RIMS-1730

Quandle cocycle invariants of Lefschetz fibrations over the 2-sphere

By

Takefumi NOSAKA

August 2011

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

Quandle cocycle invariants of Lefschetz fibrations over the 2-sphere

Takefumi Nosaka

Abstract

We introduce quandle cocycle invariants of Lefschetz fibrations over the 2-sphere, using quandle cocycles of Dehn quandles. We find a cocycle with non-abelian coefficients so that the associated invariant is equivalent to the signature of Lefschetz fibrations. We compute the second quandle homologies of some Dehn quandles, and also study cocycle invariants with local coefficients.

Keywords Mapping class group, quandle homology, signature, link, Lefschetz fibration

1 Introduction

A quandle is a set with a binary operation whose definition was motivated by studies of oriented $(n, n-2)$-dimensional C^∞-manifold pairs $N \subset M$. In fact, for any such pair, the fundamental quandle $Q(M, N)$ was defined [Joy] in analogy of the fundamental groups. Given a quandle X, Fenn, Rourke and Sanderson [FRS1] defined a classifying space BX, as an analog of that of groups. By modifying the cohomology $H^*(BX; A)$, Carter et al. [CJKLS] defined cohomology groups $H^*_Q(X; A)$ of quandles X, and introduced quandle cocycle invariants of links $L \subset S^3$. Roughly speaking, the invariant is defined by the pairing between "the fundamental class" of the fundamental quandle $Q(S^3, L)$ and a 2-cocycle in $H^2_Q(X; A)$. In [CEGS], the invariants are generalized for 2-cocycles with local coefficients and with non-abelian coefficients.

The fundamental quandles are also suitable for 4-dimensional Lefschetz fibrations. To explain this, for an oriented closed surface Σ_g of genus g, we equip the following set $\mathcal{D}_g$ with a quandle structure, called Dehn quandle [Y, Zab] (see Definition 2.3):

$$\mathcal{D}_g := \{ \text{ isotopy classes of (unoriented) essential simple closed curves } \gamma \text{ in } \Sigma_g \ \}, \qquad (1)$$

where "essential" means that no disk in Σ_g bounds γ. Further set up the fundamental quandle $Q(\mathcal{B}, \overline{\mathbf{m}})$, where $\mathcal{B}$ is either the 2-disk or the 2-sphere and $\overline{\mathbf{m}} := \{b_1, \dots, b_m\}$ are m-points in $\mathcal{B}$. As is shown in [Y], the isotopy class of Lefschetz fibrations over $\mathcal{B}$ is determined by quandle homomorphisms $Q(\mathcal{B}, \overline{\mathbf{m}}) \to \mathcal{D}_g$. This work arises from the fact, due to Matsumoto [M], that Lefschetz fibrations can be interpreted from monodromies using the mapping class group $\mathcal{M}_g$. Furthermore, from the viewpoint of monodromy, Zablow [Zab] combinatorially constructed an invariant of Lefschetz fibrations over the disk; however the invariant is valued in the quandle homology $H^Q_2(\mathcal{D}_g; \mathbb{Z})$ with trivial coefficients and there was no computation of $H^Q_2(\mathcal{D}_g; \mathbb{Z})$.

In this paper, we introduce and study quandle cocycle invariants of Lefschetz fibrations over the 2-sphere. Let $T_{m,m}$ be the (m, m)-torus link in S^3. The point is to show an isomorphism between two fundamental quandles $Q(S^2, \overline{\mathbf{m}})$ and $Q(S^3, T_{m,m})$ (Lemma 2.6). Hence, by the observation of Yetter [Y], Lefschetz fibrations over S^2 are reduced to quandle homomorphisms $Q(S^3, T_{m,m}) \to \mathcal{D}_g$. We then define quandle cocycle invariants of Lefschetz fibrations to be those of the (m, m)-torus link using the quandle $\mathcal{D}_g$. Hence the definition is applicable for 2-cocycles with local coefficients and with non-abelian coefficients (Definitions 2.9, 2.11).

However, to compute the invariants, we have to find presentations of quandle cocycles or determine the quandle (co)homologies. We first show that a cocycle invariant produces the signature of Lefschetz fibrations. Precisely, we find a non-abelian cocycle so that the associated invariant is equivalent to the signature (Theorem 3.1). A key of the proof is to present explicitly the "global monodromy" over S^2 from the view of non-abelian cocycles; Then our result follows from Endo and Nagami [EN] who gave a formula of the signature obtained from the global monodromy by "Meyer 2-cocycle". In other word, our result implies a translation of Meyer 2-cocycle to a non-abelian cocycle of $\mathcal{D}_g^{\rm ns}$.

We next compute quandle homologies with trivial coefficients. To begin, we show an epimorphism from $H_2^Q(\mathcal{D}_g;\mathbb{Z})$ onto $\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/10$ for $g\geq 3$ (Corollary 4.8). It is conjectured that $H_2^Q(\mathcal{D}_g;\mathbb{Z})$ are not so large. Furthermore, we consider the following subquandle of $\mathcal{D}_g$:

$$\mathcal{D}_g^{\rm ns} := \{\gamma\in\mathcal{D}_g \mid \gamma \text{ does not separate } \Sigma_g \}, \tag{2}$$

which plays a key role for understanding Lefschetz fibrations of "type mI" (see Remark 2.13 in details): Then we determine its homology $H_2^Q(\mathcal{D}_g^{\rm ns};\mathbb{Z})\cong\mathbb{Z}/2$ for $g\geq 5$ (Proposition 4.4), by using the Harer-Ivanov stability theorem [H1, Iva]. So far these homologies have such non-trivial parts, meanwhile we observe that the parts capture no non-trivial cocycle invariants (Remark 4.9).

By the observation of $H_2^Q(\mathcal{D}_g;\mathbb{Z})$, to discover non-trivial invariants, we shall study cocycles with local coefficients. We then estimate the cocycle invariants for the class of Lefschetz fibrations of type mI. The homotopy group $\pi_2(BX)$ of the space BX mentioned above is known to be universal among such cocycle invariants [FRS1]. By computations of $\pi_2(BX)$ [see Theorem 5.5], we conclude that if the coefficient group A satisfies $A\otimes\mathbb{Z}/6=0$, then the cocycle invariant is shown to be trivial (Corollary 5.6). In summary, for non-trivial cocycle invariants, we have to set up coefficients with 2- or 3-torsions.

Accordingly we give an easy algorithm to produce some 2-cocycles with the local coefficients: These 2-cocycles are obtained from the abelianizations of subgroups of $\mathcal{M}_g$ (Proposition 5.2). Unfortunately the author finds no non-trivial invariants using such 2-cocycles. However, in knot theory, some quandle cocycle invariants with local coefficients are more affluent than those with trivial coefficients [CEGS, IK, Kab, No2]. So a problem for the future is to find cocycles of the Dehn quandle $\mathcal{D}_g$ such that the resulting cocycle invariant develops new objects, except for the signature of 4-manifolds.

This paper is organized as follows. Section 2 reviews quandles and introduces the cocycle invariants of Lefschetz fibrations. Section 3 discusses quandle cocycles with non-abelian coefficients and the signature. Section 4 computes some quandle homologies. Section 5 studies quandle cocycle invariants with local coefficients.

Notation Throughout this paper Σ_g means the oriented closed surface of genus g, and $\Sigma_{g,k}$ is the surface of genus g with k boundaries. Let $\mathcal{M}_g$ denote the mapping class group of Σ_g, and $\mathcal{M}_{g,k}$ that of $\Sigma_{g,k}$ which are the identity on the k-boundaries. Also we often use the notation $\mathcal{D}_g$ and $\mathcal{D}_g^{\rm ns}$ described above. For a group G, we denote its abelianization by $G_{\rm ab}$.

2 Quandle cocycle invariant of Lefschetz fibrations

Our purpose in this section is to introduce quandle cocycle invariants of Lefschetz fibrations over the 2-sphere. For this, in §2.1, we review quandles and quandle homologies. After this review, in §2.2 and §2.3 we interpret Lefschetz fibrations from quandle theory. In §2.4 we define two cocycle invariants.

2.1 Review of quandle and of quandle homology

A *rack* is a set, X, with a binary operation $\triangleleft : X \times X \to X$ such that

- The map $(\bullet \triangleleft a) : X \to X$ defined by $x \mapsto x \triangleleft a$ is bijective for any $a \in X$.
- $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$ for any $a, b, c \in X$.

Further, a rack is called *quandle*, if $a \triangleleft a = a$ for any $a \in X$. A map $X \to Y$ between quandles is called *a quandle homomorphism*, if $f(a \triangleleft b) = f(a) \triangleleft f(b)$ for any $a, b \in X$. Let $\mathrm{Hom}_{\mathrm{Qd}}(X, Y)$ denote the set of quandle homorsphisms $f : X \to Y$. We will give some examples of quandles:

Example 2.1. (Conjugacy quandle) A union of some conjugacy classes of a group G is a quandle given with the conjugacy operation $x \triangleleft y := y^{-1}xy$ for any $x, y \in G$.

Example 2.2. (Fundamental quandle) Let M be a connected C^∞-manifold without boundaries, and $N \subset M$ a submanifold of codimention 2. For simplisity, assume that a tubular neighborhood of $N \subset M$ is diffeomorphic to $D^2 \times N$, where D^2 is the 2-disk. Fix two points $p_0 \in M \setminus N$ and $d_0 \in \partial D^2$. We define $Q(M, N)$ by the set of homotopy classes of all pairs $(D^2 \times \{y\}, \alpha)$, where y runs over N and α is a path in $M \setminus N$ starting from p_0 and ending at d_0 (see, eg, Figure 2). The set $Q(M, N)$ carries a quandle, called *fundamental quandle*, with an operation defined by

$$[(D_1 \times \{y_1\}, \alpha_1)] \triangleleft [(D_2 \times \{y_2\}, \alpha_2)] = [(D_1 \times \{y_1\}, \alpha_1 \cdot_{p_0} \alpha_2^{-1} \cdot_{d_0} \overline{\partial D_2} \cdot_{d_0} \alpha_2)],$$

Here the symbol $\overline{\partial D}$ means the path obtained from the boundary ∂D by cutting at d_0, and the symbol $\cdot_y$ means the connection of two paths at a point y. See [Joy, Y] for details.

Example 2.3. (Dehn quandle) Recall the set $\mathcal{D}_g$ consisting of essential simple closed curves in Σ_g given (1). For $\alpha,\ \beta \in \mathcal{D}_g$, we define $\alpha \triangleleft \beta \in \mathcal{D}_g$ by $\tau_\beta(\alpha)$, where $\tau_\beta \in \mathcal{M}_g$ is the positive Dehn twist along β. The pair $(\mathcal{D}_g, \triangleleft)$ is a quandle, and called *Dehn quandle* (see [NP, Y, Zab]). As is known [FM, Chapter 3.3], $\mathcal{D}_g$ injects into the mapping class group $\mathcal{M}_g$ by

$$\tau_\bullet : \mathcal{D}_g \longrightarrow \mathcal{M}_g \qquad (\alpha \longmapsto \tau_\alpha). \tag{3}$$

Let us provide $\mathcal{M}_g$ with a conjugacy quandle structure considered in Example 2.1. Then, by the injection (3), the Dehn quandle $\mathcal{D}_g$ can be turn into a subquandle of $\mathcal{M}_g$.

Finally we note the case $g = 1$. As is shown in [NP], $\mathcal{D}_g$ is quandle isomorphic to $Q(S^3, K_{3_1})$, where K_{3_1} is the trefoil knot. In this paper we thus mainly deal with the case $g \geq 2$.

We review the *associated group* of a rack X, denoted by $\mathrm{As}(X)$. This group $\mathrm{As}(X)$ is the abstract group defined by generators e_x labeled by $x \in X$ and relations $e_x \cdot e_y = e_y \cdot e_{x \triangleleft y}$ for $x,\ y \in X$. Note that a quandle homomorphism $X \to Y$ between quandles induces a group homomorphism $\mathrm{As}(X) \to \mathrm{As}(Y)$. Furthermore, we call a set acted on by $\mathrm{As}(X)$ *an X-set*. For example,

Example 2.4. Any rack X is an X-set obtained from the action of $\mathrm{As}(X)$ defined by $x \cdot e_y := x \triangleleft y \in X$ for $x, y \in X$. This action will be often used.

We briefly review the quandle (co)homologies with local coefficients (our formulas are a special case of those in [CEGS, §2]. See also [IK, §2.2]). Let X be a rack, Y an X-set and A a commutative ring. We set $C_n^R(X, Y; A)$ by the free right A-module generated by $Y \times X^n$. Define a boundary $\partial_n^R : C_n^R(X, Y; A) \to C_{n-1}^R(X, Y; A)$ to be

$$\partial_n^R(y, x_1, \ldots, x_n) = \sum_{1 \le i \le n} (-1)^i \big((y \triangleleft x_i, x_1 \triangleleft x_i, \ldots, x_{i-1} \triangleleft x_i, x_{i+1}, \ldots, x_n) - (y, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \big).$$

The composite $\partial_{n-1}^R \circ \partial_n^R$ is known to be zero. The homology is denoted by $H_n^R(X; A)$ and is called *rack homology*. In addition, when X is a quandle, let $C_n^D(X, Y; A)$ be a submodule of $C_n^R(X, Y; A)$ generated by $(n+1)$-tuples $(y, x_1, \ldots, x_n)$ with $x_i = x_{i+1}$ for some $i \in \{1, \ldots, n-1\}$. Note that $\partial_n^R(C_n^D(X, Y; A))$ is contained in $C_{n-1}^D(X, Y; A)$. Then the *quandle homology* $H_n^Q(X, Y; A)$ is defined by the homology of the quotient complex $C_n^R(X, Y; A)/C_n^D(X, Y; A)$. Dually, we can define the cohomology. If Y is a single point, we suppress the symbol Y and the quandle cohomology $H_Q^n(X; A)$ coincides with the original one in [CJKLS]

Remark 2.5. Let X be a quandle. As the special case $Y = X$ in Example 2.4, the homomorphism $C_n^R(X, X; \mathbb{Z}) \to C_{n+1}^R(X; \mathbb{Z})$ induced by the identification $X \times X^n \simeq X^{n+1}$ is known to be a complex isomorphism (see [FRS2, Theorem 5.14.]).

2.2 The fundamental quandle of the 2-sphere with m-points

Let $\overline{\mathbf{m}}$ be the set of m-points $\{b_1, \ldots, b_m\}$ in S^2 with the same orientations. We will observe the fundamental quandle $Q(S^2, \overline{\mathbf{m}})$ explained in Example 2.2, in order to study Lefschetz fibrations from $Q(S^2, \overline{\mathbf{m}})$ in the next subsection.

To begin, we focus on the fundamental quandle $Q(S^3, T_{m,m})$ of the (m, m)-torus link $T_{m,m}$ in S^3. We regard $T_{m,m}$ as the closure of a braid Δ^m (see Figure 1), where Δ^m is defined by $(\sigma_1 \cdots \sigma_{m-1})^m$ in the braid group B_m. In a view of the link-diagram by the closure, following from Wirtinger presentation (see [Joy, §15]), the quandle $Q(S^3, T_{m,m})$ is presented by

$$\begin{aligned} &\text{generators :} && a_i && (i \in \mathbb{Z}), \\ &\text{relations :} && (\cdots (a_i \triangleleft a_{i+1}) \triangleleft \cdots) \triangleleft a_{i+m-1} = a_i, && a_i = a_{m+i} && (i \in \mathbb{Z}). \end{aligned} \tag{4}$$

Here the generator a_i is represented by the path in the left of Figure 1.

We will show a quandle isomorphism $Q(S^3, T_{m,m}) \cong Q(S^2, \overline{\mathbf{m}})$. For this, consider m-elements $c_i \in Q(S^2, \overline{\mathbf{m}})$ shown in the right of Figure 1 for $1 \le i \le m$. Notice that, by the presentation (4), the correspondence from a_i to c_i gives rise to a quandle homomorphism $\mu_m : Q(S^3, T_{m,m}) \to Q(S^2, \overline{\mathbf{m}})$ illustrated as Figure 1. Moreover

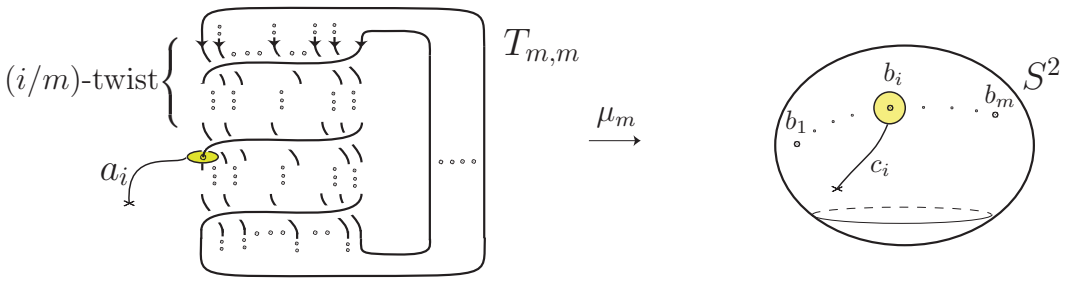

Figure 1: The quandle homomorphism μ_m.

Lemma 2.6. *This map μ_m is quandle isomorphic. Namely $Q(S^3, T_{m,m}) \cong Q(S^2, \overline{\mathbf{m}})$.*

Proof. We will show this by examining $Q(S^2, \overline{\mathbf{m}})$ for more details. Choose a disk D^2 in S^2 containing $\overline{\mathbf{m}}$. By definitions the inclusion $D^2 \subset S^2$ induces a quandle epimorphism $\kappa : Q(D^2, \overline{\mathbf{m}}) \to Q(S^2, \overline{\mathbf{m}})$. Put the element $\overline{c_i} \in Q(D^2, \overline{\mathbf{m}})$ drawn in the left of Figure 2. By van-Kampen theorem for fundamental quandles (see [Joy, Theorem 13.1]), remark that $Q(D^2, \overline{\mathbf{m}})$ is the free quandle generated by $\overline{c_i}$ (shown by induction on m), and that the kernel of κ is generated by the equivalences in Figure 2. The kernel corresponds with the relation in (4); thus the presentaions of $Q(S^3, T_{m,m})$ and $Q(S^2, \overline{\mathbf{m}})$ imply the bijectivity of μ_m. □

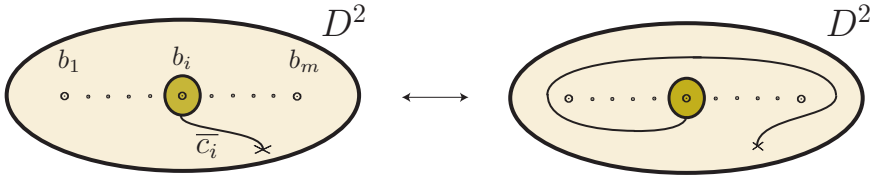

Figure 2: The relation in $Q(D^2, \overline{\mathbf{m}})$.

We will observe that $Q(S^2, \overline{\mathbf{m}})$ is a kind of a conjugacy quandle. Let $\gamma_i \in \pi_1(S^2 \setminus \overline{\mathbf{m}})$ be the loop which circulates around $b_i \in S^2$ for $1 \leq i \leq m-1$ (see the left in Figure 3). Let us equip $\pi_1(S^2 \setminus \overline{\mathbf{m}})$ with a conjugacy quandle structure. Since $Q(S^2, \overline{\mathbf{m}})$ is generated by the m pathes c_i (see Lemma 2.6), we define a quandle homomorphism $\iota : Q(S^2, \overline{\mathbf{m}}) \to \pi_1(S^2 \setminus \overline{\mathbf{m}})$ by

$$\iota(c_m) = (\gamma_1 \gamma_2 \cdots \gamma_{m-1})^{-1}, \quad \iota(c_i) = \gamma_i \quad \text{for} \ \ 1 \leq i \leq m-1.$$

The image is the set of elements of m-conjugacy classes of $\gamma_1, \ldots, \gamma_{m-1}$ and of $(\gamma_1 \gamma_2 \cdots \gamma_{m-1})^{-1}$.

Lemma 2.7. *The quandle homomorphism ι from $Q(S^2, \overline{\mathbf{m}})$ to $\pi_1(S^2 \setminus \overline{\mathbf{m}})$ is injective.*

Proof. Since $\pi_1(S^2 \setminus \overline{\mathbf{m}})$ is a free group of rank $m-1$ generated by $\gamma_1, \ldots, \gamma_{m-1}$, it is easily seen to check the injectivity by elementary calculations. □

Finally we investigate the associated group of $Q(S^2, \overline{\mathbf{m}})$. Recall $Q(S^2, \overline{\mathbf{m}}) \cong Q(S^3, T_{m,m})$. By Wirtinger presentation, $\mathrm{As}(Q(S^3, T_{m,m}))$ is isomorphic to $\pi_1(S^3 \setminus T_{m,m})$ (see also [Joy, §15]). Hence $\pi_1(S^3 \setminus T_{m,m})$ is presented by generators $a_1, \ldots, a_m$ subject to relations

$$a_1 a_2 \cdots a_m = a_m a_1 a_2 \cdots a_{m-1} = a_{m-1} a_m a_1 a_2 \cdots a_{m-2} = \cdots = a_2 \cdots a_m a_1. \tag{5}$$

Note that the quotient group of $\pi_1(S^3 \setminus T_{m,m})$ by $a_1 a_2 \cdots a_m = 1$ is isomorphic to $\pi_1(S^2 \setminus \overline{\mathbf{m}})$.

2.3 Lefschetz fibrations from quandle theory

Yetter [Y] studied a relation between $Q(S^2, \overline{\mathbf{m}})$ and Lefschetz fibrations over S^2 in terms of "augmented quandles". Our objective in this subsection is to interpret Lefschetz fibrations in the sense of quandle homomorphisms, and to show Proposition 2.8. This discussion follows *a fortiori* from theorems of Matsumoto [M].

We begin setting up some notation. Let $E \to S^2$ be a Lefschetz fibration with fiber Σ_g, where E is a closed 4-manifold (see, e.g., [EN, M, Y] for the definition). Throughout this paper, no fiber contains a (-1)-sphere, i.e., a smoothly embedded 2-sphere with self-intersection number -1. Let $\overline{\mathbf{m}} = \{b_1, \dots, b_m\} \subset S^2$ be the set of the critical values. Let $\gamma_i \in \pi_1(S^2 \setminus \overline{\mathbf{m}})$ be the loop which circulates around b_i for $1 \leq i \leq m$. As is known (see [M, Theorem 2.6]), the fibration $E \to S^2$ yields the monodromy $\rho : \pi_1(S^2 \setminus \overline{\mathbf{m}}) \to \mathcal{M}_g$, and that ρ sends each γ_i to positive Dehn twists of some essential simple closed curves. Thus, recalling the quandle monomorphisms $\mathcal{D}_g \hookrightarrow \mathcal{M}_g$ in (3) and $Q(S^2, \overline{\mathbf{m}}) \hookrightarrow \pi_1(S^2, \overline{\mathbf{m}})$ in Lemma 2.7, the restriction of the monodromy ρ to $Q(S^2, \overline{\mathbf{m}})$ turns out to be a quandle homomorphism $f_E : Q(S^2, \overline{\mathbf{m}}) \to \mathcal{D}_g$. Here notice that the global monodromy is the identity, i.e., the map f_E satisfies $f_E(c_1) \cdots f_E(c_m) = 1_{\mathcal{M}_g} \in \mathcal{M}_g$, where $f_E(c_i) \in \mathcal{D}_g$ is regarded as being in $\mathcal{M}_g$. Then we define a set by

$$\mathrm{Hom}^{\mathrm{gl}}_{\mathrm{Qd}}(Q(S^2, \overline{\mathbf{m}}),\ \mathcal{D}_g) := \{ f \in \mathrm{Hom}_{\mathrm{Qd}}(Q(S^2, \overline{\mathbf{m}}),\ \mathcal{D}_g) \mid f(c_1) \cdots f(c_m) = 1_{\mathcal{M}_g} \in \mathcal{M}_g \ \}.$$

We remark that the set $\mathrm{Hom}^{\mathrm{gl}}_{\mathrm{Qd}}(Q(S^2, \overline{\mathbf{m}}),\ \mathcal{D}_g)$ is a proper subset of $\mathrm{Hom}_{\mathrm{Qd}}(Q(S^2, \overline{\mathbf{m}}),\ \mathcal{D}_g)$ [1].

With these notation in mind, we mention an equivalence relation on the set $\mathrm{Hom}^{\mathrm{gl}}_{\mathrm{Qd}}(Q(S^2, \overline{\mathbf{m}}),\ \mathcal{D}_g)$. Let $B_m(S^2)$ be the m-string braid group on the 2-sphere S^2. Recall the standard group presentation of $B_m(S^2)$, that is, the generators are given by $\sigma_1, \dots \sigma_{m-1}$ and the relations are given by

$$\sigma_i \sigma_j = \sigma_i \sigma_j \qquad (|i-j| > 1), \qquad\qquad \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \qquad (|i-j| = 1),$$

$$(\sigma_1 \sigma_2 \cdots \sigma_{m-1})(\sigma_{m-1} \sigma_{m-2} \cdots \sigma_1) = 1 = (\sigma_1 \sigma_2 \cdots \sigma_{m-1})^m. \tag{6}$$

For $\sigma_k \in B_m(S^2)$, we consider a group isomorphism $E_k : \pi_1(S^2 \setminus \overline{\mathbf{m}}) \to \pi_1(S^2 \setminus \overline{\mathbf{m}})$ defined by

$$E_k(\gamma_k) = \gamma_{k+1}, \qquad E_k(\gamma_{k+1}) = \gamma_{k+1}^{-1} \gamma_k \gamma_{k+1}, \qquad E_k(\gamma_j) = \gamma_j \ \text{ for } j \neq k, k+1. \tag{7}$$

This E_k is called *elementary transformation* in [M, §4] (see also Figure 3). This then yields an action of $B_m(S^2)$ on $\pi_1(S^2 \setminus \overline{\mathbf{m}})$. Further, it can be easily seen that the isomorphism E_k is closed under $Q(S^2, \overline{\mathbf{m}}) \subset \pi_1(S^2 \setminus \overline{\mathbf{m}})$; Thus the action is regarded as that of $B_m(S^2)$ on $Q(S^2, \overline{\mathbf{m}})$. Using these actions, two group homomorphisms $\rho,\ \rho' : \pi_1(S^2 \setminus \overline{\mathbf{m}}) \to \mathcal{M}_g$ (or quandle homomorphisms $\rho,\ \rho' : Q(S^2, \overline{\mathbf{m}}) \to \mathcal{D}_g$) are said to be *equivalent* if there are an element $\alpha \in \mathcal{M}_g$ and an isomorphism $h : \pi_1(S^2 \setminus \overline{\mathbf{m}}) \to \pi_1(S^2 \setminus \overline{\mathbf{m}})$ derived from a braid of $B_m(S^2)$ so that $\alpha^{-1} \cdot \rho \cdot \alpha = \rho' \circ h$. Then we denote by $\overline{\mathrm{Hom}}^{\mathrm{gl}}_{\mathrm{Qd}}(Q(S^2, \overline{\mathbf{m}}),\ \mathcal{D}_g)$ the quotient set of $\mathrm{Hom}^{\mathrm{gl}}_{\mathrm{Qd}}(Q(S^2, \overline{\mathbf{m}}),\ \mathcal{D}_g)$ modulo the equivalence relation.

[1] For example, when $m = 2$, $Q(S^2, \overline{\mathbf{m}})$ is composed of two points, c_1, c_2, and satisfies $c_1 \triangleleft c_2 = c_1$ and $c_2 \triangleleft c_1 = c_2$; hence, there are many homomorphisms $f : Q(S^2, \overline{\mathbf{m}}) \to \mathcal{D}_g$ with $f(c_1) f(c_2) \neq 1_{\mathcal{M}_g}$.

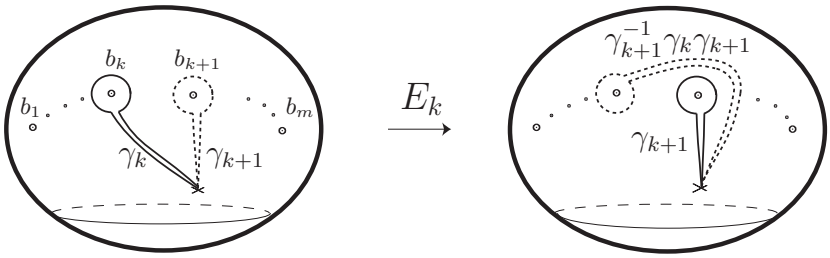

Figure 3: The elementary transformation E_k in $\pi_1(S^2 \setminus \overline{\mathbf{m}})$.

We will construct a map below (8). Recall here Theorem 2.4 in [M], i.e., two Lefschetz fibrations E and E' with fiber Σ_g and the critical values $\overline{\mathbf{m}} \subset S^2$ are isomorphic, if and only if the two monodromies $\pi_1(S^2 \setminus \overline{\mathbf{m}}) \to \mathcal{M}_g$ are equivalent (In particular, the resulting quandle homomorphisms $f_E, f_{E'} : Q(S^2, \overline{\mathbf{m}}) \to \mathcal{D}_g$ are equivalent). In summary, we obtain a map

$$\{ \text{ Lefschetz fibration over } (S^2, \overline{\mathbf{m}}) \text{ with fiber } \Sigma_g \}/\text{isotopy} \longrightarrow \overline{\mathrm{Hom}}^{\mathrm{gl}}_{\mathrm{Qd}}(Q(S^2, \overline{\mathbf{m}}),\ \mathcal{D}_g). \quad (8)$$

Proposition 2.8 (cf. [Y, Proposition 33])**.** *For $g \geq 2$, this map is bijective.*

Proof. We now construct an inverse map of (8). Given $f \in \mathrm{Hom}^{\mathrm{gl}}_{\mathrm{Qd}}(Q(S^2, \overline{\mathbf{m}}),\ \mathcal{D}_g)$, it induces a group homomorphism $f_* : \mathrm{As}(Q(S^2, \overline{\mathbf{m}})) \to \mathrm{As}(\mathcal{D}_g)$. Note that the map $\mathcal{D}_g \hookrightarrow \mathcal{M}_g$ in (3) yields a group homomorphism $\tau_* : \mathrm{As}(\mathcal{D}_g) \to \mathcal{M}_g$. Put the composite $f_* \circ \tau_* : \pi_1(S^3 \setminus T_{m,m}) = \mathrm{As}(Q(S^2, \overline{\mathbf{m}})) \to \mathcal{M}_g$. The presentation (5) of $\pi_1(S^3 \setminus T_{m,m})$ admits a quotient map $\overline{f} : \pi_1(S^2 \setminus \overline{\mathbf{m}}) \to \mathcal{M}_g$, since $f(c_1) \cdots f(c_m) = 1_{\mathcal{M}_g}$. Further, by construction, $\overline{f}(\gamma_i) \in \mathcal{D}_g$ for any $\gamma_i \in \pi_1(S^2 \setminus \overline{\mathbf{m}})$. Hence, it immediately follows from [M, Theorem 2.6] that this monodromy $\overline{f}$ recovers a Lefschetz fibration, which takes the bijectivity of (8) by construction. □

Finally, we mention a type of Lefschetz fibrations. Let $f \in \mathrm{Hom}^{\mathrm{gl}}_{\mathrm{Qd}}(Q(S^2, \overline{\mathbf{m}}), \mathcal{D}_g)$ be a quandle homomorphism associated to a Lefschetz fibration $E \to S^2$. Let $c_1, c_2, \ldots, c_m \in Q(S^2, \overline{\mathbf{m}})$ be the elements drawn in Figure 1. The fibration is said to be *of type nI*, if $n = \#\{i \mid f(c_i) \in \mathcal{D}_g^{\mathrm{ns}}\}$, where $\mathcal{D}_g^{\mathrm{ns}}$ is the subquandle of $\mathcal{D}_g$ given in (2). As is known, the number n must be larger than 0.

2.4 Definitions of two quandle cocycle invariants

Zablow [Zab, Theorem 5.3] combinatorially defined a certain invariant of Lefschetz fibrations over the disk. This invariant is valued in the second homology $H_2^Q(\mathcal{D}_g; \mathbb{Z})$ of the Dehn quandle $\mathcal{D}_g$. In this paper, we deal with Lefschetz fibrations over the sphere S^2 in a view of Propositions 2.8, and define quandle cocycle invariants, more generally, with the local coefficients and with non-abelian coefficients.

To describe the invariant with local coefficients, we first construct a 2-cycle $\mu_{Y,y} \in C_2(Q(S^2, \overline{\mathbf{m}}), Y; \mathbb{Z})$ for a $Q(S^2, \overline{\mathbf{m}})$-set Y and $y \in Y$ as follows. Recall from Lemma 2.6 that $Q(S^2, \overline{\mathbf{m}}) \cong Q(S^3, T_{m,m})$. According to [CEGS, §6], given a link-diagram of a link $L \subset S^3$, they constructed a 2-cycle of the fundamental quandle of (S^3, L) with local coefficients (see also [IK, §4.2]). As is known, the homology class of the 2-cycle depends only on L.

For our purpose, we here confine ourselves to the diagram of the torus link $L = T_{m,m}$ in the left of Figure 1. Then the 2-cycle is explicitly expressed as

$$\mu_{Y,y} := \sum_{j=1}^{m} \sum_{i=1}^{m-1} y \cdot e_{c_{j-i+1}} \cdots e_{c_{j-1}} \{ (\cdots (c_{j-i} \triangleleft c_{j-i+1}) \cdots) \triangleleft c_{j-1},\ c_j \} \in C_2^Q(Q(S^2, \overline{\mathbf{m}}), Y; \mathbb{Z}). \quad (9)$$

Definition 2.9. Let $f \in \mathrm{Hom}_{\mathrm{Qd}}^{\mathrm{gl}}(Q(S^2, \overline{\mathbf{m}}), \mathcal{D}_g)$ be a quandle homomorphism associated to a Lefschetz fibration in Proposition 2.8, and Y a set acted on transitively by $\mathrm{As}(\mathcal{D}_g)$. Let us regard Y as a $Q(S^2, \overline{\mathbf{m}})$-set induced by f. Put a quandle 2-cocycle $\psi \in H_Q^2(\mathcal{D}_g, Y; A)$. We define *the quandle cocycle invariant* of E using ψ by the formula $\Phi_{\psi,y}(E) := \langle f^*(\psi),\ \mu_{Y,y} \rangle \in A$.

The pairing $\Phi_{\psi,y}(E) := \langle f^*(\psi),\ \mu_{Y,y} \rangle$ is an invariant of Lefschetz fibrations as follows.

Proposition 2.10. *Let X, Y and ψ be as above. Let $y \in Y$. If two quandle homomorphisms $f, f' \in \mathrm{Hom}_{\mathrm{Qd}}^{\mathrm{gl}}(Q(S^2, \overline{\mathbf{m}}), \mathcal{D}_g)$ are equivalent, then $\langle f^*(\psi),\ \mu_{Y,y} \rangle = \langle f'^*(\psi),\ \mu_{Y,y} \rangle \in A$. Further, the value $\langle f^*(\psi),\ \mu_{Y,y} \rangle$ does not depend on $y \in Y$.*

This will be shown later.

Changing the subject, we next construct invariants for non-abelian cocycles as follows. Let G be a group, and X a quandle. A map $\theta : Y \times X \to G$ is called *non-abelian cocycle*, if

$$\theta(y, x_1) \cdot \theta(y \cdot e_{x_1}, x_2) = \theta(y, x_2) \cdot \theta(y \cdot e_{x_2}, x_1 \triangleleft x_2) \in G, \quad (10)$$

for any $y \in Y$ and $x_1, x_2 \in X$. Remark that, when $Y = X$ in Example 2.4, such a cocycle was called "rack 2-cocycle" in [CEGS, §5], and that when G is abelian, the condition (10) is a 1-cocycle condition in the cochain group $C_R^1(X, Y; G)$ in §2.1.

Let L be a link of m-components, and $\theta : X \times X \to G$ a non-abelian cocycle. According to [CEGS], using a link-diagram of L so that each framings of link-components are presented by 1, we can define a certain map $\Phi_{\theta,j} : \mathrm{Hom}_{\mathrm{Qd}}(Q(S^3, L), X) \to G_{\mathrm{conj}}$ for $1 \leq j \leq m$. Here G_{conj} is a quotient of G by conjugacy operations. Remark that the map $\Phi_{\theta,j}$ does not depend on the choice of such framed link-diagrams of L (see [CEGS, Theorem 5.5]).

Of particular interest to us is the case $X = \mathcal{D}_g$ and the diagram in Figure 4 of $L = T_{m,m}$. We now formulate explicitly the resulting map $\Phi_{\theta,j}$. Let $f \in \mathrm{Hom}_{\mathrm{Qd}}(Q(S^3, L), \mathcal{D}_g)$. For the generators $a_i \in Q(S^3, T_{m,m})$ in (4), we denote $f(a_i) \in \mathcal{D}_g$ by d_i. Further, let $d_{j,k}$ denote $(\cdots (d_j \triangleleft d_{j+1}) \triangleleft \cdots) \triangleleft d_{j+k} \in \mathcal{D}_g$ for short. Note $d_{j,m} = d_j$ by (4). The value $\Phi_{\theta,j}(f)$ is expressed by

$$\Phi_{\theta,j}(f) := \theta(d_j, d_{j+1}) \cdot \big(\theta(d_{j,1}, d_{j+2}) \cdots \theta(d_{j,m-1}, d_{j+m}) \big) \cdot \theta(d_j, d_j) \in G. \quad (11)$$

Noting the symmetric group $\mathfrak{S}_m$ canonically acts on the product of m-copies of G_{conj}, we consider the quotient $(G_{\mathrm{conj}})^m / \mathfrak{S}_m$. Then an invariant of Lefschetz fibrations is defined as follows.

Definition 2.11. Let $\theta : \mathcal{D}_g \times \mathcal{D}_g \to G$ be a non-abelian cocycle. For a quandle homomorphism $f \in \mathrm{Hom}_{\mathrm{Qd}}^{\mathrm{gl}}(Q(S^2, \overline{\mathbf{m}}), \mathcal{D}_g)$ associated with a Lefschetz fibration $E \to S^2$, we define the *(non-abelian) cocycle invariant* of E by the values $\big(\Phi_{\theta,1}(f), \ldots, \Phi_{\theta,m}(f) \big) \in (G_{\mathrm{conj}})^m / \mathfrak{S}_m$.

Proposition 2.12. *Let f, θ be as above. If another element $f' \in \mathrm{Hom}^{\mathrm{gl}}_{\mathrm{Qd}}(Q(S^2, \overline{\mathbf{m}}), \mathcal{D}_g)$ is equivalent to f, then the values $\big(\Phi_{\theta,1}(f), \ldots, \Phi_{\theta,m}(f)\big) = \big(\Phi_{\theta,1}(f'), \ldots, \Phi_{\theta,m}(f')\big) \in (G_{\mathrm{conj}})^m/\mathfrak{S}_m$.*

We now prove Propositions 2.10 and 2.12 in an easy way using link-diagrams (cf. the combinatorial proof in [Zab, Theorem 5.3] to show an invariance).

Proof of Proposition 2.10. By assumption, let $\alpha \in \mathcal{M}_g$ and $h \in B_m(S^2)$ satisfy that $\alpha^{-1} \cdot f \cdot \alpha = f' \circ h$. To begin, we consider the case $h = 1_{B_m(S^2)}$. Since $\mathcal{M}_g$ is generated by $\mathcal{D}_g$, we may assume $\alpha \in \mathcal{D}_g$. However, in general, it is shown [IK, Lemmas 4.5 and 4.6] that if such two cycles are conjugate, then they are homologous, and that $f_*(\mu_{Y,y}) \in H_2^Q(Q(S^3, T_{m,m}); \mathbb{Z})$ does not depend on the choice of $y \in Y$.

On the other hand, we impose a condition $\alpha = 1_{\mathcal{M}_g}$. By the presentation of $B_m(S^2)$ in (6), it is sensible to concentrate on $h = E_k : Q(S^2, \overline{\mathbf{m}}) \to Q(S^2, \overline{\mathbf{m}})$ in (7). Consider the link-diagram D of $T_{m,m}$ in the left of Figure 4, where any entry of the linking matrix is 1. Here the band in Figure 4 means the $(m-2)$-parallel strands, and the blackline is the k-th strand and the dotted line indicates the $(k+1)$-th strand,. Put another diagram D' in the right of Figure 4. Then we can see that the 2-cycle $\mu'_{Y,y}$ obtained from D' such as (9) exactly coincides with $(E_k)_*(\mu_{Y,y})$. Notice that the moves in Figure 4 mean that a finite sequence of Reidemeister II and III moves give an exchange between the k-th strand and $(k+1)$-th one. Therefore we deduce that

$$\langle f^*(\psi),\ \mu_{Y,y}\rangle = \langle f^*(\psi),\ \mu'_{Y,y}\rangle = \langle f^*(\psi),\ (E_k)_*(\mu_{Y,y})\rangle = \langle f'^*(\psi),\ \mu_{Y,y}\rangle \in A.$$

□

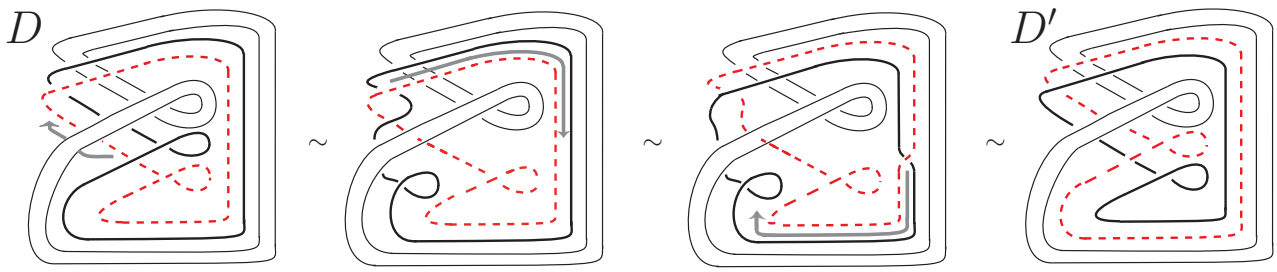

Figure 4: Exchange between the k-th strand and the $(k+1)$-th one of the (m,m)-torus link.

Proof of Proposition 2.12. Similarly, let $\alpha \in \mathcal{M}_g$ and $h \in B_m(S^2)$ satisfy that $\alpha^{-1} \cdot f \cdot \alpha = f' \circ h$. In the case $\alpha = 1_{\mathcal{M}_g}$, the proof follows from a discussion using Reidemeister moves similar to the proof of Proposition 2.10. Hence we may assume $h = 1_{B_m(S^2)}$ and $\alpha \in \mathcal{D}_g$. Notice that $\Phi_{\theta,j}(f')$ is obtained from $\Phi_{\theta,j}(f)$ after replacing d_{j+k} by $d_{j+k} \lhd \alpha$ from the definitions. By the cocycle condition (10), we have

$$\theta(x \lhd z, y \lhd z) = \theta(x,z)^{-1}\theta(x,y)\theta(x \lhd y, z), \tag{12}$$

for any $x, y, z \in \mathcal{D}_g$. Consequently, using (11) and (12), an elementary calculation shows $\Phi_{\theta,j}(f') = \theta(d_1, \alpha)^{-1} \cdot \Phi_{\theta,j}(f) \cdot \theta(d_1, \alpha)$ as required. □

Remark 2.13. Finally, we remark cocycle invariants of Lefschetz fibrations of type mI, i.e. the images of the associated homomorphisms $f \in \mathrm{Hom}_{\mathrm{Qd}}(Q(S^2, \overline{\mathbf{m}}), \mathcal{D}_g)$ are contained in $\mathcal{D}_g^{\mathrm{ns}}$. By substituting $\mathcal{D}_g$ with $\mathcal{D}_g^{\mathrm{ns}}$, the above discussion holds. Namely, given non-abelian cocycles $\mathcal{D}_g^{\mathrm{ns}} \times \mathcal{D}_g^{\mathrm{ns}} \to G$ and 2-cocycles $Y \times \mathcal{D}_g^{\mathrm{ns}} \times \mathcal{D}_g^{\mathrm{ns}} \to A$ we can define quandle cocycle invariants of Lefschetz fibrations of type mI in a similar fashion.

3 Quandle cocycle invariants with non-abelian coefficients

This section focuses on cocycles with non-abelian coefficients. We will construct a non-abelian cocycle of $\mathcal{D}_g$, and show that the associated cocycle invariant includes the signature of Lefschetz fibrations. More precisely,

Theorem 3.1. *Let $g \geq 3$ and $1 \leq j \leq m$. There exists a non-abelian cocycle $\theta_j : \mathcal{D}_g \times \mathcal{D}_g \to \mathrm{As}(\mathcal{D}_g)$ such that, for any $f \in \mathrm{Hom}^{\mathrm{gl}}_{\mathrm{Qd}}(Q(S^2, \overline{\mathbf{m}}), \mathcal{D}_g)$, the invariant $\Phi_{\theta_j, j}(f)$ lies in the center $Z(\mathrm{As}(\mathcal{D}_g))$. Further, there exists an epimorphism $p_Z : Z(\mathrm{As}(\mathcal{D}_g)) \to \mathbb{Z}$ so that $p_Z\big(\Phi_{\theta_j,j}(f)\big) = \mathrm{sign}(E) + m - n_f$ in $\mathbb{Z}$, where E is the total space of the Lefschetz fibration associated with f, and n_f is the type.*

To prove the theorem, we will mention Proposition 3.2 and Theorem 3.4 below.

We first construct many non-abelian cocycles, by slightly modifying [AG, CEGS, E1]: in these papers some constructions of quandle 2-cocycles with non-abelian coefficients were pointed out. Let X be a quandle, and Y an X-set. We give the orbit decomposition of Y, that is, $Y = \sqcup_{i \in I} Y_i$. Fix an element $y_i \in Y_i$. Then we consider a surjection $\mathrm{As}(X) \to Y_i$ sending g to $y_i \cdot g$. Choose an arbitrary section $s_i : Y_i \to \mathrm{As}(X)$ of the surjection. Define a map $\theta_i : Y_i \times X \to \mathrm{As}(X)$ by

$$\theta_i(y, x) := \mathfrak{s}_i(y) \cdot e_x \cdot \big(\mathfrak{s}_i(y \cdot e_x)\big)^{-1} \in \mathrm{As}(X), \tag{13}$$

for $y \in Y,\ x \in X$. Let us extend θ_i to a map $\vartheta_i : Y \times X \to \mathrm{As}(X)$ defined by $\vartheta_i(y, x) = 1_{\mathrm{As}(X)}$ for any $y \in Y \setminus Y_i$. Notice that the image of ϑ_i is contained in the stabilizer group $\mathrm{Stab}(y_i)$.

Proposition 3.2 (cf. [AG, Example 2.13] [E1, Lemma 9.6])**.** *Let X be a quandle, and Y an X-set. Then the map $\vartheta_i : Y \times X \to \mathrm{Stab}(y_i)$ mentioned above is a non-abelian cocycle.*

Proof. By the definition in (13), for $y \in Y_i$ and $x_1,\ x_2 \in X$, we have

$$\theta_i(y, x_1) \cdot \theta_i(y \cdot e_{x_1}, x_2) = \mathfrak{s}_i(y) \cdot e_{x_1} \cdot e_{x_2} \cdot \big(\mathfrak{s}_i((y \cdot e_{x_1}) \cdot e_{x_2})\big)^{-1} \in \mathrm{Stab}(y_i),$$

$$\theta_i(y, x_2) \cdot \theta_i(y \cdot e_{x_2}, x_1 \triangleleft x_2) = \mathfrak{s}_i(y) \cdot e_{x_2} \cdot e_{x_1 \triangleleft x_2} \cdot \big(\mathfrak{s}_i((y \cdot e_{x_2}) \cdot e_{x_1 \triangleleft x_2})\big)^{-1} \in \mathrm{Stab}(y_i).$$

The self-distribute condition immediately shows the cocycle condition in (10). □

Remark 3.3. For another section $\mathfrak{s}_i'$, the resulting cocycle $\vartheta_i' : X \times Y \to \mathrm{Stab}(y_i)$ is cohomologous to the previous ϑ_i. Namely, a function $c : Y \to \mathrm{As}(X)$ defined by $c(y) = \mathfrak{s}_i'(y)\mathfrak{s}_i(y)^{-1}$ satisfies $\vartheta_i'(y, x) = c(y) \cdot \vartheta_i(y, x) \cdot c(y \cdot e_x)^{-1}$ for any $x \in X$ and $y \in Y$. Thereby the cocycle invariant using the cocycle ϑ_i does not depend on the choice of sections $\mathfrak{s}_i$.

Besides, we further review a result in [EN]. Put the free group $F_{\mathcal{D}_g}$ generated by $\mathcal{D}_g$. Then, the map $\mathcal{D}_g \to \mathcal{M}_g$ in (3) passes to an epimorphism $F_{\mathcal{D}_g} \to \mathcal{M}_g$. Let $\mathcal{R}$ denote the kernel. With notation f in Theorem 3.1 and $d_i = f(c_i) \in \mathcal{D}_g$, notice that $e_{d_1} \cdots e_{d_m} \in F_{\mathcal{D}_g}$ lies in the kernel $\mathcal{R}$. Then Endo and Nagami [EN, Definition 3.3] constructed a homomorphism $\mathcal{C}_g : \mathcal{R} \to \mathbb{Z}$ so that

Theorem 3.4 ([EN, Theorem 4.2])**.** *With notation f, E, n_f in Theorem 3.1, let $d_i \in \mathcal{D}_g$ denote $f(c_i)$. The homomorphism $\mathcal{C}_g$ satisfies $\mathcal{C}_g(e_{d_1} \cdots e_{d_m}) = \mathrm{sign}(E) + m - n_f$.*

We further set the following group homomorphism induced by the map $\mathcal{D}_g \to \mathcal{M}_g$ in (3):

$$\pi : \mathrm{As}(\mathcal{D}_g) \longrightarrow \mathcal{M}_g. \tag{14}$$

As is well-known, π is surjective, i.e., $\mathcal{M}_g$ is generated by Dehn twists. Furthermore

Lemma 3.5. *For $g \geq 3$, the kernel of π in* (14) *coincides with the center of* $\mathrm{As}(\mathcal{D}_g)$.

Proof. Let $Z(\mathrm{As}(\mathcal{D}_g))$ denote the center. We first show $\mathrm{Ker}(\pi) \subset Z(\mathrm{As}(\mathcal{D}_g))$. Put an element $e_{x_1}^{\epsilon_1} \cdots e_{x_n}^{\epsilon_n}$ in the kernel, where $\epsilon_i \in \{\pm 1\}$ and $x_i \in \mathcal{D}_g$. Then for any $y \in \mathcal{D}_g$, we conclude $(e_{x_1}^{\epsilon_1} \cdots e_{x_n}^{\epsilon_n})^{-1} e_y e_{x_1}^{\epsilon_1} \cdots e_{x_n}^{\epsilon_n} = e_{(\cdots (y \lhd^{\epsilon_1} x_1) \cdots) \lhd^{\epsilon_n} x_n)} = e_y$ as required. Here we denote by $\bullet \lhd^{-1} x_i$ the inverse map of $\bullet \lhd x_i : \mathcal{D}_g \to \mathcal{D}_g$. Conversely, the center $Z(\mathrm{As}(\mathcal{D}_g))$ is readily contained in $\mathrm{Ker}(\pi)$, since $\mathcal{M}_g$ is centerless (see [FM, Chapter 3.4]). □

Proof of Theorem 3.1. By applying $X = Y = \mathcal{D}_g$ to Proposition 3.2, we have a non-abelian cocycle $\theta_j : (\mathcal{D}_g)^2 \to \mathrm{As}(\mathcal{D}_g)$. We will calculate the value $\Phi_{\theta,j}(f)$. By definition, note that

$$\theta_j\big(d_{j,k}, d_{j+k+1}\big) = \mathfrak{s}_j(d_{j,k})^{-1} e_{d_{j+k+1}} \mathfrak{s}_j(d_{j,k+1}),$$

where we employ the notation $d_{j,k} \in \mathcal{D}_g$ in §2.4. Hence, from the definition in (11), we have

$$\begin{aligned}
\Phi_{\theta,j}(f) &= \big(\theta_j(d_j, d_{j+1}) \cdot \theta_j(d_{j,1}, d_{j+2}) \cdots \theta_j(d_{j,m-1}, d_{j+m})\big) \cdot \theta_j(d_j, d_j) \\
&= \Big(\big(\mathfrak{s}_j(d_j)^{-1} e_{d_1} \mathfrak{s}_j(d_{j,1})\big) \cdots \big(\mathfrak{s}_j(d_{j,m-1})^{-1} e_{d_m} \mathfrak{s}_j(d_{j,m})\big)\Big) \cdot \big(\mathfrak{s}_j(d_j)^{-1} e_{d_m} \mathfrak{s}_j(d_j)\big) \\
&= \mathfrak{s}_j(d_j)^{-1} e_{d_1} e_{d_2} \cdots e_{d_m} \mathfrak{s}_j(d_j).
\end{aligned} \tag{15}$$

Here Lemma 3.5 implies that $e_{d_1} e_{d_2} \cdots e_{d_m}$ belongs to the center $Z(\mathrm{As}(\mathcal{D}_g))$ from the definition of $f \in \mathrm{Hom}^{\mathrm{gl}}_{\mathrm{Qd}}(Q(S^2, \overline{\mathbf{m}}), \mathcal{D}_g)$. In conclusion, the value $\Phi_{\theta,j}(f) \in Z(\mathrm{As}(\mathcal{D}_g))$ as desired.

Next, we will construct a projection $p_Z : Z(\mathrm{As}(\mathcal{D}_g)) = \mathrm{Ker}(\pi) \to \mathbb{Z}$ as follows. According to [EN, Proposition 3.6], the above homomorphism $\mathcal{C}_g : \mathcal{R} \to \mathbb{Z}$ satisfies $\mathcal{C}_g(e_b^{-1} e_a e_b e_{a \lhd b}^{-1}) = 0$ for any $a, b \in \mathcal{D}_g$. By recalling π in (14), this $\mathcal{C}_g$ induces a homomorphism $p_Z : \mathrm{Ker}(\pi) \to \mathbb{Z}$ as required. Further, following Theorem 3.4, we reach the conclusion

$$p_Z(\Phi_{\theta,j}(f)) = \mathcal{C}_g(e_{d_1} e_{d_2} \cdots e_{d_m}) = \mathrm{sign}(E) + m - n_f. \qquad \square$$

Remark 3.6. However, as seen in the proof, this cocycle invariant $\Phi_{\theta_j,j}(f)$ is not suitable for computation of the signature.

4 Second quandle homology groups of Dehn quandles.

We will compute or estimate the quandle homologies of the Dehn quandles in §4.2. For this, in §4.1, we give a method to compute quandle homologies, modifying that in [E1].

4.1 First homology groups of quandles.

We give a general algorithm to compute the first rack homology as follows.

Theorem 4.1. *Let X be a rack, and Y an X-set. Decompose Y into the orbits as $Y = \sqcup_{i\in I} Y_i$. For i, choose arbitrary elements $y_i \in Y_i$, and denote by $\mathrm{Stab}(y_i) \subset \mathrm{As}(X)$ the stabilizer subgroup of y_i. Then the direct sum of the abelianizations of $\mathrm{Stab}(y_i)$ is isomorphic to $H_1^R(X, Y; \mathbb{Z})$. That is, $\oplus_{i\in I}(\mathrm{Stab}(y_i))_{\mathrm{ab}} \cong H_1^R(X, Y; \mathbb{Z})$.*

This will be proved later. We next discuss an application. Eisermann [E1] gave a method to compute the second quandle homologies with trivial $\mathbb{Z}$-coefficients. While he showed topologically the result using a certain CW-complex, we will give another proof as a corollary of Theorem 4.1. To see this, let us decompose X into the orbits as $X = \sqcup_{i\in I} X_i$ under the action of $\mathrm{As}(X)$ in Example 2.4, and defin a homomorphism

$$\epsilon_i : \mathrm{As}(X) \to \mathbb{Z} \quad \text{defined by} \quad \begin{cases} \epsilon_i(e_x) = 1 \in \mathbb{Z} & \text{if } \ x \in X_i, \\ \epsilon_i(e_x) = 0 \in \mathbb{Z} & \text{if } \ x \in X \setminus X_i. \end{cases} \tag{16}$$

Corollary 4.2 ([E1, Theorem 9.9])**.** *Let X be a quandle. Decompose $X = \sqcup_{i\in I} X_i$ as the orbits by the action of $\mathrm{As}(X)$. Fix $x_i \in X_i$ for each $i \in I$. Then the quandle homology $H_2^Q(X; \mathbb{Z})$ is isomorphic to $\bigoplus_{i\in I}\big(\mathrm{Stab}(x_i) \cap \mathrm{Ker}(\epsilon_i)\big)_{\mathrm{ab}}$.*

From now on, we prove Theorem 4.1 and Corollary 4.2 as a slight application of the discussion in [FRS1, FRS2]. For the proofs, we review the (action) rack space introduced by Fenn-Rourke-Sanderson [FRS1, Example 3.1.1]. Fix a rack X and an X-set Y. Equipping X and Y with their discrete topology, we start with $\bigcup_{n\geq 0}\big(Y \times ([0,1] \times X)^n\big)$, and put equivalence relations given by

$$(y, t_1, x_1, \ldots, x_{j-1}, 1, x_j, t_{j+1}, \ldots, t_n, x_n) \sim (y{\cdot}x_j, t_1, x_1 \triangleleft x_j, \ldots, t_{j-1}, x_{j-1} \triangleleft x_j, t_{j+1}, x_{j+1}, \ldots, t_n, x_n),$$

$$(y, t_1, x_1, \ldots, x_{j-1}, 0, x_j, t_{j+1}, \ldots, t_n, x_n) \sim (y, t_1, x_1, \ldots t_{j-1}, x_{j-1}, t_{j-1}, x_{j+1}, \ldots, t_n, x_n).$$

Then the *rack space* $B(X, Y)$ is defined by the quotient space. When Y is a single point, we denote it by BX. By construction, we have a cell decomposition of $B(X, Y)$, regarding the projection $\bigcup_{n\geq 0}\big(Y \times ([0,1] \times X)^n\big) \to B(X, Y)$ as the characteristic maps. Then the complex $(C_*^R(X, Y; \mathbb{Z}), \partial_*)$ is known to be chain isomorphic to the cellular complex of $B(X, Y)$.

Proof of Theorem 4.1. We easily see, by the 1-skeleton of $B(X, Y)$, that the orbits of the action of $\mathrm{As}(X)$ on Y are bijective to the connected components of $B(X, Y)$. Via the decomposition $Y = \sqcup_{i\in I} Y_i$, we regard $B(X, Y)$ as a disjoint union of the connected spaces $B(X, Y_i)$.

We investigate the first homology of the space $B(X, Y_i)$ for each $i \in I$, from covering theory. Recall from [FRS1, Theorem 3.7] that the projection $B(X, Y_i) \to BX$ is a covering with fiber

Y_i. Thereby the universal covering $\widetilde{BX} \to BX$ factors through $B(X, Y_i)$. Since $\pi_1(BX) \cong \mathrm{As}(X)$ ([FRS2, Proposition 5.1]), the covering transformation coincides with the transitive action of $\mathrm{As}(X)$ on Y_i. In addition, it can be readily seen that the covering transformation group of $\widetilde{BX} \to B(X, Y_i)$ is the stabilizer $\mathrm{Stab}(y_i)$ $\big($see [FRS1, Proposition 4.5] for details$\big)$. Thereby $H_1(B(X, Y_i); \mathbb{Z}) \cong \pi_1(B(X, Y_i))_{\mathrm{ab}} \cong \mathrm{Stab}(y_i)_{\mathrm{ab}}$.

Finally, by considering all the connected components of $B(X, Y)$, we conclude

$$H_1^R(X, Y; \mathbb{Z}) \cong H_1(B(X, Y); \mathbb{Z}) \cong \bigoplus_{i \in I} H_1(B(X, Y_i); \mathbb{Z}) \cong \bigoplus_{i \in I} \big(\mathrm{Stab}(y_i)\big)_{\mathrm{ab}}.$$ □

Proof of Corollary 4.2. We first show (17) below. Assume $Y = X$ such as Example 2.4. For each $x_i \in X_i$, since $x_i \triangleleft x_i = x_i$, we have $e_{x_i} \in \mathrm{Stab}(x_i)$. Hence the restriction of $\epsilon_i : \mathrm{As}(X) \to \mathbb{Z}$ on $\mathrm{Stab}(x_i)$ is also surjective, and permits a section $\mathfrak{s} : \mathbb{Z} \to \mathrm{Stab}(x_i)$ defined by $\mathfrak{s}(1) = e_{x_i}$. Here we remark that the action of $\mathbb{Z}$ on $\mathrm{Stab}(x_i) \cap \mathrm{Ker}(\epsilon_i)$ induced by the section is trivial. Indeed, $g^{-1} e_{x_i} g = e_{x_i} \in \mathrm{As}(X)$ for any $g \in \mathrm{Stab}(x_i)$. We therefore have $\mathrm{Stab}(x_i)_{\mathrm{ab}} \cong \big(\mathrm{Stab}(x_i) \cap \mathrm{Ker}(\epsilon_i)\big)_{\mathrm{ab}} \oplus \mathbb{Z}$. Hence it follows from Theorem 4.1 that

$$H_1(X, X; \mathbb{Z}) \cong \bigoplus_{i \in I} H_1(B(X, X_i); \mathbb{Z}) \cong \bigoplus_{i \in I} \big(\mathrm{Stab}(x_i)\big)_{\mathrm{ab}} \cong \mathbb{Z}^I \oplus \bigoplus_{i \in I} \big(\mathrm{Stab}(x_i) \cap \mathrm{Ker}(\epsilon_i)\big)_{\mathrm{ab}}. \quad (17)$$

Further recall $H_2^R(X; \mathbb{Z}) \cong H_1^R(X, X; \mathbb{Z})$ in Remark 2.5. It is known [LN, Theorem 2.1] that $H_2^R(X; \mathbb{Z}) \cong H_2^Q(X; \mathbb{Z}) \oplus \mathbb{Z}^I$, and that the basis of $\mathbb{Z}^I$ is represented by $(x_i, x_i) \in C_2^R(X; \mathbb{Z})$ for $i \in I$. By comparing the basis with the isomorphisms in (17), we complete the proof. □

Remark 4.3. By the proof, we can see that the 1-cocycles in Proposition 3.2 detect $H_1^R(X, Y; \mathbb{Z})$.

4.2 Second quandle homology of Dehn quandles

Returning to the Dehn quandle $\mathcal{D}_g$, we will determine the second quandle homology of the subquandle $\mathcal{D}_g^{\mathrm{ns}}$ for $g \geq 5$ (see (2) for the definition of $\mathcal{D}_g^{\mathrm{ns}} \subset \mathcal{D}_g$). After that, we estimate the second quandle homology of the Dehn quandle $\mathcal{D}_g$. First of all, let us show

Proposition 4.4. *For $g \geq 2$, $H_2^Q(\mathcal{D}_g^{\mathrm{ns}}; \mathbb{Z})$ surjects onto $\mathbb{Z}/2$. Further, if $g \geq 5$, then $H_2^Q(\mathcal{D}_g^{\mathrm{ns}}; \mathbb{Z}) \cong \mathbb{Z}/2$.*

Remark 4.5. In the case $g = 1$, recall from Example 2.3 that the Dehn quandle $\mathcal{D}_g = \mathcal{D}_g^{\mathrm{ns}}$ is isomorphic to $Q(S^3, K_{3_1})$. Eisermann [E2] showed that $H_2^Q(Q(S^3, K_{3_1}); \mathbb{Z}) \cong \mathbb{Z}$.

To prove Proposition 4.4, we review a result of Gervais [Ge] on the associated group $\mathrm{As}(\mathcal{D}_g^{\mathrm{ns}})$. Let $g \geq 3$. As is well-known, the second group cohomology $H^2(\mathcal{M}_g; \mathbb{Z}) \cong \mathbb{Z}$ (see [FM, Kor]). We then set the central extension associated with a generator of $H^2(\mathcal{M}_g; \mathbb{Z})$:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{T}_g \xrightarrow{\text{proj.}} \mathcal{M}_g \longrightarrow 0. \quad (18)$$

Theorem 4.6 ([Ge, Theorems C and 1.3])**.** *Let $g \geq 3$. Then $\mathrm{As}(\mathcal{D}_g^{\mathrm{ns}})$ is isomorphic to $\mathcal{T}_g \times \mathbb{Z}$, where the second projection $\epsilon : \mathrm{As}(\mathcal{D}_g^{\mathrm{ns}}) \to \mathbb{Z}$ is given by $\epsilon(e_\beta) = 1$ for any $\beta \in \mathcal{D}_g^{\mathrm{ns}}$.*

In the following proof, we often use the fact that, in general, an epimorphism $G \to H$ between groups induces an epimorphism $G_{\mathrm{ab}} \to H_{\mathrm{ab}}$.

Proof of Proposition 4.4. We first observe the stabilizer $\mathrm{Stab}(\alpha) \subset \mathrm{As}(\mathcal{D}_g^{\mathrm{ns}})$ for $\alpha \in \mathcal{D}_g^{\mathrm{ns}}$. The inclusion $\mathcal{D}_g^{\mathrm{ns}} \to \mathcal{D}_g$ induces $\mathrm{As}(\mathcal{D}_g^{\mathrm{ns}}) \to \mathrm{As}(\mathcal{D}_g)$. Composing this with (14) yields an epimorphism $\pi : \mathrm{As}(\mathcal{D}_g^{\mathrm{ns}}) \to \mathcal{M}_g$, which we use the same notation π. Further, by Theorem 4.6, the restriction of π to $\mathrm{Ker}(\epsilon) \cong \mathcal{T}_g$ coincides with the projection in (18). In particular, for $\alpha \in \mathcal{D}_g^{\mathrm{ns}}$, we thus have $\pi(\mathrm{Stab}(\alpha)) = \pi(\mathrm{Stab}(\alpha) \cap \mathrm{Ker}(\epsilon)) \subset \mathcal{M}_g$.

We next construct a surjection from $\pi(\mathrm{Stab}(\alpha)) \subset \mathcal{M}_g$ to $\mathbb{Z}/2$ for $g \geq 2$. As is shown [PR, Proposition 7.4], we have the following exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{M}_{g-1,2} \xrightarrow{\xi} \pi(\mathrm{Stab}(\alpha)) \xrightarrow{\lambda} \mathbb{Z}/2 \tag{19}$$

Here ξ is the homomorphism induced by the gluing $(\Sigma_{g-1,2}, \partial(\Sigma_{g-1,2})) \to (\Sigma_g, \alpha)$, and λ is defined by the transposition of the connected components of boundaries of $\Sigma_g \backslash \alpha$. By the hyperelliptic involution illustrated in the left of Figure 5, λ is surjective. Hence $\pi(\mathrm{Stab}(\alpha) \cap \mathrm{Ker}(\epsilon))$ surjects onto $\mathbb{Z}/2$. Therefore, by Corollary 4.2, we obtain an epimorphism $H_2^Q(\mathcal{D}_g^{\mathrm{ns}}; \mathbb{Z}) \to \mathbb{Z}/2$.

Finally we show $H_2^Q(\mathcal{D}_g^{\mathrm{ns}}; \mathbb{Z}) \cong \mathbb{Z}/2$ for $g \geq 5$. By Corollary 4.2 again, recall that $\big(\mathrm{Stab}(\alpha) \cap \mathrm{Ker}(\epsilon)\big)_{\mathrm{ab}} \cong H_2^Q(\mathcal{D}_g^{\mathrm{ns}}; \mathbb{Z})$. To compute this, put the inclusion $\iota : \pi(\mathrm{Stab}(\alpha)) \to \mathcal{M}_g$. By the Harer-Ivanov stability theorem (see [Iva]), the composition $\iota \circ \xi : \mathcal{M}_{g-1,2} \to \mathcal{M}_g$ induces an epimorphism

$$(\iota \circ \xi)_* : H_2(\mathcal{M}_{g-1,2}; \mathbb{Z}) \to H_2(\mathcal{M}_g; \mathbb{Z}) \qquad \text{for} \quad g \geq 5. \tag{20}$$

Since $H_2(\mathcal{M}_{g-1,2}; \mathbb{Z}) \cong H_2(\mathcal{M}_g; \mathbb{Z}) \cong \mathbb{Z}$ is known [Kor], the epimorphism is isomorphic. Let $(\iota \circ \xi)^*(\mathcal{T}_g)$ denote the central extension of $\mathcal{M}_{g-1,2}$ obtained by $\iota \circ \xi$. Since $\mathcal{M}_g$ and $\mathcal{M}_{g-1,2}$ are perfect, the group $(\iota \circ \xi)^*(\mathcal{T}_g)$ is also perfect by the isomorphism (20). Note that the group $\mathrm{Stab}(\alpha) \cap \mathrm{Ker}(\epsilon)$ is isomorphic to $\iota^*(\mathcal{T}_g)$. Hence the abelianization $\big(\mathrm{Stab}(\alpha) \cap \mathrm{Ker}(\epsilon)\big)_{\mathrm{ab}}$ never be bigger than $\mathbb{Z}/2$. thus we arrive at the conclusion $H_2^Q(\mathcal{D}_g^{\mathrm{ns}}; \mathbb{Z}) \cong \mathbb{Z}/2$. □

We next discuss the Dehn quandle $\mathcal{D}_g$. However it is not easy to determine the second homology $H_2^Q(\mathcal{D}_g; \mathbb{Z})$ in general, since so is the calculation of $\mathrm{As}(\mathcal{D}_g)$ in contract with $\mathcal{D}_g^{\mathrm{ns}}$. Then we will estimate $H_2^Q(\mathcal{D}_g; A)$ from lower bounds as follows. By the homomorphism $\pi : \mathrm{As}(\mathcal{D}_g) \to \mathcal{M}_g$, the action on $\mathcal{D}_g$ of $\mathcal{M}_g$ gives rise to that of $\mathrm{As}(\mathcal{D}_g)$. It is widely known that the orbit decomposition of $\mathcal{D}_g \curvearrowleft \mathcal{M}_g$ is given by $\mathcal{D}_g = \mathcal{D}_g^{\mathrm{ns}} \cup \big(\sqcup_{0<k\leq g/2} \mathcal{D}_g^{(k)}\big)$, where

$$\mathcal{D}_g^{(k)} := \{\alpha \in \mathcal{D}_g \mid \Sigma_g \setminus \alpha \text{ is homeomorphic to } \Sigma_{k,1} \sqcup \Sigma_{g-k,1}\ \}.$$

Note that $\mathcal{D}_g^{\mathrm{ns}}$ and $\mathcal{D}_g^{(k)}$ are subquandles of $\mathcal{D}_g$. Inspired by Corollary 4.2, we will show

Proposition 4.7. *Let $g \geq 2$. Let us decompose $\mathcal{D}_g = \mathcal{D}_g^{\mathrm{ns}} \cup \big(\sqcup_{0<k\leq g/2} \mathcal{D}_g^{(k)}\big)$ as above. Put the homomorphism $\epsilon_k : \mathrm{As}(\mathcal{D}_g) \to \mathbb{Z}$ given* (16)*. For $\alpha_k \in \mathcal{D}_g^{(k)}$, let $\mathrm{Stab}(\alpha_k) \subset \mathrm{As}(\mathcal{D}_g)$ be the stabilizer subgroup of α_k.*

(i) If $k = 1$, then $\mathrm{Stab}(\alpha_k) \cap \mathrm{Ker}(\epsilon_k)$ surjects onto $\mathbb{Z}$.

(ii) If $k = 2$, then $\mathrm{Stab}(\alpha_k) \cap \mathrm{Ker}(\epsilon_k)$ surjects onto $\mathbb{Z}/10$.

(iii) If $g = 2k$, then $\mathrm{Stab}(\alpha_k) \cap \mathrm{Ker}(\epsilon_k)$ surjects onto $\mathbb{Z}/2$.

Before proving this, we give an application.

Corollary 4.8. *Let $g \geq 2$. $H_2^Q(\mathcal{D}_g; \mathbb{Z})$ surjects onto $\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/10$. Further, if g is even, then $H_2^Q(\mathcal{D}_g; \mathbb{Z})$ surjects onto $\mathbb{Z} \oplus \mathbb{Z}/10 \oplus (\mathbb{Z}/2)^2$.*

Proof. Remark that, for $\alpha_0 \in \mathcal{D}_g^{\mathrm{ns}}$, we have $\mathrm{Stab}(\alpha_0) \cap \mathrm{Ker}(\epsilon) \to \mathbb{Z}/2$ similar to Proposition 4.4. Therefore, by combing Corollary 4.2 with Proposition 4.7, we complete the proof. □

Proof of Proposition 4.7. We first observe the subgroup $\pi(\mathrm{Stab}(\alpha_k)) \subset \mathcal{M}_g$. Notice that, by Lemma 3.5, $\pi(\mathrm{Stab}(\alpha_k))$ is exactly the stabilizer subgroup of α_k by the action $\mathcal{D}_g \curvearrowleft \mathcal{M}_g$. Hence, by the inclusions $(\Sigma_{k,1}, \partial(\Sigma_{k,1})) \hookrightarrow (\Sigma_g, \alpha_k)$ and $(\Sigma_{g-k,1}, \partial(\Sigma_{g-k,1})) \hookrightarrow (\Sigma_g, \alpha_k)$, we obtain a homomorphism $\mathcal{M}_{k,1} \times \mathcal{M}_{g-k,1} \longrightarrow \pi(\mathrm{Stab}(\alpha_k))$. Further, according to [PR, Proposition 7.4 (iii)] again, the kernel and cokernel are calculated as

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{M}_{k,1} \times \mathcal{M}_{g-k,1} \longrightarrow \pi(\mathrm{Stab}(\alpha_k)) \xrightarrow{\lambda} \mathbb{Z}/2 \qquad \text{(exact)}. \tag{21}$$

Here λ is defined by the transposition of the connected components of boundaries of $\Sigma_g \setminus \alpha_k$. That is, if $g = 2k$, then λ is surjective by the hyper-elliptic involution illustrated in the right of Figure 5; otherwise, λ is the zero map. On the other hand, the kernel $\mathbb{Z}$ is generated by $(\tau_{\delta_k}^{-1}, \tau_{\delta_{g-k}}) \in \mathcal{M}_{k,1} \times \mathcal{M}_{g-k,1}$, where δ_i is a simple closed curves around the boundary of $\Sigma_{i,1}$.

Next, we claim that $\pi(\mathrm{Stab}(\alpha_k))$ is contained in $\pi(\mathrm{Ker}(\epsilon_k)) \subset \mathcal{M}_g$. This is because that $\pi(\mathrm{Stab}(\alpha_k))$ is generated by some non-separating Dehn twists by (21), and $\epsilon_k(e_\alpha) = 0 \in \mathbb{Z}$ for any $\alpha \in \mathcal{D}_g^{\mathrm{ns}}$ by definition. In particular, we obtain $\pi(\mathrm{Stab}(\alpha_k)) = \pi(\mathrm{Stab}(\alpha_k) \cap \mathrm{Ker}(\epsilon_k))$.

Finally, we will estimate $\pi(\mathrm{Stab}(\alpha_k)) \subset \mathcal{M}_g$. First of all, the sequence (21) immediately means (iii). Furthermore recall from [Kor, Theorem 5.1] the abelianizations of $\mathcal{M}_{k,r}$:

$$H_1(\mathcal{M}_{1,1}; \mathbb{Z}) \cong \mathbb{Z}, \qquad H_1(\mathcal{M}_{2,1}; \mathbb{Z}) \cong \mathbb{Z}/10, \qquad H_1(\mathcal{M}_{g,1}; \mathbb{Z}) \cong 0 \quad \text{for } g \geq 3.$$

Here the first and second isomorphisms are induced by homomorphisms $\mathcal{M}_{1,1} \to \mathbb{Z}$ and $\mathcal{M}_{2,1} \to \mathbb{Z}/10$ sending each Dehn twists to 1, respectively. Therefore, by composing the abelianizations with the sequence (21) we complete the proof of (i) and (ii). □

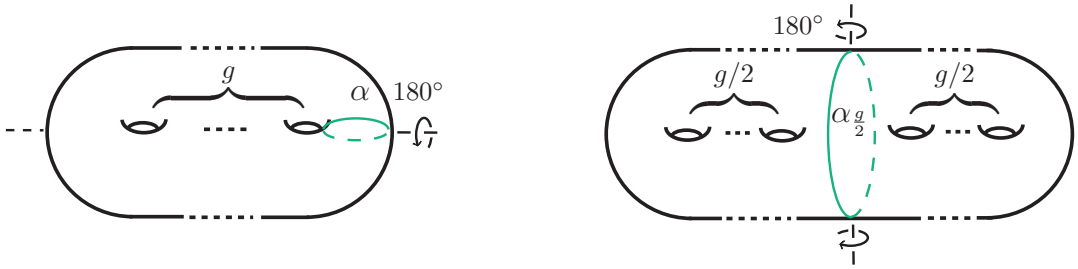

Figure 5: Hyper-elliptic involutions.

Remark 4.9. Although these homologies $H_2^Q(\mathcal{D}_g;\mathbb{Z})$ have seemingly such non-trivial parts by Propositions 4.4 and 4.7, we now explain roughly the reason why the parts capture no non-trivial cocycle invariants. The cocycle invariant with trivial coefficients can be regarded as an abelianization of the invariant with non-abelian coefficients using Proposition 3.2. However, by the equalities (15) in the proof of Theorem 3.1, the latter invariant is derived from the kernel of π in (14). Since we obtained the non-trivial parts from some subgroups of $\mathcal{M}_g$ in the proofs of the propositions, the parts therefore develop no non-trivial cocycle invariants.

5 Quandle cocycle invariants with local coefficients

This section studies the quandle cocycle invariants with local coefficients defined in Definition 2.9. In §5.1 we give an algorithm to produce some 2-cocycles with the local coefficients. In §5.2, for Lefschetz fibrations of type mI, we estimate the invariants from upper bounds.

5.1 Quandle 2-cocycles from subgroups of the mapping class group

We will give examples of 2-cocycles with local coefficient. To see this, let X be a quandle, and Y an X-set. For a map $\vartheta : Y \times X \to A$, we define a map $\overline{\vartheta} : Y \times X^2 \to A$ by

$$\overline{\vartheta}(y, x_1, x_2) := \vartheta(y, x_1) - \vartheta(y, x_2).$$

We give a method to produce 2-cocycles arising from 1-cocycles as follows:

Lemma 5.1. *If ϑ is a 1-cocycle with local coefficients, then $\overline{\vartheta}$ is a 2-cocycle.*

Proof. From the definition of δ_1, for any $y \in Y$, and $x_1, x_2 \in X$, we can see

$$\delta_1(\vartheta)(y, x_1, x_2) = \vartheta(y, x_2) - \vartheta(y \cdot e_{x_1}, x_2) - \vartheta(y, x_1) + \vartheta(y \cdot e_{x_2}, x_1 \triangleleft x_2).$$

For any $x_3 \in X$, a direct calculation shows the equality

$$\delta_1(\vartheta)(y, x_1, x_2) - \delta_1(\vartheta)(y, x_1, x_3) + \delta_1(\vartheta)(y, x_2, x_3) = \delta_2(\overline{\vartheta})(y, x_1, x_2, x_3).$$

Since ϑ is a 1-cocycle, $\overline{\vartheta}$ is a 2-cocycle as desired. □

In applications of the cocycle invariants, let us assume that $X = \mathcal{D}_g$ and the right action of $\mathrm{As}(\mathcal{D}_g)$ on Y is transitive. Hence, Y is a left quotient of $\mathrm{As}(\mathcal{D}_g)$, i.e., $Y = H\backslash\mathrm{As}(\mathcal{D}_g)$ for some subgroup $H \subset \mathrm{As}(\mathcal{D}_g)$. Here recall from Proposition 3.2 that the abelianization of H produces 1-cocycles. In conclusion, one has obtained 2-cocycles from the abelianizations of subgroups of $\mathcal{M}_g$ or $\mathrm{As}(\mathcal{D}_g)$ as follows:

Proposition 5.2. *Let G be either $\mathcal{M}_g$ or $\mathrm{As}(\mathcal{D}_g)$ For a subgroup H of G, let Y be a left quotient $H\backslash G$. Choose a section $\mathfrak{s} : Y \to G$. Then the map $\theta : Y \times \mathcal{D}_g \times \mathcal{D}_g \to (H)_{\mathrm{ab}}$ defined by $\theta(y, x_1, x_2) := e_{x_1} \cdot e_{x_2}^{-1}\mathfrak{s}(y \triangleleft x_1)^{-1}\mathfrak{s}(y \triangleleft x_2)$ is a 2-cocycle.*

Proof. By combing Lemma 5.1 with Proposition 3.2 strightforward. □

We now observe some examples of the subgroups and these abelianizations.

Example 5.3. Let us regard $\mathrm{Sp}(2g;\mathbb{Z})$ as a left quotient $\mathcal{I}_{g,1}\backslash\mathcal{M}_{g,1}$, where $\mathcal{I}_{g,1}$ is the Torelli group. The group $\mathcal{M}_{g,1}$ is presented by a quotient of $\mathrm{As}(\mathcal{D}_g)$ (see, e.g., [Ge, FM]). Here recall that the abelianization of $\mathcal{I}_{g,1}$ is given by a homomorphism

$$\tau \oplus \mathcal{BC} : H_1(\mathcal{I}_{g,1};\mathbb{Z}) \longrightarrow \big(\wedge^3 H_1(\Sigma_g;\mathbb{Z})\big) \oplus \mathbb{Z}/2\mathbb{Z}^{n_g},$$

for some $n_g \in \mathbb{Z}$ (see, e.g., [FM] for details)

Furthermore, we consider the level d mapping class group. For $d \in \mathbb{Z}$, put the epimorphism $\mathrm{Sp}(2g;\mathbb{Z}) \to \mathrm{Sp}(2g;\mathbb{Z}/d\mathbb{Z})$ induced by the projection $\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$. Denote $\mathcal{I}_g^d$ by the kernel of the composite $\mathcal{M}_g \to \mathrm{Sp}(2g;\mathbb{Z}) \to \mathrm{Sp}(2g;\mathbb{Z}/d\mathbb{Z})$. When d is either odd or 2, Sato [Sat] completely determined the abelianization of $\mathcal{I}_g^d$.

Example 5.4. Let G be a finite group. As is well-known (see [FM, Chapter 7]), G injects into $\mathcal{M}_g$ for some g. Hence, we obtain a 2-cocycle valued in G_{ab}.

So we have obtained many 2-cocycles with local coefficients; however, the author finds no examples of non-trivial cocycle invariants using such 2-cocycles.

5.2 An estimate of the quandle cocycle invariants

In this subsection, we restrict ourselves to Lefschetz fibrations of type mI. By Remark 2.13, for the study of quandle cocycle invariants, we may focus on the non-separating Dehn quandle $\mathcal{D}_g^{\mathrm{ns}}$. In what follows, fix $X = \mathcal{D}_g^{\mathrm{ns}}$.

Recall the rack space BX considered in §4.1. Our purpose in this subsection is to calculate the homotopy group $\pi_2(BX)$, and give an estimate of our cocycle invariants as a corollary. To state Theorem 5.5, we prepare some notation. Since it is known that $\pi_2(BX)$ contains a direct summand $\mathbb{Z}$ (see [No1, Proposition 3.12][2]), we denote the quotient by $\pi_2(BX)'$, i.e., $\pi_2(BX) \cong \mathbb{Z} \oplus \pi_2(BX)'$. For a prime $p \in \mathbb{Z}$, let $\pi_2(B\mathcal{D}_g^{\mathrm{ns}})'_{(p)}$ be the p-localization.

Theorem 5.5. *For $g \geq 7$, then $\pi_2(B\mathcal{D}_g^{\mathrm{ns}})'_{(p)} \cong 0$ for any prime p with $p \neq 2, 3$. Furthermore, if $g \geq 6$, then $\pi_2(B\mathcal{D}_g^{\mathrm{ns}})$ is of finite order.*

Before proving this, we will explain an application.

Corollary 5.6. *Let $\psi : Y \times \mathcal{D}_g^{\mathrm{ns}} \times \mathcal{D}_g^{\mathrm{ns}} \to A$ be a quandle cocycle of $\mathcal{D}_g^{\mathrm{ns}}$. If $A \otimes \mathbb{Z}/6 \cong 0$ and $g \geq 7$, then the cocycle invariant is trivial, i.e., $\langle f^*(\psi), \mu_{Y,y}\rangle = 0$.*

As a conclusion (Corollary 5.6), in order to discover non-trivial cocycle invariants, we have to set up coefficients with 2- and 3-torsions.

This corollary is easily shown as follows. Let $X = \mathcal{D}_g^{\mathrm{ns}}$, and let L be the torus link $T_{m,m}$ (see [No1, §2] for general lines). Given a quandle homomorphism $f : Q(S^3, L) \to X$, we can define an element $\Xi_X(f) \in \pi_2(BX)'$, which was first considered in [FRS1] (see [No1, §2] for details, although $\Xi_X(f)$ is not used later.). As is known, $\Xi_X(f) \in \pi_2(BX)'$ is universal among quandle cocycle invariants with local coefficients in Definition 2.9. Precisely, any cocycle $\psi : Y \times X \times X \to A$ and $y \in Y$ admit a homomorphism $H_\psi : \pi_2(BX)' \to A$ so that

$$\langle f^*(\psi), \mu_{Y,y}\rangle = H_\psi\big(\Xi_X(f)\big) \in A,$$

[2]In [No1], the homotopy group of the rack space was denoted by $\pi_2(\widehat{BX})$, and $\pi_2(BX)'$ was denoted by $\pi_2(BX)$.

(See the equations (6) and (7) in [No1] for details). Hence, Corollary 5.6 immediately follows from Theorem 5.5.

Proof of Theorem 5.5. By a routine discussion of Postnikov tower on BX, we have an exact sequence (see, e.g., Propositions 3.2, 3.3 and 3.12 in [No1] for details)

$$H_3(\mathrm{As}(\mathcal{D}_g^{\mathrm{ns}});\mathbb{Z}) \longrightarrow \pi_2(BX)' \longrightarrow H_2^Q(\mathcal{D}_g^{\mathrm{ns}};\mathbb{Z}) \longrightarrow H_2(\mathrm{As}(\mathcal{D}_g^{\mathrm{ns}});\mathbb{Z}) \longrightarrow 0. \tag{22}$$

To calculate the first term, we need a lemma, which is shown later.

Lemma 5.7. *If $g \geq 7$, then $H_3(\mathrm{As}(\mathcal{D}_g^{\mathrm{ns}});\mathbb{Z})_{(p)} \cong H_3(\mathcal{M}_g;\mathbb{Z})_{(p)}$ without $p = 2$. If $g = 6$, then $H_3(\mathrm{As}(\mathcal{D}_g^{\mathrm{ns}});\mathbb{Z})$ is of finite order.*

Recall $H_2^Q(\mathcal{D}_g^{\mathrm{ns}};\mathbb{Z}) \cong \mathbb{Z}/2$ for $g \geq 5$ (Proposition 4.7). By localizing (22) at $p \neq 2$, we have

$$H_3(\mathcal{M}_g;\mathbb{Z})_{(p)} \longrightarrow \pi_2(BX)'_{(p)} \longrightarrow 0 \qquad \text{(exact)}, \tag{23}$$

for $g \geq 7$. Further, $H_3(\mathcal{M}_g;\mathbb{Z}) \otimes \mathbb{Z}/p \cong 0$ with $p \neq 2, 3$ is shown in [Ga] (Further $H_3(\mathcal{M}_g;\mathbb{Z}) \cong \mathbb{Z}/12$ is claimed in [Eb]). Hence we have proven the statement in Theorem 5.5 for $g \geq 7$.

Finally, we consider the case $g = 6$. Likewise, $\pi_2(B\mathcal{D}_g^{\mathrm{ns}})'$ is finite as required, from which we deduce Lemma 5.7 and the sequence (22). □

Finally we will prove Lemma 5.7 by an elementary argument of spectral sequences and by facts of the stable cohomology of the mapping class group.

Proof of Lemma 5.7. Recall from Theorem 4.6 that $\mathrm{As}(\mathcal{D}_g^{\mathrm{ns}})$ is isomorphic to $\mathcal{T}_g \times \mathbb{Z}$ for $g \geq 3$. Thereby the homology of $\mathrm{As}(\mathcal{D}_g^{\mathrm{ns}})$ is finite generated.

To analyze the homology in more details, we consider the Serre-Hochschild spectral sequence of the central extension (18), which converges to $E_\infty \cong H^*(\mathcal{T}_g;\mathbb{Z})$. Further the E_2-term is given by

$\mathbb{Z}$	0	$H^2(\mathcal{M}_g;\mathbb{Z})$	$H^3(\mathcal{M}_g;\mathbb{Z})$	$H^4(\mathcal{M}_g;\mathbb{Z})$	$\cdots$
$\mathbb{Z}$	0	$H^2(\mathcal{M}_g;\mathbb{Z})$	$H^3(\mathcal{M}_g;\mathbb{Z})$	$H^4(\mathcal{M}_g;\mathbb{Z})$	$\cdots$

In particular, the E_3-term is isomorphic to the E_∞-term. Moreover notice that the differential $d_2 : E_2^{1,0} = \mathbb{Z} \to H^2(\mathcal{M}_g;\mathbb{Z}) = E_2^{2,1}$ is isomorphic, since $\mathcal{T}_g$ is the universal central extension. Hence $H^1(\mathcal{T}_g;\mathbb{Z}) \cong H^2(\mathcal{T}_g;\mathbb{Z}) \cong 0$.

Next, we focus on $E_2^{s,t}$ for $s \geq 2$. Since $H_3(\mathcal{M}_g;\mathbb{Q}) = 0$ and $H_4(\mathcal{M}_g;\mathbb{Q}) \cong \mathbb{Q} \oplus \mathbb{Q}$ (see [H2]), it follows from the universal coefficient theorem that

$$H^3(\mathcal{M}_g;\mathbb{Z}) \cong 0, \qquad H^4(\mathcal{M}_g;\mathbb{Z}) \cong \mathbb{Z}^2 \oplus H_3(\mathcal{M}_g;\mathbb{Z}).$$

We claim that, when localized at $p \neq 2$, the cokernel of d_2 is expressed by

$$\mathrm{Coker}(d_2 : E_2^{2,1} \to E_2^{4,0})_{(p)} \cong \mathbb{Z}_{(p)} \oplus H_3(\mathcal{M}_g;\mathbb{Z})_{(p)}, \tag{24}$$

where $g \geq 7$. To show this, put the inclusion $\mathcal{M}_{g,1} \to \mathcal{M}_{g+1,1}$ obtained by gluing $\Sigma_{1,2}$ along one of its boundary components. Let $\mathcal{M}_\infty := \lim_{g\to\infty} \mathcal{M}_{g,1}$. Further put an epimorphism

$\delta_g : \mathcal{M}_{g,1} \to \mathcal{M}_g$ induced by gluing a disc to the boundary component of $\Sigma_{g,1}$. According to the Harer-Ivanov stability theorem improved by [RW], the inclusion $\mathcal{M}_{g,1} \to \mathcal{M}_\infty$ induces an isomorphism $H^4(\mathcal{M}_{g,1};\mathbb{Z}) \cong H^4(\mathcal{M}_\infty;\mathbb{Z})$, and δ_g gives rise to an isomorphism $H^4(\mathcal{M}_g;\mathbb{Z}) \cong H^4(\mathcal{M}_{g,1};\mathbb{Z})$. Further, it is shown [GMT] that a ring isomorphism

$$\big(H^*(\mathcal{M}_\infty;\mathbb{Z})/\text{torsion}\big)_{(p)} \cong \mathbb{Z}_{(p)}[[\kappa_2, \kappa_4, \kappa_6, \dots]],$$

where $\kappa_{2i} \in H^{2i}(\mathcal{M}_g;\mathbb{Z})_{(p)}$ is roughly defined by a certain divisor of the Mumford-Morita-Miller class. Hence, by the multiplication of the differential d_2, we obtain (24) as required.

Notice that $H^4(\mathcal{T}_g;\mathbb{Z})_{(p)} \cong (E_\infty^4)_{(p)} \cong \mathrm{Coker}(d_2 : E_2^{2,1} \to E_2^{4,0})_{(p)}$. By the universal coefficient theorem, we obtain $H_3(\mathcal{T}_g;\mathbb{Z})_{(p)} \cong H_3(\mathcal{M}_g;\mathbb{Z})_{(p)}$. Since $\mathrm{As}(\mathcal{D}_g^{\mathrm{ns}}) \cong \mathcal{T}_g \times \mathbb{Z}$, the Kunneth theorem concludes the desired isomorphism $H_3(\mathrm{As}(\mathcal{D}_g^{\mathrm{ns}});\mathbb{Z})_{(p)} \cong H_3(\mathcal{M}_g;\mathbb{Z})_{(p)}$ for $g \geq 7$.

Finally, we deal with $g = 6$. Similarly, the inclusion $\mathcal{M}_{g,1} \to \mathcal{M}_\infty$ and the projection $\mathcal{M}_{g,1} \to \mathcal{M}_g$ give rise to a rational isomorphism $H^4(\mathcal{M}_g;\mathbb{Q}) \cong H^4(\mathcal{M}_\infty;\mathbb{Q})$. By iterating the above process after rationalization, we can show that $H_3(\mathcal{T}_g;\mathbb{Q}) \cong 0$. Therefore $H_3(\mathrm{As}(\mathcal{D}_g^{\mathrm{ns}});\mathbb{Z})$ is finite. □

Acknowledgment

The author expresses his gratitude to Tomotada Ohtsuki for helpful suggestions. He also thank Masatoshi Sato, Naoyuki Monden and Yoshiro Yaguchi for useful comments on the mapping class group. This research was partially supported by JSPS Research Fellowships for Young Scientists.

References

[AG] N. Andruskiewitsch, M. Graña: *From racks to pointed Hopf algebras*, Adv. in Math. **178** (2003), 177–243.

[CEGS] J. S. Carter, J. S. Elhamdadi, M. Graña, M. Saito, *Cocycle knot invariants from quandle modules and generalized quandle homology*, Osaka J. Math. **42** (2005), 499–541.

[CJKLS] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford, M. Saito, *Quandle cohomology and state-sum invariants of knotted curves and surfaces*, Trans. Amer. Math. Soc. **355** (2003) 3947–3989.

[E1] M. Eisermann, *Quandle coverings and their Galois correspondence*, arXiv:math/0612459v3

[E2] ———, *Homological characterization of the unknot*, J. Pure Appl. Algebra **177** (2003), no. 2, 131–157.

[Eb] J.F. Ebert, *The low-dimensional homotopy of the stable mapping class group*, arXiv:0707.1014

[EN] H. Endo, S. Nagami, *Signature of relations in mapping class groups and non-holomorphic Lefschetz fibrations,* Trans. Amer. Math. Soc. **357** (2005), no. 8, 3179–3199.

[FM] B. Farb and D. Margalit, *A primer on mapping class groups*, version 5, preprint 2010.

[FRS1] R. Fenn, C. Rourke, B. Sanderson, *Trunks and classifying spaces*, Appl. Categ. Structures **3** (1995) 321–356.

[FRS2] ———, *The rack space*, Trans. Amer. Math. Soc. **359** (2007), no. 2, 701–740.

[Ga] S. Galatius, *Mod p homology of the stable mapping class group*, Topology **43** (2004), 1105–1132.

[Ge] S. Gervais, *Presentation and central extension of mapping class groups*, Trans. Amer. Math. Soc. **348** (1996) 3097–3132.

[GMT] S. Galatius, I. Madsen, U. Tillmann, *Divisibility of the stable Miller-Morita-Mumford classes*, Journal of Amer. Math. Soc., **19** (2006), 759–779.

[H1] J. L. Harer, *Stability of the homology of the mapping class groups of orientable surfaces*, Ann. of Math., **121** (2), (1985) 215–249.

[H2] ———, *The third homology group of the moduli space of curves*, Duke Mathematical Journal **63** (1991), 25–55.

[IK] A. Inoue, Y. Kabaya, *Quandle homology and complex volume*, arXiv:math/1012.2923.

[Iva] N. V. Ivanov, *Stabilization of the homology of Teichmüller modular groups*, Algebrai Analiz **1** (1989), 110–126 (Russian); translation in Leningrad Math. J. **1** (1990) 675–691.

[Joy] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra **23** (1982) 37–65.

[Kor] M. Korkmaz, *Low-dimensional homology groups of mapping class groups: a survey.* Turkish J. Math. 26 (2002), no. 1, 101–114.

[LN] R. Litherland and S. Nelson, *The Betti numbers of some finite racks.* J. Pure Appl. Algebra **178** (2003), no. 2, 187–202.

[Kab] Y. Kabaya, *Cyclic branched coverings of knots and quandle homology,* arXiv:math/1012.3729

[M] Y. Matsumoto, *Lefschetz fibrations of genus two – a topological approach*, Proceedings of the 37th Taniguchi Symposium on Topology and Teichmuller Spaces, ed. Sadayoshi Kojima et al., World Scientific (1996), 123–148.

[No1] T. Nosaka, *On homotopy groups of quandle spaces and the quandle homotopy invariant of links*, Topology and its Applications **158** (2011), 996–1011.

[No2] ———, *Quandle cocycle invariants of knots using Mochizuki's 3-cocycles and Dijkgraaf-Witten invariants of 3-manifolds*, preprint, arXiv:1103.3839.

[NP] M. Niebrzydowski, J. H. Przytycki, *The quandle of the trefoil as the Dehn quandle of the torus*, Osaka Journal of Mathematics **46** (2009) 645–659.

[PR] L. Paris, D. Rolfsen, *Geometric subgroups of mapping class groups,* J. Reine Angew. Math. **521** (2000), 47–83.

[RW] O. Randal-Williams, *Resolutions of moduli spaces and homological stability*, arXiv:0909.4278.

[Sat] M. Sato, *The abelianization of the level d mapping class group*, J. Topol. **3** (2010), no. 4, 847–882.

[Y] D. N. Yetter, *Quandles and monodromy*, J. Knot Theory Ramifications **12** (2003), 523–541.

[Zab] J. Zablow, *On relations and homology of the Dehn quandle.* Algebr. Geom. Topol. **8** (2008), no. 1, 19–51.

Research Institute for Mathematical Sciences, Kyoto University, Sakyo-ku, Kyoto, 606-8502, Japan

E-mail address: nosaka@kurims.kyoto-u.ac.jp