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Inversion formula for the growth function of a cancellative monoid

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Abstract

Let $(M, \deg)$ be a cancellative monoid M equipped with a discrete degree map $\deg : M \to \mathbb{R}_{\geq 0}$, and let $P_{M,\deg}(t) := \sum_{u \in M/\sim} t^{\deg(u)}$ be its generating series, called the *growth function* of M. We show the *inversion formula*

$$P_{M,\deg}(t) \cdot N_{M,\deg}(t) = 1$$

where the second factor of the formula is given by

$$N_{M,\deg}(t) := 1 + \sum_{T \in \mathrm{Tmcm}(M,I_0)} (-1)^{\#J_1 + \cdots + \#J_n - n + 1} \sum_{\Delta \in |T|} t^{\deg(\Delta)},$$

which we call the *skew-growth function* of M, and is a signed generating series of certain tree $\mathrm{Tmcm}(M, I_0)$ of towers of minimal common multiple sets constructed in M of the minimal generator system $I_0 := \min\{M/\sim \setminus \{1\}\}$.

If the monoid is $(M, \deg) = (\mathbb{Z}_{>0}, \log)$, we get Riemann's zeta function $P_{\mathbb{Z}_{\geq 0},\log}(\exp(-s)) = \zeta(s)$ as the growth function. Then the inversion formula turns out to be the Euler product formula $\zeta(s) \prod_{p \in I_0} (1 - p^{-s}) = 1$.

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[1]

1 Introduction

Let M be a monoid, i.e. a semigroup with the unit 1, and let $\deg : M \to \mathbb{R}_{\geq 0}$ be a discretely valued degree map on M (see §4). Then the (spherical) growth function of M with respect to *deg* is defined as the generating series:

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$$P_{M,\deg}(t) := \sum_{u \in M/\sim} t^{\deg(u)},$$

where $u \sim v$ for $u, v \in M$ means $u \mid_l v$ and $v \mid_l u$. Even though the definition of the growth function of a monoid looks a simple generalization of that for a group, not much work seems to be available except for some case studies and some general works on language ([A-N][B][C][G-P][I1][S1234][S-I]), and we know little about its general nature. The purpose of the present paper is to give a new approach to the growth function of a monoid by giving a presentation of its inversion function $\frac{1}{P_{M,I}(t)}$ by a certain *"skew generating function" of some common multiple sets in the monoid* M (which is not a group!).

Let us explain the idea of the inversion function in the most naive case studied in [A-N] and [S23]. Let M be a monoid generated by a finite set I with positive homogeneous relations. It admits naturally an integral degree map by giving weight 1 to each generator in I. Suppose, further, that M is cancellative and that any subset J of I admits either the least right common multiple Δ_J or no common multiple in M (typically the case for Artin monoids [B-S]). Then the inversion function of the growth function $P_{M,I}(t)$ is given by the formula:

$$N_{M,I}(t) := \sum_{J \subset I} (-1)^{\#J} t^{\deg(\Delta_J)}$$

where the summation index J runs over all subsets of I whose least right common multiple exists. [2]

This formula says that the inversion function $P_{M,I}(t)^{-1}$ of $P_{M,I}(t)$ for this class of monoids can be described by the datum $\{\Delta_I\}_{J \subset I}$ of the least common multiples for subsets J of I. However, in general, a monoid may not admit the least common multiple $\Delta_J = \mathrm{lcm}(J)$ for a given finite subset J of M. More precisely, even if there exist common right-multiples of J, there may not exist the unique least element among them: i.e. the unique minimal element among the common right-multiples of J with respect to the partial order defined by the division (from the left) relation may not exist in general. That is, the lack of the least common multiples in M may look an obstruction to generalize the above inversion formula to a wider class of monoids.

The purpose of the present paper is to resolve this problem as follows. By assuming the descending chain condition on M with respect to the partial order induced by left-divisibility relation, for any given finite set $J \subset M$, we are able to consider the set $\mathrm{mcm}(J)$ of *minimal common right-multiples* for J instead of considering the least common right-multiple Δ_J. However, still the datum $\{\mathrm{mcm}(J)\}_{J \subset I}$ is not sufficient to recover the inversion formula, since in general (as we shall see in examples), a subset J' of $\mathrm{mcm}(J)$ may have common right-multiples. So we need to consider the set $\mathrm{mcm}(J')$ for any subset J' of $\mathrm{mcm}(J)$. Then again, we may need to consider $\mathrm{mcm}(J'')$ for a subset $J'' \subset \mathrm{mcm}(J')$, and so on. Repeating this process, we are necessarily lead to consider a *tower*: a

[2] In this case, $N_{M,I}(t)$ is a polynomial. Actually, it is shown that the coefficients of $N_{M,I}(t)$ give the recursion relation on the sequence γ_n $(n \in \mathbb{Z}_{\geq 0})$ of coefficients of $P_{M,\deg}(t)$ (which is generalized in §5 of the present paper). Zero loci of the polynomial $N_{M,I}(t)$ plays a quite important role in the study of limit functions in [S1234], which motivated the present work.

finite sequence $I \supset J, J', J'', \cdots$ of subsets of M such that $J' \subset \mathrm{mcm}(J)$, $J'' \subset \mathrm{mcm}(J')$, $J''' \subset \mathrm{mcm}(J'')$, $\cdots$. It is convenient to put an *oriented graph structure* on the set $\mathrm{Tmcm}(M)$ of all towers, by putting an arrow from a tower T to its immediate successor T'. The graph decomposes into a disjoint union of rooted trees $\mathrm{Tmcm}(M) = \sqcup_{I \subset M} \mathrm{Tmcm}(M, I)$ where the label I, as the ground of towers, runs over all subsets of M satisfying the minimality condition $I = \min(I)$.

If M has a discrete degree map $\deg\colon M \to \mathbb{R}_{\geq 0}$,[3] we can define not only the growth function $P_{M,\deg}(t)$ as above, but also the "*skew generating function*"

$$N_{M,I}(t) := 1 + \sum_{T \in \mathrm{Tmcm}(M,I)} (-1)^{\#J_1 + \cdots + \#J_n - n + 1} \sum_{\Delta \in |T|} t^{\deg(\Delta)}$$

with respect to deg for the tree $\mathrm{Tmcm}(M, I)$ of all towers grounded over the label set I. Here, the summation index T runs over the vertex of the tree $\mathrm{Tmcm}(M, I)$, $J_1, \cdots, J_n$ are the stages of the tower T and $|T| := \mathrm{mcm}(J_n)$ is the set of the minimal common multiples on the top stage of the tower T (see §4).

In particular, for the label set $I_0 = \min(M/\sim \setminus \{1\})$ (the set of minimal elements of M with respect to the partial ordering induced by the left division relation), we set $N_{M,\deg}(t) := N_{M,I_0}(t)$. Then, as the main result of the present paper, we obtain an inversion formula (§5 Theorem):

$$P_{M,\deg}(t) \cdot N_{M,\deg}(t) = 1.$$

We stress that *the first factor $P_{M,\deg}(t)$ describes the growth nature of the monoid M and the second factor $N_{M,\deg}(t)$ describes the multiplicative nature of the monoid M, which are combined as in the formula.* In order to illustrate this nature of the formula, let us consider the monoid $M = \mathbb{Z}_{>0}$ with the ordinary product structure and take "log" to be the degree map (§5 Example 2). Then, by a change of variable $t = \exp(-s)$, we have $P_{\mathbb{Z}_{>0},\log}(\exp(-s)) = \zeta(s)$ (Riemann's zeta-function) and the inversion formula turns out to be the Euler product formula:

$$\zeta(s) \cdot \prod_{\substack{p:\text{prime} \\ \text{numbers}}} (1 - p^{-s}) = 1.$$

The construction of the paper is as follows. In §2, we fix basic concepts and notation on division theory on a monoid assuming a descending chain condition, where we introduce two operations "cm" (common multiple), "min" (minimal) and their composition "mcm=min·cm" (minimal common multiple) on subsets of a monoid. The concept of a tower of minimal common multiples, and the graph structure on the set of all towers are introduced in §3. Introducing a concept of a discrete degree map on a monoid, we introduce, in §4, the generating functions and the skew generating functions. The inversion formula is formulated and proven in §5. In §6, we describe presentations of monoids satisfying the assumptions of the main theorem, and, using it, we give an example where the growth function and the skew-growth function belong to the Novikov ring.

[3]The range of the degree map may not necessarily be contained in an arithmetic progression, but we assume only the discreteness (see §4 Definition). Therefore, the (skew-)growth functions are not necessarily power series in the usual sense, but they may better be regarded as Dirichlet series (see Remark 4.3 and §6 Example).

2 Minimal common multiples

We recall some basic concepts on monoids and fix notations.

Definition. 1. A semigroup M with the unit element 1 is called a *monoid*.

2. A monoid M is called *cancellative* if a relation $aub = avb$ for elements $a, b, u, v \in M$ implies a relation $u = v$.

3. For two elements u, v of M, we denote

$$u \mid_l v$$

if there exists an element $x \in M$ such that $v = ux$, and say that u divides v from the left, or, v is a multiple of u from the right.

4. A monoid M is called (left) *conical* if relations $u \mid_l v$ and $v \mid_l u$ for elements $u, v \in M$ imply $u = v$.

For a monoid M which may not be conical, one can define an equivalence relation on M by putting $u \sim v \Leftrightarrow_{def.} u \mid_l v$ and $v \mid_l u$. We shall often confuse $u \in M$ with its class in $M/\sim$. What is important is that by this equivalence relation, the left divisibility relation are preserved (i.e. if $u \sim u'$, $v \sim v'$ and $u \mid_l v$ then $u' \mid_l v'$). Thus, we obtain a partial order set structure "$\leq$" (or $\geq$) on the quotient set $M/\sim$ induced by the left division relation $\mid_l$, i.e. $u \mid_l v \Leftrightarrow u \leq v \Leftrightarrow v \geq u$. We shall denote by $u < v$ (or $v > u$) if $u \mid_l v$ and $u \not\sim v$.

In case when M is cancellative, the equivalence relation $\sim$ has much simpler interpretation as follows.

Assertion 2.1. *Let M be a cancellative monoid. Then, the set of right invertible elements of M coincides with the set of left invertible elements of M and they form the largest subgroup, denote by G, of M. Then, $u \sim v$ for $u, v \in M$ if and only if $uG = vG$, that is:*

$$M/\sim = M/G.$$

Proof. This is immediate from the definition and left to the reader. □

Remark 2.2. In the sequel of the present paper, the enumerations in the growth functions and skew growth functions are done in the level of the quotient set $M/\sim$ but not of M, since we use only the division relations among the elements in $M/\sim$ but not the product structure on M.

In fact, the product structure on M may not be preserved on the quotient $M/\sim$ (i.e. $u \sim u'$ and $v \sim v'$ does not imply $uv \sim u'v'$ in general). If the product structure is preserved (i.e. $\sim$ is a normal relation and G is a normal subgroup) then the quotient $M/\sim$ is automatically a conical monoid.

Above Remarks and Assertion all together indicate that the growth functions and their inversions, which we are studying in the present paper, live essentially in the world of non invertible monoids (but not of the groups) where only the poset structure on $M/\sim$ coming from division relation equipped with the left M-action: $M \times M/\sim \to M/\sim$ plays the role. This causes a question: what is the relationship between the growth function of a monoid M and that of the group $G = M_M$ obtained by localizing the monoid?

In the rest of the paper, we assume the following chain condition on $M/\sim$.

Descending chain condition. There does not exist an infinite strictly decreasing sequence $u_1 > u_2 > u_3 > \cdots$ of elements in $M/\sim$.

We consider two operations on the set of subsets of $M/\sim$: *common multiple set* and *minimal set*. For a subset J of $M/\sim$ (which may not necessarily be finite), put

$$\begin{array}{ll} \mathrm{cm}(J) & := \{\ u \in M/\sim \ \mid \quad j \mid_l u \quad \forall j \in J\ \} \\ \min(J) & := \{\quad u \in J \quad \mid \nexists v \in J \text{ s.t. } v < u\ \}, \end{array}$$

and their composition: the *set of minimal common multiples* of the set J by

$$\mathrm{mcm}(J) := \min(\mathrm{cm}(J)).$$

Actually, $\mathrm{cm}(J)$ may be the empty set. However, due to the **Descending chain condition**, if $J \neq \emptyset$ then $\min(J) \neq \emptyset$. More precisely, we have the following fact.

Fact. *For any $u \in J$ there exists an element $v \in \min(J)$ such that $v \mid_l u$ holds.*

3 Tower of minimal common multiples

Let M be, as in §2, a monoid satisfying **Descending chain condition**. In this section, we introduce towers of minimal common multiples for M.

Definition. A *tower* of M of *height* $n \in \mathbb{Z}_{\geq 0}$ is a sequence

$$T := (I_0, J_1, J_2, \cdots, J_n)$$

of subsets of $M/\sim \setminus \{1\}$ satisfying the following:

i) $I_0 \neq \emptyset$ and $I_0 = \min(I_0)$.
ii) $\mathrm{mcm}(J_k) \neq \emptyset$ and we put $I_k := \mathrm{mcm}(J_k)$ for $k = 1, \cdots, n$.
iii) $J_k \subset I_{k-1}$ such that $1 < \#J_k$ for $k = 1, \cdots, n$.

We call I_0 the *ground* of the tower T, J_k and I_k the kth *stage* and the *set of minimal common multiples on the kth stage* of the tower T, respectively. In particular, the set $\mathrm{mcm}(J_n)$ of minimal common multiples on the top stage is denoted by $|T| := I_n$. The set of all towers of M shall be denote by $\mathrm{Tmcm}(M)$.

Definition. We put an *oriented graph structure on* $\mathrm{Tmcm}(M)$ as follows.

i) The set of vertices is equal to the set $\mathrm{Tmcm}(M)$ of all towers.

ii) An oriented edge from a tower T of height n to a tower T' of height n' is given if and only if $n' = n + 1$ with $T = (I_0, J_1, \cdots, J_n)$ and $T' = (I_0, J_1, \cdots, J_{n+1})$ (or, we shall write $T' = (T, J_{n+1})$). That is, T' is a tower obtained by just adding one stage above to the tower T.

We denoted again by $\mathrm{Tmcm}(M)$ the set equipped with this graph structure.

Since any tower of height $n \geq 1$ has exactly one immediate predecessor and any tower of height 0 has no predecessor, each connected component of the graph $\mathrm{Tmcm}(M)$ is a rooted tree whose root is given by a tower of height 0 of the form $T_0 = (I_0)$ for a ground set $I_0 \subset M$ with $I_0 = \min(I_0)$. Denoting the tree (component) by $\mathrm{Tmcm}(M, I_0)$, we have the disjoint decomposition:

$$\mathrm{Tmcm}(M) = \bigsqcup_{\substack{I_0 \in\ 2^M \setminus \{\emptyset\} \\ I_0 = \min(I_0)}} \mathrm{Tmcm}(M, I_0).$$

Example. 1. Let M be a free monoid. Then, any tower of M is of height 0.

2. Let M be a monoid, which admits least common right-multiples, that is, for any subset J of M, the set $\mathrm{mcm}(J)$ is either empty or consisting of a single element (in $M/\sim$). Then, any tower has height at most 1. (*Proof.* For any tower T of height ≥ 1, $I_1 = \mathrm{mcm}(J_1) \neq \emptyset$ consists of a single element so that the Definition iii) of a tower prohibits to have J_2.)

Thus, each component $\mathrm{Tmcm}(M, I_0)$ is star-shaped, consisting of the vertex (I_0) (the ground) and the vertices of the form (I_0, J_1) where J_1 is a finite subset of I_0 having more than two elements which have the common multiple.

4 Generating functions $P_{M,I_0}(t)$ and $N_{M,I_0}(t)$

From present section, we fix a discrete degree map deg defined on a monoid M. Using it, we introduce a growth function $P_{M,I_0}(t)$ and a skew growth function $N_{M,I_0}(t)$ labeled by a set $I_0 \subset M$ satisfying $I_0 = \min(I_0)$. In particular, if the label set I_0 is the set $\min\{\mathrm{M}/\sim \setminus \{1\}\}$ of all minimal elements of M, we call them the growth function and the skew-growth function of $(M, \deg)$ and denote them by $P_{M,\deg}(t)$ and $N_{M,\deg}(t)$, respectively.

Definition. A *discrete degree map* on a monoid M is a map

$$\deg\ :\ M \quad \longrightarrow \quad \mathbb{R}_{\geq 0}$$

such that i) $\deg(u) = 0$ if and only if $u \sim 1$,
ii) $\deg(uv) = \deg(u) + \deg(v)$ for any $u, v \in M$,
iii) $\#\{u \in M/\sim \mid\ \deg(u) \leq r\} < \infty$ for any $r \in \mathbb{R}_{>0}$.

If $u \,|_l\, v$ then ii) implies $\deg(u) \leq \deg(v)$, and, hence, if $u \sim v$ then $\deg(u) = \deg(v)$ so that deg induces a map $M/\sim \to \mathbb{R}_{\geq 0}$, denoted by the same notation "deg". Thus, the condition iii) has a meaning. The iii), in particular, implies that the range $\deg(M)$ is a discrete subset of $\mathbb{R}$ and

$$d_{min} := \inf\{\ \deg(u)\ \mid\ u \in (M/\sim \setminus \{1\})\ \}$$

is a positive constant. This implies that the additive sub-semigroup of $\mathbb{R}_{\geq 0}$ generated by the image set $\deg(M)$ is also a discrete subset of $\mathbb{R}_{\geq 0}$.

For any subset A of $M/\sim$, put $\deg(A) := \inf\{\deg(u) \mid u \in A\}$. In particular, we call

$$\deg(|T|) := \min\{\deg(\Delta) \mid \Delta \in \mathrm{mcm}(J_n)\}$$

the *minimal degree of the tower* T of height n.

Assertion 4.1. *Let T be a tower of height n, then we have*

$$\deg(|T|) \geq (n+1)d_{min}.$$

Proof. Put $T = (I_0, J_1, \cdots, J_n)$. Using the condition §3 Def. iii) for a tower, we see that

$$\begin{array}{rcllll} \deg(|T|) & = & \deg(I_n) & \geq & \deg(J_n) + d_{min} & (\S3 \text{ Def.iii})) \\ & \geq & \deg(I_{n-1}) + d_{min} & \geq & \deg(J_{n-1}) + 2d_{min} & (\S3 \text{ Def.iii})) \\ & \geq & \cdots & \geq & \cdots & (\cdots) \\ & \geq & \deg(I_1) + (n-1)d_{min} & \geq & \deg(J_1) + nd_{min} & (\S3 \text{ Def.i})) \\ & \geq & \deg(I_0) + nd_{min} & \geq & (n+1)d_{min} & (4.2). \end{array}$$

□

Remark 4.2. i) The existence of a discrete degree map implies automatically that the monoid M satisfies the descending chain condition.

ii) The existence of a discrete degree map implies automatically $\#(J_k) < \infty$ for any tower T and $k \leq$ height of T, since $\mathrm{mcm}(J_k) \neq \emptyset$ is assumed.

Next, associated with any algebra A (we shall use only the case $A = \mathbb{Z}$ in the present paper), let us introduce an algebra R_A over A equipped with a formal topology (see Remark 4.3 below). Set a topological A-module by

$$R_A := \Big\{ \sum_{n=0}^{\infty} a_n t^{d_n} \Big| \begin{array}{l} a_n \in A \ (\forall n \in \mathbb{Z}_{\geq 0}), \text{ and } \{d_n\}_{n \in \mathbb{Z}_{\geq 0}} \text{ is} \\ \text{a sequence in } \mathbb{R}_{\geq 0} \text{ divergent to } +\infty. \end{array} \Big\}$$

where the system of (formal) neighbourhoods of $0 \in R_A$ are given by

$$t^d R_A := \Big\{ \sum_{n=0}^{\infty} a_n t^{d_n} \in R_A \Big| \min\{d_n \mid n \in \mathbb{Z}_{\geq 0} \text{ and } a_n \neq 0\} \geq d \Big\}$$

for $d \in \mathbb{R}_{\geq 0}$. Then, the product w.r.t. this topology is well-defined by setting

$$\sum_{d \in \mathbb{R}_{\geq 0}} \Big(\sum_{\substack{n,m \in \mathbb{Z}_{\geq 0} \\ d_n + e_m = d}} a_n b_m \Big) t^d := \Big(\sum_{n=0}^{\infty} a_n t^{d_n} \Big) \Big(\sum_{m=0}^{\infty} b_m t^{e_m} \Big)$$

so that R_A becomes an A-algebra.

Remark 4.3. In some literature, from a topological view point, the ring R_A is called a *Novikov ring*. However, for our later applications, it is convenient to regard it as the ring of *formal Dirichlet series* (see [H-R]) as we explain below.

For any $f(t) \in R_A$, by a change $t = \exp(-s)$ of the variable, we consider a formal series

$$f(\exp(-s)) = \sum_{n=0}^{\infty} a_n \exp(-d_n s).$$

Assume $a_n \in \mathbb{C}$ $(n \in \mathbb{Z}_{\geq 0})$. The expression is called a Dirichlet series (of exponential type) for $\lim_{n \to \infty} d_n = +\infty$. If the series converges (absolutely) at $s_0 \in \mathbb{C}$, then it also converges (absolutely) for all $\{s \in \mathbb{C} \mid \Re(s) > \Re(s_0)\}$ and defines a holomorphic function on that half plane. The product in the ring $R_{\mathbb{C}}$ is compatible with the product as holomorphic function (if both factors are absolutely convergent). The holomorphic function may extends meromorphically (including branches) to a region covering $\mathbb{C}$. For our application ([S1,2,3,4]), we are interested in the locations of the poles and the zeros of these extended functions.

We return to the construction of growth and skew-growth functions, which, due to the discreteness of the degree map, belong to the ring $R_{\mathbb{Z}}$ of integral Dirichlet series.

1. Growth function $P_{M,\deg}(t)$ of $(M,\deg)$.

For any subset I_0 of $M/\sim\,\setminus\{1\}$ with $I_0=\min(I_0)$, consider the submonoid $M(I_0)$ of M generated by I_0, that is, the smallest submonoid N of M satisfying

i) an element of M whose equivalence class belongs to I_0 belongs to N,

ii) if $u,v\in M$ satisfies $u\in N$ and $u\sim v$ then $v\in N$.

Then, in this new monoid $M(I_0)$, we can consider the division theory and define the equivalence relation as in §2. Then, $M(I_0)/\sim$ is naturally embedded into $M/\sim$. Thus, the restriction of the degree map on M to $M(I_0)$ induces also a degree map which we shall denote again by deg.

Then, the generating function of the degree map on $M(I_0)$, which sometimes is also called a growth function labeled by I_0, is defined as follows.

$$P_{M,I_0}(t):=\sum_{u\in M(I_0)/\sim} t^{\deg(u)}=\sum_{d\in\mathbb{R}_{\geq 0}}\#((M(I_0)/\sim)_d)\ t^d.$$

Here, we put

$$(M(I_0)/\sim)_d:=\{u\in M(I_0)/\sim\ \mid\ \deg(u)=d\}$$

for any real number $d\in\mathbb{R}_{\geq 0}$, which is a finite set due to the assumption on deg, and therefore $P_{M,I_0}(t)\in R_{\mathbb{Z}}$. In particular, by choosing I_0 to be the minimal generating set $I_0=\min(M/\sim\setminus\{1\})$ of M ($\Leftrightarrow M(I_0)=M$), we define the *growth function of the monoid M with respect to the degree map* deg by

$$P_{M,\deg}(t):=\sum_{u\in M/\sim} t^{\deg(u)}=\sum_{d\in\mathbb{R}_{\geq 0}}\#((M/\sim)_d)\ t^d.$$

2. Skew growth function $N_{M,\deg}(t)$ of $(M,\deg)$.

For any tower T of height n, consider the sum

$$\sum_{\Delta\in|T|} t^{\deg(\Delta)}.$$

It is well-defined in the ring $R_{\mathbb{Z}}$ for any tower T due to the the finiteness iii) on the degree map, and it belongs to the ideal $t^{\deg(|T|)}R_{\mathbb{Z}}$.

Assertion 4.4. *To any non-empty set $I_0\subset M/\sim\,\setminus\{1\}$ with $I_0=\min(I_0)$, we define a skew generating function labeled by I_0 by*

$$N_{M,I_0}(t):=1+\sum_{T\in\mathrm{Tmcm}(M,I_0)}(-1)^{\#J_1+\cdots+\#J_n-n+1}\sum_{\Delta\in|T|}t^{\deg(\Delta)}$$

where we recall that $\mathrm{Tmcm}(M,I_0)$ is the connected component of the graph $\mathrm{Tmcm}(M)$ (consisting of all towers whose ground is the set I_0) and the summation index T is a tower of the form $(I_0,J_1,\cdots,J_n)$ on the ground I_0. Then, the formal sum is convergent in the ring $R_{\mathbb{Z}}$.

Proof. In order to show that the sum with respect to the running index $T \in \mathrm{Tmcm}(M, I_0)$ is convergent in the ring $R_{\mathbb{Z}}$, it is sufficient to show that for any positive number $r \in \mathbb{R}_{\geq 0}$, the set $\{T \in \mathrm{Tmcm}(M, I_0) \mid \deg(|T|) \leq r\}$ is finite.

a) We first recall the inequality $\deg(|T|) \geq (n+1)d_{min}$ for any tower T of height n. Thus the height of a tower whose minimal degree is bounded by a constant r is bounded by $(r/d_{min}) - 1$.

b) Next, let us show by induction on $n \in \mathbb{Z}_{\geq 0}$ that the number of towers in $\#\{T \in \mathrm{Tmcm}(M, I_0) \mid$ the height of T is equal to n & $\deg(|T|) \leq r\} < \infty$. Since there is only one tower $T = (I_0)$ of height 0, the first induction hypothesis is satisfied. Assume the result for n. Any tower T' of height $n+1$ has the form (T, J_{n+1}) for the predecessor T of height n. The requirement $r \geq \deg(|T'|)$ implies the boundedness $r - d_{min} \geq \deg(|T|)$. By induction hypothesis, the number of such T is finite. Therefore, it is sufficient to show that for any tower $T \in \mathrm{Tmcm}(M, I_0)$ of height n, the number of its successors T' such that $r \geq \deg(|T'|)$ is finite. The choice of T' is determined by the choice of the subset J_{n+1} of I_n, where we have the equality: $\deg(|T'|) = \min\{\deg(\Delta) \mid \Delta \in |T'| = \mathrm{mcm}(J_{n+1})\}$. Since, for any $\Delta \in I_{n+1} := \mathrm{mcm}(J_{n+1})$ and any $j \in J_{n+1}$, one has $j < \Delta$ (*proof.* That $j \leq \Delta$ is obvious by $\Delta \in \mathrm{cm}(J_{n+1})$. But $\Delta \leq j$ is impossible, if else, then any element of J_{n+1} is less of equal than j which contradicts that J_{n+1} consists of more than two elements (§3 Def. ii)) so that $\deg(\Delta) \geq d_{min} + \max\{\deg(j) \mid j \in J_{n+1}\}$, and, hence, $\deg(j) \leq r - d_{min}$. This means that J_{n+1} is a subset of $I'_n := \{j \in I_n \mid \deg(j) \leq r - d_{min}\}$. However, by the discreteness condition (§4 Def. iii)) on deg, the number of elements of I'_n is finite. This means the freedom of the choice of J_n is also finite.

Combining a) and b), the proof of Assertion is completed. □

In particular, we define the *skew-growth function of* $(M, \deg)$ by

$$N_{M,\deg}(t) := N_{M,\min(M/\sim\setminus\{1\})}(t).$$

5 Inversion formula for the growth function

The main result of the present paper is formulated in the following theorem.

Theorem. *Let M be a cancellative monoid equipped with a discrete degree map. Then we have the inversion formula in the ring $R_{\mathbb{Z}}$:*

$$P_{M,\deg}(t) \cdot N_{M,\deg}(t) = 1.$$

Proof. For $d \in \mathbb{R}_{\geq 0}$, put $m_d := \#(\{u \in M/\sim \mid \deg(u) = d\})$ so that $P_{M,\deg}(t) = \sum_{d \in \mathbb{R}_{\geq 0}} m_d t^d$. Then we have to show the "infinite recursion" relation

$$*: \qquad m_d \ + \sum_{T \in \mathrm{Tmcm}(M, I_0)} (-1)^{\#J_1 + \cdots + \#J_n - n + 1} \sum_{\Delta \in |T|} m_{d - \deg(\Delta)} = 0$$

for all $d \in \mathbb{R}_{>0}$ (which is an analogous of Möius inversion formula).

For any subset I and J of $M/\sim$, let us introduce two sets:

$$\begin{array}{rcl} M^I & := & \{u \in M/\sim \mid \exists \Delta \in I \text{ s.t. } \Delta|_l u\} \\ M_J & := & \{u \in M/\sim \mid \forall \Delta \in J \text{ s.t. } \Delta|_l u\}. \end{array}$$

[4] Note that we have an obvious relation:

$$M_J = M^{\mathrm{mcm}(J)} \quad \text{and} \quad M^I = \cup_{J\in 2^I\setminus\{\emptyset\}} M_J$$

for any subset J and I of $M/\sim$. In particular, we have $M^\emptyset = \emptyset$ and $M_\emptyset = M/\sim$.

We also put $M_d^I := M_d \cap M^I$ and $M_{d,J} := M_d \cap M_J$.

Since $M_d = M_d^{I_0}$, we express $M_d = \cup_{\Delta\in I_0} M_{d,\{\Delta\}} = \cup_{J_1\in 2^{I_0}\setminus\{\emptyset\}} M_{d,J_1}$, where we have $M_{d,J_1} = \cap_{\Delta\in J_1} M_{d,\{\Delta\}}$ for any $J_1 \neq \emptyset \subset I_0$, and M_{d,J_1} may be an empty set. We remark that $\{J_1 \in 2^{I_0} \mid M_{d,J_1} \neq \emptyset\}$ is a finite set, since J_1 should be a subset of $\{\Delta \in I_0 \mid \deg(\Delta) \leq d\}$ which is a finite set due to the discreteness of the degree map deg. Then, the following is a finite sum and has a meaning.

$$\#(M_d) = \sum_{J_1\in 2^{I_0}\setminus\{\emptyset\}} (-1)^{\#J_1-1}\#(M_{d,J_1}).$$

Since we have $M_{d,J_1} = M_d^{\mathrm{mcm}(J_1)}$, by putting $I_1 := \mathrm{mcm}(J_1)$, we have $M_{d,J_1} = \cup_{\Delta\in I_1} M_{d,\{\Delta\}}$, and again decompose

$$M_{d,J_1} = \cup_{J_2\in 2^{I_1}\setminus\{\emptyset\}} M_{d,J_1,J_2},$$

where $M_{d,J_1,J_2} := M_{d,J_1} \cap M_{J_2} (= M_{d,J_2})$. Therefore

$$\#(M_{d,J_1}) = \sum_{J_2\in 2^{I_1}\setminus\{\emptyset\}} (-1)^{\#J_2-1}\#(M_{d,J_1,J_2}).$$

Repeating the same process n-times, we obtain a formula

$$\bigstar:\ \#(M_d) = \sum_{J_1\in 2^{I_0}\setminus\{\emptyset\}} \sum_{J_2\in 2^{I_1}\setminus\{\emptyset\}} \cdots \sum_{J_n\in 2^{I_{n-1}}\setminus\{\emptyset\}} (-1)^{\#J_1+\#J_2+\cdots+\#J_n-n}\#(M_{d,J_1,J_2,\cdots,J_n})$$

where we put $I_k = \mathrm{mcm}(J_k)$ $(k = 1, \cdots, n-1)$. Some of $M_{d,J_1,J_2,\cdots,J_n} = M_d \cap M_{J_1} \cap \cdots \cap M_{J_n} (= M_{d,J_n})$ may be an empty set.

Assertion 5.1. *If a running index J_k of the formula $\bigstar$ for some $1 \leq k \leq n$ consists only of a single element, say Δ, then we have $J_k = J_{k+1} = \cdots = J_n = \{\Delta\}$ and $M_{d,J_1,\cdots,J_k} = M_{d,J_1,\cdots,J_k,J_{k+1}} = \cdots = M_{d,J_1,\cdots,J_k,\cdots,J_n} = \hat{\Delta}\cdot M_{d-\deg(\Delta)}$, where $\hat{\Delta}$ is a representative in M of the equivalence class $\Delta \in M/\sim$.*

ii) If $n \geq d/d_{min}$, then, for any running index $(J_1, \cdots, J_n)$ of the formula $\bigstar$, either the set J_n consists of a single element, or the set $M_{d,J_1,J_2,\cdots,J_n}$ is empty.

Proof. i) The fact $J_k = \{\Delta\}$ implies that $M_{d,J_1,\cdots,J_k} = \hat{\Delta}\cdot M_{d-\deg(\Delta)}$. On the other hand, we have $I_k := \mathrm{mcm}(J_k) = \{\Delta\}$. Therefore, if $k < n$, then the only possible choice of J_{k+1} is the only non-empty subset of I_k, i.e. $\{\Delta\}$.

ii) If $\#J_n \geq 2$. Then, due to above i), we should have all of $J_1, \cdots, J_n$ must have the cardinality greater or equal than 2. That is, for any $j \in J_{k+1} \subset$

[4]The notation M_J is confusing with M_d given in the previous section. However, we shall only use the suffixes d and J (or $J_1, J_2, \cdots$) so that they should be distinguished.

$\mathrm{mcm}(J_k)$, we have $\deg(j) \geq \max\{\deg(\Delta) \mid \Delta \in J_k\} + d_{min} \geq \deg(J_k) + d_{min}$ for $k = 1, \cdots, n-1$ so that we have $\deg(J_n) \geq \deg(J_{n-1}) + d_{min} \geq \cdots \geq \deg(J_1) + (n-1)d_{min} \geq nd_{min}$. Therefore, any element of $M_{d,J_1,\cdots,J_n} = M_{d,J_n}$ (if it exists) has degree at least $\deg(J_n) \geq nd_{min} > d$, which is impossible. □

Assertion 5.2. *Let $n \in \mathbb{Z}$ such that $n \geq d/d_{min}$. Then we have a bijection*

$$\begin{array}{rl} & \{(\Delta, T) \in (M/\sim \times \mathrm{Tmcm}(I_0)) \mid \Delta \in |T|, \deg(\Delta) \leq d\} \\ \simeq & \{(J_1, \cdots, J_n) \mid J_1 \subset I_0,\ J_k \subset \mathrm{mcm}(J_{k-1})\ (2 \leq k \leq n),\ M_{d,J_1,\cdots,J_n} \neq \emptyset\}. \end{array}$$

If $(\Delta, T = (I_0, J_1, \cdots, J_{n_0}))$ and $(J_1, \cdots, J_n)$ are corresponding to each other, then we have the equality

$$(-1)^{\#J_1+\#J_2+\cdots+\#J_{n_0}-n_0} m_{d-\deg(\Delta)} = (-1)^{\#J_1+\#J_2+\cdots+\#J_n-n} \#(M_{d,J_1,J_2,\cdots,J_n}).$$

Proof. a) Let (Δ, T) be an element in LHS, and let $T = (I_0, J_1, \cdots, J_{n_0})$. The condition implies $d \geq \deg(\Delta) \geq \deg(|T|) \geq (n_0+1)d_{min}$, and hence $n_0 < d/d_{min} \leq n$. Then $(J_1, \cdots, J_{n_0}, \{\Delta\}, \cdots, \{\Delta\})$ belongs to RHS.

b) Let $(J_1, \cdots, J_n)$ be an element in RHS such that $M_{d,J_1,\cdots,J_n} \neq \emptyset$. Due to Assertion ii), in such case, we have $\#(J_n) = 1$. For such index, put $n_0 + 1 = \inf\{1 \leq k \leq n \mid \#J_k = 1\}$ for some $0 \leq n_0 < n$. Then $J_{n_0+1} = \{\Delta\}$ and $\Delta \in \mathrm{mcm}(J_{n_0})$. Since $\#(J_{n_0}) \geq 2$, $T := (I_0, J_1, \cdots, J_{n_0})$ is a tower such that (Δ, T) belongs to LHS.

It is clear that a) and b) are inverse to each other. Suppose $(J_1, \cdots, J_n) \leftrightarrow (\Delta, T)$, then we have the equality

$$\begin{array}{rl} & (-1)^{\#J_1+\#J_2+\cdots+\#J_n-n} \#(M_{d,J_1,J_2,\cdots,J_n}) \\ = & (-1)^{\#J_1+\#J_2+\cdots+\#J_{n_0+1}-(n_0+1)} \#(M_{d,J_1,J_2,\cdots,J_{n_0+1}}) \\ = & (-1)^{\#J_1+\#J_2+\cdots+\#J_{n_0}-n_0} \#(M_{d,J_{n_0+1}}) \\ = & (-1)^{\#J_1+\#J_2+\cdots+\#J_{n_0}-n_0} \#(\hat{\Delta} \cdot M_{d-\deg(\Delta)}). \end{array}$$

The cacellativity of the monoid M implies the bijection $\hat{\Delta} \cdot M_{d-\deg(\Delta)} \simeq M_{d-\deg(\Delta)}$. Here, the bijection $\hat{\Delta} \cdot u \leftrightarrow u$ depends on a choice of $\hat{\Delta}$. However it does not effect on the enumeration of the cardinality of both sides, so that the last term is equal to $(-1)^{\#J_1+\#J_2+\cdots+\#J_{n_0}-n_0} \#(M_{d-\deg(\Delta)})$. □

Last Assertion shows that the formula ★ is the same as the recursion formula $*$. This completes the proof of the recursion formula and of Theorem. □

Corollary 5.3. For any subset I_0 of $M/\sim \setminus\{1\}$ with $I_0 = \min(I_0)$, we have

$$P_{M,I_0}(t) \cdot N_{M,I_0}(t) = 1.$$

Remark 5.4. If the both Dirichlet series $P_{M,I_0}(\exp(-s))$ and $N_{M,I_0}(\exp(-s))$ converge absolutely in some half plane $\{s \in \mathbb{C} \mid \Re(s) > c\}$ for some $c \in \mathbb{R}$, then the inversion formula gives a functional equation $P_{M,I_0}(\exp(-s))N_{M,I_0}(\exp(-s)) = 1$ on the half plane. This implies, in particular, that they do have neither zeros nor poles on the half plane. Let us denote by the same $P_{M,I_0}(\exp(-s))$ and

$N_{M,I_0}(\exp(-s))$ their meromorphic continuations (including algebraic branches), respectively. Obviously, the functional equation extends meromorphically, i.e. one meromorphic continuation determines the other as the inverse function, where the poles and the zeros of them interchange to each other. Motivated by a trace formula for thermo-dynamical limit functions ([S1-4]), we are interested in zeros of $N_{M,I_0}(\exp(-s))$. In particular, we ask the followings (c.f. [T]).

Conjecture 5.5. If the Dirichlet series $P_{M,I_0}(\exp(-s))$ converges absolutely on some half plane $\{s\in\mathbb{C} \mid \Re(s)>\sigma\}$ for some $\sigma\in\mathbb{R}$, then $N_{M,I_0}(\exp(-s))$ also converges absolutely on the same half plane.

Problem 5.6. Show following **Assertion** for a suitable class (which should yet be clarified) of a pair $(M,\deg)$ of a monoid with a discrete deree map:

Assertion. *Let $\sigma_0\in\mathbb{R}$ be the minimal of σ's in* **Conjecture 5.5**. *Then the Dirichlet series $N_{M,\deg}(\exp(-s))$ continues holomorphically on a open neighborhood of the point $\sigma_0\in\mathbb{C}$ where the function takes the value 0 at σ_0.* [5]

What is the meaning of the order of zeros of $N_{M,I_0}(\exp(-s))$ at σ_0?

All the following Examples 1. i),ii),iii),iv),v), 2. and 3., satisfy positively above Conjecture 5.5 and belong to the class in Problem 5.6. In next section §6, we shall give a series of examples where Conjecture 5.5 is satisfied by all examples, but they may not always belong to the class in Problem 5.6.

Example. 1. For a monoid M (cancellative and satisfying the descending chain condition) and a subset $I_0\subset(M/\sim\setminus\{1\})$ with $I_0=\min(I_0)$, set

$$h(M,I_0) := \max\{\text{height of } T\in \mathrm{Tmcm}(M,I_0)\}.$$

i) It is clear that M is a free monoid if and only if $h(M,I_0)=0$ for any I_0.
ii) An Artin monoid (or, more generally, a monoid, any of whose finite subsets admits the least common multiple) has $h(M,I_0)\leq 1$ (see following 2.).
iii) Ishibe [I2] gave an example of $N_{M,\deg}(t)$ with $h(M,\deg)=2$.
iv) In general, if $h(M,I_0)<\infty$, then $N_{M,I_0}(t)$ is a polynomial in t.
v) Ishibe [I2] has determined explicitly $N_{M,\deg}(t)$ for monoid of type $\mathrm{B_{ii}}$ ([I1],[S-I]) and certain Zariski-van Kampen monoids, which have $h(M,\deg)=\infty$.

2. Let $M:=\mathbb{Z}_{>0}$ with the ordinary product structure. We remark that the unit group of this case is a trivial group so that $\mathbb{Z}_{>0}/\sim=\mathbb{Z}_{>0}$. As for the degree map, we take the logarithm function $\deg(n):=\log(n)$ for $n\in\mathbb{Z}_{>0}$, which is discrete since $\lim_{n\to\infty}\log(n)=\infty$. Then, by a change $t=\exp(-s)$ of variable, the growth function is equal to Riemann's zeta-function (in the region $\Re(s)>1$)

$$P_{\mathbb{Z}_{>0},\log}(t) \;:=\; \sum_{n=1}^{\infty} t^{\log(n)} = \sum_{n=1}^{\infty} n^{\log(t)} = \sum_{n=1}^{\infty} n^{-s} \;=\; \zeta(s),$$

[5] This condition on the class of $(M,\deg)$ comes from a view point of the limit functions [S1]. One would like to think more stronger class of $(M,\deg)$, where $N_{M,\deg}(\exp(-s))$ continues holomorphically on a open neighbourhood of the axis $\{\Re(s)=\sigma_0\}\subset\mathbb{C}$, where the order of zeros at $s=\sigma_0$ is the maximal among all zeros on $\Re(s)=\sigma_0$.

which is welknow to extend meromorphically to the whole plane $\mathbb{C}$ with a simple pole at $s=1$.

On the other hand, the skew-growth function for $(\mathbb{Z}_{>0}, \log)$ is determined as follows. The ground set is given by $I_0 := \min(\mathbb{Z}_{>0} \setminus \{1\}) = \{\text{prime numbers}\}$. Since there exists always the least common multiple for any finite subset of I_0, all non-trivial towers have height 1. Thus the skew-growth function is given by

$$N_{\mathbb{Z}_{>0},\log}(t) \ := \sum_{\substack{J:\text{finite set of} \\ \text{prime numbers}}} (-1)^{\#J} \prod_{p\in J} t^{\log(p)} \ = \prod_{p:\text{prime numbers}} (1-p^{-s}).$$

Thus, the inversion formula turns out to be the Euler product formula:

$$\zeta(s) \ \cdot \prod_{p:\text{prime}} (1-p^{-s}) = 1$$

for the Riemann zeta function.

3. We give an example whose towers can have arbitrarily large heights.

Consider the monoid $M := \langle a, b \mid a^2 = b^2,\ ab = ba \rangle_{mo}$ (for a precise definition of this notation, see the next section §6). The condition $\deg(a) = \deg(b) = 1$ uniquely determine a degree map on M. Put $c := a^2 = b^2$ in M. Then, it is easy to see that any element of M is uniquely expressed in the form

$$a^{\varepsilon_1} b^{\varepsilon_2} c^n \quad \text{for some } \varepsilon_1, \varepsilon_2 \in \{0,1\},\ n \in \mathbb{Z}_{\geq 0}.$$

Therefore, the growth function is given by

$$P_{W,\deg}(t) = \frac{1}{1-t^2} + \frac{t}{1-t^2} + \frac{t}{1-t^2} + \frac{t^2}{1-t^2} = \frac{1+t}{1-t}.$$

On the other hand, put $J_n = \begin{cases} \{ac^{[n/2]}, bc^{[n/2]}\} & \text{for odd } n \in \mathbb{Z}_{>0} \\ \{c^{[n/2]}, abc^{[n/2]-1}\} & \text{for even } n \in \mathbb{Z}_{>0}. \end{cases}$ Then, one shows easily $\mathrm{mcm}(J_n) = J_{n+1}$ $(n \in \mathbb{Z}_{>0})$. This implies that there exists a unique tower $T_n = (I_0, J_1, \cdots, J_n)$ of height $n \in \mathbb{Z}_{>0}$ with the ground set $I_0 = \{a, b\}$. Therefore, the skew growth function is given by

$$N_{W,\deg}(t) = 1 + \sum_{n:\ \text{odd}\geq 1}^{\infty} 2t^{n+1} - \sum_{n:\ \text{even}\geq 1}^{\infty} 2t^{n-1} = \frac{1-t}{1+t}.$$

6 Positive homogeneous presentation of a monoid

In this section, we discuss presentations of monoids by infinite generators and relations, which are natural extension of the class of positive homogeneously presented monoids studied in [S-I]. Using the presentation, we give examples of monoids, where the Novikov ring $R_{\mathbb{Z}}$ is necessarily used in their growth functions.

Including the Zariski-van Kampen monoids of type $\mathrm{B_{ii}}$, Ishibe has shown the cancellativity for several monoids in this class and calculated the skew-growth function $N_{M,\deg}(t)$ explicitly (see a forthcoming paper [I2]).

Definition. 1. We call a pair $\langle L \mid R\rangle_{mo}$ a presentation of a monoid defined below, if i) L is a set, called generators, and ii) R is a set, called relations, consisting of expressions $R_i = S_i$ with R_i and S_i are positive words $\in L^*$. Both sets L and R may not necessarily be finite.

2. The monoid associated with the presentation $\langle L \mid R\rangle_{mo}$ is defined as the quotient of the free monoid L^* generated by L by the equivalence relation $\simeq$ defined by the following 1) and 2).

1) two words U and V in L^* are called *elementarily equivalent* if either $U = V$ or V is obtained from U by substituting a substring R_i of U by S_i where $R_i = S_i$ is a relation of R ($S_i = R_i$ is also a relation if $R_i = S_i$ is a relation),

2) two words U and V in L^* are called *equivalent*, denoted by $U \simeq V$ if there exists a sequence $U = W_0, W_1, \cdots, W_n = V$ of words in L^* for $n \in \mathbb{Z}_{\geq 0}$ such that W_i is elementarily equivalent to W_{i-1} for $i = 1, \cdots, n$.

3) If $U_1 \simeq V_1$ and $U_2 \simeq V_2$, then $U_1U_2 \simeq V_1V_2$. Thus, we define the product between the equivalence classes.

3. A map $\deg : L \longrightarrow \mathbb{R}_{\geq 0}$, which naturally extends to a map $L^* \to \mathbb{R}_{\geq 0}$ additively denoted again by deg, is called a degree map, if i) $\deg(R_i) = \deg(S_i)$ for any relation $R_i = S_i$ in R, and ii) if $\deg(a) = 0$ for some $a \in L$ then there exists $a' \in L^*$ and a relation $aa' = \emptyset$. The condition i) implies that deg induces an additive map $\langle L \mid R\rangle_{mo} \to \mathbb{R}_{\geq 0}$, denoted again by deg called a degree map. The condition ii) requires that $\deg(u) = 0$ implies u is invertible in the monoid. If, further, the inverse image $\deg^{-1}(0, r)$ of an interval $(0, r) \subset \mathbb{R}$ for any $r \in \mathbb{R}_{>0}$ intersect with L by a finite set, then deg gives a discrete degree map on the monoid $\langle L \mid R\rangle_{mo}$.

Example. For any sequence $\mathbf{p} = \{p_k\}_{k\in\mathbb{Z}_{\geq 0}}$ with $p_k \in \mathbb{Z}_{\geq 0}$ ($k \in \mathbb{Z}_{\geq 0}$) and $p_1 \in 2\mathbb{Z}_{>0}$, let us consider a pair $(M_{\mathbf{p}}, \deg)$ presented as follows:

$$L = \{a_k\}_{k\in\mathbb{Z}_{\geq 0}} \quad \text{and} \quad R = \{a_k^2 = a_0^{p_k} a_{k-1}\}_{k\in\mathbb{Z}_{\geq 1}} \cup \{a_k a_l = a_l a_k\}_{k,l\in\mathbb{Z}_{\geq 0}},$$

$$\deg : L \to \mathbb{R}_{>0}, \quad a_k \mapsto d_k := \tfrac{1}{2^k} + \textstyle\sum_{i=1}^{k} \frac{p_i}{2^{k-i+1}},$$

The degree map is discrete if the sequence $d_0 = 1, d_1, d_2, \cdots$ diverges to $+\infty$.

Assertion 6.1. *i) Any element u of the monoid $M_{\mathbf{p}}$ has a unique expression:*

$$*) \qquad a_0^n \prod_{k=1}^{\infty} a_k^{\varepsilon_k}$$

for suitable $n \in \mathbb{Z}_{\geq 0}$ and $\varepsilon \in E := \{(\varepsilon_k)_{k\in\mathbb{Z}_{\geq 1}} \in \{0,1\}^{\mathbb{Z}_{\geq 1}} \mid \varepsilon_k = 0 \text{ for } k >> 0\}$. The degree map deg *gives an embedding of $M_{\mathbf{p}}$ into $\mathbb{R}_{\geq 0}$.*

ii) There exists an additive function $m(\mathbf{p}, \delta) \in \mathbb{Z}$ on $\mathbf{p}$ depending $\delta \in E$ such that a number $r = m + \sum_{k=1}^{\infty} \delta_k/2^k \in \mathbb{R}$ for $m \in \mathbb{Z}$ and $\delta \in E$ belongs to the image $\deg(M_{\mathbf{p}})$ *if and only if $m \geq m(\mathbf{p}, \delta)$.*

Terminology. We shall call the expression $*)$ the *normal form* of u, a_0^n and $\prod_{k=1}^{\infty} a_k^{\varepsilon_k}$ the *integral part* and *2-decimal part* of u, respectively. The number $\mathrm{depth}(u) := \max\{k \in \mathbb{Z}_{\geq 1} \mid \varepsilon_k \neq 0\}$ shall be called the *depth* of $u \in M_{\mathbf{p}}$.

Proof. i) Using the relations R, it is clear that any element u of $\langle L, R\rangle_{mo}$ can be expressed in the form $*$). Then, we have

$$\deg(u) = n + \sum_{k=1}^{\infty} \varepsilon_k \big(\frac{1}{2^k} + \sum_{i=1}^{k} \frac{p_i}{2^{k-i+1}}\big).$$

Note that RHS is a finite sum w.r.t. the index k, and expressed as a 2-decimal number. In view of the fact that the leading term (the deepest part) of the 2-decimal expansion of d_k is equal to $\frac{1}{2^k}$ ($k \in \mathbb{Z}_{\geq 0}$), the 2-decimal part of the number $\deg(u)$ mod 1 determines the part $\varepsilon = (\varepsilon_k)_{k=1}^{\infty}$ of the word $*$). So the exponent n of the integral part a_0^n of $*$) is given by $\deg(u) - \sum_{k=1}^{\infty} \varepsilon_k d_k$. Thus the expression $*$) is uniquely determined from $\deg(u)$, and this implies that the correspondence $u \mapsto \deg(u)$ is injective.

ii) For the given $r = m + \sum_{k=1}^{\infty} \delta_k / 2^k \in \mathbb{R}$, we want to solve the equation

$$n + \sum_{k=1}^{\infty} \varepsilon_k \big(\frac{1}{2^k} + \sum_{i=1}^{k} \frac{p_i}{2^{k-i+1}}\big) = m + \sum_{k=1}^{\infty} \frac{\delta_k}{2^k}$$

on $(n, \varepsilon) \in \mathbb{Z}_{\geq 0} \times E$. The comparison of 2-decimal parts of both hand sides, says that ε is uniquely determined from δ and $\mathbf{p}$. Then, in view of the equality: $n + \Big[\sum_{k=1}^{\infty} \varepsilon_k \big(\frac{1}{2^k} + \sum_{i=1}^{k} \frac{p_i}{2^{k-i+1}}\big) \Big] = m$ (here, "$[\ \cdot\ ]$" is the Gauss symbol), the condition: $n \geq 0$ is transformed to $m - \Big[\sum_{k=1}^{\infty} \varepsilon_k \big(\frac{1}{2^k} + \sum_{i=1}^{k} \frac{p_i}{2^{k-i+1}}\big) \Big] \geq 0$. Rewriting ε in terms of δ, we obtain the result. □

Corollary 6.2. *Using the normal form, the growth function is given by*

$$P_{M_{\mathbf{p}},\deg}(t) \ = \ \sum_{n=0}^{\infty} t^n \cdot \prod_{k=1}^{\infty} \sum_{\varepsilon_k=0}^{1} t^{\deg(a_k)\varepsilon_k} \ = \ \frac{\prod_{k=1}^{\infty}(1 + t^{\deg(a_k)})}{1-t}.$$

The Dirichlet series $P_{M_{\mathbf{p}},\deg}(\exp(-s))$ *(up to the factor* $1-\exp(-s)$*) converges on the domain* $\{s \in \mathbb{C} \mid \Re(s) > \overline{\lim_{m\to\infty}} \frac{1}{m} \log(\#\{n \in \mathbb{Z}_{\geq 0} \mid [m] \leq d_n < m\})\}$ *(Kojima).*

Applying this formula to the inversion formula, we have the following description of the skew-growth function.

$$\begin{aligned} N_{M_{\mathbf{p}},L}(t) &= \frac{1-t}{\prod_{k=1}^{\infty}(1 + t^{\deg(a_k)})} \\ &= \sum_{\mathbf{n}} (-1)^{|\mathbf{n}|} \ t^{\deg(\Delta_{\mathbf{n}})} + \sum_{\mathbf{n}} (-1)^{|\mathbf{n}|+1} \ t^{\deg(a_0 \Delta_{\mathbf{n}})}, \end{aligned}$$

where we put $\Delta_{\mathbf{n}} := \prod_{k=1}^{\infty} a_k^{n_k}$ for any sequence $\mathbf{n} = \{n_k\}_{k=1}^{\infty}$ of non-negative integer where only finite n_k are non-zero, and we put $|\mathbf{n}| = \sum_{k=1}^{\infty} n_k$.

Remark 6.3. Let r be the radius of absolute convergence of $\prod_{k=1}^{\infty}(1+t^{\deg(a_k)})$. Since $d_k = \deg(a_k)$ is a divergent sequence, one has $r \leq 1$ and, hence, the radius of absolute convergence of $P_{M_{\mathbf{p}},L}(t)$ is equal to r. On the other hand, it may be clear that $N_{M_{\mathbf{p}},L}(t)$ converges absolutely on the disc of radius r so that Conjecture 5.5 is satisfied by these examples. However, whether $(M_{\mathbf{p}}, \deg)$ belongs to the class in Problem 5.6. or not is an open question.

We do not know how to obtain this formula directly through the towers $\mathrm{Tmcm}(M_{\mathbf{p}}, L)$. As a first step to determine $\mathrm{Tmcm}(M_{\mathbf{p}})$, we describe algorithms i), ii) and iii) to obtain the set $\mathrm{mcm}(J)$ for any finite subset J of $M_{\mathbf{p}}$.

i) If the elements of J have depth bounded by $k \in \mathbb{Z}_{>0}$, then the depth of the elements of $\mathrm{mcm}(J)$ is also bounded by the same k.

Proof. Let v has depth $K > k$ and let $v = v' \prod_{j=k+1}^{K} a_j^{\varepsilon_j}$ be the normal form of v where v' has depth at most k. If v is divisible by $u \in J$, then the decimal expansion $\deg(v/u) = \deg(v) - \deg(u)$ deeper than k is unchanged from that of v. That is, $v/u = w \prod_{j=k+1}^{K} a_j^{\varepsilon_j}$ for some $w \in M_{\mathbf{p}}$. That is, $v'/u = w \in M$. In other words, $v' = v/\prod_{j=k+1}^{K} a_j^{\varepsilon_j}$ is still divisible by $u \in J$, i.e. v was not minimal. □

ii) For each $\varepsilon \in E$, consider the set $M_\varepsilon := \{a_0^n \prod_{k=1}^{\infty} a_k^{\varepsilon_k} \mid n \in \mathbb{Z}_{\geq 0}\}$. Then, there exists a unique minimal element of $M_\varepsilon \cap \mathrm{cm}(J)$, denoted by $\mathrm{mcm}_\varepsilon(J)$.

Proof. The intersection $M_\varepsilon \cap \mathrm{cm}(J)$ is non-empty since a_0^N is always divisible by a given finite set J for sufficiently large N. Since M_ε is inductively ordered by $\mathbb{Z}_{\geq} 0$, the intersection always has the minimal element. □

iii) In view of i) and ii), $\mathrm{mcm}(J) \subset \{\mathrm{mcm}_\varepsilon(J) \mid \mathrm{depth}(\varepsilon) \leq \mathrm{depth}(J)\}$. Since RHS is a finite set, we can choose its minimal subset $\mathrm{mcm}(J)$ by finite steps.

Using the above i)-iii), for any finite subset $J \subset L$ of the generator set L and a positive integer $n \in \mathbb{Z}_{\geq 0}$, we are algorithmically able to determine all towers having J as its first stage and height less or equal than n, and to check that minimal common multiples which appear in these towers have expressions of the form either $\Delta_{\mathbf{n}}$ or $a_0 \Delta_{\mathbf{n}}$ for some $\mathbf{n}$ (which are independent of the structure constant $\mathbf{p}$). However, an explicit list of all towers on the ground L (i.e. the explicit description of $\mathrm{Tmcm}(M_{\mathbf{p}}, L)$) is still unknown.

Remark 6.4. In [S-I], we have studied a particular subclass of presentation of monoids, where the set L is finite and the set R consists of relations of the form $R_i = S_i$ where R_i and S_i are words in L of the same length. In this case, the map $\deg : L \to \{1\}$ automatically defines a discrete degree map on the monoid. Actually, Artin monoids [B-S] and many Zariski-van Kampen monoids (see e.g. [S-I], [I1],[I2]) belong to this class of monoids. In a forthcoming paper [I2], skew growth functions for some monoids in this class, in particular for the monoid of type B_{ii} [I1], and also some other examples shall be determined explicitly.

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