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1 Abstract

We present constructive a posteriori estimates of inverse operators for initial value problems in linear ordinary differential equations on a bounded interval. Here, “constructive” indicates that we can obtain the bounds of the operator norm in which all constants are explicitly given or are represented in a numerically computable form. In general, it is difficult for us to obtain this estimates of this inverse operators in the meaning of a priori. We propose the technique for obtaining the a posteriori estimates by using Galerkin approximation of inverse operators. This type of estimates will play an important role in the numerical verification of solutions for initial value problems in nonlinear ODEs.

2 Introduction

In the present paper, we consider the positive constant $C_{L^2,L^p}$ in a posteriori estimates of the form

$$\|\left(\frac{d}{dt} + B\right)^{-1}\|_{L^2(J^n \to L^p(J^n))} \leq C_{L^2,L^p},$$

where $J := (0,T) \subset \mathbb{R}$, $(T < \infty)$ is a bounded interval, $n$ is a positive integer, $A$ is a symmetric positive definite matrix in $\mathbb{R}^{n \times n}$, $B$ is an element of $L^\infty(J)^{n \times n}$ and $p$ is an arbitrary constants which satisfying $2 \leq p \leq \infty$. For arbitrary $f \in L^2(J)^n$, we consider the following initial value problems in linear ODEs,

$$\begin{cases}
Au' + Bu = f, & \text{in } J, \\
u(0) = 0.
\end{cases}$$

The problem of obtaining the estimates of (1) and the problem of obtaining the estimates for the solution $u$ of the equations (2a) and (2b) become equivalent.
In the case of \( n = 1 \), the solution of (2a) and (2b) is explicitly written by
\[
 u(t) = \frac{1}{A} e^{-\frac{1}{A} \int_0^t B(s) \, ds} \int_0^t \left( e^{\frac{1}{A} \int_0^r B(r) \, dr} \right) f(s) \, ds.
\]
(3)

From Schwarz inequality, we have
\[
 |u(t)| \leq \frac{1}{A} e^{-\frac{1}{A} \int_0^t B(s) \, ds} \int_0^t \left( e^{\frac{1}{A} \int_0^r B(r) \, dr} \right) |f(s)| \, ds
\]
\[
 \leq \frac{1}{A} e^{-\frac{1}{A} \int_0^t B(s) \, ds} \left\| \frac{1}{A} \int_0^r B(r) \, dr \right\|_{L^2(0,t)} \left\| f \right\|_{L^2(0,t)}.
\]

Then we have the a priori estimates of \( u \) by
\[
 \left\| u \right\|_{L^p(J)} \leq \frac{1}{A} \left( \int_J e^{-\frac{1}{A} \int_0^r B(s) \, ds} \left\| e^{\frac{1}{A} \int_0^r B(r) \, dr} \left\|_{L^2(0,t)}^p \, dt \right\|_{L^2(J)} \right)^{\frac{1}{p}}.
\]
(4)

Thus, we can obtain constant \( C_{L^2,L^p} \) which satisfies (1) for \( n = 1 \). However, the value of \( C_{L^2,L^p} \) often becomes large in this a priori estimates. We define the Galerkin approximate operator of \( (A \frac{d}{dt} + B)^{-1} \), and we propose the technique for obtaining a posteriori estimates of (1) that is expected smaller than (4). In general case of \( n \), the solution \( u \) cannot be written in explicitly like (3). Therefore, it is not easy to obtain the a priori estimates of \( \left\| u \right\|_{L^p} \). On the other hand, our proposed methods can similarly obtain the estimates of \( \left\| u \right\|_{L^p} \) for general integer \( n \).

In section 3, we introduce some function spaces and finite element space. And, we calculate constructive a priori error estimates of finite element approximation. In section 4, we propose a posteriori estimates of (1). In section 5, we show a posteriori error estimates for the exact solution of (2a) and (2b) and its finite element solution. Here, the meaning of a posteriori error estimates is operator norm one for integral operators. Namely, this error estimates can be calculated for the given finite element space, this is independent of \( f \). In section 6, we show several numerical results.

3 Finite element space

In this section, we introduce the function spaces, projections to finite dimensional subspaces and these error estimates. Let \( 0 < T < \infty \) be a finite open subset of \( \mathbb{R} \) which defined by \( J := (0, T) \). \( J \) is divided in \( m_e \). Let \( t_i \in J \) be the nodal points satisfying \( 0 = t_0 < t_1 < \cdots < t_{m_e} = T \). Let \( J_i := (t_{i-1}, t_i) \) be each elements. We define the element size by \( |J_i| := t_i - t_{i-1} \). And, we denote the mesh size by \( k := \max_{1 \leq i \leq m_e} |J_i| \).

3.1 Constructive a priori error estimates for scalar functions

Let \( \mathcal{S}(J_i, N_i) \) be a finite dimensional subspace of \( H^1_0(J_i) \) depend on the parameter \( N_i \). For example, \( N_i \) is the polynomial degree when we employ the finite element method.
We define the $H_0^1$-projection on $J_i$, $\tilde{P}_{h_i} : H_0^1(J_i) \to \tilde{S}(J_i, N_i)$ by

$$(u - \tilde{P}_{h_i} u, v_h)_{H_0^1(J_i)} = 0, \quad \forall v_h \in \tilde{S}(J_i, N_i),$$

where $(\cdot, \cdot)_{H_0^1(J_i)}$ is an inner product of Hilbert space $H_0^1(J_i)$ which defined by $(u, v)_{H_0^1(J_i)} = (u', v')_{L^2(J_i)}$. In this paper, we assume the following estimates about $\tilde{P}_{h_i}$ holds.

**Assumption 3.1** There exist a positive constant $C([J_i] \cup N_i) > 0$ satisfying,

$$
\begin{align*}
\|u - \tilde{P}_{h_i} u\|_{H_0^1(J_i)} &\leq C([J_i] \cup N_i) \|u''\|_{L^2(J_i)}, \quad \forall u \in H_0^1(J_i) \cap H^2(J_i), \\
\|u - \tilde{P}_{h_i} u\|_{L^2(J_i)} &\leq C([J_i] \cup N_i) \|u - \tilde{P}_{h_i} u\|_{H_0^1(J_i)}, \quad \forall u \in H_0^1(J_i).
\end{align*}
$$

Assumption 3.1 is the most basic error estimates in the finite element method. For example, in the case of linear polynomial approximation of $H_0^1(J_i)$, the value $C([J_i] \cup 1)$ of Assumption 3.1 are known by $C([J_i] \cup 1) = \frac{|J_i|}{h_i}$. In the case of quadratic polynomial approximation, Assumption 3.1 is satisfied by $C([J_i] \cup 2) = \frac{|J_i|}{h_i^2}$. Moreover, these constants are optimal constants(e.g. [3]). In the case of $N_i$ degree polynomial approximation, Assumption 3.1 is satisfied by $C([J_i] \cup N_i) = O([J_i]^{-1})$. However, the optimal constants in these case are not known(e.g. [2]).

Let $V^1(J)$ be a subspace of $H^1(J)$ defined by $V^1(J) := \{u \in H^1(J) : u(0) = 0\}$. Then $V^1(J)$ is a Hilbert space with inner product $(u, v)_{V^1(J)} := (u', v')_{L^2(J)}$. Let $\tilde{S}_k(J)$ be a finite dimensional subspace of $V^1(J)$ depend on the parameter $k$. Let $\tilde{m}$ be a degree of freedom for $\tilde{S}_k(J)$, $\psi_i$ be the basis functions of $\tilde{S}_k(J)$. Namely, $\tilde{S}_k(J) := \text{span}_{1 \leq i \leq \tilde{m}} \{\psi_i\}$.

We denote the $V^1$-projection $\tilde{P}_k^1 : V^1(J) \to \tilde{S}_k(J)$ by

$$(u - \tilde{P}_k^1 u, v_k)_{V^1(J)} = 0, \quad \forall v_k \in \tilde{S}_k(J).$$

We have the following equalities corresponding to $V^1$-projection.

**Lemma 3.2** Let $\tilde{S}_k(J)$ be an Lagrange type finite element subspace of $V^1(J)$. Then, for arbitrary $u \in V^1(J)$ is satisfied

$$u(t_i) = \tilde{P}_k^1 u(t_i), \quad i = 0, 1, \cdots, m_e.$$  

**Proof.** First, it is clearly at $i = 0$ because $u(0) = \tilde{P}_k^1 u(0) = 0$. Next, we choose the test function $v_k(t) = t$ in (7), thus we have

$$0 = \int_0^T (u(s) - \tilde{P}_k^1 u(s))' ds = u(T) - \tilde{P}_k^1 u(T).$$

Therefore, we get $u(T) = \tilde{P}_k^1 u(T)$, $(t_{m_e} = T)$. Finally, for arbitrary $t_i$, $(i = 1, 2, \cdots, m_e - 1)$, we choose the test function as

$$v_k(t) = \begin{cases} 
(1 - t_i) t, & 0 \leq t \leq t_i, \\
t_i (1 - t), & t_i \leq t \leq T.
\end{cases}$$
Thus, from (7), we have

\[
0 = \int_0^T (u(s) - P^1_k u(s))' (1 - t_i) \, ds + \int_{t_i}^T (u(s) - P^1_k u(s))' (-t_i) \, ds
\]

\[
= u(t_i) - P^1_k u(t_i).
\]

Here, we use \( u(T) = P^1_k u(T). \) □

When we use the piecewise linear polynomial as \( \hat{S}_k(J), \) Lemma 3.2 is shown that \( V^1 \)-projection is equal to interpolation operator. Moreover, we use the piecewise high degree polynomials of Lagrange type, it is at least satisfied that the function and its \( V^1 \)-projections of values are equal in the nodal points.

Here, we introduce the following error estimates corresponding to \( V^1 \)-projection.

**Theorem 3.3** Under the Assumption 3.1, and same assumptions in Lemma 3.2, we have the following error estimates,

\[
\| u - P^1_k u \|_{V^1(J)} \leq C_J(k) \| u'' \|_{L^2(J)}, \quad \forall u \in V^1(J) \cap H^2(J),
\]

\[(9)\]

\[
\| u - P^1_k u \|_{L^2(J)} \leq C_J(k) \| u - P^1_k u \|_{V^1(J)}, \quad \forall u \in V^1(J),
\]

\[(10)\]

where \( C_J(k) := \max_{1 \leq i \leq m_v} C(J_i, N_i). \)

**Proof.** — First, we show (9). For arbitrary \( u \in V^1(J) \cap H^2(J) \), we denote the piecewise linear interpolation of \( u \) as \( \Pi_k u \in \hat{S}_k(J) \). We put \( \tilde{u} := u - \Pi_k u \). Let \( \tilde{u}|_{J_i} \) be the restrictions of \( \tilde{u} \) on \( J_i \). Then, \( \tilde{u}|_{J_i} \) is satisfying \( \tilde{u}|_{J_i} \in H^1_0(J_i) \cap H^2(J_i) \). Similarly, we denote \( \hat{S}(J_i, N_i) \) by the restrictions of \( \hat{S}_k(J) \) on \( J_i \). Therefore, we can define \( H^1_0 \)-projection \( \hat{P}_k \tilde{u}|_{J_i} \in \hat{S}(J_i, N_i) \). Let \( v_k \) be

\[
v_k := \begin{cases} 
\hat{P}_k \tilde{u}|_{J_i}, & \text{in } J_i, \\
0, & \text{otherwise},
\end{cases}
\]

then, \( v_k \in \hat{S}_k(J) \). From \( \hat{P}_k u \) is the best approximation for \( V^1 \) norm, the following equation is satisfied

\[
\| u - \hat{P}_k u \|_{V^1(J)} = \inf_{v_k \in \hat{S}_k(J)} \| u - v_k \|_{V^1(J)}.
\]

In particular, we choose \( v_k := \Pi_k u + \sum_{i=1}^{m_v} v_k \), then we have the following estimates by Assumption 3.1,

\[
\| u - \hat{P}_k u \|_{V^1(J)}^2 \leq \sum_{i=1}^{m_v} \| \tilde{u}|_{J_i} - \hat{P}_k \tilde{u}|_{J_i} \|_{V^1(J_i)}^2
\]

\[
\leq \sum_{i=1}^{m_v} C(J_i, N_i) \| \tilde{u}'' \|_{L^2(J_i)}^2
\]

\[
\| u - \hat{P}_k u \|_{V^1(J)} \leq \max_{1 \leq i \leq m_v} C(J_i, N_i) \| u'' \|_{L^2(J_i)},
\]
where we use \((\Pi_k u)^n = 0\) because \(\Pi_k u\) are linear polynomials on each \(J_i\). Next, we show (10). This error estimates is well known so called Aubin-Nitsche trick. For arbitrary \(u \in V^1(J)\), we put \(g := u - \Pi_k^1 u \in V^1(J)\). Then, there exists a unique solution \(w \in V^1(J)\) such that

\[
(w', v)_{L^2(J)} = (g, v)_{L^2(J)}, \quad \forall v \in V^1(J),
\]

by the Lax-Milgram theorem. In particular, for arbitrary test functions \(v \in C^0_0(J) \subset V^1(J)\) in (11), we have

\[
(-w'', v)_{L^2(J)} = (g, v)_{L^2(J)}, \quad \forall v \in C^0_0(J).
\]

From \(C^0_0(J)\) is dense of \(L^2(J)\), we have

\[-w'' = g, \quad \text{in } L^2(J).\]

Therefore, for arbitrary test functions \(v \in V^1(J)\) in (11), we have

\[
(-w'', v)_{L^2(J)} + w'(T)v(T) = (g, v)_{L^2(J)}, \quad \forall v \in V^1(J)
\]

\[
w'(T)v(T) = 0.
\]

From \(v(T)\) is arbitrary, we have \(w'(T) = 0\). Finally, we choose \(v = g\) in (11), we have

\[
\|g\|_{L^2(J)}^2 = \|(u - \Pi_k^1 u)', (w - \Pi_k^1 w)\|_{L^2(J)}^2 \leq C_f(k) \|w''\|_{L^2(J)}^2
\]

where we use (9). □

### 3.2 Constructive a priori error estimates for vector functions

Let \(n\) be a positive integer. Let \(V^1(J)^n := V^1(J) \times \cdots \times V^1(J)\) with inner product \((u, v)_{V^1(J)^n} := \sum_{i=1}^n (u_i, v_i)_{V^1(J)}\). Let \(S_k(J)^n\) be a finite dimensional subspace of \(V^1(J)^n\) defined by \(S_k(J)^n := \tilde{S}_k(J) \times \cdots \times \tilde{S}_k(J)\). Let \(m := mn\) be a degree of freedom for \(S_k(J)^n\). \(\Psi_i\) be the basis functions of \(S_k(J)^n\). Namely, \(S_k(J)^n := \text{span}_{i \leq m} \{\Psi_i\}\). We denote the \(V^1\)-projection \(P_k^1: V^1(J)^n \to S_k(J)^n\) by

\[
(u - P_k^1 u, v_k)_{V^1(J)^n} = 0, \quad \forall v_k \in S_k(J)^n.
\]

From the definition of \(S_k(J)^n\), \(P_k^1\) is satisfied

\[
P_k^1 u = (\bar{P}_k^1 u_1, \cdots, \bar{P}_k^1 u_n)^T,
\]

where \(\bar{P}_k^1\) is the scalar \(V^1\)-projection defined in (7).

Here, we introduce the following error estimates corresponding to \(V^1\)-projection.
Theorem 3.4 Under the same assumptions in Theorem 3.3, we have the following error estimates,

\[
\left\| u - P_k u \right\|_{V^1(J)^n} \leq C(k) \left\| u'' \right\|_{L^2(J)^n}, \quad \forall u \in V^1(J)^n \cap H^2(J)^n, \tag{14}
\]
\[
\left\| u - P_k^1 u \right\|_{L^2(J)^n} \leq C(k) \left\| u - P_k^1 u \right\|_{V^1(J)^n}, \quad \forall u \in V^1(J)^n. \tag{15}
\]

Proof. —— It is clear from (13) and Theorem 3.3. □

4 A posteriori estimates for inverse ODEs operator

In this section, we consider the positive constant $C_{L^2,L^p}$ in (1). Let $A \in \mathbb{R}^{n,n}$ be a symmetric positive definite matrix. Let $\sigma(A) \subset \mathbb{R}$ be the set of the eigenvalues of $A$. Let $A_c \in \mathbb{R}$ be a minimum eigenvalue of $A$, i.e. $A_c := \min \sigma(A)$. Let $B \in L^\infty(J)^{n,n}$. The $L^\infty$ norm of $B$ is defined by $\|B\|_{L^\infty(J)^{n,n}} := \max \sigma(B^TB)$. Let $G$ be a minimum eigenvalue of $A$, i.e. $G \in \mathbb{R}$. Then, the regularity of $G$ and $A_c$ is defined by

\[
\sigma(A) \cdot \sigma(B) \cdot \sigma(C) \cdot \sigma(D) \cdot \sigma(E) \cdot \sigma(F) \cdot \sigma(G) \cdot \sigma(H) = G. \tag{19}
\]

Let $G_\psi$ be a matrix in $\mathbb{R}^{m,m}$ which each elements are defined by

\[
G_{\psi,i,j} := (A\psi'_j, \psi'_i)_{L^2(J)^n} \quad \forall i,j \leq m. \tag{17}
\]

Then, the regularity of $G_\psi$ and the uniquely existence of the solution $u_k$ for (16) become equivalent. Therefore, we assume the regularity of $G_\psi$ in this paper. When you actually apply the a posteriori estimates that we propose, it is necessary to confirm the regularity of $G_\psi$ by validated computations.

We define the symmetric positive definite matrices $D_\psi$ and $L_\psi$ in $\mathbb{R}^{m,m}$ by

\[
D_{\psi,i,j} := (\psi'_j, \psi'_i)_{L^2(J)^n}, \quad 1 \leq i,j \leq m, \tag{18}
\]
\[
L_{\psi,i,j} := (\psi_j, \psi_i)_{L^2(J)^n}, \quad 1 \leq i,j \leq m. \tag{19}
\]

Let $D_{\psi}^{1/2}$ and $L_{\psi}^{1/2}$ be the Cholesky factors of $D_\psi$ and $L_\psi$ respectively, i.e.

\[
D_\psi = D_{\psi}^{1/2} D_{\psi}^{T/2}, \quad \text{and} \quad L_\psi = L_{\psi}^{1/2} L_{\psi}^{T/2}. \tag{20}
\]

We define a positive constant $M_\psi^{01}$ by

\[
M_\psi^{01} := \left\| L_{\psi}^{T/2} G^{-1} D_{\psi}^{1/2} \right\|_2, \tag{20}
\]

where $\| \cdot \|_2$ is the two norm of matrix (i.e. maximum singular value). We have the following a posteriori estimates for inverse operators.
Theorem 4.1  Under the same assumptions in Theorem 3.3. Let \( \kappa_\psi > 0 \) be satisfied
\[
\kappa_\psi := C_f(k) \| B \|_{L^\infty(J)^n} \left( 1 + M_{\psi^1} \| B \|_{L^\infty(J)^n} \right) < A_c.
\] (21)

Then we have following estimates,
\[
\left\| \left( A \frac{d}{dt} + B \right)^{-1} \right\|_{L^\infty(L^2(J)^n)} \leq \frac{A_c M_{\psi^1} + C_f(k) \left( 1 + M_{\psi^1} \| B \|_{L^\infty(J)^n} \right)}{A_c - \kappa_\psi}.
\] (22)

**Proof.** —— For arbitrary \( f \in L^2(J)^n \), we put \( u := (A \frac{d}{dt} + B)^{-1} f \in V^1(J)^n \). Then \( u \) is satisfied following integral equation,
\[
Au = \int_0^t (-B(s)u(s) + f(s)) \, ds.
\] (23)

In shortly, we denote the integral operator of (23) as \( \partial_t^{-1} : L^2(J)^n \to V^1(J)^n \). We separate (23) by \( V^1 \)-projection as finite and infinite part,
\[
\begin{cases}
AP_k^1 u = P_k^1 \partial_t^{-1}(-Bu + f) \\
A(I - P_k^1)u = (I - P_k^1)\partial_t^{-1}(-Bu + f)
\end{cases}
\] (24a)
\[
\begin{cases}
AP_k^1 u = P_k^1 \partial_t^{-1}(-Bu + f) \\
A(I - P_k^1)u = (I - P_k^1)\partial_t^{-1}(-Bu + f)
\end{cases}
\] (24b)

where there is commutative in \( A \) and \( P_k^1 \) because \( A \) is the constant matrix and (13). In shortly, we denote \( u_\perp := u - P_k^1 u \).

From (24a), for arbitrary \( v_k \in S_k(J)^n \), we have
\[
(.AP_k^1 u, v_k)_{V^1(J)^n} = (P_k^1 \partial_t^{-1}(-Bu + f), v_k)_{V^1(J)^n}
\]
\[
(A(P_k^1 u), v_k)_{L^2(J)^n} = (-Bu + f, v_k)_{L^2(J)^n}
\]
\[
(A(P_k^1 u), v_k)_{L^2(J)^n} + (BP_k^1 u, v_k)_{L^2(J)^n} = \left( \frac{d}{dt} \partial_t^{-1}(-Bu_\perp + f), v_k \right)_{L^2(J)^n}
\]
\[
= (P_k^1 \partial_t^{-1}(-Bu_\perp + f), v_k)_{V^1(J)^n}.
\] (25)

From \( P_k^1 u \) and \( P_k^1 \partial_t^{-1}(-Bu_\perp + f) \) are elements of \( S_k(J)^n \), these are expressible by linear combination of the basis of \( S_k(J)^n \). Namely, there exists \( \alpha := (\alpha_1, \cdots, \alpha_m) \in \mathbb{R}^m \) and \( \beta := (\beta_1, \cdots, \beta_m) \in \mathbb{R}^m \) such that
\[
P_k^1 u = \sum_{i=1}^m \alpha_i \psi_i, \quad P_k^1 \partial_t^{-1}(-Bu_\perp + f) = \sum_{i=1}^m \beta_i \psi_i.
\]

(25) is rewritten by using \( \alpha \) and \( \beta \), then we have
\[
G_\psi \alpha = D_\psi \beta,
\] (26)
where the matrices $G_y$ and $D_y$ are defined by (17) and (18), respectively. From (26), the $L^2$ norm of $P_k^1 u$ is satisfied that
\[
\|P_k^1 u\|_{L^2(J)^n}^2 = \alpha^2 L_y \alpha \\
= \left( T_{1/2}^{T/2} \alpha \right)^T L_y^{T/2} G_y^{-1/2} \left( D_y^{T/2} \beta \right) \\
\leq \|P_k^1 u\|_{L^2(J)^n} \left\| L_y^{T/2} G_y^{-1/2} D_y^{T/2} \right\|_2 \|P_k^1 \partial_t^{-1} (-Bu + f)\|_{V^1(J)^n}.
\]
From the fact that $P_k^1$ is $V^1$-projection, it is easy to see $\|P_k^1\|_{X(V^1)} = 1$. Therefore, we have
\[
\|P_k^1 u\|_{L^2(J)^n} \leq M_y^0 \|\partial_t^{-1} (-Bu + f)\|_{V^1(J)^n} \\
\leq M_y^0 \left( \|B\|_{L^\infty(J)^n} \|u\|_{L^2(J)^n} + \|f\|_{L^2(J)^n} \right) .
\]

Next, we calculate $L^2(J)^n$ norm inner product (24b) and $u_\perp$, we have
\[
(Au_\perp, u_\perp)_{L^2(J)^n} = (I - P_k^1) \partial_t^{-1} (-Bu + f), u_\perp)_{L^2(J)^n} \\
A_c \|u_\perp\|_{L^2(J)^n} \leq \|I - P_k^1\|_{L^2(J)^n} \|u_\perp\|_{L^2(J)^n} \|u_\perp\|_{L^2(J)^n},
\]
where $A_c$ is the minimum eigenvalue of $A$. From (15) and (27), we have
\[
A_c \|u_\perp\|_{L^2(J)^n} \leq C_J(k) \|Bu + f\|_{L^2(J)^n} \\
\leq C_J(k) \|B\|_{L^\infty} \left( M_y^0 \|B\|_{L^2} \|u\|_{L^2} + M_y^0 \|f\|_{L^2} + \|u\|_{L^2} \right) + C_J(k) \|f\|_{L^2} \\
\|u_\perp\|_{L^2(J)^n} \leq C_J(k) \frac{1 + M_y^0 \|B\|_{L^\infty(J)^n}}{A_c - \kappa_y} \|f\|_{L^2(J)^n} ,
\]
where we put $\kappa_y := C_J(k) \|B\|_{L^\infty} \left( 1 + M_y^0 \|B\|_{L^2} \right)$. From (28) and (27), we have
\[
\|P_k^1 u\|_{L^2(J)^n} \leq M_y^0 \|B\|_{L^\infty(J)^n} C_J(k) \frac{1 + M_y^0 \|B\|_{L^\infty(J)^n}}{A_c - \kappa_y} \|f\|_{L^2(J)^n} + M_y^0 \|f\|_{L^2(J)^n} \\
= M_y^0 \frac{A_c}{A_c - \kappa_y} \|f\|_{L^2(J)^n} .
\]

Finally, from (28) and (29), we have
\[
\|u\|_{L^2(J)^n} \leq \|P_k^1 u\|_{L^2(J)^n} + \|u_\perp\|_{L^2(J)^n} \\
\leq \frac{A_c M_y^0}{A_c - \kappa_y} \left( C_J(k) \left( 1 + M_y^0 \|B\|_{L^\infty(J)^n} \right) \right) \|f\|_{L^2(J)^n} .
\]

Therefore, this proof is completed. $\square$

The $V^1$ estimates is obtained by doing as well as the proof of Theorem 4.1. We define a positive constant $M_y^1$ by
\[
M_y^1 := \left\| D_y^{T/2} G_y^{-1} D_y^{1/2} \right\|_2 .
\]
Theorem 4.2  Under the same assumptions in Theorem 4.1, then we have the following estimates,

\[
\left\| \left( A \frac{d}{dt} + B \right)^{-1} \right\|_{L^2(J)^n, V^1(J)^n} \leq \frac{\sqrt{\left( A_M^{11} \right)^2 + (1 + M_{11}^{01}) \| B \|_{L^\infty(J)\times n}^2}}{A_c - \kappa_{\psi}}. \quad (31)
\]

Proof. —— From (26), the \( V^1 \) norm of \( P_k^1 u \) is satisfied that

\[
\| P_k^1 u \|_{V^1(J)^n}^2 = \alpha^T D_{\psi} \alpha \\
= \left( D_{\psi}^{T/2} \alpha \right)^T D_{\psi}^{T/2} G_{\psi}^{-1} D_{\psi}^{T/2} \left( D_{\psi}^{T/2} \beta \right) \\
\leq \| P_k^1 u \|_{V^1(J)^n} \left\| D_{\psi}^{T/2} G_{\psi}^{-1} D_{\psi}^{T/2} \right\|_2 \| P_k^1 \partial_t^{-1} (-Bu + f) \|_{V^1(J)^n}.
\]

From (15), we have

\[
\| P_k^1 u \|_{V^1(J)^n} \leq M_{11}^{11} \| -Bu + f \|_{L^2(J)^n} \\
\leq M_{11}^{11} \left( C_j(k) \| B \|_{L^\infty(J)\times n} \| u_\perp \|_{V^1(J)^n} + \| f \|_{L^2(J)^n} \right).
\]

Next, we calculate \( V^1(J)^n \) inner product (24b) and \( u_\perp \), we have

\[
(A u_\perp, u_\perp)_{V^1(J)^n} = \left( (I - P_k^1) \partial_t^{-1} (-Bu + f), u_\perp \right)_{V^1(J)^n} \\
A_c \| u_\perp \|_{V^1(J)^n}^2 \leq \| -Bu + f \|_{L^2(J)^n} \| u_\perp \|_{V^1(J)^n}.
\]

From (27) and (15), we have

\[
A_c \| u_\perp \|_{V^1(J)^n} \leq \| B \|_{L^\infty} \left( \| P_k^1 u \|_{L^2} + \| u_\perp \|_{L^2} \right) + \| f \|_{L^2} \\
\leq \| B \|_{L^\infty} \left( M_{11}^{01} \| B \|_{L^\infty} \| u_\perp \|_{L^2} + M_{11}^{01} \| f \|_{L^2} + \| u_\perp \|_{L^2} \right) + \| f \|_{L^2} \\
\| u_\perp \|_{V^1(J)^n} \leq \frac{1 + M_{11}^{01} \| B \|_{L^\infty(J)\times n} \| f \|_{L^2(J)^n}}{A_c - \kappa_{\psi}} \| f \|_{L^2(J)^n}, \quad (32)
\]

where we put \( \kappa_{\psi} := C_j(k) \| B \|_{L^\infty} \left( 1 + M_{11}^{01} \| B \|_{L^\infty} \right) \). We apply (32) to the estimates of \( \| P_k^1 u \|_{V^1} \), therefore

\[
\| P_k^1 u \|_{V^1(J)^n} \leq M_{11}^{11} C_j(k) \| B \|_{L^\infty} \frac{1 + M_{11}^{01} \| B \|_{L^\infty} \| f \|_{L^2(J)^n}}{A_c - \kappa_{\psi}} \| f \|_{L^2(J)^n} + M_{11}^{11} \| f \|_{L^2(J)^n} \\
= \frac{A_c M_{11}^{11}}{A_c - \kappa_{\psi}} \| f \|_{L^2(J)^n}.
\]
Finally, we have
\[
\|u\|_{V^1(J)^n}^2 \leq \left( \frac{A_c M_{W^1}^1}{A_c - \kappa_W} \right)^2 \|f\|_{L^2(J)^n}^2 + \left( \frac{1 + M_{W^1}^0 \|B\|_{L^2(J)^{n,n}}}{A_c - \kappa_W} \right)^2 \|f\|_{L^2(J)^n}.
\]

Therefore, this proof is completed. \(\square\)

(31) is expected to converge to \(\sqrt{(M_{W^1}^1)^2 + A_c^{-2}(1 + M_{W^1}^0 \|B\|_{L^2}^2)}\) as \(k \to 0\). This estimates looks over estimates, because we cannot give the order for \(k\) to the estimates of \(\|u_\perp\|_{V^1}\). It is necessary to assume the regularities of \(B\) to improve the overestimation of (31) a little.

**Theorem 4.3** Under the same assumptions in Theorem 4.1, let \(B \in W^{1,\infty}(J)^n,n\). Let \(\kappa_W > 0\) be satisfied
\[
\kappa_W := C_f(k) \left( C_f(k) \|B\|_{L^2} (1 + \|B\|_{L^2} M_{W^1}^0) + \|B\|_{L^2} (1 + C_f(k) \|B\|_{L^2} M_{W^1}^0) \right) < A_c.
\]

Then we have the following estimates,
\[
\left\| \left( A \frac{d}{dt} + B \right)^{-1} \right\|_{L^2(J)^n, V^1(J)^n} \leq \sqrt{\frac{(A_c - C_f(k) \|B\|_{L^2})^2 (M_{W^1}^1)^2 + (1 + C_f(k) \|B\|_{L^2} M_{W^1}^0 + C_f(k) \|B\|_{L^2} M_{W^1}^0)^2}{A_c - \kappa_W}}.
\]  

(33)

**Proof.** — For arbitrary \(f \in L^2(J)^n\), we put \(u := (A \frac{d}{dt} + B)^{-1} f \in V^1(J)^n\). We calculate \(V^1(J)^n\) inner product (24b) and \(u_\perp\), we have
\[
(A u_\perp, u_\perp)_{V^1(J)^n} = \left( (I - P_k^1) \frac{d}{dt} (-Bu + f), u_\perp \right)_{V^1(J)^n}.
\]

Therefore, we get
\[
\left| (P_k^1 u, u_\perp)_{V^1(J)^n} \right| \leq \frac{A_c (1 + C_f(k) \|B\|_{L^2} M_{W^1}^0 + C_f(k) \|B\|_{L^2} M_{W^1}^0)^2}{A_c - \kappa_W} \|B\|_{L^2} \|B\|_{L^2} M_{W^1}^0 \|f\|_{L^2(J)^n}.
\]  

(34)
We obtain
\[ \|u\|^2_{V^1(J)^p} = \|P_1^1 u\|^2_{V^1(J)^p} + \|u\|^2_{V^1(J)^p} \]
\[ \leq \left( M_{\psi}^1 A_c - C_f(k)^2 \|B\|_{L^2} \right)^2 \|f\|^2_{L^2(J)^p} \]
\[ + \left( 1 + C_f(k) \|B\|_{L^2} M_{\psi}^{11} + C_f(k) \|B\|_{L^2} M_{\psi}^{01} \right)^2 \|f\|^2_{L^2(J)^p}. \]

Therefore, this proof is completed. \( \square \)

(33) is expected to converge to \( \sqrt{(M_{\psi}^1)^2 + A_c^{-2}} \) as \( k \to 0 \). Therefore, we obtain an estimates that was smaller than (31). Furthermore, since we cannot obtain the order for \( k \) in the estimates of \( \|u\|_{V^1} \), the improvement more than this is difficult.

To obtain the \( L^p \) estimates, the following theorem becomes important.

**Theorem 4.4 (Gagliardo-Nirenberg)** Let \( \Omega \) be a bounded domain on \( \mathbb{R}^d \) where \( d = 1, 2 \) or 3. Let constants \( p \) and \( q \) be satisfied \( 1 \leq p \leq q^* \leq \infty \) where \( q^* \) is Sobolev conjugate index defined by \( q^* := \frac{dq}{d-q} \). Then, for arbitrary \( 0 \leq \theta \leq 1 \), there exists positive constant \( C_{g,r,p,q} > 0 \) such that
\[
\|u\|_{L^2(\Omega)} \leq C_{g,r,p,q} \|u\|_{L^q(\Omega)}^\theta \|u\|_{W^{1,q}(\Omega)}^{1-\theta}, \quad \forall u \in W^{1,q}(\Omega),
\]
where \( \frac{1}{p} = \frac{\theta}{r} + \frac{1-\theta}{q} \).

It is known that the optimal constants of \( C_{g,r,p,q} \) in Theorem 4.4 become the minimum eigenvalue of the certain nonlinear elliptic boundary value problems (e.g. [1]). Moreover, we can obtain the upper bounds of \( C_{g,r,p,q} \) by Sobolev constants. For example, if we can calculate the Sobolev constants \( C_{\infty} > 0 \) such that
\[
\|u\|_{L^2(\Omega)} \leq C_{\infty} \|u\|_{V^1(\Omega)}, \quad \forall u \in V^1(\Omega),
\]
then for arbitrary \( 2 \leq p \leq \infty \), we have
\[
\|u\|_{L^p(J)} \leq \|u\|_{L^2(J)}^{\frac{2}{p}} \|u\|_{L^2(J)}^{1-\frac{2}{p}} \]
\[
\leq C_{\infty}^{1-\frac{2}{p}} \|u\|_{L^2(J)}^{\frac{2}{p}} \|u\|_{V^1(J)^p}^{1-\frac{2}{p}}.
\]

Therefore, we obtain \( C_{g,p,2,2} \leq C_{\infty}^{1-\frac{2}{p}} \).

Finally in this section, we present the estimate in \( L^p \).

**Corollary 4.5** Assume that following two inequalities are provided,
\[
\left\| \left( A \frac{d}{dt} + B \right)^{-1} \right\|_{\mathcal{L}(L^2(J)^p, L^2(J)^p)} \leq C_{L^2,L^2}
\]
\[
\left\| \left( A \frac{d}{dt} + B \right)^{-1} \right\|_{\mathcal{L}(L^2(J)^p, V^1(J)^p)} \leq C_{L^2,V^1}
\]
then, for arbitrary $2 \leq p \leq \infty$, we have
\[
\left\| \left( A \frac{d}{dt} + B \right)^{-1} \right\|_{L^p(J^p) \to L^p(J^p)} \leq C_{g,p,2,2} C_{L^2L^2}^{1-p} C_{L^2V^1}^{1-p} C_{g,p,2,2}.
\] (36)

**Proof.** —— For arbitrary $f \in L^2(J)^n$, we put $u := (A \frac{d}{dt} + B)^{-1} f \in V^1(J)^n$. For arbitrary index $1 \leq i \leq n$, from Gagliardo-Nirenberg’s inequality, we have
\[
\|u_i\|_{L^p(J)} \leq C_{g,p,2,2} \|u_i\|_{L^2(J)}^{\frac{1}{2}} \|u_i\|_{V^1(J)}^{\frac{1}{2}}.
\]
From Hölder’s inequality, we have
\[
\|u\|_{L^p(J)^n}^2 = \sum_{i=1}^n \|u_i\|_{L^p(J)}^2 \leq C_{g,p,2,2}^2 \sum_{i=1}^n \|u_i\|_{L^2(J)}^{\frac{1}{2}} \|u_i\|_{V^1(J)}^{\frac{1}{2}} \left( \sum_{i=1}^n \|u_i\|_{V^1(J)}^2 \right) \leq C_{g,p,2,2}^2 \|u\|_{L^2(J)^n}^{\frac{1}{2}} \left( \sum_{i=1}^n \|u_i\|_{V^1(J)}^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^n \|u_i\|_{V^1(J)}^2 \right)^{-\frac{1}{2}} \|u\|_{L^2(J)^n}^{2(1-\frac{1}{p})}. \]

By the assumptions, we obtain
\[
\|u\|_{L^p(J)^n} \leq C_{g,p,2,2} |u|_{L^2(J)^n} \|u\|_{V^1(J)^n}^{1-\frac{1}{p}} \leq C_{g,p,2,2} C_{L^2L^2}^{\frac{1}{2}} \|f\|_{L^2(J)^n}^{\frac{1}{2}} \|f\|_{L^2(J)^n}^{1-\frac{1}{p}}. \]

Therefore, this proof is completed. \(\Box\)

## 5 A posteriori error estimates for inverse ODEs operator

In this section, we consider the error estimates for (2a) and (2b).

First of all, we prove the following lemma.

**Lemma 5.1** Let $A \in \mathbb{R}^{n,n}$. For any $v_k \in S_k(J)^n$, there exist a $w_k \in S_k(J)^n$ satisfying,
\[
w_k = Av_k.
\] (37)

**Proof.** —— Let $\psi_i$, $(1 \leq i \leq \bar{m})$ be the basis of finite dimensional subspace $\tilde{S}_k(J) \subset V^1(J)$. From $S_k(J)^n = \tilde{S}_k(J) \times \cdots \times \tilde{S}_k(J)$, we set $\psi_i$ be the basis of $S_k(J)^n$ in ordered
by,

\[
\psi_1 := \begin{pmatrix} \psi_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \psi_n := \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \psi_{n+1} := \begin{pmatrix} \psi_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \psi_m := \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},
\]

where \( m = n \bar{m} \).

For any integer \( i \) with \( 1 \leq i \leq m \), we denote two integers \( j \) and \( k \) be a integer quotient and a reminder of \( i/n \), i.e. \( i = jn + k \). Since

\[
A \psi_i = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \sum_{l=1}^{n} a_{k,l} \psi_{nj+l},
\]

\( A \psi_i \) can be written as the linear combination of other basis. Thus, for any \( \psi_k \in S_k(\mathcal{J}^n) \), there exist \( \psi_k \in S_k(\mathcal{J}^n) \) satisfying (37). □

We denote a bilinear form \( L : V^1(\mathcal{J})^n \times V^1(\mathcal{J})^n \to \mathbb{R} \) by,

\[
L(u, v) := (Au, v)_{L^2(\mathcal{J})^n} + (Bu, v)_{L^2(\mathcal{J})^n} \quad \forall u, v \in V^1(\mathcal{J})^n.
\]

Define a \( L \)-projection \( P_{k} : V^1(\mathcal{J})^n \to S_k(\mathcal{J})^n \) by,

\[
P_{k}L(u, v) = 0, \quad \forall v_k \in S_k(\mathcal{J})^n,
\]

(38)

From non-singularity of \( G_\psi \), \( P_{k} \) is well defined. We define the operator \( \mathcal{L}_{i,k}^{-1} : L^2(\mathcal{J})^n \to V^1(\mathcal{J})^n \) which gives \( u \), a solution of (2a) and (2b) for any \( f \in L^2(\mathcal{J})^n \). And define the operator \( \mathcal{L}_{i,k}^{-1} : L^2(\mathcal{J})^n \to S_k(\mathcal{J})^n \) gives \( u_k \), the solution of (16). We have \( \mathcal{L}_{i,k}^{-1} = P_{k}L_{\mathcal{J}}^{-1} \) from definition of \( P_{k} \).

We have a following estimate corresponding to \( P_{k} \) and \( P_{k}L_{\mathcal{J}}^{-1} \).

**Lemma 5.2** Let \( A \in \mathbb{R}^{n,n} \) be a symmetric positive definite matrix. Then, following inequalities hold,

\[
\|P_{k}^L u - P_{k} u\|_{L^2(\mathcal{J})^n} \leq M_{\psi}^{01} \|B(u - P_{k}^L u)\|_{L^2(\mathcal{J})^n}, \quad \forall u \in V^1(\mathcal{J})^n,
\]

(40)

\[
\|P_{k}^L u - P_{k} u\|_{V^1(\mathcal{J})^n} \leq M_{\psi}^{11} \|B(u - P_{k}^L u)\|_{L^2(\mathcal{J})^n}, \quad \forall u \in V^1(\mathcal{J})^n.
\]

(41)

**Proof.** — We denote \( u_\perp := u - P_{k}^L u \), for any \( u \in V^1(\mathcal{J})^n \). For any \( \psi_k \in S_k(\mathcal{J})^n \), we have,

\[
L(P_{k}^L u - P_{k}^L u, \psi_k) = L(u - P_{k}^L u, \psi_k)
= (A(u_\perp)'(u_\perp))_{L^2(\mathcal{J})^n} + (Bu_\perp, \psi_k)_{L^2(\mathcal{J})^n}
= ((u_\perp)', (Av_k)'(u_\perp))_{L^2(\mathcal{J})^n} + (Bu_\perp, \psi_k)_{L^2(\mathcal{J})^n},
\]
from definition of $P^L_k$ and symmetry of $A$. From Lemma 5.1, there exist a $w_k \in S_k(J)^n$ satisfying $w_k = Av_k$ and we have,

$$
((u_\perp)', (Av_k))_{L^2(J)^n} = (u_\perp, w_k)_{V^1(J)^n} = 0.
$$

Thus, we have,

$$
L(P^L_k u - P^L_k u, v_k) = (Bu_\perp, v_k')_{L^2(J)^n} = (\partial_t^{-1} Bu_\perp, v_k')_{L^2(J)^n} = (P^L_k \partial_t^{-1} Bu_\perp, v_k)_{V^1(J)^n}.
$$

(42)

Since $P^L_k u - P^L_k u$ and $P^L_k \partial_t^{-1} Bu_\perp$ are in $S_k(J)^n$, these can be written as the linear combination of the basis. Therefore, there exists a set of coefficients $a := (a_1, \ldots, a_m)$ and $b := (b_1, \ldots, b_m) \in \mathbb{R}^m$ such that

$$
P^L_k u - P^L_k u = \sum_{i=1}^m a_i \psi_i, \quad P^L_k \partial_t^{-1} Bu_\perp = \sum_{i=1}^m b_i \psi_i.
$$

Here, (42) can be written as

$$
G_\psi a = D_\psi b,
$$

where $D_\psi$ and $G_\psi$ are matrices defined in (18) and (17), respectively. Moreover, we have the following inequality,

$$
||P^L_k u - P^L_k u||^2_{L^2(J)^n} = \alpha^T L_\psi \alpha
\leq \left(\frac{L_{\psi}^{1/2}}{L_{\psi}^{1/2} G_\psi D_{\psi}^{1/2} \left(D_{\psi}^{1/2} \beta\right)}\right) ||P^L_k u - P^L_k u||_{L^2(J)^n} M_{\psi}^{01} ||P^L_k \partial_t^{-1} Bu_\perp||_{V^1(J)^n}.
$$

Since $||P^L_k||_{\mathcal{L}(V^1)} \leq 1$, we have,

$$
||P^L_k u - P^L_k u||_{L^2(J)^n} \leq M_{\psi}^{01} ||\partial_t^{-1} Bu_\perp||_{V^1(J)^n} \leq M_{\psi}^{01} ||Bu_\perp||_{L^2(J)^n}.
$$

Thus (40) is obtained. (41) can be proved in similar way. □

**Theorem 5.3** Under the same assumption in Theorem 4.1, the following inequality holds,

$$
\left\| \mathcal{L}^{-1}_t - \mathcal{L}^{-1}_{1,k} \right\|_{\mathcal{L}(L^2(J)^n, L^2(J)^n)} \leq C_J(k) \left(1 + M_{\psi}^{01} \left\|B\right\|_{L^2(J)^n}^2\right) \frac{1}{A_c - \kappa_\psi}.
$$

(43)
Proof. —— For any \( f \in L^2(J)^n \), let \( u := \mathcal{L}^{-1}_1 f \in V^1(J)^n \) and \( u_k := \mathcal{L}^{-1}_{i,k} f \in S_k(J)^n \).

Let \( P_k \) denote the \( V^1 \)-projection of \( u \). By the triangle inequality, we obtain,

\[
\|u - u_k\|_{L^2(J)^n} \leq \|u - P_k u\|_{L^2(J)^n} + \|P_k u - P_k u_k\|_{L^2(J)^n}.
\]

From (40), we have,

\[
\|P_k u - P_k u_k\|_{L^2(J)^n} \leq M^{i,j}_W \|B(u - P_k u)\|_{L^2(J)^n} \leq M^{i,j}_W \|B\|_{L^w(J)^n,n} \|u - P_k u\|_{L^2(J)^n}.
\]

It is easy to prove that \( u - P_k u \) satisfies (28) by referring the proof of Theorem 4.1. Therefore, we obtain

\[
\|u - u_k\|_{L^2(J)^n} \leq \|u - P_k u\|_{L^2(J)^n} + M^{i,j}_W \|B\|_{L^w(J)^n,n} \|u - P_k u\|_{L^2(J)^n} \leq \left(1 + M^{i,j}_W \|B\|_{L^w(J)^n,n}\right) C_j(k) \frac{1 + M^{i,j}_W \|B\|_{L^w(J)^n,n}}{A_c - \kappa_W} \|f\|_{L^2(J)^n}.
\]

Therefore, this proof is completed. □

Theorem 5.4 Under the same assumption in Theorem 4.1, we have

\[
\|L^{-1}_1 - L^{-1}_{i,k}\|_{L^2(J)^n,V^1(J)^n} \leq \left(1 + M^{i,j}_W \|B\|_{L^w(J)^n,n}\right) \sqrt{1 + \left(C_j(k)M^{i,j}_W \|B\|_{L^w(J)^n,n}\right)^2} \frac{1 + M^{i,j}_W \|B\|_{L^w(J)^n,n}}{A_c - \kappa_W} \|f\|_{L^2(J)^n}.
\]

Proof. —— By using the triangular inequality and (41), we have,

\[
\|u - u_k\|_{V^1(J)^n} = \|u - P_k u\|_{V^1(J)^n} + \|P_k u - P_k u_k\|_{V^1(J)^n} \leq \|u - P_k u\|_{V^1(J)^n} + \left(M^{i,j}_W \|B\|_{L^w(J)^n,n}\right)^2 \|u - P_k u\|_{L^2(J)^n},
\]

\[
\|u - u_k\|_{V^1(J)^n} \leq \sqrt{1 + \left(C_j(k)M^{i,j}_W \|B\|_{L^w(J)^n,n}\right)^2} \|u - P_k u\|_{V^1(J)^n}.
\]

From (32), we obtain,

\[
\|u - u_k\|_{V^1(J)^n} \leq \frac{1 + M^{i,j}_W \|B\|_{L^w(J)^n,n}}{A_c - \kappa_W} \|f\|_{L^2(J)^n}.
\]

Therefore, this proof is completed. □

Next, we consider the more accurately estimate by assuming that \( B \) is sufficiently smooth.

Theorem 5.5 Under the same assumption in Theorem 4.3, we have a following inequality,

\[
\|L^{-1}_1 - L^{-1}_{i,k}\|_{L^2(J)^n,V^1(J)^n} \leq \frac{1 + C_j(k)\|B\|_{L^w(J)^n,n} + C_j(k)\|B\|_{L^w(J)^n,n} \|B\|_{L^w(J)^n,n}}{A_c - \kappa_W} \|f\|_{L^2(J)^n}.
\]
Proof. —— (46) is obtained by (45) and (34). □

Finally, we present the error estimate in $L^p$.

**Corollary 5.6** Assume that following two inequalities are provided,

$$
\left\| \mathcal{L}_t^{-1} - \mathcal{L}_{t,k}^{-1} \right\|_{L^p(J^a,L^p(J^b))} \leq E_{L^2,L^2},
$$

$$
\left\| \mathcal{L}_t^{-1} - \mathcal{L}_{t,k}^{-1} \right\|_{L^2(J^a,V^1(J^b))} \leq E_{L^2,V^1},
$$

then, for arbitrary $2 \leq p \leq \infty$, we have

$$
\left\| \mathcal{L}_t^{-1} - \mathcal{L}_{t,k}^{-1} \right\|_{L^p(J^a,L^p(J^b))} \leq C_{g,p,2} E_{L^2,L^2} E_{L^2,V^1}^{-\frac{1}{2}},
$$

(47)

The proof is similarly to Corollary 4.5.

### 6 Numerical example

In this section, we present numerical experiments on some easy test problems. We compared a priori estimates (4) with a posteriori estimate proposed in Corollary 4.5. We consider the test problems that $B$ are sufficiently smooth. Then, estimates on $V^1$ can be computed from Theorem 4.3. To compute the numerical results, we use the P1 finite element approximation on the uniform mesh.

#### 6.1 Case of $n = 1$, $A = 1$ and $B = 1$

Here, we consider a case of $n = 1$, $A = 1$ and $B = 1$. From (3), we have an exact solution of (2a) and (2b),

$$
u(t) = e^{-t} \int_0^t e^s f(s) ds,$$

for any $f \in L^2(J)$.

**Case of $p = 2$**

In case of $p = 2$, a priori estimate by (4) is,

$$
\left\| \left( \frac{d}{dt} + 1 \right)^{-1} \right\|_{L^p(J^a,L^2(J))} \leq \frac{1}{2} \sqrt{2T + e^{-2T} - 1}.
$$

(48)

It implies that $C_{L^2,L^2} = O(\sqrt{T})$.

Table 1 shows the validated values of $M_{ij}^{01}$ with various $k$ and $T$. In tables, the row “Order” indicate the mean value of order over $k$. $M_{ij}^{01}$ is expected to converge to a constant as $k \to 0$. However, the validated computations of matrix 2-norm will be difficult when matrix size is large. Table 1 show that these properties. Also, the column “Order” indicate the mean value of order over $T$. The order is expected to be smaller than a priori estimate, however there are no big differences.
### Table 1: Validated computational result of $M_0^1$

<table>
<thead>
<tr>
<th>$k\setminus T$</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0E-1</td>
<td>0.054986</td>
<td>0.261386</td>
<td>0.441610</td>
<td>0.883341</td>
<td>0.961404</td>
<td>0.569535</td>
</tr>
<tr>
<td>5.0E-2</td>
<td>0.059504</td>
<td>0.262358</td>
<td>0.441993</td>
<td>0.883315</td>
<td>0.961391</td>
<td>0.556685</td>
</tr>
<tr>
<td>1.0E-2</td>
<td>0.061093</td>
<td>0.262669</td>
<td>0.442117</td>
<td>0.883491</td>
<td>0.961458</td>
<td>0.552391</td>
</tr>
<tr>
<td>5.0E-3</td>
<td>0.061143</td>
<td>0.262680</td>
<td>0.442157</td>
<td>0.883489</td>
<td>0.961790</td>
<td>0.552398</td>
</tr>
<tr>
<td>1.0E-3</td>
<td>0.061160</td>
<td>0.262911</td>
<td>0.442477</td>
<td>0.926966</td>
<td>1.287028</td>
<td>0.647522</td>
</tr>
<tr>
<td>order</td>
<td>-0.032916</td>
<td>-0.001674</td>
<td>-0.000501</td>
<td>-0.007481</td>
<td>-0.045379</td>
<td></td>
</tr>
</tbody>
</table>

The values of $M_0^1$ are shown in Table 1. From (21), if $M_0^1 < 1$, then we have a posteriori estimate of norm of inverse operator by using Theorem 4.1. In Table 1, $M_0^1 < 1$ for all $k$ and $T$, so that we can apply the estimates (22).

### Table 2: Validated computational result of $\kappa_0$

<table>
<thead>
<tr>
<th>$k\setminus T$</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0E-1</td>
<td>0.033581</td>
<td>0.040151</td>
<td>0.045888</td>
<td>0.059949</td>
<td>0.062433</td>
<td>0.132090</td>
</tr>
<tr>
<td>5.0E-2</td>
<td>0.016863</td>
<td>0.020091</td>
<td>0.022950</td>
<td>0.029974</td>
<td>0.031217</td>
<td>0.131323</td>
</tr>
<tr>
<td>1.0E-2</td>
<td>0.003378</td>
<td>0.004019</td>
<td>0.004590</td>
<td>0.005995</td>
<td>0.006244</td>
<td>0.131051</td>
</tr>
<tr>
<td>5.0E-3</td>
<td>0.001689</td>
<td>0.002010</td>
<td>0.002295</td>
<td>0.002998</td>
<td>0.003122</td>
<td>0.131108</td>
</tr>
<tr>
<td>1.0E-3</td>
<td>0.000338</td>
<td>0.000402</td>
<td>0.000459</td>
<td>0.000613</td>
<td>0.000728</td>
<td>0.181754</td>
</tr>
<tr>
<td>order</td>
<td>0.998206</td>
<td>0.999652</td>
<td>0.999846</td>
<td>0.999946</td>
<td>0.997610</td>
<td></td>
</tr>
</tbody>
</table>

The values of $\kappa_0$ are shown in Table 2. From (21), if $\kappa_0 < 1$, then we have a posteriori estimate of norm of inverse operator by using Theorem 4.1. In Table 2, $\kappa_0 < 1$ for all $k$ and $T$, so that we can apply the estimates (22).

### Table 3: Validated computational result of $C_{L^2;L^2}$

<table>
<thead>
<tr>
<th>$k\setminus T$</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0E-1</td>
<td>0.091645</td>
<td>0.314150</td>
<td>0.510944</td>
<td>1.003445</td>
<td>1.092015</td>
<td>0.502140</td>
</tr>
<tr>
<td>5.0E-2</td>
<td>0.077677</td>
<td>0.288240</td>
<td>0.475864</td>
<td>0.941509</td>
<td>1.024592</td>
<td>0.520992</td>
</tr>
<tr>
<td>1.0E-2</td>
<td>0.064689</td>
<td>0.267765</td>
<td>0.448767</td>
<td>0.894851</td>
<td>0.973781</td>
<td>0.544597</td>
</tr>
<tr>
<td>5.0E-3</td>
<td>0.062938</td>
<td>0.265223</td>
<td>0.445475</td>
<td>0.889152</td>
<td>0.967934</td>
<td>0.548445</td>
</tr>
<tr>
<td>1.0E-3</td>
<td>0.061518</td>
<td>0.263419</td>
<td>0.443139</td>
<td>0.928149</td>
<td>1.288694</td>
<td>0.646730</td>
</tr>
<tr>
<td>a priori</td>
<td>0.068430</td>
<td>0.303265</td>
<td>0.532761</td>
<td>1.500004</td>
<td>2.179449</td>
<td>0.730030</td>
</tr>
</tbody>
</table>

Table 3 show that $C_{L^2;L^2}$ by (22) and (48). It is thought that the rounding error is a cause in the reason why the value is large in the Table 3 at $k = 1.0E-3$ and $N = 10$. Our proposed a posteriori estimates has been better than a priori estimates results.
**6 NUMERICAL EXAMPLE**

Case of $p = \infty$

In case of $p = \infty$, from (4), we have a priori estimate

$$
\left\| \left( \frac{d}{dt} + 1 \right)^{-1} \right\|_{L^{2}(\mathcal{J}), L^{\infty}(\mathcal{J})} \leq \sqrt{\frac{1 - e^{-2T}}{2}}.
$$

(49)

<table>
<thead>
<tr>
<th>$k \backslash T$</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0E-1</td>
<td>0.952381</td>
<td>0.999870</td>
<td>0.999969</td>
<td>0.999999</td>
<td>1.000000</td>
<td>0.007599</td>
</tr>
<tr>
<td>5.0E-2</td>
<td>0.999695</td>
<td>0.999992</td>
<td>0.999998</td>
<td>1.000000</td>
<td>1.000000</td>
<td>0.000049</td>
</tr>
<tr>
<td>1.0E-2</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.000001</td>
<td>0.000000</td>
</tr>
<tr>
<td>5.0E-3</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.157722</td>
<td>0.052822</td>
</tr>
<tr>
<td>1.0E-3</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.131759</td>
<td>fail</td>
<td>fail</td>
<td>0.089283</td>
</tr>
<tr>
<td>order</td>
<td>-0.017535</td>
<td>-0.000045</td>
<td>-0.019237</td>
<td>-0.000001</td>
<td>-0.070492</td>
<td></td>
</tr>
</tbody>
</table>

The validated values of $M^{11}_{\psi}$ are shown in Table 4. In this table, “fail” means we couldn’t compute a validated upper bound of $M^{11}_{\psi}$, from effects of round-off errors. Also, $M^{11}_{\psi}$ is expected to converge to a constant as $k \to 0$. Table 4 shows this property when the matrix sizes are not large. The orders over $T$ are near to the order of a priori estimate (49).

Table 5: Validated computational result of $\hat{k}_{\psi}$

<table>
<thead>
<tr>
<th>$k \backslash T$</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0E-1</td>
<td>0.032796</td>
<td>0.032844</td>
<td>0.032844</td>
<td>0.032844</td>
<td>0.032844</td>
<td>0.000229</td>
</tr>
<tr>
<td>5.0E-2</td>
<td>0.016169</td>
<td>0.016169</td>
<td>0.016169</td>
<td>0.016169</td>
<td>0.016169</td>
<td>0.000001</td>
</tr>
<tr>
<td>1.0E-2</td>
<td>0.003193</td>
<td>0.003193</td>
<td>0.003193</td>
<td>0.003193</td>
<td>0.003193</td>
<td>0.000000</td>
</tr>
<tr>
<td>5.0E-3</td>
<td>0.001594</td>
<td>0.001594</td>
<td>0.001594</td>
<td>0.001594</td>
<td>0.001594</td>
<td>0.000090</td>
</tr>
<tr>
<td>1.0E-3</td>
<td>0.000318</td>
<td>0.000318</td>
<td>0.000318</td>
<td>0.000318</td>
<td>fail</td>
<td>0.000030</td>
</tr>
<tr>
<td>order</td>
<td>1.007807</td>
<td>1.008335</td>
<td>1.008329</td>
<td>1.010851</td>
<td>1.010731</td>
<td></td>
</tr>
</tbody>
</table>

Table 5 shows $\hat{k}_{\psi}$. In this table, $\hat{k}_{\psi} < 1$ then we can apply Theorem 4.3 to obtain a posteriori estimates.

Table 6 shows a posteriori estimates by (33) and a priori estimate (49) of $C_{L_2, L_{\infty}}$.

In tables, $M^{01}_{\psi}$ and $M^{11}_{\psi}$ are hardly depends on $k$. Then, the relative errors of $C_{L_2, L_2}$ and $C_{L_2, L_{\infty}}$ are $O(k)$. In other problems, we show the numerical results only when $k = 1.0E-2$. 

Table 6 shows a posteriori estimates by (33) and a priori estimate (49) of $C_{L_2, L_{\infty}}$. 

In tables, $M^{01}_{\psi}$ and $M^{11}_{\psi}$ are hardly depends on $k$. Then, the relative errors of $C_{L_2, L_2}$ and $C_{L_2, L_{\infty}}$ are $O(k)$. In other problems, we show the numerical results only when $k = 1.0E-2$. 

In case of $p = \infty$, from (4), we have a priori estimate

$$
\left\| \left( \frac{d}{dt} + 1 \right)^{-1} \right\|_{L^{2}(\mathcal{J}), L^{\infty}(\mathcal{J})} \leq \sqrt{\frac{1 - e^{-2T}}{2}}.
$$

(49)
Table 6: Validated computational result of $C_{L^2,L^\infty}$

<table>
<thead>
<tr>
<th>$k/T$</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0E-1</td>
<td>0.458731</td>
<td>1.050476</td>
<td>1.485671</td>
<td>3.322111</td>
<td>4.698177</td>
<td>0.503720</td>
</tr>
<tr>
<td>5.0E-2</td>
<td>0.458125</td>
<td>1.024551</td>
<td>1.448938</td>
<td>3.239927</td>
<td>4.581949</td>
<td>0.500024</td>
</tr>
<tr>
<td>1.0E-2</td>
<td>0.449361</td>
<td>1.004801</td>
<td>1.421004</td>
<td>3.177461</td>
<td>4.493610</td>
<td>0.500000</td>
</tr>
<tr>
<td>5.0E-3</td>
<td>0.448284</td>
<td>1.002394</td>
<td>1.417599</td>
<td>3.169849</td>
<td>4.849173</td>
<td>0.528335</td>
</tr>
<tr>
<td>1.0E-3</td>
<td>0.447427</td>
<td>1.000478</td>
<td>1.421004</td>
<td>3.177461</td>
<td>4.493610</td>
<td>0.500000</td>
</tr>
<tr>
<td>a priori</td>
<td>0.301056</td>
<td>0.562192</td>
<td>0.657520</td>
<td>0.707091</td>
<td>0.707107</td>
<td>0.164805</td>
</tr>
</tbody>
</table>

6.2 Case of $n = 1$, $A = 1$ and $B = -1$

In case of $n = 1$, $A = 1$ and $B = -1$, for any $f \in L^2(J)$, the exact solution

$$u(t) = e^t \int_0^t e^{-s} f(s) \, ds,$$

is obtained from (3). Then, we have following a priori estimates,

$$\left\| \frac{d}{dt} - 1 \right\|_{L^2(L^2(J),L^2(J))} \leq \frac{1}{2} \sqrt{e^{2T} - 2T - 1},$$

(50)

$$\left\| \frac{d}{dt} - 1 \right\|_{L^\infty(L^2(J),L^\infty(J))} \leq \frac{e^T - e^{-T}}{2}.$$  

(51)

These a priori estimates are $O(e^T)$.

Table 7: Validated computational result of $B = -1$, $k = 1.0E-2$

<table>
<thead>
<tr>
<th>$T$</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M^0_w$</td>
<td>0.066362</td>
<td>0.394230</td>
<td>1.000025</td>
<td>76.83091</td>
<td>1.716179</td>
<td></td>
</tr>
<tr>
<td>$k_\psi$</td>
<td>0.003394</td>
<td>0.004438</td>
<td>0.006366</td>
<td>0.247744</td>
<td>0.987375</td>
<td></td>
</tr>
<tr>
<td>a posteriori $C_{L^2,L^2}$</td>
<td>0.069894</td>
<td>0.400445</td>
<td>1.012839</td>
<td>102.4633</td>
<td>3.064492</td>
<td></td>
</tr>
<tr>
<td>a priori $C_{L^2,L^2}$</td>
<td>0.073148</td>
<td>0.423758</td>
<td>1.047504</td>
<td>74.18805</td>
<td>11013.23</td>
<td></td>
</tr>
<tr>
<td>$M^1_w$</td>
<td>1.052164</td>
<td>1.312844</td>
<td>1.810207</td>
<td>fail</td>
<td>3.00495</td>
<td></td>
</tr>
<tr>
<td>$k_\psi$</td>
<td>0.003194</td>
<td>0.003196</td>
<td>0.003201</td>
<td>fail</td>
<td>0.001393</td>
<td></td>
</tr>
<tr>
<td>a posteriori $C_{L^2,L^\infty}$</td>
<td>0.461229</td>
<td>1.172494</td>
<td>2.077498</td>
<td>74.20322</td>
<td>2.996783</td>
<td></td>
</tr>
<tr>
<td>a priori $C_{L^2,L^\infty}$</td>
<td>0.100167</td>
<td>0.521095</td>
<td>1.175201</td>
<td>11013.24</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 7 show that the numerical results for $k = 1.0E-2$ and various $T$. In this table, “a posteriori $C_{L^2,L^2}$” are the estimate by Theorem 4.1, and “a priori $C_{L^2,L^2}$” the values of (50). Also, “a posteriori $C_{L^2,L^\infty}$” and “a priori $C_{L^2,L^\infty}$” are the values of estimates by Corollary 4.5 and (51), respectively.
6.3 Case of \( n = 1, A = 1 \) and \( B = \cos t \)

In case of \( n = 1, A = 1 \) and \( B = \cos t \), for any \( f \in L^2(J) \), we have an exact solution of (2a) and (2b),

\[
u(t) = e^{-\sin t} \int_0^t e^{\sin s} f(s) \, ds,
\]

from (3). Then, we have the following a priori estimates,

\[
\left\| \frac{d}{dt} + \cos t \right\|_{L^2(J), L^2(J)} \leq \frac{e^2}{\sqrt{2}} T, \quad (52)
\]

\[
\left\| \frac{d}{dt} + \cos t \right\|_{L^2(J), L^\infty(J)} \leq e^2 \sqrt{T}. \quad (53)
\]

Numerical results are shown in Table 8.

### Table 8: Validated computational result of \( B = \cos t, k = 1.0 \times 10^{-2} \)

<table>
<thead>
<tr>
<th>( T )</th>
<th>0.1</th>
<th>0.5</th>
<th>1.0</th>
<th>5.0</th>
<th>10.0</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M^{(1)}_\psi )</td>
<td>0.061096</td>
<td>0.264437</td>
<td>0.463939</td>
<td>11.92004</td>
<td>26.38994</td>
<td>1.221238</td>
</tr>
<tr>
<td>( K_\psi )</td>
<td>0.003378</td>
<td>0.004025</td>
<td>0.004660</td>
<td>0.041126</td>
<td>0.087185</td>
<td>0.689345</td>
</tr>
<tr>
<td>( a ) ( \text{a posteriori} ) ( C ) ( L^2, L^2 )</td>
<td>0.064692</td>
<td>0.269547</td>
<td>0.470793</td>
<td>12.47417</td>
<td>29.00602</td>
<td>1.236196</td>
</tr>
<tr>
<td>( a ) ( \text{a priori} ) ( C ) ( L^2, L^2 )</td>
<td>0.522485</td>
<td>2.612426</td>
<td>5.224852</td>
<td>26.12426</td>
<td>52.24852</td>
<td>1.000000</td>
</tr>
<tr>
<td>( M^{(1)}_\psi )</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.000000</td>
<td>fail</td>
<td>0.000000</td>
<td></td>
</tr>
<tr>
<td>( K_\psi )</td>
<td>0.003194</td>
<td>0.003199</td>
<td>0.003206</td>
<td>0.001920</td>
<td>0.001920</td>
<td></td>
</tr>
<tr>
<td>( a ) ( \text{a posteriori} ) ( C ) ( L^2, L^\infty )</td>
<td>0.449366</td>
<td>1.005010</td>
<td>1.421904</td>
<td>0.500368</td>
<td>0.500368</td>
<td></td>
</tr>
<tr>
<td>( a ) ( \text{a priori} ) ( C ) ( L^2, L^\infty )</td>
<td>2.336625</td>
<td>5.224852</td>
<td>7.389056</td>
<td>16.52244</td>
<td>23.36625</td>
<td>0.500000</td>
</tr>
</tbody>
</table>

6.4 Case of linearized equations

Our a posterior estimate can be applied to the more general problems in case of \( n > 1 \). Here, we consider a linearized problem of the following nonlinear initial value problem,

\[
\begin{align*}
    \begin{cases} 
        u' = g(u), & \text{in } J, \\
        u(0) = u^0,
    \end{cases} \\
    (54a)
\end{align*}
\]

where \( g(u) := (u_2, u_1 - u_1^3)^T \) and \( u^0 := (0, 4)^T \). Let \( u_k \) be an approximate solution of (54a) and (54b). \( u_k \) draws like a periodic orbit of period \( T = \pi \). We perform the our a posteriori estimates to the linearized inverse operator \( -\frac{d}{dt} - g'(u_k) \), where,

\[
g'(u_k) = \begin{pmatrix} 0 & 1 \\ 1 - 3u_k^2 & 0 \end{pmatrix}.
\]
In this problem, apparently, to authors’ knowledge, a priori estimate is not known, so we cannot compare with a posteriori estimate.

Results in case of $n=2$, $A=I$, $B=g'(u_k)$, $k=1.0E-2$ are shown in Table 9.

**Remark 6.1 (Computer environment)** All computations are carried out on a PC (Intel Core i7 860, 16GB of DDR3 memory) by using INTLAB version 6.0, a toolbox in MATLAB 2010a developed by Rump [4] for self-validating algorithms. Therefore, all numerical values in these tables are verified data in the sense of strictly rounding error control.

### 7 Conclusions

We presented a posteriori (error) estimates of inverse operators for initial value problems in linear ODEs. Our estimates can be applied to the simultaneous linear ODEs. Actually, we introduced some numerical example that used our a posteriori estimates.

### References


