RIMS-1738

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By

Takefumi NOSAKA

January 2012

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

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Takefumi Nosaka

**Abstract**

Let $G$ be a group. Any $G$-module $M$ has an algebraic structure called "$G$-family of Alexander quandles". Given a 2-cocycle of a cohomology of this $G$-family, topological invariants of (handlebody-)knots in the 3-sphere were defined. This paper develops a simple algorithm to algebraically construct $n$-cocycles of this $G$-family from $G$-invariant group $n$-cocycles of the abelian group $M$. We provide many examples of 2-cocycles of these $G$-families by facts in (modular) invariant theory.

**Keywords** group homology, quandle, invariant theory, modular representation, knots

## 1 Introduction

A quandle is a set with a binary operation whose definition was motivated from knot theory. A particularly interesting class is that of "the associated quandles to $G$-families of Alexander quandles" proposed by Ishii et al. [IIJO]: Specifically, each quandle in the class is defined to be the product $M \times G$ of a group $G$ and a right $G$-module $M$ with the binary operation

$$(M \times G) \times (M \times G) \longrightarrow M \times G, \qquad (a, g, b, h) \longmapsto (\ (a - b) \cdot h + b,\ h^{-1}gh\ ). \tag{1}$$

For any quandle $X$, Fenn, Rourke and Sanderson defined rack space $BX$ (see [FRS] and references therein), in analogy to the classifying spaces of groups. By slightly modifying its cohomology $H^*(BX)$, Carter et al. introduced quandle cohomologies $H^*_Q(X; A)$; further, if $X$ is of finite order, with respect to its 2-, 3-cocycles, they defined quandle cocycle invariants of links in the 3-sphere $L \subset S^3$ and of knotted-surfaces in the 4-sphere $\Sigma_g \hookrightarrow S^4$ (see [CJKLS, CKS]). Furthermore, Ishii et al. [IIJO] showed that, if $X$ is a quandle of the form $M \times G$ above and its quandle 2-cocycle satisfies certain strong conditions, then the cocycle invariant is generalized for "handlebody-knots" in the 3-sphere, i.e., embeddings of handlebodies $H_g \hookrightarrow S^3$.

However, there are a few methods to find cocycles in the quandle cohomology $H^*_Q(X; A)$, compared with group cohomology theory. We here refer to two results: first, for any quandle $X$, Inoue and Kabaya [IK] constructed a map from the homology $H_*(BX)$ to a "simplicial homology of $X$" in order to interpret the Chern-Simons class as a quandle cocycle. On the other hand, most of known quandle 3-cocycles is obtained by Mochizuki [Moc]: he determined all the 2-, 3-cocycles of some "Alexander quandles", which are subquandles of the forms $M \times \{1\} \subset M \times G$ with $G = \mathbb{Z}$ and $M = \mathbb{F}_q$; further the proof was to solve carefully all the cocycle-conditions through differential equations.

This paper deals generally with arbitrary groups $G$ and right $G$-modules $M$, and develops a simple algorithm to algebraically construct $n$-cocycles of the above quandle on $M \times G$ (Theorem 3.2). Actually, after a review of quandles in Section 2, we construct a chain map $\varphi_n$ from the quandle complex of the quandle $M \times G$ to the $G$-coinvariants of the group complex of the abelian group $M$ (Proposition 3.1), where we define this map $\varphi_n$ by modifying the Inoue-Kabaya map above (see Remark 3.4). Moreover, Section 4 shows that the pullback using the chain map $\varphi_n$ permits the strong conditions in [IIJO] mentioned above (see Propositions 4.3, 4.4 for details); Also Section 4.2 particularly gives simple formulae of such quandle 2-cocycles, while it was not easy to obtain $n$-cocycles of the $G$-family so far even if $G$ is abelian (cf. [II, Proposition 12] in abelian case). In conclusion, if we find a $G$-invariant group $n$-cocycle of $M$, then we obtain a quandle $n$-cocycle as the pullback via the chain map $\varphi_n$ (Theorem 3.2), which will consequently enable us to compute the associated cocycle invariants of tame links, of knotted surfaces and of handlebody-knots.

In sections 5 and 6, for the case $G$ and $M$ are of finite order, we seek $G$-invariant group $n$-cocycles of $M$ in the forms of $G$-invariant multilinear maps $M^n \to A$. Our approach is based on known results in (modular) invariant theory. For example, with respect to finite groups of Lie type, Chern-Weil theory and the Dickson theorem produce many $G$-invariant multilinear maps (Examples 6.3, 6.4); However, in general, as is known in modular representation theory, it is not easy to pick $G$-invariant multilinear maps, even if $G$ is a cyclic group. Meanwhile we address two modular representations: first, in respect to the standard action of $G = SL(2; \mathbb{F}_p)$ on $M = (\mathbb{F}_q)^2$, following from the work [CSW] to determine the $G$-invariant polynomial-ring $\mathbb{F}_q[M^{\oplus n}]^G$, one can list all the $G$-invariant multilinear maps $M^n \to \mathbb{F}_q$ with $n = 2$ or 3; further using the Bockstein map, we succeed in discovering its quandle $n$-cocycles (Propositions 6.6 and 6.8). On the other hand, we also work with the indecomposable representation of the cyclic group $\mathbb{Z}_p$ on $(\mathbb{F}_q)^2$ in §6.3. In summary, thanks to invariant theory, we describe explicitly many expressions of quandle cocycles of the quandle $M \times G$.

In doing so, we can obtain more examples of $n$-cocycles of the quandles $M \times G$, which we hope will be applicable in the study of knot theory; furthermore, our results will provide a motivation to find $G$-invariant polynomials.

## 2 Reviews of quandles and of quandle homologies

In this section, we review $G$-families of quandles and quandle homologies.

A *quandle*, $X$, is a non-empty set with a binary operation $(x, y) \to x \lhd y$ such that, for any $x, y, z \in X$, $x \lhd x = x$, $(x \lhd y) \lhd z = (x \lhd z) \lhd (y \lhd z)$ and there exists uniquely $w \in X$ satisfying $w \lhd y = x$. For example, a $\mathbb{Z}[T^{\pm}]$-module $M$ has a quandle structure given by $x \lhd y := Tx + (1 - T)y$, called *Alexander quandle*. Another example is a group $X = G$ with conjugation as the quandle operation: $g \lhd h = h^{-1}gh \in G$.

We next review $G$-families of quandles introduced in [IIJO]. Let $G$ be a group with identity $e$, and let $\mathcal{S}$ be a set. Given a map $\lhd^G : \mathcal{S}^2 \times G \to \mathcal{S}$, let us denote $\lhd^G(x, y, g) \in \mathcal{S}$ by $x \lhd^g y$ for any $(g, x, y) \in \mathcal{S}^2 \times G$ for short. A triple consisting of $(\mathcal{S}, G, \lhd^G)$ is referred to as *a $G$-family of quandles* [IIJO], if it satisfies the following three axioms:

- For any $g \in G$ and $x \in \mathcal{S}$, we have $x \lhd^g x = x$.
- For any $g, h \in G$ and $x, y \in \mathcal{S}$, we have $x \lhd^{gh} y = (x \lhd^g y) \lhd^h y$ and $x \lhd^e y = x$.
- For any $g, h \in G$ and $x, y, z \in \mathcal{S}$, we have $(x \lhd^g y) \lhd^h z = (x \lhd^h z) \lhd^{h^{-1}gh} (y \lhd^h z)$.

In a typical example of the $G$-family of quandles, for any right $\mathbb{Z}[G]$-module $M$, let us define a map $\lhd^G : M^2 \times G \to M$ sending $(x, y, g)$ to $(x - y) \cdot g + y$. Then its triple $(M, G, \lhd^G)$ is a $G$-family of quandles. Let us call such a triple $(M, G, \lhd^G)$ *$G$-family of Alexander quandles.* Furthermore, we remark that, for a $G$-families of quandles $(\mathcal{S}, G, \lhd^G)$, the direct product $\mathcal{S} \times G$ has a quandle structure with the operation defined by $(x, g) \lhd (y, h) := (x \lhd^h y, h^{-1}gh)$ [IIJO, Lemma 2.2]. The quandle on $\mathcal{S} \times G$ is called *the associated quandle* $X$ with the $G$-family $(\mathcal{S}, G, \lhd^G)$ [recall also (1) in Alexander case]; Hence, the class of *$G$-families of quandles* can be considered to be a subclass of *quandles*. Meanwhile, note that, for any $g \in G$, the subset $\mathcal{S} \times \{g\}$ is a subquandle of $\mathcal{S} \times G$. For instance, in Alexander case of the $G$-family, the subquandle on $\mathcal{S} \times \{g\}$ is an Alexander quandle mentioned above.

We set up the *associated group* of a quandle $X$, denoted by $\mathrm{As}(X)$ [FRS]. This group $\mathrm{As}(X)$ is the abstract group defined by the generators $e_x$ labeled by $x \in X$ and the relations $e_x \cdot e_y = e_y \cdot e_{x \lhd y}$ for $x,\ y \in X$. In addition, *an $X$-set* is a set acted on by $\mathrm{As}(X)$. For example, any quandle $X$ is an $X$-set obtained from the action of $\mathrm{As}(X)$ defined by $x \cdot e_y := x \lhd y \in X$ for $x, y \in X$; We call it *the primitive $X$-set*. As another example, a single point is an $X$-set.

We briefly review the quandle (co)homologies with local coefficients (our formula is based on that in [IK, §2.2]). Let $X$ be a quandle, $Y$ an $X$-set and $A$ a commutative ring. We set $C_n^R(X, Y; A)$ by the free $A$-module generated by the elements $(y, x_1, \dots, x_n)$ of $Y \times X^n$. Define a boundary $\partial_n^R : C_n^R(X, Y; A) \to C_{n-1}^R(X, Y; A)$ to be

$$\partial_n^R(y, x_1, \dots, x_n) :=$$
$$\sum_{1 \le i \le n} (-1)^i \big( (y \cdot e_{x_i}, x_1 \lhd x_i, \dots, x_{i-1} \lhd x_i, x_{i+1}, \dots, x_n) - (y, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \big).$$

The composite $\partial_{n-1}^R \circ \partial_n^R$ is known to be zero; The complex $(C_*^R(X, Y; A),\ \partial_*^R)$ is called *rack complex*[1], and $H_n^R(X, Y; A)$ denotes its homology. In addition, let $C_n^H(X, Y; A)$ be a submodule of $C_n^R(X, Y; A)$ generated by $(n+1)$-tuples $(y, x_1, \dots, x_n)$ with $x_i = x_{i+1}$ for some $i \in \{1, \dots, n-1\}$. Since $\partial_n^R(C_n^H(X, Y; A))$ is known to be contained in $C_{n-1}^H(X, Y; A)$, we can define a complex $\big(C_*^Q(X, Y; A), \partial_*^R\big)$ by the quotient $C_*^R(X, Y; A)/C_*^H(X, Y; A)$, and set its homology $H_*^Q(X, Y; A)$ called *quandle homology*. Dually, we can define the cohomologies $H_R^n(X, Y; A)$ and $H_Q^n(X, Y; A)$. Furthermore, a representative cocycle of an element in $H_R^n(X, Y; A)$ (resp. $H_Q^n(X, Y; A)$) is called *rack cocycle* (resp. *quandle cocycle*). Hereafter, in the case where $Y$ is a single point, we will suppress the symbol $Y$; we note that the quandle cohomology $H_R^n(X; A)$ coincides with the original one in [CJKLS].

**Remark 2.1.** Let $X$ be a quandle, and let $Y$ be the primitive $X$-set. As is known (see [FRS, Theorem 5.14]), we can easily verify that the homomorphism $C_n^R(X, X; \mathbb{Z}) \to C_{n+1}^R(X; \mathbb{Z})$ induced by the identification $X \times X^n \simeq X^{n+1}$ is a complex isomorphism.

As mentioned in the introduction, for applications of knot theory, it is significant to find concrete expressions of quandle 2-, 3-cocycles (see [CJKLS, CKS, IIJO] for details).

## 3 From $G$-invariant group cocycles to quandle cocycles

In this paper, our main concern is with the class of $G$-families of Alexander quandles; we shall establish some notation in what follows.

**Notation** We denote by $G$ a group, by $\mathbb{Z}[G]$ the group ring over $\mathbb{Z}$ and by $M$ a right $\mathbb{Z}[G]$-module. We fix the associated quandle $X$ $(= M \times G)$ with the operation (1). Furthermore, commutative rings with unit are often denoted by $A$.

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[1] As is known [CJKLS, CKS], the chain $(C_n^R(X, Y; A), \partial_*^R)$ coincide with the cellular complex of the rack space $BX$ [FRS], by regarding an $X$-set $Y$ as a local system.

Our goal in this section is to describe a simple algorithm to obtain *quandle* cocycles of $X$ from $G$-invariant *group* cocycles (Theorem 3.2), inspired by [IK] (see Remark 3.4 for details).

For this, we briefly review $G$-coinvariants of *non*-homogeneous chains of groups with trivial coefficients as follows (see [Bro, §II.2]). Let us regard the $G$-module $M$ as an abelian group. Then we can construct a complex $C_*^{\mathrm{gr}}(M;\mathbb{Z})$ by putting the free $\mathbb{Z}$-module $C_n^{\mathrm{gr}}(M;\mathbb{Z})$ spanned by $(x_1,\dots,x_n)\in M^n$ and letting its boundary map $\partial_n^{\mathrm{gr}}(a_1,\dots,a_n)\in C_{n-1}^{\mathrm{gr}}(M;\mathbb{Z})$ be

$$(a_2,\dots,a_n)+\sum_{1\le i\le n-1}(-1)^i(a_1,\dots,a_{i-1},a_i+a_{i+1},a_{i+2},\dots,a_n)+(-1)^n(a_1,\dots,a_{n-1}).$$

Under the diagonal action of $G$ on $C_*^{\mathrm{gr}}(M;\mathbb{Z})$, the *$G$-coinvariant part* of $C_n^{\mathrm{gr}}(M;\mathbb{Z})_G$ is defined by $C_n^{\mathrm{gr}}(M;\mathbb{Z})\otimes_{\mathbb{Z}[G]}\mathbb{Z}$. Let $H_n^{\mathrm{gr}}(M;\mathbb{Z})_G$ denote its homology. In addition, we set up the subcomplex $C_n^{\mathrm{N}}(M;\mathbb{Z})_G$ of $C_n^{\mathrm{gr}}(M;\mathbb{Z})_G$ generated by the elements $(a_1,\dots,a_n)\in M^n$ such that $a_i=0$ for some $1\le i<n$. As is well-known [Bro, §I.5], the canonical quotient chain map $C_*^{\mathrm{gr}}(M;\mathbb{Z})_G\to C_*^{\mathrm{gr}}(M;\mathbb{Z})_G/C_*^{\mathrm{N}}(M;\mathbb{Z})_G$ induces an isomorphism between the cohomologies of their $G$-invariant parts $H^n_{\mathrm{gr}}(M;A)^G$, $H^n_{\mathrm{nor}}(M;A)^G$. Furthermore, a representative group $n$-cocycle $\kappa: M^n\to A$ of an element of $H^n_{\mathrm{nor}}(M;A)^G$ is called *normalized group cocycle.*

For a parallel discussion, we now reformulate the rack complex $C_n^R(X;\mathbb{Z})$ in *non*-homogeneous coordinates (cf. [Moc, §2.1.2]). In this section, a symbol $\vec{g}$ means a $n$-tuple $(g_1,\dots,g_n)\in G^n$ for short; further we use the following symbols: for $i\le n-1$,

$$\vec{g}_{\{i\}}:=(g_1,\dots,g_i,g_{i+2},\dots,g_n)\in G^{n-1},$$

$$\vec{g}_{\{\lhd i\}}:=(g_{i+1}^{-1}g_1g_{i+1},\dots,g_{i+1}^{-1}g_ig_{i+1},g_{i+2},g_{i+3},\dots,g_n)\in G^{n-1}.$$

Define a module $C_*^{R_U}(X;\mathbb{Z})$ to be the free $\mathbb{Z}$-module generated by the elements of $G^n\times M^n$, and let its boundary map be

$$\partial_n^{R_U}(\vec{g};U_1,\dots,U_n)=\sum_{i\le n-1}(-1)^i\big((\vec{g}_{\{i\}};U_1,\dots,U_{i-1},U_i+U_{i+1},U_{i+2},\dots,U_n)-$$
$$(\vec{g}_{\{\lhd i\}};U_1\cdot g_{i+1},\dots,U_{i-1}\cdot g_{i+1},U_i\cdot g_{i+1}+U_{i+1},U_{i+2},\dots,U_n)\big)\in C_{n-1}^{R_U}(X;\mathbb{Z}),$$

for any generator $(\vec{g};U_1,\dots,U_n)\in C_n^{R_U}(X;\mathbb{Z})$. As a conclusion, we can see that a bijection

$$X^n=(M\times G)^n\to G^n\times M^n,\qquad (x_1,g_1,\dots,x_n,g_n)\mapsto(\vec{g};x_1-x_2,\dots,x_{n-1}-x_n,x_n)\quad (2)$$

gives rise to a chain isomorphism $\Upsilon_*:(C_*^R(X;\mathbb{Z}),\partial_*^R)\cong(C_*^{R_U}(X;\mathbb{Z}),\partial_*^{R_U})$.

We next construct a chain map $\varphi_n$ from the complex $C_n^{R_U}(X;\mathbb{Z})$ to the coinvariant part $C_n^{\mathrm{gr}}(M;\mathbb{Z})_G$. For this, put a set $\mathcal{K}_n:=\{\Bbbk_n=(k_1,\dots,k_n)\in\{0,1\}^n\mid k_1=0\}$ of order $2^{n-1}$. For $\Bbbk_n\in\mathcal{K}_n$, let $|\Bbbk_n|\in\mathbb{Z}$ be $k_1+\cdots+k_n$. Furthermore, given $\vec{g}=(g_1,\dots,g_n)\in G^n$, we define notation $\vec{g}_{\Bbbk_n,i}:=g_{i+1}^{k_{i+1}}g_{i+2}^{k_{i+2}}\cdots g_n^{k_n}\in G$ for $i\le n-1$, and $\vec{g}_{\Bbbk_n,i}:=e\in G$ for $i=n$. We then define the required map $\varphi_n(g_1,\dots,g_n;U_1,\dots,U_n)$ by

$$\sum_{\Bbbk_n=(k_1,\dots,k_n)\in\mathcal{K}_n}(-1)^{|\Bbbk_n|}\big(U_1\cdot\vec{g}_{\Bbbk_n,1},U_2\cdot\vec{g}_{\Bbbk_n,2},\dots,U_n\cdot\vec{g}_{\Bbbk_n,n}\big)\in C_n^{\mathrm{gr}}(M;\mathbb{Z})_G.\quad (3)$$

For example, when $n=2$ and $n=3$, the definition of the map $\varphi_n$ is rewritten in

$$\varphi_2(f,g;a,b)=(a,b)-(a\cdot g,b),\quad (4)$$

$$\varphi_3(f,g,h;a,b,c)=(a,b,c)-(a\cdot g,b,c)-(a\cdot h,b\cdot h,c)+(a\cdot(gh),b\cdot h,c).\quad (5)$$

**Proposition 3.1.** *Let $G$ be a group, $M$ a right $\mathbb{Z}[G]$-module, and $X$ the quandle on $M \times G$ with the operation* (1)*. Then the map $\varphi_n : C_n^{R_U}(X;\mathbb{Z}) \to C_n^{\mathrm{gr}}(M;\mathbb{Z})_G$ is a chain map.*

*Proof.* This will be proven by direct calculation. See Appendix A for details. □

In conclusion, we can obtain rack and quandle $n$-cocycles via the map $\varphi_n$. Namely

**Theorem 3.2.** *Let $G$, $M$ and $X$ be as above. Then for any $G$-invariant group $n$-cocycle $\theta : M^n \to A$ of the abelian group $M$, the pullback $\varphi_n^*(\theta)$ is a rack $n$-cocycle of the quandle $X$. Moreover, if the cocycle $\theta$ is normalized, then the pullback $\varphi_n^*(\theta)$ is a quandle $n$-cocycle of $X$.*

*Proof.* The former part is the dual of Proposition 3.1. The latter is easily seen from (3). □

**Remark 3.3.** As a result, we obtain the induced map from the $G$-invariant part $\varphi^* : H^*_{\mathrm{gr}}(M;A)^G \to H^*_Q(X;A)$. We now discuss its kernel. There are chances that its pullback $\varphi^*(\kappa)$ is zero for some group cocycle $\kappa \in H^*_{\mathrm{gr}}(M;A)^G$, e.g., see §6.2 and 6.3. However, other group cocycles $\kappa$ afford non-trivial quandle cocycles $\varphi^*(\kappa)$. To verify these non-trivialness, the cocycle invariants of links with respect to $\varphi^*(\kappa)$ are useful. Actually, as is shown [CJKLS, CKS], if the cocycle $\varphi^*(\kappa)$ is null-cohomologous, then the invariant is trivial.

**Remark 3.4.** Finally, we roughly explain a relation between our map $\varphi_n$ and a chain map $\varphi_{\mathrm{IK}}$ introduced by [IK, §3] in some detail. For any quandle $Q$, Inoue and Kabaya defined a "simplicial complex $C_n^{\Delta}(Q;\mathbb{Z})$" by a certain quotient of $\mathbb{Z}\langle Q^{n+1}\rangle$ in its homogeneous coordinate, and constructed a chain map $\varphi_{\mathrm{IK}} : C_n^R(Q;\mathbb{Z}) \to C_n^{\Delta}(Q;\mathbb{Z})$ by using shuffle products. Their motivation was from the Chern-Simons invariant using a special quandle on $(\mathbb{C}^2 \setminus \{0\})/\{\pm\}$.

Returning to our subject, let $Q$ be the associated quandle $X = M \times G$. Then a certain canonical projection $X^{n+1} \to M^n$ induces an epimorphism $\pi : C_n^{\Delta}(X;\mathbb{Z}) \to C_n^{\mathrm{gr}}(M;\mathbb{Z})_G$. We can further see that the composite $\pi \circ \varphi_{\mathrm{IK}}$ coincides with our map $\varphi_n$. In summary, Proposition 3.1 means that the composite $\pi \circ \varphi_{\mathrm{IK}}$ is a chain map, and simply relates the rack homology of $X$ to the $G$-invariant group homology in inhomogeneous term.

# 4 Cocycles of the $G$-family of Alexander quandles

Moreover, this section observes that the chain map $\varphi_n$ is practicable to obtain examples of "$n$-cocycles of $G$-families of quandles" defined in [IIJO]. Readers who are interested not in general discussions on $n$-cocycles but in only such 2-cocycles may skip to §4.2.

To explain the definition of such cocycles, we first review a quotient of the rack complex $C_n^R(X;\mathbb{Z})$ defined in [IIJO, §4]. Let $X$ be the associated quandle on $M \times G$. Let an $X$-set $Y$ satisfy the equality $(y \cdot e_{a,g}) \cdot e_{a,h} = y \cdot e_{a,gh}$ for any $y \in Y$, $(a,g)$, $(a,h) \in X$. Let $C_n^D(X,Y;A)$ be the submodule of $C_n^R(X,Y;A)$ generated by the elements of the following two set:

$$\bigcup_{1\le i\le n-1}\Big\{(y;q_1,\dots,q_{i-1},(a,g),(a,h),q_{i+2},\dots,q_n)\mid y\in Y,\ q_1,\dots,q_n\in X,\ g,h\in G,\ a\in M.\Big\},$$

$$\bigcup_{i=1}^{n}\left\{\begin{array}{c}(y;q_1,\dots,q_{i-1},(a,gh),q_{i+1},\dots,q_n)-(y;q_1,\dots,q_{i-1},(a,g),q_{i+1},\dots,q_n)\\ -(y\cdot e_{(a,g)};q_1\lhd(a,g),\dots,q_{i-1}\lhd(a,g),(a,h),q_{i+1},\dots,q_n)\end{array}\middle|\begin{array}{c}q_j\in X,\ y\in Y\\ g,h\in G,\ a\in M\end{array}\right\}.$$

Then the submodule $C_*^D(X,Y;\mathbb{Z})$ is known to be a subcomplex, i.e., $\partial_n^R\big(C_n^D(X,Y;A)\big)$ is contained in $C_{n-1}^D(X,Y;A)$ [IIJO, Lemma 4.1]. Hence, considering the quotient complex

$C^R_*(X,Y;\mathbb{Z})/C^D_*(X,Y;\mathbb{Z})$, we have its (co)homology $H^{\mathrm{Gf}}_*(X,Y;A)$, $H^*_{\mathrm{Gf}}(X,Y;A)$. A representative map $\phi : Y \times X^n \to A$ in the cohomology $H^n_{\mathrm{Gf}}(X,Y;A)$ is called *a cocycle of the $G$-family $(M,G)$ with an $X$-set $Y$*. According to [IIJO], if finding such a 2-cocycle, we can define and compute a topological invariant of handlebody-knots in the 3-sphere.

**Remark 4.1.** Any $n$-cocycle of the $G$-family $\phi : Y \times X^n \to A$ is also a usual quandle $n$-cocycle, since the subcomplex $C^D_n(X,Y;\mathbb{Z})$ above includes the subcomplex $C^H_n(X,Y;\mathbb{Z})$ defined in §2. Compare [II, Proposition 12] where Ishii and Iwakiri conversely suggested a sum-formula of cocycles of the $G$-family obtained from quandle cocycles under some conditions, when $G$ is abelian. However, we will give simple formulae of cocycles of $G$-families, including cases of non-abelian groups.

### 4.1 A relation between the subcomplex $C^D_*(X,Y;\mathbb{Z})$ and the chain map $\varphi_*$

In this paper, to involve cocycles of the $G$-family from the chain map $\varphi_n$ (Theorem 4.5), we restrict ourselves to the case where an $X$-set $Y$ is either the single $X$-set or the primitive $X$-set. Roughly speaking, the theorem 4.5 says that the map $\varphi_n$ yields two homomorphisms from the proceeding homologies: $H^{\mathrm{Gf}}_n(X,X;A) \to H^{\mathrm{gr}}_{n+1}(M;A)_G$ and $H^{\mathrm{Gf}}_n(X;A) \to H^{\mathrm{gr}}_n(M;A)_G$.

Under the restriction, we first observe the subcomplex $C^D_n(X,Y;\mathbb{Z})$ as that of the complex $C^{R_U}_*(X;\mathbb{Z})$ via the bijection (2). For $s = 1$ or 2, define $C^{D_U}_{n,s}(X;\mathbb{Z})$ to be a submodule of $C^{R_U}_*(X;\mathbb{Z})$ generated by the elements of the following two sets:

$$\bigcup_{1\le i\le n-1} \Big\{ (\ \vec{g}\ ; U_1,\dots,U_{i-1},0,U_{i+1},\dots,U_n) \ \Big|\ \vec{g}\in G^n,\ U_1,\dots,U_{i-1},U_{i+1},\dots,U_n \in M. \Big\},$$

$$\bigcup_{i=s}^{n} \left\{ \begin{array}{l} (g_1,\dots,g_n;U_1,\dots,U_n) - (g_1,\dots,g_{i-1},g_ih,g_{i+1},\dots,g_n;U_1,\dots,U_n) \\ +(g_i^{-1}g_1g_i,\dots,g_i^{-1}g_{i-1}g_i,h,g_{i+1},\dots,g_n;U_1g_i,\dots,U_{i-1}g_i,U_i,\dots,U_n) \end{array} \ \middle|\ \begin{array}{l} g_1,\dots,g_n,h\in G \\ U_1,\dots,U_n \in M \end{array} \right\}.$$

**Lemma 4.2.** *Let $\Upsilon_n : C^R_n(X;\mathbb{Z}) \to C^{R_U}_n(X;\mathbb{Z})$ and $\Theta_n : C^R_n(X,X;\mathbb{Z}) \to C^R_{n+1}(X;\mathbb{Z})$ be the chain isomorphisms induced by* (2) *and mentioned in Remark 2.1, respectively. Then the restriction of $\Upsilon_n$ on $C^D_n(X;\mathbb{Z})$ gives an isomorphism $C^D_n(X;\mathbb{Z}) \cong C^{D_U}_{n,1}(X;\mathbb{Z})$. Furthermore, when $Y$ is the primitive $X$-set, the restriction of the composite $\Upsilon_n \circ \Theta_n$ on $C^D_n(X,X;\mathbb{Z})$ is an isomorphism $C^D_n(X,X;\mathbb{Z}) \cong C^{D_U}_{n+1,2}(X;\mathbb{Z})$.*

*Proof.* This is shown by elementary calculations and definitions; so we omit the details. □

Next, we study the images of the subcomplexes $C^{D_U}_{*,s}(X;\mathbb{Z})$ via the chain map $\varphi_*$ in (3). To begin, let $s = 2$. Recalling the (acyclic) subcomplex $C^{\mathrm{N}}_n(M;\mathbb{Z})_G$ of the group complex $C^{\mathrm{gr}}_n(M;\mathbb{Z})_G$ explained in §3, we later show the following:

**Proposition 4.3.** *Then the image $\varphi_n(C^{D_U}_{n,2}(X;\mathbb{Z}))$ is included in $C^{\mathrm{N}}_n(M;\mathbb{Z})_G$.*

However, for the case $s = 1$, the similar discussion does not hold for integer coefficients. To modify this, we fix a ring $A$, and an additive homomorphism $\lambda : G \to A$. Further, define a map $\widetilde{\lambda} : C^{R_U}_n(X;\mathbb{Z}) \to A$ by $\widetilde{\lambda}(g_1,\dots,g_n,U_1,\dots,U_n) = \lambda(g_1)$. In addition, define a map $\varphi_{n,\lambda} : C^{R_U}_n(X;\mathbb{Z}) \otimes A \to C^{\mathrm{gr}}_n(M,\mathbb{Z})_G \otimes A$ by $(\varphi_n \otimes_{\mathbb{Z}} A) \cdot \widetilde{\lambda}$.

**Proposition 4.4.** *Let $s = 2$. Then the image $\varphi_{n,\lambda}(C^{D_U}_{n,1}(X;\mathbb{Z}) \otimes A)$ is contained in $C^{\mathrm{N}}_n(M;A)_G$.*

We later give the proofs of Propositions 4.3 and 4.4 in Appendix A.

In conclusion, composing Propositions 4.3 and 4.4 with Lemma 4.2 readily shows

**Theorem 4.5.** *Let $\Theta_n$ and $\Upsilon_n$ be the chain isomorphisms in Lemma 4.3. Let $\varphi_n : C_n^{R_U}(X;\mathbb{Z}) \to C_n^{\mathrm{gr}}(M;\mathbb{Z})_G$ be the chain map (3). Then, regarding $X$ as an $X$-set, the composite $\varphi_{n+1} \circ \Theta_n \circ \Upsilon_n : C_n^R(X,X;\mathbb{Z}) \to C_{n+1}^{\mathrm{gr}}(M;\mathbb{Z})_G$ induces a homomorphism $H_n^{\mathrm{Gf}}(X,X;\mathbb{Z}) \to H_{n+1}^{\mathrm{gr}}(M;\mathbb{Z})_G$. Moreover, for a group homomorphism $\lambda : G \to A$, the composite $\varphi_{n,\lambda} \circ \Upsilon_n : C_n^R(X;A) \to C_n^{\mathrm{gr}}(M;A)_G$ induces a homomorphism $H_n^{\mathrm{Gf}}(X;A) \to H_n^{\mathrm{gr}}(M;A)_G$.*

### 4.2 Concrete expressions of 2-cocycles of the $G$-family of quandles $(M,G)$.

As a special case $n = 2$, we now give a more useful description of Theorem 4.5, for users of the quandle cocycle invariants of links.

First, we consider the primitive $X$-set $X$. Let $\kappa : M^3 \to A$ be a $G$-invariant normalized group 3-cocycle. Let $\phi_\kappa : X^3 \to A$ denote the pullback of $\kappa$ via the composite $\varphi_3 \circ \Theta_2 \circ \Upsilon_2$ in Theorem 4.5. Precisely, recalling $\varphi_3$ in (5), the map $\phi_\kappa \in C_R^3(X;A)$ is then expressed by

$$\phi_\kappa\big((a,f),(b,g),(c,h)\big) := \kappa\big(a-b, b-c, c\big) - \kappa\big((a-b)\cdot g, b-c, c\big)$$
$$-\kappa\big((a-b)\cdot h, (b-c)\cdot h, c\big) + \kappa\big((a-b)\cdot gh, (b-c)\cdot h, c\big),$$

for $a,b,c \in M,\ f,g,h \in G$. Then, Theorem 4.5 with $n = 2$ concisely means

**Corollary 4.6.** *The map $\phi_\kappa$ is a 2-cocycle of the $G$-family $(M,G)$ with the primitive $X$-set. Further, by Remarks 2.1 and 4.1, the map $\phi_\kappa$ is also a usual quandle 3-cocycle in $H_Q^3(X;A)$.*

On the other hand, we work with the $X$-set $Y$ consisting of a single point. As mentioned above, fix an additive homomorphism $\lambda : G \to A$. For a $G$-invariant group 2-cocycle $\theta : M^2 \to A$, let $\phi_{\theta,\lambda} : X^2 \to A$ denote the pullback of $\theta$ via the composite $\varphi_{2,\lambda} \circ \Upsilon_2$ in Theorem 4.5: That is, the map $\phi_{\theta,\lambda} \in C_R^2(X;A)$ is expressed by

$$\phi_{\theta,\lambda}\big((a,g),(b,h)\big) := \big(\theta(a-b,b) - \theta(a\cdot h - b\cdot h, b)\big)\cdot\lambda(g) \in A, \qquad \text{for } g,h \in G,\ a,b \in M.$$

Then the latter part of Theorem 4.5 with $n = 2$ is reduced to

**Corollary 4.7.** *The map $\phi_{\theta,\lambda}$ is a 2-cocycle of the $G$-family $(M,G)$ with the single $X$-set.*

**Remark 4.8.** Notice that the homomorphism $\lambda : G \to A$ factors through its abelianization $G_{\mathrm{ab}}$; thus, to obtain non-trivial 2-cocycles, we should choose appropriate groups $G$ such that $|G_{\mathrm{ab}}| \neq 0$ and the coefficient ring $A$ is annihilated by $|G_{\mathrm{ab}}|$.

## 5 Preliminaries for $G$-invariant group cocycles and invariant theory

In §5 and 6, we will give examples of $G$-invariant group $n$-cocycles; As a result, following Corollaries 4.6 and 4.7, we will describe explicitly quandle cocycles of the quandle $X$ and of the $G$-family $(M,G)$. Furthermore, for applications to the cocycle invariant of links (see [CJKLS, CKS, IIJO]), in §5, 6 we focus mainly on group 2-, 3-cocycles, and we should assume finiteness of $G$, $M$ and $A$: To be specific,

**Assumption** Groups $G$, $G$-modules $M$ and coefficient rings $A$ are of finite order.

### 5.1 Easy cases where $|G|$ is coprime to $|M|$

To begin, we assume that the orders $|G|$ and $|M|$ are coprime. To study the cohomology of the finite group $M$, it is sensible to assume that $|G| \in \mathbb{Z}$ is invertible in the coefficient ring $A$. Then we can define an $A$-homomorphism from the cochain group of the abelian group $M$ to its $G$-invariant part by

$$C^n_{\rm gr}(M;A) \to C^n_{\rm gr}(M;A)^G, \qquad f \longmapsto \frac{1}{|G|}\sum_{g\in G} f\cdot g.$$

As is well-known, we see that this is a chain map and surjective; thus we easily have a $G$-invariant group cocycles, since it is easy to construct cocycles of the abelian group $M$. Furthermore, we remark that, by the transfer map (see [Bro, §III.9]), the $G$-invariant part of the cohomology $H^n_{\rm gr}(M;A)$ is isomorphic to $H^n_{\rm gr}(M \rtimes G;A)$.

**Example 5.1.** From our viewpoint, we now observe the quandle 3-cocycles found by Mochizuki [Moc]. Let $M$ be a finite field $\mathbb{F}_q$ of order $q = p^h$, and let $\omega \in \mathbb{F}_q \setminus \{0,1\}$. Let $\ell$ denote the order of $\omega$. Consider the case where $G = \mathbb{Z}_\ell$ acts on $\mathbb{F}_q$ by multiplication of $\omega$. Since the order $\ell$ is coprime to $q$, we can easily describe $G$-invariant group $n$-cocycles $\theta$ of the abelian group $M = (\mathbb{Z}_p)^h$. Thus, by Theorem 3.2, we obtain the resulting quandle $n$-cocycles $\varphi_n^*(\theta)$ of the quandle $X = M \times G = \mathbb{F}_q \times \mathbb{Z}_\ell$. When $n = 2$ or 3, it can be seen that the restricted $n$-cocycle $\varphi_n^*(\theta)$ on the subquandle $\mathbb{F}_q \times \{1\}$ coincide with some $n$-quandle cocycles found in [Moc].

To avoid such easy cases, we later discuss the case where $|G|$ and $|M|$ are not coprime.

### 5.2 Quandle cocycles from $G$-invariant multilinear maps

To obtain $G$-invariant group cocycles of $M$, we find it the most convenient to study $G$-invariant multilinear maps. Here, for an $A$-module $M$, say $A = \mathbb{Z}/|M|\mathbb{Z}$, an $A$-multilinear map $f : M^n \to A$ is said to be $G$-*invariant*, if it satisfies

$$f(a_1,\dots,a_n) = f(a_1\cdot g,\dots,a_n\cdot g), \qquad \text{for any } (a_1,\dots,a_n)\in M^n,\ g\in G.$$

We now summarize descriptions of quandle cocycles from $G$-invariant multilinear maps:

**Theorem 5.2.** *Any $G$-invariant $A$-multilinear map $f : M^n \to A$ is a normalized $G$-invariant group $n$-cocycle of $M$. Further, the pullback $\varphi_n^*(f)$ by* (3) *is a quandle $n$-cocycle of the quandle $X$. In particular, if $n = 3$, the resulting quandle 3-cocycle $\phi_f \in C^3_Q(X;A)$ forms*

$$\phi_f\big((a_1,g_1),(a_2,g_2),(a_3,g_3)\big) = f\big((a_1-a_2)\cdot(1-g_2),\ a_2-a_3,\ a_3\cdot(1-g_3^{-1})\big), \tag{6}$$

*for $a_1,a_2,a_3 \in M$ and $g_1,g_2,g_3 \in G$. Furthermore, if $n = 2$, the resulting 2-cocycle forms*

$$\phi_f\big((a_1,g_1),(a_2,g_2)\big) = f\big(a_1-a_2,\ a_2\cdot(1-g_2^{-1})\big), \tag{7}$$

Therefore, until the end of §6, we seek $G$-invariant multilinear maps with respect to some $G$-modules $M$. To begin, we present two simple examples arising from Chern-Weil theory:

**Example 5.3.** Take $M$ to be $A^{\oplus n}$. Let $G = SL(n;A)$ act on $M$ canonically. Notice that the determinant $\det : M^{\oplus n} \to A$ is an $SL(n;A)$-invariant $A$-multilinear map. In particular, when $n = 3$, the resulting quandle 3-cocycle $\phi \in C^3_Q(X;A)$ is represented as

$$\phi((v_1,g_1),(v_2,g_2),(v_3,g_3)) := \det\big((v_1-v_2)\cdot(1-g_2),\ v_2-v_3,\ v_3\cdot(1-g_3^{-1})\big). \tag{8}$$

**Example 5.4.** Let $GL(n;A)$ act on the matrix ring $\mathrm{Mat}(n \times n; A)$ by conjugation. Given a subgroup $G$ of $GL(n;A)$, we regard $\mathrm{Mat}(n \times n; A)$ as a $G$-module. Let $M$ be a $G$-submodule of $\mathrm{Mat}(n \times n; A)$. Then we put known two $G$-invariant multilinear maps $c_3 : M^3 \to A$ and $c_2 : M^2 \to A$ defined by $c_3(S_1, S_2, S_3) = \mathrm{Tr}(S_1S_2S_3)$ and $c_2(S_1, S_2) = \mathrm{Tr}(S_1S_2)$ for $S_i \in M$, respectively; thus the resulting quandle cocycles $c_3', c_2' \in C_Q^*(X; A)$ are of the forms

$$c_3'((S_1, g_1), (S_2, g_2), (S_3, g_3)) := \mathrm{Tr}\big((S_1 - S_2 - g_2^{-1}S_1g_2 + g_2^{-1}S_2g_2)(S_2 - S_3)(S_3 - g_3S_3g_3^{-1})\big),$$

$$c_2'((S_1, g_1), (S_2, g_2)) := \mathrm{Tr}\big((S_1 - S_2)(S_2 - g_2S_2g_2^{-1})\big), \qquad \text{for } S_i \in M, \ g_i \in G.$$

## 6 Some quandle 2-, 3-cocycles from modular invariant theory.

Inspired by Theorem 5.2, in this section, we ask for some ways of finding $G$-invariant multilinear maps from modular representation theory. For this, we now assume that the coefficient ring $A$ is the finite field $\mathbb{F}_q$ of order $q$. Namely, the $G$-module $M$ is a representation of $G$ over $\mathbb{F}_q$; we often denote $M$ by $V$ as a $\mathbb{F}_q$-vector space, in what follows.

In §6.1 we review a classical method, the full polarization, to obtain $G$-invariant multilinear maps from $G$-invariant polynomial rings. In §6.2 and §6.3, in the case $G$ is $SL(2;\mathbb{F}_p)$ or $\mathbb{Z}/p\mathbb{Z}$, we deal with $G$-invariant group cocycles not derived from the full polarization.

### 6.1 From $G$-invariant polynomials to $G$-invariant multilinear maps

We review a full polarization (see, e.g., [CW, §1.9]). There is nothing new in this subsection. Set the $(\dim_{\mathbb{F}_q} V)$-variable polynomial ring $\mathbb{F}_q[V]$. For $d \in \mathbb{Z}$, let $\mathbb{F}_q[V]_d$ denote the subspace consisting homogenous polynomials of degree $d$; Further, for $(\lambda_1, \dots, \lambda_m) \in \mathbb{N}^m$, we define a subspace of $\mathbb{F}_q[V^{\oplus m}] = \mathbb{F}_q[V]^{\otimes m}$ of degree $(\lambda_1, \dots, \lambda_m)$ by

$$\mathbb{F}_q[V^{\oplus m}]_{(\lambda_1,\dots,\lambda_m)} := \mathbb{F}_q[V]_{\lambda_1} \otimes \mathbb{F}_q[V]_{\lambda_2} \otimes \cdots \otimes \mathbb{F}_q[V]_{\lambda_m},$$

where the symbols $\otimes$ temporarily mean symmetric tensor products. Put the canonical projection $\pi_{(\lambda_1,\dots,\lambda_m)} : \mathbb{F}_q[V^{\oplus m}] \to \mathbb{F}_q[V^{\oplus m}]_{(\lambda_1,\dots,\lambda_m)}$. Under the action of $G$ on $V$, we let $\mathbb{F}_q[V]_d^G$ be the $G$-invariant subspace of $\mathbb{F}_q[V]_d$ of degree $d$. For its element $f \in \mathbb{F}_q[V]_d^G$, consider a $d$-variable polynomial of the form $f(v_1 + \cdots + v_d) \in \mathbb{F}_q[V^{\oplus d}]$, where $(v_1, \dots, v_d) \in V^{\oplus d}$. Then the *full polarization* of $f$ is defined by

$$\mathcal{P}(f) := \pi_{(1,\dots,1)}\big(f(v_1 + \cdots + v_d)\big) \in \mathbb{F}_q[V^{\oplus d}]_{(1,\dots,1)}.$$

By definition, the full polarization $\mathcal{P}(f)$ is a $G$-invariant $\mathbb{F}_q$-multilinear map. In summary

**Lemma 6.1.** *For any $G$-invariant polynomial $f \in \mathbb{F}_q[V]_d^G$ of degree $d$, the full polarization $\mathcal{P}(f) : V^{\oplus d} \to \mathbb{F}_q$ is a $G$-invariant $\mathbb{F}_q$-multilinear map.*

**Remark 6.2.** Notice $\mathcal{P}(f)(v, \dots, v) = d^d f(v)$ by definition; thereby, in general, the polarization $\mathcal{P} : \mathbb{F}_q[V]_d \to \mathbb{F}_q[V^{\oplus d}]_{(1,\dots,1)}$ is not always surjective (cf. Example 5.3).

To obtain $G$-invariant multilinear maps, we now sketch some examples of $\mathbb{F}_q[V]_d^G$.

**Example 6.3** (Chern-Weil theory)**.** Let $G$ be an algebraic group scheme over $A$, and $\mathfrak{g}$ the associated Lie algebra. The invariant parts $A[\mathfrak{g}]^G$ under the adjoint actions of $G$ on $\mathfrak{g}$ are much studied. For instance, considering special cases of $G = GL(n;\mathbb{F}_q)$ or $O(n;\mathbb{F}_q)$, the invariant polynomials represented as the $i$-th coefficients of $\det(I_n t - B) \in \mathbb{F}_q[\mathfrak{g}][t]$ relate to the orbit Chern, Pontrjagin and Euler classes, where $B \in \mathfrak{g}$.

**Example 6.4** (Dickson theorem)**.** Fortunately, for some modular representations $\rho : G \to GL(V;\mathbb{F}_q)$, these $G$-invariant parts $\mathbb{F}_q[V]^G_d$ are determined (see, e.g., [CW], [NS]). For example, when $G = GL(V;\mathbb{F}_q)$ and $\rho = \mathrm{id}$, thanks to Dickson theorem (see, e.g., [NS, §6]), the $G$-invariant polynomial ring $\mathbb{F}_q[\mathfrak{g}]^G$ is generated by "Dickson polynomials" which are derived from the orbits Chern classes and the Euler class of $G$. Accordingly, in the case $G \subset GL(V)$, the part $\mathbb{F}_q[V]^G_d$ is partially related to such classes as well (see [NS, §7.4]).

### 6.2 The standard representation of $SL(2;\mathbb{F}_p)$

As mentioned in Remark 6.2, the polarization $\mathcal{P} : \mathbb{F}_q[V]^G_3 \to \mathbb{F}_q[V^{\oplus 3}]^G_{(1,1,1)}$ is not always surjective. However, in general, it is not easy to find generators of the cokernel. Actually, the study to decompose the tensor representation $V \otimes V \otimes V$ is a hard problem in modular representations theory, even if $G$ is cyclic (see, e.g., [CW, §4 and §7]). Hereafter we fix a field extension $\mathbb{F}_p \subset \mathbb{F}_q$ and a notation $n \in \mathbb{Z}$ such that $q = p^n$ (possibly $n = 1$).

We now observe the cokernel under the standard action of $G = SL(2;\mathbb{F}_p)$ on $V = \mathbb{F}_q \oplus \mathbb{F}_q$ canonically, as an interesting example (cf. the Dickson theorem which implies that, for $m \leq 3$, the invariant part $\mathbb{F}_q[V]^{SL(2;\mathbb{F}_p)}_m$ is zero with $p \neq 2$.). From the cokernel, we will find two cocycles of the associated quandle $V \times G$ (Propositions 6.6 and 6.8).

To see this, we review the work [CSW, Theorem 8.1]: fortunately, they have presented generators of the $G$-invariant $\mathbb{F}_q$-polynomial ring $\mathbb{F}_q[V^{\oplus m}]^G$. However, if describing explicitly their results, we had to set up many notation and polynomials; we here consider their result with $m = 2$ and $m = 3$, and describe the $G$-invariant $\mathbb{F}_q$-multilinear maps (Theorem 6.5 below). To describe them, we put $e_1, e_1', e_2, e_2', e_3, e_3'$ as the canonical basis of $V^{\oplus 3}$, and denote elements $v$ of $V^{\oplus 3}$ by $v = \sum_{k=1}^{3} x_k e_k + y_k e_k'$ with $x_k,\ y_k \in \mathbb{F}_q$. Set a polynomial defined by

$$\mathcal{G}_3^{stu} = (x_1 + y_1)^{p^s}(x_2 + y_2)^{p^t}(x_3 + y_3)^{p^u} - x_1^{p^s} x_2^{p^t} x_3^{p^u} - y_1^{p^s} y_2^{p^t} y_3^{p^u}.$$

**Theorem 6.5** (A special case of [CSW, Theorem 8.1])**.** *Let $m$ be either 2 or 3. Let $q = p^n$ for some $n \in \mathbb{Z}$. Without $(p, m) = (2, 3)$, the space composed of $G$-invariant $\mathbb{F}_p$-multilinear maps $V^{\oplus m} \to \mathbb{F}_q$ is spanned by the following set $I_m$ consisting of polynomials $\mathcal{F}_{ij}^{st}$:*

$$I_m := \{\mathcal{F}_{ij}^{st} := x_i^{p^s} y_j^{p^t} - x_j^{p^t} y_i^{p^s} \ \mid\ 0 \leq s, t \leq n-1, \ \textit{and}\ 1 \leq i < j \leq m.\ \}.$$

*Furthermore, if $p = 2$ and $m = 3$, the space composed of $G$-invariant $\mathbb{F}_p$-multilinear maps $V^{\oplus m} \to \mathbb{F}_q$ is spanned by $I_3 \cup \{\mathcal{G}_3^{stu} \mid 0 \leq s, t, u \leq n-1\ \}$.*

*Proof.* This can be shown by observing carefully the multidegree parts $\mathbb{F}_q[V^{\oplus m}]^G_{(\lambda_1,\dots,\lambda_m)}$ with $\lambda_j = p^s$ (possibly $\lambda_j = 0$). See [CSW, Theorem 8.1] for details. □

Meanwhile, we now describe 2-cocycles of the $G$-family $(V, SL(2;\mathbb{F}_p))$ with the *single* $X$-set. Considering Remark 4.8, we note that $SL(2;\mathbb{F}_p)$ is perfect for $p \geq 5$, and that the abelianization of $SL(2;\mathbb{F}_p)$ is $\mathbb{Z}_p$ for $p = 2$ or 3. Recall from Theorem 6.5, the space composed of $G$-invariant $\mathbb{F}_q$-bilinear maps is generated by $I_2$. By applying this case to Corollary 4.7, we easily have a 2-cocycle of the $G$-family as follows:

**Proposition 6.6.** *Let* $p = 2$ *or* $3$*. Let* $q = p^n$*. Let* $\lambda : SL(2;\mathbb{F}_p) \to \mathbb{F}_p$ *be its abelianization. For* $0 \le s,t \le n-1$*, we define a map* $\phi_{s,t} : (V \times SL(2;\mathbb{F}_p))^2 \to \mathbb{F}_q$ *by*

$$\phi_{s,t}((x_1,y_1,g),(x_2,y_2,h)) := \lambda(g)\Big((x_1-x_2)^{p^s}\big((1-a)y_2+bx_2\big)^{p^t}+(y_1-y_2)^{p^t}\big((1-d)x_2+cy_2\big)^{p^s}\Big),$$

*where* $a,b,c,d \in \mathbb{F}_p$ *are defined by* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = h \in SL(2;\mathbb{F}_p)$*. Then the map* $\phi_{s,t} \in C^2_R(X;\mathbb{F}_q)$ *is a 2-cocycle of the* $G$*-family* $(V, SL(2;\mathbb{F}_p))$ *with the single* $X$*-set.*

**Remark 6.7.** If $s = t$ and $p = 2$, then the map $\phi_{s,s}$ is cohomologous to zero. Namely we can verify $\phi_{s,s} = \delta_1^R\big(\lambda(g)(x_1^2+y_1^2+x_1y_1)^{p^{s-1}}\big)$.

Next, we discuss 2-cocycles of the $G$-family with the *primitive* $X$-set. Following Theorem 6.5, we now calculate quandle 3-cocycles obtained from the polynomials $\mathcal{F}_{ij}^{st}$ in $I_3$, although unfortunately the resulting cocycles are trivial as follows. By the definition of $\varphi_3^*$ [see (5)], it is verified that the pullback $\varphi_3^*(\mathcal{F}_{ij}^{st})$ is zero without $(i,j) = (1,3)$; Further, the remaining cocycles $\varphi_3^*(\mathcal{F}_{13}^{st})$ are null-cohomologous: precisely, $\delta_2^R(x_1^{p^s}y_2^{p^t} - x_1^{p^t}y_2^{p^s}) = \varphi_3^*(\mathcal{F}_{13}^{st}) \in C^3_R(X;\mathbb{F}_q)$.

Incidentally, in respect to $p = 2$, we briefly comment the reminder polynomial $\mathcal{G}_3^{stu}$ in Theorem 6.5. The pullback $\varphi_3^*(\mathcal{G}_3^{stu})$ is not a null-cohomologous cocycle. Indeed, it is seen that the cocycle invariant of the trefoil knot using $\varphi_3^*(\mathcal{G}_3^{stu})$ is non-trivial (recall Remark 3.3).

Although we can not obtain many quandle 3-cocycles directly from Theorem 6.5, we now construct another group 3-cocycle; For any prime $p$, consider the following polynomial [2]:

$$\Theta_{s,t} := \big(|v_1^{p^s}+v_2^{p^s}, v_3^{p^t}|^p - |v_1^{p^s}, v_2^{p^t}+v_3^{p^t}|^p - |v_2^{p^s}, v_3^{p^t}|^p + |v_1^{p^s}, v_2^{p^t}|^p\big)/p \in \mathbb{F}_q[V^{\oplus 3}], \tag{9}$$

where $0 \le s,t \le n-1$ and we denote by $|\bullet,\bullet|$ the determinant map $V \otimes V \to \mathbb{F}_q$, and denote by $\bullet^{p^s} : V \to V$ the Frobenius map sending $v = (x,y)$ to $(x^{p^s}, y^{p^s})$. By direct calculation, we see that the polynomial $\Theta_{s,t}$ is a $G$-invariant group 3-cocycle. Hence, by computing the pullback $\Phi_{s,t} := \varphi_3^*(\Theta_{s,t})$, Corollary 4.6 immediately amounts to

**Proposition 6.8.** *Let* $p$ *be a prime. Let* $q = p^n$*. For* $0 \le s,t \le n-1$*, we take a map* $\overline{\Phi}_{s,t} : (V \times SL(2;\mathbb{F}_p))^3 \to \mathbb{F}_q$ *defined by* $\overline{\Phi}_{s,t}\big((v_1,f),(v_2,g),(v_3,h)\big) =$

$$\big(|v_1^{p^s}+v_2^{p^s}, v_3^{p^t}|^p - |v_1^{p^s}\cdot g+v_2^{p^s}, v_3^{p^t}|^p - |v_1^{p^s}+v_2^{p^s}, v_3^{p^t}\cdot h^{-1}|^p + |v_1^{p^s}\cdot g+v_2^{p^s}, v_3^{p^t}\cdot h^{-1}|^p$$

$$-|v_1^{p^s}, v_2^{p^t}+v_3^{p^t}|^p + |v_1^{p^s}\cdot g, v_2^{p^t}+v_3^{p^t}|^p + |v_1^{p^s}, v_2^{p^t}+v_3^{p^t}\cdot h^{-1}|^p - |v_1^{p^s}\cdot g, v_2^{p^t}+v_3^{p^t}\cdot h^{-1}|^p\big)/p.$$

*Further, define another map* $\Phi_{s,t}$ *by the variable change* $\Phi_{s,t}\big((v_1,f),(v_2,g),(v_3,h)\big) := \overline{\Phi}_{s,t}\big((v_1-v_2,f),(v_2-v_3,g),(v_3,h)\big)$*. Then the map* $\Phi_{s,t}$ *is a 2-cocycle of the* $G$*-family* $(V,G)$ *with the primitive* $X$*-set. Furthermore, this* $\Phi_{s,t}$ *is also a quandle 3-cocycle contained in* $C^3_R(X;\mathbb{F}_q)$*.*

**Remark 6.9.** The cocycle $\Phi_{s,t}$ is expected to be not null-cohomologous, except $s = t$ and $p = 2$, although we here have no proof to verify the non-trivialness.

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[2] Let $\beta_m : H^m_{\mathrm{gr}}(M;\mathbb{F}_q)^G \to H^{m+1}_{\mathrm{gr}}(M;\mathbb{F}_q)^G$ be the Bockstein map. Then the polynomial map $\Theta_{s,t}$ is obtained by $\beta_2(x_1^{p^s}y_2^{p^t} - x_2^{p^t}y_1^{p^s})$ from the definition of the Bockstein map, although we omit the detailed calculation.

### 6.3 2-dimensional modular representation of the cyclic group $\mathbb{Z}_p$

Finally, we study the right upper triangular subgroup $G$ of $SL(2;\mathbb{F}_p)$ and the restricted action on $V = \mathbb{F}_q \oplus \mathbb{F}_q$. This $G$ is isomorphic to the cyclic group $\mathbb{Z}_p$. As is known, the minimal algebra generating set of $\mathbb{F}_q[V^{\oplus m}]^G$ are determined (see [CW, §7.4]). For simplicity, we confirm ourselves to the subspace $\mathbb{F}_q[V^{\oplus m}]^G_{(1,\dots,1)}$ with $m = 2, 3$. To state this, we define a set by

$$J_3 := \{\ x_1x_2x_3,\ x_1(y_2x_3 - y_3x_2),\ x_2(y_3x_1 - y_1x_3),\ x_3(y_1x_2 - y_2x_1)\ \}. \tag{10}$$

**Theorem 6.10** (see also [CSW, Theorem 7.4.1])**.** *When $m = 2$, the space $\mathbb{F}_q[V^{\oplus 2}]^G_{(1,1)}$ is spanned by two polynomials $x_1x_2$ and $x_1y_2 - y_1x_2$. Furthermore, when $m = 3$ and $p \neq 2$, the space $\mathbb{F}_q[V^{\oplus 3}]^G_{(1,1,1)}$ is spanned by the set $J_3$ in* (10)*. In addition, when $p = 2$, the space $\mathbb{F}_q[V^{\oplus 3}]^G_{(1,1,1)}$ is spanned by the set $J_3$ and by a polynomial $(x_1 + y_1)(x_2 + y_2)(x_3 + y_3) + y_1y_2y_3$.*

For the application, we first consider $m = 2$. Note that $\varphi_2^*(x_1x_2) = 0$ using (4). Hence, by calculating the pullback of the polynomial $x_1y_2 - y_1x_2$, Corollary 4.7 gives

**Proposition 6.11.** *Define a map $\phi \in C^2_R(X;\mathbb{F}_q)$ from $(V \times \mathbb{Z}_p)^2$ to $\mathbb{F}_q$ by setting*

$$\phi\big((x_1, y_1, g),\ (x_2, y_2, h)\big) := g\big((x_1 - x_2)h + y_1 - y_2\big)x_2.$$

*Then the map $\phi$ is a 2-cocycle of the $\mathbb{Z}_p$-family of quandles $(V, \mathbb{Z}_p)$ with the single $X$-set.*

On the other hand, unfortunately the action of the group $G = \mathbb{Z}_p$ on $V$ is incompatible with 2-cocycles of the $G$-family with the *primitive* $X$-set. Actually, with respect to any polynomial $f$ in (10), the pullback $\varphi_3^*(f)$ vanishes. Furthermore, when $p = 2$, the remainder polynomial $(x_1 + y_1)(x_2 + y_2)(x_3 + y_3) + x_1x_2x_3 + y_1y_2y_3$ in Theorem 6.10 is the restriction on $G = \mathbb{Z}_p$ of a polynomial $\mathcal{G}_3^{111}$ in Theorem 6.5. In addition, if we construct group 3-cocycles similar to Proposition 6.8, then the results are the restricted 3-cocycles on $G$ arising from $SL(2;\mathbb{F}_p)$.

## A Appendix. Proofs of Propositions 3.1, 4.3 and 4.4

We will use the notation $\vec{g}_{\{i\}}$, $\vec{g}_{\{\lhd g_i\}} \in G^{n-1}$ and $\vec{g}_{\Bbbk_n,i} \in G^n$ defined in §3. Furthermore, for $i \leq n-1$, we set up the following two subsets of $\mathcal{K}_n$:

$$\mathcal{K}^+_{n,i} := \{(k_1,\dots,k_n) \in \{0,1\}^n \mid k_1 = 0,\ k_{i+1} = 1\},\quad \mathcal{K}^0_{n,i} := \{(k_1,\dots,k_n) \in \{0,1\}^n \mid k_1 = k_{i+1} = 0\}.$$

Note that $|\mathcal{K}^+_{n,i}| = |\mathcal{K}^0_{n,i}| = 2^{n-2}$ and $\mathcal{K}_n = \mathcal{K}^+_{n,i} \cup \mathcal{K}^0_{n,i}$ for any $i \leq n-1$.

*Proof of Proposition 3.1.* Our goal is to show the required equality $\partial_n^{\mathrm{gr}} \circ \varphi_n = \varphi_{n-1} \circ \partial_n^{R_U}$. We now start computing $\varphi_{n-1} \circ \partial_n^{R_U}$: for $\vec{g} \in G^n$ and $(U_1,\dots,U_n) \in M^n$,

$$\begin{aligned}
&\varphi_{n-1} \circ \partial_n^{R_U}(\vec{g}; U_1,\dots,U_n) = \\
&= \varphi_{n-1}\Big(\sum_{i\leq n-1}(-1)^i\big((\vec{g}_{\{i\}}; U_1,\dots,U_{i-1}, U_i{+}U_{i+1}, U_{i+2},\dots,U_n) - (\vec{g}_{\{\lhd i\}}; U_1{\cdot}g_{i+1},\dots,U_i{\cdot}g_{i+1}{+}U_{i+1}, U_{i+2},\dots,U_n)\big)\Big) \\
&\quad = \sum_{i\leq n-1}\Big(\sum_{\Bbbk_n\in\mathcal{K}^0_{n,i}}(-1)^{|\Bbbk_n|+i}\big(U_1\cdot\vec{g}_{\Bbbk_n,1},\dots,U_{i-1}\cdot\vec{g}_{\Bbbk_n,i-1},(U_i+U_{i+1})\cdot\vec{g}_{\Bbbk_n,i+1},U_{i+2}\cdot\vec{g}_{\Bbbk_n,i+2},\dots,U_n\cdot\vec{g}_{\Bbbk_n,n}\big) \\
&\qquad - \sum_{\Bbbk_n\in\mathcal{K}^+_{n,i}}(-1)^{|\Bbbk_n|+i}\big(U_1\cdot\vec{g}_{\Bbbk_n,1},\dots,U_{i-1}\cdot\vec{g}_{\Bbbk_n,i-1},(U_i\cdot g_{i+1}+U_{i+1})\cdot\vec{g}_{\Bbbk_n,i+1},U_{i+2}\cdot\vec{g}_{\Bbbk_n,i+2},\dots,U_n\cdot\vec{g}_{\Bbbk_n,n}\big)\Big)
\end{aligned}$$

$$= \sum_{i\leq n-1}\Big(\sum_{\mathbb{k}_n\in\mathcal{K}_n}(-1)^{|\mathbb{k}_n|+i}\big(U_1\cdot\vec{g}_{\mathbb{k}_n,1},\dots,U_{i-1}\cdot\vec{g}_{\mathbb{k}_n,i-1},(U_i\cdot g_{i+1}^{k_i}+U_{i+1})\cdot\vec{g}_{\mathbb{k}_n,i+1},U_{i+2}\cdot\vec{g}_{\mathbb{k}_n,i+2},\dots,U_n\cdot\vec{g}_{\mathbb{k}_n,n}\big)\Big). \quad (11)$$

On the other hand, we next compute $\partial_n^{\mathrm{gr}}\circ\varphi_n(\vec{g};U_1,\dots,U_n)$ as

$$\begin{aligned}
&= \partial_n^{\mathrm{gr}}\Big(\sum_{\mathbb{k}_n\in\mathcal{K}_n}(-1)^{|\mathbb{k}_n|}\big(U_1\cdot\vec{g}_{\mathbb{k}_n,1},\dots,U_n\cdot\vec{g}_{\mathbb{k}_n,n}\big)\Big)\\
&= \sum_{\mathbb{k}_n\in\mathcal{K}_n}(-1)^{|\mathbb{k}_n|}\Big(\big(U_2\cdot\vec{g}_{\mathbb{k}_n,2},\dots,U_n\cdot\vec{g}_{\mathbb{k}_n,n}\big)+(-1)^n\big(U_1\cdot\vec{g}_{\mathbb{k}_n,1},\dots,U_{n-1}\cdot\vec{g}_{\mathbb{k}_n,n-1}\big)\\
&\quad +\big(\sum_{i\leq n-1}(-1)^i\big(U_1\cdot\vec{g}_{\mathbb{k}_n,1},\dots,U_{i-1}\cdot\vec{g}_{\mathbb{k}_n,i-1},(U_i\cdot g_{i+1}^{k_i}+U_{i+1})\cdot\vec{g}_{\mathbb{k}_n,i+1},U_{i+2}\cdot\vec{g}_{\mathbb{k}_n,i+2},\dots,U_n\cdot\vec{g}_{\mathbb{k}_n,n}\big)\Big)\\
&= \big(\sum_{\mathbb{k}_n\in\mathcal{K}^+_{n,1}}+\sum_{\mathbb{k}_n\in\mathcal{K}^0_{n,1}}\big)(-1)^{|\mathbb{k}_n|+n}\big(U_2\cdot\vec{g}_{\mathbb{k}_n,2},\dots,U_n\cdot\vec{g}_{\mathbb{k}_n,n}\big)\\
&\quad +\big(\sum_{\mathbb{k}_n\in\mathcal{K}^+_{n,n-1}}+\sum_{\mathbb{k}_n\in\mathcal{K}^0_{n,n-1}}\big)(-1)^{|\mathbb{k}_n|}\big(U_1\cdot\vec{g}_{\mathbb{k}_n,1},\dots,U_{n-1}\cdot\vec{g}_{\mathbb{k}_n,n-1}\big)+\big(\varphi_{n-1}\circ\partial_n^{R_U}\big)(\vec{g};U_1,\dots,U_n), \quad (12)
\end{aligned}$$

where we use (11) for the third term in the last equality. For the proof it suffices to show that the first and second terms are zero. For this, we define the bijection $\mathcal{K}^0_{n,i}\longrightarrow\mathcal{K}^+_{n,i}$ by

$$(k_1,\dots,k_{i-1},0,k_{i+1},\dots,k_n)\longmapsto(k_1,\dots,k_{i-1},1,k_{i+1},\dots,k_n).$$

By considering the case $i=n-1$, we can formulate the second term as

$$\sum_{\mathbb{k}_n\in\mathcal{K}^0_{n,n-1}}(-1)^{|\mathbb{k}_n|}\big((U_1\cdot\vec{g}_{\mathbb{k}_n,1},\dots,U_{n-1}\cdot\vec{g}_{\mathbb{k}_n,n-1})-(U_1\cdot\vec{g}_{\mathbb{k}_n,1}\cdot g_n,\dots,U_{n-1}\cdot\vec{g}_{\mathbb{k}_n,n-1}\cdot g_n)\big)=0\in C^{\mathrm{gr}}_{n-1}(M;\mathbb{Z})_G.$$

Finally, it readily follows from the bijection with $i=1$ that the first term in (12) vanishes. □

*Proof of Proposition 4.3.* Recall that the subcomplex $C^{D_U}_{n,2}(X)$ is generated by the elements of the two sets defined in §4.2. Concerning the former set, by definitions, it is evident that

$$\varphi_n\big((\ \vec{g}\ ;U_1,\dots,U_{i-1},0,U_{i+1},\dots,U_n)\big)\subset C^N_n(M;\mathbb{Z})_G.$$

Next, we claim that, for the generators of the latter set are sent to zero via the map $\varphi_n$. Actually, for $2\leq i\leq n$, using a notation $\vec{gh}_{\mathbb{k}_n,j}:=g_{j+1}^{k_{j+1}}\cdots g_i^{k_i}h^{k_i}g_{i+1}^{k_{i+1}}\cdots g_n^{k_n}$ with $j<i$, we now compute

$$\begin{aligned}
&\varphi_n\big((g_1,\dots,g_n;U_1,\dots,U_n)+(g_i^{-1}g_1g_i,\dots,g_i^{-1}g_{i-1}g_i,h,g_{i+1},\dots,g_n;U_1g_i,\dots,U_{i-1}g_i,U_i,\dots,U_n)\big)\\
&=\big(\sum_{\mathbb{k}_n\in\mathcal{K}^+_{n,i-1}}+\sum_{\mathbb{k}_n\in\mathcal{K}^0_{n,i-1}}\big)(-1)^{|\mathbb{k}_n|}\big(U_1\cdot\vec{g}_{\mathbb{k}_n,1},\dots,U_n\cdot\vec{g}_{\mathbb{k}_n,n}\big)\\
&\quad+\sum_{\mathbb{k}_n\in\mathcal{K}^+_{n,i-1}}(-1)^{|\mathbb{k}_n|}\big(U_1\cdot\vec{gh}_{\mathbb{k}_n,1},\dots,U_{i-1}\cdot\vec{gh}_{\mathbb{k}_n,i-1},U_i\cdot\vec{g}_{\mathbb{k}_n,i},U_{i+1}\cdot\vec{g}_{\mathbb{k}_n,i+1},\dots,U_n\cdot\vec{g}_{\mathbb{k}_n,n}\big)\\
&\quad+\sum_{\mathbb{k}_n\in\mathcal{K}^+_{n,i-1}}(-1)^{|\mathbb{k}_n|+1}\big(U_1\cdot\vec{g}_{\mathbb{k}_n,1},\dots,U_i\cdot\vec{g}_{\mathbb{k}_n,i},U_{i+1}\cdot\vec{g}_{\mathbb{k}_n,i+1},\dots,U_n\cdot\vec{g}_{\mathbb{k}_n,n}\big)\big)\\
&=\big(\sum_{\mathbb{k}_n\in\mathcal{K}^+_{n,i-1}}+\sum_{\mathbb{k}_n\in\mathcal{K}^0_{n,i-1}}\big)(-1)^{|\mathbb{k}_n|}\big(U_1\cdot\vec{gh}_{\mathbb{k}_n,1},\dots,U_{i-1}\cdot\vec{gh}_{\mathbb{k}_n,i-1},U_i\cdot\vec{g}_{\mathbb{k}_n,i},U_{i+1}\cdot\vec{g}_{\mathbb{k}_n,i+1},\dots,U_n\cdot\vec{g}_{\mathbb{k}_n,n}\big)\\
&=\varphi_n(g_1,\dots,g_{i-1},g_ih,g_{i+1}\dots,g_n;U_1,\dots,U_n), \quad (13)
\end{aligned}$$

where the first and third equalities are derived from the definitions of $\varphi_n$ and of $\mathcal{K}_{n,i}=\mathcal{K}^0_{n,i}\sqcup\mathcal{K}^+_{n,i}$; further the second equality is obtained by offsetting the second term by the forth. In conclusion, all the generators of the subcomplex $C^{D_U}_{n,1}(X)$ are sent in $C^N_n(M;\mathbb{Z})_G$; thus the image $\varphi_n(C^{D_U}_{n,1}(X;\mathbb{Z}))$ is included in $C^{\mathrm{N}}_n(M;\mathbb{Z})_G$ as desired. □

*Proof of Proposition 4.4.* Remark that the homomorphism $\lambda : G \to A$ satisfies $\lambda(h^{-1}gh) = \lambda(g)$ for any $g, h \in G$. From the definitions of $\widetilde{\lambda} : C_n^{R_U}(X; A) \to A$ and of $\varphi_n$, it can be easily seen that if we replace $\varphi_n$ by $\varphi_{n,\lambda}$, then the equality $\partial_n^{\mathrm{gr}} \circ \varphi_{n,\lambda} = (\varphi_{n-1,\lambda}) \circ \partial_n^{R_U}$ and the equalities (13) hold in a similar fashion. Finally, as the remainder case $i = 1$, it suffices to show the following equality:

$$\begin{aligned}
&\varphi_{n,\lambda}\big((g_1, \dots, g_n; U_1, \dots, U_n) + (h, g_2, \dots, g_n; U_1 \dots, U_n)\big) \\
&= \varphi_n\big(g_1, \dots, g_n; U_1, \dots, U_n\big) \cdot \lambda(g_1) + \varphi_n\big(h, g_2, \dots, g_n; U_1, \dots, U_n\big) \cdot \lambda(h) \\
&= \varphi_n\big(g_1 h, g_2, \dots, g_n; U_1, \dots, U_n\big) \cdot \lambda(g_1 h) = \varphi_{n,\lambda}\big(g_1 h, g_2, \dots, g_n; U_1, \dots, U_n\big),
\end{aligned}$$

where we use the equality $\varphi_n\big(g_1, g_2, \dots, g_n; U_1, \dots, U_n\big) = \varphi_n\big(k, g_2, \dots, g_n; U_1, \dots, U_n\big)$ for any $k \in G$ □

## Acknowledgment

The author expresses his gratitude to J. Scott Carter for many discussions and teaching him about $G$-families of quandles, while he was visiting the University of South Alabama. He also thanks Masahide Iwakiri, Kengo Kishimoto, Kanako Oshiro and Masahico Saito for giving him useful comments. He is particularly grateful to Atsushi Ishii for valuable advice and discussions. This work was partially supported by “JSPS Research Fellowships for Young Scientists”.

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Research Institute for Mathematical Sciences, Kyoto University, Sakyo-ku, Kyoto, 606-8502, Japan

E-mail address: `nosaka@kurims.kyoto-u.ac.jp`