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Dual Consistent Systems of Linear Inequalities and Cardinality Constrained Polytopes

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Abstract

We introduce a concept of dual consistency of systems of linear inequalities with full generality. We show that a cardinality constrained polytope is represented by a certain system of linear inequalities if and only if the systems of linear inequalities associated with the cardinalities are dual consistent. Typical dual consistent systems of inequalities are those which describe polymatroids, generalized polymatroids, and dual greedy polyhedra with certain choice functions. We show that the systems of inequalities for cardinality-constrained ordinary bipartite matching polytopes are not dual consistent in general, and give additional inequalities to make them dual consistent. Moreover, we show that ordinary systems of inequalities for the cardinality-constrained (poly)matroid intersection are not dual consistent, which disproves a conjecture of Maurras, Spiegelberg, and Stephan about a linear representation of the cardinality-constrained polymatroid intersection.

1. Introduction

Cardinality constrained polyhedra and their linear representations were first investigated by Maurras [7] and Camion and Maurras [1], and later rediscovered by Grötschel [5] for what is called a cardinality homogeneous set system (also see related recent work by

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Kaibel and Stephan [6], Stephan [10], Maurras and Stephan [9], and Maurras, Spiegelberg, and Stephan [8]).

Given a finite nonempty set S, a combinatorial optimization problem Π on S, and an increasing sequence $c = (c_1, \ldots, c_m)$ of nonnegative integers c_i $(i = 1, \ldots, m)$, the cardinality constrained version Π_c of Π has the set of feasible solutions consisting of all feasible solutions of the original problem with the property that the cardinality (i.e. the number of elements) of every solution is equal to c_i for some $i \in \{1, \ldots, m\}$. In [7, 1, 5] they introduced *forbidden cardinality inequalities* of the form

$$\begin{aligned} &(c_{p+1} - c_p)x(U) - (|U| - c_p)x(S) \leq c_p(c_{p+1} - |U|) \\ &\text{for all } U \subseteq S \text{ with } c_p < |U| < c_{p+1} \text{ for some } p \in \{1, \ldots, m\}, \end{aligned} \tag{1}$$

where $x(U) = \sum_{u \in U} x(u)$ for $U \subseteq S$, and showed that the inequalities hold for Π_c. Usually these inequalities are not facet-defining for the polyhedron associated with Π_c.

Recently Maurras and Stephan [9] derived strong valid inequalities that give a complete linear description for cardinality constrained matroids. This result has been generalized by Maurras, Spiegelberg, and Stephan [8, 11] to cardinality constrained polymatroids as follows. Given a polymatroid rank function $f : 2^S \to \mathbb{R}$ and an increasing sequence $(c_1, \ldots, c_m)$ of nonnegative integers c_i $(i = 1, \ldots, m)$, they aim for the convex hull of all vectors x of the polymatroid associated with f of cardinality c_i for some $i \in \{1, \ldots, m\}$, i.e. $x(S) = c_i$. The cardinality constrained polymatroid is shown to be determined by the following system of inequalities:

$$\begin{aligned} &x(U) \leq f(U) \quad (U \subseteq S), \\ &(c_{p+1} - c_p)x(U) - (f(U) - c_p)x(S) \leq c_p(c_{p+1} - f(U)) \\ &\qquad (U \subseteq S \ \text{ with } \ c_p < f(U) < c_{p+1} \ \text{ for some } \ p \in \{1, \ldots, m-1\}), \\ &c_1 \leq x(S) \leq c_m, \qquad x \geq \mathbf{0}. \end{aligned} \tag{2}$$

In the present paper we introduce the concept of dual consistent systems of linear inequalities and formulate the cardinality constrained problem in a more general setting. In Section 2 we give a characterization of certain complete systems of linear inequalities expressing cardinality constrained polytopes with two cardinalities, where an essential role is played by the concept of *dual consistency* of systems of linear inequalities that we introduce in the present paper. Section 3 is concerned with multiple cardinality constraints. In Section 4 we show how the inequalities given in [9, 8, 11] are derived from our result. We also show that the systems of inequalities for the cardinality-constrained ordinary bipartite matching polytopes and for the cardinality-constrained (poly)matroid intersection are not dual consistent in general. The latter implies that a conjecture of Maurras, Spiegelberg, and Stephan [8, 11] about a linear representation of the cardinality-constrained intersection of polymatroids does not hold in general.

2. Cardinality Constrained Polytopes

In this section we consider the case where we have two cardinalities $c_1 < c_2$ (i.e., $m = 2$). The multiple cardinality case (i.e. $m > 2$) will be discussed in Section 3.

2.1. Dual Consistent Systems of Inequalities

Let S be a finite nonempty set and $\mathcal{Z}$ be a finite nonempty set of non-zero vectors in $\mathbb{R}^S$. Choose and fix a vector $z_0 \in \mathcal{Z}$. Then, consider two functions $f_i : \mathcal{Z} \to \mathbb{R}$ $(i = 1, 2)$ with $c_1 := f_1(z_0) < f_2(z_0) =: c_2$. Note that for the cardinality constrained polymatroid, $\mathcal{Z}$ is the set of characteristic vectors χ_X of all nonempty subsets X of S and z_0 is given by χ_S, the all-one vector in $\mathbb{R}^S$. (For each $U \subseteq S$ the characteristic vector $\chi_U \in \mathbb{R}^S$ is defined by $\chi_U(u) = 1$ for $u \in U$ and $\chi_U(u) = 0$ for $u \in S \setminus U$.)

For each $i = 1, 2$ define the polyhedron

$$P_{f_i}^{c_i} = \{x \in \mathbb{R}^S \mid \forall z \in \mathcal{Z} : \langle z, x\rangle \leq f_i(z),\ \langle z_0, x\rangle = c_i\}, \tag{3}$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical inner product defined by $\langle z, x\rangle = \sum_{u\in S} z(u)x(u)$. We assume that the $P_{f_i}^{c_i}$ $(i = 1, 2)$ are nonempty and bounded. $P_{f_i}^{c_i}$ can be regarded as a polyhedron restricted to vectors of *cardinality* c_i where the *cardinality* of a vector x is given by $\langle z_0, x\rangle$ (we may have $z_0 = \chi_S$ (the all-one vector) in the ordinary case).

We are interested in obtaining a complete system of linear inequalities for the convex hull of $P_{f_1}^{c_1} \cup P_{f_2}^{c_2}$. To this end we introduce a concept of *dual consistent* systems of inequalities. We will show that if, and only if, the systems of linear inequalities appearing in (3) for $i = 1, 2$ are dual consistent, the convex hull is represented by the inequalities

$$(c_2 - c_1)\langle z, x\rangle - (f_2(z) - f_1(z))\langle z_0, x\rangle \leq c_2 f_1(z) - c_1 f_2(z) \quad (z \in \mathcal{Z}), \tag{4}$$

$$c_1 \leq \langle z_0, x\rangle \leq c_2 \tag{5}$$

(see Theorem 1 to be shown below).

Remark 1: It should be noted that for each $i = 1, 2$, if we add the constraint $\langle z_0, x\rangle = c_i$ to (4), the system of inequalities (4) together with the added constraint is equivalent to

$$\langle z, x\rangle \leq f_i(z) \quad (z \in \mathcal{Z}), \qquad \langle z_0, x\rangle = c_i. \tag{6}$$

This is exactly the system of inequalities defining $P_{f_i}^{c_i}$ in (3). □

Now, for any $w \in \mathbb{R}^S$ and $i = 1, 2$ consider the following problem

$$(\mathbf{P}_i^w) \quad \text{Maximize} \quad \langle w, x\rangle \quad \text{subject to} \quad x \in P_{f_i}^{c_i}. \tag{7}$$

Let $\hat{x}_i$ be an optimal solution of Problem $(\mathbf{P}_i^w)$ for $i = 1, 2$ and define

$$\mathcal{Z}_i(\hat{x}_i) = \{z \in \mathcal{Z} \mid \langle z, \hat{x}_i \rangle = f_i(z)\} \qquad (i = 1, 2), \tag{8}$$

which represents the set of active (or tight) constraints of (6) at $\hat{x}_i$ for $i = 1, 2$. For each $i = 1, 2$ a set $\mathcal{B} \subseteq \mathcal{Z}$ is called a *dual optimal base* for Problem $(\mathbf{P}_i^w)$ if there exists an optimal solution $\hat{x}_i$ of Problem $(\mathbf{P}_i^w)$ such that

$$\mathcal{B} \subseteq \mathcal{Z}_i(\hat{x}_i), \tag{9}$$

$$\operatorname{rank} \mathcal{B} = |S|, \tag{10}$$

where $\operatorname{rank} \mathcal{B}$ is the rank of the matrix formed by the vectors in $\mathcal{B}$.

By definition $\hat{x}_i$ is an extreme point of $P_{f_i}^{c_i}$. It follows from (9) and (10) that $\hat{x}_i$ is a unique solution of the system of equations

$$\langle z, x \rangle = f_i(z) \qquad (z \in \mathcal{B}). \tag{11}$$

We assume that for every dual optimal base $\mathcal{B}$ appearing in the following arguments we have $z_0 \in \mathcal{B}$. The systems of linear inequalities (6) for $i = 1, 2$ are called *dual consistent* if for every $w \in \mathbb{R}^S$ there exists a common dual optimal base $\mathcal{B}$ for $(\mathbf{P}_1^w)$ and $(\mathbf{P}_2^w)$. If there is no possibility of confusion, we also simply call the pair (f_1, f_2) *dual consistent* in the sequel. Recall that the dual consistency depends on the choice of c_i $(i = 1, 2)$ and z_0 besides f_i $(i = 1, 2)$.

Examples: If f_1 and f_2 are submodular functions on 2^S with $f_1(S) = c_1 < c_2 = f_2(S)$ and $f_1(\emptyset) = f_2(\emptyset) = 0$, the pair (f_1, f_2) is dual consistent due to the greedy algorithm ([2] and also see, e.g., [4]). More generally, dual greedy polyhedra with a common choice function give us a dual consistent pair. This follows directly by their definitions (see [3]). □

Remark 2: We have assumed that $P_{f_i}^{c_i}$ $(i = 1, 2)$ are nonempty and bounded. We can extend the concept of dual consistency to systems of linear inequalities such that $P_{f_i}^{c_i}$ $(i = 1, 2)$ are pointed and have a common characteristic cone, by considering only weight vectors w that give finite optimal values for Problem $(\mathbf{P}_i^w)$. (We may also call (f_1, f_2) *totally dual consistent with respect to* z_0 if (f_1, f_2) is dual consistent for z_0 and every choice of c_i $(i = 1, 2)$ such that $P_{f_i}^{c_i} \neq \emptyset$ $(i = 1, 2)$. Polymatroids give typical examples of totally dual consistent systems with respect to the all-one vector as z_0.) □

2.2. The Convex-hull Polyhedron

Define the polyhedron (polytope) $\hat{P}$ by

$$(c_2 - c_1)\langle z, x \rangle - (f_2(z) - f_1(z))\langle z_0, x \rangle \leq c_2 f_1(z) - c_1 f_2(z) \quad (z \in \mathcal{Z}), \tag{12}$$

$$c_1 \leq \langle z_0, x \rangle \leq c_2. \tag{13}$$

(Recall Remark 1 given in Section 2.1.)

Let $P^{c_1,c_2}_{f_1,f_2}$ denote the convex hull of $P^{c_1}_{f_1} \cup P^{c_2}_{f_2}$. We will show that $P^{c_1,c_2}_{f_1,f_2} = \hat{P}$ (defined by (12) and (13)) if and only if the pair (f_1, f_2) is dual consistent. Before we show this, we analyze another (infinite) system of linear inequalities.

Denoting the optimal objective function value of (7) by ζ^w_i, we introduce the condition that every $x \in \hat{P}$ satisfies

$$(c_2 - c_1)\langle w, x\rangle - (\zeta^w_2 - \zeta^w_1)\langle z_0, x\rangle \leq c_2\zeta^w_1 - c_1\zeta^w_2 \qquad (\forall w \in \mathbb{R}^S). \tag{14}$$

Lemma 1: *If for all $w \in \mathbb{R}^S$ and all $x \in \hat{P}$ inequality* (14) *holds, then the pair (f_1, f_2) is dual consistent.*

(Proof) If we consider ζ^w_i for each $i = 1, 2$ as a function in $w \in \mathbb{R}^S$, it is what is called the *support function* of polytope $P^{c_i}_{f_i}$. Hence, the inequalities of (14) together with $c_1 \leq \langle z_0, x\rangle \leq c_2$ determine the convex hull $P^{c_1,c_2}_{f_1,f_2}$ of $P^{c_1}_{f_1} \cup P^{c_2}_{f_2}$. (Note that inequalities of (14) are exactly those which support both $P^{c_i}_{f_i}$ $(i = 1, 2)$.) It follows from the assumption that $\hat{P} \subseteq P^{c_1,c_2}_{f_1,f_2}$. We can easily see that the converse inclusion relation also holds true, due to Remark 1. Consequently,

$$\hat{P} = P^{c_1,c_2}_{f_1,f_2}. \tag{15}$$

Since it suffices to consider an arbitrary generic $w \in \mathbb{R}^S$ to show the dual consistency, it follows from (15) that for a generic $w \in \mathbb{R}^S$ the unique optimal solutions $\hat{x}_i$ for problem $\mathbf{P}^w_i$ $(i = 1, 2)$ are adjacent vertices of $\hat{P}(= P^{c_1,c_2}_{f_1,f_2})$, which implies that there exists a dual optimal base for problem $\mathbf{P}^w_i$ $(i = 1, 2)$ for all w. □

Remark 3: Here we do not need that f_1 and f_2 have a common domain $\mathcal{Z}$. For different domains of f_1 and f_2 the above proof is valid to show that constraint (14) implies the dual consistency of (f_1, f_2). □

Remark 4: As noted in the above proof, the inequalities of (14) together with inequalities $c_1 \leq \langle z_0, x\rangle \leq c_2$ determine the convex hull $P^{c_1,c_2}_{f_1,f_2}$ of $P^{c_1}_{f_1} \cup P^{c_2}_{f_2}$. Since $P^{c_1,c_2}_{f_1,f_2}$ is a polytope, we need only a finite number of inequalities from (14) besides (13) to obtain a representation of $P^{c_1,c_2}_{f_1,f_2}$ by linear inequalities, but the number of required inequalities could be much larger than $|\mathcal{Z}|$. Note that every inequality of (14) gives a hyperplane that supports both $P^{c_1}_{f_1}$ and $P^{c_2}_{f_2}$, and *vice versa*. □

Next we show the following lemma.

Lemma 2: *For any dual consistent pair (f_1, f_2) with $f_1(z_0) = c_1 < c_2 = f_2(z_0)$ the convex hull $P^{c_1,c_2}_{f_1,f_2}$ of $P^{c_1}_{f_1} \cup P^{c_2}_{f_2}$ is expressed by* (12) *and* (13).

(Proof) Recall that $\hat{P}$ is the polytope defined by (12) and (13) and that $P^{c_1,c_2}_{f_1,f_2} \subseteq \hat{P}$. Suppose that $P^{c_1,c_2}_{f_1,f_2} \neq \hat{P}$. Then there exists an edge L of $\hat{P}$ connecting a vertex x_1 and a vertex x_2 of $\hat{P}$ such that one of the two is a vertex of $P_{f_1}^{c_1}$ or of $P_{f_2}^{c_2}$ and that the other belongs to $\hat{P} \setminus P_{f_1,f_2}^{c_1,c_2}$. We assume without loss of generality that x_1 is a vertex of $P_{f_1}^{c_1}$ and $x_2 \in \hat{P} \setminus P_{f_1,f_2}^{c_1,c_2}$.

Let $\langle w, x\rangle = b$ be a supporting hyperplane of $\hat{P}$ that defines the edge L. Then x_1 is the unique optimal solution of Problem $(\mathbf{P}_1^w)$. Let y_2 be an optimal solution of Problem $(\mathbf{P}_2^w)$. We can assume that w is (generically) chosen so that y_2 is a unique optimal solution as well. Because of the dual consistency there exists a base $\mathcal{B}$ such that $\mathcal{B}$ is a dual optimal base for both problems $(\mathbf{P}_i^w)$ $(i = 1, 2)$. It follows from Remark 1 that x_1 and y_2 lie on the line L' determined by the system of equations given by (12) for all $z \in \mathcal{B} \setminus \{z_0\}$, each of (12) for such z holding with equality. It follows that x_2 must coincide with y_2, which contradicts the assumption on x_2. □

Lemma 3: *If* $P_{f_1,f_2}^{c_1,c_2} = \hat{P}$*, then inequalities* (14) *hold for all* $x \in \hat{P}$.

(Proof) Since $P_{f_1,f_2}^{c_1,c_2}$ is determined by (14) and inequalities $c_1 \le \langle z_0, x\rangle \le c_2$, the present theorem follows. □

Now it follows from Lemmas 1, 2, and 3 that

Theorem 1: *The following three statements are equivalent:*

(a) *We have* $P_{f_1,f_2}^{c_1,c_2} = \hat{P}$.

(b) *Inequalities* (14) *hold for all* $x \in \hat{P}$.

(c) *The pair* (f_1, f_2) *is dual consistent.* □

Remark 5: When the domains of f_1 and f_2 are different and given by $\mathcal{Z}_1$ and $\mathcal{Z}_2$, we can always obtain a common domain $\mathcal{Z}_1 \cup \mathcal{Z}_2$ by adding redundant constraints. □

Remark 6: For any two polytopes P_1 and P_2 lying on two distinct parallel hyperplanes, let z_0 be a common normal vector of the hyperplanes, and let $\mathcal{Z}$ be a finite set of normal vectors of hyperplanes (linear inequalities) that define the convex hull P of $P_1 \cup P_2$, connecting the two polytopes. Then we get two functions $f_i : \mathcal{V} \to \mathbb{R}$ $(i = 1, 2)$ such that $P_i = P_{f_i}^{c_i}$ with $f_i(z_0) = c_i$ $(i = 1, 2)$ and the pair (f_1, f_2) is dual consistent.

This means that the systems of inequalities for any such two polytopes can be made dual consistent by adding some redundant inequalities. (Also see Remark 4.) □

3. Multiple Cardinality Constrained Polytopes

In Section 2 we have considered cardinality constrained polytopes with only two cardinalities c_1 and c_2. In the multiple cardinality case where $m > 2$ there are a finite sequence of cardinalities $(c_1, \ldots, c_m)$ with $c_1 < c_2 < \cdots < c_m$ and functions $f_1, \ldots, f_m : \mathcal{Z} \to \mathbb{R}$ with $f_i(z_0) = c_i$ $(i = 1, \ldots, m)$, where S, $\mathcal{Z}$, and z_0 are the same as those in Section 2. We assume that each pair of f_i and f_{i+1} is dual consistent for $i = 1, \ldots, m-1$. It should be noted that the relation of dual consistency on such pairs is not an equivalence relation, and it is not transitive, in particular.

Again we consider nonempty polytopes $P_{f_i}^{c_i}$ $(i = 1, \ldots, m)$ defined as in (3) and aim for a linear inequality representation of the convex hull $P_{f_1,\ldots,f_m}^{c_1,\ldots,c_m}$ of $P_{f_1}^{c_1} \cup P_{f_2}^{c_2} \cup \cdots \cup P_{f_m}^{c_m}$.

In the most general case it will be hard to derive inequalities for the convex hull if the inequalities $\langle z, x\rangle \leq f_i(z)$ of (3) ($1 \leq i \leq m$ and $z \in \mathcal{Z}$) are not valid for all points $x \in P_{f_1,\ldots,f_m}^{c_1,\ldots,c_m}$ with $\langle z_0, x\rangle = c_i$ for every $i = 1, \ldots, m$. Hence we assume

$$P_{f_1,\ldots,f_m}^{c_1,\ldots,c_m} \cap \{x \in \mathbb{R}^S | \, \langle z_0, x\rangle = c_i\} = P_{f_i}^{c_i} \qquad (i = 1, \ldots, m). \tag{16}$$

We also assume

(T) each inequality in (3) $(i = 1, \ldots, m)$ defines a face (or supports the polytope with equality).

Here (T) is the tightness condition for each f_i and c_i. It should be noted that the tightness condition (T) is not required when $m = 2$.

Remark 7: Let $P_* \subset \mathbb{R}^S$ be a polyhedron, $z_0 \in \mathbb{R}^S \setminus \{\mathbf{0}\}$, $c_1 < \cdots < c_m$ a sequence of cardinalities, and $P_*^{c_i} = P_* \cap \{x \in \mathbb{R}^S | \, \langle z_0, x\rangle = c_i\}$ (nonempty and bounded). Then there is a finite set $\mathcal{Z} \subset \mathbb{R}^S \setminus \{\mathbf{0}\}$ and functions $f_i : \mathcal{Z} \to \mathbb{R}$ $(1 \leq i \leq m)$ such that $P_*^{c_i} = P_{f_i}^{c_i}$ for all $i = 1, \ldots, m$. Due to the convexity of the polyhedron P_* equations (16) hold true. □

Under assumption (16) we immediately get

$$P_{f_1,\ldots,f_m}^{c_1,\ldots,c_m} = \bigcup_{1 \leq i \leq m-1} \mathrm{Conv}\left(P_{f_i}^{c_i} \cup P_{f_{i+1}}^{c_{i+1}}\right), \tag{17}$$

where $\mathrm{Conv}(\cdot)$ is the convex hull operator in $\mathbb{R}^S$.

We can easily generalize Theorem 1 to the multiple cardinality case as follows. Define a polyhedron (polytope) $\hat{P}_{f_1,\ldots,f_m}^{c_1,\ldots,c_m}$ by

$$(c_{i+1} - c_i)\langle z, x\rangle - (f_{i+1}(z) - f_i(z))\langle z_0, x\rangle \leq c_{i+1} f_i(z) - c_i f_{i+1}(z) \qquad (z \in \mathcal{Z}, \ i = 1, \ldots, m-1), \tag{18}$$

$$c_1 \leq \langle z_0, x\rangle \leq c_m. \tag{19}$$

For each $i = 1, \ldots, m{-}1$ and $z \in \mathcal{Z} \setminus \{z_0\}$ denote the inequality in (18) by H_i^z. We see from the tightness condition (T) and Theorem 1 that inequality H_i^z supports the following three polytopes:

$$P_{f_i}^{c_i}, \quad P_{f_{i+1}}^{c_{i+1}}, \quad \mathrm{Conv}\left(P_{f_i}^{c_i} \cup P_{f_{i+1}}^{c_{i+1}}\right).$$

It follows from assumption (16) and the convexity of $P_{f_1,\ldots,f_m}^{c_1,\ldots,c_m}$ that inequality H_i^z is also valid for other polytopes $P_{f_j}^{c_j}$ $(j \in \{1, \ldots, m\} \setminus \{i, i{+}1\})$. It should be noted that (convex) polytopes $\mathrm{Conv}(P_{f_i}^{c_i} \cup P_{f_{i+1}}^{c_{i+1}})$ $(i = 1, \ldots, m{-}1)$ and $P_{f_1,\ldots,f_m}^{c_1,\ldots,c_m}$ have the same dimension.

Because of this argument and Theorem 1 we then get

Theorem 2: *Under assumption* (16) *and the tightness condition* (T) *the following statements are equivalent:*

(i) *We have* $P_{f_1,\ldots,f_m}^{c_1,\ldots,c_m} = \hat{P}_{f_1,\ldots,f_m}^{c_1,\ldots,c_m}$. *That is, the system of inequalities in* (18) *and* (19) *represents the cardinality constrained polytope* $P_{f_1,\ldots,f_m}^{c_1,\ldots,c_m}$.

(ii) *Functions* f_i *and* f_{i+1} *are dual consistent for all* $i = 1, \ldots, m{-}1$. □

4. Examples and Counterexamples

4.1. Polymatroids

For each $U \subseteq S$ we identify U with the characteristic vector $\chi_U \in \mathbb{R}^S$.

We now show how the forbidden cardinality inequalities of [9] and [11] can be derived from (12). To this end let $f : 2^S \to \mathbb{R}_{\geq 0}$ be a polymatroid rank function and let $\mathcal{Z} = 2^S \setminus \{\emptyset\}$ and $z_0 = S$. Also let $0 \leq c_1 < \cdots < c_m \leq f(S)$. Now define functions $f_i : \mathcal{Z} \cup \{\emptyset\} \to \mathbb{R}$ $(i = 1, \ldots, m)$ by $f_i(U) = \min\{c_i, f(U)\}$ for $U \in \mathcal{Z} \cup \{\emptyset\}$. Consider polytopes $P_{f_i}^{c_i}$ defined by (3) for all $i = 1, \ldots, m$.

Note that for each $i = 1, \ldots, m$ f_i is the rank function of the truncation, by c_i, of the underlying polymatroid with rank function f. Due to the submodularity of f_i $(i = 1, \ldots, m)$, the functions f_i and f_{i+1} are dual consistent for all $i = 1, \ldots, m{-}1$. Moreover, the tightness condition (T) holds for all f_i and c_i and (16) also holds.

Hence by Theorem 2 the system of inequalities in (18) and (19) defines the convex hull of $P_{f_1}^{c_1} \cup \cdots \cup P_{f_m}^{c_m}$. Note that Remark 7 applies to the current polymatroid case.

Inequalities (18) can be written as

$$(c_{i+1} - c_i)x(U) - (f_{i+1}(U) - f_i(U))x(S) \leq c_{i+1}f_i(U) - c_i f_{i+1}(U) \qquad (U \subseteq S,\ i = 1, \ldots, m{-}1). \quad (20)$$

For any $i \in \{1, \ldots, m{-}1\}$ consider any subset $U \subseteq S$ such that $c_i \leq f(U) \leq c_{i+1}$. Then by definition of f_i and f_{i+1} we get $f_i(U) = c_i$ and $f_{i+1}(U) = f(U)$. Hence inequality

(20) reduces to

$$(c_{i+1} - c_i)x(U) - (f(U) - c_i)x(S) \le c_i(c_{i+1} - f(U)) \tag{21}$$

for such U. These are exactly the f-induced forbidden cardinality inequalities shown in [8, 9, 11] (see (2)). It should be noted that if $c_{i+1} < f(U)$, (20) becomes $x(U) \le f(U)$ and if $f(U) < c_i$, then $0 \le x(S \setminus U)$, both being valid inequalities for the original polymatroid polytope. More precisely, (20) together with $c_1 \le x(S) \le c_m$ implies (2).

4.2. Bipartite Matchings

Let $G = (V^+, V^-; E)$ be a bipartite graph with a vertex bipartition (V^+, V^-) and a set E of edges between V^+ and V^-. For any vertex $v \in V^+ \cup V^-$ denote by δv the set of edges incident to v.

Let w be a weight vector generically chosen from $\mathbb{R}^E$ and c_i $(i = 1, 2)$ be positive integers with $c_1 < c_2$ such that there exists at least one matching M in G of size $|M| = c_2$. Then for each $i = 1, 2$ consider a maximum-weight matching problem with a cardinality constraint, relaxed in $\mathbb{R}^E$ as follows.

$$
\begin{aligned}
(\mathbf{P}_i^w) \qquad & \text{Maximize} && \sum_{e \in E} w(e)x(e) \\
& \text{subject to} && \sum_{e \in \delta v^+} x(e) \le 1 \qquad (v^+ \in V^+), \\
& && \sum_{e \in \delta v^-} x(e) \le 1 \qquad (v^- \in V^-), \\
& && 0 \le x(e) \le 1 \qquad (e \in E), \\
& && \sum_{e \in E} x(e) = c_i.
\end{aligned} \tag{22}
$$

Here, we have $z_0 = \chi_E \in \mathbb{R}^E$, and $\mathcal{Z}$ is the set of the coefficient vectors of the inequalities and the equation appearing in (22), where $0 \le x(e)$ should be regarded as an inequality $-x(e) \le 0$ for all $e \in E$. Also, for each $i = 1, 2$ function $f_i : \mathcal{Z} \to \mathbb{R}$ is defined so as to take the values specified by the right-hand sides of (22).

For each $i = 1, 2$ let $\hat{x}_i$ be the unique optimal solution of Problem $(\mathbf{P}_i^w)$, where the uniqueness is due to the choice of generic w. Then, due to the integrality of (22), for each $i = 1, 2$ there is a matching $M_i \subseteq E$ in G such that $\hat{x}_i = \chi_{M_i}$.

Consider the symmetric difference $M_1 \Delta M_2 \equiv (M_1 \setminus M_2) \cup (M_2 \setminus M_1)$. Then $M_1 \Delta M_2$ can be decomposed into vertex-disjoint paths and possible cycles. Note that such paths and cycles are formed by alternating edges of M_1 and M_2. Because of the uniqueness of the optimal solutions there does not exist any such alternating cycle or path of even length (even number of edges). Suppose that the vertex-disjoint paths are then given by $Q^{(k)}$

($k = 1, \ldots, \ell$), each of which satisfies one of the following two. We denote by $E(Q^{(k)})$ the edge set of $Q^{(k)}$.

$$|M_2 \cap E(Q^{(k)})| = |M_1 \cap E(Q^{(k)})| + 1, \tag{23}$$
$$|M_2 \cap E(Q^{(k)})| = |M_1 \cap E(Q^{(k)})| - 1. \tag{24}$$

Let n_+ and n_-, respectively, be the number of paths $Q^{(k)}$ of type (23) and that of type (24). Then we see that $\ell = n_+ + n_-$ and $n_+ - n_- = c_2 - c_1 \geq 1$. Suppose that $n_- \geq 1$, and then consider a pair of a path of type (23) and a path of type (24). The pair contains the same number of arcs from M_1 and from M_2 in total, which contradicts the uniqueness of the optimal solutions. It follows that we have $n_- = 0$, i.e., $n_+ = c_2 - c_1 = \ell$.

For each path $Q^{(k)}$ denote by $\tilde{V}(Q^{(k)})$ the set of intermediate (inner) vertices of $Q^{(k)}$, its initial and terminal vertices being discarded.

The tight inequalities (equations) in (22) common for $i = 1, 2$ are given as follows.

(i) For all $e \in M_1 \cap M_2$ we have $x(e) = 1$.

(ii) For all $e \in E \setminus (M_1 \cup M_2)$ we have $x(e) = 0$.

(iii) For each $k = 1, \ldots, c_2 - c_1$, associated with (23), we have

$$\sum_{e \in \delta v^+} x(e) = 1 \qquad (v^+ \in \tilde{V}(Q^{(k)}) \cap V^+), \tag{25}$$
$$\sum_{e \in \delta v^-} x(e) = 1 \qquad (v^- \in \tilde{V}(Q^{(k)}) \cap V^-). \tag{26}$$

For each $k = 1, \ldots, c_2 - c_1$ the total number of equations appearing in (25) and (26) is equal to $|\tilde{V}(Q^{(k)})| = |E(Q^{(k)})| - 1$.

Since equations of type (i) $x(e) = 0$ and type (ii) $x(e) = 1$ can always be taken into a dual base, we delete the arcs of $(M_1 \cap M_2) \cup (E \setminus (M_1 \cup M_2))$ from G, and assume that $M_1 \cap M_2 = \emptyset$ and $E = M_1 \cup M_2$ in the sequel.

If $c_2 - c_1 = 1$, then the symmetric difference $M_1 \Delta M_2$ must form a single path. We can see that the system of exactly $|E|$ equations of (i), (ii), and (iii) (with $c_2 - c_1 = 1$) together with the cardinality constraint uniquely determines the optimal solution $\hat{x}_i$ for each $i = 1, 2$. Hence $(\mathbf{P}_i^w)$ $(i = 1, 2)$ have a common optimal dual base. It follows that the systems of inequalities for $(\mathbf{P}_i^w)$ $(i = 1, 2)$ is dual consistent. In the present case the cardinality constrained polytope is represented by (22) with the last equation being replaced by

$$c_1 \ \leq \ \sum_{e \in E} x(e) \ \leq \ c_2 \ (= c_1 + 1). \tag{27}$$

The present fact is closely related to the primal-dual, augmenting path algorithm for the maximum-weight matching problem, and is well known.

On the other hand, if $c_2 - c_1 \geq 2,$ there are $c_2 - c_1$ (at least two) paths of (23), so that the number of tight equations common for $i = 1, 2$ is at most $|E| - 2$. Hence $(\mathbf{P}_i^w)$ $(i = 1, 2)$ cannot have any common dual optimal base even if we take the cardinality constraint into account. That is, the systems of inequalities for $(\mathbf{P}_i^w)$ $(i = 1, 2)$ are not dual consistent. This implies that we need some additional redundant inequalities for $(\mathbf{P}_i^w)$ $(i = 1, 2)$ to express the cardinality constrained polytope $P_{f_1,f_2}^{c_1,c_2} = \mathrm{Conv}(P_{f_1}^{c_1} \cup P_{f_2}^{c_2})$. Such additional inequalities can be given in a form of (14). A set of additional inequalities is, for example, given as follows.

For each $k = 1, \ldots, c_2 - c_1$ let $e^{(k)}$ be an edge of M_2 in path $Q^{(k)}$. For any $F \subseteq E$ let $\sigma(F)$ be the maximum size of a matching in G contained in F. Put $M := M_1 \cup M_2$. By construction we have $\sigma(M) = |M_2| = c_2$. For each $k = 1, \ldots, c_2 - c_1$ consider set $M \setminus \{e^{(k)}\}$. We see that $M_2 \setminus \{e^{(k)}\}$ is a matching in $M \setminus \{e^{(k)}\}$ and there cannot be a larger one within $M \setminus \{e^{(k)}\}$. Hence we have $\sigma(M \setminus \{e^{(k)}\}) = c_2 - 1$.

It follows that each inequality

$$\sum_{e \in M \setminus \{e^{(k)}\}} x(e) \ \leq \ c_1 \tag{28}$$

is valid for $(\mathbf{P}_1^w)$ and is tight for $x = \chi_{M_1}$, while each inequality

$$\sum_{e \in M \setminus \{e^{(k)}\}} x(e) \ \leq \ c_2 - 1 \tag{29}$$

is valid for $(\mathbf{P}_2^w)$ and is tight for $x = \chi_{M_2}$. Note that inequalities (28) (or (29)) together with the other tight inequalities (25) and (26) are linearly independent since $c_2 - c_1 \geq 2$. (One of these inequalities can be deleted, if we take into account the cardinality constraint $x(E) = c_i$ for $i = 1$ or 2.) Adding inequalities (28) to $(\mathbf{P}_1^w)$ and (29) to $(\mathbf{P}_2^w)$, we have a common dual optimal base formed by these inequalities.

Any generic weight w determines a pair of optimal matchings M_1 for $(\mathbf{P}_1^w)$ and M_2 for $(\mathbf{P}_2^w)$. Let us call such a pair (M_1, M_2) an *admissible pair*. Then, adding inequalities (28) to $(\mathbf{P}_1^w)$ and (29) to $(\mathbf{P}_2^w)$ for all admissible pairs (M_1, M_2) makes the systems of inequalities for $(\mathbf{P}_i^w)$ $(i = 1, 2)$ dual consistent, i.e., it makes them have a common dual base for any w. It should be noted that for a non-generic w, even if optimal matchings M_1 and M_2 are not unique, we can always find optimal matchings M_1' and M_2' with $|M_1'| = c_1$ and $|M_2'| = c_2$ such that (M_1', M_2') is admissible.

4.3. Matroid intersection

Suppose we are given two matroids $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ on a ground set S with rank functions r_1 and r_2, respectively. Define the function $f : 2^S \to \mathbb{R}$ by

$$f(U) = \min\{r_1(T) + r_2(U \setminus T) \mid T \subseteq U\} \qquad (\forall U \subseteq S). \tag{30}$$

Note that $f(U)$ is equal to the maximum size of a common independent set of $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ restricted on $U \subseteq S$. Consider the matroid intersection polytope represented by

$$x(U) \leq f(U) \qquad (U \subseteq S), \tag{31}$$

$$x \geq \mathbf{0}. \tag{32}$$

Taking into account the nonnegativity constraint, define

$$\mathcal{Z} = \mathcal{Z}_a \cup \mathcal{Z}_b, \tag{33}$$

$$\mathcal{Z}_a = \{\chi_U \mid U \subseteq S, U \neq \emptyset\}, \quad \mathcal{Z}_b = \{-\chi_e \mid e \in S\}. \tag{34}$$

Let c_1 and c_2 with $c_1 < c_2 \leq f(S)$ be two given positive integers (the cardinalities) and define $f_i : \mathcal{Z} \to \mathbb{R}$ for each $i = 1, 2$ by

$$f_i(z) = \begin{cases} \min\{f(U), c_i\} & (z = \chi_U,\ \emptyset \neq U \subseteq S) \\ 0 & (z = -\chi_e,\ e \in S) \end{cases} \qquad (\forall z \in \mathcal{Z}). \tag{35}$$

The cardinality-constrained polytopes $P_{f_1}^{c_1}$ and $P_{f_2}^{c_2}$ are given by (3).

Let us examine whether the pair (f_1, f_2) is dual consistent in general, i.e. whether the convex hull of $P_{f_1}^{c_1} \cup P_{f_2}^{c_2}$ is described by (4) and (5):

$$(c_2 - c_1)\langle z, x\rangle - (f_2(z) - f_1(z))\langle z_0, x\rangle \leq c_2 f_1(z) - c_1 f_2(z) \quad (z \in \mathcal{Z}),$$
$$c_1 \leq \langle z_0, x\rangle \leq c_2,$$

where z_0 is given by χ_S, the all-one vector in $\mathbb{R}^S$. Actually we will show that the pair (f_1, f_2) for matroid intersection is not dual consistent in general.

Remark 8: In Section 4.2 we have seen that ordinary systems of linear inequalities for cardinality-constrained bipartite matchings are not dual consistent. However, this does not imply that the linear representations of the cardinality-constrained matroid intersection are not dual consistent in general, though the bipartite matching problem is a special case of the matroid intersection problem. Note that $\mathcal{Z} \supseteq 2^S \setminus \{\emptyset\}$ for matroid intersection and that this is not the case for ordinary bipartite matching polytopes. (We identify a subset of S with its characteristic vector as before.) □

Now let $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ be the graphic matroids on the ground set $S = \{1, 2, 3, 4, 5\}$ represented by the graphs G_1 and G_2 given in Figure 1.

Suppose $c_1 = 1$ and $c_2 = 4$. For an appropriately given weight vector w we have

$$I_{c_1} = \{5\}, \qquad I_{c_2} = \{1, 2, 3, 4\} \tag{36}$$

as the unique maximum-weight common independent sets of size $c_1 (= 1)$ and $c_2 (= 4)$, respectively, which give the unique optimal solutions $\hat{x}_1 = \chi_{I_{c_1}}$ and $\hat{x}_2 = \chi_{I_{c_2}}$ of Problems

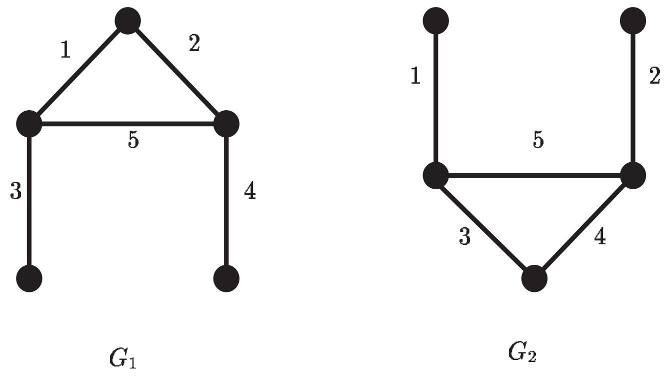

Figure 1: The graphs G_1 and G_2 representing the graphic matroids $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$.

$(\mathbf{P}_1^w)$ and $(\mathbf{P}_2^w)$, respectively, due to the integrality of the matroid intersection polytope with a single cardinality constraint. For such a weight vector w we can easily see that a common dual optimal base is given by the following five:

$$S(=\{1,2,3,4,5\}), \qquad S \setminus \{e\} \quad (e \in \{1,2,3,4\}). \tag{37}$$

Next, copy each of $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ on the ground set $S' = \{1',2',3',4',5'\}$. For $i = 1,2$ consider the direct sum of $\mathbf{M}^{(i)}$ and its copy and denote it by $\mathbf{M}^{(i)}$ again, so that $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ are defined on the ground set $S \cup S' = \{1,2,3,4,5,1',2',3',4',5'\}$. Put $S \leftarrow S \cup S'$ and let $c_1 = 2$ and $c_2 = 8$. For an appropriate weight vector w we get $I_{c_1} = \{5,5'\}$ as the unique maximum-weight common independent set of size $c_1(=2)$ and $I_{c_2} = \{1,2,3,4,1',2',3',4'\}$ as the unique maximum-weight common independent set of size $c_2(=8)$.

We can see that a maximum rank of common tight sets of $\hat{x}_1 = \chi_{I_{c_1}}$ and $\hat{x}_2 = \chi_{I_{c_2}}$ for the inequalities and equation in (6) for $i = 1,2$ is attained by the following nine sets:

$$S\ (=\{1,2,3,4,5,1',2',3',4',5'\}), \tag{38}$$

$$S \setminus \{e\} \quad e \in \{1,2,3,4,1',2',3',4'\}. \tag{39}$$

Since there are ten variables, we do not have a common dual optimal base, i.e. the pair (f_1, f_2) is not dual consistent. An additional valid inequality that yields a common dual base with respect to the present w is given, for example, by

$$6x(\{5\}) + x(\{1,2,3,4,5,1',2',3',4',5'\}) \le 8. \tag{40}$$

It is left open to give a finite set of additional inequalities in a systematic way that makes the systems for cardinality-constrained (poly)matroid intersection dual consistent.

It is conjectured in [8, 11] that the convex hull of $P_{f_1}^{c_1} \cup P_{f_2}^{c_2}$ is determined by

$$\begin{aligned} & x(U) \leq f(U) \quad (U \subseteq S), \\ & (c_2 - c_1)x(U) - (f(U) - c_1)x(S) \leq c_1(c_2 - f(U)) \\ & \qquad (U \subseteq S \ \text{with} \ c_1 < f(U) < c_2), \\ & c_1 \leq x(S) \leq c_2, \qquad x \geq \mathbf{0}. \end{aligned} \tag{41}$$

Similarly as discussed in Section 4.1 we can see that inequalities (41) are implied by inequalities (4) and (5), so that the polytope $\hat{P}$ determined by (4) and (5) is included in the polytope P' determined by (41). Since in our example the pair (f_1, f_2) is not dual consistent, it follows from Theorem 1 that the convex hull $P_{f_1,f_2}^{c_1,c_2}$ of $P_{f_1}^{c_1} \cup P_{f_2}^{c_2}$ is strictly included in $\hat{P}$. Hence $P_{f_1,f_2}^{c_1,c_2} \neq P'$ and our example given above disproves a conjecture of Maurras, Spiegelberg, and Stephan [8, 11] for the cardinality-constrained polymatroid intersection.

5. Concluding Remarks

We have introduced a new concept of dual consistency of systems of inequalities and have revealed that the concept of dual consistency plays a crucial role in the linear representation of cardinality constrained polytopes. We have also shown that the ordinary systems of inequalities for the cardinality-constrained bipartite matching polytopes are not dual consistent in general and have given a set of additional inequalities to make the system of inequalities dual consistent.

Moreover, we have shown that ordinary systems of inequalities for the cardinality-constrained (poly)matroid intersection are not dual consistent in general, which disproves a conjecture of Maurras, Spiegelberg, and Stephan [8, 11] about a linear representation of the cardinality-constrained polymatroid intersection.

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