrak{g})$ (resp. $\mathbf{U}_q^-(\mathfrak{g}) f_i$). The following result is due to Saito and Lusztig.

Theorem 4.15 ([52, Prospoition 3.4.7, Corollary 3.4.8], [43, Theorem 1.2])**.** For $b \in \mathscr{B}(\infty)$ with $\varepsilon_i^*(b) = 0$, we have

$$T_i(\pi^i G^{\mathrm{low}}(b)) = {}^i\pi(G^{\mathrm{low}}(\Lambda_i(b))) \in \mathscr{L}(\infty),$$

where $\Lambda_i \colon \{b \in \mathscr{B}(\infty); \varepsilon_i^*(b) = 0\} \to \{b \in \mathscr{B}(\infty); \varepsilon_i(b) = 0\}$ is the bijection given by $\Lambda_i(b) = \widetilde{f}_i^{*\varphi_i(b)} \widetilde{e}_i^{\varepsilon_i(b)} b$ and its inverse is given by $\Lambda_i^{-1}(b) = \widetilde{f}_i^{\varphi_i^*(b)} \widetilde{e}_i^{*\varepsilon_i^*(b)} b$.

By Theorem 4.15, we obtain the following result.

Theorem 4.16 ([52, Theorem 4.1.2], [43, Proposition 8.2]). For $w \in W$, $\widetilde{w} = (i_1, i_2, \cdots, i_l) \in R(w)$ and $e \in \{\pm 1\}$,

(1) we have $F_e(\mathbf{c}, \widetilde{w}) \in \mathscr{L}(\infty)$ and

$$b_e(\mathbf{c}, \widetilde{w}) = F_e(\mathbf{c}, \widetilde{w}) \bmod q\mathscr{L}(\infty) \in \mathscr{B}(\infty).$$

(b) The map $\mathbb{Z}^l_{\geq 0} \to \mathscr{B}(\infty)$ which is defined by $\mathbf{c} \mapsto b_e(\mathbf{c}, \widetilde{w})$ is injective. We denote the image by $\mathscr{B}(w, e)$ and this does not depend on a choice of $\widetilde{w} \in R(w)$.

For fixed $\widetilde{w} \in R(w)$, we denote the inverse of $\mathbf{c} \mapsto b_e(\mathbf{c}, \widetilde{w})$ by $L_{e,\widetilde{w}} \colon \mathscr{B}(w, e) \to \mathbb{Z}^l_{\geq 0}$. This map is called *Lusztig data* of b associated with $\widetilde{w}$.

4.4.1. As a corollary of the above description, we have the following properties.

Corollary 4.17. (1) We have $\Lambda_i S_m(b) = S_m \Lambda_i(b)$ for $b \in \{b \in \mathscr{B}(\infty); \varepsilon_i^*(b) = 0\}$.

(2) We have $S_m \colon \mathscr{B}(w, e) \to \mathscr{B}(w, e)$ for any $m \geq 1$ and $S_m(b_e(\mathbf{c}, \widetilde{w})) = b_e(m\mathbf{c}, \widetilde{w})$.

(3) We have $*(\mathscr{B}(w, e)) = \mathscr{B}(w, -e)$ and $*b_e(\mathbf{c}, \widetilde{w}) = b_{-e}(\mathbf{c}, \widetilde{w})$.

4.5. **Inner products of PBW basis.** By 2.12, we have the following modification of [41, Proposition 38.2.1].

Proposition 4.18. For $x, y \in \mathbf{U}_q^-(\mathfrak{g})_\xi$ with $x, y \in \mathbf{U}_q^-[i]$ (resp. with $x, y \in {}^*\mathbf{U}_q^-[i]$), we have

$$(x, y)_K = (1 - q_i^2)^{-\langle h_i, \xi \rangle}(T_i^{-1}x, T_i^{-1}y)_K \text{ (resp. } (1 - q_i^2)^{-\langle h_i, \xi \rangle}(T_i x, T_i y)_K).$$

4.5.1. We have the following formula for inner product of PBW basis with respect to Lusztig's bilinear form $(\ , \ )_L$. For more details, see [41, Propsition 38.2.3].

Proposition 4.19. Let $w \in W$ and $\widetilde{w} \in R(w)$ with $l = \ell(w)$. We have

$$(F(\mathbf{c}, \widetilde{w}), F(\mathbf{c}', \widetilde{w}))_L = \prod_{k=1}^{l} \delta_{c_k, c'_k} \prod_{s=1}^{c_k} \frac{1}{1 - q_i^{2s}} = \prod_{k=1}^{l} \delta_{c_k, c'_k} (-1)^{c_k} \frac{q_{i_k}^{-\frac{c_k(c_k+1)}{2}}}{(q_{i_k} - q_{i_k}^{-1})^{c_k}[c_k]^!_{i_k}}.$$

4.6. **Compatibility with T_i and the dual canonical base.**

4.6.1. By using the above results, we obtain the following compatibility between the dual canonical basis and Lusztig's braid group symmetry T_i.

Theorem 4.20. For $b \in \mathscr{B}(\infty)$ wiht $\varepsilon_i^*(b) = 0$, we have

$$(1 - q_i^2)^{\langle h_i, \xi \rangle} T_i G^{\mathrm{up}}(b) = G^{\mathrm{up}}(\Lambda_i b).$$

Proof. We shall prove that $((1 - q_i^2)^{\langle h_i, \xi \rangle} T_i G^{\mathrm{up}}(b), G^{\mathrm{low}}(b'))_K = \delta_{b', \Lambda_i(b)}$. By Lemma 4.13, $(1 - q_i^2)^{\langle h_i, \xi \rangle}(T_i G^{\mathrm{up}}(b), G^{\mathrm{low}}(b'))_K$ is equal to $(1 - q_i^2)^{\langle h_i, \xi \rangle}(T_i G^{\mathrm{up}}(b), {}^i\pi G^{\mathrm{low}}(b'))_K$. By the Proposition 4.18, this is equal to $(G^{\mathrm{up}}(b), T_i^{-1} {}^i\pi G^{\mathrm{low}}(b'))_K$. Using Theorem 4.15, we have

$$\begin{aligned}(G^{\mathrm{up}}(b), T_i^{-1} {}^i\pi G^{\mathrm{low}}(b'))_K &= (G^{\mathrm{up}}(b), \pi^i G^{\mathrm{low}}(\Lambda_i^{-1} b'))_K \\ = (G^{\mathrm{up}}(b), G^{\mathrm{low}}(\Lambda_i^{-1} b'))_K &= \delta_{b, \Lambda_i^{-1}(b')}.\end{aligned}$$

Then we obtain the assertion. q.e.d

As a corollary, we obtain the following multiplicative properties.

Corollary 4.21. (1) For $b_1, b_2 \in \mathscr{B}(\infty)$ with $\varepsilon_i^*(b_1) = \varepsilon_i^*(b_2) = 0$ (resp. $\varepsilon_i(b_1) = \varepsilon_i(b_2) = 0$) and $b_1 \perp b_2$, we have $\Lambda_i(b_1) \perp \Lambda_i(b_2)$ (resp. $\Lambda_i^{-1}(b_1) \perp \Lambda_i^{-1}(b_2)$).

(2) If $b \in \mathscr{B}(\infty)$ with $\varepsilon_i^*(b) = 0$ (resp. $\varepsilon_i(b) = 0$) is (strongly) real, then $\Lambda_i(b)$ (resp. $\Lambda_i^{-1}(b)$) is also (strongly) real.

4.7. **Compatibility with the dual canonical basis.** In this subsection, we prove the compatibility of the dual canonical basis with the $\mathbb{Q}(q)$-subalgebra $\mathbf{U}_q^-(w,e)$. This is a straightforward generalization of [9, 2.2 Proposition] and [32, Theorem 4.1]. Here we fix $w \in W$ and $\widetilde{w} \in R(w)$.

Theorem 4.22. For $w \in W$ and $e \in \{\pm 1\}$, the triple $(\mathbf{U}_q^-(w,e) \cap \mathscr{L}(\infty), \mathbf{U}_q^-(w,e) \cap \sigma(\mathscr{L}(\infty)), \mathbf{U}_q^-(w,e) \cap \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}})$ is balanced, in particular we have

$$\mathbf{U}_q^-(w,e) \cap \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} = \bigoplus_{b \in \mathscr{B}(w,e)} \mathcal{A} G^{\mathrm{up}}(b).$$

The proof of this theorem occupies the rest of this subsection. As in [36, 3.4], [34, Proposition 31, Corollary 41] and [9], we first prove that the dual root vectors are contained in the dual canonical basis and then prove the unitriangular property of upper global basis with respect to the dual PBW basis. The compatibility with the dual canonical basis is its direct consequence.

Our proof needs an extra step from ones in [36, 34, 9], as it is not known that the PBW basis is an $\mathcal{A}$-basis of $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}} \cap \mathbf{U}_q^-(w)$ unless $\mathfrak{g}$ is of finite or affine type.

4.7.1.

Proposition 4.23. (1) For $i \in I$ and $n \geq 1$, let

$$F^{\mathrm{up}}(n\alpha_i) := \frac{f_i^{(n)}}{(f_i^{(n)}, f_i^{(n)})_K}.$$

Then we have $F^{\mathrm{up}}(n\alpha_i) \in \mathbf{B}^{\mathrm{up}}$, $(F_{\alpha_i}^{\mathrm{up}})^n \in q^{\mathbb{Z}}\mathbf{B}^{\mathrm{up}}$ and $F^{\mathrm{up}}(n\alpha_i)F^{\mathrm{up}}(m\alpha_i) = q_i^{mn}F^{\mathrm{up}}((m+n)\alpha_i)$

(2) For $n \geq 1$ and $1 \leq k \leq l$, let

$$F^{\mathrm{up}}(n\beta_k) := \frac{F(n\beta_k)}{(F(n\beta_k), F(n\beta_k))_K}.$$

Then we have $F^{\mathrm{up}}(n\beta_k) \in \mathbf{B}^{\mathrm{up}}$, $F^{\mathrm{up}}(\beta_k)^n \in q^{\mathbb{Z}}\mathbf{B}^{\mathrm{up}}$ and $F^{\mathrm{up}}(n\beta_k)F^{\mathrm{up}}(m\beta_k) = q_{i_k}^{mn}F^{\mathrm{up}}((m+n)\beta_k)$

Proof. Since $F_i^{(n)}$ are the canonical base elements and $\dim \mathbf{U}_q^-(\mathfrak{g})_{-n\alpha_i} = 1$ for any $n \geq 1$, $F^{\mathrm{up}}(n\alpha_i)$ are the dual canonical base elements by its definition. By Proposition 4.19 and Lemma 2.12, we have

$$(F_i^{(n)}, F_i^{(n)})_K = (1-q_i^2)^n / \prod_{j=1}^{n}(1-q_i^{2j}) = \frac{1}{[n]_i!}q_i^{\frac{n(n-1)}{2}}.$$

Therefore we have $(F^{\mathrm{up}}(\alpha_i))^n = q_i^{\frac{n(n-1)}{2}}F^{\mathrm{up}}(n\alpha_i) \in \mathbf{B}^{\mathrm{up}}$, in particular $F^{\mathrm{up}}(\alpha_i)$ is a strongly real element. Applying Theorem 4.20, we obtain the result for $F^{\mathrm{up}}(n\beta_k)$ for $1 \leq k \leq l$. q.e.d

4.7.2. For the computation of the action of the dual bar involution σ, we need an integrality property of the Levendorskii-Soibelman's formula for the dual root vectors and its multiple. For $w \in W$, $\widetilde{w} \in R(w)$ and $e \in \{\pm 1\}$, we set

$$F_e^{\mathrm{up}}(\mathbf{c}, \widetilde{w}) := \frac{1}{(F_e(\mathbf{c}, \widetilde{w}), F_e(\mathbf{c}, \widetilde{w}))_K}F_e(\mathbf{c}, \widetilde{w}).$$

This is the dual basis of $\{F_e(\mathbf{c}, \widetilde{w})\}$ with respect to Kashiwara's bilinear form $(\cdot,\cdot)_K$. As before, when we consider only the $e = 1$ case, we omit the subscript e.

Theorem 4.24 (dual Levendorskii-Soibelman formula)**.** For $j < k$, we write

$$F^{\mathrm{up}}(c_k\beta_k)F^{\mathrm{up}}(c_j\beta_j) - q^{-(c_j\beta_j, c_k\beta_k)}F^{\mathrm{up}}(c_j\beta_j)F^{\mathrm{up}}(c_k\beta_k) = \sum f^*_{\mathbf{c}'}F^{\mathrm{up}}(\mathbf{c}', \widetilde{w}).$$

Then $f^*_{\mathbf{c}'} \in \mathcal{A}$ and if $f^*_{\mathbf{c}'} \neq 0$, then $c'_j < c_j$ and $c'_k < c_k$ with $\sum_{j \le m \le k} c'_m\beta_m = c_j\beta_j + c_k\beta_k$.

Proof. Firstly, a weaker statement that $f^*_{\mathbf{c}'} \in \mathbb{Q}(q)$ with the above conditions follows from Theorem 4.10 and Proposition 4.19. Let us prove that $f^*_{\mathbf{c}'} \in \mathcal{A}$. Since the twisted coproduct r preserves the $\mathcal{A}$-form $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}$, the dual integral form $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$ is an $\mathcal{A}$-subalgebra of $\mathbf{U}_q^-(\mathfrak{g})$. Therefore the left hand side belongs to $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$ by Proposition 4.23. Taking the inner product with $F(\mathbf{c}', \widetilde{w})$, we find $f^*_{\mathbf{c}'} \in \mathcal{A}$ thanks to Theorem 4.9. q.e.d

In particular, we have the $\mathcal{A}$-subalgebra $\mathbf{U}_q^-(w,e)_{\mathcal{A}}^{\mathrm{up}}$ of $\mathbf{U}_q^-(w,e)$ generated by $\{F^{\mathrm{up}}(c\beta_k)\}_{1 \le k \le l, c \ge 1}$. This subalgebra has a free $\mathcal{A}$-basis $\{F_e^{\mathrm{up}}(\mathbf{c}, \widetilde{w}); \mathbf{c} \in \mathbb{Z}^l_{\ge 0}\}$. We call this base the *dual PBW basis*.

4.7.3. We compute the action of the dual bar involution σ on the dual PBW basis. The following is straightforward generalization of [9, 2.1 Corollary (i)], and follows from (3.7) and Theorem 4.24.

Proposition 4.25. We have

$$\sigma(F^{\mathrm{up}}(\mathbf{c}, \widetilde{w})) = F^{\mathrm{up}}(\mathbf{c}, \widetilde{w}) + \sum_{\mathbf{c}' < \mathbf{c}} f^*_{\mathbf{c},\mathbf{c}'}(q)F^{\mathrm{up}}(\mathbf{c}', \widetilde{w}),$$

where $f^*_{\mathbf{c},\mathbf{c}'}(q) \in \mathcal{A}$.

4.7.4.

Theorem 4.26. (1) Let $w \in W$ and $\widetilde{w} \in R(w)$. Then there exists a unique $\mathcal{A}$-basis $\{B^{\mathrm{up}}(\mathbf{c}, \widetilde{w}); \mathbf{c} \in \mathbb{Z}^l_{\ge 0}\}$ of $\mathbf{U}_q^-(w,e)_{\mathcal{A}}^{\mathrm{up}}$ with the following properties:

$$\sigma(B^{\mathrm{up}}(\mathbf{c}, \widetilde{w})) = B^{\mathrm{up}}(\mathbf{c}, \widetilde{w}), \tag{4.27a}$$

$$F^{\mathrm{up}}(\mathbf{c}, \widetilde{w}) = B^{\mathrm{up}}(\mathbf{c}, \widetilde{w}) + \sum_{\mathbf{c}' < \mathbf{c}} \varphi_{\mathbf{c},\mathbf{c}'}B^{\mathrm{up}}(\mathbf{c}', \widetilde{w}), \varphi_{\mathbf{c},\mathbf{c}'} \in q\mathbb{Z}[q]. \tag{4.27b}$$

(2) We have $B^{\mathrm{up}}(\mathbf{c}, \widetilde{w}) = G^{\mathrm{up}}(b(\mathbf{c}, \widetilde{w}))$.

Proof. The proof of (1) is the same as one for the existence of Kazhdan-Lusztig polynomials. The only claim we need is Proposition 4.25.

(2) Since we have $f_{\mathbf{c}}(q) = (F(\mathbf{c}, \widetilde{w}), F(\mathbf{c}, \widetilde{w}))_K \in \mathcal{A}_0$ and $f_{\mathbf{c}}(0) = 1$, we obtain

$$B^{\mathrm{up}}(\mathbf{c}, \widetilde{w}) \equiv F^{\mathrm{up}}(\mathbf{c}, \widetilde{w}) \equiv b(\mathbf{c}, \widetilde{w}) \bmod q\mathscr{L}(\infty).$$

Therefore (2) follows from (1) and Proposition 3.4. q.e.d

As a corollary, we have $\mathbf{U}_q^-(w,e)_{\mathcal{A}}^{\mathrm{up}} = \mathbf{U}_q^-(w,e) \cap \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$ since $\{G^{\mathrm{up}}(b)\}_{b \in \mathscr{B}(\infty)}$ is an $\mathcal{A}$-basis of $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$. Together with this result, Theorem 4.26 implies Theorem 4.22.

4.8. In this subsection, we study basic commutation relation among the dual canonical basis of $\mathbf{U}_q^-(w, e = 1)$. The following is a generalization of [50, Proposition 4.2], and follows from the characterization of dual canonical basis in terms of dual PBW basis. For $\mathbf{c}, \mathbf{c}' \in \mathbb{Z}_{\geq 0}^l$, we set

$$c_{\widetilde{w}}(\mathbf{c}, \mathbf{c}') = \sum_{l<k} (c_k\beta_k, c_l'\beta_l) - \frac{1}{2}\sum_k c_k c_k' (\beta_k, \beta_k).$$

Proposition 4.28. We have

$$G^{\mathrm{up}}(b(\mathbf{c}, \widetilde{w}))G^{\mathrm{up}}(b(\mathbf{c}', \widetilde{w})) = q^{-c_{\widetilde{w}}(\mathbf{c},\mathbf{c}')}G^{\mathrm{up}}(b(\mathbf{c}+\mathbf{c}', \widetilde{w})) + \sum d_{\mathbf{c},\mathbf{c}'}^{\mathbf{d}}(q) G^{\mathrm{up}}(b(\mathbf{d}, \widetilde{w})),$$

where $\mathbf{d} < \mathbf{c} + \mathbf{c}'$ and $d_{\mathbf{c},\mathbf{c}'}^{\mathbf{d}}(q) \in \mathcal{A}$.

Corollary 4.29. If $b(\mathbf{c}, \widetilde{w}) \perp b(\mathbf{c}', \widetilde{w})$, we have

$$G^{\mathrm{up}}(b(\mathbf{c}, \widetilde{w}))G^{\mathrm{up}}(b(\mathbf{c}, \widetilde{w})) \simeq G^{\mathrm{up}}(b(\mathbf{c}+\mathbf{c}', \widetilde{w})),$$

that is $b(\mathbf{c}, \widetilde{w}) \circledast b(\mathbf{c}', \widetilde{w}) = b(\mathbf{c}+\mathbf{c}', \widetilde{w})$.

4.8.1. Using Proposition 4.28, we have the following expression of q-power of the q-commuting dual canonical basis elements in $\mathscr{B}(w, e = 1)$ as in [36, Proposition 18].

Proposition 4.30. If $G^{\mathrm{up}}(b(\mathbf{c}, \widetilde{w}))G^{\mathrm{up}}(b(\mathbf{c}', \widetilde{w})) = q^{-N_{\widetilde{w}}(\mathbf{c},\mathbf{c}')}G^{\mathrm{up}}(b(\mathbf{c}', \widetilde{w}))G^{\mathrm{up}}(b(\mathbf{c}, \widetilde{w}))$, then we have

$$N_{\widetilde{w}}(\mathbf{c}, \mathbf{c}') = c_{\widetilde{w}}(\mathbf{c}, \mathbf{c}') - c_{\widetilde{w}}(\mathbf{c}', \mathbf{c}). \tag{4.31}$$

4.9. In this subsection, we recall the specialization of $\mathbf{U}_q^-(w, e)$ at $q = 1$.

4.9.1. We have the following property of the specialization of $\mathbf{U}_q^-$ at $q = 1$.

Theorem 4.32 ([41, §33.1])**.** There is an isomorphism of algebras:

$$\Phi \colon U(\mathfrak{n}) \xrightarrow{\sim} \mathbb{C} \otimes_{\mathcal{A}} \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}$$

which sends f_i to f_i.

Let $r \colon U(\mathfrak{n}) \to U(\mathfrak{n}) \otimes U(\mathfrak{n})$ be the coproduct defined by $r(f) = f \otimes 1 + 1 \otimes f$ for $f \in \mathfrak{n}$. Here we note that $U(\mathfrak{n})$ is generated by $\{f_i\}_{i\in I}$ as algebra. Since the specializatioin of the twisted coproduct satisfies this relation on the generators, the above is an isomorphism of bialgebras.

4.9.2. Let $\mathbb{C}[N]$ be the restricted dual of the universal enveloping algebra $U(\mathfrak{n})$ of the Lie algebra $\mathfrak{n}$, that is

$$\mathbb{C}[N] := \bigoplus_{\xi \in Q} U(\mathfrak{n})_\xi^*.$$

We take the dual $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$ of $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}$ as before. Since the multiplication of $\mathbf{U}_q^-(\mathfrak{g})$ preserves $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}$, the twisted coproduct r preserves the dual integral form $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$, that is $r(\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}) \subset \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} \otimes \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$.

Let $r^* \colon \mathbb{C}[N] \otimes \mathbb{C}[N] \to \mathbb{C}[N]$ be a product so that $\langle r^*(\varphi \otimes \varphi'), x\rangle = \langle \varphi \otimes \varphi', r(x)\rangle$ holds for any $x \in U(\mathfrak{n})$ and $\mu^* \colon \mathbb{C}[N] \to \mathbb{C}[N] \otimes \mathbb{C}[N]$ be a coproduct so that $\langle \mu^*(\varphi), x \otimes x'\rangle = \langle \varphi, \mu(x \otimes x')\rangle$ holds for any $x, x' \in U(\mathfrak{n})$, where $\mu \colon U(\mathfrak{n}) \otimes U(\mathfrak{n}) \to U(\mathfrak{n})$ is the product on $U(\mathfrak{n})$. The above isomorphism Φ induces the following.

Proposition 4.33. There is an isomorphism of bialgebras

$$\Phi^{\mathrm{up}} \colon \mathbb{C} \otimes_{\mathcal{A}} \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} \xrightarrow{\sim} \mathbb{C}[N],$$

that is we have

$$\mu^* \circ \Phi^{\mathrm{up}} = (\Phi^{\mathrm{up}} \otimes \Phi^{\mathrm{up}}) \circ r,$$
$$r^* \circ (\Phi^{\mathrm{up}} \otimes \Phi^{\mathrm{up}}) = \Phi^{\mathrm{up}} \circ \mu.$$

4.9.3. Let

$$\begin{aligned}\sigma_i :=&\ \exp(-f_i)\exp(e_i)\exp(-f_i)\\ =&\ \exp(e_i)\exp(-f_i)\exp(e_i),\end{aligned}$$

for $i \in I$. Then we have

$$\begin{aligned}(\sigma_i)^{-1} =&\ \exp(f_i)\exp(-e_i)\exp(f_i)\\ =&\ \exp(-e_i)\exp(f_i)\exp(-e_i).\end{aligned}$$

(This $(\sigma_i)^{-1}$ is equal to $\overline{s}_i$ used in [23, 7.1].) The action of σ_i is well-defined on integrable $\mathfrak{g}$-modules, especially on the adjoint representation of $\mathfrak{g}$. Under the specialization at $q = 1$, we have $\sigma_i = S_i|_{q=1}$.

4.9.4. For $\widetilde{w} \in R(w)$ and $e \in \{\pm 1\}$, let

$$f_e(\beta_k) := \sigma_{i_1}^e \cdots \sigma_{i_{k-1}}^e (f_{i_k}).$$

Then we have $f_e(\beta_k) \in \mathfrak{g}_{-\beta_k}$ and

$$\mathfrak{n}(w) = \bigoplus_{1 \le k \le l} \mathbb{C} f_e(\beta_k).$$

By the definition, $f_e(\beta_k)$ is the specialization of $F_e(\beta_k)$.

4.9.5. Let $\mathbb{C}[N(w)]$ be the restricted dual of the universal enveloping algebra $U(\mathfrak{n}(w))$ associated with $\mathfrak{n}(w)$. We consider a basis of $\mathfrak{n}(w)$ given by $\{f_e(\beta_k)\}_{1 \le k \le l}$ and also a basis $\{f_e(\beta_k)\}_{1 \le k \le l} \cup \{f'_k\}$ of $\mathfrak{g}$ which includes $\{f_e(\beta_k)\}_{1 \le k \le l}$ as in [23, 4.3]. Here we fix a total order on the basis of $\mathfrak{g}$ by

$$f_e(\beta_1) < \cdots < f_e(\beta_k) < f'_1 < f'_2 < \cdots .$$

By the Poincaré-Birkhoff-Witt basis theorem, we have a basis of $U(\mathfrak{n})$ given by

$$f_e((\mathbf{c}, \mathbf{d}), \widetilde{w}) := \begin{cases} f_e(\beta_1)^{(c_1)} \cdots f_e(\beta_l)^{(c_l)} f_1'^{(d_1)} \cdots & \text{when } e = 1, \\ \cdots f_1'^{(d_1)} f_e(\beta_l)^{(c_l)} \cdots f_e(\beta_1)^{(c_1)} & \text{when } e = -1, \end{cases}$$

and also a basis of $U(\mathfrak{n}(w))$ given by

$$f_e(\mathbf{c}, \widetilde{w}) := \begin{cases} f_e(\beta_1)^{(c_1)} \cdots f_e(\beta_l)^{(c_l)} & \text{when } e = 1, \\ f_e(\beta_l)^{(c_l)} \cdots f_e(\beta_1)^{(c_1)} & \text{when } e = -1, \end{cases}$$

where $x^{(c)} = x^c / c!$ for $x \in \mathfrak{g}$ and $c \in \mathbb{Z}_{\ge 0}$. We have $\Phi(f_e(\mathbf{c}, \widetilde{w})) = F_e(\mathbf{c}, \widetilde{w})|_{q=1}$.

4.9.6. Let $\{f_e^*(\mathbf{c},\widetilde{w})\}$ (resp. $\{f_e^*((\mathbf{c},\mathbf{d}),\widetilde{w})\}$) be the dual basis of $\{f_e(\mathbf{c},\widetilde{w})\}$ (resp. $\{f_e((\mathbf{c},\mathbf{d}),\widetilde{w})\}$). Using these, we obtain a section of $\mathbb{C}[N]\to\mathbb{C}[N(w)]$ as algebras.

Lemma 4.34. Let $\widetilde{\pi}_w^*\colon \mathbb{C}[N(w)]\to\mathbb{C}[N]$ be a $\mathbb{C}$-linear homomorphism defined by

$$\widetilde{\pi}_w^*(f_e^*(\mathbf{c},\widetilde{w})):=f_e^*((\mathbf{c},0),\widetilde{w}).$$

Then it is an algebra embedding.

Proof. First $\langle\widetilde{\pi}_w^*(f_e^*(\mathbf{c}_1,\widetilde{w}))\cdot\widetilde{\pi}_w^*(f_e^*(\mathbf{c}_2,\widetilde{w})),f_e((\mathbf{c}',\mathbf{d}'),\widetilde{w})\rangle$ is equal to

$$\left\langle\widetilde{\pi}_w^*(f_e^*(\mathbf{c}_1,\widetilde{w}))\otimes\widetilde{\pi}_w^*(f_e^*(\mathbf{c}_2,\widetilde{w})),r(f_e((\mathbf{c}',\mathbf{d}'),\widetilde{w}))\right\rangle.$$

We note that

$$r(f_e((\mathbf{c}',\mathbf{d}'),\widetilde{w}))=\sum_{\mathbf{c'}_1+\mathbf{c'}_2=\mathbf{c}',\mathbf{d}'_1+\mathbf{d}'_2=\mathbf{d}'}f_e((\mathbf{c'}_1,\mathbf{d}'_1),\widetilde{w}))\otimes f_e((\mathbf{c'}_2,\mathbf{d}'_2),\widetilde{w})). \tag{4.35}$$

Hence the above is equal to $\delta_{\mathbf{c}_1+\mathbf{c}_2,\mathbf{c}'}\delta_{0,\mathbf{d}'}$. On the other hand, we consider

$$\left\langle\widetilde{\pi}_w^*(f_e^*(\mathbf{c}_1,\widetilde{w})\cdot f_e^*(\mathbf{c}_2,\widetilde{w})),f_e((\mathbf{c}',\mathbf{d}'),\widetilde{w})\right\rangle.$$

By (4.35), we have $f_e^*(\mathbf{c}_1,\widetilde{w})\cdot f_e^*(\mathbf{c}_2,\widetilde{w}):=r^*(f_e^*(\mathbf{c}_1,\widetilde{w})\otimes f_e^*(\mathbf{c}_2,\widetilde{w}))=f_e^*(\mathbf{c}_1+\mathbf{c}_2,\widetilde{w})$. Then the above is equal to $\langle\widetilde{\pi}_w^*(f_e^*(\mathbf{c}_1+\mathbf{c}_2,\widetilde{w})),f_e((\mathbf{c}',\mathbf{d}'),\widetilde{w})\rangle=\delta_{\mathbf{c}_1+\mathbf{c}_2,\mathbf{c}'}\delta_{0,\mathbf{d}'}$. Then the assertion holds. q.e.d

By [23, Proposition 8.2], this embedding does not depend on the choice of $\widetilde{w}\in R(w)$ and of the basis of $\mathfrak{g}$.

4.9.7. We study the image of $\mathbf{U}_q^-(w,e)_{\mathcal{A}}^{\mathrm{up}}\otimes_{\mathcal{A}}\mathbb{C}$ under the isomorphism Φ^{up}.

Lemma 4.36. Let $f\in\mathfrak{g}_\alpha$ with $\alpha\in\Delta_+\setminus\Delta_+(w)$, we have

$$\langle f,\Phi^{\mathrm{up}}(G^{\mathrm{up}}(b)|_{q=1})\rangle=0$$

for $b\in\mathscr{B}(w,e)$.

Proof. Suppose that $b\in\mathscr{B}(w,e)$ and $f\in\mathfrak{g}_\alpha$ with $\langle f,\Phi^{\mathrm{up}}(G^{\mathrm{up}}(b)|_{q=1})\rangle\neq 0$. Then we have

$$\alpha=\sum_{1\le k\le l}a_k\beta_k$$

for some $a_k\in\mathbb{Z}_{\ge 0}$. By the definition of $\Delta_+(w)$, we have $w^{-1}\alpha\in\Delta_+$ and $w^{-1}(\sum_{1\le k\le l}a_k\beta_k)\in Q_-$. This is a contradiction. Hence we get the assertion. q.e.d

4.9.8. We have the following formula of the (twisted) coproduct of the root vectors $F(\beta_k)$, see [10, 3.5 Corollary 3].

Proposition 4.37. We have the following expansion:

$$r(F(\beta_k))-(1\otimes F(\beta_k)+F(\beta_k)\otimes 1)=\sum_{\mathbf{c}}x_{\mathbf{c}}\otimes F(\mathbf{c},\widetilde{w}),$$

where $x_{\mathbf{c}}\in\mathbf{U}_q^-(\mathfrak{g})$ and if $x_{\mathbf{c}}\neq 0$, then $c_{k'}=0$ for $k'\ge k$.

We have the compatibility of the twisted coproduct r with $\mathbf{U}_q^-(w,e)$ (cf. [37, 2.4.2 Theorem c)]).

Proposition 4.38. We have

$$r(\mathbf{U}_q^-(w,+1)_{\mathcal{A}}^{\mathrm{up}}) \subset \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} \otimes \mathbf{U}_q^-(w,+1)_{\mathcal{A}}^{\mathrm{up}},$$
$$r(\mathbf{U}_q^-(w,-1)_{\mathcal{A}}^{\mathrm{up}}) \subset \mathbf{U}_q^-(w,-1)_{\mathcal{A}}^{\mathrm{up}} \otimes \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}},$$

that is $\mathbf{U}_q^-(w,+1)_{\mathcal{A}}^{\mathrm{up}}$ (resp. $\mathbf{U}_q^-(w,-1)_{\mathcal{A}}^{\mathrm{up}}$) is a left (resp. right) $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$-comodule.

Proof. Recall that we proved $\mathbf{U}_q^-(w,e)_{\mathcal{A}}^{\mathrm{up}} = \mathbf{U}_q^-(w,e) \cap \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$ during the proof of Theorem 4.22. Since r preserves the dual $\mathcal{A}$-form $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$, it suffices to prove a weaker statement, that is

$$r(\mathbf{U}_q^-(w,+1)) \subset \mathbf{U}_q^-(\mathfrak{g}) \otimes \mathbf{U}_q^-(w,+1),$$
$$r(\mathbf{U}_q^-(w,-1)) \subset \mathbf{U}_q^-(w,-1) \otimes \mathbf{U}_q^-(\mathfrak{g}).$$

Moreover if we apply the $*$-involution, we obtain the claim for the $e=-1$ case from the claim for the $e=1$ case. So it is enough to prove the $e=1$ case. This assertion is a consequence of Proposition 4.11 and Proposition 4.37. q.e.d

4.9.9.

Theorem 4.39. Under the algebra homomorphism Φ^{up}, we have

$$\mathbb{C} \otimes_{\mathcal{A}} \mathbf{U}_q^-(w,e)_{\mathcal{A}}^{\mathrm{up}} \simeq \mathbb{C}[N(w)].$$

In view of this theorem, the quantum nilpotent subalgebra $\mathbf{U}_q^-(w,e)$ can be considered as the "quantum coordinate ring" of the corresponding unipotent subgroup $N(w)$, so we call it the *quantum unipotent subgroup* and denote it by $\mathcal{O}_q[N(w)]$.

Proof. We compute the following inner product:

$$\left\langle \Phi^{\mathrm{up}}(F_e^{\mathrm{up}}(\mathbf{c},\widetilde{w})|_{q=1}), f_e((\mathbf{c}',\mathbf{d}'),\widetilde{w}) \right\rangle.$$

First we have

$$\begin{aligned}
&\left\langle \Phi^{\mathrm{up}}(F_e^{\mathrm{up}}(\mathbf{c},\widetilde{w})|_{q=1}), f_e((\mathbf{c}',\mathbf{d}'),\widetilde{w}) \right\rangle \\
&= \begin{cases} \langle \mu^*(\Phi^{\mathrm{up}}(F_e^{\mathrm{up}}(\mathbf{c},\widetilde{w})|_{q=1}), f_e((\mathbf{c}',0),\widetilde{w}) \otimes f_e((0,\mathbf{d}'),\widetilde{w}) \rangle & \text{when } e=1 \\ \langle \mu^*(\Phi^{\mathrm{up}}(F_e^{\mathrm{up}}(\mathbf{c},\widetilde{w})|_{q=1}), f_e((0,\mathbf{d}'),\widetilde{w}) \otimes f_e((\mathbf{c}',0),\widetilde{w}) \rangle & \text{when } e=-1 \end{cases} \\
&= \begin{cases} \langle (\Phi^{\mathrm{up}} \otimes \Phi^{\mathrm{up}})(r(F_e^{\mathrm{up}}(\mathbf{c},\widetilde{w})|_{q=1})), f_e((\mathbf{c}',0),\widetilde{w}) \otimes f_e((0,\mathbf{d}'),\widetilde{w}) \rangle & \text{when } e=1 \\ \langle (\Phi^{\mathrm{up}} \otimes \Phi^{\mathrm{up}})(r(F_e^{\mathrm{up}}(\mathbf{c},\widetilde{w})|_{q=1})), f_e((0,\mathbf{d}'),\widetilde{w}) \otimes f_e((\mathbf{c}',0),\widetilde{w}) \rangle & \text{when } e=-1 \end{cases} \\
&= 0
\end{aligned}$$

if $\mathbf{d}' \neq 0$. This follows from Lemma 4.36 and Proposition 4.38. Hence it suffices to compute the following form

$$\left\langle \Phi^{\mathrm{up}}(F_e^{\mathrm{up}}(\mathbf{c},\widetilde{w})|_{q=1}), f_e(\mathbf{c}',\widetilde{w}) \right\rangle.$$

This is equal to $\langle F_e^{\mathrm{up}}(\mathbf{c},\widetilde{w})|_{q=1}, \Phi(f_e(\mathbf{c}',\widetilde{w}))\rangle = \langle F^{\mathrm{up}}(\mathbf{c},\widetilde{w})|_{q=1}, F_e(\mathbf{c}',\widetilde{w})|_{q=1}\rangle = \delta_{\mathbf{c},\mathbf{c}'}$. Hence we have $\Phi^{\mathrm{up}}(F_e^{\mathrm{up}}(\mathbf{c},\widetilde{w})|_{q=1}) = f_e^*((\mathbf{c},0),\widetilde{w})$ and the assertion. q.e.d

5. Quantum closed unipotent cell and the dual canonical basis

5.1. **Demazure-Schubert filtration $\mathbf{U}_w^-$.** We recall the definition of the Demazure-Schubert filtration $\mathbf{U}_w^-$ associated with a Weyl group element $w \in W$.

5.1.1. Let $\mathbf{i} = (i_1, \cdots, i_l)$ be a sequence in I and $\mathbf{U}_{\mathbf{i}}^-$ the $\mathbb{Q}(q)$-linear subspace spanned by the monomials $F_{i_1}^{(a_1)} \cdots F_{i_1}^{(a_l)}$ for all $(a_1, a_2, \cdots, a_l) \in \mathbb{Z}_{\geq 0}^l$, that is

$$\mathbf{U}_{\mathbf{i}}^- := \sum_{a_1, a_2, \cdots, a_l \in \mathbb{Z}_{\geq 0}} \mathbb{Q}(q) F_{i_1}^{(a_1)} \cdots F_{i_1}^{(a_l)}.$$

By its definition, this is a $\mathbb{Q}(q)$-subcoalgebra of $\mathbf{U}_q^-$. We have the following compatibility with the canonical base.

Proposition 5.1 ([42, 4.2])**.** The subcoalgebra $\mathbf{U}_{\mathbf{i}}^-$ is compatible with the canonical basis $\mathbf{B}$, that is there exists a subset $\mathscr{B}_{\mathbf{i}}(\infty)$ of $\mathscr{B}(\infty)$ such that

$$\mathbf{U}_{\mathbf{i}}^- = \bigoplus_{b \in \mathscr{B}_{\mathbf{i}}(\infty)} \mathbb{Q}(q) G^{\mathrm{low}}(b).$$

Remark 5.2. If we consider the $\mathcal{A}$-subspace $(\mathbf{U}_{\mathbf{i}}^-)_{\mathcal{A}}$ spanned by the monomials $F_{i_1}^{(a_1)} \cdots F_{i_l}^{(a_l)}$, then $(\mathbf{U}_{\mathbf{i}}^-)_{\mathcal{A}}$ is a $\mathcal{A}$-subcoalgebra of $\mathbf{U}_q^-$ and we have

$$(\mathbf{U}_{\mathbf{i}}^-)_{\mathcal{A}} = \bigoplus_{b \in \mathscr{B}_{\mathbf{i}}(\infty)} \mathcal{A} G^{\mathrm{low}}(b).$$

Remark 5.3. By the construction of $\mathbf{U}_{\mathbf{i}}^-$, it is clear that

$$*(\mathbf{U}_{\mathbf{i}}^-) = \mathbf{U}_{\mathbf{i}^{\mathrm{opp}}}^-,$$
$$*(\mathscr{B}_{\mathbf{i}}(\infty)) = \mathscr{B}_{\mathbf{i}^{\mathrm{opp}}}(\infty),$$

where $\mathbf{i}^{opp} = (i_l, i_{l-1}, \cdots, i_1)$ for $\mathbf{i} = (i_1, i_2, \cdots, i_l)$.

5.1.2. For $w \in W$, we consider $\mathbf{U}_{\widetilde{w}}^-$ associated with $\widetilde{w} = (i_1, \cdots, i_l) \in R(w)$. Then it is known that $\mathbf{U}_{\widetilde{w}}^-$ does not depend on the choice of the reduced expression $\widetilde{w}$ ([42, 5.3]). Therefore we denote $\mathbf{U}_{\widetilde{w}}^-$ by $\mathbf{U}_w^-$ and also $\mathscr{B}_{\mathbf{i}}(\infty)$ by $\mathscr{B}_w(\infty)$ by abuse of notations. By their constructions, we have

$$*(\mathbf{U}_w^-) = \mathbf{U}_{w^{-1}}^-, \tag{5.4a}$$
$$*(\mathscr{B}_w(\infty)) = \mathscr{B}_{w^{-1}}(\infty). \tag{5.4b}$$

5.1.3. Following [4, 9.3], we define the *quantum closed unipotent cell* $\mathcal{O}_q[\overline{N_w}]$ associated with w by

$$\mathcal{O}_q[\overline{N_w}] := \mathbf{U}_q^-(\mathfrak{g})/(\mathbf{U}_w^-)^{\perp} = \mathbf{U}_q^-(\mathfrak{g})/\bigoplus_{b \notin \mathscr{B}_w(\infty)} \mathbb{Q}(q) G^{\mathrm{up}}(b).$$

Let $\iota_w^* \colon \mathbf{U}_q^-(\mathfrak{g}) \to \mathcal{O}_q[\overline{N_w}]$ be the natural projection. Since $(\mathbf{U}_w^-)^{\perp} = \bigoplus_{b \notin \mathscr{B}_w(\infty)} \mathbb{Q}(q) G^{\mathrm{up}}(b)$ is compatible with $\mathbf{B}^{\mathrm{up}}$, the natural projection induces an bijection $\{G^{\mathrm{up}}(b); b \in \mathscr{B}_w(\infty)\} \simeq \{\iota_w^*(G^{\mathrm{up}}(b)); b \in \mathscr{B}_w(\infty)\}$. Moreover, $(\mathbf{U}_w^-)^{\perp}$ is a two-sided ideal since $\mathbf{U}_w^-$ is a subcoalgebra. Thus $\mathcal{O}_q[\overline{N_w}]$ has an induced algebra structure.

5.2. **Demazure module and its crystal.** In this subsection, we recall the definition of the extremal vector $u_{w\lambda}$ and the associated Demazure module $V_w(\lambda)$. In particular, we remind that $\mathscr{B}_w(\infty)$ can be considered as a certain limit of the Demazure crystal.

5.2.1. For $i \in I$, we consider the subalgebra $\mathbf{U}_q(\mathfrak{g})_i$ generated by e_i, f_i, t_i. Consider the $(l+1)$-dimensional irreducible representation of $\mathbf{U}_q(\mathfrak{g})_i$ with a highest weight vector $u_0^{(l)}$, let $u_k^{(l)} := f_i^{(k)} u_0^{(l)}$ $(1 \leq k \leq l)$. We have

$$S_i(u_k^{(l)}) = (-1)^{l-k} q_i^{(l-k)(k+1)} u_{l-k}^{(l)}. \tag{5.5}$$

In particular, we have

$$S_i(u_l^{(l)}) = u_0^{(l)}, \tag{5.6a}$$

$$S_i(u_0^{(l)}) = (-q_i)^l u_l^{(l)}. \tag{5.6b}$$

5.2.2. For $\lambda \in P$ and $w \in W$, let us denote by $u_{w\lambda}$ the canonical basis element of weight $w\lambda$. We have the following description ([26, 3.2] and [41, Lemma 39.1.2]):

$$\begin{aligned} u_{w\lambda} &= u_\lambda && \text{if } w = 1, \\ u_{s_i w\lambda} &= f_i^{(m)} u_{w\lambda} = S_i^{-1} u_{w\lambda} && \text{if } m = \langle h_i, w\lambda \rangle \geq 0. \end{aligned}$$

Recall that $u_{w\lambda}$ is also the dual canonical basis element. For $\widetilde{w} \in R(w)$, we have

$$u_{w\lambda} = S_{i_1}^{-1} \cdots S_{i_l}^{-1} u_\lambda. \tag{5.7}$$

5.2.3. We recall basic properties of the Demazure module, see [26, §3] or [28, Chapitre 9] for more details. Let $\lambda \in P_+$ and $V(\lambda)$ be the integrable highest weight $\mathbf{U}_q(\mathfrak{g})$-module with a highest weight vector u_λ of weight λ. Let $V_w(\lambda) := \mathbf{U}_q^+(\mathfrak{g}) u_{w\lambda}$. This $\mathbf{U}_q^+(\mathfrak{g})$-module is called the *Demazure module* associated with w and λ. We have the following properties of the Demazure module $V_w(\lambda)$.

Proposition 5.8. Let $w \in W$ and $\widetilde{w} = (i_1, \cdots, i_l) \in R(w)$ be a reduced expression of w.

(1) We have

$$V_w(\lambda) = \sum_{a_1, \cdots, a_l \in \mathbb{Z}_{\geq 0}} \mathbb{Q}(q) F_{i_1}^{(a_1)} \cdots F_{i_l}^{(a_l)} u_\lambda.$$

(2) We define $\mathscr{B}_w(\lambda) \subset \mathscr{B}(\lambda)$ by

$$\mathscr{B}_w(\lambda) := \{\widetilde{f}_{i_1}^{a_1} \cdots \widetilde{f}_{i_r}^{a_l} u_\lambda \in \mathscr{B}(\lambda); (a_1, \cdots, a_l) \in \mathbb{Z}_{\geq 0}^l \setminus \{0\}\} \tag{5.9a}$$

$$= \{b \in \mathscr{B}(\lambda); \widetilde{e}_{i_l}^{\max} \cdots \widetilde{e}_{i_1}^{\max} b = u_\lambda\}. \tag{5.9b}$$

Then we have

$$V_w(\lambda) = \bigoplus_{b \in \mathscr{B}_w(\lambda)} \mathbb{Q}(q) G_\lambda^{\mathrm{low}}(b).$$

(3) For $i \in I$, we have

$$\widetilde{e}_i \mathscr{B}_w(\lambda) \subset \mathscr{B}_w(\lambda) \sqcup \{0\}.$$

We call $\mathscr{B}_w(\lambda)$ the *Demazure crystal*.

5.2.4. We have a similar description of $\mathscr{B}_w(\infty)$ as $\mathscr{B}_w(\lambda)$. Thus $\mathscr{B}_w(\infty)$ can be interpreted as certain limit of the Demazure crystal.

Proposition 5.10 ([26, Corollary 3.2.2])**.** Let $w \in W$ and $(i_1, \cdots, i_l) \in R(w)$ be its reduced expression.

(1) We have

$$\mathscr{B}_w(\infty) = \{\widetilde{f}_{i_1}^{a_1} \cdots \widetilde{f}_{i_r}^{a_l} u_\infty \in \mathscr{B}(\infty); (a_1, \cdots, a_l) \in \mathbb{Z}_{\geq 0}^l \setminus \{0\}\} \tag{5.11a}$$

$$= \{b \in \mathscr{B}(\infty); \widetilde{e}_{i_l}^{\max} \cdots \widetilde{e}_{i_1}^{\max} b = u_\infty\}. \tag{5.11b}$$

(2) For $i \in I$, we have

$$\widetilde{e}_i \mathscr{B}_w(\infty) \subset \mathscr{B}_w(\infty) \sqcup \{0\}. \tag{5.12}$$

5.3. To study multiplicative properties of $\mathbf{U}_q^-(w, e)$, we relate it to the quantum closed unipotent cell $\mathcal{O}_q[\overline{N_w}]$. The following is a generalization of [9, 3.2 Lemma]. This can be considered as a quantum analogue of [23, Corollary 15.7].

Theorem 5.13. For $w \in W$ and $e \in \{\pm 1\}$, we have the following embedding of algebras:

$$\mathbf{U}_q^-(w, e) \hookrightarrow \mathcal{O}_q[\overline{N_{w^{-e}}}].$$

Proof. We consider the composite of the inclusion $\mathbf{U}_q^-(w, e) \hookrightarrow \mathbf{U}_q^-(\mathfrak{g})$ and the natural projection $\iota_{w^{-e}}^* \colon \mathbf{U}_q^-(\mathfrak{g}) \to \mathcal{O}_q[\overline{N_{w^{-e}}}]$. Since both homomorphisms are algebra homomorphisms, we obtain an algebra homomorphism

$$\mathbf{U}_q^-(w, e) \to \mathcal{O}_q[\overline{N_{w^{-e}}}].$$

Since $\mathbf{U}_q^-(w, e)$ is compatible with $\mathbf{B}^{\mathrm{up}}$ and $\iota_{w^{-e}}^*$ induces an bijection $\{G^{\mathrm{up}}(b); b \in \mathscr{B}_{w^{-e}}(\infty)\} \simeq \{\iota_{w^{-e}}^*(G^{\mathrm{up}}(b)); b \in \mathscr{B}_{w^{-e}}(\infty)\}$, it suffices to prove the corresponding assertion for the crystals, that is $\mathscr{B}(w, e) \hookrightarrow \mathscr{B}_{w^{-e}}(\infty)$. Since we have $*(\mathscr{B}(w, e)) = \mathscr{B}(w, -e)$ and $*(\mathscr{B}_w(\infty)) = \mathscr{B}_{w^{-1}}(\infty)$, it is enough to prove the claim for the $e = 1$ case.

We prove $\mathscr{B}(w, 1) \subset \mathscr{B}_{w^{-1}}(\infty)$ by the induction on $l = \ell(w)$. For $l = 1$ case, by the constructions of $\mathscr{B}(s_i, e)$ and $\mathscr{B}_{s_i}(\infty)$, we have $\mathscr{B}(s_i, e) = \mathscr{B}_{s_i}(\infty)$ for any $i \in I$ and $e \in \{\pm 1\}$. Let $\widetilde{w} = (i_1, \cdots, i_l) \in R(w)$ be a reduced expression. For $l \geq 2$, we can assume that, for $w_{\geq 2} := s_{i_2} \cdots s_{i_l} \in W$, $\widetilde{w}_{\geq 2} = (i_2, i_3, \cdots, i_l) \in R(w_{\geq 2})$ and $\mathbf{c}_{\geq 2} = (c_2, c_3, \cdots, c_l) \in \mathbb{Z}_{\geq 0}^{l-1}$, we have

$$b_{\geq 2} := F(\mathbf{c}_{\geq 2}, \widetilde{w}_{\geq 2}) \bmod q\mathscr{L}(\infty) \in \mathscr{B}_{w_{\geq 2}^{-1}}(\infty)$$

by the induction hypothesis. Note that $\widetilde{e}_{i_1}^{\max} b_{\geq 2} \in \mathscr{B}_{w_{\geq 2}^{-1}}(\infty)$ by Lemma 5.10 (2). Since $*(\mathscr{B}_w(\infty)) = \mathscr{B}_{w^{-1}}(\infty)$, we have $\widetilde{f}_{i_1}^{*\varphi_{i_1}(b_{\geq 2})} \widetilde{e}_{i_1}^{\max} b_{\geq 2} \in \mathscr{B}_{w^{-1}}(\infty)$. In view of [26, Theorem 3.3.2], it suffices to prove $\widetilde{f}_{i_1}(\widetilde{f}_{i_1}^{*\varphi_{i_1}(b_{\geq 2})} \widetilde{e}_{i_1}^{\max} b_{\geq 2}) \in \mathscr{B}_{w^{-1}}(\infty)$. We consider the image of it under the Kashiwara embedding $\Psi_{i_1} \colon \mathscr{B}(\infty) \to \mathscr{B}(\infty) \otimes \mathscr{B}_{i_1}$ and show that the image of it is contained in $\mathscr{B}_{w_{\geq 2}^{-1}}(\infty) \otimes \mathscr{B}_{i_1}$. Since Ψ_{i_1} is a strict embedding, we have

$$\begin{aligned}\Psi_{i_1}(\widetilde{f}_{i_1}(\widetilde{f}_{i_1}^{*\varphi_{i_1}(b_{\geq 2})} \widetilde{e}_{i_1}^{\max} b_{\geq 2})) &= \widetilde{f}_{i_1}(\Psi_{i_1}(\widetilde{f}_{i_1}^{*\varphi_{i_1}(b_{\geq 2})} \widetilde{e}_{i_1}^{\max} b_{\geq 2})) \\ &= \widetilde{f}_{i_1}(\widetilde{e}_{i_1}^{\max} b_{\geq 2} \otimes \widetilde{f}_{i_1}^{\varphi_{i_1}(b_{\geq 2})} b_{i_1}).\end{aligned}$$

If $\varphi_{i_1}(\widetilde{e}_{i_1}^{\max} b_{\geq 2}) \leq \varepsilon_{i_1}(\widetilde{f}_{i_1}^{\varphi_{i_1}(b_{\geq 2})})$, we have $\widetilde{f}_{i_1}(\widetilde{e}_{i_1}^{\max} b_{\geq 2} \otimes \widetilde{f}_{i_1}^{\varphi_{i_1}(b_{\geq 2})} b_{i_1}) = \widetilde{e}_{i_1}^{\max} b_{\geq 2} \otimes \widetilde{f}_{i_1}^{\varphi_{i_1}+1(b_{\geq 2})} b_{i_1}$. This is contained in $\mathscr{B}_{w_{\geq 2}^{-1}}(\infty) \otimes \mathscr{B}_{i_1}$.

Suppose $\varphi_{i_1}(\widetilde{e}_{i_1}^{\max} b_{\geq 2}) > \varepsilon_{i_1}(\widetilde{f}_{i_1}^{\varphi_{i_1}(b_{\geq 2})})$. This means that $\varepsilon_{i_1}(b_{\geq 2}) > 0$. Let S be the i_1-string which contains $b_{\geq 2}$ and $\widetilde{e}_{i_1}^{\max}(b_{\geq 2})$. Note that $b_{\geq 2} \neq \widetilde{e}_{i_1}^{\max}(b_{\geq 2})$. Since both $b \geq 2$ and $\widetilde{e}_{i_1}^{\max} b_{\geq 2}$ are in $\mathscr{B}_{w_{\geq 2}^{-1}}(\infty)$, we have $S \cap \mathscr{B}_{w_{\geq 2}^{-1}}(\infty) = S$ by [26, Proposition 3.3.4]. Hence we get the assertion. q.e.d

6. Construction of initial seed: Quantum flag minors

In this section, we give a construction of the quantum initial seed in Conjecture 1.1 which corresponds to the initial seed in [23]. We only consider the $e = -1$ case, but the other case follows by applying the $*$-involution.

6.1. Quantum generalized minors.

6.1.1. We first define a quantum generalized minor. This is a q-analogue of a (restricted) generalized minor $D_{w\lambda} = D_{w\lambda,\lambda}$ which is defined in [23, 7.1].

Definition 6.1 (quantum generalized minor)**.** For $\lambda \in P_+$ and $w \in W$, let

$$\Delta_{w\lambda} = \Delta_{w\lambda,\lambda} := j_\lambda(u_{w\lambda}).$$

We call it a *quantum generalized minor*. When λ is a fundamental weight, we call it a *quantum flag minor*.

By its definition, it is given by a matrix coefficient as

$$(\Delta_{w\lambda,\lambda}, P) = (u_{w\lambda}, P u_\lambda).$$

6.1.2. The following result for extremal vectors is well-known.

Lemma 6.2 ([48, Lemma 8.6])**.** For $\lambda, \mu \in P_+$ and $w \in W$, we have

$$\Phi(\lambda, \mu)(u_{w(\lambda+\mu)}) = u_{w\lambda} \otimes u_{w\mu}.$$

It follows that

$$q_{\lambda,\mu}(u_{w\lambda} \otimes u_{w\mu}) = u_{w\lambda + w\mu}.$$

Therefore we get

$$q^{(w\mu-\mu,\lambda)} \Delta_{w\lambda} \Delta_{w\mu} = \Delta_{w(\lambda+\mu)}$$

by Proposition 3.31. In particular, $\Delta_{w,\lambda}$ is strongly real for any $w \in W$ and $\lambda \in P_+$.

6.1.3. We describe extremal vectors in terms of the PBW basis. This is a straight forward generalization of [7]. For $1 \leq k \leq l$, we define the following operations as in [23, 9.8],

$$k^- := \max(0, \{1 \leq s \leq k-1; i_s = i_k\}),$$
$$k_{\max} := \max\{1 \leq s \leq l; i_s = i_k\}.$$

Proposition 6.3. For $0 \leq k \leq l$, we define $\mathbf{n}_k$ by

$$\mathbf{n}_k(j) := \begin{cases} 1 & \text{if } i_j = i_k, j \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

If $i = i_k$ (here we understand $i = i_k$ holds for any i if $k = 0$), we have $F_{e=-1}(m\mathbf{n}_k; \widetilde{w}) u_{m\varpi_i} = u_{s_{i_1} \cdots s_{i_k} m\varpi_i}$ for $m \geq 1$.

Proof. We follow the argument in [7, 2.1 Lemma]. We prove the assertion by an induction on k. The assertion is trivial when $k=0$. Note that

$$F_{-1}(m\mathbf{n}_k,\widetilde{w})=T_{i_1}^{-1}\cdots T_{i_{k-1}}^{-1}(F_{i_k}^{(m)})F_{-1}(m\mathbf{n}_{k^-},\widetilde{w}).$$

Therefore we have

$$F_{-1}(m\mathbf{n}_k,\widetilde{w})u_{m\varpi}=T_{i_1}^{-1}\cdots T_{i_{k-1}}^{-1}(F_{i_k}^{(m)})u_{s_{i_1}\cdots s_{i_{k^-}}m\varpi_i}$$

by the induction hypothesis. By (4.6), this is equal to

$$\begin{aligned}&S_{i_1}^{-1}\cdots S_{i_{k-1}}^{-1}(F_{i_k}^{(m)})S_{i_{k-1}}\cdots S_{i_1}S_{i_1}^{-1}\cdots S_{i_{k^-}}^{-1}u_{m\varpi_i}\\ =&S_{i_1}^{-1}\cdots S_{i_{k-1}}^{-1}(F_{i_k}^{(m)})S_{i_{k-1}}\cdots S_{i_{k^-+1}}u_{m\varpi_i}.\end{aligned}$$

Since none of $i_{k^-+1},\cdots,i_{k-1}$ is i, this is equal to

$$S_{i_1}^{-1}\cdots S_{i_{k-1}}^{-1}(F_{i_k}^{(m)})u_{m\varpi_i}.$$

By (5.7), this is nothing but $S_{i_1}^{-1}\cdots S_{i_k}^{-1}u_{m\varpi_i}$. Therefore the assertion also holds for k. q.e.d

By the above proposition, we have $\pi_{m\varpi_{i_k}}(b_{-1}(m\mathbf{n}_k;\widetilde{w}))\neq 0$ for any $1\leq k\leq l$ and $m\geq 1$. Hence $j_{m\varpi_{i_k}}(u_{s_{i_1}\cdots s_{i_k}m\varpi_{i_k}})=G^{\mathrm{up}}(b_{-1}(m\mathbf{n}_k,\widetilde{w}))$ for any $1\leq k\leq l$. As a special case, we obtain the following result.

Corollary 6.4. For $w\in W$ and fix $\widetilde{w}\in R(w)$. For $i\in I$, we set $\mathbf{n}^i$ by $\mathbf{n}_{k_{\max}}$ with $i_k=i$. For $\lambda\in P_+$, we set $\mathbf{n}^\lambda:=\sum_{i\in I}\lambda_i\mathbf{n}^i\in\mathbb{Z}_{\geq 0}^l$. Then we have

$$\Delta_{w\lambda}=G^{\mathrm{up}}(b_{-1}(\mathbf{n}^\lambda,\widetilde{w})). \tag{6.5}$$

Proof. By Proposition 6.3, we have

$$\Delta_{wm\varpi_i}=G^{\mathrm{up}}(b_{-1}(m\mathbf{n}^i,\widetilde{w})) \tag{6.6}$$

for any $i\in I$. Then by (6.6), Lemma 6.2 and Corollary 4.29, we obtain the assertion. q.e.d

6.2. **Commutativity relations.** In this subsection, we prove that quantum generalized minors $\{\Delta_{w\lambda}\}$ q-commute with $G^{\mathrm{up}}(b)$ for $b\in\mathscr{B}_w(\infty)$ in the quotient $\mathcal{O}_q[\overline{N_w}]$. It means that $\Delta_{w\lambda}$ and $G^{\mathrm{up}}(b)$ q-commute up to $(\mathbf{U}_w^-)^\perp$. By Theorem 5.13, they literally q-commute when $b\in\mathscr{B}_w(w,-1)$. We denote the projection of $G^{\mathrm{up}}(b)$ to $\mathcal{O}_q[\overline{N_w}]$ also by $G^{\mathrm{up}}(b)$ for brevity.

6.2.1. For the proof of certain q-commutativity relation, we need to use the quasi $\mathcal{R}$-matrix. We recall its properties.

First we consider another coproduct $\overline{\Delta}$ defined by $(^-\otimes^-)\circ\Delta\circ^-$. We have an analogue of Lemma 2.5:

$$\overline{\Delta}(q^h)=q^h\otimes q^h, \tag{6.7a}$$

$$\overline{\Delta}(e_i)=e_i\otimes t_i+1\otimes e_i, \tag{6.7b}$$

$$\overline{\Delta}(f_i)=f_i\otimes 1+t_i^{-1}f_i. \tag{6.7c}$$

We consider the following completion

$$\mathbf{U}_q^+(\mathfrak{g})\widehat{\otimes}\mathbf{U}_q^-(\mathfrak{g})=\bigoplus_{\xi\in Q}\prod_{\xi=\xi'+\xi''}\mathbf{U}_q^+(\mathfrak{g})_{\xi'}\otimes\mathbf{U}_q^-(\mathfrak{g})_{\xi''}.$$

Note that the counit ε extends to the completion. In [41, Chapter 4], Lusztig has shown that there exists a unique intertwiner $\Xi \in \mathbf{U}_q^+(\mathfrak{g})\widehat{\otimes}\mathbf{U}_q^-(\mathfrak{g})$ such that

$$\Xi \circ \Delta(x) = \overline{\Delta}(x) \circ \Xi \text{ for any } x \in \mathbf{U}_q(\mathfrak{g}), \tag{6.8}$$

$\varepsilon(\Xi) = 1$, and $\Xi \circ \overline{\Xi} = \overline{\Xi} \circ \Xi = 1$. We have an analogue of Lemma 2.5:

$$\overline{\Delta}(x) = \sum q^{-(\mathrm{wt}\, x_{(1)}, \mathrm{wt}\, x_{(2)})} x_{(2)} t_{\mathrm{wt}\, x_{(1)}} \otimes x_{(1)}, \tag{6.9}$$

for any $x \in \mathbf{U}_q^-(\mathfrak{g})$ with $r(x) = \sum x_{(1)} \otimes x_{(2)}$. In particular, we have

$$\overline{\Delta}(x)(u_\lambda \otimes u_\mu) = \sum q^{-(\mathrm{wt}\, x_{(1)}, \mathrm{wt}\, x_{(2)})} x_{(2)} t_{\mathrm{wt}\, x_{(1)}} u_\lambda \otimes x_{(1)} u_\mu, \tag{6.10}$$

for such $x \in \mathbf{U}_q^-(\mathfrak{g})$.

6.2.2.

Proposition 6.11. For $b \in \mathscr{B}_w(\mu)$ and $u_{w\lambda} \in \mathscr{B}_w(\lambda)$, we have the following q-commutation relation in $\mathcal{O}_q[\overline{N_w}]$:

$$j_\lambda(u_{w\lambda}) G^{\mathrm{up}}(j_\mu(b)) \simeq G^{\mathrm{up}}(j_\mu(b)) j_\lambda(u_{w\lambda}).$$

Proof. Since we only consider the equality in quantum closed unipotent cell $\mathcal{O}_q[\overline{N}_w]$, it is enough to check that inner products with $x \in \mathbf{U}_w^-$ are the same up to some q-shifts, and the q-shifts do not depend on choice of x. By Proposition 3.31, this is equal to

$$(u_{w\lambda} \otimes G_\mu^{\mathrm{up}}(b), \Delta(x)(u_\lambda \otimes u_\mu))_{\lambda,\mu},$$

where $(\cdot,\cdot)_{\lambda,\mu}$ denotes the inner product on $V(\lambda) \otimes V(\mu)$ defined by $(u \otimes u', v \otimes v')_{\lambda,\mu} := (u, v)_\lambda (u', v')_\mu$. We use the quasi $\mathcal{R}$-matrix to rewrite this as

$$(u_{w\lambda} \otimes G_\mu^{\mathrm{up}}(b), (\overline{\Xi} \circ \overline{\Delta}(x) \circ \Xi)(u_\lambda \otimes u_\mu))_{\lambda,\mu}.$$

Since the action of the quasi $\mathcal{R}$-matrix is trivial on the highest weight vector, this is equal to

$$(u_{w\lambda} \otimes G_\mu^{\mathrm{up}}(b), (\overline{\Xi} \circ \overline{\Delta}(x))(u_\lambda \otimes u_\mu))_{\lambda,\mu}.$$

Since the inner product has an adjoint property for φ, this is equal to

$$((\varphi \otimes \varphi)(\overline{\Xi})(u_{w\lambda} \otimes G_\mu^{\mathrm{up}}(b)), (\overline{\Delta}(x))(u_\lambda \otimes u_\mu))_{\lambda,\mu}.$$

Note that $(\overline{\Delta}(x))(u_\lambda \otimes u_\mu)$ is contained in the tensor product of Demazure modules $V_w(\lambda) \otimes V_w(\mu)$ by §5.2.3. By the form of quasi $\mathcal{R}$-matrix (6.8) and the definition of φ, the nontrivial part of $(\varphi \otimes \varphi)(\overline{\Xi})(u_{w\lambda} \otimes G_\mu^{\mathrm{up}}(b))$ is not contained in the tensor product $V_w(\lambda) \otimes V_w(\mu)$, therefore the above is equal to

$$(u_{w\lambda} \otimes G_\mu^{\mathrm{up}}(b), (\overline{\Delta}(x))(u_\lambda \otimes u_\mu))_{\lambda,\mu}.$$

By (6.9), this is equal to

$$(u_{w\lambda} \otimes G_\mu^{\mathrm{up}}(b), (\mathrm{flip} \circ \Delta(x))(u_\lambda \otimes u_\mu))_{\lambda,\mu},$$

and also to

$$(j_\lambda(u_{w\lambda}) \otimes G^{\mathrm{up}}(j_\mu b), (\mathrm{flip} \circ r(x))_K$$

up to some q-shifts, where $\mathrm{flip}(P \otimes Q) := Q \otimes P$. Therefore we get

$$(j_\lambda(u_{w\lambda}) G^{\mathrm{up}}(j_\mu(b)), x)_K \simeq (G^{\mathrm{up}}(j_\mu(b)) j_\lambda(u_{w\lambda}), x)_K$$

for any $x \in \mathbf{U}_w^-$. Here we note that q-shifts depend only on the weights of $u_{w\lambda}$ and $j_\mu(b)$, and is independent of x. Then we obtain the assertion. q.e.d

Restricting the above equality, we obtain the following q-commutativity relations in $\mathcal{O}_q[N(w)]$.

Corollary 6.12. For $\mathbf{c} \in \mathbf{Z}^l_{\geq 0}$, we have

$$G^{\mathrm{up}}(b_{-1}(\mathbf{c}, \widetilde{w}))\Delta_{w\lambda} \simeq \Delta_{w\lambda} G^{\mathrm{up}}(b_{-1}(\mathbf{c}, \widetilde{w})).$$

6.3. **Factorization of q-center.** In this subsection, we prove the multiplicative property of $\mathscr{B}_w(\infty)$ with respect to the quantum minors $\{\Delta_{w\lambda}\}_{\lambda \in P_+}$ in $\mathcal{O}_q[\overline{N_w}]$. This is a generalization of [8, 3.1], [9] and [32, Lemma 4.2]. This result can be considered as a q-analogue of [23, Lemma 15.8].

6.3.1. Using Corollary 3.17 and (3.18) inductively, we obtain the following lemma.

Lemma 6.13. Let $w \in W$ and $\widetilde{w} = (i_1, \cdots, i_l) \in R(w)$ as above. We define

$$\varepsilon_{\widetilde{w}}(b) := (\varepsilon_{i_1}(b), \varepsilon_{i_2}(\widetilde{e}_i^{\max} b), \cdots, \varepsilon_{i_l}(\widetilde{e}_{i_{l-1}}^{\max} \cdots \widetilde{e}_{i_1}^{\max} b))$$

for $b \in \mathscr{B}(\infty)$. For $b_1, b_2 \in \mathscr{B}(\infty)$, let us write

$$G^{\mathrm{up}}(b_1)G^{\mathrm{up}}(b_2) = \sum d^b_{b_1,b_2}(q) G^{\mathrm{up}}(b)$$

with $d^b_{b_1,b_2}(q) \in \mathcal{A}$. If $d^b_{b_1,b_2}(q) \neq 0$, then $\varepsilon_{\widetilde{w}}(b) \leq \varepsilon_{\widetilde{w}}(b_1) + \varepsilon_{\widetilde{w}}(b_2)$, where $\leq$ is the lexicographic order on $\mathbb{Z}^l_{\geq 0}$ as in §4.3.5.

Let $b \in \mathscr{B}_w(\infty)$ with $\varepsilon_{\widetilde{w}}(b) = \varepsilon_{\widetilde{w}}(b_1) + \varepsilon_{\widetilde{w}}(b_2)$ for $b_1, b_2 \in \mathscr{B}_w(\infty)$. Then we have $d^b_{b_1,b_2}(q) = q^N$ for some $N \in \mathbb{Z}$.

6.3.2.

Proposition 6.14. Let $w \in W$ and $\lambda, \mu \in P_+$. For $b \in \mathscr{B}_w(\mu)$ and $u_{w\lambda} \in \mathscr{B}_w(\lambda)$, there exists $b' \in \mathscr{B}_w(\lambda + \mu)$ such that

$$\Phi(\lambda, \mu)(b') = u_{w\lambda} \otimes b,$$

and we have an equality in $\mathcal{O}_q[\overline{N}_w]$:

$$\Delta_{w\lambda} G^{\mathrm{up}}(j_\mu(b)) \simeq G^{\mathrm{up}}(j_{\lambda+\mu}(b')).$$

Proof. Fix $\widetilde{w} = (i_1, \cdots, i_l) \in R(w)$, we have

$$\widetilde{e}_{i_1}^{\max}(u_{w\lambda} \otimes b) = u_{w_{\geq 2}\lambda} \otimes \widetilde{e}_{i_1}^{\max} b$$

by the tensor product rule (2.34a) for crystal operators and $\varphi_{i_1}(u_{w\lambda}) = 0$. Using this recursively, we get

$$\widetilde{e}_{i_l}^{\max} \cdots \widetilde{e}_{i_1}^{\max}(u_{w\lambda} \otimes b) = u_\lambda \otimes u_\mu.$$

In particular, there exists $b' \in \mathscr{B}_w(\lambda+\mu)$ such that $\Phi(\lambda, \mu)(b') = u_{w\lambda} \otimes b$. By Proposition 3.31 and Proposition 3.29, we have

$$q^{(\mathrm{wt}\, b - \mu, \lambda)} \Delta_{w\lambda} G^{\mathrm{up}}(j_\mu b) = G^{\mathrm{up}}(j_{\lambda+\mu}(b')) + \sum f^{b''}_{b,w\lambda}(q) G^{\mathrm{up}}(b'') \tag{6.15}$$

for some $f^{b''}_{b,w\lambda}(q) \in q\mathbb{Z}[q]$. By the second assertion of Lemma 6.13, we have $f^{j_{\lambda+\mu}(b')}_{b,w\lambda}(q) = 0$ as in [8, 1.8 Proposition (i)].

Applying the dual bar-involution σ, we obtain

$$q^{-(\mathrm{wt}\, b - \mu, \lambda) + (\mathrm{wt}\, b - \mu, w\lambda - \lambda)} G^{\mathrm{up}}(j_\mu b) \Delta_{w\lambda} = G^{\mathrm{up}}(j_{\lambda+\mu}(b')) + \sum f^{b''}_{b,w\lambda}(q^{-1}) G^{\mathrm{up}}(b''). \tag{6.16}$$

By Proposition 6.11, we have $G^{\mathrm{up}}(j_\mu b)\Delta_{w\lambda} = q^m \Delta_{w\lambda} G^{\mathrm{up}}(j_\mu b)$ for some $m \in \mathbb{Z}$ in $\mathcal{O}_q[\overline{N_w}]$ It is equal to

$$(6.17)\quad q^{-(\mathrm{wt}\, b-\mu,\lambda)+(\mathrm{wt}\, b-\mu,w\lambda-\lambda)+m}\Delta_{w\lambda}G^{\mathrm{up}}(j_\mu b) = G^{\mathrm{up}}(j_{\lambda+\mu}(b')) + \sum f^{b''}_{b,w\lambda}(q^{-1})G^{\mathrm{up}}(b'').$$

Therefore we obtain $f^{b''}_{b,w\lambda}(q) = 0$ for any b'' by comparing (6.15) and (6.17). q.e.d

Since there exists $\mu \in P_+$ such that $\pi_\mu(b) \neq 0$, we obtain the following theorem.

Theorem 6.18. Let $b \in \mathscr{B}_w(\infty)$ and $\lambda \in P_+$. There exists $b' \in \mathscr{B}_w(\infty)$ such that

$$\Delta_{w\lambda}G^{\mathrm{up}}(b) \simeq G^{\mathrm{up}}(b')$$

in $\mathcal{O}_q[\overline{N_w}]$.

Taking b from $\mathscr{B}(w,-1)$, we obtain the following theorem by Corollary 4.29.

Theorem 6.19. For $\mathbf{c} \in \mathbf{Z}^l_{\geq 0}$ and $\lambda \in P_+$, we have

$$\Delta_{w\lambda}G^{\mathrm{up}}(b_{-1}(\mathbf{c},\widetilde{w})) \simeq G^{\mathrm{up}}(b_{-1}(\mathbf{c}+\mathbf{n}^\lambda,\widetilde{w})).$$

6.3.3. The following is a generalization of Caldero's result [8, 2.1 Lemma, 2.2 Theorem]. It follows from Theorem 6.18 by an induction on the length of w.

Theorem 6.20. Let $w \in W$ and fix $\widetilde{w} \in R(w)$. We set

$$\Delta_{\widetilde{w},k} := \Delta_{s_{i_1}\cdots s_{i_k}\varpi_{i_k}}$$

for $1 \leq k \leq l$. Then $\{\Delta_{\widetilde{w},k}\}_{1\leq k\leq l}$ forms a strongly compatible subset.

6.3.4. Following [23, 15.5], we call $\mathbf{c} \in \mathbb{Z}^l_{\geq 0}$ *interval-free* if $\mathbf{c}$ satisfies the following conditions:

$$c^{(i)} := \min\{c_k; i_k = i\} = 0$$

for any $i \in I$. By definition, ${}^{\varphi}\mathbf{c} := \mathbf{c} - \sum_{i\in I} c^{(i)}\mathbf{n}^i \in \mathbb{Z}^l_{\geq 0}$ is interval free. We have the following factorization property with respect to the extremal vectors $\{\Delta_{w\lambda}\}_{\lambda\in P_+}$.

Theorem 6.21. For $\mathbf{c} \in \mathbb{Z}^l_{\geq 0}$, we set $\lambda(\mathbf{c}) := \sum_{i\in I} c^{(i)}\varpi_i \in P_+$. Then we have

$$G^{\mathrm{up}}(b_{-1}(\mathbf{c},\widetilde{w})) \simeq G^{\mathrm{up}}(b_{-1}({}^{\varphi}\mathbf{c},\widetilde{w}))\Delta_{w\lambda(\mathbf{c})}.$$

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