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On the Spin-refined Reshetikhin-Turaev SU(2) Invariants of Lens Spaces

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ON THE SPIN-REFINED RESHETIKHIN-TURAEV SU(2) INVARIANTS OF LENS SPACES

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Abstract. We give an explicit presentation of the value of the spin-refined Reshetikhin-Turaev $SU(2)$ invariants of lens spaces. Using this result, we also present the value of spin-refined Turaev-Viro $SU(2)$ invariants of lens spaces.

0. Introduction

The quantum invariants of closed oriented 3-manifolds associated with a semisimple Lie group were proposed by Witten [Wit89], and rigorously constructed by Reshetikhin-Turaev [ReTu91], called the *Reshetikhin-Turaev invariants.*

When the Lie group is $SU(2)$, two kinds of refinements of the Reshetikhin-Turaev invariants were defined by Kirby-Melvin [KiMe91] at r-th root of unity for even r: the invariants of 3-manifolds with first $(\mathbb{Z}/2\mathbb{Z})$-cohomology classes for $r \equiv 2 \pmod 4$, and with spin structures for $r \equiv 0 \pmod 4$.

For lens spaces, the Reshetikhin-Turaev $SU(2)$ invariants were calculated by Kirby-Melvin [KiMe91], Jeffrey [Jef92], Yamada [Yam95], and Li-Li [LiLi96], and the refined Reshetikhin-Turaev $SU(2)$ invariants associated with first $(\mathbb{Z}/2\mathbb{Z})$-cohomology classes by Sato [Sat06].

In this paper, we calculate the refined Reshetikhin-Turaev $SU(2)$ invariants of lens spaces associated with spin structures. To see this, we set up some notation. Let ζ_n denote the n-th root of unity $\exp(2\pi\sqrt{-1}/n)$. Fix a positive odd integer p. Let p^* be the inverse of p modulo 8. For a rational number n/m with m coprime to p, let $(n/m)^\vee$ denote $n\overline{m} \in \mathbb{Z}/p\mathbb{Z}$, where $\overline{m}$ is the inverse of m modulo p.

Theorem 1. *Let $a > 0$ and b be integers with a even, $(a, b) = 1$ and $(a, p) = 1$. Let Θ be a spin structure of the lens space $L(a, b)$. Then the spin-refined Reshetikhin-Turaev $SU(2)$ invariant of $L(a, b)$ at $4p$-th root of unity $\tau_{4p}(L(a, b), \Theta)$ is presented by*

$$\tau_{4p}(L(a,b),\Theta) = \left(\frac{a}{p}\right) \zeta_p^{-(\frac{3}{4}\mathrm{s}(b,a)+\frac{\delta}{8a})^\vee}\, \zeta_{16}^{-3p^*\mu(L(a,b),\Theta)}\, \frac{\zeta_8^{p^*} - \zeta_8^{-p^*}}{\zeta_{8p} - \zeta_{8p}^{-1}}\,,$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol, $\mathrm{s}(b, a)$ the Dedekind sum, and $\mu(M, \Theta)$ the μ-invariant of the spin manifold (M, Θ). We will define the sign $\delta = \pm 1$ by (10) in §2.

On the other hand, Roberts [Rob97] defined the *spin-refined Turaev-Viro invariants*[1] of 3-manifolds associated with spin structures and second $(\mathbb{Z}/2\mathbb{Z})$-homology classes, and pointed out the relation with the spin-refined Reshetikhin-Turaev invariants (see [Rob97, Theorem 3.6] and (11)).

Using Theorem 1 and the equality (11), we can derive these invariants of lens spaces as follows:

Corollary 2. *Under the same assumption as in Theorem* 1 *and for* $y \in H_2(L(a,b), \mathbb{Z}/2\mathbb{Z})$, *the refined Turaev-Viro invariants at* $4p$*-th root of unity* $TV_{4p}(L(a,b), \Theta, y)$ *is presented by*

$$TV_{4p}(L(a,b),\Theta,0) = \frac{2}{|\zeta_{8p} - \zeta_{8p}^{-1}|^2},$$

$$TV_{4p}(L(a,b),\Theta,y) = \frac{2\,\zeta_{16}^{-3p^*(\mu-\mu')}}{|\zeta_{8p} - \zeta_{8p}^{-1}|^2} \cdot \begin{cases} 1 & a \equiv 0 \mod 4 \\ \zeta_p^{-\delta\cdot\overline{4a}} & a \equiv 2 \mod 4 \end{cases} \quad \text{for } y \neq 0.$$

where we put $\mu = \mu(L(a,b),\Theta)$, $\mu' = \mu(L(a,b),\Theta')$ *with* Θ' *a spin structure of* $L(a,b)$ *distinct from* Θ.

The paper is organized as follows: In §1 we review the definition of the spin-refined Reshetikhin-Turaev invariants. In §2 we prove Theorem 1 and in §3 we derive Corollary 2.

I am deeply grateful to Tomotada Ohtsuki whose insightful comments made enormous contribution to my work. I would also like to thank Takefumi Nosaka whose constructive comments and warm encouragements.

1. Review of the invariants

The spin-refined Reshetikhin-Turaev $SU(2)$ invariants of spin 3-manifolds were defined by Kirby-Melvin [KiMe91]. By using linear skein, combinatorial definition of these invariants were given by Blanchet [Bla92]. In this section, we review the Blanchet's definition.

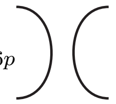
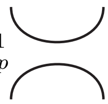
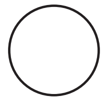
Fix a positive odd integer p. We put $[i] = \dfrac{\zeta_{8p}^{i} - \zeta_{8p}^{-i}}{\zeta_{8p} - \zeta_{8p}^{-1}}$ for non-negative integer i. Let $(F, 2i)$ denote a surface F with ordered $2i$ points on ∂F. We define the vector space $\mathcal{S}(F, 2i)$ over $\mathbb{C}$ by

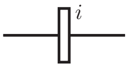
$$\mathcal{S}(F,2i) = \mathrm{span}_{\mathbb{C}}\{\text{tangle diagrams on } F\}/\sim .$$

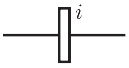
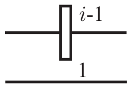
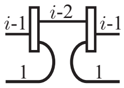
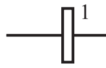
Here D is tangle diagrams on $(F, 2i)$ if D is tangle diagrams on F, and ∂D is equal to the fixed $2i$ points. The equivalent relation "$\sim$" is generated by the isotopies of

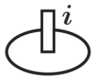
[1]in Roberts' paper, this invariants are denoted by $CH(M,\Theta,y)$, and $\tau_r(M,\Theta)$ by $I(M,\Theta)$.

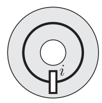
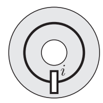
tangle diagrams on the surface F and the skein relations below:

 $= \zeta_{16p}$  $+ \zeta_{16p}^{-1}$  ,  $= (-\zeta_{8p} - \zeta_{8p}^{-1})\, \emptyset\,,$

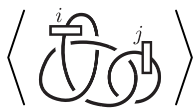
where $\emptyset$ means an empty diagram. We denote $\mathcal{S}(F, 0)$ by $\mathcal{S}(F)$ for short. Remark that $\mathcal{S}(S^2)$ is isomorphic to $\mathbb{C}$ by the isomorphism $D \mapsto \langle D \rangle$, where $\langle D \rangle$ is the Kauffman bracket of D. We define inductively the *Jones-Wenzl idempotents*  $\in \mathcal{S}([0,1]^2, 2i)$ as follows [Lic93]:

 $=$  $+ \dfrac{[i-1]}{[i]}$  $(i \geq 2)\,,$  $=$,

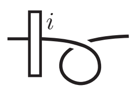
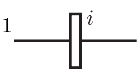
where the strand with a number i stands for the union of i parallel copies of one strand. We can show that $\langle$  $\rangle = (-1)^i [i+1]$ $(i \geq 1)$ by induction on i. Define two elements ω_0 and ω_1 in $\mathcal{S}(S^1 \times [0,1])$ by

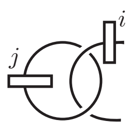
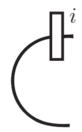
$\omega_0 = \sum_{\substack{0 \leq i < 4p \\ i:\text{even}}} [i+1]$  , $\quad \omega_1 = -\sum_{\substack{0 \leq i < 4p \\ i:\text{odd}}} [i+1]$  .

Let $L = L_1 \cup \cdots \cup L_N$ be a framed link on a surface F with a link diagram $D = D_1 \cup \cdots \cup D_N$. For $x_1, \ldots, x_N \in \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, define $\langle L_1^{\omega_{x_1}} \cup \cdots \cup L_N^{\omega_{x_N}} \rangle \in \mathbb{C}$ by substituting ω_{x_ℓ} for each component D_ℓ. For example,

 $= -\sum_{\substack{0 \leq i,j < 4p \\ i:\ \text{even},\ j:\ \text{odd}}} [i+1][j+1]$  .

Remark that this definition is independent of the choice of a diagram D (see [Bla92]). The following equalities are known [Lic93].

(1)  $= (-1)^i \zeta_{16p}^{(i+1)^2 - 1}$  ,

(2)  $= (-1)^j \dfrac{[(i+1)(j+1)]}{[i+1]}$  .

Let M be a closed connected oriented 3-manifold, and Θ a spin structure of M. M is obtained from S^3 by a surgery along some framed link $L = L_1 \cup \cdots \cup L_N$ in S^3 (see [Kir78]). Let B be the linking matrix of L. It is known [KiMe91] that the set of spin structures on M can be identified with the set $\{\boldsymbol{x} \in (\mathbb{Z}/2\mathbb{Z})^N | B\boldsymbol{x} \equiv {}^t(b_{11}, \ldots, b_{NN}) \pmod 2\}$, where $b_{11}, \ldots, b_{NN}$ are the diagonal entries of B. Let $\boldsymbol{x} = {}^t(x_1, \ldots, x_N)$ be the element in this set corresponding to Θ. Then the *spin-refined Reshetikhin-Turaev $SU(2)$ invariant* $\tau_{4p}(M,\Theta)$ is defined as follows:

$$\tau_{4p}(M,\Theta)=c_+^{-\sigma_+}c_-^{-\sigma_-}\langle L_1^{\omega_{x_1}}\cup\cdots\cup L_N^{\omega_{x_N}}\rangle\,, \tag{3}$$

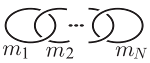
where $c_\pm=\langle U_\pm^{\omega_1}\rangle$ ($U_\pm$ is the trivial knot with framing ±1), and $\sigma_\pm$ is a number of the positive/negative eigenvalues of B. It is known that the right-hand side of (3) is independent of the choice of the framed link L [KiMe91, Bla92].

2. Proof of Theorem 1

In this section, we calculate the value of $\tau_{4p}^{SU(2)}(L(a,b),\Theta)$ and prove Theorem 1. We choose a continued fraction expansion of a/b:

$$\frac{a}{b}=m_1-\cfrac{1}{m_2-\cfrac{}{\ddots-\cfrac{1}{m_N}}}\qquad(|m_k|\geq 2)\,.$$

It is known that $L(a,b)$ is obtained from S^3 by a surgery along the framed link



in S^3, where m_i on each component means the framing. We put

$$B=\begin{pmatrix} m_1 & 1 & & \\ 1 & m_2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & m_N\end{pmatrix},\quad \boldsymbol{e}_1=\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix},\ \boldsymbol{e}_N=\begin{pmatrix}0\\ \vdots\\0\\1\end{pmatrix}. \tag{4}$$

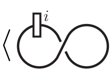
Remark that B is the linking matrix of L, and $\det B=a$.

Let Θ be the spin structure of the lens space $L(a,b)$. We set $\boldsymbol{x}={}^t(x_1,\ldots,x_N)\in(\mathbb{Z}/2\mathbb{Z})^N$ corresponds to Θ, satisfying

$$B\boldsymbol{x}\equiv{}^t(m_1,\ldots,m_N)\pmod 2. \tag{5}$$

Proof of Theorem 1. We first calculate the normalization constant $c_+^{\sigma_+}c_-^{\sigma_-}$. By definition and the formula (1),

$$c_+=-\sum_{\substack{0\leq i<4p\\ i:\text{odd}}}[i+1]\langle$$



$$\rangle=-\sum_{\substack{0<i\leq 4p\\ i:\text{odd}}}\zeta_{16p}^{(i+1)^2-1}[i+1]^2$$

$$=-\frac{\zeta_{16p}^{-1}}{(\zeta_{8p}-\zeta_{8p}^{-1})^2}\sum_{j\in\mathbb{Z}/2p\mathbb{Z}}\zeta_{4p}^{j^2}(\zeta_{4p}^{2j}-2+\zeta_{4p}^{-2j})=\frac{2\,\zeta_{16p}^{-3}}{\zeta_{8p}-\zeta_{8p}^{-1}}\sum_{j\in\mathbb{Z}/2p\mathbb{Z}}\zeta_{4p}^{j^2}$$

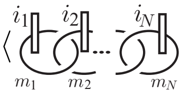
$$=\frac{2\,\zeta_{16p}^{-3}}{\zeta_{8p}-\zeta_{8p}^{-1}}\sum_{\gamma\in\mathbb{Z}/2\mathbb{Z}}\zeta_4^{p\gamma^2}\,G_p,$$

where we put $i = 2j-1$ in the third equality, $j = 2k + p\gamma$ in the fifth equality, and denote Gaussian sum $\displaystyle\sum_{k\in\mathbb{Z}/p\mathbb{Z}} \zeta_p^{k^2}$ by G_p. Since $a = \det B > 0$ is equal to the product of the eigenvalues of B, σ_- is even. Remark that c_- is equal to the complex conjugate of c_+ and that $\displaystyle\sum_{k\in\mathbb{Z}/p\mathbb{Z}} \zeta_p^{-k^2} = \left(\frac{-1}{p}\right) G_p$. Thus we get

$$c_+^{\sigma_+} c_-^{\sigma_-} = \frac{2^N \zeta_{16p}^{-3\,\sigma}}{(\zeta_{8p} - \zeta_{8p}^{-1})^N} \Big(\sum_{\gamma\in\mathbb{Z}/2\mathbb{Z}} \zeta_4^{p\gamma^2}\Big)^{\sigma_+} \Big(\sum_{\gamma\in\mathbb{Z}/2\mathbb{Z}} \zeta_4^{-p\gamma^2}\Big)^{\sigma_-} G_p^N\,,$$

where $\sigma = \sigma_+ - \sigma_-$ is the signature of B.

Now we calculate $\tau_{4p}(L(a,b),\Theta)$. By the definition (3),

$$\tau_{4p}(L(a,b),\Theta) = c_+^{-\sigma_+} c_-^{-\sigma_-} \sum_{\substack{0\le i_\ell < 4p \\ i_\ell - x_\ell\,:\,\text{even}}} (-1)^{x_1+\cdots+x_N} [i_1+1]\ldots[i_N+1]\langle$$  $$\rangle\,.$$

Applying the formulae (1) and (2) repeatedly and substituting the indices i_ℓ for $i_\ell - 1$ ($\ell = 1,\ldots,N$), this is equal to

$$(-1)^\kappa c_+^{-\sigma_+} c_-^{-\sigma_-} \sum_{\substack{0< i_\ell \le 4p \\ i_\ell - x_\ell\,:\,\text{odd}}} \zeta_{16p}^{m_1 i_1^2+\cdots+m_N i_N^2 - \mathrm{tr}B} [i_1][i_1 i_2]\ldots[i_{N-1} i_N][i_N]\,,$$

where $\kappa = m_1 x_1 + \cdots + m_N x_N$. Since this formula is symmetric with respect to the substitutions $i_1 \to -i_1, \ldots, i_N \to -i_N$, this is equal to

$$\begin{aligned}
&\frac{(-1)^\kappa\, 2^N c_+^{-\sigma_+} c_-^{-\sigma_-}}{(\zeta_{8p}-\zeta_{8p}^{-1})^{N+1}} \sum_{\substack{0\le i_\ell < 4p \\ i_\ell - x_\ell\,:\,\text{odd}}} \zeta_{16p}^{m_1 i_1^2+\cdots+m_N i_N^2 + 2(i_1 + i_1 i_2 + \cdots + i_{N-1} i_N)} (\zeta_{16p}^{2i_N} - \zeta_{16p}^{-2i_N}) \\
&\qquad = C \sum_{\pm} \pm \sum_{\substack{\boldsymbol{i}\in(\mathbb{Z}/4p\mathbb{Z})^N \\ \boldsymbol{i}\equiv \boldsymbol{y}\ (\mathrm{mod}\ 2)}} \zeta_{16p}^{{}^t\boldsymbol{i}B\boldsymbol{i} + 2\,{}^t(\boldsymbol{e}_1 \pm \boldsymbol{e}_N)\boldsymbol{i}},
\end{aligned}$$

where we put $\boldsymbol{i} = {}^t(i_1,\ldots,i_N)$, $\boldsymbol{y} = \boldsymbol{x} + {}^t(1,\ldots,1)$, and

$$C = \frac{(-1)^\kappa \zeta_{16p}^{3\sigma - \mathrm{tr}B}}{\zeta_{8p} - \zeta_{8p}^{-1}} \Big(\sum_{\gamma\in\mathbb{Z}/2\mathbb{Z}} \zeta_4^{p\gamma^2}\Big)^{-\sigma_+} \Big(\sum_{\gamma\in\mathbb{Z}/2\mathbb{Z}} \zeta_4^{-p\gamma^2}\Big)^{-\sigma_-} G_p^{-N}.$$

Thus

$$\begin{aligned}
(6)\qquad \tau_{4p}(L(a,b),\Theta) &= C \sum_{\pm} \pm \zeta_{16p}^{{}^t\boldsymbol{y}B\boldsymbol{y} + 2\,{}^t(\boldsymbol{e}_1 \pm \boldsymbol{e}_N)\boldsymbol{y}} \sum_{\boldsymbol{j}\in(\mathbb{Z}/2p\mathbb{Z})^N} \zeta_{4p}^{{}^t\boldsymbol{j}B\boldsymbol{j} + 2\,{}^t\boldsymbol{u}_\pm \boldsymbol{j}} \\
&= C \sum_{\pm} \pm \zeta_{16p}^{{}^t\boldsymbol{y}B\boldsymbol{y} + 2\,{}^t(\boldsymbol{e}_1 \pm \boldsymbol{e}_N)\boldsymbol{y}} \sum_{\boldsymbol{\gamma}\in(\mathbb{Z}/2\mathbb{Z})^N} \zeta_4^{p({}^t\boldsymbol{\gamma}B\boldsymbol{\gamma} + 2\,{}^t\boldsymbol{u}_\pm\boldsymbol{\gamma})} \sum_{\boldsymbol{k}\in(\mathbb{Z}/p\mathbb{Z})^N} \zeta_p^{{}^t\boldsymbol{k}B\boldsymbol{k} + {}^t\boldsymbol{u}_\pm \boldsymbol{k}} \\
&= \delta\, C\, \zeta_{16p}^{{}^t\boldsymbol{y}B\boldsymbol{y} + 2\,{}^t(\boldsymbol{e}_1 + \delta\boldsymbol{e}_N)\boldsymbol{y}} \sum_{\boldsymbol{\gamma}\in(\mathbb{Z}/2\mathbb{Z})^N} \zeta_4^{p({}^t\boldsymbol{\gamma}B\boldsymbol{\gamma} + 2\,{}^t\boldsymbol{u}_\delta\boldsymbol{\gamma})} \sum_{\boldsymbol{k}\in(\mathbb{Z}/p\mathbb{Z})^N} \zeta_p^{{}^t\boldsymbol{k}B\boldsymbol{k} + {}^t\boldsymbol{u}_\delta \boldsymbol{k}},
\end{aligned}$$

where we put $\boldsymbol{i} = 2\boldsymbol{j} + \boldsymbol{y}$ in the first equality, $\boldsymbol{j} = 2\boldsymbol{k} + p\,\boldsymbol{\gamma}$, $\boldsymbol{u}_\pm = \frac{1}{2}(B\boldsymbol{y} + \boldsymbol{e}_1 \pm \boldsymbol{e}_N)$ in the second equality, and use Lemma 3 below in the last equality. Remark that, since $B\boldsymbol{y} \equiv \boldsymbol{e}_1 + \boldsymbol{e}_N \pmod 2$ by the definition of B and the equality (5), $\boldsymbol{u}_\pm$ is a vector whose entries are integers.

In a similar way, we can show that

$$\tau_4(L(a,b),\Theta)\big|_{\zeta_{16}^{p^*}} = \delta C' \zeta_{16}^{p^*\{{}^t\boldsymbol{y}B\boldsymbol{y}+2^t(\boldsymbol{e}_1+\delta\boldsymbol{e}_N)\boldsymbol{y}\}} \sum_{\boldsymbol{\gamma}\in(\mathbb{Z}/2\mathbb{Z})^N} \zeta_4^{p({}^t\boldsymbol{\gamma}B\boldsymbol{\gamma}+2^t\boldsymbol{u}_\delta\boldsymbol{\gamma})}, \tag{7}$$

where $\tau_4(L(a,b),\Theta)\big|_{\zeta_{16}^{p^*}}$ denotes the value obtained from $\tau_4(L(a,b),\Theta)$ by replacing ζ_{16} with $\zeta_{16}^{p^*}$, and we put

$$C' = \frac{(-1)^\kappa \zeta_{16}^{p^*(3\sigma-\mathrm{tr}B)}}{\zeta_8^{p^*}-\zeta_8^{-p^*}} \Big(\sum_{\gamma\in\mathbb{Z}/2\mathbb{Z}} \zeta_4^{p\gamma^2}\Big)^{-\sigma_+} \Big(\sum_{\gamma\in\mathbb{Z}/2\mathbb{Z}} \zeta_4^{-p\gamma^2}\Big)^{-\sigma_-}.$$

Comparing (6) and (7), we get

$$\begin{aligned} &(8)\quad \tau_{4p}(L(a,b),\Theta) \\ &= \delta\Big(\frac{a}{p}\Big) \frac{\zeta_8^{p^*}-\zeta_8^{-p^*}}{\zeta_{8p}-\zeta_{8p}^{-1}} \zeta_p^{\overline{16}\{{}^t\boldsymbol{y}B\boldsymbol{y}+2(\boldsymbol{e}_1+\delta\boldsymbol{e}_N)\boldsymbol{y}+3\sigma-\mathrm{tr}B\}^\vee} G_p^{-N} \sum_{\boldsymbol{k}\in(\mathbb{Z}/p\mathbb{Z})^N} \zeta_p^{{}^t\boldsymbol{k}B\boldsymbol{k}+{}^t\boldsymbol{u}_\delta\boldsymbol{k}} \tau_4(L(a,b),\Theta)\big|_{\zeta_{16}^{p^*}}, \end{aligned}$$

Since B is symmetric, there exists $P \in \mathrm{GL}(N,\mathbb{Z}/p\mathbb{Z})$ such that ${}^tPBP$ is diagonal. Thus by putting ${}^tPBP = (b_1)\oplus\cdots\oplus(b_N)$, $P\boldsymbol{u}_\delta = {}^t(c_1,\ldots,c_N)$, we see

$$\begin{aligned} \sum_{\boldsymbol{k}\in(\mathbb{Z}/p\mathbb{Z})^N} \zeta_p^{{}^t\boldsymbol{k}B\boldsymbol{k}+{}^t\boldsymbol{u}_\delta\boldsymbol{k}} &= \sum_{\boldsymbol{k}'\in(\mathbb{Z}/p\mathbb{Z})^N} \zeta_p^{{}^t\boldsymbol{k}'({}^tPBP)\boldsymbol{k}'+{}^t(P\boldsymbol{u}_\delta)\boldsymbol{k}'} \\ &= \prod_{\ell=1}^N \sum_{j_\ell\in\mathbb{Z}/p\mathbb{Z}} \zeta_p^{b_\ell j_\ell^2+c_\ell j_\ell} = \Big(\frac{b_1\ldots b_N}{p}\Big) \zeta_p^{-(\overline{4b_1}c_1^2+\cdots+\overline{4b_N}c_N^2)} G_p^N \\ &= \Big(\frac{\det({}^tPBP)}{p}\Big) \zeta_p^{-\overline{4}\{{}^t(P\boldsymbol{u}_\delta)({}^tPBP)^{-1}(P\boldsymbol{u}_\delta)\}^\vee} G_p^N \\ &= \Big(\frac{a}{p}\Big) \zeta_p^{-\overline{4}({}^t\boldsymbol{u}_\delta B^{-1}\boldsymbol{u}_\delta)^\vee} G_p^N \\ &= \Big(\frac{a}{p}\Big) \zeta_p^{-\overline{16}\{{}^t\boldsymbol{y}B\boldsymbol{y}+2^t(e_1+\delta e_N)\boldsymbol{y}+{}^t\boldsymbol{e}_1B^{-1}\boldsymbol{e}_1+{}^t\boldsymbol{e}_NB^{-1}\boldsymbol{e}_N+\delta\frac{2}{a}\}^\vee} G_p^N, \end{aligned} \tag{9}$$

where we put $\boldsymbol{k} = P\boldsymbol{k}'$ in the first equality.

Substituting (9) into (8) and using Lemma 4 below, we get

$$\tau_{4p}(L(a,b),\Theta) = \Big(\frac{a}{p}\Big) \frac{\zeta_8^{p^*}-\zeta_8^{-p^*}}{\zeta_{8p}-\zeta_{8p}^{-1}} \zeta_p^{-(\frac{3}{4}\mathrm{s}(b,a)+\frac{\delta}{8a})^\vee} \tau_4(L(a,b),\Theta)\big|_{\zeta_{16}^{p^*}}.$$

Further, since [KiMe91, Example8.31] says that

$$\tau_4(L(a,b),\Theta) = \zeta_{16}^{-3\mu(L(a,b),\Theta)},$$

we obtain the required formula. $\square$

Lemma 3. *Let $\boldsymbol{d} \in (\mathbb{Z}/2\mathbb{Z})^N$ be a non-zero vector satisfying $B\boldsymbol{d} \equiv \boldsymbol{0} \pmod 2$. Recall that $\boldsymbol{u}_\pm = \frac{1}{2}(B\boldsymbol{y} + \boldsymbol{e}_1 \pm \boldsymbol{e}_N)$, and its entries are integers. Put a sign δ by*

$$\delta = \zeta_4^{{}^t\boldsymbol{d}B\boldsymbol{d}+2{}^t\boldsymbol{u}_+\boldsymbol{d}} = (-1)^{\frac{1}{2}{}^t\boldsymbol{d}B\boldsymbol{d}+{}^t\boldsymbol{u}_+\boldsymbol{d}}. \tag{10}$$

Then it holds

$$\sum_{\boldsymbol{\gamma}\in(\mathbb{Z}/2\mathbb{Z})^N} \zeta_4^{p({}^t\boldsymbol{\gamma}B\boldsymbol{\gamma}+2{}^t\boldsymbol{u}_{-\delta}\boldsymbol{\gamma})} = 0.$$

Proof. Since $({}^t\boldsymbol{d}B\boldsymbol{d} + 2{}^t\boldsymbol{u}_+\boldsymbol{d}) - ({}^t\boldsymbol{d}B\boldsymbol{d} + 2{}^t\boldsymbol{u}_-\boldsymbol{d}) \equiv 2{}^t\boldsymbol{e}_N\boldsymbol{d} \equiv 2 \pmod 4$, we can check $\zeta_4^{{}^t\boldsymbol{d}B\boldsymbol{d}+2{}^t\boldsymbol{u}_{-\delta}\boldsymbol{d}} = -1$. Thus

$$\begin{aligned}
\sum_{\boldsymbol{\gamma}\in(\mathbb{Z}/2\mathbb{Z})^N} \zeta_4^{p({}^t\boldsymbol{\gamma}B\boldsymbol{\gamma}+2{}^t\boldsymbol{u}_{-\delta}\boldsymbol{\gamma})} &= \sum_{\boldsymbol{\gamma}'\in(\mathbb{Z}/2\mathbb{Z})^N} \zeta_4^{p\{{}^t(\boldsymbol{\gamma}'+\boldsymbol{d})B(\boldsymbol{\gamma}'+\boldsymbol{d})+2{}^t\boldsymbol{u}_{-\delta}(\boldsymbol{\gamma}'+\boldsymbol{d})\}} \\
&= \zeta_4^{p({}^t\boldsymbol{d}B\boldsymbol{d}+2{}^t\boldsymbol{u}_{-\delta}\boldsymbol{d})} \sum_{\boldsymbol{\gamma}'\in(\mathbb{Z}/2\mathbb{Z})^N} \zeta_4^{p({}^t\boldsymbol{\gamma}'B\boldsymbol{\gamma}'+2{}^t\boldsymbol{u}_{-\delta}\boldsymbol{\gamma}')} \\
&= -\sum_{\boldsymbol{\gamma}\in(\mathbb{Z}/2\mathbb{Z})^N} \zeta_4^{p({}^t\boldsymbol{\gamma}B\boldsymbol{\gamma}+2{}^t\boldsymbol{u}_{-\delta}\boldsymbol{\gamma})},
\end{aligned}$$

where we put $\boldsymbol{\gamma}' = \boldsymbol{\gamma} - \boldsymbol{d}$ in the first equality. Thus we obtain the lemma. $\square$

Lemma 4. *With the above notation, the following equality holds:*

$$3\sigma - \mathrm{tr}B = {}^t\boldsymbol{e}_1B^{-1}\boldsymbol{e}_1 + {}^t\boldsymbol{e}_NB^{-1}\boldsymbol{e}_N - 12\mathrm{s}(b,a)\,.$$

Proof. In the same fashion as [KiMe91, §2.3], we can derive

$$3\{\mathrm{sgn}(a_0a_1) + \cdots + \mathrm{sgn}(a_{N-1}a_N)\} - \mathrm{tr}B = {}^t\boldsymbol{e}_1B^{-1}\boldsymbol{e}_1 + {}^t\boldsymbol{e}_NB^{-1}\boldsymbol{e}_N - 12\mathrm{s}(b,a),$$

where

$$a_0 = 1, \quad a_i = \det\begin{pmatrix} m_1 & 1 & & \\ 1 & m_2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & m_i \end{pmatrix} \ (i = 1, \ldots, N).$$

Since it is known that $\sigma = \mathrm{sgn}(a_0a_1) + \cdots + \mathrm{sgn}(a_{N-1}a_N)$ (for example, see [Mac46, Theorem 34.3]), we obtain the lemma. $\square$

3. Proof of Corollary 2

In this section we prove the Corollary 2. For a positive integer p, an oriented closed 3-manifold M, a spin structure Θ of M, and $y \in H_2(M, \mathbb{Z}/2\mathbb{Z})$, [Rob97, Theorem3.6] says that

$$TV_{4p}(M, \Theta, y) = \tau_{4p}(M, \Theta)\overline{\tau_{4p}(M, \Theta + D(y))}, \tag{11}$$

where $D(y) \in H^1(M, \mathbb{Z}/2\mathbb{Z})$ is the Poincaré dual of y. Remark that the set of spin structures is affinely isomorphic to $H^1(M, \mathbb{Z}/2\mathbb{Z})$, so $\Theta + D(y)$ makes sense. Let

Θ, Θ' be the distinct spin structures of the lens space $L(a,b)$. Using this formula and Theorem 1, we obtain

$$TV_{4p}(L(a,b),\Theta,0) = |\tau_{4p}(L(a,b),\Theta)|^2 = \frac{2}{|\zeta_{8p} - \zeta_{8p}^{-1}|^2},$$

and

$$\begin{aligned} TV_{4p}(L(a,b),\Theta,y) &= \tau_{4p}(L(a,b),\Theta)\overline{\tau_{4p}(L(a,b),\Theta')} \\ &= \frac{2}{|\zeta_{8p} - \zeta_{8p}^{-1}|^2}\zeta_{16}^{-3p^*(\mu-\mu')}\zeta_p^{-\overline{8a}(\delta-\delta')} \end{aligned}$$

for $y \neq 0$, where δ' is the "Θ'-version" of δ, similarly defined as (10). Now we can check by definition that $\delta' = (-1)^{a/2}\delta$. This completes the proof of the corollary.

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