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On the Spin-refined Reshetikhin-Turaev SU(2) Invariants of Lens Spaces

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Abstract. We give an explicit presentation of the value of the spin-refined Reshetikhin-Turaev SU(2) invariants of lens spaces. Using this result, we also present the value of spin-refined Turaev-Viro SU(2) invariants of lens spaces.

0. Introduction

The quantum invariants of closed oriented 3-manifolds associated with a semisimple Lie group were proposed by Witten [Wit89], and rigorously constructed by Reshetikhin-Turaev [ReTu91], called the Reshetikhin-Turaev invariants.

When the Lie group is SU(2), two kinds of refinements of the Reshetikhin-Turaev invariants were defined by Kirby-Melvin [KiMe91] at $r$-th root of unity for even $r$: the invariants of 3-manifolds with first $(\mathbb{Z}/2\mathbb{Z})$-cohomology classes for $r \equiv 2 \pmod{4}$, and with spin structures for $r \equiv 0 \pmod{4}$.

For lens spaces, the Reshetikhin-Turaev SU(2) invariants were calculated by Kirby-Melvin [KiMe91], Jeffrey [Jef92], Yamada [Yam95], and Li-Li [LiLi96], and the refined Reshetikhin-Turaev SU(2) invariants associated with first $(\mathbb{Z}/2\mathbb{Z})$-cohomology classes by Sato [Sat06].

In this paper, we calculate the refined Reshetikhin-Turaev SU(2) invariants of lens spaces associated with spin structures. To see this, we set up some notation. Let $\zeta_n$ denote the $n$-th root of unity $\exp(2\pi\sqrt{-1}/n)$. Fix a positive odd integer $p$. Let $p^*$ be the inverse of $p$ modulo 8. For a rational number $n/m$ with $m$ coprime to $p$, let $(n/m)^\vee$ denote $n\overline{m} \in \mathbb{Z}/p\mathbb{Z}$, where $\overline{m}$ is the inverse of $m$ modulo $p$.

**Theorem 1.** Let $a > 0$ and $b$ be integers with $a$ even, $(a, b) = 1$ and $(a, p) = 1$. Let $\Theta$ be a spin structure of the lens space $L(a, b)$. Then the spin-refined Reshetikhin-Turaev SU(2) invariant of $L(a, b)$ at $4p$-th root of unity $\tau_{4p}(L(a, b), \Theta)$ is presented by

$$
\tau_{4p}(L(a, b), \Theta) = \left(\frac{a}{p}\right) \zeta_p^{-\left(\frac{s(b, a) + \frac{1}{8}}{m}\right)^\vee} \zeta_{16}^{-3p^*\mu(L(a, b), \Theta)} \frac{\zeta_8^p - \zeta_8^{-p^*}}{\zeta_{8p} - \zeta_{8p}^{-1}},
$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol, $s(b, a)$ the Dedekind sum, and $\mu(M, \Theta)$ the $\mu$-invariant of the spin manifold $(M, \Theta)$. We will define the sign $\delta = \pm 1$ by (10) in §2.
On the other hand, Roberts [Rob97] defined the spin-refined Turaev-Viro invariants\(^1\) of 3-manifolds associated with spin structures and second \((\mathbb{Z}/2\mathbb{Z})\)-homology classes, and pointed out the relation with the spin-refined Reshetikhin-Turaev invariants (see [Rob97, Theorem 3.6] and (11)).

Using Theorem 1 and the equality (11), we can derive these invariants of lens spaces as follows:

**Corollary 2.** Under the same assumption as in Theorem 1 and for \(y \in H_2(L(a,b),\mathbb{Z}/2\mathbb{Z})\), the refined Turaev-Viro invariants at \(4p\)-th root of unity \(TV_{4p}(L(a,b),\Theta, y)\) is presented by

\[
TV_{4p}(L(a,b),\Theta, y) = 2 \frac{\zeta \cdot \zeta^{-1}}{\zeta^2 - \zeta^{-1}} \cdot \begin{cases} 
1 & a \equiv 0 \mod 4 \\
\zeta^{-3a} & a \equiv 2 \mod 4 
\end{cases} \text{ for } y \neq 0.
\]

where we put \(\mu = \mu(L(a,b),\Theta)\), \(\mu' = \mu(L(a,b),\Theta')\) with \(\Theta'\) a spin structure of \(L(a,b)\) distinct from \(\Theta\).

The paper is organized as follows: In §1 we review the definition of the spin-refined Reshetikhin-Turaev invariants. In §2 we prove Theorem 1 and in §3 we derive Corollary 2.

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1. **Review of the invariants**

The spin-refined Reshetikhin-Turaev \(SU(2)\) invariants of spin 3-manifolds were defined by Kirby-Melvin [KiMe91]. By using linear skein, combinatorial definition of these invariants were given by Blanchet [Bla92]. In this section, we review the Blanchet’s definition.

Fix a positive odd integer \(p\). We put \([i] = \frac{\zeta^i}{\zeta^p - \zeta^{-1}}\) for non-negative integer \(i\).

Let \((F,2i)\) denote a surface \(F\) with ordered \(2i\) points on \(\partial F\). We define the vector space \(S(F,2i)\) over \(\mathbb{C}\) by

\[
S(F,2i) = \text{span}_\mathbb{C}\{\text{tangle diagrams on } F\}/ \sim .
\]

Here \(D\) is tangle diagrams on \((F,2i)\) if \(D\) is tangle diagrams on \(F\), and \(\partial D\) is equal to the fixed \(2i\) points. The equivalent relation \(\sim\) is generated by the isotopies of

\(^1\)in Roberts’ paper, this invariants are denoted by \(CH(M,\Theta,y)\), and \(\tau_r(M,\Theta)\) by \(I(M,\Theta)\).
tangle diagrams on the surface $F$ and the skein relations below:

\[
\begin{align*}
\begin{array}{c}
\cross = \zeta_{16p}
\end{array}
\end{align*}
\]

where $\emptyset$ means an empty diagram. We denote $\mathcal{S}(F,0)$ by $\mathcal{S}(F)$ for short. Remark that $\mathcal{S}(S^2)$ is isomorphic to $\mathbb{C}$ by the isomorphism $D \mapsto \langle D \rangle$, where $\langle D \rangle$ is the Kauffman bracket of $D$. We define inductively the Jones-Wenzl idempotents

\[
\begin{align*}
\begin{array}{c}
\omega_0 = \sum_{0 \leq i < 4, \text{i: even}} [i+1] \prod_{i} \bigcirc, \\
\omega_1 = - \sum_{0 \leq i < 4, \text{i: odd}} [i+1] \bigcirc
\end{array}
\end{align*}
\]

Let $L = L_1 \cup \cdots \cup L_N$ be a framed link on a surface $F$ with a link diagram $D = D_1 \cup \cdots \cup D_N$. For $x_1, \ldots, x_N \in \mathbb{Z}/2\mathbb{Z} = \{0,1\}$, define $\langle L_1^{x_1} \cup \cdots \cup L_N^{x_N} \rangle \in \mathbb{C}$ by substituting $\omega_{x_i}$ for each component $D_i$. For example,

\[
\begin{align*}
\begin{array}{c}
\langle \omega_0 \omega_3 \rangle = - \sum_{0 \leq i,j < 4, \text{i: even, j: odd}} [i+1][j+1] \langle \omega_i \omega_j \rangle
\end{array}
\end{align*}
\]

Remark that this definition is independent of the choice of a diagram $D$ (see [Bla92]). The following equalities are known [Lic93]. The following equalities are known [Lic93].

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\end{align*}
\]

(1)

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\end{align*}
\]

(2)

Let $M$ be a closed connected oriented 3-manifold, and $\Theta$ a spin structure of $M$. $M$ is obtained from $S^3$ by a surgery along some framed link $L = L_1 \cup \cdots \cup L_N$ in $S^3$ (see [Kir78]). Let $B$ be the linking matrix of $L$. It is known [KiMe91] that the set of spin structures on $M$ can be identified with the set $\{x \in (\mathbb{Z}/2\mathbb{Z})^N | Bx \equiv t(b_{11}, \ldots, b_{NN}) \pmod{2}\}$, where $b_{11}, \ldots, b_{NN}$ are the diagonal entries of $B$. Let $x = t(x_1, \ldots, x_N)$ be
the element in this set corresponding to Θ. Then the spin-refined Reshetikhin-Turaev $SU(2)$ invariant $\tau_{4p}(M, \Theta)$ is defined as follows:

$$\tau_{4p}(M, \Theta) = c_+^{\sigma_+} c_-^{\sigma_-} \langle L^{\omega_1 \cup \cdots \cup L^{\omega_N}} \rangle,$$

where $c_{\pm} = \langle U_{\pm}^{\omega_i} \rangle$ ($U_{\pm}$ is the trivial knot with framing $\pm 1$), and $\sigma_{\pm}$ is a number of the positive/negative eigenvalues of $B$. It is known that the right-hand side of (3) is independent of the choice of the framed link $L$ [KiMe91, Bla92].

2. PROOF OF THEOREM 1

In this section, we calculate the value of $\tau_{SU(2)}^{4p}(L(a, b), \Theta)$ and prove Theorem 1. We choose a continued fraction expansion of $a/b$:

$$\frac{a}{b} = m_1 - \frac{1}{m_2 - \frac{1}{m_3 - \cdots - \frac{1}{m_N}}} \quad (|m_k| \geq 2).$$

It is known that $L(a, b)$ is obtained from $S^3$ by a surgery along the framed link $\bigotimes_{m_1} \bigotimes_{m_2} \bigotimes_{m_N}$ in $S^3$, where $m_i$ on each component means the framing. We put

$$B = \begin{pmatrix} m_1 & 1 \\ 1 & m_2 \\ \vdots & \vdots \\ 1 & m_N \end{pmatrix}, \ e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ e_N = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Remark that $B$ is the linking matrix of $L$, and det $B = a$.

Let $\Theta$ be the spin structure of the lens space $L(a, b)$. We set $x = \iota(x_1, \ldots, x_N) \in (\mathbb{Z}/2\mathbb{Z})^N$ corresponds to $\Theta$, satisfying

$$Bx \equiv \iota(m_1, \ldots, m_N) \pmod{2}.$$  

Proof of Theorem 1. We first calculate the normalization constant $c_+^{\sigma_+} c_-^{\sigma_-}$. By definition and the formula (1),

$$c_+ = - \sum_{0 \leq i < 4p} \frac{\zeta_{16p}^{i+1} - \zeta_{16p}^{-i}}{(\zeta_{8p} - \zeta_{8p}^{-1})^2} \sum_{j \in \mathbb{Z}/2p\mathbb{Z}} \zeta_{4p}^j (\zeta_{4p}^{2j} - 2 + \zeta_{4p}^{-2j}) = \frac{2 \zeta_{16p}^{-3}}{\zeta_{8p} - \zeta_{8p}^{-1}} \sum_{j \in \mathbb{Z}/2p\mathbb{Z}} \zeta_{4p}^{2j} \sum_{\gamma \in \mathbb{Z}/2\mathbb{Z}} \zeta_{4p}^{2\gamma} G_p,$$

$$c_- = - \sum_{0 \leq i < 4p} \frac{\zeta_{16p}^{i+1} - \zeta_{16p}^{-i}}{(\zeta_{8p} - \zeta_{8p}^{-1})^2} \sum_{j \in \mathbb{Z}/2p\mathbb{Z}} \zeta_{4p}^j (\zeta_{4p}^{-2j} - 2 + \zeta_{4p}^{2j}) = \frac{2 \zeta_{16p}^{-3}}{\zeta_{8p} - \zeta_{8p}^{-1}} \sum_{j \in \mathbb{Z}/2p\mathbb{Z}} \zeta_{4p}^{-2j} \sum_{\gamma \in \mathbb{Z}/2\mathbb{Z}} \zeta_{4p}^{-2\gamma} G_p.$$
where we put \( i = 2j - 1 \) in the third equality, \( j = 2k + p\gamma \) in the fifth equality, and denote Gaussian sum \( \sum_{k \in \mathbb{Z}/p\mathbb{Z}} \zeta_p^{k^2} \) by \( G_p \). Since \( a = \det B > 0 \) is equal to the product of the eigenvalues of \( B \), \( \sigma_- \) is even. Remark that \( c_- \) is equal to the complex conjugate of \( c_+ \) and that \( \sum_{k \in \mathbb{Z}/p\mathbb{Z}} \zeta_p^{-k^2} = \left( -\frac{1}{p} \right) G_p \). Thus we get

\[
c_+^{\sigma} c_-^{\sigma} = \frac{2N \zeta_{16p}}{(\zeta_{8p} - \zeta_{8p}^{-1})^{N+1}} \left( \sum_{\gamma \in \mathbb{Z}/2\mathbb{Z}} \zeta_4^{\gamma^2} \right)^{\sigma} \left( \sum_{\gamma \in \mathbb{Z}/2\mathbb{Z}} \zeta_4^{-\gamma^2} \right)^{\sigma} G_p^N,
\]

where \( \sigma = \sigma_+ - \sigma_- \) is the signature of \( B \).

Now we calculate \( \tau_{4p}(L(a, b), \Theta) \). By the definition (3),

\[
\tau_{4p}(L(a, b), \Theta) = c_{\sigma}^{\sigma} c_{-\sigma}^{\sigma} \sum_{0 < i_1 < 4p} (\zeta_4^{m_1^2 + \cdots + m_N^2 \gamma \left( i_1 i_2 \right) \cdots [i_N - i_N] N \left( 2i_1^N - \zeta_4^{-2i_1^N} \right) \left( \sum_{\gamma \in \mathbb{Z}/2\mathbb{Z}} \zeta_4^{\gamma^2} \right)^{-\sigma} \left( \sum_{\gamma \in \mathbb{Z}/2\mathbb{Z}} \zeta_4^{-\gamma^2} \right)^{-\sigma} G_p^{-N}.
\]

Applying the formulae (1) and (2) repeatedly and substituting the indices \( i_\ell \) for \( i_\ell - 1 \) \((\ell = 1, \ldots, N)\), this is equal to

\[
(\zeta_4^{m_1^2 + \cdots + m_N^2 \gamma \left( i_1 i_2 \right) \cdots [i_N - i_N] N \left( 2i_1^N - \zeta_4^{-2i_1^N} \right) \left( \sum_{\gamma \in \mathbb{Z}/2\mathbb{Z}} \zeta_4^{\gamma^2} \right)^{-\sigma} \left( \sum_{\gamma \in \mathbb{Z}/2\mathbb{Z}} \zeta_4^{-\gamma^2} \right)^{-\sigma} G_p^{-N}.
\]

Thus

\[
\tau_{4p}(L(a, b), \Theta) = C \sum_{\pm} \zeta_{16p}^{\pm i \gamma i y B y + 2^i (e_1 \pm e_N) y} \sum_{j \in (\mathbb{Z}/2\mathbb{Z})^N} \zeta_{4p}^{i j B j + 2^i u_\pm j}
\]

\[
= C \sum_{\pm} \zeta_{16p}^{\pm i \gamma i y B y + 2^i (e_1 \pm e_N) y} \sum_{\gamma \in (\mathbb{Z}/2\mathbb{Z})^N} \zeta_{4p}^{i \gamma B \gamma + 2^i u_\pm \gamma} \sum_{k \in (\mathbb{Z}/p\mathbb{Z})^N} \zeta_p^{i k B k + 2^i u_\pm k},
\]

where we put \( i = i_1, \ldots, i_N \), \( y = x + i_\ell (1, \ldots, 1) \), and

\[
C = \frac{(\zeta_4^{m_1^2 + \cdots + m_N^2 \gamma \left( i_1 i_2 \right) \cdots [i_N - i_N] N \left( 2i_1^N - \zeta_4^{-2i_1^N} \right) \left( \sum_{\gamma \in \mathbb{Z}/2\mathbb{Z}} \zeta_4^{\gamma^2} \right)^{-\sigma} \left( \sum_{\gamma \in \mathbb{Z}/2\mathbb{Z}} \zeta_4^{-\gamma^2} \right)^{-\sigma} G_p^{-N}.
\]
Comparing (6) and (7), we get

\[ (7) \quad \tau_4(L(a, b), \Theta)|_{\zeta_{16}^{p^*}} = \delta C' \zeta_{16}^{(\mu B y + 2^i (e_1 + e_{N}) y)} \sum_{\gamma \in (\mathbb{Z}/2\mathbb{Z})^N} \zeta_4^{p(\gamma B y + 2^i u_\gamma)} \]

where we put

\[ \zeta \]

Further, since [KiMe91, Example 8.31] says that \( B_\gamma \) whose entries are integers.

In a similar way, we can show that

\[ (9) \quad \tau_4(L(a, b), \Theta) = \delta C' \zeta_{16}^{(\mu B y + 2^i (e_1 + e_{N}) y)} \sum_{\gamma \in (\mathbb{Z}/2\mathbb{Z})^N} \zeta_4^{p(\gamma B y + 2^i u_\gamma)} \]

Comparing (6) and (7), we get

\[ (8) \quad \tau_4(L(a, b), \Theta) = \delta (a/p) \zeta_{16}^{p^*} \sum_{k \in (\mathbb{Z}/p\mathbb{Z})^N} \zeta_k^{i b \cdot u_\delta} \]

Since \( B \) is symmetric, there exists \( P \in \text{GL}(N, \mathbb{Z}/p\mathbb{Z}) \) such that \( t^{P BP} \) is diagonal. Thus by putting \( t^{P BP} = (b_1) \oplus \cdots \oplus (b_N), P u_\delta = (c_1, \ldots, c_N) \), we see

\[ (9) \quad \sum_{k \in (\mathbb{Z}/p\mathbb{Z})^N} \zeta_k^{i b \cdot u_\delta} = \sum_{k' \in (\mathbb{Z}/p\mathbb{Z})^N} \zeta_k^{i k' \cdot t^{P BP} k'} \]

where we put \( k = P k' \) in the first equality.

Substituting (9) into (8) and using Lemma 4 below, we get

\[ \tau_4(L(a, b), \Theta) = \left( \frac{a}{p} \right) \zeta_{16}^{p^*} \sum_{k \in (\mathbb{Z}/p\mathbb{Z})^N} \zeta_k^{i k' \cdot t^{P BP} k'} \tau_4(L(a, b), \Theta)|_{\zeta_{16}^{p^*}} \]

Further, since [KiMe91, Example 8.31] says that

\[ \tau_4(L(a, b), \Theta) = \zeta_{16}^{3 \mu(L(a, b), \Theta)} \]
we obtain the required formula.

**Lemma 3.** Let $d \in (\mathbb{Z}/2\mathbb{Z})^N$ be a non-zero vector satisfying $Bd \equiv 0 \pmod{2}$. Recall that $u_{\pm} = \frac{1}{2}(By + e_1 \pm e_N)$, and its entries are integers. Put a sign $\delta$ by

$$\delta = \zeta_4^{t'd Bd + 2'u_+ d} = (-1)^{t'd Bd + t'u_+ d}.$$  

Then it holds

$$\sum_{\gamma \in (\mathbb{Z}/2\mathbb{Z})^N} \zeta_4^{p(\gamma B + 2'u_- \gamma)} = 0.$$  

**Proof.** Since $(t'd Bd + 2'tu_+ d) - (t'd Bd + 2'tu_- d) \equiv 2'te_N d \equiv 2 \pmod{4}$, we can check $\zeta_4^{t'd Bd + 2'u_- d} = -1$. Thus

$$\sum_{\gamma \in (\mathbb{Z}/2\mathbb{Z})^N} \zeta_4^{p(\gamma B + 2'u_- \gamma)} = \sum_{\gamma' \in (\mathbb{Z}/2\mathbb{Z})^N} \zeta_4^{p(\gamma' B + 2'u_- \gamma')}$$

$$= \zeta_4^{p(d Bd + 2'u_- d)} \sum_{\gamma' \in (\mathbb{Z}/2\mathbb{Z})^N} \zeta_4^{p(\gamma' B + 2'u_- \gamma')}$$

$$= - \sum_{\gamma \in (\mathbb{Z}/2\mathbb{Z})^N} \zeta_4^{p(\gamma B + 2'u_- \gamma)},$$

where we put $\gamma' = \gamma - d$ in the first equality. Thus we obtain the lemma.

**Lemma 4.** With the above notation, the following equality holds:

$$3\sigma - \text{tr}B = t'e_1B^{-1}e_1 + t'e_NB^{-1}e_N - 12s(b, a).$$

**Proof.** In the same fashion as [KiMe91, §2.3], we can derive

$$3\{\text{sgn}(a_0a_1) + \cdots + \text{sgn}(a_{N-1}a_N)\} - \text{tr}B = t'e_1B^{-1}e_1 + t'e_NB^{-1}e_N - 12s(b, a),$$

where

$$a_0 = 1, \quad a_i = \det \begin{pmatrix} m_1 & 1 \\ 1 & m_2 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \cdots & \ddots \\ \cdots & \cdots & \ddots & \ddots & \cdots \\ 1 & \cdots & \cdots & \cdots & 1 \end{pmatrix} (i = 1, \ldots, N).$$

Since it is known that $\sigma = \text{sgn}(a_0a_1) + \cdots + \text{sgn}(a_{N-1}a_N)$ (for example, see [Mac46, Theorem 34.3]), we obtain the lemma.

## 3. Proof of Corollary 2

In this section we prove the Corollary 2. For a positive integer $p$, an oriented closed 3-manifold $M$, a spin structure $\Theta$ of $M$, and $y \in H_2(M, \mathbb{Z}/2\mathbb{Z})$, [Rob97, Theorem 3.6] says that

$$TV_{4p}(M, \Theta, y) = \tau_4p \tau_4 \tau_4(M, \Theta + D(y)),$$

where $D(y) \in H^1(M, \mathbb{Z}/2\mathbb{Z})$ is the Poincaré dual of $y$. Remark that the set of spin structures is affinely isomorphic to $H^1(M, \mathbb{Z}/2\mathbb{Z})$, so $\Theta + D(y)$ makes sense. Let
\( \Theta, \Theta' \) be the distinct spin structures of the lens space \( L(a, b) \). Using this formula and Theorem 1, we obtain

\[
TV_{4p}(L(a, b), \Theta, 0) = |\tau_{4p}(L(a, b), \Theta)|^2 = \frac{2}{|\zeta_{8p} - \zeta_{8p}^{-1}|^2},
\]

and

\[
TV_{4p}(L(a, b), \Theta, y) = \tau_{4p}(L(a, b), \Theta)\tau_{4p}(L(a, b), \Theta')
\]

\[
= \frac{2}{|\zeta_{8p} - \zeta_{8p}^{-1}|^2} \zeta_{8p}^{-3p^*(\mu - \mu')} \zeta_{8p}^{-8\delta - \delta'}
\]

for \( y \neq 0 \), where \( \delta' \) is the “\( \Theta' \)-version” of \( \delta \), similarly defined as (10). Now we can check by definition that \( \delta' = (-1)^{a/2}\delta \). This completes the proof of the corollary.

**References**


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