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On the universal sl_2 invariant of Brunnian bottom tangles

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Abstract

A link L is called Brunnian if every proper sublink of L is trivial. Similarly, a bottom tangle T is called Brunnian if every proper subtangle of T is trivial. In this paper, we give a small subalgebra of the n-fold completed tensor power of $U_h(sl_2)$ in which the universal sl_2 invariant of n-component Brunnian bottom tangles takes values. As an application, we give a divisibility property of the colored Jones polynomial of Brunnian links.

1 Introduction

The universal invariant of tangles associated with a ribbon Hopf algebra [3, 5, 6, 7, 8, 9, 13, 14] has the universality property for the colored link invariants which are defined by Reshetikhin and Turaev [14].

The universal sl_2 invariant J_T of an n-component bottom tangle T takes values in the n-fold completed tensor powers $U_h(sl_2)^{\hat{\otimes} n}$ of $U_h(sl_2)$, and we can obtain the colored Jones polynomial of the closure link $\mathrm{cl}(T)$ from J_T by taking the quantum traces. Here, a *bottom tangle* is a tangle in a cube consisting of only arc components such that each boundary point is on the bottom and the two boundary points of each arc are adjacent to each other, see Figure 1 (a) for example. The closure of a bottom tangle is defined as in Figure 1 (b).

Our interest is in the relationship between *topological properties* of tangles and links and *algebraic properties* of the universal sl_2 invariant and the colored Jones polynomial. Habiro [3] proved that the universal sl_2 invariant of n-component, algebraically-split, 0-framed bottom tangles takes values in a subalgebra $(\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$ of $U_h(sl_2)^{\hat{\otimes} n}$ (Theorem 4.4). The present author proved improvements of this result with a smaller subalgebra $(\bar{U}_q^{\mathrm{ev}})^{\hat{}\hat{\otimes} n} \subset (\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$ in the special case of ribbon bottom tangles [15] and boundary bottom tangles [16] (Theorem 4.5). Here, the result for boundary bottom tangles had been conjectured by Habiro [3].

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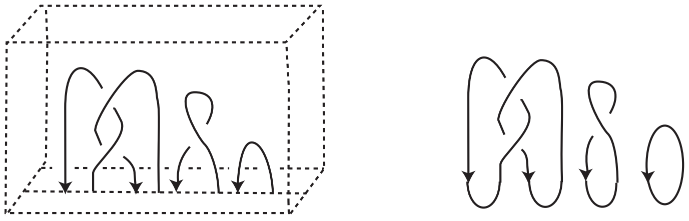


(a) (b)

Figure 1: (a) A bottom tangle T, (b) The closure link $\mathrm{cl}(T)$ of T

A link L is called *Brunnian* if every proper sublink of L is trivial. Similarly, a bottom tangle T is called *Brunnian* if every proper subtangle of T is *trivial*, i.e., looks like $\cap \cdots \cap$. Habiro [4, Proposition 12] proved that for every Brunnian link L, there is a Brunnian bottom tangle whose closure is isotopic to L.

In the present paper, we give a subalgebra $U_{Br}^{(n)}$ of $U_h(sl_2)^{\hat{\otimes} n}$ such that $(\bar{U}_q^{\mathrm{ev}})^{\hat{}\,\hat{\otimes} n} \subset U_{Br}^{(n)} \subset (\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$ in which the universal sl_2 invariant of n-component Brunnian bottom tangles takes values (Theorem 4.7). As an application, we prove a divisibility property of the colored Jones polynomial of Brunnian links (Theorem 5.4).

The rest of this paper is organized as follows. In Section 2, we recall basic facts of the quantized enveloping algebra $U_h(sl_2)$. In Section 3, we define the universal sl_2 invariant of bottom tangles. In Section 4, we give the main result for the universal sl_2 invariant of Brunnian bottom tangles. In Section 5, we give an application for the colored Jones polynomial of Brunnian links. Section 6 is devoted to the proofs of the results.

2 Quantized enveloping algebra $U_h(sl_2)$

In this section, we recall the definition of $U_h(sl_2)$ and its subalgebras. We follow the notations in [3, 16].

2.1 Quantized enveloping algebra $U_h(sl_2)$

We recall the definition of the universal enveloping algebra $U_h(sl_2)$.

We denote by $U_h = U_h(sl_2)$ the h-adically complete $\mathbb{Q}[[h]]$-algebra, topologically generated by H, E, and F, defined by the relations

$$HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}},$$

where we set

$$q = \exp h, \quad K = q^{H/2} = \exp \frac{hH}{2}.$$

We equip U_h with the topological $\mathbb{Z}$-graded algebra structure such that $\deg E = 1$, $\deg F = -1$, and $\deg H = 0$. For a homogeneous element x of U_h, the degree of x is denoted by $|x|$.

2.2 $\mathbb{Z}[q,q^{-1}]$-subalgebras of $U_h(sl_2)$

We recall $\mathbb{Z}[q,q^{-1}]$-subalgebras of U_h from [3, 16].

In what follows, we use the following q-integer notations.

$$\{i\}_q = q^i - 1, \quad \{i\}_{q,n} = \{i\}_q\{i-1\}_q \cdots \{i-n+1\}_q, \quad \{n\}_q! = \{n\}_{q,n},$$
$$[i]_q = \{i\}_q/\{1\}_q, \quad [n]_q! = [n]_q[n-1]_q \cdots [1]_q, \quad \begin{bmatrix} i \\ n \end{bmatrix}_q = \{i\}_{q,n}/\{n\}_q!,$$

for $i \in \mathbb{Z}, n \geq 0$.

Set

$$\tilde{E}^{(n)} = (q^{-1/2}E)^n/[n]_q!, \quad \tilde{F}^{(n)} = F^nK^n/[n]_q! \quad \in U_h, \tag{1}$$
$$e = (q^{1/2} - q^{-1/2})E, \quad f = (q-1)FK \quad \in U_h, \tag{2}$$

for $n \geq 0$.

Let $U_{\mathbb{Z},q} \subset U_h$ denote the $\mathbb{Z}[q,q^{-1}]$-subalgebra generated by $K, K^{-1}, \tilde{E}^{(n)}$, and $\tilde{F}^{(n)}$ for $n \geq 1$, which is a $\mathbb{Z}[q,q^{-1}]$-version of the Lusztig's integral form (cf. [10, 15]).

Let $\mathcal{U}_q \subset U_{\mathbb{Z},q}$ denote the $\mathbb{Z}[q,q^{-1}]$-subalgebra generated by K, K^{-1}, e, and $\tilde{F}^{(n)}$ for $n \geq 1$.

Let $\bar{U}_q \subset \mathcal{U}_q$ denote the $\mathbb{Z}[q,q^{-1}]$-subalgebra generated by K, K^{-1}, e and f, which is a $\mathbb{Z}[q,q^{-1}]$-version of the integral form defined by De Concini and Procesi (cf. [1, 15]).

For $X = U_{\mathbb{Z},q}$, $\mathcal{U}_q$, $\bar{U}_q$, let X^{ev} denote the $\mathbb{Z}[q,q^{-1}]$-subalgebra of U_h defined by the same generators as X except that $K^{\pm 2}$ replaces $K^{\pm 1}$, i.e., $U^{\mathrm{ev}}_{\mathbb{Z},q} \subset U_{\mathbb{Z},q}$ denotes the $\mathbb{Z}[q,q^{-1}]$-subalgebra generated by $K^2, K^{-2}, \tilde{E}^{(n)}$, $\tilde{F}^{(n)}$, $n \geq 1$; $\mathcal{U}^{\mathrm{ev}}_q \subset \mathcal{U}_q$ denotes the $\mathbb{Z}[q,q^{-1}]$-subalgebra generated by K^2, K^{-2}, e, $\tilde{F}^{(n)}$, $n \geq 1$; and $\bar{U}^{\mathrm{ev}}_q \subset \bar{U}_q$ denotes the $\mathbb{Z}[q,q^{-1}]$-subalgebra generated by K^2, K^{-2}, e, f.

To summarize, we have the following inclusions of the subalgebras of U_h.

$$\begin{array}{ccccccc}
\bar{U}^{\mathrm{ev}}_q & \subset & \mathcal{U}^{\mathrm{ev}}_q & \subset & U^{\mathrm{ev}}_{\mathbb{Z},q} & & \\
\cap & & \cap & & \cap & & \\
\bar{U}_q & \subset & \mathcal{U}_q & \subset & U_{\mathbb{Z},q} & \subset & U_h
\end{array}$$

2.3 Completion

In this section, we recall from [3] the completion $\tilde{\mathcal{U}}^{\mathrm{ev}}_q$ of $\mathcal{U}^{\mathrm{ev}}_q$ in U_h and its completed tensor powers $(\tilde{\mathcal{U}}^{\mathrm{ev}}_q)^{\tilde{\otimes} n}$ for $n \geq 0$.

First, we define $\tilde{\mathcal{U}}^{\mathrm{ev}}_q$. For $p \geq 0$, let $\mathcal{F}_p(\mathcal{U}^{\mathrm{ev}}_q)$ be the two-sided ideal in $\mathcal{U}^{\mathrm{ev}}_q$ generated by e^p. Let $\tilde{\mathcal{U}}^{\mathrm{ev}}_q$ be the completion of $\mathcal{U}^{\mathrm{ev}}_q$ in U_h with respect to the decreasing filtration $\{\mathcal{F}_p(\mathcal{U}^{\mathrm{ev}}_q)\}_{p\geq 0}$, i.e., we define $\tilde{\mathcal{U}}^{\mathrm{ev}}_q$ as the image of the homomorphism

$$\varprojlim_{p\geq 0} \mathcal{U}^{\mathrm{ev}}_q/\mathcal{F}_p(\mathcal{U}^{\mathrm{ev}}_q) \to U_h$$

induced by $\mathcal{U}^{\mathrm{ev}}_q \subset U_h$.

We define $(\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$ for $n \geq 0$. For $n = 0$, we define $(\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} 0} = \mathbb{Z}[q, q^{-1}]$. For $n \geq 1$, we define $(\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$ as the completion of $(\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}$ in $U_h^{\hat{\otimes} n}$ with respect to the decreasing filtration $\{\mathcal{F}_p\big((\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}\big)\}_{p\geq 0}$, where we set

$$\mathcal{F}_p\big((\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}\big) = \sum_{i=1}^{n} (\mathcal{U}_q^{\mathrm{ev}})^{\otimes (i-1)} \otimes \mathcal{F}_p(\mathcal{U}_q^{\mathrm{ev}}) \otimes (\mathcal{U}_q^{\mathrm{ev}})^{\otimes (n-i)}, \quad p \geq 0,$$

i.e., we define

$$(\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n} = \mathrm{Im}\Big(\varprojlim_{p\geq 0} (\mathcal{U}_q^{\mathrm{ev}})^{\otimes n} / \mathcal{F}_p\big((\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}\big) \to U_h^{\hat{\otimes} n}\Big).$$

For a $\mathbb{Z}[q, q^{-1}]$-subalgebra A of $(\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}$, $n \geq 0$, we denote by $\{A\}\hat{}$ the *closure* of A in $(\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$, which is the completion of A in $U_h^{\hat{\otimes} n}$ with respect to the decreasing filtration $\mathcal{F}_p\big((\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}\big) \cap A$, i.e.,

$$\{A\}\hat{} = \mathrm{Im}\Big(\varprojlim_{p\geq 0} (A / \big(\mathcal{F}_p\big((\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}\big) \cap A\big) \to U_h^{\hat{\otimes} n}\Big).$$

In particular, we denoted by $(\bar{U}_q^{\mathrm{ev}})^{\tilde{}\,\tilde{\otimes} n}$ the closure $\{(\bar{U}_q^{\mathrm{ev}})^{\otimes n}\}\hat{}$ of $(\bar{U}_q^{\mathrm{ev}})^{\otimes n}$ in $(\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$.

3 Universal sl_2 invariant of bottom tangles

In this section, we recall the definition of the universal sl_2 invariant of bottom tangles.

3.1 Bottom tangles

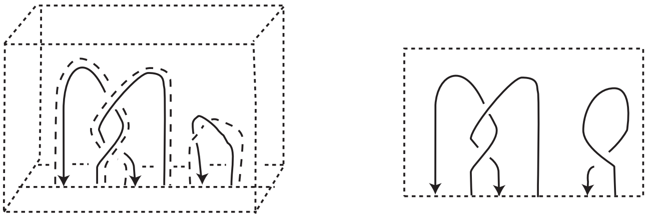
A *bottom tangle* (cf. [2, 3]) is an oriented, framed tangle in a cube consisting of arc components such that each boundary point is on a line on the bottom, and the two boundary points of each component are adjacent to each other. We give a preferred orientation of the tangle so that each component runs from its right boundary point to its left boundary point. For example, see Figure 2 (a), where the dotted lines represent the framing. We draw a diagram of a bottom tangle in a rectangle assuming the blackboard framing, see Figure 2 (b).

The *closure link* $\mathrm{cl}(T)$ of a bottom tangle T is defined as the link in $\mathbb{R}^3$ obtained from T by closing, see Figure 1 again. For each n-component link L, there is an n-component bottom tangle whose closure is L. For a bottom tangle, we can define its linking matrix as that of the closure link.

3.2 Universal R-matrix of U_h

Set

$$D = q^{\frac{1}{4} H \otimes H} = \exp\big(\frac{h}{4} H \otimes H\big) \in U_h^{\hat{\otimes} 2}.$$



(a) (b)

Figure 2: (a) A bottom tangle T, (b) a diagram of T

We use the following *universal R-matrix* of U_h,

$$R^{\pm 1} = \sum_{n\geq 0} \alpha_n^{\pm} \otimes \beta_n^{\pm} \in U_h^{\hat{\otimes} 2},$$

where we set formally

$$\begin{aligned} \alpha_n \otimes \beta_n (= \alpha_n^{+} \otimes \beta_n^{+}) &= D\Big(q^{\frac{1}{2}n(n-1)}\tilde{F}^{(n)}K^{-n} \otimes e^n\Big), \\ \alpha_n^{-} \otimes \beta_n^{-} &= D^{-1}\Big((-1)^n \tilde{F}^{(n)} \otimes K^{-n}e^n\Big). \end{aligned}$$

(Note that the right hand sides are sums of infinitely many tensors of the form $x \otimes y$ with $x, y \in U_h$. We denote them by $\alpha_n^{\pm} \otimes \beta_n^{\pm}$ for simplicity.)

3.3 Universal sl_2 invariant of bottom tangles

For an n-component bottom tangle $T = T_1 \cup \cdots \cup T_n$, we define the universal sl_2 invariant $J_T \in U_h^{\hat{\otimes} n}$ in four steps as follows. We follow the notation in [16].

Step 1. Choose a diagram. We choose a diagram $\tilde{T}$ of T obtained from the copies of the fundamental tangles depicted in Figure 3, by pasting horizontally and vertically. We denote by $C(\tilde{T})$ the set of the crossings of $\tilde{T}$. For example, for the bottom tangle B depicted in Figure 4 (a), we can take a diagram $\tilde{B}$ with $C(\tilde{B}) = \{c_1, c_2\}$ as depicted in Figure 4 (b). We call a map

$$s\colon\ C(\tilde{T}) \ \ \rightarrow \ \ \{0, 1, 2, \ldots\}$$

a *state*. We denote by $\mathcal{S}(\tilde{T})$ the set of states of the diagram.

Step 2. Attach labels. Given a state $s \in \mathcal{S}(\tilde{T})$, we attach labels on the copies of the fundamental tangles in the diagram following the rule described in Figure 5, where "S'" should be replaced with id if the string is oriented downward, and with S otherwise. For example, for a state $t \in \mathcal{S}(\tilde{B})$, we put labels on $\tilde{B}$ as in Figure 4 (c), where we set $m = t(c_1)$ and $n = t(c_2)$.

Step 3. Read the labels. We read the labels we have just put on $\tilde{T}$ and define an element $J_{\tilde{T},s} \in U_h^{\hat{\otimes} n}$ as follows. Let $\tilde{T} = \tilde{T}_1 \cup \cdots \cup \tilde{T}_1$, where $\tilde{T}_i$ corresponds to T_i. We

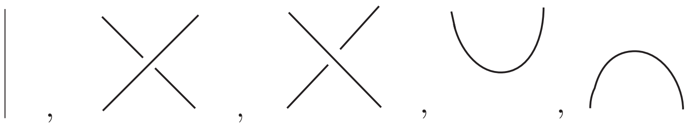
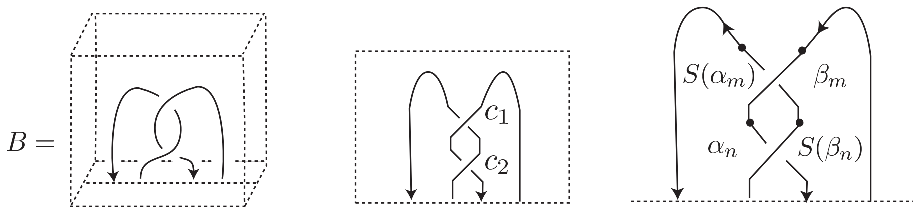


Figure 3: Fundamental tangles, where the orientations of the strands are arbitrary


$B =$
c_1
c_2
$S(\alpha_m)$
β_m
α_n
$S(\beta_n)$

(a) (b) (c)

Figure 4: (a) A bottom tangle B, (b) A diagram $\tilde{B}$ of B, (c) The labels associated to a state $t \in \mathcal{S}(B)$

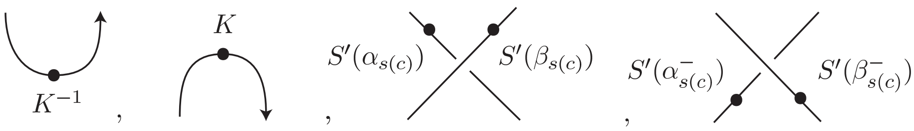

K^{-1}
K
$S'(\alpha_{s(c)})$
$S'(\beta_{s(c)})$
$S'(\alpha^{-}_{s(c)})$
$S'(\beta^{-}_{s(c)})$

Figure 5: How to place labels on the fundamental tangles

define the ith tensorand of $J_{\tilde{T},s}$ as the product of the labels on $\tilde{T}_i$, where the labels are read off along T_i reversing the orientation, and written from left to right. For example, for the bottom tangle B and the state $t \in \mathcal{S}(\tilde{B})$ in Figure 4, we have

$$J_{\tilde{B},t} = S(\alpha_m)S(\beta_n) \otimes \alpha_n\beta_m.$$

Here, we identify the labels $S'(\alpha_i^{\pm})$ and $S'(\beta_i^{\pm})$ with the first and the second tensorands, respectively, of the element $S'(\alpha_i^{\pm}) \otimes S'(\beta_i^{\pm}) \in U_h^{\hat{\otimes}2}$. Also we identify the label $K^{\pm 1}$ with the element $K^{\pm 1} \in U_h$. Thus $J_{\tilde{T},s}$ is a well-defined element in $U_h^{\hat{\otimes}n}$. For example, we have

$$\begin{aligned} J_{\tilde{B},t} &= S(\alpha_m)S(\beta_n) \otimes \alpha_n\beta_m \\ &= \sum q^{\frac{1}{2}m(m-1)}q^{\frac{1}{2}n(n-1)}S(D_1'\tilde{F}^{(m)}K^{-m})S(D_2''e^n) \otimes D_2'\tilde{F}^{(n)}K^{-n}D_1''e^m \\ &= (-1)^{m+n}q^{-n+2mn}D^{-2}(\tilde{F}^{(m)}K^{-2n}e^n \otimes \tilde{F}^{(n)}K^{-2m}e^m) \in U_h^{\hat{\otimes}2}, \end{aligned}$$

where $D = \sum D_1' \otimes D_1'' = \sum D_2' \otimes D_2''$. Note that $J_{\tilde{T},s}$ *depends* on the choice of the diagram.

Step 4. Take the state sum. Set

$$J_T = \sum_{s \in \mathcal{S}(\tilde{T})} J_{\tilde{T},s}.$$

For example, we have

$$J_B = \sum_{t \in \mathcal{S}(\tilde{B})} J_{\tilde{B},t} = \sum_{m,n \geq 0} (-1)^{m+n}q^{-n+2mn}D^{-2}(\tilde{F}^{(m)}K^{-2n}e^n \otimes \tilde{F}^{(n)}K^{-2m}e^m).$$

As is well known [13], J_T does not depend on the choice of the diagram, and defines an isotopy invariant of bottom tangles.

4 Results for the universal sl_2 invariant of bottom tangles

In this section, we give the main result for the universal sl_2 invariant of Brunnian bottom tangles.

4.1 Universal sl_2 invariant of algebraically-split bottom tangles, ribbon bottom tangles and boundary bottom tangles

We recall several results for the value of the universal sl_2 invariant of bottom tangles. Recall the sequence of the subalgebras $\bar{U}_q^{\text{ev}} \subset \mathcal{U}_q^{\text{ev}} \subset U_{\mathbb{Z},q}^{\text{ev}} \subset U_h$.

For an n-component bottom tangle T, let $\text{Lk}(T)$ denote the linking matrix of T. Set

$$\tilde{D}^{\text{Lk}(T)} = \prod_{1 \leq i \leq n} K_i^{m_{ii}} \prod_{1 \leq i \leq j \leq n} D_{ij}^{2m_{ij}} \in U_h^{\hat{\otimes}n},$$

where, $K_i = 1^{\otimes i-1} \otimes K \otimes 1^{\otimes n-i}$ for $1 \leq i \leq n$ and

$$\begin{aligned} D_{ij} &= \sum 1^{\otimes i-1} \otimes D' \otimes 1^{\otimes j-i-1} \otimes D'' \otimes 1^{\otimes n-j}, \\ D_{kk} &= \sum 1^{\otimes k-1} \otimes D'D'' \otimes 1^{\otimes n-k}, \end{aligned}$$

for $1 \leq i < j \leq n$, $1 \leq k \leq n$, where $D = \sum D' \otimes D''$.

Theorem 4.1 ([15, Proposition 4.2, Remark 4.7])**.** *Let T be an n-component bottom tangle. For every diagram $\tilde{T}$ of T and every state $s \in \mathcal{S}(\tilde{T})$, we have*

$$J_{\tilde{T},s} \in \tilde{D}^{\mathrm{Lk}(T)} (\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}.$$

More precisely, the proof of [15, Proposition 4.2] implies the following.

Proposition 4.2. *Let T be an n-component bottom tangle. For any diagram $\tilde{T}$ and any state $s \in \mathcal{S}(\tilde{T})$, we have*

$$J_{\tilde{T},s} \in \tilde{D}^{\mathrm{Lk}(T)} \mathcal{F}_{|s|}((\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}),$$

where we set $|s| = \max\{s(c) \mid c \in C(\tilde{T})\}$.

Theorem 4.1 and Proposition 4.2 imply the following.

Theorem 4.3 ([15, Proposition 4.2, Remark 4.7])**.** *For an n-component bottom tangle T, we have*

$$J_T \in \tilde{D}^{\mathrm{Lk}(T)} (\tilde{\mathcal{U}}_q^{ev})^{\tilde{\otimes} n}.$$

The following is the special case of Theorem 4.3 for algebraically-split bottom tangle with 0-framing (i.e., a bottom tangle with 0-linking matrix), which was proved first by Habiro [3].

Theorem 4.4 (Habiro [3])**.** *Let T be an n-component algebraically-split bottom tangle with 0-framing. Then we have*

$$J_T \in (\tilde{\mathcal{U}}_q^{ev})^{\tilde{\otimes} n}.$$

In [15] and [16], we defined a refined completion $(\bar{U}_q^{\mathrm{ev}})^{\hat{}\hat{\otimes} n} \subset (\bar{U}_q^{\mathrm{ev}})^{\tilde{}\tilde{\otimes} n}$, and proved the following theorem, which is an improvement of Theorem 4.4 in the case of ribbon bottom tangles and boundary bottom tangles.

Theorem 4.5 ([15, 16])**.** *Let T be an n-component ribbon or boundary bottom tangle with 0-framing. Then we have*

$$J_T \in (\bar{U}_q^{\mathrm{ev}})^{\hat{}\hat{\otimes} n}.$$

Remark 4.6. Theorem 4.5 with $(\bar{U}_q^{\mathrm{ev}})^{\tilde{}\tilde{\otimes} n}$ replaced with $(\bar{U}_q^{\mathrm{ev}})^{\hat{}\hat{\otimes} n}$ for boundary bottom tangles had been conjectured by Habiro [3, Conjecture 8.9]. Here, we do not know whether the inclusion $(\bar{U}_q^{\mathrm{ev}})^{\hat{}\hat{\otimes} n} \subset (\bar{U}_q^{\mathrm{ev}})^{\tilde{}\tilde{\otimes} n}$ is proper or not, but the definition of $(\bar{U}_q^{\mathrm{ev}})^{\hat{}\hat{\otimes} n}$ is more natural than that of $(\bar{U}_q^{\mathrm{ev}})^{\tilde{}\tilde{\otimes} n}$ in the settings in [15, 16].

4.2 Result for the universal sl_2 invariant of Brunnian bottom tangles

The main result of this paper is the following, which is an improvement of Theorem 4.4 in the case of Brunnian bottom tangles.

Theorem 4.7. *Let T be an n-component Brunnian bottom tangle with $n \geq 3$. We have*

$$J_T \in U_{Br}^{(n)},$$

where we set

$$U_{Br}^{(n)} = \bigcap_{i=1}^{n} \Big\{ \Big((\bar{U}_q^{\mathrm{ev}})^{\otimes i-1} \otimes U_{\mathbb{Z},q}^{\mathrm{ev}} \otimes (\bar{U}_q^{\mathrm{ev}})^{\otimes n-i} \Big) \cap (\mathcal{U}_q^{\mathrm{ev}})^{\otimes n} \Big\}\hat{}.$$

Here, since a trivial bottom tangle has 0-framing, a Brunnian bottom tangle also has 0-framing by the definition. To compare Theorem 4.7 with Theorems 4.4 and 4.5, for $n \geq 3$, we have the following.

$$\begin{array}{ccc}
\{n\text{-comp. alg. split bottom tangles with 0-framing}\} & \overset{J}{\to} & (\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n} \\
\cup & & \cup \\
\{n\text{-comp. Brunnian bottom tangles}\} & \overset{J}{\to} & U_{Br}^{(n)} \\
 & & \cup \\
\{n\text{-comp. ribbon or boundary bottom tangles with 0-framing}\} & \overset{J}{\to} & (\bar{U}_q^{\mathrm{ev}})^{\hat{}\hat{\otimes} n}
\end{array}$$

We can define the Milnor μ invariants [11, 12] of a bottom tangle as that of the corresponding string link described in [2, Section 13]. It is known that the Milnor μ invariants of ribbon bottom tangles and boundary bottom tangles vanish. It is also known that the Milnor μ invariants of length $\leq n-1$ of n-component Brunnian bottom tangles vanish. Thus we have the following conjecture.

Conjecture 4.8. *(i) Let T be an n-component bottom tangle with 0-framing. If the Milnor μ invariants of T vanish, then we have $J_T \in (\bar{U}_q^{\mathrm{ev}})^{\hat{}\hat{\otimes} n}$.*

(ii) For $n \geq 3$, let T be an n-component bottom tangle with 0-framing. If the Milnor μ invariants of T of length $\leq n-1$ vanish, then we have $J_T \in U_{Br}^{(n)}$.

Theorem 4.7 is derived from the following proposition, which we prove in Section 6.1.

Proposition 4.9. *Let T be an n-component Brunnian bottom tangle with $n \geq 3$. For each $i = 1, \ldots, n$, there is a diagram $\tilde{T}^{(i)}$ of T such that*

$$J_{\tilde{T}^{(i)},s} \in (\bar{U}_q^{\mathrm{ev}})^{\otimes i-1} \otimes U_{\mathbb{Z},q}^{\mathrm{ev}} \otimes (\bar{U}_q^{\mathrm{ev}})^{\otimes n-i}$$

for any state $s \in \mathcal{S}(\tilde{T}^{(i)})$.

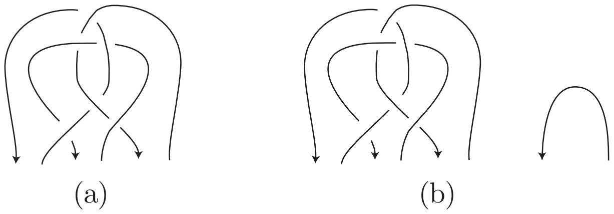


Figure 6: (a) The Borromean bottom tangle T_B, (b) A bottom tangle T'_B

Proof of Theorem 4.7 by assuming Proposition 4.9. For each $i = 1, \ldots, n$, by Theorem 4.7 and Proposition 4.2, there is a diagram $\tilde{T}^{(i)}$ of T such that

$$J_{\tilde{T}^{(i)},s} \in \Big((\bar{U}_q^{\mathrm{ev}})^{\otimes i-1} \otimes U_{\mathbb{Z},q}^{\mathrm{ev}} \otimes (\bar{U}_q^{\mathrm{ev}})^{\otimes n-i} \Big) \cap \mathcal{F}_{|s|}((\mathcal{U}_q^{\mathrm{ev}})^{\otimes n})$$

for any state $s \in \mathcal{S}(\tilde{T}^{(i)})$. Hence we have

$$J_T \in \Big\{ \Big((\bar{U}_q^{\mathrm{ev}})^{\otimes i-1} \otimes U_{\mathbb{Z},q}^{\mathrm{ev}} \otimes (\bar{U}_q^{\mathrm{ev}})^{\otimes n-i} \Big) \cap (\mathcal{U}_q^{\mathrm{ev}})^{\otimes n} \Big\}^{\hat{}}$$

for all $i = 1, \ldots, n$. $\square$

Example 4.10. *For the Borromean bottom tangle T_B depicted in Figure 6 (a), we have*

$$\begin{aligned} J_T \in & \Big\{ \Big(U_{\mathbb{Z},q}^{\mathrm{ev}} \otimes (\bar{U}_q^{\mathrm{ev}})^{\otimes 2} \Big) \cap (\mathcal{U}_q^{\mathrm{ev}})^{\otimes 3} \Big\}^{\hat{}} \\ & \cap \Big\{ \Big(\bar{U}_q^{\mathrm{ev}} \otimes U_{\mathbb{Z},q}^{\mathrm{ev}} \otimes \bar{U}_q^{\mathrm{ev}} \Big) \cap (\mathcal{U}_q^{\mathrm{ev}})^{\otimes 3} \Big\}^{\hat{}} \\ & \cap \Big\{ \Big((\bar{U}_q^{\mathrm{ev}})^{\otimes 2} \otimes U_{\mathbb{Z},q}^{\mathrm{ev}} \Big) \cap (\mathcal{U}_q^{\mathrm{ev}})^{\otimes 3} \Big\}^{\hat{}}. \end{aligned}$$

See Example 6.2 for explicit expressions of J_T.

Example 4.11. *Let us add a trivial arc to the Borromean bottom tangle as in Figure 6 (b), and denote it by T'_B. Note that the bottom tangle T'_B is not Brunnian but algebraically-split. We have*

$$J_{T'_B} = J_{T_B} \otimes 1 \notin \Big\{ \Big((\bar{U}_q^{\mathrm{ev}})^{\otimes 3} \otimes U_{\mathbb{Z},q}^{\mathrm{ev}} \Big) \cap (\mathcal{U}_q^{\mathrm{ev}})^{\otimes 4} \Big\}^{\hat{}}.$$

5 Application to the colored Jones polynomial

In this section, we give an application of Theorem 4.7 to the colored Jones polynomial of Brunnian links (Theorem 5.4).

5.1 Colored Jones polynomials of algebraically-split links, ribbon links and boundary links

We recall results for the colored Jones polynomials of algebraically-split links, ribbon links, and boundary links.

For $m \geq 1$, let V_m denote the m-dimensional irreducible representation of U_h. Let $\mathcal{R}$ denote the representation ring of U_h over $\mathbb{Q}(q^{\frac{1}{2}})$, i.e., $\mathcal{R}$ is the $\mathbb{Q}(q^{\frac{1}{2}})$-algebra

$$\mathcal{R} = \mathrm{Span}_{\mathbb{Q}(q^{\frac{1}{2}})}\{V_m \mid m \geq 1\}$$

with the multiplication induced by the tensor product. It is well known that $\mathcal{R} = \mathbb{Q}(q^{\frac{1}{2}})[V_2]$.

For an n-component link L with 0-framing, take a bottom tangle T whose closure is L. For $X_1, \ldots, X_n \in \mathcal{R}$, the colored Jones polynomial $J_{L;X_1,\ldots,X_n}$ of L with the ith component L_i colored by X_i is given by

$$J_{L;X_1,\ldots,X_n} = (\mathrm{tr}_q^{X_1} \otimes \cdots \otimes \mathrm{tr}_q^{X_n})(J_T) \in \mathbb{Q}(q^{\frac{1}{2}}),$$

where, for $Y = \sum_j y_j V_j \in \mathcal{R}$ and $u \in U_h$, we set

$$\mathrm{tr}_q^Y(u) = \mathrm{tr}^Y(K^{-1}u) = \sum_j y_j \, \mathrm{tr}^{V_j}(K^{-1}u).$$

Habiro [3] studied the following elements in $\mathcal{R}$

$$P_l = \prod_{i=0}^{l-1}(V_2 - q^{i+\frac{1}{2}} - q^{-i-\frac{1}{2}}) \in \mathcal{R}, \tag{3}$$

$$\tilde{P}'_l = \frac{q^{\frac{1}{2}l}}{\{l\}_q!} P_l \in \mathcal{R}, \tag{4}$$

for $l \geq 0$, which are used in an important technical step in his construction of the unified Witten-Reshetikhin-Turaev invariants for integral homology spheres.

Recall the notation $\{l\}_{q,i} = \{l\}_q\{l-1\}_q \cdots \{l-i+1\}_q$ for $l \in \mathbb{Z}$, $i \geq 0$. Theorem 4.4 implies the following.

Theorem 5.1 (Habiro [3]). *Let L be an n-component algebraically-split link with 0-framing. For $l_1, \ldots, l_n \geq 0$, we have*

$$J_{L;\tilde{P}'_{l_1},\ldots,\tilde{P}'_{l_n}} \in Z_a^{(l_1,\ldots,l_n)}. \tag{5}$$

Here we set

$$Z_a^{(l_1,\ldots,l_n)} = \frac{\{2l_{\max}+1\}_{q,l_{\max}+1}}{\{1\}_q}\mathbb{Z}[q,q^{-1}],$$

where $l_{\max} = \max(l_1, \ldots, l_n)$.

For $l \geq 0$, let I_l denote the ideal in $\mathbb{Z}[q,q^{-1}]$ generated by $\{l-k\}_q!\{k\}_q!$ for $k = 0, \ldots, l$. Theorem 4.5 implies the following improvement of Theorem 5.1.

Theorem 5.2 ([15, 16]). *Let L be an n-component ribbon or boundary link with 0-framing. For $l_1, \ldots, l_n \geq 0$, we have*

$$J_{L;\tilde{P}'_{l_1},\ldots,\tilde{P}'_{l_n}} \in Z_{r,b}^{(l_1,\ldots,l_n)}. \tag{6}$$

Here we set

$$\begin{aligned} Z_{r,b}^{(l_1,\dots,l_n)} &= (\prod_{1\leq i\leq n, i\neq i_M} I_{l_i}) \cdot Z_a^{(l_1,\dots,l_n)} \\ &= \frac{\{2l_{\max}+1\}_{q,l_{\max}+1}}{\{1\}_q} \prod_{1\leq i\leq n, i\neq i_M} I_{l_i}, \end{aligned}$$

where $l_{\max} = \max(l_1,\dots,l_n)$ *and* i_M *is an integer such that* $l_{i_M} = l_{\max}$.

For $m \geq 1$, let $\Phi_m = \prod_{d|m}(q^d-1)^{\mu(\frac{m}{d})} \in \mathbb{Z}[q]$ denote the mth cyclotomic polynomial, where $\prod_{d|m}$ denotes the product over all positive divisors d of m, and μ is the Möbius function. For $r \in \mathbb{Q}$, we denote by $\lfloor r \rfloor$ the largest integer smaller than or equal to r.

In [17], we study the ideal I_l and prove the following result, which we use later.

Proposition 5.3 ([17])**.** *For* $l \geq 0$, *the ideal* I_l *is the principal ideal generated by*

$$g_l = \prod_{m\geq 1} \Phi_m^{t_{l,m}}, \tag{7}$$

where

$$t_{l,m} = \begin{cases} \lfloor \frac{l+1}{m} \rfloor - 1 & \text{for } 1 \leq m \leq l, \\ 0 & \text{for } l < m. \end{cases}$$

5.2 Result for the colored Jones polynomial of Brunnian links

The following is an application of Theorem 4.7 to the colored Jones polynomial of Brunnian links, which we prove in Section 6.2.

Theorem 5.4. *Let* L *be an* n*-component Brunnian link with* $n \geq 3$. *For* $l_1,\dots,l_n \geq 0$, *we have*

$$J_{L;\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}} \in Z_{Br}^{(l_1,\dots,l_n)}. \tag{8}$$

Here we set

$$Z_{Br}^{(l_1,\dots,l_n)} = \frac{\{2l_{\max}+1\}_{q,l_{\max}+1}}{\{1\}_q\{l_{\min}\}_q!} \prod_{1\leq i\leq n, i\neq i_M, i_m} I_{l_i},$$

where $l_{\max} = \max(l_1,\dots,l_n)$, $l_{\min} = \min(l_1,\dots,l_n)$ *and* i_M, i_m, $i_M \neq i_m$, *are two integers such that* $l_{i_M} = l_{\max}$, $l_{i_m} = l_{\min}$, *respectively.*

Since a Brunnian link L with $n \geq 3$ components is algebraically-split with 0-framing, L satisfies both (5) and (8). Note that there is no inclusion which satisfies for all $l_1,\dots,l_n \geq 0$ between $Z_a^{(l_1,\dots,l_n)}$ and $Z_{Br}^{(l_1,\dots,l_n)}$. For example, we have $Z_a^{(2,2,2,2)} \not\subset$

$Z_{Br}^{(2,2,2,2)}$ and $Z_{Br}^{(2,2,2,2)} \not\subset Z_a^{(2,2,2,2)}$ since

$$\begin{aligned}
Z_a^{(2,2,2,2)} &= \frac{\{5\}_{q,3}}{\{1\}_q}\mathbb{Z}[q,q^{-1}] \\
&= (q-1)^2(q+1)(q^2+q+1)(q^2+1)(q^4+q^3+q^2+q^1+1)\mathbb{Z}[q,q^{-1}], \\
Z_{Br}^{(2,2,2,2)} &= \frac{\{5\}_{q,3}}{\{1\}_q\{2\}_q!}\{1\}_q^4\mathbb{Z}[q,q^{-1}] \\
&= (q-1)^4(q^2+q+1)(q^2+1)(q^4+q^3+q^2+q^1+1)\mathbb{Z}[q,q^{-1}].
\end{aligned}$$

For $l_1, \dots, l_n \geq 0$, set

$$\tilde{Z}_{Br}^{(l_1,\dots,l_n)} = Z_a^{(l_1,\dots,l_n)} \cap Z_{Br}^{(l_1,\dots,l_n)}.$$

The above argument implies the following refinement of Theorem 5.4.

Theorem 5.5. *Let L be an n-component Brunnian link with $n \geq 3$. For $l_1, \dots, l_n \geq 0$, we have*

$$J_{L;\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}} \in \tilde{Z}_{Br}^{(l_1,\dots,l_n)}.$$

For $n \geq 3$, we have

$$\begin{aligned}
Z_{r,b}^{(l_1,\dots,l_n)} =&\Big(\prod_{1\leq i\leq n, i\neq i_M} I_{l_i}\Big) \cdot Z_a^{(l_1,\dots,l_n)} \\
=&\big(\{l_{\min}\}_q! I_{l_{\min}}\big) \cdot Z_{Br}^{(l_1,\dots,l_n)}.
\end{aligned}$$

Thus, comparing Theorem 5.5 with Theorems 5.1 and 5.2, we have the following for $n \geq 3$.

$$\begin{tikzcd}
\{n\text{-comp. alg. split links with 0-framing}\} \arrow[r, "J_{*;\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}}"] & Z_a^{(l_1,\dots,l_n)} \\
\cup & \cup \\
\{n\text{-comp. Brunnian links}\} \arrow[r, "J_{*;\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}}"] & \tilde{Z}_{Br}^{(l_1,\dots,l_n)} \\
 & \cup \\
\{n\text{-comp. ribbon or boundary links with 0-framing}\} \arrow[r, "J_{*;\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}}"] & Z_{r,b}^{(l_1,\dots,l_n)}
\end{tikzcd}$$

Remark 5.6. By Proposition 5.3, the ideals $Z_a^{(l_1,\dots,l_n)}$, $Z_{r,b}^{(l_1,\dots,l_n)}$, $Z_{Br}^{(l_1,\dots,l_n)}$ and $\tilde{Z}_{Br}^{(l_1,\dots,l_n)}$ are principal, each generated by a product of cyclotomic polynomials. See [17] for details and examples.

6 Proofs

In this section, we prove Proposition 4.9 and Theorem 5.4.

6.1 Proof of Proposition 4.9

We use the following lemma.

Lemma 6.1. *For $m \geq 0$ and $k, l \in \mathbb{Z}$, we have*

$$S^k(\alpha^{\pm}_m) \otimes S^l(\beta^{\pm}_m) \in D^{\pm(-1)^{k+l}}\big((U_{\mathbb{Z},q} \otimes \bar{U}_q) \cap (\bar{U}_q \otimes U_{\mathbb{Z},q})\big),$$

Proof. For $m \geq 0$, we have

$$\begin{aligned} \alpha_m \otimes \beta_m &= D\big(q^{\frac{1}{2}m(m-1)} \tilde{F}^{(m)} K^{-m} \otimes e^m\big) \\ &= D\big(q^{m(m-1)} f^m K^{-m} \otimes \tilde{E}^{(m)}\big) \\ &\in D\big((U_{\mathbb{Z},q} \otimes \bar{U}_q) \cap (\bar{U}_q \otimes U_{\mathbb{Z},q})\big), \end{aligned} \tag{9}$$

$$\begin{aligned} \alpha^-_m \otimes \beta^-_m &= D^{-1}\big((-1)^m \tilde{F}^{(m)} \otimes K^{-m} e^m\big) \\ &= D^{-1}\big((-1)^n q^{\frac{1}{2}m(m-1)} f^m \otimes K^{-m} \tilde{E}^{(m)}\big) \\ &\in D^{-1}\big((U_{\mathbb{Z},q} \otimes \bar{U}_q) \cap (\bar{U}_q \otimes U_{\mathbb{Z},q})\big). \end{aligned} \tag{10}$$

For $k, l \in \mathbb{Z}$, we have

$$(S^k \otimes S^l)(D^{\pm 1}) = D^{\pm(-1)^{k+l}}, \tag{11}$$

$$(S^k \otimes S^l)\big((U_{\mathbb{Z},q} \otimes \bar{U}_q) \cap (\bar{U}_q \otimes U_{\mathbb{Z},q})\big) = (U_{\mathbb{Z},q} \otimes \bar{U}_q) \cap (\bar{U}_q \otimes U_{\mathbb{Z},q}). \tag{12}$$

For $x \in U_h$ homogeneous, we have

$$(x \otimes 1)D^{\pm 1} = D^{\pm 1}(x \otimes K^{\mp|x|}). \tag{13}$$

Now, (9)–(13) imply the assertion. For example, we have

$$\begin{aligned} S(\alpha_m) \otimes S(\beta_m) &= (S \otimes S)(\alpha_m \otimes \beta_m) \\ &\in (S \otimes S)\Big(D\big((U_{\mathbb{Z},q} \otimes \bar{U}_q) \cap (\bar{U}_q \otimes U_{\mathbb{Z},q})\big)\Big) \\ &\subset \big((U_{\mathbb{Z},q} \otimes \bar{U}_q) \cap (\bar{U}_q \otimes U_{\mathbb{Z},q})\big)D \\ &= D\big((U_{\mathbb{Z},q} \otimes \bar{U}_q) \cap (\bar{U}_q \otimes U_{\mathbb{Z},q})\big). \end{aligned}$$

□

Proof of Proposition 4.9. Let $T = T_1 \cup \cdots \cup T_n$ be an n-component Brunnian bottom tangle with $n \geq 3$. We prove the assertion for $i = 1$, i.e., we prove that there is a diagram $\tilde{T}$ of T such that

$$J_{\tilde{T},s} \in U^{\mathrm{ev}}_{\mathbb{Z},q} \otimes \bar{U}^{\mathrm{ev}}_q \otimes \bar{U}^{\mathrm{ev}}_q \otimes \cdots \otimes \bar{U}^{\mathrm{ev}}_q \tag{14}$$

for any state $s \in \mathcal{S}(\tilde{T})$. The other cases $2 \leq i \leq n$ are similar.

Since T is Brunnian, the subtangle $T_2 \cup \cdots \cup T_n$ is trivial. Thus T has a diagram $\tilde{T} = \tilde{T}_1 \cup \tilde{T}_2 \cup \cdots \cup \tilde{T}_n$ whose subdiagram $\tilde{T}_2 \cup \cdots \cup \tilde{T}_n$ has no crossing. See Figure 7 for an example of such a diagram for the Borromean rings T_B.

We prove that $\tilde{T}$ satisfies (14). Note that $\tilde{T}$ has only two types of crossings as follows.

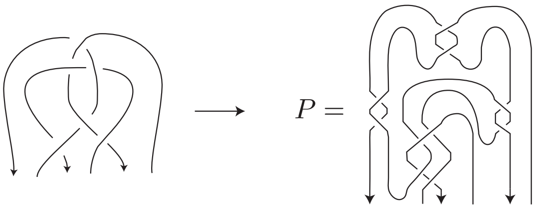


Figure 7: Borromean rings T_B and its diagram $P = P_1 \cup P_2 \cup P_3$ such that $P_2 \cup P_3$ has no crossing

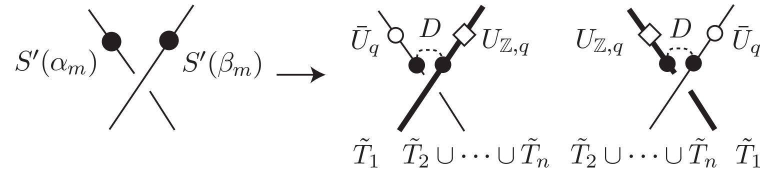

$S'(\alpha_m)$
$S'(\beta_m)$
$\bar{U}_q$
D
$U_{\mathbb{Z},q}$
$\tilde{T}_1$
$\tilde{T}_2 \cup \cdots \cup \tilde{T}_n$
$U_{\mathbb{Z},q}$
D
$\bar{U}_q$
$\tilde{T}_2 \cup \cdots \cup \tilde{T}_n$
$\tilde{T}_1$
$U_{\mathbb{Z},q}$
D
$U_{\mathbb{Z},q}$
$\tilde{T}_1$
$\tilde{T}_1$

Figure 8: The labels on a crossing

Type A: Crossings between $\tilde{T}_1$ and $\tilde{T}_2 \cup \cdots \cup \tilde{T}_n$

Type B: Self crossings of $\tilde{T}_1$

Recall from the definition of $J_{\tilde{T},s}$ in Section 3.3 the labels which are put on the diagram. For the crossings of type A, by Lemma 6.1, we can assume that the labels on $\tilde{T}_1$ are legs of copies of $D^{\pm 1}$ and elements of $U_{\mathbb{Z},q}$, and the labels on $\tilde{T}_2 \cup \cdots \cup \tilde{T}_n$ are legs of copies of $D^{\pm 1}$ and elements of $\bar{U}_q$. For the crossings of type B, we assume that the labels on $\tilde{T}_1$ are legs of copies of $D^{\pm 1}$ and elements of $U_{\mathbb{Z},q}$. See Figure 8 for example, where $\diamond$s denote elements in $U_{\mathbb{Z},q}$ and $\circ$s denote elements in $\bar{U}_q$.

Now, except copies of $D^{\pm 1}$, all labels on $\tilde{T}_1$ are elements of $U_{\mathbb{Z},q}$, and all labels on $\tilde{T}_2 \cup \cdots \cup \tilde{T}_n$ are elements of $\bar{U}_q$, see Figure 9 for example. We gather every copy of $D^{\pm 1}$

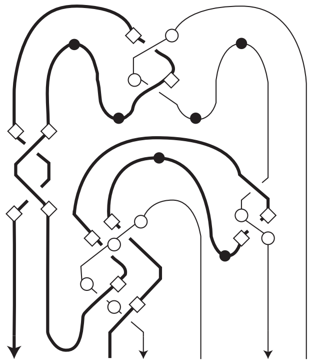


Figure 9: Labels except $D^{\pm 1}$s, where the black dots are $K^{\pm 1}$

to the leftmost of the expression of $J_{\tilde{T},s}$ by using (13), and cancel these as $\tilde{D}^{\mathrm{Lk}(T)} = 1$ since the matrix $\mathrm{Lk}(T)$ is 0. Then we have

$$J_{\tilde{T},s} \in U_{\mathbb{Z},q} \otimes \bar{U}_q \otimes \bar{U}_q \otimes \cdots \otimes \bar{U}_q.$$

By Theorem 4.4, we have

$$\begin{aligned} J_{\tilde{T},s} \in & \big(U_{\mathbb{Z},q} \otimes \bar{U}_q \otimes \bar{U}_q \otimes \cdots \otimes \bar{U}_q\big) \cap (\mathcal{U}_q^{\mathrm{ev}})^{\otimes n} \\ \subset & U_{\mathbb{Z},q}^{\mathrm{ev}} \otimes \bar{U}_q^{\mathrm{ev}} \otimes \bar{U}_q^{\mathrm{ev}} \otimes \cdots \otimes \bar{U}_q^{\mathrm{ev}}. \end{aligned}$$

This completes the proof. □

Example 6.2. *The following is the universal sl_2 invariant of the Borromean bottom tangle calculated by using the diagram Figure 6 (a) (cf. [3]).*

$$\begin{aligned} J_{T_B} = & \sum_{m_1,m_2,m_3,n_1,n_2,n_3 \geq 0} q^{m_3+n_3}(-1)^{n_1+n_2+n_3} q^{\sum_{i=1}^{3}\left(-\frac{1}{2}m_i(m_i+1)-n_i+m_im_{i+1}-2m_in_{i-1}\right)} \\ & \tilde{F}^{(n_3)}e^{m_1}\tilde{F}^{(m_3)}e^{n_1}K^{-2m_2} \otimes \tilde{F}^{(n_1)}e^{m_2}\tilde{F}^{(m_1)}e^{n_2}K^{-2m_3} \otimes \tilde{F}^{(n_2)}e^{m_3}\tilde{F}^{(m_2)}e^{n_3}K^{-2m_1} \\ & \in (\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} 3}, \end{aligned}$$

where the index i should be considered modulo 3.

By using the diagram P in Figure 7, after the two self-crossings of the leftmost strand cancel each other out, we also have the following expression of J_{T_B}.

$$\begin{aligned} J_{T_B} = & \sum_{g,h,k,l,m,n \geq 0} \sum_{i=0}^{m} \sum_{j=0}^{n} t_{g,h,i,j,k,l,m,n}(q) \\ & K^{-2(h+k)}\tilde{F}^{(g)}\tilde{E}^{(h)}\tilde{E}^{(j)}\tilde{F}^{(i)}\tilde{E}^{(k)}\tilde{F}^{(l)}\tilde{F}^{(m-i)}\tilde{E}^{(n-j)} \otimes K^{2(k-l-m)}f^ne^m \otimes K^{-2(h-i+j+k)}e^lf^{h+k}e^g \\ & \in \Big\{\Big(U_{\mathbb{Z},q}^{\mathrm{ev}} \otimes (\bar{U}_q^{\mathrm{ev}})^{\otimes 2}\Big) \cap (\mathcal{U}_q^{\mathrm{ev}})^{\otimes 3}\Big\}^{\hat{}}, \end{aligned}$$

where

$$\begin{aligned} t_{g,h,i,j,k,l,m,n}(q) = (-1)^{g+h+m+n+i+j} & q^{-2g(3h+k)+\frac{1}{2}h(h-1)+h(2l-1)+k(2l+2n-i-j-1)-\frac{1}{2}l(l-1)} \\ & \cdot q^{-l(2n+i-3j)-m(6n-i-j)+\frac{1}{2}n(n-1)-n(j+1)-\frac{1}{2}i(i-1)+\frac{1}{2}j(j-1)}. \end{aligned}$$

6.2 Proof of Theorem 5.4

In this section, we prove Theorem 5.4.

First of all, we recall generators of $\bar{U}_q^{\mathrm{ev}}$ and $U_{\mathbb{Z},q}^{\mathrm{ev}}$ as $\mathbb{Z}[q, q^{-1}]$-modules. The following Lemma is a variant of a well-known fact about the integral form of De Concini and Procesi (cf. [1, 3]).

Lemma 6.3. *$\bar{U}_q^{\mathrm{ev}}$ is freely $\mathbb{Z}[q, q^{-1}]$-spanned by the elements $f^iK^{2j}e^k$ with $i, k \geq 0$ and $j \in \mathbb{Z}$.*

Set

$$\begin{aligned}\begin{bmatrix}H\\ s\end{bmatrix}_q =&\frac{(K^2-1)(q^{-1}K^2-1)\cdots(q^{-s+1}K^2-1)}{\{s\}_q!}\\ =&\frac{1}{\{s\}_q!}\sum_{p=0}^{s}(-1)^{s-p}q^{\frac{1}{2}p(p+1)-sp}\begin{bmatrix}s\\ p\end{bmatrix}_q K^{2p}\end{aligned}$$

for $s \geq 0$.

The following Lemma is a variant of a well-known fact about Lusztig's integral form (cf. [10]).

Lemma 6.4. *$U_{\mathbb{Z},q}^{\mathrm{ev}}$ is $\mathbb{Z}[q,q^{-1}]$-spanned by the elements $\tilde{F}^{(i)}K^{2j}\begin{bmatrix}H\\ s\end{bmatrix}_q \tilde{E}^{(k)}$ with $i,k,s \geq 0$ and $j \in \mathbb{Z}$.*

For the elements $P_l, \tilde{P}'_l \in \mathcal{R}$ defined in (3), (4) in Section 5.1, we have the following results.

Lemma 6.5 (Habiro [3, Lemma 8.8])**.** (1) *If $l, i, i' \geq 0$, $i \neq i'$, and $j \in \mathbb{Z}$, then we have $\mathrm{tr}_q^{P_l}(\tilde{F}^{(i)}K^{2j}e^{i'}) = 0$.*

(2) *If $0 \leq l < i$ and $j \in \mathbb{Z}$, then we have $\mathrm{tr}_q^{P_l}(\tilde{F}^{(i)}K^{2j}e^{i}) = 0$.*

(3) *If $0 \leq i \leq l$ and $j \subset \mathbb{Z}$, then we have*

$$\mathrm{tr}_q^{P_l}(\tilde{F}^{(i)}K^{2j}e^{i}) = q^{\frac{1}{2}l-lj+2ij+i^2-il}\{l\}_q!\{l-i\}_q!\begin{bmatrix}j+l-1\\ l-i\end{bmatrix}_q\begin{bmatrix}j-1\\ l-i\end{bmatrix}_q.$$

For $l \geq 0$, recall the ideal I_l in $\mathbb{Z}[q,q^{-1}]$, which is generated by $\{l-i\}_q!\{i\}_q!$ for $i = 0, \ldots, l$.

Corollary 6.6 (Habiro [3])**.** *For $l \geq 0$, we have $\mathrm{tr}_q^{\tilde{P}'_l}(\bar{U}_q^{\mathrm{ev}}) \subset I_l$.*

Proof. The assertion follows from Lemma 6.3, Lemma 6.5 (1), (2), and

$$\begin{aligned}\mathrm{tr}_q^{\tilde{P}'_l}(f^iK^{2j}e^i) =&q^{-\frac{1}{2}i(i-1)}\{i\}_q!\,\mathrm{tr}_q^{\tilde{P}'_l}(\tilde{F}^{(i)}K^{2j}e^i)\\ =&q^{-\frac{1}{2}i(i-1)}\{i\}_q!\frac{q^{\frac{1}{2}l}}{\{l\}_q!}\,\mathrm{tr}_q^{P_l}(\tilde{F}^{(i)}K^{2j}e^i)\\ =&q^{-\frac{1}{2}i(i-1)+l-lj+2ij+i^2-il}\{i\}_q!\{l-i\}_q!\begin{bmatrix}j+l-1\\ l-i\end{bmatrix}_q\begin{bmatrix}j-1\\ l-i\end{bmatrix}_q \in I_l\end{aligned}$$

for $0 \leq i \leq l$ and $j \in \mathbb{Z}$. □

Proposition 6.7. *For $l \geq 0$, we have $\{l\}_q!\,\mathrm{tr}_q^{\tilde{P}'_l}(U_{\mathbb{Z},q}^{\mathrm{ev}}) \in \mathbb{Z}[q,q^{-1}]$.*

Proof. By Lemma 6.4 and Lemma 6.5 (1), (2), it is enough to check

$$\{l\}_q! \operatorname{tr}_q^{\tilde{P}'_l}(\tilde{F}^{(i)} K^{2j} \begin{bmatrix} H \\ s \end{bmatrix}_q \tilde{E}^{(i)}) \in \mathbb{Z}[q, q^{-1}]$$

for $0 \leq i \leq l$, $s \geq 0$ and $j \in \mathbb{Z}$.

For $s = 0$, by Lemma 6.6 (3), we have

$$\begin{aligned}
\{l\}_q! \operatorname{tr}_q^{\tilde{P}'_l}(\tilde{F}^{(i)} K^{2j} \tilde{E}^{(i)}) =& \frac{\{l\}_q!}{\{i\}_q!} \operatorname{tr}_q^{\tilde{P}'_l}(\tilde{F}^{(i)} K^{2j} e^i) \\
=& \frac{q^{\frac{1}{2}l}}{\{i\}_q!} \operatorname{tr}_q^{P_l}(\tilde{F}^{(i)} K^{2j} e^i) \\
=& q^{l-lj+2ij+i^2-il} \{l\}_{q,l-i} \{l-i\}_q! \begin{bmatrix} j+l-1 \\ l-i \end{bmatrix}_q \begin{bmatrix} j-1 \\ l-i \end{bmatrix}_q .
\end{aligned} \tag{15}$$

For $s \geq 0$, by using the case $s = 0$, we have

$$\begin{aligned}
&\{l\}_q! \operatorname{tr}_q^{\tilde{P}'_l}(\tilde{F}^{(i)} K^{2j} \begin{bmatrix} H \\ s \end{bmatrix}_q \tilde{E}^{(i)}) \\
&\quad= \{l\}_q! \frac{1}{\{s\}_q!} \sum_{p=0}^{s} (-1)^{s-p} q^{\frac{1}{2}p(p+1)-sp} \begin{bmatrix} s \\ p \end{bmatrix}_q \operatorname{tr}_q^{\tilde{P}'_l}(\tilde{F}^{(i)} K^{2j+2p} \tilde{E}^{(i)}) \\
&\quad= \frac{1}{\{s\}_q!} \sum_{p=0}^{s} (-1)^{s-p} q^{\frac{1}{2}p(p+1)-sp} \begin{bmatrix} s \\ p \end{bmatrix}_q q^{l-lj+2i(j+p)+i^2-il} \{l\}_{q,l-i} \{l-i\}_q! \begin{bmatrix} (j+p)+l-1 \\ l-i \end{bmatrix}_q \begin{bmatrix} (j+p)-1 \\ l-i \end{bmatrix}_q \\
&\quad= \frac{1}{\{s\}_q!} \sum_{p=0}^{s} (-1)^{s-p} q^{\frac{1}{2}p(p+1)-sp} \begin{bmatrix} s \\ p \end{bmatrix}_q q^{l-lj+2i(j+p)+i^2-il} \begin{bmatrix} l \\ l-i \end{bmatrix}_q \{(j+p)+l-1\}_{q,l-i} \{(j+p)-1\}_{q,l-i} \\
&\quad= \frac{1}{\{s\}_q!} \sum_{p=0}^{s} (-1)^{s-p} q^{\frac{1}{2}p(p+1)-sp} \begin{bmatrix} s \\ p \end{bmatrix}_q q^{l-lj+2i(j+p)+i^2-il} \begin{bmatrix} l \\ l-i \end{bmatrix}_q \\
&\qquad \Big(\sum_{t=0}^{l-i} (-1)^{l-i-t} q^{\frac{1}{2}t(t+1)+(j+p+i-1)t} \begin{bmatrix} l-i \\ t \end{bmatrix}_q \Big) \Big(\sum_{u=0}^{l-i} (-1)^{l-i-u} q^{\frac{1}{2}u(u+1)+(j+p-l+i-1)u} \begin{bmatrix} l-i \\ u \end{bmatrix}_q \Big) \\
&\quad= \sum_{t=0}^{l-i} \sum_{u=0}^{l-i} (-1)^{t+u} q^{l-lj+2ij+i^2-il+\frac{1}{2}t(t+1)+(j+i-1)t+\frac{1}{2}u(u+1)+(j-l+i-1)u} \\
&\qquad \Big(\frac{1}{\{s\}_q!} \sum_{p=0}^{s} (-1)^{s-p} q^{\frac{1}{2}p(p+1)+(i+t+u-s)p} \begin{bmatrix} s \\ p \end{bmatrix}_q \Big) \begin{bmatrix} l \\ l-i \end{bmatrix}_q \begin{bmatrix} l-i \\ t \end{bmatrix}_q \begin{bmatrix} l-i \\ u \end{bmatrix}_q \\
&\quad= \sum_{t=0}^{l-i} \sum_{u=0}^{l-i} (-1)^{t+u} q^{l-lj+2ij+i^2-il+\frac{1}{2}t(t+1)+(j+i-1)t+\frac{1}{2}u(u+1)+(j-l+i-1)u} \\
&\qquad \begin{bmatrix} i+t+u \\ s \end{bmatrix}_q \begin{bmatrix} l \\ l-i \end{bmatrix}_q \begin{bmatrix} l-i \\ t \end{bmatrix}_q \begin{bmatrix} l-i \\ u \end{bmatrix}_q \quad \in \mathbb{Z}[q, q^{-1}].
\end{aligned}$$

Here, the first equality follows from the definition of $\begin{bmatrix} H \\ s \end{bmatrix}_q$, the second equality follows from (15), and the other equalities follow from straightforward calculations, where we use

$$\{k\}_{q,n} = \sum_{r=0}^{n} (-1)^{n-r} q^{\frac{1}{2}r(r+1)+r(k-n)} \begin{bmatrix} n \\ r \end{bmatrix}_q$$

for $k \in \mathbb{Z}$ and $n \geq 0$. Hence we have the assertion. $\square$

We use the following proposition.

Proposition 6.8. *Let T be an n-component Brunnian bottom tangle with $n \geq 3$. For $1 \leq i \leq n$ and $l_i \geq 0$, we have*

(i) $(\mathrm{id}^{\otimes i-1} \otimes \mathrm{tr}_q^{\tilde{P}'_{l_i}} \otimes \mathrm{id}^{\otimes n-i})(J_T) \in (\mathcal{U}_q^{\mathrm{ev}})^{\otimes n-1}$,

(ii) $\{l_i\}_q!(\mathrm{id}^{\otimes i-1} \otimes \mathrm{tr}_q^{\tilde{P}'_{l_i}} \otimes \mathrm{id}^{\otimes n-i})(J_T) \in (\bar{U}_q^{\mathrm{ev}})^{\otimes n-1}$.

Proof. We prove the assertion with $i = 1$. The other cases are similar. Let $\tilde{T} = \tilde{T}^{(1)}$ be a diagram of T as in Proposition 4.9. By the proof of Proposition 4.9, we can assume that $\tilde{T} = \tilde{T}_1 \cup \cdots \cup \tilde{T}_1$ has only two types of crossings as follows.

Type A: Crossings between $\tilde{T}_1$ and $\tilde{T}_2 \cup \cdots \cup \tilde{T}_n$

Type B: Self crossings of $\tilde{T}_1$

Let $s \in \mathcal{S}(\tilde{T})$. Set $|s| = \max\{s(c) \mid c \in C(\tilde{T})\}$. Note that, for $0 \leq m < p$, the elements E^p and F^p act as 0 on the m-dimensional irreducible representation V_m of U_h. Since each crossing of either type involves the strand $\tilde{T}_1$, there is a crossing c on $\tilde{T}_1$ such that $s(c) = |s|$. Since $\tilde{P}'_{l_1} \in \mathrm{Span}_{\mathbb{Q}(q^{1/2})}\{V_0, \ldots, V_{l_1}\}$, if $|s| > l_1$, we have

$$(\mathrm{tr}_q^{\tilde{P}'_{l_1}} \otimes \mathrm{id}^{\otimes n-1})(J_{\tilde{T},s}) = 0. \tag{16}$$

By (16), Theorem 4.1 with $\mathrm{Lk}(T) = 0$ implies (i), and Propositions 4.9, 6.7 imply (ii). $\square$

For a subalgebra X of U_h, let $Z(X)$ denote the center of X. Habiro [3, Proposition 8.6] proved that for an n-component algebraically-split bottom tangle with 0-framing, we have

$$(\mathrm{id} \otimes \mathrm{tr}_q^{\tilde{P}'_{l_2}} \otimes \mathrm{tr}_q^{\tilde{P}'_{l_3}} \otimes \cdots \otimes \mathrm{tr}_q^{\tilde{P}'_{l_n}})(J_T) \in Z(\tilde{\mathcal{U}}_q^{\mathrm{ev}}).$$

We can improve this result for Brunnian bottom tangles as follows.

Proposition 6.9. *Let T be an n-component Brunnian bottom tangle with $n \geq 3$. For $l_2, \ldots, l_n \geq 0$, we have*

$$(\mathrm{id} \otimes \mathrm{tr}_q^{\tilde{P}'_{l_2}} \otimes \mathrm{tr}_q^{\tilde{P}'_{l_3}} \otimes \cdots \otimes \mathrm{tr}_q^{\tilde{P}'_{l_n}})(J_T) \in Z(\mathcal{U}_q^{\mathrm{ev}}).$$

Proof. By Proposition 6.8 (i) and $\mathrm{tr}_q^{\tilde{P}'_l}(\mathcal{U}_q^{\mathrm{ev}}) \subset \mathbb{Z}[q, q^{-1}]$ for $l \geq 0$, we have

$$\begin{aligned}(\mathrm{id} \otimes \mathrm{tr}_q^{\tilde{P}'_{l_2}} \otimes \mathrm{tr}_q^{\tilde{P}'_{l_3}} \otimes \cdots \otimes \mathrm{tr}_q^{\tilde{P}'_{l_n}})(J_T) &\in (\mathrm{id} \otimes \mathrm{tr}_q^{\tilde{P}'_{l_3}} \otimes \cdots \otimes \mathrm{tr}_q^{\tilde{P}'_{l_n}})((\mathcal{U}_q^{\mathrm{ev}})^{\otimes n-1}) \\ &\subset \mathcal{U}_q^{\mathrm{ev}}.\end{aligned}$$

It is well-known that J_T is contained in the invariant part of $U_h^{\hat{\otimes} n}$ (cf. [3, Proposition 4.2]). This fact implies

$$(\mathrm{id} \otimes \mathrm{tr}_q^{\tilde{P}'_{l_2}} \otimes \mathrm{tr}_q^{\tilde{P}'_{l_3}} \otimes \cdots \otimes \mathrm{tr}_q^{\tilde{P}'_{l_n}})(J_T) \in Z(U_h).$$

Since $\mathcal{U}_q^{\mathrm{ev}} \cap Z(U_h) \subset Z(\mathcal{U}_q^{\mathrm{ev}})$, we have the assertion. □

Let $C = (q^{1/2} - q^{-1/2})^2 FE + q^{1/2}K + q^{-1/2}K^{-1}$ denote the Casimir element. Recall from [3] that $Z(\mathcal{U}_q^{\mathrm{ev}})$ is freely generated by C^2 as a $\mathbb{Z}[q, q^{-1}]$-algebra, and thus, freely spanned by the following monic polynomials in C^2 as a $\mathbb{Z}[q, q^{-1}]$-module.

$$\sigma_p = \prod_{i=1}^{p} \big(C^2 - (q^i + 2 + q^{-i})\big), \quad p \geq 0.$$

Habiro proved the following.

Proposition 6.10 (Habiro [3, Proposition 6.3])**.** *For $l, m \geq 0$, we have*

$$\mathrm{tr}_q^{P''_l}(\sigma_m) = \delta_{l,m},$$

where $P''_l = q^{l(l+1)} \frac{\{1\}_q}{\{2l+1\}_{q,l+1}} \tilde{P}'_l$.

Proposition 6.10 implies the following.

Corollary 6.11 (Habiro [3])**.** *For $l \geq 0$, we have*

$$\mathrm{tr}_q^{\tilde{P}'_l}\big(Z(\mathcal{U}_q^{\mathrm{ev}})\big) \subset \frac{\{2l+1\}_{q,l+1}}{\{1\}_q} \mathbb{Z}[q, q^{-1}].$$

Now, we prove Theorem 5.4.

Proof of Theorem 5.4. For $n \geq 3$, let L be an n-component Brunnian link and T a Brunnian bottom tangle whose closure is L. Let $l_1, \ldots, l_n \geq 0$. Without loss of generality, we assume $l_1 = \max(l_1, \ldots, l_n)$ and $l_2 = \min\{l_i \mid 1 \leq i \leq n\}$. By Proposition 6.8 (ii) and Corollary 6.6, we have

$$\begin{aligned}&\{l_2\}_q!(\mathrm{id} \otimes \mathrm{tr}_q^{\tilde{P}'_{l_2}} \otimes \mathrm{tr}_q^{\tilde{P}'_{l_3}} \otimes \cdots \otimes \mathrm{tr}_q^{\tilde{P}'_{l_n}})(J_T) \\ &\in (\mathrm{id} \otimes \mathrm{tr}_q^{\tilde{P}'_{l_3}} \otimes \mathrm{tr}_q^{\tilde{P}'_{l_3}} \otimes \cdots \otimes \mathrm{tr}_q^{\tilde{P}'_{l_n}})((\bar{U}_q^{\mathrm{ev}})^{\otimes n-1}) \\ &\subset \Big(\prod_{3 \leq i \leq n} I_i \Big) \cdot \bar{U}_q^{\mathrm{ev}} \\ &\subset \Big(\prod_{3 \leq i \leq n} I_i \Big) \cdot \mathcal{U}_q^{\mathrm{ev}} \\ &= g_{l_3} \cdots g_{l_n} \mathcal{U}_q^{\mathrm{ev}},\end{aligned} \tag{17}$$

where the last equation is follows from Proposition 5.3.

Since $\mathcal{U}_q^{\mathrm{ev}}$ has no non-trivial zero divisor, we have

$$\left(g_{l_3}\cdots g_{l_n}\mathcal{U}_q^{\mathrm{ev}}\right)\cap Z(\mathcal{U}_q^{\mathrm{ev}})\subset g_{l_3}\cdots g_{l_n}Z(\mathcal{U}_q^{\mathrm{ev}}). \tag{18}$$

By (17), (18) and Proposition 6.9, we have

$$\{l_2\}_q!(\mathrm{id}\otimes \mathrm{tr}_q^{\tilde{P}'_{l_2}}\otimes \mathrm{tr}_q^{\tilde{P}'_{l_3}}\otimes\cdots\otimes \mathrm{tr}_q^{\tilde{P}'_{l_n}})(J_T)\subset g_{l_3}\cdots g_{l_n}Z(\mathcal{U}_q^{\mathrm{ev}}). \tag{19}$$

By (19) and Corollary 6.11, we have

$$\begin{aligned}\{l_2\}_q!J_{L;\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}} &= \{l_2\}_q!(\mathrm{tr}_q^{\tilde{P}'_{l_1}}\otimes \mathrm{tr}_q^{\tilde{P}'_{l_2}}\otimes \mathrm{tr}_q^{\tilde{P}'_{l_3}}\otimes\cdots\otimes \mathrm{tr}_q^{\tilde{P}'_{l_n}})(J_T)\\ &\in \mathrm{tr}_q^{\tilde{P}'_{l_1}}\left(g_{l_3}\cdots g_{l_n}Z(\mathcal{U}_q^{\mathrm{ev}})\right)\\ &\subset \frac{\{2l_1+1\}_{q,l_1+1}}{\{1\}_q}g_{l_3}\cdots g_{l_n}\mathbb{Z}[q,q^{-1}]\\ &= \frac{\{2l_1+1\}_{q,l_1+1}}{\{1\}_q}\prod_{3\le i\le n} I_{l_i}.\end{aligned}$$

Hence we have

$$J_{L;\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}} = \frac{\{2l_1+1\}_{q,l_1+1}}{\{1\}_q\{l_2\}_q!}\prod_{3\le i\le n} I_{l_i}.$$

This completes the proof. □

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