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Comparison of some quotients of fundamental groups of algebraic curves over p-adic fields

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Introduction Basic problems related to lifting and reduction of etale covers of curves had been treated in the fundamental work of Grothendieck [4], which established unique liftability, as well as good reduction property for Galois covers with degrees not divisible by the residue characteristic p. This applies also to tame covers, say, of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$, in which case Raynaud [16] proved a partial but yet unsurpassed result for Galois covers of degree divisible by p but not by p^2. Historically, there is another line of investigations started mainly by Shimura and Igusa. In [5], Igusa made a basic contribution to the case of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ by geometric method, proving that the modular tower of levels not divisible by p (the Galois degrees can be divisible by p) has good reduction. The theory of Shimura curves [19, 20] provided extremely rich arithmetic systems of curves and source of further studies. In connection with fundamental groups, we just recall here the following; each tower obtained by reduction of modular or Shimura curves can be characterized, inside the tower of curves with prescribed tame ramifications, only by the complete splitting of "special $\mathbf{F}_{q^2}$-rational points" [6, 8, 9, 10]. As for developments after 1980's related to the study of the algebraic fundamental groups of curves, we shall leave their descriptions to other articles of this Volume.

Now, here, we take up the following question. Let $p > 2$, let $\bar{\mathbf{F}}_p$ be an algebraic closure of $\mathbf{F}_p$, and $R_0 = \mathbb{W}[[\bar{\mathbf{F}}_p]]$ be the ring of Witt vectors. Note that R_0 does not contain the group μ_p of p-th roots of unity. Let k_0 be the quotient field of R_0, and G_{k_0} be the absolute Galois group $G_{k_0} = \mathrm{Gal}(\bar{k_0}/k_0)$. Let X be a proper smooth R_0-scheme whose fibers $X_\eta = X \otimes k_0$ and $X_s = X \otimes \bar{\mathbf{F}}_p$ are geometrically irreducible curves. Pick an R_0-section $x = (x_\eta, x_s) \in X(R_0)$. Then G_{k_0} acts on the fundamental group $\pi_1(X_\eta \otimes \bar{k}_0, x_\eta)$, and the (surjective) specialization homomorphism

$$\pi_1(X_\eta \otimes \bar{k}_0, x_\eta) \to \pi_1(X_s, x_s)$$

factors through the G_{k_0}-coinvariant of the group on the left, inducing

$$\psi : \pi_1(X_\eta \otimes \bar{k}_0, x_\eta)_{G_{k_0}} \to \pi_1(X_s, x_s).$$

Our questions and partial results are related to its kernel, $\mathrm{Ker}(\psi)$. After some vain trials to find a non-trivial element in the kernel (first for the case of $X = \mathbf{P}^1 \setminus \{0, 1, \infty\}$), I

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turned direction, inclining to think that the kernel may reduce to $\{1\}$, and decided to pose this question in this article (instead of giving the content of my talk which is more or less direct consequences of my previous results [6, 10]). Whether $\mathrm{Ker}(\psi) = \{1\}$, to be called question (Q1'), is equivalent to the following (Q1): *is it true that a finite etale cover $f_\eta : Y_\eta \to X_\eta$ has good reduction if one k_0-rational point x_η of X_η splits completely in $Y_\eta(k)$?* It is almost obvious that these questions make sense only for a *fixed* base ring as "small" as R_0 (§1.1,§4.1). These are questions on 2-dimensional absolute "surfaces", and not on relative curves.

We add here a few remarks to avoid misunderstandings. We have just referred to *complete splitting* of *one* point, but it should be noted that (i) splitting is equivalent to p-adic unramifiedness in the fiber (because non-trivial finite extensions of k_0 are totally ramified), and (ii) the complete splitting of one point means that of all points sufficiently nearby. As for (i), it may be more impressive to use the term like "p-adically unramified", but this can also be confusing. As for (ii), it is sometimes more reasonable to look at the effect of complete splitting of one whole "disk". This is in fact so in the case of bigger base rings (§4.1), but for the case of small base rings, splitting of one point seems to be an appropriate starting point, in connection with the Galois action on fundamental groups with a given base point, and for rigidifying related covers.

The main purpose of this article it to draw attention to the above (equivalent) questions; however, we shall also present partial results. We first prove that $\mathrm{Ker}(\psi)$ does not have non-trivial pro-solvable quotients (Corollary 1-1 §1.4). Let $f : Y \to X$ denote the integral closure of X in the function field of Y_η, so that Y is a normal R_0-scheme with $f_\eta : Y_\eta \to X_\eta$ etale. We shall then prove that (Q1) is valid when the cover f is "locally realizable in a relatively 1-parameter space" (Theorem 2 §1.5.). This has some application to non-splitting of points in the "level $\mathfrak{p}$" covers of Shimura curves.

In §1, we pose the basic questions and present Theorems 1,2 with their Corollaries. They will be proved in §2,3. In §4, we discuss some related subjects, for example, the case of more general base rings, and the case of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$. Examples will also be given to indicate that (i) if the question has general affirmative answer, it cannot be proved only by local methods (Example 1 §3.4), and that (ii) for a bigger base ring, even the complete splitting of one whole disk does not imply good reduction of covers (Example 2 §4.2). The local tools we use (for Theorem 2) consist of (a) a simple upper bound of the different exponent in the mixed characteristic case (Lemma A §3), and (b) a lower bound for elementwise relative discriminants, under the existence of a "splitting section" in some 2-dimensional local R_0-algebras. The global tools we use (for Theorem 1, etc.) are the abelian group schemes, as in Raynaud [16], and also the "associated differential". In §5, as an Appendix, we shall give a general definition of the associated differential ω, which is a differential of multiple degree on the special fiber canonically associated to each non-etale cover $f : Y \to X$. The special case of ω related to Shimura curves was studied in e.g.,[7], and now some other cases became relevant.

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1 Questions and main results

1.1 – Let R be a complete discrete valuation ring of mixed characteristics $(0,p)$ with quotient field k whose residue field is $\kappa = \bar{\mathbf{F}}_p$ (an algebraic closure of $\mathbf{F}_p$). It is either $R_0 = \mathbb{W}[[\bar{\mathbf{F}}_p]]$ itself, or a finite totally ramified extension of R_0. We denote by π a prime element of R, and by $\mathrm{ord} = \mathrm{ord}_\pi$ the corresponding normalized additive valuation. As usual, write $\mathrm{Spec}(R) = \{\eta, s\}$ (η the generic point, s the closed point). Note that $\mathrm{Spec}(R)$ has no non-trivial connected finite etale covers. For any R-scheme Z, $Z_\eta = Z \otimes k$ (resp. $Z_s = Z \otimes \kappa$) denotes its general (resp. special) fiber, and $Z(R)$ (resp. $Z_\eta(k)$) denotes the set of sections $\mathrm{Spec}(R) \to Z$ (resp. k-rational points of Z_η). We shall consider only proper flat R-schemes Z; hence Z_η, Z_s are non-empty. The local ring and the maximal ideal at $z \in Z$ will be denoted by $\mathcal{O}_{z,Z}$, $\mathfrak{m}_{z,Z}$ respectively. When Z is an integral scheme, its function field will be denoted by $k(Z)$ ($= k(Z_\eta)$ when Z/R is flat).

Let X be a proper smooth R-scheme whose fibers X_η, X_s are geometrically irreducible curves. We denote its function field by $K = k(X) = k(X_\eta)$. Each $x_\eta \in X_\eta(k)$ uniquely determines its specialization $x_s \in X_s(\kappa)$ and the section $x = (x_\eta, x_s) \in X(R)$. Given x_s, such x_η and x will be called a lifting of x_s. The collection of all k-rational points of X_η that lift a given point x_s will be called the *disk above* x_s(denoted by $Dsk(\eta/x_s)$), or simply a disk. It carries p-adic topology. Let L be a finite extension of K, and let $f : Y \to X$ be the integral closure of X in L. We shall exclusively study the case where *the general fiber* $f_\eta : Y_\eta \to X_\eta$ *of* f *is etale* but f itself may not be so.

1.2 – Assume temporarily that f is also etale. Then $f_\eta : Y_\eta \to X_\eta$ is not only finite, etale and connected, but also satisfies an extra strong property; namely, all points x_η of $X_\eta(k)$ split completely in $Y_\eta(k)$. In fact, if $x \in X(R)$ denotes the closure of x_η in X, then $f^{-1}(x)$ is finite and etale over $\mathrm{Spec}(R)$; hence it must be a disjoint union of copies of $\mathrm{Spec}(R)$; in particular, $f_\eta^{-1}(x_\eta)$ is a disjoint union of copies of $\mathrm{Spec}(k)$.

Conversely, start with X as in §1.1 and a connected finite etale cover $f_\eta : Y_\eta \to X_\eta$, and suppose that there exists a non-empty subset $S_{Y/X}(k) \subseteq X_\eta(k)$ such that all points x_η of $S_{Y/X}(k)$ split completely in $Y_\eta(k)$. In which cases can we conclude that $f : Y \to X$ itself is etale? The first thing to be noted is: we can conclude so when either Y_η/X_η has potential good reduction (§2.1), or if $S_{Y/X}(k)$ is large in some sense (Proposition 2 (§4.1), Cor. 2-4 (§1.5)). But for the general base ring R (outside the case of potential good reduction), the splitting assumption for a finite non-empty set $S_{Y/X}(k)$ cannot imply the etaleness of f. This is simply because if we take any ramified and not potentially unramified cover

f, take any finite subset S of $X_\eta(k)$, and replace k by $k' = k(f^{-1}(x_\eta); x_\eta \in S)$, then f remains ramified and the splitting condition for $S_{Y/X}(k) = S$ will be trivially satisfied. Such an example is made at the cost of expanding the base ring. But when we fix the base ring as, say, $R_0 = \mathbb{W}[[\bar{\mathbf{F}}_p]]$ (the ring of Witt vectors) with $p > 2$, or a little more generally, impose the condition $\operatorname{ord} p < p-1$, then I found no counterexamples to the following question, and am inclined to think that the answer may be affirmative.

Question (Q1) *Assume that* $\operatorname{ord} p < p-1$, *and let* $f : Y \to X$ *be as in §1.1. Suppose that at least one point* x_η *of* $X_\eta(k)$ *splits completely in* $Y_\eta(k)$*. Then is* f *necessarily etale?*

When we assert that (Q1) has an affirmative answer for some specified class of f, we shall say that the *assertion* (Q1Sp) *holds* (specifying the class each time).

Note that the formulation fits well with forming towers and with taking the Galois closure; the answer to (Q1) is positive if and only if (Q1Sp) holds for those L/K having no proper intermediate subfields, and if and only if (Q1Sp) holds for Galois extensions L/K. By the purity of branch locus, f is etale if and only if the discrete valuation of K defined by the generic point of the special fiber X_s of X is unramified in L. Note also that if there exists a point $x_s \in X(\kappa)$ such that f is etale at all points of $f^{-1}(x_s)$, then f is etale on Y. For $x = (x_\eta, x_s) \in X(R)$, the splitting of x_η in $Y_\eta(k)$ implies that $f^{-1}(x)$ consists of distinct irreducible components each isomorphic to $\operatorname{Spec}(R)$. But $f^{-1}(x)$ may possibly be connected; two distinct irreducible components may specialize to the same point of $f^{-1}(x_s)$. The goal is to show, for some specified classes of f, that $f^{-1}(x)$ must be totally disconnected, which implies the etaleness of f.

But before giving these, we shall reformulate (Q1) in terms of fundamental groups.

1.3 – Let $\bar{k}$ be an algebraic closure of the quotient field k of R, write $X_{\bar{k}} = X \otimes \bar{k}$, and let $\pi_{\mathbf{1}} := \pi_1(X_{\bar{k}}, x_\eta)$ be the algebraic (profinite) fundamental group of $X_{\bar{k}}$ with a given k-rational base point $x_\eta \in X_\eta(k)$. The absolute Galois group $G_k = \operatorname{Gal}(\bar{k}/k)$ acts on $\pi_{\mathbf{1}}$ in the usual manner. Let $(\pi_{\mathbf{1}})_{G_k}$ denote the G_k-coinvariant, i.e., the largest quotient on which G_k acts trivially. The fundamental group of the geometric special fiber $\pi_1(X_s, x_s)$ with base point x_s (the specialization of x_η) is also a canonical quotient of $\pi_{\mathbf{1}}$ which factors through $(\pi_{\mathbf{1}})_{G_k}$, inducing a surjective homomorphism

$$\psi : \pi_1(X_{\bar{k}}, x_\eta)_{G_k} \to \pi_1(X_s, x_s). \tag{1.3.1}$$

One sees easily (cf. §2.1) that (Q1) is equivalent to

Question (Q1') *Assume that* $\operatorname{ord} p < p-1$*. Then, is* ψ *an isomorphism ?*

If not, what is the meaning of the difference ?.

1.4 – **Theorem 1** *Notations being as in §1.1, let G denote the Galois group of the Galois closure of L/K. The assertion (Q1Sp) holds for the following classes of G; either G is solvable, or more generally the order of each composition factor of G is either equal to p or not divisible by p.*

This will be proved in two ways; (i) by using a global abelian R-group scheme argument as in Raynaud [16] (§2.3), (ii) by a more elementary treatment of normal relative curves which makes it clear why a local counterexample cannot extend to a global one (§4.1).

Corollary 1-1 *Let* $\operatorname{ord} p < p-1$. *Then* $\operatorname{Ker}(\psi)$ *has no non-trivial prosolvable quotient, or more strongly, if a finite simple group G appears as its quotient, then G must be non-cyclic and with order divisible by p.*

1.5 – We shall give another type of results on (Q1Sp). Let $f : Y \to X$ be as in §1.1, so that f is etale on the general fiber. Let $y_s \in Y_s(\kappa)$ be any point on the special fiber. Let us call f *a cover in a 1-parameter space* at y_s, if the vertical component in the tangent space at y_s is at most 1-dimensional;

$$\dim \operatorname{Ker}(T_{y_s}(Y_s) \to T_{x_s}(X_s)) \leq 1 \tag{1.5.1}$$

($x_s = f(y_s)$), or equivalently, putting $(\mathcal{O}', \mathfrak{m}') = (\mathcal{O}_{y_s,Y_s}, \mathfrak{m}_{y_s,Y_s})$ and $(\mathcal{O}, \mathfrak{m}) = (\mathcal{O}_{x_s,X_s}, \mathfrak{m}_{x_s,X_s})$, if

$$\dim_\kappa(\mathfrak{m}'/(\mathfrak{m}\mathcal{O}' + \mathfrak{m}'^2)) \leq 1. \tag{1.5.2}$$

This is also equivalent to that the κ-algebra $\mathcal{O}'/\mathfrak{m}\mathcal{O}'$ is generated by one element, or (by Krull-Azumaya's lemma) to that the $\mathcal{O}^\wedge$-algebra $\mathcal{O}'^\wedge$ is generated by a single element ($\wedge$ the completion).

Consider the case where y_s extends to a section $y = (y_\eta, y_s) \in Y(R)$. Then, the local ring $\mathcal{O}_{y_s,Y}$ is regular if and only if the prime ideal corresponding to y is principal (say, (t'), which gives $\mathfrak{m}_{y_s,Y} = (t', \pi)$), and when this is also satisfied, f is a cover in a 1-parameter space at y_s, because $\pi \in \mathfrak{m}_{x_s,X} \setminus \mathfrak{m}^2_{y_s,Y}$.

Theorem 2 *Notations being as in §1.1 and (Q1), denote by x_s and $x = (x_\eta, x_s)$ the specialization of x_η and the corresponding section in $X(R)$, respectively. The assertion (Q1Sp) holds for the following classes of f; either each section $y = (y_\eta, y_s) \in Y(R)$ above x is locally defined by a single equation, or more generally, f is a cover in a 1-parameter space at each point y_s above x_s.*

This is obtained by combining an estimate from below of the p-adic order of *elementwise* discriminants using this splitting of $f^{-1}(x_\eta)$, with an estimate from above of the order of the discriminant itself. The geometric assumption confirms that the discriminant is equal to the greatest common divisor of elementwise discriminants (§3).

Corollary 2-1 *(Q1Sp) holds when Y is a regular scheme.*

Corollary 2-2 *Let $W \to X$ be a proper smooth relatively 1-dimensional X-scheme, and suppose that $f : Y \to X$ can be obtained from a closed integral flat X-subscheme $T \subset W$ by a "small modification", i.e., by the normalization $\mu : Y \to T$ which is assumed to be unramified ("net") at each point of Y. Then (Q1Sp) has an affirmative answer for such Y/X.*

The special case where $W = X \times_R Z$ with some proper smooth relatively 1-dimensional R-scheme Z and where $T \subset X \times_R Z$ is the "graph" of an algebraic correspondence, will be applied in the following.

We have also tried to find a counterexample for (Q1). For this aim, we have especially looked at Shimura curve analogues of the modular curve of level p usually called $X_0(p)$ which has bad reduction. Let F be a totally real number field and B a quaternion algebra over F in which all but one archimedean primes are ramified. Shimura [19] constructed and studied a canonical tower of curves over abelian extensions of F, the Shimura curves associated to B of all levels. If $\mathfrak{p}$ is any non-archimedean prime divisor of F not dividing the discriminant of B, we obtain from (the levels coprime with $\mathfrak{p}$-part of) his system a $\mathfrak{p}$-canonical system of triples of relative curves (cf. [8, 9] and [10](the author's notes 2008)):

$$\{X \xleftarrow{f} X_0(\mathfrak{p}) \xrightarrow{f'} X'\} \tag{1.5.3}$$

over $\mathfrak{o}_\mathfrak{p}^{(2)}$. Here, $\mathfrak{o}_\mathfrak{p}$ is the $\mathfrak{p}$-adic completion of the ring of integers of F, $\mathfrak{o}_\mathfrak{p}^{(2)}$ is its unique quadratic unramified extension, X, X' are proper smooth relative curves over $\mathfrak{o}_\mathfrak{p}^{(2)}$ that are mutually conjugate over $\mathfrak{o}_\mathfrak{p}$, and $X_0(\mathfrak{p})$ is the normalization of the Hecke correspondence $T(\mathfrak{p}) \subset X \times_{\mathfrak{o}_\mathfrak{p}^{(2)}} X'$ (which is formal locally a closed immersion; cf [8]). Moreover, (1.5.3) is a CR-system, i.e., the special fiber $T(\mathfrak{p})_s$ consists of two irreducible components Π, Π' meeting transversely above each $\mathbf{F}_{q^2}$-rational point of X_s ($q = N(\mathfrak{p})$), and the component Π (resp. Π') is the graph of the q-th power morphism $X_s \to X'_s$ (resp. $X'_s \to X_s$).

As for ramifications, unless $B \simeq M_2(\mathbf{Q})$ and unless the level of X is too small, f_η, f'_η are etale, because the corresponding discrete subgroups of $PSL_2(\mathbf{R})$ are cocompact and torsion-free. On the special fiber, f_s, f'_s involve inseparable morphisms and hence f, f' cannot be etale. Although a basic property of these CR-systems is that the fundamental groups of the special fiber and the general fiber are strictly isomorphic, here, we shall forget this aspect and lift the base scheme to the completion R of the maximal unramified extension of $\mathfrak{o}_\mathfrak{p}$. The Galois group of the Galois closure of L/K in this case is either $PGL_2(\mathbf{F}_q)$ or $PSL_2(\mathbf{F}_q)$, and L/K corresponds to the Borel subgroup (index $q+1$).

As a direct consequence of Corollary 2-2 we obtain

Corollary 2-3 *When* $\operatorname{ord} p < p - 1$, *no point of $X_\eta(k)$ splits completely in $X_0(\mathfrak{p})_\eta(k)$; in other words, (Q1Sp) holds for the cover $f : X_0(\mathfrak{p}) \to X$.*

Theorems 1,2 tell us that in order to find a counterexample, we need to look at coverings with non-solvable Galois groups of the Galois closure which are not defined locally by a single equation in two variables (at the crucial points).

The following weaker form of (Q1) is a direct consequence of Theorem 2.

Corollary 2-4 *Let the ring R and $f : Y \to X$ be as in §1.1, and assume* $\operatorname{ord} p < p - 1$. *Suppose f is not etale. Then there exists a finite subset Σ_s of $X_s(\kappa)$ satisfying the following: Let Σ_η denote the set of all liftings of Σ_s to points of $X_\eta(k)$. Then no point of $X_\eta(k) \setminus \Sigma_\eta$ splits completetly in $Y_\eta(k)$.*

(A weaker form for the general case of the base ring R will be given in §4.1)

Proof The two dimensional normal scheme Y is regular outside a finite set Σ'_s of points of $Y_s(\kappa)$. (Indeed, by localization one can make each prime ideal of height 1 in the algebra of sections of $\mathcal{O}(Y)$ principal. The rest is obvious.) Let Σ_s be the projection of Σ'_s to $X_s(\kappa)$, and let $x_s \in X_s(\kappa) \setminus \Sigma_s$. Then each point $y_s \in f^{-1}(x_s)$ is regular on Y. If x_η (with the specialization x_s) were splitting completely in $Y_\eta(k)$, then as we have seen above, f must be a cover in a 1-parameter space at y_s; hence by our assumption $\operatorname{ord} p < p - 1$ and by Theorem 2, f must be etale, contrary to our assumption. □

2 Proof of Theorem 1 and other basic statements.

2.1 – Notations being as in §1.1, the finite etale cover $f_\eta : Y_\eta \to X_\eta$ corresponding to the function field extension L/K is said to have potential good reduction, if there exists a finite extension k'/k such that the cover $f' : Y' \to X'$ obtained by taking the integral closures of X in the field extension Lk'/Kk' is etale. Recall that this is equivalent to the unramifiedness in Lk'/Kk' of the discrete valutaion v' of Kk' defined by the generic point of X'_s, and that in this case, extension of v' in Lk' is unique. First we shall prove

Proposition 1 *The assertion (Q1Sp) holds when $f_\eta : Y_\eta \to X_\eta$ has potential good reduction.*

Proof We may assume that L/K is Galois. Let v be the discrete valuation of K defined by the generic point of X_s. By assumption there exists a finite extension k'/k, which we may also assume Galois, such that the extension v' of v to Kk' is unramified in Lk'/Kk'. By applying the purity of branch locus and Zariski's connectedness theorem to the integral closures Y'/X' of X in Lk'/Kk', we see that the extension of v' to Lk' is unique; hence the extension of v to Lk' is also unique. But Lk'/K being Galois, this implies that Lk'/K is linearly disjoint with the v-adic completion of K. Hence can take the maximal unramified subfield L^*/K in Lk'/K. It is clear that $L^*k' = Lk'$. Every $x_\eta \in X_\eta(k)$ splits completely in $Y^*_\eta(k)$, Y^* being the integral closure of X in L^*; hence if x_η splits completely also in $Y_\eta(k)$ and if $L^* \neq L$, then it must split in a non-trivial subextension of k'/k, a contradiction. Therefore, $L^* = L$; hence $f : Y \to X$ must be etale. □

2.2 – We shall show that (Q1) and (Q1') are equivalent. Each k-rational point $x_\eta \in X_\eta(k)$ defines a splitting $s : G_k \to \pi_1(X_\eta, x_\eta)$ of the exact sequence

$$1 \to \pi_1(X_{\bar{k}}, x_\eta) \to \pi_1(X_\eta, x_\eta) \to G_k \to 1. \tag{2.2.1}$$

This splitting defines a subtower of connected finite etale covers of X_η having one projective system of k-rational points above x_η. If N denotes the kernel of the projection $\pi_1(X_{\bar{k}}, x) \to \pi_1(X_{\bar{k}}, x)_{G_k}$, then the semi-direct product $s(G_k) \cdot N$ is the smallest normal subgroup of $\pi_1(X_\eta, x_\eta)$ containing $s(G_k)$. This corresponds to the tower of all connected finite etale covers of X_η in which x_η splits completely. □

2.3 – **Proof of Theorem 1** Let L/K be the function fields of Y_η/X_η. We may assume L/K to be Galois with degree n, with either $n \not\equiv 0 (\operatorname{mod} p)$ or $n = p$. The first case reduces to Proposition 1 by Abhyankar's lemma. When $n = p$, let J be the (proper smooth) abelian scheme over R obtained as the Jacobian of X, and $\phi : X \to J$ be the canonical morphism which maps x to the origin 0_J of J. Since the covering f_η is abelian, there exists a k-isogeny $F_\eta : A_\eta \to J_\eta$ of abelian varieties over k such that $Y_\eta = X_\eta \times_{J_\eta} A_\eta$ (cf.[17]; the descent argument to k is easy). Being isogenous to an abelian variety having good reduction, A_η also has good reduction, i.e., extends to a proper smooth abelian scheme A over R, and F_η extends to an R-morphism $F : A \to J$ of abelian schemes. Put $Y^* = X \times_J A$, and let $f^* : Y^* \to X$ be the projection. If $x = (x_\eta, x_s) \in X(R)$ denotes the unique extension of x_η, then $(f^*)^{-1}(x)$, as R-scheme, is isomorphic to $F^{-1}(0_J)$ which is a finite commutative group scheme of order p on R. But $(f^*)^{-1}(x)$ has $[L : K]$ distinct sections, and hence $F^{-1}(0_J)$ has as many distinct R-sections. Therefore, by the classification of finite commutative group schemes of order p over R [14, 15], this must be etale over R when $\operatorname{ord} p < p - 1$. Therefore, $(f^*)^{-1}(x)$ is etale over R, which means $f^* : Y^* \to X$ etale above x_s; hence etale everywhere, and this must be the integral closure Y of X in L. □

3 Proof of Theorem 2 and its corollaries

3.1 – We start with a lemma which is basic and elementary but does not seem to be well-known.

Lemma A *Let (K, v) be a complete discrete valuation field of mixed characteristics $(0, p)$, and (L, V) be any finite extension. Let $\mathfrak{D}(V/v)$ be the different ideal. Then*

$$\operatorname{ord}_V \mathfrak{D}(V/v) \le e - 1 + \operatorname{ord}_V(n^*). \tag{3.1.1}$$

Here, ord_V *is the normalized additive form of V, e is the ramification index, $n = [L : K]$, and $n^* = n/f_s$, f_s being the degree of the separable part of the residue field extension.*

Remark When the residue field extension is separable, this can be found in a classical literature, e.g.[2]. Deeper results for separable residue extension case (resp. more general case) are exposed in [18](resp.[1]). But even the fact that a simple upper bound exists in the mixed characteristic case does not seem to be singled out explicitly in the literatures that the author has met.

Corollary A *Let (K, v) be a discrete valuation field of mixed characteristics $(0, p)$, and L be any finite extension. Let A be the valuation ring of (K, v), B be its integral closure in L, and $D(B/A)$ be the discriminant of B/A which is an integral ideal of A. Then*

$$\text{ord}_v D(B/A) \leq \sum_{V/v} n_V \left(\text{ord}_v n_V^* + 1 - \frac{1}{e_V} \right), \tag{3.1.2}$$

where V runs over all distinct extensions of v to L and n_V, n_V^*, e_V are as in Lemma A for the completion L_V/K_v.

Proof of Lemma A (i) (Reduction to $n = p$ cases) Let $\mathfrak{P}$ denote the valuation ideal for (L, V), and put

$$\mathfrak{D}'(V/v) = n^* \mathfrak{P}^{e-1}. \tag{3.1.3}$$

The lemma is equivalent to that $\mathfrak{D}(V/v)$ divides $\mathfrak{D}'(V/v)$. A point is that (not only $\mathfrak{D}$ but also) $\mathfrak{D}'$ satisfies the transitivity condition for towers;

$$\mathfrak{D}'(V_2/V_1)\mathfrak{D}'(V_1/v) = \mathfrak{D}'(V_2/v) \tag{3.1.4}$$

for any extensions $V_2/V_1/v$. This is obvious from the definition of $\mathfrak{D}'(V/v)$. So, when L/K is a tower of subextensions, it suffices to prove the inequality for each step of the tower. Since it is obviously satisfied for tamely ramified extensions, and since L/K always possesses the maximal tamely ramified subextension (including the case where inseparable residue field extensions are allowed; cf. e.g. [2]), it suffices to prove it in the wildly totally ramified case. Moreover, since the *equality* $\mathfrak{D} = \mathfrak{D}'$ holds in the tame cases, we see that if L/K is wildly totally ramified and K'/K is tamely ramified, the inequality for LK'/K' will imply that for L/K.

Now let M be the Galois closure of L/K and K^{tr} be the maximal tamely ramified subextension in M. As we have seen above, it suffices to prove the inequality for the extension $L.K^{tr}/K^{tr}$. But M/K^{tr} is a Galois extension whose Galois group is a p-group P, and being a p-group, for each subgroup P_0 of P, there exists an increasing sequence of subgroups, starting with P_0 ending with P, with index $= p$ for each adjacent subgroups. Hence the extension $L.K^{tr}/K^{tr}$ has a filtration by extensions of degree p. Thus the proof is reduced to the case of $n = p$.

(iia) (The case $n = e = p$) Let $\mathfrak{O}_v \subset \mathfrak{O}_V$ be the valuation rings and Π be a prime element of V. Then $\mathfrak{O}_V = \mathfrak{O}_v[\Pi]$. Therefore, if $f(X) = \text{Irr}(\Pi, K, X)$ denotes the monic irreducible polynomial giving the equation for Π over K, then $\mathfrak{D}(V/v) = (f'(\Pi))$. Write $f(X) = \sum_{0\le i\le p} a_i X^{p-i}$ ($a_i \in \mathfrak{O}_v$, $a_0 = 1$). Then $f'(\Pi)$ is the sum of $(p-i)a_i\Pi^{p-i-1}$ over $0 \le i \le p-1$. Since the additive order ord_V of these terms are mutually distinct $\mod p$,

$$\mathrm{ord}_V\mathfrak{D}(V/v) = \mathrm{Min}_{0\le i\le p-1}\left(\mathrm{ord}_V((p-i)a_i\Pi^{p-i-1})\right) \tag{3.1.5}$$

$$\le \mathrm{ord}_V(p\Pi^{p-1}) = \mathrm{ord}_V n^* + p - 1, \tag{3.1.6}$$

as desired.

(iib)(The case $n = p$ and $e = 1$) Here we shall use the symbols $\bar{L}$, $\bar{\omega}$, etc, for the residue field of L, the residue class of $\omega \in \mathfrak{O}_V$, etc. In this case, $\bar{L}/\bar{K}$ is a purely inseparable extension with prime degree p; hence $\bar{L} = \bar{K}(\bar{\omega})$ with some $\bar{\omega}$ such that $\bar{\omega}^p \in \bar{K}, \notin \bar{K}^p$. Let $\omega \in \mathfrak{O}_V$ be a lifting of $\bar{\omega}$, so that $\mathfrak{O}_V = \mathfrak{O}_v[\omega]$; hence $\mathfrak{D}(V/v) = (f'(\omega))$ where $f(X) = \mathrm{Irr}(\omega, K, X) = \sum_{0\le i\le p} a_i X^{p-i}$. Note that $\bar{a_i} = 0$ $(1 \le i \le p-1)$. Put

$$\nu = \mathrm{Min}(\mathrm{ord}_v p, \mathrm{ord}_v((p-i)a_i), (1 \le i \le p-1)), \tag{3.1.7}$$

so that $0 < \nu \le \mathrm{ord}_v p$. Let $\pi \in K$ be a prime element of v and express $f'(\omega)$ as

$$f'(\omega) = \pi^\nu(b_0\omega^{p-1} + ... + b_{p-1}), \tag{3.1.8}$$

with $b_i \in \mathfrak{O}_v$, $(\overline{b_0}, ..., \overline{b_{p-1}}) \ne (0, ..., 0)$. Then since $\bar{\omega}^{p-1}, .., \bar{\omega}, 1$ are linearly independent over $\bar{K}$, we have

$$\mathrm{ord}_V(f'(\omega)) = \nu \le \mathrm{ord}_v p = \mathrm{ord}_V p = \mathrm{ord}_V n^* + e - 1, \tag{3.1.9}$$

as desired. $\square$

3.2 – We shall also need the following

Lemma B *Let R be a complete discrete valuation ring of mixed characteristics $(0, p)$ with prime element π such that $\kappa = R/\pi$ is algebraically closed. Let $(A, \mathfrak{m})$ be a 2-dimensional regular local domain dominating (R, π), and let K be the quotient field of A. Let L/K be a finite field extension of degree n, and B be the integral closure of A, so that B is a semi-local ring with maximal ideals denoted by $\mathfrak{m}_1, ..., \mathfrak{m}_r$. Suppose that there exists an R-homomorphism $\phi : A \to R$ that extends to n distinct R-homomorphisms*

$$\phi_j^{(i)} : B \to R \qquad (1 \le j \le r,\ 1 \le i \le d_j) \tag{3.2.1}$$

where the index j indicates that the maximal ideal of B defined as the kernel of the composite $\overline{\phi_j^{(i)}} = \phi_j^{(i)}(\mathrm{mod}\pi)$ is $\mathfrak{m}_j$. For each $b \in B$, let $D_{L/K}(b) \in A$ denote its discriminant. Then

$$\mathrm{ord}_\pi(\phi(D_{L/K}(b))) \ge \sum_{j=1}^{r} d_j(d_j - 1) \qquad \textit{for any } b \in B. \tag{3.2.2}$$

Proof of Lemma B Let M/K be the Galois closure of L/K, G be the Galois group, and C be the integral closure of B in M. Then ϕ extends to an R-homomorphism $C \to R$, because by assumption and by the definition of M it extends at least to a k-homomorphism $C \otimes k \to k$, but the image of C is integral over R; hence it is R itself. Pick one such extension $\phi_{C,1} : C \to R$, and denote by $\bar{\phi}_{C,1}$ the composite with the reduction map $R \to \kappa$. Let $\mathfrak{m}_{C,1}$ be the maximal ideal of C corresponding to the kernel of $\bar{\phi}_{C,1}$. The Galois group G acts on C and acts simply transitively $\phi_C \to \phi_C \circ g$ ($g \in G$) on all extensions ϕ_C of ϕ to C. [1] The distinct extensions of ϕ to B thus correspond, via $\phi_{C,1} \circ g|_B \longleftrightarrow gH$ with the right coset space G/H, where $H = \mathrm{Gal}(M/L)$. Let I denote the stabilizer of $\mathfrak{m}_{C,1}$ in G. (Note that elements of I act trivially also on $C/\mathfrak{m}_{C,1} = \kappa$.) Note that $\phi_{C,1} \circ g|_B$ and $\phi_{C,1} \circ g'|_B$ belong to the same maximal ideal of B if and only if suitable H-conjugates of $\phi_{C,1} \circ g$ and $\phi_{C,1} \circ g'$ belong to the same maximal ideal of C, and hence if and only if $IgH = Ig'H$. Thus, $\{\mathfrak{m}_j\}_{1 \leq j \leq r}$ correspond bijectively with $I \backslash G/H$, and d_j is the number of elements of G/H contained in this double coset. Note that $n = (G:H) = \sum_{j=1}^{r} d_j$.

Now, by definition,

$$D_{L/K}(b) = \pm \prod_{\substack{\sigma H, \tau H \in G/H \\ \sigma H \neq \tau H}} (\sigma b - \tau b) \in A; \tag{3.2.3}$$

hence

$$\phi(D_{L/K}(b)) = \phi_{C,1}(D_{L/K}(b)) = \pm \prod_{\substack{\sigma H, \tau H \in G/H \\ \sigma H \neq \tau H}} \phi_{C,1}(\sigma b - \tau b) \tag{3.2.4}$$

$$\equiv 0 \Big(\mathrm{mod} \prod_{\substack{\sigma H, \tau H \in G/H \\ \sigma H \neq \tau H,\, I\sigma H = I\tau H}} \phi_{C,1}(\sigma b - \tau b)\Big), \tag{3.2.5}$$

where mod refers to divisibility in R. But when $I\sigma H = I\tau H$, $\sigma b - \tau b = (\sigma b) - i(\sigma b) \in \mathfrak{m}_{C,1}$ with some $i \in I$; hence $\phi_{C,1}(\sigma b - \tau b) \in \pi R$ for any such σ, τ. Therefore, the product inside the mod sign is divisible by

$$\pi^{\sum_{j=1}^{r} d_j(d_j - 1)}.$$

This proves lemma B.

3.3 – By combining Lemmas A and B, we obtain the following theorem, of which Theorem 2 (§1.5) is a direct consequence (apply Theorem C to $A = \mathfrak{O}^{\wedge}_{x_s,X}, B = \mathfrak{O}^{\wedge}_{y_s,Y}$).

[1] Recall that for any Galois extension R'/R of a normal ring R and a prime ideal $\mathfrak{p}$ of R, the Galois group acts transitively on the set of all prime ideals of R' lying over $\mathfrak{p}$; cf. e.g. [13].

Theorem C *Let* (R,π) *be as in Lemma B, and let* $A = R[[t]]$ *be the ring of formal power series in one variable over* R. *Let* B *be the integral closure of* A *in a finite extension* $Ł$ *of the quotient field* $\mathfrak{K}$ *of* A, *and put* $n = [Ł : \mathfrak{K}]$. *Assume:*

(i) *for any prime ideal* $\mathfrak{q}$ *of* A *with height* 1 *other than* (π), *the discrete valuation of* $\mathfrak{K}$ *defined by the localization at* $\mathfrak{q}$ *is unramified in* $Ł$;

(ii) *the* R*-homomorphism* $\phi_0 : A \to R$ *defined by* $t \mapsto 0$ *extends to* n *distinct homomorphisms* $B \to R$;

(iii) $\operatorname{ord} p < p-1$;

(iv) *the* A *-algebra* B *is generated by a single element.*

Then $B = A$.

Proof Note first that B is a complete normal local domain. By assumption (ii) we have, by Lemma B (for A complete; hence $r = 1, d_1 = n$),

$$\operatorname{ord}_\pi \phi_0(D_{Ł/\mathfrak{K}}(b)) \geq n(n-1) \qquad \textit{for any } b \in B. \tag{3.3.1}$$

By assumption (iv), $B = A[b_0]$ with some $b_0 \in B$; hence $D(B/A)$ is a principal A-ideal generated by $D_{Ł/\mathfrak{K}}(b_0)$. By abuse of notations, we use the symbol $D(B/A) \sim D_{Ł/\mathfrak{K}}(b_0)$ also for any generator, determined up to $A^\times$-multiples (denoted by $\sim$). Therefore, $\operatorname{ord}_\pi \phi_0(D(B/A)) \geq n(n-1)$, i.e.,

$$D(B/A) \in (t, \pi^{n(n-1)}). \tag{3.3.2}$$

On the other hand, A is a unique factorization domain and by assumption (i), $D(B/A)$ cannot be divisible by any prime of A other than that $\sim \pi$; hence

$$D(B/A) \sim \pi^\delta, \tag{3.3.3}$$

with

$$\delta \geq n(n-1). \tag{3.3.4}$$

Now, on the other hand, if $A^{(\pi)}$ denotes the discrete valuation ring obtained by localization at (π), and if we put $B^{(\pi)} = B \otimes_A A^{(\pi)}$ which is nothing but the integral closure of $A^{(\pi)}$ in $Ł$, then

$$\delta = \operatorname{ord}_\pi D(B^{(\pi)}/A^{(\pi)}); \tag{3.3.5}$$

hence by Corollary A of Lemma A, we have (noting that $n_\mathfrak{p}^*$ is a factor of $n_\mathfrak{p}$)

$$\delta \leq \sum_{\mathfrak{p}/(\pi)} n_\mathfrak{p} \left(\operatorname{ord}_\pi n_\mathfrak{p} + 1 - \frac{1}{e_\mathfrak{p}} \right), \tag{3.3.6}$$

where $\mathfrak{p}$ runs over all extensions of (π) to $B^{(\pi)}$, $n_{\mathfrak{p}}$ is the local degree and $e_{\mathfrak{p}}$ is the ramification index; hence by (3.3.4),

$$n(n-1) \leq \sum_{\mathfrak{p}/(\pi)} n_{\mathfrak{p}} \left(\mathrm{ord}_{\pi} n_{\mathfrak{p}} + 1 - \frac{1}{e_{\mathfrak{p}}} \right). \tag{3.3.7}$$

But since n is the sum of local degrees $n = \sum_{\mathfrak{p}/(\pi)} n_{\mathfrak{p}}$, we have

$$\sum_{\mathfrak{p}/(\pi)} n_{\mathfrak{p}}(n_{\mathfrak{p}} - 1) \leq n(n-1); \tag{3.3.8}$$

hence by (3.3.7)

$$\sum_{\mathfrak{p}/(\pi)} n_{\mathfrak{p}}(n_{\mathfrak{p}} - 1) \leq \sum_{\mathfrak{p}/(\pi)} n_{\mathfrak{p}} \left(\mathrm{ord}_{\pi} n_{\mathfrak{p}} + 1 - \frac{1}{e_{\mathfrak{p}}} \right); \tag{3.3.9}$$

or equivalently,

$$\sum_{\substack{\mathfrak{p}/(\pi) \\ n_{\mathfrak{p}}>1}} n_{\mathfrak{p}}(n_{\mathfrak{p}} - 1) \leq \sum_{\substack{\mathfrak{p}/(\pi) \\ n_{\mathfrak{p}}>1}} n_{\mathfrak{p}} \left(\mathrm{ord}_{\pi} n_{\mathfrak{p}} + 1 - \frac{1}{e_{\mathfrak{p}}} \right). \tag{3.3.10}$$

We now appeal to the following small sublemma:

(Sublemma) *Let p be a prime number > 2, and a, m be integers satisfying $1 \leq a \leq p-2$ and $2 \leq m$. Then*

$$a.\mathrm{ord}_{p} m + 1 \leq m - 1. \tag{3.3.11}$$

To verify this, first note that the inequality holds for $\mathrm{ord}_p m = 0$ because $m \geq 2$, and also for $\mathrm{ord}_p m = 1$ because then $a + 1 \leq p - 1 \leq m - 1$. So let $b := \mathrm{ord}_p m \geq 2$. Then $m \geq p^b$; hence

$$m - 1 \geq p^b - 1 \geq (p-1)^b + b(p-1) > 1 + b(p-1) > 1 + ab. \tag{3.3.12}$$

Now returning to the proof of Theorem C, by applying this (3.3.11) to $m = n_{\mathfrak{p}}$, $a = \mathrm{ord}_{\pi} p$, we obtain (for $n_{\mathfrak{p}} > 1$)

$$\mathrm{ord}_{\pi} n_{\mathfrak{p}} + 1 \leq n_{\mathfrak{p}} - 1; \tag{3.3.13}$$

hence by (3.3.10) and (3.3.13),

$$\sum_{\substack{\mathfrak{p}/(\pi) \\ n_{\mathfrak{p}}>1}} n_{\mathfrak{p}} \left(\mathrm{ord}_{\pi} n_{\mathfrak{p}} + 1 \right) \leq \sum_{\substack{\mathfrak{p}/(\pi) \\ n_{\mathfrak{p}}>1}} n_{\mathfrak{p}}(n_{\mathfrak{p}} - 1) \leq \sum_{\substack{\mathfrak{p}/(\pi) \\ n_{\mathfrak{p}}>1}} n_{\mathfrak{p}} \left(\mathrm{ord}_{\pi} n_{\mathfrak{p}} + 1 - \frac{1}{e_{\mathfrak{p}}} \right). \tag{3.3.14}$$

This can be satisfied only when the sum is empty, i.e., when there is no $\mathfrak{p}/(\pi)$ with $n_{\mathfrak{p}} > 1$. Therefore, $n_{\mathfrak{p}} = 1$ holds for all $\mathfrak{p}/(\pi)$; in particular, $\mathfrak{p}/(\pi)$ must be unramified. Together with the assumption (i), this means that all primes of A of height 1 are unramified in B. Since A is regular and B is normal, this implies that B/A is etale (purity of branch loci; cf. e.g. [13]). Since κ is algebraically closed and the local ring A is complete, this means that $B = A$. □

Remark The readers may (reasonably) wonder, on looking at the above inequality (3.3.14), why "the $e_{\mathfrak{p}} = 1$" case should look more contradictory than the case where $e_{\mathfrak{p}}$ is close to ∞; especially whether even an unramified extension should lead, absurdly, to a contradiction. But this is not the case. In Theorem C, we consider an integral extension of completed local rings where the degree does not contain any contributions from unramified extensions.

3.4 – We note that if Theorem C holds without the assumption (iv), then it will give (Q1) an affirmative answer, together with a local proof. Indeed, (iv) corresponds to the assumption in Theorem 2 that f be a cover in a 1-parameter space (at the specializations of splitting sections). But the fact is that the assumption (iv) cannot be dropped. In this subsection, we shall first show this by an example. It is cyclic of degree p. Recall that there is no counterexample to (Q1) for such extensions (Theorem 1), so this local counterexample should not extend to a global extension. We shall proceed to give an explanation of this situation, and give an alternative proof of Theorem 1 which does not (at least directly) rely on classification of finite commutative group schemes of order p.

For simplicity, let $\operatorname{ord} p = 1$, i.e., $R = \mathbb{W}[[\bar{\mathbf{F}}_p]]$ $(p > 2)$. Put $R' = R[\mu_p]$. As is well-known, R' (in fact, $\mathbf{Z}_p[\mu_p]$) contains a prime element π' defined by $\pi'^{p-1} = -p$ (up to μ_{p-1}-multiples), which is useful because it is an eigenvector of the $\Delta = \mathrm{Gal}(k(\mu_p)/k)$ action corresponding to the Teichmüller lifting. As in Theorem C, let $A = R[[t]]$, and $\mathfrak{K}$ be its quotient field. Put $A' = A[\pi'] = R'[[t]]$. An element of A' congruent to 1 $(\operatorname{mod} \pi')$ is a p-th power if and only if it is 1 $(\operatorname{mod} \pi'^p)$. Let $\exp^*$ denote the truncated exponential series up to degree $p - 1$.

Example 1 Let $\mathfrak{K}' = \mathfrak{K}(\pi')$. The Kummer extension $Ł' = \mathfrak{K}'(\exp^*(\pi' t)^{1/p})$ over $\mathfrak{K}'$ of degree p descends (uniquely) to a cyclic extension $Ł/\mathfrak{K}$ of degree p. The integral closure B of A in $Ł$ satisfies the assumptions (i)(ii)(iii) but not the conclusion of Theorem C, because $B \neq A$. In fact, the discrete valuation of $\mathfrak{K}$ defined by the prime ideal (p) of A is ramified in $Ł$, with ramification index p. Accordingly, B does not satisfy (iv).

In general, for each $a = a(t) \in A$, let $g_a = \exp^*(a\pi') \in A'$. This is not a p-th power if and only if $a \not\equiv 0 (\operatorname{mod} p)$. This being assumed, the Kummer extension $Ł'_a = \mathfrak{K}'(g_a^{1/p})$ descends uniquely to a cyclic extension $Ł_a/\mathfrak{K}$ of degree p, and every cyclic extension $Ł/\mathfrak{K}$ of degree p is obtained this way. The point is that the class of g_a in the multiplicative group $A'^{\times}/(A'^{\times})^p$ is again a $\Delta \backsimeq \mathrm{Gal}(\mathfrak{K}'/\mathfrak{K})$-eigenvector with the correct eigenvalue, i.e., $\delta \in \Delta$

which maps ζ to ζ^r ($\zeta \in \mu_p$) raises this class also to its r-th power. The discrete valuation of $\mathfrak{K}$ defined by (p) is ramified in $\L_a$. (In fact, the equation for $g_a^{1/p}-1$ over $\mathfrak{K}'$ is Eisenstein; hence the ramification index is p for $\L_a'/\mathfrak{K}'$ and hence also for $\L_a/\mathfrak{K}$.) It is potentially unramified if and only if $a \equiv a(0) (\operatorname{mod} p)$. All primes of A of height 1 other than (p) are unramified in $\L_a$ (because the residue class of g_a is non-zero). Finally, the prime (t) splits completely in the integral closure B_a of A in $\L_a$ if and only if $a(0) \equiv 0 (\operatorname{mod} p)$, hence in particular if $a = t$.

Now let $f : Y \to X$ be as in §1.1 for $R = \mathbb{W}[[\bar{\mathbf{F}}_p]]$ ($p > 2$). Suppose that the function field extension L/K is cyclic with degree p. We shall give a direct proof for potential unramifiedness of f, which is the main content of Theorem 1.

Let v be the discrete valuation of K defined by the generic point of the special fiber X_s. Let V be an extension of v to L, and suppose that V/v is ramified. Note that the extension V/v is then unique. We shall denote by $\bar{*}$ the residue class of $*$ (elements, fields, etc.), and by $*'$ the adjunction of μ_p to $*$; for example, $K' = K(\mu_p)$, $\bar{K}' = \bar{K}$, $X_s' \backsimeq X_s$ (canonically).

(**Claims**) (i) There exists $g \in K'$ such that $L' = K'(g^{1/p})$ and $\bar{g} = 1$;

(ii) $\mathbf{a} := \overline{(g-1)/\pi'} \in \bar{K}' = \bar{K}$ is independent of the choice of such g as in (i);

(iii) $\mathbf{a}$ is holomorphic everywhere on X_s and hence is a constant $\in \kappa$;

(iv) V is potentially unramified in L/K.

(**Proof of the Claims**) (i) By Kummer theory, there exists $g \in K'$ with $L' = K'(g^{1/p})$ and we may replace g by a multiple of any element of $(K'^{\times})^p$. Since $L' = LK'$ with L/K abelian, the class of g in the multiplicative group $K'^{\times}/(K'^{\times})^p$ must be a Δ-eigenvector with the correct eigenvalues; i.e., if $\delta \in \Delta$ raises each primitive p-th root of unity to its r-th power, then it also raises the class of g to its r-th power. The point in the following is that there exists δ with which $r \not\equiv 1 (\operatorname{mod} p)$. Since Δ leaves the valuation V' invariant and acts trivially on the residue field $\bar{K}'$, this first shows that the V'-adic order of g must be divisible by p and hence we may assume it is 0; and then that the residue class $\bar{g}$ must be a p-th power in $\bar{K}'^{\times}$ and hence we may assume it is 1.

(ii) If we replace g by $g' = gh^p$ with $\bar{h} = 1$, i.e., $h = 1 + \pi' h_0$ with $\operatorname{ord}_{v'}(h_0) \geq 0$, then $h^p \equiv 1 (\operatorname{mod} \pi'^p)$. Hence $g' \equiv g (\operatorname{mod} \pi'^p)$, whence (ii).

(iii) Since Y_η/X_η, and hence also Y_η'/X_η' is etale, the divisor of g must be a p-th power; $(g) = D^p$ on the curve X_η'. (Note that the k'-rationality of D^p implies that of D.) Since $\bar{g} = 1$, the specialization $D_s = \bar{D}$ on X_s is trivial; $D_s = (1)$. (The poles and the zeros cancel with each other after specialization.) For any divisor D' on X_η', we denote (as usual) by $L(D')$ the k'-linear space of rational functions F on K' satisfying $(F) \cdot D' \geq 1$, and by $\ell(D')$ its dimension. We use the suffix s to denote the specialization, and use similar notations for divisors on X_s. If $\mathfrak{O}_{v'}$ denotes the valuation ring of v', then there is a reduction map from $L(D') \cap \mathfrak{O}_{v'}$ (which is a free R'-module of rank $\ell(D')$) into $L(D_s')$. It is not generally surjective, but it is so when $\ell(D') = \ell(D_s')$.

Now, we shall show that $\mathbf{a}$, as a rational function on X_s, is holomorphic at any given point (which we write like a divisor here as) P_s. For this purpose, pick any point Q of $X'_\eta(k')$ such that $Q_s \neq P_s$. Let $g_X > 0$ denote the common genus of X_η and X_s, and take any $n > 2g_X - 2$, so that (by Riemann-Roch, noting that $\deg(D) = 0$ and that $D_s = (1)$):

$$\ell(Q^n D) = \ell(Q_s^n D_s) = \ell(Q_s^n) = n - g_X + 1 > 0;$$

Hence $L(Q^n D) \cap \mathfrak{O}_{v'}$ maps surjectively to $L(Q_s^n)$ which contains the constant 1; hence there exists $h \in L(Q^n D)$ such that $\bar{h} = 1$. Replace g by $g' = gh^p$, so that $(g') = (D(h))^p \geq Q^{-np}$; hence the only pole of g' on X'_η is at Q. Therefore, $(g' - 1)/\pi'$ is finite at every generalization of P_s on X'; hence it must be finite at P_s; in other words, $\mathbf{a}$ must be finite at P_s. Since P_s was an arbitrary point of $X_s(\kappa)$, $\mathbf{a}$ must be a constant. This settles (iii).

(iv) Now pick any $x_s \in X_s(\kappa)$, let t be a local parameter at x_s, and identify the complete local ring $\mathfrak{O}^\wedge_{x_s,X}$ with $A = R[[t]]$. Then as shown above, the extension Ł of the fraction field $\mathfrak{K}$ of A corresponding to L must be of the form Ł $=$ Ł$_a$, with some $a = a(t) \in A$. But since $g_a = \exp^*(\pi' a)$, we obtain

$$\mathbf{a} = \overline{(g_a - 1)/\pi'} = a(t) (\operatorname{mod} p). \tag{3.4.1}$$

Therefore, by claim (iii), we obtain $a(t) \equiv a(0) (\operatorname{mod} p)$; hence the discrete valuation (p) is potentially unramified in Ł, and hence also in L. □

4 Other related subjects

4.1 – Over general base rings Let R and $f : Y \to X$ be as in §1.1. So far, we discussed the question whether the complete splitting of one point of $X_\eta(k)$ in $Y_\eta(k)$ implies etaleness of f, under the assumption $\operatorname{ord} p < p - 1$. This question stands on a narrow unstable spot, and so we are trying to find natural generalizations. One direction for consideration is, to remove the assumption on R and see how the situation changes; this we shall discuss in the present and the following subsection. Another direction is related to the consideration of X/p or $X/4$ instead of $X_s = X/\pi$, of relevant infinitesimal automorphisms, and questions related to the regularity of Y. These are mutually related, and certainly also related to whether R had been chosen to be the minimal base ring for X. This direction requires further studies and will not be discussed here.

The situation changes *drastically* if we change R so as to contain μ_p. To see this, it suffices to look at any isogeny $\lambda : E' \to E$ of (proper smooth models over R of) elliptic curves having ordinary reduction, with λ_s inseparable of degree (say) p. This cover can possess a complete splitting k-rational point only if each point of $\operatorname{Ker}(\lambda)$ is k-rational, and this imposes $\mu_p \subset R$. This also makes the cover Galois. Now this being assumed, we see that *each* disk on $E_\eta(k)$ contains *both* splitting (completely in $E'_\eta(k)$) points *and* non-splitting points. Indeed the k-rational torsion points (and points in their small neighborhood) give splitting points, but on the other hand, the induced isogeny of formal groups $R \to R$ cannot be surjective; hence non-splitting points certainly exist inside the same disk.

What we can show, in the case of general base ring R, is the following:

Proposition 2 *Let R and $f : Y \to X$ be as in §1.1. Suppose f is not etale. Then there exists a finite subset Σ_s of $X_s(\kappa)$ such that every disk $Dsk(\eta/x_s)$ with $x_s \notin \Sigma_s$ contains a point which does not split completely in $Y_\eta(k)$. If Y is regular, then this holds with $\Sigma_s = \phi$.*

For isogenies $\lambda : E' \to E$ of elliptic curves, we can take $\Sigma_s = \phi$, because E' is regular. An example of higher genus case, where we need a non-empty exceptional set Σ_s, will be given in §4.2.

Proof As in the proof of Corollary 2-4, we remove from $X_s(\kappa)$ the (finite) set Σ_s consisting of projections of all points of $Y_s(\kappa)$ that are not regular on Y. Let $x_s \notin \Sigma_s$ and $y_s \in f^{-1}(x_s)$, so that y_s is regular on Y. As noted earlier, f not being etale, there exists $y_s \in f^{-1}(x_s)$ at which f is not etale. Choose such a point y_s. If y_s does not extend to any section in $Y(R)$, then it implies that no k-rational point x_η in the disk corresponding to x_s can split completely in $Y_\eta(k)$. So, suppose that y_s extends to a section $y = (y_\eta, y_s) \in Y(R)$, and put $f(y) = x = (x_\eta, x_s) \in X(R)$. In the same disk we shall find a non-splitting point. Since the local rings $B = \mathfrak{O}_{y_s,Y}$, $A = \mathfrak{O}_{x_s,X}$ are regular, each section y (resp. x) is defined by a single equation $T = 0$ (resp. $t = 0$). Since the maximal ideal of R is generated by (π), the maximal ideals of B, A have the generators $\mathfrak{m}_B = (T, \pi)$, $\mathfrak{m}_A = (t, \pi)$, respectively. Hence B/π is also regular; hence y_s is smooth on Y_s, lying on just one irreducible component Π of Y_s of multiplicity one. Since f is not etale at y_s and hence also on the generic point of Π, $f_s : \Pi \to X_s$ must be inseparable. Therefore, by completion we obtain $B^\wedge = R[[T]] \supset A^\wedge = R[[t]]$ which is a finite integral extension, where

$$t = F(T) = \sum_{n \geq 0} a_n T^n, \tag{4.1.1}$$

with $a_0 = 0$ (because the section $T = 0$ lies above $t = 0$), $a_n \equiv 0 (\operatorname{mod} \pi)$ for all n with $(n, p) = 1$ (because of the inseparability mentioned above); but not all the coefficients are divisible by π. Let m be the smallest positive integer such that $a_m \not\equiv 0 (\operatorname{mod} \pi)$. Now take, say, $t_0 = \pi$, and decompose the power series $F(T) - t_0$ into the product of an Eisenstein polynomial $E(T)$ of degree m and a unit of $B^\times$. Then since $E(T)$ is irreducible over k, the R-section of X defined by $t = t_0$ cannot extend to an R-section of Y through y_s. Hence it cannot split completely in $Y_\eta(k)$. $\square$

4.2 – Kummer covers of degree p and their global invariants

Let us study in more detail the following case; $\mu_p \subset R$, f_η is cyclic with degree p, and f_s is *purely inseparable*. Then Y_s is irreducible, reduced, and has some cuspidal singularities. In this case, the "safety" exceptional set Σ_s in Proposition 2 will not be chosen in connection with irregular points as in its proof, but from a different reason. By using the "associated invariant differential", we can give an upper bound of the cardinality of Σ_s for this case:

$$|\Sigma_s| \leq 2(g_X - 1)/p. \tag{4.2.1}$$

To show this, let $\mu : Y_s^* \to Y_s$ be the normalization. Then the p-th power ($\mathbf{F}_p$-)isomorphism Φ_p maps X_s onto Y_s^*;

$$Y_s^* \xrightarrow{\mu} Y_s \xrightarrow{f_s} X_s, \tag{4.2.2}$$

$$X_s \overset{\Phi_p}{\backsimeq} Y_s^*. \tag{4.2.3}$$

Let $(Q_s^*,\, Q_s,\, P_s)$ be a triple of corresponding points of $(Y_s^*,\, Y_s,\, X_s)$. Their completed local rings and inclusion relations can be expressed as

$$\kappa[[T]] \supset B = \mathcal{O}^{\wedge}_{Q_s, Y_s} \supset \kappa[[T^p]]. \tag{4.2.4}$$

As in [17](IV-1), call $n = n_{Q_s}$ the conductor exponent, i.e., the smallest non-negative integer satisfying $B \supset T^n\kappa[[T]]$, and put $\delta = \delta_{Q_s} = \dim(\kappa[[T]]/B)$. Call $M = M_{Q_s}$ the collection of all such $m' \in \mathbf{N}$ that B contains a power series starting with $T^{m'}$, which forms a submonoid of $\mathbf{N}$. Note that $\delta = |\mathbf{N} \setminus M|$ and $n - 1$ is the largest element of $\mathbf{N} \setminus M$. Clearly, M contains p, and some other element $\not\equiv 0 (\mathrm{mod}\, p)$, of which let $m = m_{Q_s}$ be the smallest. Since M contains the monoid $\langle p, m\rangle$ generated by p and m, we have

$$(p-1)(m_{Q_s} - 1) \geq 2\delta_{Q_s}, \tag{4.2.5}$$

where the equality holds if and only if $M = \langle p, m\rangle$. We are going to show that

$$n_{Q_s} = 2\delta_{Q_s} = (p-1)(m_{Q_s} - 1), \tag{4.2.6}$$

$$2(g_X - 1) = \sum_{Q_s} (m_{Q_s} - 1). \tag{4.2.7}$$

Let L/K be the corresponding function field extension. By our assumption $\mu_p \subset R$, L/K is a Kummer extension, $L = K(g^{1/p})$ with some $g \in K^\times$ determined up to $(K^\times)^p$-multiples. In our case where an inseparable residue extension arises, we can choose g such that $\bar{g} = g_{X_s} \neq 0, \infty$. There are two cases;

(Case 1) $\bar{g}^{1/p}$ generates $\bar{L}$;

(Case 2) One can choose such g that satisfies $g \equiv 1 (\mathrm{mod}\, \pi^{dp})$, with some $d < (\mathrm{ord}_\pi p)/(p-1)$ (choose d to be as large as possible).

Let us mainly consider Case 1. Then the differential $\omega_1 = d\bar{g}/\bar{g}$ is independent of the choice of g. By the etaleness of f_η, the order of $\bar{g}$ at each point of X_s must be divisible by p; hence $d\bar{g}/\bar{g}$ cannot have any poles on X_s; i.e., ω_1 is a differential of the first kind on X_s (invariant by the Cartier operator, being of $d\log$-type).

Now, given (Q_s^*, Q_s, P_s), by an argument similar to that in §3.4, we may change g and assume that $g \in \mathcal{O}_{P_s,X}$ with $g \equiv 1 (\operatorname{mod} \mathfrak{m}_{P_s,X})$; hence $g^{1/p} \in \mathcal{O}_{Q_s,Y}$; hence $\bar{g}^{1/p} \in B$, where we use $T = \bar{t}^{1/p}$. Let $m' = m'_{Q_s}$ be the smallest positive integer not divisible by p such that the term $T^{m'}$ appears in the series of $\bar{g}^{1/p}$ in $B \subset \kappa[[T]]$, or equivalently, such that $\bar{t}^{m'}$ appears in the series of $\bar{g}$ in $\kappa[[\bar{t}]] = \mathcal{O}^{\wedge}_{P_s,X_s}$. This means that the order of $d\bar{g}$; hence that of ω_1 at P_s is equal to $m'_{Q_s} - 1$;

$$\begin{aligned} \text{(4.2.8)} \qquad\qquad \operatorname{ord}_{Q_s}(\omega_1) &= m'_{Q_s} - 1, \\ \text{(4.2.9)} \qquad\qquad 2(g_X - 1) = \deg(\omega_1) &= \sum_{Q_s} (m'_{Q_s} - 1). \end{aligned}$$

By their definitions, we also have

$$\text{(4.2.10)} \qquad\qquad m'_{Q_s} \geq m_{Q_s}.$$

Now, since $f_\eta : Y_\eta \to X_\eta$ is etale with $\deg f_\eta = p$, the invariance of the arithmetic genus for the two fibers of Y gives $p(g_X - 1) + 1 = g_X + \sum_{Q_s} \delta_{Q_s}$, i.e.,

$$\text{(4.2.11)} \qquad\qquad (p-1)(g_X - 1) = \sum_{Q_s} \delta_{Q_s}.$$

From (4.2.9)(4.2.10), we obtain

$$\text{(4.2.12)} \qquad 2(p-1)(g_X - 1) = \sum_{Q_s} (p-1)(m'_{Q_s} - 1) \geq \sum_{Q_s} (p-1)(m_{Q_s} - 1);$$

and by (4.2.5) and (4.2.11),

$$\text{(4.2.13)} \qquad \sum_{Q_s} (p-1)(m_{Q_s} - 1) \geq 2 \sum_{Q_s} \delta_{Q_s} = 2(p-1)(g_X - 1).$$

Therefore, all "$\geq$" involved must be equalities, and we have

$$\begin{aligned} \text{(4.2.14)} \qquad\qquad m'_{Q_s} &= m_{Q_s}, \ 2\delta_{Q_s} = (p-1)(m_{Q_s} - 1), \\ \text{(4.2.15)} \qquad\qquad M_{Q_s} &= \langle p, m_{Q_s} \rangle, \ n_{Q_s} = 2\delta_{Q_s}. \end{aligned}$$

We have also proved the equalities (4.2.6)(4.2.7).

Now let us show that if $m'_{Q_s} < p$, then the disk ($\subset X_\eta(k)$) above P_s contains a non-splitting point. By the definition of $m' = m'_{Q_s}$, $g = g(t) \in \mathcal{O}^\wedge_{P_s,X} = R[[t]]$ can be expressed as $g(t) \equiv 1 + cG(t)(\mathrm{mod}\, t^{m'+1})$, with some $c \in R^\times$ and a monic polynomial $G(t)$ of degree m' whose lower degree coefficients are all divisible by π. It is easy to see that there exists $t_0 \in \pi R$ such that $\mathrm{ord}_\pi G(t_0) = m'$. (Indeed, for each fixed zero τ of $G(t)$ in a finite extension of R, the inequality $\mathrm{ord}(t_0 - \tau) > 1$ for $t_0 \in \pi R$ can hold on at most one residue class mod π^2.) For such t_0, we have $g(t_0) - 1 \notin \pi^{m'+1}R$; hence $g(t_0) - 1 \notin \pi^p R$, which implies that $g(t_0)$ cannot be a p-th power element of $R^\times$. Hence the point of $X_\eta(k)$ defined by $t = t_0$ cannot split completely in $Y_\eta(k)$. Now if H denotes the number of points Q_s satisfying $m'_{Q_s} \geq p+1$, then (4.2.7) gives $H \leq 2(g_X - 1)/p$, as desired.

In Case 2, the invariant differential on X_s is $\omega_1 = d\mathbf{a}$, where $\mathbf{a} = \overline{((g-1)/\pi^{dp})}$. and $\mathbf{a}^{1/p}$ is a generator of $\bar{L}$. In this case, ω_1 is killed by the Cartier operator, being an exact differential.

Example 2 ($p = g_X = 3$) Let $\kappa = \bar{\mathbf{F}}_3$, $R = \mathbb{W}[[\kappa]][\mu_3]$, and k be the quotient field of R. Choose $i \in R$ with $i^2 = -1$. Let $X \subset \mathbf{P}^2 = \{(x:y:z)\}$ be the smooth plane quartic

$$x^3y - xy^3 + (4xy - 3y^2)z^2 + z^4 = 0 \qquad (4.2.16)$$

over R. Then $P' = (i : i : 1)$ and $P" = (-i : -i : 1)$ are points of inflexion on $X_\eta = X \otimes k$, with the tangent lines $\ell' : z + ix = 0$, $\ell" : z - ix = 0$, respectively. Moreover, $X_\eta \cap \ell' \setminus \{P'\} = X_\eta \cap \ell" \setminus \{P"\} = \{P\}, P = (0:1:0)$. Thus, the function $g = (z+ix)/(z-ix) \in K = k(X)$ has divisor

$$(g) = (P'/P")^3.$$

The cover $f : Y \to X$ corresponding to $L = K(g^{1/3})$ is cyclic of order 3, with f_η etale and f_s purely inseparable. The differential $\omega_1 = d\bar{g}/\bar{g}$ on the reduced curve X_s over κ has the divisor

$$(\omega_1) = P_s^4.$$

Hence Y_s has a unique cuspidal singularity Q_s above P_s, with $m_{Q_s} = 5$ and $\delta_{Q_s} = 4$. It is easy to see that all points on the disk $Dsk(P_s) \subset X_\eta(k)$ split completely in $Y_\eta(k)$. Thus, the exceptional set Σ_s in Proposition 2 is really necessary.

Incidentally, the curve X_s has p-rank $= 3 = g_X$, all points are points of inflexion, i.e., meet the tangent line with the order of contact ≥ 3, which is equal to 4 (the Weierstrass point) if and only if the point lies on $z = 0$ (P_s being one of such). In connection with the abelian argument in §2.3, we note the following. The canonical map $\phi : X \to J$ into the Jacobian with $\phi(P) = 0_J$ induces a mapping from the disk $Dsk(P_s)$ into the disk neighborhood $Dsk(0_J)$ of 0_J, but in the present example, this local image cannot generate $Dsk(0_J)$ (intuitively, the image of the curve disk has the highest possible contact with the hyperplane corresponding to ω_1; it is too much like a straight line to be able to generate the whole disk $Dsk(0_J)$.

4.3 – $\mathbf{P^1} - \{\mathbf{0}, \mathbf{1}, \infty\}$. The questions and results discussed so far should naturally be generalized to the case of tame covers. Here, we shall only formulate the case of $\mathbf{X} = \mathbf{P}^1 - \{0, 1, \infty\}$, the projective t-line $\tilde{\mathbf{X}} = \mathbf{P}^1$ over R, *minus* 3 sections defined by $t = 0, 1, \infty$ (as usual, to be called *cusps*). Actually, the question arose from the consideration of this case. Let the base ring R be as in §1.1, $K = k(t)$ be the function field of $\mathbf{X}$, L/K be a finite extension, and $\tilde{f} : \tilde{\mathbf{Y}} \to \tilde{\mathbf{X}}$ (resp. $f : \mathbf{Y} \to \mathbf{X}$) be the integral closures in L of $\tilde{\mathbf{X}}$ (resp. $\mathbf{X}$). Assume that f_η is etale, and that the ramification indices of $\tilde{f}_\eta$ above each cusp are not divisible by p.

Question (Q2) *Assume* $\operatorname{ord} p < p - 1$. *Let* $x = (x_\eta, x_s) \in \tilde{\mathbf{X}}(R)$ *be such that* x *is either a cuspidal section, i.e.,* $x_\eta = 0, 1,$ *or* ∞, *or is disjoint from them, i.e.,* $x_s \neq 0, 1, \infty$. *Suppose that the following conditions are satisfied for each closed point* $y_\eta \in \tilde{\mathbf{Y}}_\eta$ *above* x_η*; (i)* y_η *is* k*-rational; (ii) when* x *is cuspidal, each* y_η *moreover has a local parameter* τ *such that* $\operatorname{ord}_\pi(\tau^e/t_x)|_{y_\eta} = 0$*, where* $e = e(y_\eta/x_\eta)$ *is the ramification index and* t_x *is a "good" local parameter at* x_η*, i.e., its reduction serves also as a local parameter at* x_s.

Then, is f *necessarily etale?*

Let b be either a tangential base point, e.g., $\overrightarrow{01}$, or a k-rational point $x_\eta \in \mathbf{X}_\eta(k)$ such that $x_s \neq 0, 1, \infty$. Consider the quotient $\pi_1^\tau(\mathbf{X}_{\bar{k}}, b)$ of the fundamental group $\pi_1(\mathbf{X}_{\bar{k}}, b)$ defined by the condition that the ramification indices above cusps are not divisible by p. The absolute Galois group $G_k = \mathrm{Gal}(\bar{k}/k)$ acts on this group, and we obtain a canonical surjective homomorphism

$$\psi_{\mathbf{X},b} : \pi_1^\tau(\mathbf{X}_{\bar{k}}, b)_{G_k} \to \pi_1^{tame}(\mathbf{X}_s, b_s), \tag{4.3.1}$$

from the G_k-coinvariant of $\pi_1^\tau(\mathbf{X}_{\bar{k}}, b)$ onto the tame fundamental group of $\mathbf{X}_s = \mathbf{P}^1_\kappa - \{0, 1, \infty\}$. When $b = \overrightarrow{01}$, $\pi_1^\tau(\mathbf{X}_{\bar{k}}, b)_{G_k}$ can be identified naturally with the Galois group $\mathrm{Gal}(M/K)$, where M is the maximal Galois extension *contained in* $k\{\{t\}\} = \cup_{N \geq 1} k((t^{1/N}))$ (the field of Puiseux series over k) which is unramified outside cusps and such that the ramification indices above cusps are not divisible by p. It is easy to see that (Q2) is equivalent to:

Question (Q2') *Assume* $\operatorname{ord} p < p - 1$. *Then, is* $\psi_{\mathbf{X},b}$ *an isomorphism?*

For each positive integer $N \not\equiv 0 (\operatorname{mod} p)$, we have the Fermat cover $f_N : \mathbf{Y}_N \to \mathbf{X}$ of level N defined by the function field extension $k(t^{1/N}, (1-t)^{1/N})$ of K. The cover f_N is abelian, with Galois group $(\mathbf{Z}/N)^2$, the ramification index above each cusp is exactly N, and is etale on $\mathbf{Y}_N$. By using the Fermat covers f_N for all $N \not\equiv 0 (\operatorname{mod} p)$, it is easy to deduce the following:

(i) *If the equivalent questions (Q1)(Q1') have affirmative answers, then so do (Q2)(Q2').*

(ii) *Theorem 1 is valid also in this case.*

(iii) *Theorem 2 remains valid if the given splitting point* $x_\eta \in \mathbf{X}_\eta(k)$ *is such that* $x_s \neq 0, 1, \infty$. *When* $x_\eta \in \{0, 1, \infty\}$, *say* $x_\eta = 0$, *it remains valid under the following modification:*

Replace $f : \mathbf{Y} \to \mathbf{X}$ by the integral closures $f' : \mathbf{Y}' \to \mathbf{X}'$ in $L(t^{1/N})/K(t^{1/N})$ (so that $\mathbf{X}' = \mathbf{P}^1 - \{0, \mu_N, \infty\}$), N being the least common multiple of ramification indices above $t = 0$ in L/K. Assume that f' is a cover in a 1-parameter space at each point of $\mathbf{Y}'_s$ above 0. Then, under the assumptions $\operatorname{ord} p < p - 1$ and $k(f'^{-1}_\eta(0)) = k$, it follows that f is etale.

5 Appendix: The associated differential.

5.1 – Let R, π, k be as in §1.1. When the special fiber f_s of the cover $f : Y \to X$ contains (potential) inseparability of degree, say q which is a power of p, we can canonically construct a certain rational differential of degree $q - 1$ on a finite separable cover of X_s. The idea goes back to [7]. (The author has not heard that something similar has appeared in a different language since then.) Although delicate phenomena are of codimension 2, the definition itself is 1-dimensional, local.

Thus, take any complete discrete valuation field $\mathbb{K}$ containing k and extending the valuation of k, such that

(i) $\mathbb{K}/k$ is absolutely unramified, i.e., π serves also as a prime element of $\mathbb{K}$;

(ii) the residue field $\bar{\mathbb{K}}$ is a finitely generated 1-dimensional extension of κ.

Since κ is perfect, $\bar{\mathbb{K}}$ is separably generated over κ, and since $\bar{\mathbb{K}}$ is 1-dimensional, the only purely inseparable extensions of $\bar{\mathbb{K}}$ are $\bar{\mathbb{K}}^{1/p^n}$, and $[\bar{\mathbb{K}}^{1/p^n} : \bar{\mathbb{K}}] = p^n$ $(n = 1, 2, ...)$. Let $\mathbb{O}$ denote the valuation ring of $\mathbb{K}$, and $\Omega = \Omega_{\mathbb{O}}$, the module of continuous differentials, which is a principal $\mathbb{O}$-module. The reduction $\bar{\Omega} = \Omega \otimes_{\mathbb{O}} \bar{\mathbb{K}}$ can be identified with $\Omega_{\bar{\mathbb{K}}}$, the module of differentials of $\bar{\mathbb{K}}$.

Let $\mathbb{L}/\mathbb{K}$ be a finite extension, with the valuation ring $\mathbb{O}'$, the module of continuous differentials Ω', and let $i : \Omega \to \Omega'$ be the $\mathbb{O}$-module homomorphism induced from the inclusion $\mathbb{O} \subset \mathbb{O}'$. Let $\mathcal{D}$ be the "different", i.e., the $\mathbb{O}'$-ideal defined by $\mathbb{O}' i(\Omega) = \mathcal{D}\Omega'$. First let us consider the case where the ramification index $e = 1$ and $\bar{\mathbb{L}}/\bar{\mathbb{K}}$ is purely inseparable, $\bar{\mathbb{L}} = \bar{\mathbb{K}}^{1/q}$. Then $\mathcal{D} = \pi^\nu \mathbb{O}'$ with some positive integer ν, and the associated differential ω, which is a non-zero element of $\Omega_{\bar{\mathbb{K}}}^{\otimes(q-1)}$, is defined as follows.

Pick any $\xi \in \Omega$ such that $\Omega = \mathbb{O}\xi$. Note that $\pi^{-\nu} i(\xi) \in \Omega', \notin \pi\Omega'$. Hence $\overline{\pi^{-\nu} i(\xi)}$ defines a non-zero element of $\bar{\Omega}' = \Omega_{\bar{\mathbb{L}}}$. The q-th power map $\bar{\mathbb{L}} \to \bar{\mathbb{K}}$ induces an equivariant morphism $\Phi : \Omega_{\bar{\mathbb{L}}} \to \Omega_{\bar{\mathbb{K}}}$, which is an $\mathbf{F}_p$-module isomorphism; hence $\bar{\zeta} := \Phi(\overline{\pi^{-\nu} i(\xi)})$ is a non-zero element of $\Omega_{\bar{\mathbb{K}}}$. Now define

$$\omega = \bar{\xi}^{\otimes q} / \bar{\zeta} \in \Omega_{\bar{\mathbb{K}}}^{\otimes(q-1)}. \tag{5.1.1}$$

The point is that this ω is independent of the choice of ξ. This is because if ξ is replaced by $g\xi$ $(g \in \mathbb{O}, \bar{g} \neq 0)$, then the numerator and the denominator in (5.1.1) are both multiplied by $\bar{g}^q$. The differential ω depends only on the choice of the initial prime element π of R. Thus in the absolutely canonical sense, it is determined up to $\kappa^\times$-multiples.

If the ramification index e in $\mathbb{L}/\mathbb{K}$ is not equal to 1, then for a suitable constant field extension k'/k, $\mathbb{L}k'/\mathbb{K}k'$ will have the ramification index equal to 1. This is due to Epp [3]. (The assumption in [3] is that $\bar{\mathbb{L}}^{p^\infty}$, which in this case is κ, be separably algebraic over κ.) If $e = 1$ but $\bar{\mathbb{L}}/\bar{\mathbb{K}}$ is not purely inseparable, then by replacing $\mathbb{K}$ by the maximal unramified subextension $\mathbb{K}^*$ in $\mathbb{L}$ we can define the differential ω which will be a differential of degree $q^* - 1$ on $\bar{\mathbb{K}}^*$, where $q^* = [\bar{\mathbb{L}} : \bar{\mathbb{K}}^*]$.

Finally, if $e = 1$ with $\bar{\mathbb{L}}/\bar{\mathbb{K}}$ purely inseparable, and if we make further constant field extension k'/k, then ($e = 1$ holds also for $\mathbb{L}k'/\mathbb{K}k'$ trivially, and) $\overline{\mathbb{L}k'}/\overline{\mathbb{K}k'}$ is again purely inseparable with the same degree, and the differential ω for $\mathbb{L}k'/\mathbb{K}k'$ is the same as that for $\bar{\mathbb{L}}/\bar{\mathbb{K}}$. Here, note that the different $\mathcal{D}$ for $\mathbb{L}k'/\mathbb{L}$ and $\mathbb{K}k'/\mathbb{K}$ have the common generators as that of k'/k, and hence $\mathcal{D}$ for $\mathbb{L}k'/\mathbb{K}k'$ and $\mathbb{L}/\mathbb{K}$ also have common generators. Thus,

Any ramified and not potentially unramified extension $\mathbb{L}$ *of* $\mathbb{K}$ *gives rise canonically to an associated differential of degree* $q-1$ (q *some power of* p) *on a finite separable extension* $\bar{\mathbb{K}}^*$ *of* $\bar{\mathbb{K}}$.

(Transitivity) When we have a tower $\mathbb{K} \subset \mathbb{L} \subset \mathbb{M}$ of finite extensions each with ramification index 1, we have the following transitivity relation, which follows directly from the definitions:

$$\omega_{\mathbb{M}/\mathbb{K}^{**}} = \Phi(\omega_{\mathbb{M}/(\mathbb{L}\mathbb{K}^{**})}) \otimes i(\omega_{\mathbb{L}/\mathbb{K}^*})^{\otimes q'} \quad \in \Omega_{\overline{\mathbb{K}^{**}}}^{\otimes (qq'-1)}. \tag{5.1.2}$$

Here, $\mathbb{K}^*$, $\mathbb{K}^{**}$ are the maximal unramified subextensions of $\mathbb{K}$ in $\mathbb{L}$, $\mathbb{M}$, respectively, so that $\mathbb{L}\mathbb{K}^{**}$ is the maximal unramified subextension of $\mathbb{L}$ in $\mathbb{M}$. Thus, $\bar{\mathbb{L}}/\overline{\mathbb{K}^*}$ and $\overline{\mathbb{L}\mathbb{K}^{**}}/\overline{\mathbb{K}^{**}}$ are purely inseparable with the same degree (denote by q), and so is $\bar{\mathbb{M}}/\overline{\mathbb{L}\mathbb{K}^{**}}$ (the degree denoted by q'). Each suffix in ω indicates the relavant extension;

$$\Phi : (\Omega_{\overline{\mathbb{L}\mathbb{K}^{**}}})^{\otimes (q'-1)} \to (\Omega_{\overline{\mathbb{K}^{**}}})^{\otimes (q'-1)} \tag{5.1.3}$$

is induced from the q-th power morphism of the base fields, and $i : \Omega_{\overline{\mathbb{K}^*}} \to \Omega_{\overline{\mathbb{K}^{**}}}$ is induced from the base field inclusion.

This can be applied, for example, to a Galois extension $\mathbb{M}/\mathbb{K}$ for various intermediate fields $\mathbb{L}$.

(An explicit description) Let $[\mathbb{L} : \mathbb{K}] = [\bar{\mathbb{L}} : \bar{\mathbb{K}}] = q$, with $\bar{\mathbb{L}}/\bar{\mathbb{K}}$ purely inseparable. Take $x \in \mathbb{K}$ such that $\bar{x} \notin \bar{\mathbb{K}}^p$, which implies $\bar{\mathbb{L}} = \bar{\mathbb{K}}(\bar{x}^{1/q})$. Take $y \in \mathbb{L}$ such that $\bar{y} = \bar{x}^{1/q}$, which implies $\mathbb{O}' = \mathbb{O}[y]$; hence $\mathcal{D} = (f'(y))$, where

$$f(y) = \sum_{i=0}^{q} a_i y^{q-i} = 0 \qquad (a_i \in \mathbb{O},\ a_0 = 1,\ \bar{a}_i = 0 (1 \leq i < q), \bar{a}_q = -\bar{x}) \tag{5.1.4}$$

is the monic irreducible equation for y over $\mathbb{K}$. Write

$$f'(y) = \pi^\nu \sum_{i=0}^{q-1} b_i y^{q-i-1} \qquad (b_i \in \mathbb{O},\ (\bar{b}_0, \cdots, \bar{b}_{q-1}) \neq (0, \cdots, 0)). \tag{5.1.5}$$

Then by

$$\pi^{\nu}\sum_{i=0}^{q-1} b_i y^{q-i-1}dy + (\sum_{i=0}^{q}(da_i/dx)y^{q-i})dx = 0, \tag{5.1.6}$$

and by the above formulas for $\bar{a}_i$ we obtain

$$\bar{\zeta} = \Phi(\sum_{i=0}^{q-1}\bar{b}_i\bar{y}^{q-i-1}d\bar{y}) = \sum_{i=0}^{q-1}\bar{b}_i^q\bar{x}^{q-i-1}d\bar{x}; \tag{5.1.7}$$

hence

$$\omega = \frac{(d\bar{x})^{\otimes(q-1)}}{\sum_{i=0}^{q-1}\bar{b}_i^q\bar{x}^{q-i-1}}. \tag{5.1.8}$$

In the two extreme cases where $(\bar{b}_0, \cdots \bar{b}_{q-1}) = (*, 0, \cdots, 0)$ (resp. $(0, \cdots 0, *)$), we have $\omega = \omega_1^{\otimes(q-1)}$, with $\omega_1 = d\bar{x}/\bar{x}$ (resp. $\omega_1 = d\bar{x}$). The examples corresponding to Case 1 (resp. Case 2) in §4.2 are of this sort.

(The case of CR systems of Shimura curves) Let

$$\{X \xleftarrow{f} X_0(\mathfrak{p}) \xrightarrow{f'} X'\} \tag{5.1.9}$$

be the system of relative curves as in §1.5, and $K \subset L \supset K'$ be the corresponding function fields. The special fiber of $X_0(\mathfrak{p})$ has two components Π, Π', with the properties that the projections $\Pi \to X_s$, $\Pi' \to X'_s$ are isomorphisms and $\Pi' \to X_s$, $\Pi \to X'_s$ are purely inseparable with degree q. The generic points of X_s, Π, Π', X'_s define discrete valuations of K, L, L, K', respectively. The completions with respect to these valuations yield

$$K_{X_s} \subset L_{\Pi'} \backsimeq K'_{X'_s}, \qquad K_{X_s} \backsimeq L_{\Pi} \supset K'_{X'_s}, \tag{5.1.10}$$

where the non-isomorphic inclusions are of degree q with purely inseparable residue extensions. We thus obtain a pair (ω, ω') of differentials of degree $q-1$ on (X_s, X'_s). They are holomorphic and the divisors are

$$(\omega) = (S_1 \cdots S_H)^2, \qquad (\omega') = (S'_1 \cdots S'_H)^2; \qquad (H = (q-1)(g_X - 1)) \tag{5.1.11}$$

where $S_1, \cdots S_H$ (resp. $S'_1, \cdots S'_H$) are the projections of $\Pi \cap \Pi' \subset X_0(\mathfrak{p})_s$ on X_s (resp. X'_s). Moreover, the two extensions have the common different exponent ν, and the local equation for $X_0(\mathfrak{p})$ at each intersection is of the form $uv = \pi^{\nu}$ with *this* ν. Therefore, $X_0(\mathfrak{p})$ is regular if and only if $\nu = 1$. In particular, it is regular when $q = p$ and $\operatorname{ord} p = 1$ (cf. [7]). This pair of differential is closely related to the first infinitesimal lifting of the special fiber of the triple (5.1.9) [11, 12].

I mention this old work, because the lifting of inseparable covers to etale covers does not seem to have been studied so much, and believe that the associated differential is at least closely related to this subject. I might also add that when the covering system has a non-compact automorphism group, as in the case of Shimura curves, the associated differential is the invariant of the whole system. There is also a system of curves over κ having non-compact automorphism group and an invariant differential for which we do not know yet anything about its liftability to characteristic 0.

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