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Abstract

Sorin Popa initiated the study of Polish groups which are embeddable into the unitary group of a separable finite von Neumann algebra. Such groups are called of finite type or said to belong to the class $\mathscr{U}_{\rm fin}$. We give necessary and sufficient conditions for Polish groups to be of finite type, and construct exmaples of such groups from I_∞ and II_∞ von Neumann algebras. We also discuss permanence properties of finite type groups under various algebraic operations. Finally we close the paper with some questions concerning Polish groups of finite type.

Keywords bi-invariant metric, class $\mathscr{U}_{\rm fin}$, finite type group, Polish group, positive definite function, SIN-group, II_1 factor

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Contents

1 Introduction

In this paper we consider the following problem. Denote by $\mathcal{U}(M)$ the unitary group of a von Neumann algebra M.

Problem 1.1.
Determine the necessary and sufficient condition for a Polish group G to be isomorphic as a topological group onto a strongly closed subgroup of some $\mathcal{U}(M)$, where M is a separable finite von Neumann algebra.

S. Popa defined a Polish group to be of finite type if it has this property. Denote by $\mathscr{U}_{\text{fin}}$ the class of all finite type Polish groups. He initiated the study of this class in an attempt to enrich the study of rapidly developping cocycle superrigidity theory (cf. [6, 14, 16]). In particular, he proposed in [16] the problem of studying and characterizing the class $\mathscr{U}_{\text{fin}}$.

Secondly, this problem is motivated from our previous work [1] on infinite-dimensional Lie algebras associated with such groups: Let M be a finite von Neumann algebra on a Hilbert space $\mathcal{H}$. Let G be a strongly closed subgroup of $\mathcal{U}(M)$ and $\overline{M}$ be a set of all densely defined closed operators on $\mathcal{H}$ which are affiliated to M. It is proved that the set

$$\mathrm{Lie}(G) := \{A^* = -A \in \overline{M}; e^{tA} \in G \text{ for all } t \in \mathbb{R}\}$$

is a complete topological Lie algebra with respect to the strong resolvent topology (see also the related work of D. Beltita [3]). Since these Lie algebras turn out to be non-locally convex in general when M is non-atomic, they are quite exotic as a Lie algebra and their properties are still unknown. Therefore it would be interesting to find non-trivial examples of such groups.

We give an answer in Theorem 2.7 to Problem 1.1 by the aid of positive definite functions on groups and their GNS representations, and charactelize locally compact groups or amenable Polish groups of finite type via compatible bi-invariant metrics (Theorem 2.20 and Theorem 2.22). Combining with Popa's result [16], Theorem 2.7 gives a necessary and sufficient condition for a Polish group to be isomorphic onto a closed subgroup of the unitary group of a separable II_1 factor. We then give examples of Polish groups G of finite type using noncommutative integration of E. Nelson [15]. Finally we discuss some hereditary properties of finite type groups and pose some questions concerning Polish groups of finite type.

Notation. In this paper we often say a von Neuman algebra M is *separable* if it has a separable predual, especially when the Hilbert space on which M acts is implicit. This is known to be equivalent to the condition that M has a faithful representation on a separable Hilbert space. We denote by $\mathrm{Proj}(M)$ the lattice of all projections in M. A von Neumann algebra is said to be *finite* if it admits no non-unitary isometry. When we consider a group G, its identity is denoted as e_G. However, we also use 1 as the identity when we consider a concrete subgroup of the unitary group of a von Neumann algebra. We always regard the unitary group of a von Neumann algebra as a topological group with the strong operator topology.

2 Polish Groups of Finite Type and its Characterization

In this section, we characterize Polish groups of finite type via positive definite functions. We then characterize when locally compact groups or amenable Polish groups are of finite type via compatible bi-invariant metrics. To this end, we review notions of SIN-groups, bi-invariant metrics and unitary representability.

2.1 Polish Groups of Finite Type

Recall that a Polish space is a separable completely metrizable topological space, and a Polish group is a topological group whose topology is Polish.

We now introduce finite type groups after Popa [16].

Definition 2.1. A Hausdorff topological group is called of *finite type* if it is isomorphic as a topological group onto a closed subgroup of the unitary group of a finite von Neumann algebra.

Remark 2.2. Popa [16] requires the topological group of finite type to be Polish, whereas our definition of finiteness does not require any countability. We will show in Theorem 2.7 that a Polish group G of finite type in our sense coincides with Popa's definition of finite type group. That is, G is isomorphic onto a closed subgroup of the unitary group of a finite von Neumann algebra acting on a separable Hilbert space.

All of second countable locally compact Hausdorff groups, the unitary group of a von Neumann algebra acting on a separable Hilbert space are Polish groups. Furthermore, separable Banach spaces are Polish groups as an additive group. We denote the class of all Polish groups of finite type by $\mathscr{U}_{\text{fin}}$.

Note that since a von Neumann algebra is finite if and only if its unitary group is complete with respect to the left uniform structure, Polish groups of finite type are necessarily complete. Thus we have the following simple consequence.

Proposition 2.3. *The unitary group of a von Neumann algebra M acting on a separable Hilbert space is of finite type if and only if M is finite.*

Another examples of Polish groups of finite type are given later.

2.2 Positive Definite Functions

A complex valued function f on a Hausdorff topological group G is called *positive definite* if for all $g_1, \cdots, g_n \in G$ and for all $c_1, \cdots, c_n \in \mathbb{C}$,

$$\sum_{i,j=1}^{n} \bar{c_i} c_j f(g_i^{-1} g_j) \geq 0.$$

Moreover if a complex valued function f is invariant under inner automorphisms, that is

$$f(hgh^{-1}) = f(g), \qquad \forall g, h \in G,$$

then f is called *a class function.*

It is well-known that there is an one-to-one correspondence between the set of all continuous positive definite functions on a topological group and the set of unitary equivalence classes of all cyclic unitary representations of it. more precisely, for each continuous positive definite function f on a topological group G, there exists a triple $(\pi_f, \mathcal{H}_f, \xi_f)$ consisting of a cyclic unitary representation π_f in a Hilbert space $\mathcal{H}_f$ and a cyclic vector ξ_f in $\mathcal{H}_f$ such that

$$f(g) = \langle \xi_f, \pi_f(g) \xi_f \rangle, \qquad g \in G,$$

and this triple is unique up to unitary equivalence. This triple is called *the GNS triple* associated to f. Note that if G is separable, then so is $\mathcal{H}_f$.

The GNS triple is of the following form for each continuous positive definite class function.

Lemma 2.4. *Let f be a continuous positive definite class function on a topological group G and $(\pi, \mathcal{H}, \xi)$ be its GNS triple. Then the von Neumann algebra M generated by $\pi(G)$ is finite and linear functional*

$$\tau(x) := \langle \xi, x\xi \rangle, \qquad x \in M,$$

is faithful normal tracial state on M. In particular M is countably decomposable.

Proof. It is clear that τ is a normal state on M. Since f is a class function, it is easy to see that τ is tracial on the strongly dense *-subalgebra of M spanned by $\pi(G)$. Therefore by normality, τ is tracial on M. Therefore we have only to check the faithfulness of τ. Assume $\tau(x^*x) = 0$. Since τ is a trace, we have

$$\|x\pi(g)\xi\|^2 = \tau(\pi(g)^* x^* x \pi(g)) = 0,$$

for all $g \in G$. By the cyclicity of ξ, x must be 0. $\square$

Example 2.5 (I. J. Schoenberg [17])**.** Let $\mathcal{H}$ be a complex Hilbert space. Note that $\mathcal{H}$ is an additive group. Then a function f defined by $f(\xi) := e^{-\|\xi\|^2}$ $(\xi \in \mathcal{H})$ is a positive definite (class) function on $\mathcal{H}$.

Example 2.6 (I. J. Schoenberg [17])**.** For all $1 \leq p \leq 2$ a function f_p defined by $f_p(a) := e^{-\|a\|_p^p}$ $(a \in l^p)$ is a positive definite (class) function on a separable Banach space l^p.

For more details about positive definite class functions, see [10].

2.3 The First Characterization

We now characterize Polish groups of finite type.

Theorem 2.7. *For a Polish group G the following are equivalent.*

(i) G is of finite type.

(ii) G is isomorphic as a topological group onto a closed subgroup of the unitary group of a finite von Neumann algebra acting on a separable Hilbert space.

(iii) A family $\mathcal{F}$ of continuous positive definite class functions on G generates a neighborhood basis of the identity e_G of G. That is, for each neighborhood V at the identity, there are functions $f_1, \cdots, f_n \in \mathcal{F}$ and open sets $\mathcal{O}_1, \cdots, \mathcal{O}_n$ in $\mathbb{C}$ such that

$$e_G \in \bigcap_{i=1}^{n} f_i^{-1}(\mathcal{O}_i) \subset V.$$

(iv) There exists a positive, continuous positive definite class function which generates a neighborhood basis of the identity of G.

(v) A family $\mathcal{F}$ of continuous positive definite class functions on G separates the identity of G and closed subsets A with $A \not\ni e_G$. That is, for each closed subset A with $A \not\ni e_G$, there exists a continuous positive definite class function $f \in \mathcal{F}$ such that

$$\sup_{x\in A} |f(x)| < |f(e_G)|.$$

(vi) There exists a positive continuous positive definite class function which separates the identity of G and closed subsets A with $A \not\ni e_G$.

Proof. (iv)⇔(vi)⇒(v)⇒(iii) and (ii)⇒(i) are trivial.

(iii)⇒(ii). Since G is first countable, there exists a countable subfamily $\{f_n\}_n$ of $\mathcal{F}$ which generates a neighborhood basis of the identity of G. Let $(\pi_n, \xi_n, \mathcal{H}_n)$ be the GNS triple associated to f_n and M_n be a von Neumann algebra generated by $\pi_n(G)$. Since each M_n is finite, the direct sum $M := \bigoplus_n M_n$ is also finite and acts on a separable Hilbert space $\mathcal{H} := \bigoplus_n \mathcal{H}_n$ (see the remark above Lemma 2.4). Put $\pi := \bigoplus_n \pi_n$, then π is an embedding of G into $\mathcal{U}(M)$. The image of π is closed in $\mathcal{U}(M)$, as both G and $\mathcal{U}(M)$ are Polish.

(i)⇒(iii). Let π be an embedding of G into the unitary group of a finite von Neumann algebra M. Since each finite von Neumann algebra is the direct sum of countably decomposable finite von Neumann algebras, we can take of a family of countably decomposable finite von Neumann algebras $\{M_i\}_{i\in I}$ with $M = \bigoplus_{i\in I} M_i$. In this case π is also of the form $\pi = \bigoplus_{i\in I} \pi_i$, where each $\pi_i : G \to \mathcal{U}(M_i)$ is a continuous group homomorphism. Let τ_i be a faithful normal tracial state on M_i and $(\rho_i, \xi_i, \mathcal{H}_i)$ be its GNS triple as a C^*-algebra. Here each ρ_i is an isomorphism from M_i into $\mathbb{B}(\mathcal{H}_i)$ and

$$\tau_i(x) = \langle \xi_i, \rho_i(x)\xi_i\rangle, \qquad x \in M_i,$$

holds. Now set $f_i := \tau_i \circ \pi_i$. Then each f_i is a continuous positive definite class functions on G and $\{f_i\}_{i\in I}$ generates a neighborhood basis of the identity e_G of G.

(iii)⇒(iv). Let $\{f_n\}_n$ be a countable family of continuous positive definite class functions generating a neighborhood basis of the identity of G with $f_n(e_G) = 1$. Set

$$\begin{aligned} f'_n(g) &:= e^{\mathrm{Re}(f_n(g))-1} \\ &= e^{-1} \sum_{k=0}^{\infty} \frac{1}{k!} \left[\mathrm{Re}(f_n(g))\right]^k, \qquad g \in G, \end{aligned}$$

then $\{f'_n\}_n$ is not only a family of continuous positive definite class functions generating a neighborhood basis of the identity of G with $f'_n(e_G) = 1$ but also a family of positive functions. Define a positive, continuous positive definite class function by $f(g) := \sum_n f'_n(g)/2^n$ ($g \in G$). It is easy to see that f generates a neighborhood basis of the identity of G. $\square$

Remark 2.8. The proof of the above theorem is inspired by Theorem 2.1 of S. Gao [8].

Remark 2.9. Popa (Lemma 2.6 of [16]) showed that a Polish group G is of finite type if and only if it is isomorphic onto a closed subgroup of the unitary group of a separable II_1 factor. Therefore Theorem 2.7 gives a necessary and sufficient condition for a Polish group to be isomorphic onto a closed subgroup of the unitary group of a separable II_1 factor.

2.4 SIN-groups and Bi-Invariant Metrics

To discuss further properties of finite type groups, we consider the following notions, say SIN-groups, bi-invariant metrics and unitarily representability.

A neighborhood V at the identity of a topological group G is called *invariant* if it is invariant under all inner automorphisms, that is, $gVg^{-1} = V$ holds for all $g \in G$. *A SIN-group* is a topological group which has a neighborhood basis of the identity consisting of invariant identity neighborhoods. Note that a locally compact Hausdorff SIN-group is unimodular.

A bi-invariant metric on a group G is a metric d which satisfies

$$d(kg, kh) = d(gk, hk) = d(g, h), \quad \forall g, h, k \in G.$$

It is known that a first countable Hausdorff topological group is SIN if and only if it admits a compatible bi-invariant metric.

As Popa [16] pointed out, one of the most important fact of Polish groups of finite type is an existence of a compatible bi-invariant metric.

Lemma 2.10. *Each Polish group of finite type has a compatible bi-invariant metric. In particular, it is SIN.*

Proof. It is enough to show that for every finite von Neumann algebra M acting on a separable Hilbert space $\mathcal{H}$ the unitary group $\mathcal{U}(M)$ has a compatible bi-invariant metric. For this let τ be a faithful normal tracial state on M. Then a metric d defined by

$$d(u, v) := \tau((u - v)^*(u - v))^{\frac{1}{2}}, \quad u, v \in \mathcal{U}(M),$$

is a compatible bi-invariant metric on $\mathcal{U}(M)$. □

2.5 Unitary Representability

A Hausdorff topological group is called *unitarily representable* if it is isomorphic as a topological group onto a subgroup of the unitary group of a Hilbert space. All locally compact Hausdorff groups are unitarily representable via the left regular representation. It is clear that a Polish group of finite type is necessarily unitarily representable. The following characterization of unitary representability has been considered by specialists and can be seen in e.g., Gao [8].

Lemma 2.11. *For a Polish group G the following are equivalent.*

(i) G is unitarily representable.

(ii) There exists a positive, continuous positive definite function which separates the identity of G and closed subsets A with $A \not\ni e_G$.

2.6 Simple Examples

All of the following examples are well-known. The first three examples are locally compact groups.

Example 2.12. Any compact metrizable group is a Polish group of fintie type. This follows from the Peter-Weyl theorem.

Example 2.13. Any abelian second countable locally compact Hausdorff group is a Polish group of finite type. Indeed its left regular representation is an embedding into the unitary group of a Hilbert space and the von Neumann algebra generated by its image is commutative (in particular, finite).

Example 2.14. Any countable discrete group is a Polish group of finite type. For its left regular representation is an embedding into the unitary group of a finite von Neumann algebra.

The following two examples suggest there are few other examples of locally compact groups of finite type.

Example 2.15. Let $G := \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{K}) \ ; \ x \in \mathbb{K}^{\times}, y \in \mathbb{K} \right\}$ be the $ax+b$ group, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. By easy computations, we have

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} x & -bx+ay+b \\ 0 & 1 \end{pmatrix},$$

so that the conjugacy class $C\left(\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\right)$ of $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ is

$$C\left(\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\right) = \begin{cases} \left\{ \begin{pmatrix} x & \sharp \\ 0 & 1 \end{pmatrix} \ ; \ \sharp \in \mathbb{K} \right\} & (x \neq 1), \\ \left\{ \begin{pmatrix} 1 & \sharp \\ 0 & 1 \end{pmatrix} \ ; \ \sharp \in \mathbb{K}^{\times} \right\} & (x = 1, \ y \neq 0), \\ \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} & (x = 1, \ y = 0). \end{cases}$$

Thus for each $n \in \mathbb{N}$ there exists a matrix $h_n \in G$ such that $h_n g_n h_n^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, where $g_n := \begin{pmatrix} 1 & 1/n \\ 0 & 1 \end{pmatrix}$. Clearly, $g_n \to 1$ and $h_n g_n h_n^{-1} \not\to 1$. This implies that $ax+b$ group does not admit a compatible bi-invariant metric. Hence it is not of finite type.

Example 2.16. The special linear group $SL(n,\mathbb{K})$ $(n \geq 2)$ is not of finite type since the map $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ is an embedding of $ax+b$ group into $SL(2,\mathbb{K})$. Thus the general linear group $GL(n,\mathbb{K})$ $(n \geq 2)$ is also not of finite type.

Next we consider abelian groups. Note that an abelian topological group is of finite type if and only if it is unitarily representable.

Example 2.17. Any separable Hilbert space is a Polish group of finite type. This follows from Example 2.5 and Theorem 2.7.

Example 2.18. A separable Banach space l^p $(1 \leq p \leq \infty)$ is a Polish group of finite type if and only if $1 \leq p \leq 2$. The "only if" part follows from Example 2.6 and Theorem 2.7, but the "if" part is non-trivial. For details, see [13].

Here is another counter example.

Example 2.19. Separable Banach space $C[0,1]$ of all continuous functions on the interval $[0,1]$ is a Polish group but not of finite type. For, since every separable Banach space is isometrically isomorphic to a closed subspace of $C[0,1]$, if $C[0,1]$ is of finite type, then any separable Banach space is a Polish group of finite type. But this is a contradiction to the previous example.

2.7 A Characterization for Locally Compact Groups

R. V. Kadison and I. Singer [12] proved that every connected locally compact Hausdorff SIN group is isomorphic as a topological group onto a topological group of the form $\mathbb{R}^n \times K$, where K is a compact Hausdorff group. Therefore such groups are of finite type. In this subsection, we show that SIN property is a necessary and sufficient condition for a locally compact group to be of finite type.

Theorem 2.20. *A second countable locally compact Hausdorff group is of finite type if and only if it is SIN.*

Proof. Let G be a second countable locally compact Hausdorff SIN-group, μ be the Haar measure on it and λ be the left-regular representation. For each compact invariant neighborhood U at the identity, we define a continuous positive definite function φ_U on G by

$$\varphi_U(g) := \langle \chi_U, \lambda(g)\chi_U \rangle = \mu\left(U \cap gU\right), \qquad g \in G.$$

Note that, for each $g,h,x \in G$, we have

$$h^{-1}x \in U \Leftrightarrow x \in hU = Uh \Leftrightarrow xh^{-1} \in U,$$

and

$$(gh)^{-1}x \in U \Leftrightarrow x \in ghU = gUh \Leftrightarrow xh^{-1} \in gU.$$

Also note that a locally compact SIN-group is unimodular. Thus we see that

$$\begin{aligned}\varphi_U(h^{-1}gh) &= \langle \lambda(h)\chi_U, \lambda(gh)\chi_U\rangle \\ &= \int_G \chi_U(h^{-1}x)\chi_U((gh)^{-1}x)d\mu(x) \\ &= \int_G \chi_U(xh^{-1})\chi_{gU}(xh^{-1})d\mu(x) \\ &= \int_G \chi_U(x)\chi_{gU}(x)d\mu(x) \\ &= \int_G \chi_U(x)\chi_U(g^{-1}x)d\mu(x) \\ &= \varphi_U(g).\end{aligned}$$

This implies φ_U is a class function. It is not hard to check that a family $\{\varphi_U\}_U$ generates a neighborhood basis of the identity of G. This completes the proof by Theorem 2.7. $\square$

Remark 2.21. K. Hofmann, S. Morris and M. Stroppel [11] proved that every totally disconnected locally compact Hausdorff group is SIN if and only if it is a strict projective limit of discrete groups.

2.8 A Characterization for Amenable Groups

Next, we also characterize (not necessarily locally compact) amenable Polish groups of finite type. Recall that a Hausdorff topological group G is amenable if LUCB(G) admits a left-translation invariant positive functional $m \in \mathrm{LUCB}(G)^*$ with $m(1) = 1$, where LUCB(G) is a complex Banach space of all left-uniformly continuous bounded functions on G. Such a m is called an *invariant mean.*

Theorem 2.22. *A unitarily representable amenable Polish group is of finite type if and only if it is SIN.*

Proof. Let G be a unitarily representable amenable Polish SIN-group and let f be a positive, continuous positive definite function on G which separates the identity of G and closed subsets A with $A \not\ni e_G$ (see Lemma 2.11). We can assume $f(e_G) = 1$. For each $x \in G$, we define a positive function $\Psi_{x,f} \in$ LUCB(G) by

$$\Psi_{x,f}(g) := f(g^{-1}xg), \qquad g \in G.$$

Let $m \in \mathrm{LUCB}(G)^*$ be an invariant mean. Put

$$\psi_f(x) := m(\Psi_{x,f}), \qquad x \in G,$$

then $\psi_f(x)$ is a continuous, positive, positive definite class function on G with $\psi_f(e_G) = 1$ and it separates the identity of G and closed subsets A with $A \not\ni e_G$ This completes the proof by Theorem 2.7. $\square$

Remark 2.23. The above proof is inspired by the proof of Theorem 2.13 of J. Galindo [7].

3 More Examples of Finite Type Groups

In this section we will give another examples of groups of finite type. To construct such examples we need to start not from finite von Neumann algeras, but from semifinite von Neumann algebras, say of type I_∞ or of type II_∞. In the end of this section we also review other known examples of Polish groups of finite type.

3.1 L^2-unitary groups $\mathcal{U}(M)_2$

Let M be a semifinite von Neumann algebra on a Hilbert space $\mathcal{H}$ equipped with a normal faithful semifinite trace τ. A densely defined, closed operator T on $\mathcal{H}$ is said to be *affiliated* to M if for all $u \in \mathcal{U}(M')$, $uTu^* = T$ holds. Denote by $\overline{M}$ the set of all densely defined, closed operators on $\mathcal{H}$ which are affiliated to M. Recall that $L^2(M,\tau)$ is a Hilbert space completion of the space $\mathfrak{n}_\tau := \{x \in M; \tau(x^*x) < \infty\}$ by the inner product

$$\langle x, y\rangle := \tau(x^*y), \qquad x, y \in \mathfrak{n}_\tau.$$

We define $||x||_2 := \tau(x^*x)^{\frac{1}{2}}$ for $x \in L^2(M,\tau)$.

Definition 3.1. We call $\mathcal{U}(M)_2 := \{u \in \mathcal{U}(M); 1 - u \in L^2(M,\tau)\}$ the L^2-*unitary group* of (M,τ).

Note that when M is not a factor, $\mathcal{U}(M)_2$ depends on the choice of τ too. In the sequel we show the following theorem.

Theorem 3.2. *Let M be a separable semifinite von Neumann algebra with a normal faithful semifinite trace τ. Then $\mathcal{U}(M)_2$ is a Polish group of finite type, where the topology is determined by the following metric d,*

$$d(u,v) := ||u - v||_2, \qquad u, v \in \mathcal{U}(M)_2.$$

To prove the theorem, we need some preparations. In the sequel we consider M to be represented on $\mathcal{H} = L^2(M,\tau)$ by left multiplication. Recall that a closed operator $T \in \overline{M}$ on $L^2(M,\tau)$ is called τ-*measurable* if for any $\varepsilon > 0$, there exists a projection $p \in M$ with $\mathrm{ran}(p) \subset \mathrm{dom}(T)$ and $\tau(1-p) < \varepsilon$. Note that $L^2(M,\tau)$ can be identified with the set of closed, densely defined and τ-measurable operators T such that

$$||T||_2^2 := \tau(|T|^2) = \int_0^\infty \lambda^2 d\tau(e(\lambda)) < \infty,$$

where $e(\cdot)$ is a spectral resolution of $|T| = (T^*T)^{\frac{1}{2}}$ and $T = u|T|$ is the polar decomposition of T (for more details about non-commutative integration, see vol II of [18]).

Lemma 3.3. *Let M be a semifinite von Neumann algebra with a normal faithful semifinite trace τ. $\mathcal{U}(M)_2$ is a topological group.*

Proof. This can be shown directly, using the equalities:

$$||x^*||_2 = ||x||_2, \quad ||uxv||_2 = ||x||_2,$$

for all $x \in L^2(M,\tau)$ and $u, v \in \mathcal{U}(M)$. □

Lemma 3.4. *Let M be a semifinite von Neumann algebra with a normal faithful semifinite trace τ. Let U be a densely defined closed τ-measurable operator on $L^2(M,\tau)$ affiliated to M. Then* $\mathrm{dom}(U) \cap M$ *is dense in $L^2(M,\tau)$.*

Proof. Let $\varepsilon > 0$. Let $\xi \in L^2(M,\tau)$. Since $M \cap L^2(M,\tau)$ is dense, there exists $\xi_0 \in M \cap L^2(M,\tau)$ such that $||\xi - \xi_0||_2 < \varepsilon$. On the other hand, the measurability of U implies the existence of an increasing sequence $\{p_n\}_{n=1}^{\infty}$ of projections in M such that $p_n L^2(M,\tau) \subset \mathrm{dom}(U)$ for all n and $p_n \nearrow 1$ strongly. Therefore there exists $n_0 \in \mathbb{N}$ such that

$$||\xi_0 - p_{n_0}\xi_0||_2 < \varepsilon.$$

By the choice of ξ_0, $p_{n_0}\xi_0 \in \mathrm{dom}(U) \cap M$ and

$$\begin{aligned} ||\xi - p_{n_0}\xi_0||_2 &\leq ||\xi - \xi_0||_2 + ||\xi_0 - p_{n_0}\xi_0||_2 \\ &\leq \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, it follows that $\mathrm{dom}(U) \cap M$ is dense in $L^2(M,\tau)$. □

Lemma 3.5. *Let M be a semifinite von Neumann algebra with a normal faithful semifinite trace τ. d is a complete metric on $\mathcal{U}(M)_2$.*

Proof. Suppose $\{u_n\}_{n=1}^{\infty}$ is a d-Cauchy sequence in $\mathcal{U}(M)_2$. Since $L^2(M,\tau)$ is complete, there exists $V \in L^2(M,\tau)$ such that $||(1-u_n) - V||_2 \to 0$. Define $U := 1 - V$. Then $||U - u_n||_2 \to 0$. We show that U is bounded and moreover $U \in \mathcal{U}(M)_2$. Since U is closed and $\mathrm{dom}(U) \cap M$ is dense by Lemma 3.4, to prove the boundedness of U it suffices to show that U is isometric on $\mathrm{dom}(U) \cap M$. Let $\xi \in \mathrm{dom}(U) \cap M$. Since ξ is bounded, we have

$$\begin{aligned} ||(U-u_n)\xi||_2^2 &= \tau(\xi^*(U-u_n)^*(U-u_n)\xi) \\ &= \tau((U-u_n)\xi\xi^*(U-u_n)^*) \\ &\leq ||\xi||^2 \tau((U-u_n)(U-u_n)^*) \\ &= ||\xi||^2 ||U-u_n||_2^2 \to 0, \end{aligned}$$

which implies

$$||U\xi||_2 = \lim_{n\to\infty} ||u_n\xi||_2 = ||\xi||_2,$$

for all $\xi \in \mathrm{dom}(U) \cap M$. Therefore $U|_{\mathrm{dom}(U)\cap M}$ is isometric and U is bounded. Since $||U^* - u_n^*||_2 = ||U - u_n||_2$, it holds that U^* is an isometry too, which means U is unitary. Finally, it is clear that $U = 1 - V \in \mathcal{U}(M)_2$. □

Proof of Theorem 3.2. Since M is separable, the separability of $\mathcal{U}(M)_2$ follows from the separability of $L^2(M,\tau)$. Therefore by Lemma 3.5, $\mathcal{U}(M)_2$ is a Polish group. By Schoenberg's theorem (see Example 2.5),

$$\varphi(u) := e^{-||1-u||_2^2}, \qquad u \in \mathcal{U}(M)_2,$$

is a continuous, positive definite class function on $\mathcal{U}(M)_2$. It is easy to see that φ generates a neighborhood basis of the identity of $\mathcal{U}(M)_2$. Therefore the claim follows from Theorem 2.7. ☐

Remark 3.6. *$\mathcal{U}(M)_2'' = M$.*

Proof. Clearly $\mathcal{U}(M)_2'' \subset M$. Let p be a finite projection in M. Then $2p \in L^2(M,\tau)$ and $1-2p \in \mathcal{U}(M)_2$. Therefore $p \in \mathcal{U}(M)_2''$. Since M is semifinite, M is generated by finite projections. Therefore $\mathcal{U}(M)_2'' = M$. ☐

When $M = \mathbb{B}(\mathcal{H})$, $\mathcal{U}(M)_2$ is the well-known example of a Hilbert-Lie group and is denoted as $\mathcal{U}(\mathcal{H})_2$.

3.2 Non-isomorphic Properties of $\mathcal{U}(M)_2$

Let $\mathcal{H}$ be an infinite dimensional Hilbert space. We show that when M is a II_∞ factor and N is a finite von Neumann algebra, then $\mathcal{U}(M)_2$, $\mathcal{U}(\mathcal{H})_2$ and $\mathcal{U}(N)$ are mutually non-isomorphic. In this subsection, no separability assumptions are required.

Proposition 3.7. *Let M be a II_∞ factor. Then $\mathcal{U}(M)_2$ is not isomorphic onto $\mathcal{U}(\mathcal{H})_2$.*

Proof. Let τ be a normal faithful semifinite trace on M, Tr be the usual operator trace on $\mathcal{H}$. We denote their corresponding trace 2-norms by $||\cdot||_{2,\tau}$ and $||\cdot||_{2,\mathrm{Tr}}$, respectively. We prove the claim by contradiction. Suppose there exists a topological group isomorphism $\varphi : \mathcal{U}(M)_2 \to \mathcal{U}(\mathcal{H})_2$. Let p be a nonzero finite-rank projection in $\mathbb{B}(\mathcal{H})$. Then $1-2p \in \mathcal{U}(\mathcal{H})_2$ and let

$$q := \frac{1}{2}(1-\varphi^{-1}(1-2p)).$$

It is easy to see that $q \in L^2(M,\tau)$ is a nonzero finite projection in M. Let $k \in \mathbb{N}$. Since M is a II_∞ factor, there exists a projection $0 < q_k \le q$ in M such that $\lim_{k\to\infty} \tau(q_k) = 0$. Then define $p_k := \frac{1-\varphi(1-2q_k)}{2}$. Since

$$||q_k||_{2,\tau}^2 = \tau(q_k) \to 0 \ (k \to \infty),$$

$1-2q_k \to 1$ holds in $\mathcal{U}(M)_2$, which in turn means

$$1-2p_k = \varphi(1-2q_k) \to \varphi(1) = 1 \ \text{ in } \mathcal{U}(\mathcal{H})_2.$$

However, since the topology of $\mathcal{U}(\mathcal{H})_2$ is given by the operator trace 2-norm, it holds that

$$2 \le ||2p_k||_{2,\mathrm{Tr}} = ||1-(1-2p_k)||_{2,\mathrm{Tr}} \to 0 \ (k \to \infty).$$

This is clearly a contradiction. Therefore $\mathcal{U}(M)_2 \ncong \mathcal{U}(\mathcal{H})_2$. ☐

Proposition 3.8. *Let M be a II_∞ factor, N be a finite von Neumann algebra. Then $\mathcal{U}(M)_2$ is not isomorphic onto $\mathcal{U}(N)$.*

Before going to the proof, recall that for projections p, q in a von Neumann algebra M, we write $p \sim q$ (resp. $p \prec q$) if p is equivalent (resp. subequivalent) to q in Murray-von Neumann sense.

Proof. Suppose there exists a topological group isomorphism $\varphi : \mathcal{U}(N) \to \mathcal{U}(M)_2$. We first show that N must be a factor. If $u \in Z(\mathcal{U}(M)_2)$ be an element of $\mathcal{U}(M)_2$ which commutes with every elements in $\mathcal{U}(M)_2$. Then for any finite projection $p \in M$, $u(1-2p) = (1-2p)u$ holds. Therefore u commutes with all finite projections in M. Since M is generated by its finite projections, $u \in Z(M) = \mathbb{C}1$ holds. Thus $Z(\mathcal{U}(M)_2)$ is equal to $\{e^{it}1; t \in [0, 2\pi)\}$. Since φ maps $Z(\mathcal{U}(N)) = \mathcal{U}(Z(N))$ onto $Z(\mathcal{U}(M)_2)$, the center $Z(N)$ must be $\mathbb{C}1$.

Let τ_M be a normal faithful semifinite trace on M, τ_N be a normal faithful tracial state on N.

Step 1 If p and q are equivalent projections in N, then $p' \sim q'$ in M, where

$$p' := \frac{1 - \varphi(1-2p)}{2}, \qquad q' := \frac{1 - \varphi(1-2q)}{2},$$

finite are projections in M.

Since N is finite, there exists a unitary $v \in \mathcal{U}(N)$ such that $u^*pu = q$ holds. Then

$$\begin{aligned} q' &= \frac{1 - \varphi(1 - 2u^*pu)}{2} \\ &= \frac{1 - \varphi(u^*(1-2p)u)}{2} \\ &= \varphi(u)^* \cdot \frac{1 - \varphi(1-2p)}{2} \cdot \varphi(u) \\ &\sim p'. \end{aligned}$$

Step 2 The map $F : [0, 1] \to [0, \infty)$ defined by

$$F(t) := \tau_M \left(\frac{1 - \varphi(1 - 2p_t)}{2} \right),$$

where p_t is an arbitrary projection N with $\tau_N(p_t) = t \in [0, 1]$, is well-defined and is continuous.

If p_t and q_t are projections in N with $\tau_N(p_t) = \tau_N(q_t) = t$, then by the factoriality of N, we have $p_t \sim q_t$. Therefore by Step 1,

$$\tau_M \left(\frac{1 - \varphi(1 - 2p_t)}{2} \right) = \tau_M \left(\frac{1 - \varphi(1 - 2q_t)}{2} \right).$$

Thus F is well-defined. Next we prove that F is left-continuous. Suppose $t_n \nearrow t$ in $[0, 1]$. Let $\{p_n\}_{n=1}^\infty$ be a sequence of projections in N such that

$\tau_N(p_n) = t_n$. Since $\{t_n\}$ is increasing, we have $p_1 \prec p_2$. Therefore there exists $p_2' \in \mathrm{Proj}(N)$ with $p_1 \leq p_2' \sim p_2$. Since $p_2 \prec p_3$, there exists $p_3' \in \mathrm{Proj}(N)$ such that $p_2' \leq p_3' \sim p_3$ holds. Continuing this, we get an increasing sequence $\{p_n'\}_{n=1}^{\infty}$ of projections in N such that $p_n' \sim p_n$ for all n. Let

$$p' := \text{s-}\lim_{n\to\infty} p_n'.$$

The limit exists because $\{p_n'\}$ is increasing. Clearly $\tau(p') = t$ holds. Then it holds that

$$1 - 2p_n' \to 1 - 2p' \text{ in } \mathcal{U}(N).$$

Therefore

$$F(t_n) = \tau_M\left(\frac{1-\varphi(1-2p_n')}{2}\right) \to \tau_M\left(\frac{1-\varphi(1-2p')}{2}\right) = F(t).$$

Hence F is left-continuous. Similarly, we can prove that F is right-continuous. Therefore F is continuous.

Step 3 Now we shall deduce a contradiction. By Step 2, $F([0,1])$ is compact and hence bounded. Therefore there exists $c > \max_{t\in[0,1]} F(t)$. On the other hand, as M is of type II_∞, there exists a projection q with $\tau_M(q) = c$. But then we have

$$F(t) = c, \quad t := \tau_N(p),$$

where

$$p := \frac{1-\varphi^{-1}(1-2q)}{2} \in \mathrm{Proj}(N).$$

This is a contradiction and we get $\mathcal{U}(M)_2 \ncong \mathcal{U}(N)$. $\square$

Proposition 3.9. *Let N be a finite von Neumann algebra, $\mathcal{H}$ be an infinite dimensional Hilbert space. Then $\mathcal{U}(\mathcal{H})_2$ is not isomorphic onto $\mathcal{U}(N)$.*

Proof. Suppose $\mathcal{U}(\mathcal{H})_2$ is isomorphic onto $\mathcal{U}(N)$. Then N is a factor (see the proof of Proposition 3.8). Suppose N is of type II_1. Then using the diffuse property of $N, \mathcal{U}(\mathcal{H})_2$ is not isomorphic onto $\mathcal{U}(N)$ (same proof as in Proposition 3.8 works), a contradiction. On the other hand, if N is a finite type I factor, then $\mathcal{U}(N)$ is compact, while $\mathcal{U}(\mathcal{H})_2$ is not. Therefore they cannot be isomorphic and we get a contradiction. $\square$

Finally, we show that a surjective homomorphism between L^2-unitary groups preserves the order $\prec$ of projections in the following case:

Proposition 3.10. *Let M, N be type II factors with normal faithful semifinite traces τ_M, τ_N, respectively. Suppose there exists a surjective continuous homomorphism $\varphi : \mathcal{U}(M)_2 \to \mathcal{U}(N)_2$. Let $p, q \in$ Proj(M) be such that $p \prec q$ and q is finite. Let*

$$p' = \frac{1-\varphi(1-2p)}{2}, \quad q' := \frac{1-\varphi(1-2q)}{2} \in Proj(N).$$

Then we have $p' \prec q'$ in N.

We need preparations. The next lemma is taken from vol III, Lemma XIV.2.1 of [18].

Lemma 3.11. *If e and f are equivalent projections in a finite von Neumann algebra M, then there exists a unitary $u \in \mathcal{U}(M)$ such that*

$$|u-1| \leq \sqrt{2}|e-f|, \qquad ueu^* = f.$$

Corollary 3.12. *Let M be a seminfinite von Neumann algebra with a faithful normal semifinite trace τ. Then for any two finite equivalent projections e and f in M, there is $u \in \mathcal{U}(M)_2$ such that $ueu^* = f$ and $||1-u||_2^2 \leq 2\sqrt{2}||e-f||_1$.*

Proof. This can be shown by a direct computation using Lemma 3.11. $\square$

Proof of Proposition 3.10. Let $c := \tau_M(1) \in [0,\infty], d := \tau_N(1) \in [0,\infty]$. Let $F : [0,c) \to [0,d)$ by

$$F(t) := \tau_N\left(\frac{1-\varphi(1-2p_t)}{2}\right),$$

for $p_t \in \mathrm{Proj}(M)$ with $\tau_M(p_t) = t$. Then F is continuous. We show that F is injective. If $F(t) = F(s)$ for $s,t \in [0,c)$, then take $p_t, p_s \in \mathrm{Proj}(M)$ with $\tau_M(p_t) = t, \tau_M(p_s) = s$. Define $p_t', p_s' \in \mathrm{Proj}(N)$ from p_t, p_s as above. Since $\tau_N(p_t') = F(t) = F(s) = \tau_N(p_s')$ and N is a factor, we have $p_t' \sim p_s'$. Thus by Corollary 3.12, there exists $u \in \mathcal{U}(N)_2$ such that $u^*p_t'u = p_s'$. Since φ is surjective, there exists $v \in \mathcal{U}(M)_2$ with $\varphi(v) = u$. Then we have, in a similar argument as before, that

$$\begin{aligned} p_s &= \frac{1-\varphi^{-1}(1-2p_s')}{2} \\ &= \frac{1-\varphi^{-1}(1-2u^*p_t'u)}{2} \\ &= v^*\frac{1-\varphi^{-1}(1-2p_t')}{2}v \\ &\sim p_t \end{aligned}$$

and thus $t = \tau_M(p_t) = \tau_M(p_s) = s$. Hence F is injective. Furthermore, we have $F(0) = 0$ and $F(t) > 0$ for some t because φ is surjective. This implies F is a monotone increasing function. Therefore $p \prec q$ in M implies $p' \prec q'$ in N. $\square$

3.3 Other Known Examples

The class $\mathscr{U}_{\mathrm{fin}}$ has not been studied well. However, there are some known examples other than the ones presented in §2.6.

Example 3.13. Normalizer groups $\mathcal{N}_M(A)$ and $\mathcal{N}(E)$
Let A be an abelian von Neumann subalgebra of a separable II_1 factor M. The *normalizer group* $\mathcal{N}_M(A)$ of A, defined by

$$\mathcal{N}_M(A) := \{u \in \mathcal{U}(M); uAu^* = A\},$$

is clearly a strongly closed subgroup of $\mathcal{U}(M)$ and hence belongs to $\mathscr{U}_{\text{fin}}$. This group has been drawn much attention to specialists, especially when A is maximal abelian and $\mathcal{N}_M(A)$ generates M as a von Neumann algebra. In such a case, A is called a *Cartan subalgebra*. Similarly, the *normalizer group* $\mathcal{N}(E)$ for a normal faithful conditional expectation $E : M \to N$ onto a von Neumann subalgebra N,

$$\mathcal{N}(E) := \{u \in \mathcal{U}(M); uE(x)u^* = E(uxu^*), \text{ for all } x \in M\},$$

is also of finite type.

Example 3.14. The full group $[\mathcal{R}]$
Let $\mathcal{R}$ be a II_1 countable equivalence relation on a standard probability space (X, μ). A. Furman showed that the full group $[\mathcal{R}]$ equipped with so-called *uniform topology* is a Polish group of finite type (see §2 of Furman [6]).

4 Hereditary Properties of Finite Type Groups

In this section, we discuss the permanence properties of the class $\mathscr{U}_{\text{fin}}$ under several algebraic operations. In summary, we will observe the following permanence properties of finite type groups.

Operation	$\mathscr{U}_{\text{fin}}$?
Closed subgroup $H < G$	YES
Countable direct product $\prod_{n\ge 1} G_n$	YES
Semidirect product $G \rtimes H$	NO
Quotient G/N	NO
Extension $1 \to N \to G \to K \to 1$	NO
Projective limit $\varprojlim G_n$	YES

As can be seen from the above table, finiteness property is suprisingly delicate and can easily be broken under natural operations.

Remark 4.1. (On the ultraproduct of metric groups) Let $\{(G_n, d_n)\}_{n=1}^{\infty}$ be a sequence of finite type Polish groups with a compatible bi-invariant metric. It is not difficult to show that the ultraproduct (G_ω, d_ω) of $\{(G_n, d_n)\}_{n=1}^{\infty}$ along a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ is a completely metrizable topological group of finite type, but not Polish in general. We will discuss topological groups which are embeddable into the unitary group of a (not necessarily separable) finite von Neumann algebra elsewhere.

4.1 Closed Subgroup and Countable Direct Product

It is clear the class $\mathscr{U}_{\text{fin}}$ is closed under taking closed (or even G_δ) subgroup. Since countable direct sum of separable finite von Neumann algebras is again separable and finite, the class $\mathscr{U}_{\text{fin}}$ is closed under countable direct product.

4.2 Extension and Semidirect Product

The class $\mathscr{U}_{\mathrm{fin}}$ is not closed under extension nor semidirect product.

Proposition 4.2. *There exits a Polish group G not of finite type, which has a closed normal subgroup N such that N and the quotient group G/N are of finite type.*

Proof. Let G be the $ax+b$ group (see Example 2.15). Since G does not have a compatible bi-invariant metric, it is not of finite type. On the other hand, G can be written as a semidirect product $G = \mathbb{K} \rtimes \mathbb{K}^{\times}$, where $\mathbb{K}^{\times}$ acts on $\mathbb{K}$ as multiplication. There fore the exact sequence

$$0 \longrightarrow \mathbb{K} \longrightarrow G \longrightarrow \mathbb{K}^{\times} \longrightarrow 1$$

gives a counter example for extension case. □

Note that the above example also shows that the class $\mathscr{U}_{\mathrm{fin}}$ is not closed under semidirect product.

4.3 Quotient

The class $\mathscr{U}_{\mathrm{fin}}$ is not closed under quotient.

Proposition 4.3. *There exists an abelian Polish groups of finite type G such that the quotient G/N of G by its closed subgroup is not of finite type.*

Proof. Consider the separable Banach space $A := l^3$ as an additive Polish group. As we saw in Example 2.18, $l^p (1 \leq p \leq \infty)$ is unitarily representable if and only if $1 \leq p \leq 2$. On the other hand, every separable Banach space is isomorphic onto a quotiant Banach space of ℓ^1 (see e.g., Theorem 5.1 of [5]). In particular, although not of finite type, $A = \ell^3$ is a quotient of $G := \ell^1$ by its closed subgroup N. □

Remark 4.4. Note that even for abelian Polish groups, the situation can be worst possible. It is known (chapter 4 of [2]) that there exists an abelian Polish group A which has no non-trivial unitary representation. Such a group is called *strongly exotic*. On the other hand, S. Gao and V. Pestov [9] proved that any abelian Polish group is a quotient of ℓ^1 by a closed subgroup N. Therefore, strongly exotic groups are also quotients of finite type Polish groups.

4.4 Projective Limit

The class $\mathscr{U}_{\mathrm{fin}}$ is closed under projective limit.

Proposition 4.5. *Let $\{G_n, j_{m,n} : G_m \to G_n (n \leq m)\}_{n,m=1}^{\infty}$ be a projective system of Polish groups of finite type. Then $G = \varprojlim G_n$ is a Polish group of finite type too.*

Proof. Since the connecting map $\{j_{m,n}\}$ is continuous, it is clear that G can be seen as a closed subgroup of $\prod_{n\in\mathbb{N}} G_n$. Since finiteness property passes to direct product, $\prod_{n\in\mathbb{N}} G_n$ is also a Polish group of finite type. Therefore its closed subgroup G is also a Polish of finite type. $\square$

5 Some Questions

Finally let us discuss some questions to which we do not have answers at this stage. Let $\mathscr{U}_{\rm inv}$ denote the class of Polish groups with a compatible bi-invariant metric. As we saw in Example 2.6, $\mathscr{U}_{\rm inv}$ is strictly larger than $\mathscr{U}_{\rm fin}$ (l^3 is in $\mathscr{U}_{\rm inv}$ but not in $\mathscr{U}_{\rm fin}$). Furthermore, there exists a more interesting example. Recently L. van den Dries and S. Gao [4] constructed a Polish group G with a compatible bi-invariant metric, which does not have Lie sum (see [4] for the definition). On the other hand, we proved in [1] that if G belongs to the class $\mathscr{U}_{\rm fin}$, then G has a complete topological Lie algebra, hence a fortiori has a Lie sum. Thus G is not of finite type. Therefore it would be desirable to consider the following questions:

Question 5.1. Is van den Dries-Gao's Polish group unitarily representable?

Question 5.2. Is a unitarily representable Polish SIN-group of finite type?

Hopefully Theorem 2.7 will play the role for solving the above questions. Also, since l^p belongs to $\mathscr{U}_{\rm fin}$ if and only if $1\leq p\leq 2$, it is worth considering whether

Question 5.3. Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space. Does $\mathcal{U}(\mathcal{H})_p:=\{u\in\mathcal{U}(\mathcal{H}); 1-u\in S^p(\mathcal{H})\}$ belong to $\mathscr{U}_{\rm fin}$ for some $1\leq p<2$? Here $S^p(\mathcal{H})$ denotes the space of Schatten p-class operators.

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