RIMS-1750

# ON THE CUSPIDALIZATION PROBLEM FOR HYPERBOLIC CURVES OVER FINITE FIELDS

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June 2012

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Abstract. In this paper, we study some group-theoretic constructions associated to arithmetic fundamental groups of hyperbolic curves over finite fields. One of the main results of this paper asserts that any Frobenius-preserving isomorphism between the geometrically pro-$l$ fundamental groups of hyperbolic curves with one given point removed induces an isomorphism between the geometrically pro-$l$ fundamental groups of the hyperbolic curves obtained by removing other points. Finally, we apply this result to obtain results concerning certain cuspidalization problems for fundamental groups of (not necessarily proper) hyperbolic curves over finite fields.

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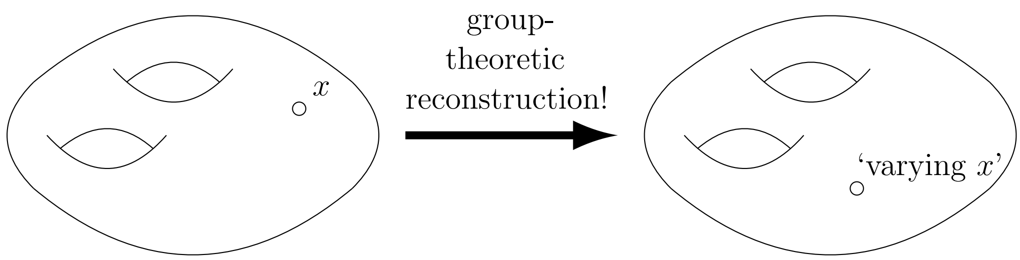
## Introduction

In the present paper, we consider the following problem:

**Problem.**

*Suppose that we are given a hyperbolic curve over a finite field in which $l$ is invertible. Then, given the geometrically pro-$l$ fundamental group of the curve obtained by removing a* **specific** *point from this hyperbolic curve, is it possible to reconstruct the geometrically pro-$l$ fundamental groups of the curves obtained by removing other points which* **vary** *"continuously" in a suitable sense?*

2010 *Mathematics Subject Classification.* Primary 14H30; Secondary 14H10.




We shall formulate the above problem mathematically.

Let $l$ be a prime number, $X$ a hyperbolic curve over a finite field $K$ in which $l$ is invertible. For $n$ a positive integer, we denote by $X_n$ the $n$-th configuration space associated to $X$ (hence, $X_1 = X$), and write $\Pi_{X_n}$ for the geometrically pro-$l$ fundamental group of $X$. Here, the fiber of $X_2 \to X$ over a $K$-rational point $x \in X$ may be naturally identified with $X \backslash \{x\}$, so we may regard $X_2 \to X$ as a *continuous family of cuspidalizations of* $X$. Therefore, the above problem can be formulated as follows (where $Y$ denotes a hyperbolic curve over a finite field $L$ in which $l$ is also invertible, and we use similar notations for $Y$ to the notations used for $X$):

**Theorem A.**

*Let*

$$\alpha : \Pi_{X\backslash\{x\}} \xrightarrow{\sim} \Pi_{Y\backslash\{y\}}$$

*be a Frobenius-preserving isomorphism [cf. Definition 3.5] which maps a specific decomposition group $D_x$ of $x$ onto a specific decomposition group $D_y$ of $y$. Here, we shall denote by $\overline{\alpha} : \Pi_X \xrightarrow{\sim} \Pi_Y$ (resp., $\overline{D}_x$, $\overline{D}_y$) the isomorphism obtained by passing to the quotients $\Pi_{X\backslash\{x\}} \twoheadrightarrow \Pi_X$, $\Pi_{Y\backslash\{y\}} \twoheadrightarrow \Pi_Y$ (resp., as the image of $D_x$, as the image of $D_y$).*

*Then there exists an isomorphism*

$$\alpha_2 : \Pi_{X_2} \xrightarrow{\sim} \Pi_{Y_2}$$

*which is uniquely determined up to composition with an inner automorphism by the condition that it be compatible with the natural switching automorphisms up to an inner automorphism and fit into a commutative diagram*

$$\begin{array}{ccc} \Pi_{X_2} & \xrightarrow{\alpha_2} & \Pi_{Y_2} \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle p_1} \\ \Pi_X & \xrightarrow{\overline{\alpha}} & \Pi_Y \end{array}$$

*that induces $\alpha$ by restricting $\alpha_2$ to the inverse images (via the vertical arrows) of $\overline{D}_x$ and $\overline{D}_y$.*

*In particular, if $x'$ (resp., $y'$) is a $K$-rational point of $X$ (resp., an $L$-rational point of $Y$), and we assume that the decomposition groups of $x'$, $y'$ correspond* *via $\alpha$, then we have an isomorphism*

$$\alpha' : \Pi_{X\setminus\{x'\}} \xrightarrow{\sim} \Pi_{Y\setminus\{y'\}}$$

*such that $\alpha$ and $\alpha'$ induce the same isomorphism $\Pi_X \xrightarrow{\sim} \Pi_Y$.*

Now let us explain the content of each section briefly. In Section 1, we recall the notion of the (log) configuration space associated to a hyperbolic curve and review group-theoretic properties of the various fundamental groups associated to such spaces. In particular, the splitting determined by the Frobenius action on the pro-$l$ étale fundamental group $\Delta_{X_n}$ of $X_n \times_K \overline{K}$ gives rise to an explicit description of the graded Lie algebra obtained by considering the weight filtration on $\Delta_{X_n}$ (cf. Definition 1.6). This explicit description will play an essential role in the proof of Theorem A.

In Section 2, we discuss a certain *specific choice* (among composites with inner automorphisms) of the morphism between geometrically pro-$l$ fundamental groups obtained by switching the two ordered marked points parametrized by the second configuration space. This choice will play a key role in the proof of Theorem A.

Section 3 is devoted to proving Theorem A. Roughly speaking, starting from a given geometrically pro-$l$ fundamental group $\Pi_{X\setminus\{x\}}$, we reconstruct group-theoretically a suitable topological group, i.e., $\Pi^{\mathrm{Lie}}_{X_2}$ (cf. Definition 3.1), which contains the geometrically pro-$l$ fundamental group of the second configuration space, by using the explicit description of graded Lie algebra studied in Section 1. Next, we reconstruct the automorphism on $\Pi^{\mathrm{Lie}}_{X_2}$ induced by the specific choice of the switching morphism studied in Section 2. Finally, we verify that $\Pi_{X_2}$ can be generated, as a subgroup of $\Pi^{\mathrm{Lie}}_{X_2}$, by the given fundamental group $\Pi_{X\setminus\{x\}}$ and the image of this fundamental group via the specific choice of the switiching morphism studied in Section 2; this allows us to reconstruct $\Pi_{X_2}$ as a subgroup of $\Pi^{\mathrm{Lie}}_{X_2}$.

In Section 4, as an application of (a slightly generalized version of) Theorem A, we give a group-theoretic construction of the cuspidalization of an affine hyperbolic curve $X$ over a finite field at a point *"infinitesimally close"* to the cusp $x$. That is to say, we give a construction, starting from the geometrically pro-$l$ fundamental group $\Pi_X$ of $X$, of the geometrically pro-$l$ fundamental group $\Pi_{\overline{X}_x^{\log}}$ of the log scheme obtained by gluing $X$ to a tripod (i.e., the projective line minus three points) at a cusp $x$ of $X$:

**Theorem B.**

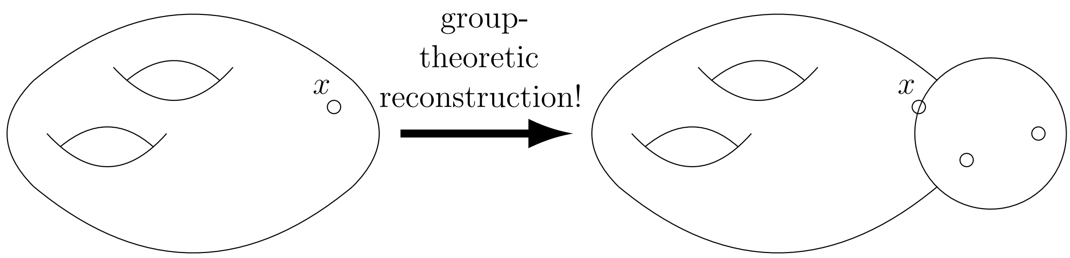
*Let $X$ (resp., $Y$) be an affine hyperbolic curve over a finite field $K$ (resp., $L$), $x$ a $K$-rational point of $\overline{X} \setminus X$ (resp., $y$ an $L$-rational point of $\overline{Y} \setminus Y$). Let*

$$\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$$

*be a Frobenius-preserving isomorphism such that the decomposition groups of $x$ and $y$ (which are well-defined up to conjugacy) correspond via $\alpha$. Then there* *exists an isomorphism*

$$\alpha_{x,y} : \Pi_{\overline{X}_x^{\log}} \xrightarrow{\sim} \Pi_{\overline{Y}_y^{\log}}$$

*which is uniquely determined up to composition with an inner automorphism by the condition it map the conjugacy class of the decomposition group of $\tilde{x}$ to the conjugacy class of decomposition group of $\tilde{y}$, and induce $\alpha$ upon passing to the quotients $\Pi_{\overline{X}_x^{\log}} \twoheadrightarrow \Pi_X$, $\Pi_{\overline{Y}_y^{\log}} \twoheadrightarrow \Pi_Y$.*




At the end of this paper, we consider the cuspidalization problem for (geometrically pro-$l$) fundamental groups of configuration spaces of (not necessarily proper) hyperbolic curves over finite fields (cf. Theorem 4.4):

**Theorem C.**
*Let $X$ (resp., $Y$) be a hyperbolic curve over a finite field $K$ (resp., $L$). Let*

$$\alpha_1 : \Pi_X \xrightarrow{\sim} \Pi_Y$$

*be a Frobenius-preserving isomorphism. Then for any $n \in \mathbb{Z}_{\geq 0}$, there exists an isomorphism*

$$\alpha_n : \Pi_{X_n} \xrightarrow{\sim} \Pi_{Y_n}$$

*which is uniquely determined up to composition with an inner automorphism by the condition that it be compatible with the natural respective outer actions of the symmetric group on $n$ letters and make the diagram*

$$\begin{array}{ccc} \Pi_{X_{n+1}} & \xrightarrow{\alpha_{n+1}} & \Pi_{Y_{n+1}} \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle p_i} \\ \Pi_{X_n} & \xrightarrow{\alpha_n} & \Pi_{Y_n} \end{array}$$

*($i = 1, \cdots, n+1$) commute.*

Finally, we make a remark on the results in the present paper. When the curves involved are of genus $\geq 2$, Theorem A may be obtained as an immediate consequence of [12], Theorem 3.1; [4], Theorem 4.1; [4], Corollary 4.1 (i). Also, Theorem C is already proved in [12] for the case where $n = 2$ and $X$ is proper, and in [4] for the case where $n \geq 3$ and $X$ is proper. On the other hand, the proof of Theorem A given in the present paper is considerably simpler and more direct than the proofs of [12] and [4]. Indeed, in the present paper, we shall apply Theorem A to give (cf. Theorem C) a substantially simpler proof of [4], Theorem 4.1, than the proof given in [4], which, moreover, includes, for the first time, the *affine case*.

**Acknowledgement**

The author would like to express his sincere gratitude to Professors Shinichi Mochizuki and Yuichiro Hoshi for their warm encouragements, suggestions, and many helpful advices, as well as to Professor Akio Tamagawa for constructive comments concerning this paper.

## Notations and Conventions

**Numbers:**

We shall denote by $\mathbb{Q}$ the field of *rational numbers*, by $\mathbb{Z}$ the ring of *rational integers*, and by $\mathbb{N} \subseteq \mathbb{Z}$ $\big($resp., $\mathbb{Z}_{\geq a} \subseteq \mathbb{Z}\big)$ the additive submonoid of integers $n \geq 0$ $\big($resp., the subset of integers $n \geq a$ for $a \in \mathbb{Z}\big)$. If $l$ is a prime number, then $\mathbb{Z}_l$ $\big($resp., $\mathbb{Q}_l\big)$ denotes the *l-adic completion* of $\mathbb{Z}$ $\big($resp., $\mathbb{Q}\big)$.

**Topological Groups:**

For an arbitrary Hausdorff topological group $G$, the notation

$$G^{\mathrm{ab}}$$

will be used to denote the *abelianization* of $G$, i.e., the quotient of $G$ by the closed subgroup of $G$ topologically generated by the commutators of $G$.

For each closed subgroups $H$ of $G$, let us write

$$N_G(H) := \big\{g \in G \mid g \cdot H \cdot g^{-1} = H\big\}$$

for the *normalizer* of $H$ in $G$. We shall say that a closed subgroup $H \subseteq G$ is *normally terminal* in $G$ if the normalizer $N_G(H)$ is equal to $H$.

If $G$ is a center-free, then we have a natural exact sequence

$$1 \longrightarrow G \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \longrightarrow 1,$$

where $\mathrm{Aut}(G)$ denotes the group of automorphisms of the topological group $G$; the injective (since $G$ is center-free) homomorphism $G \to \mathrm{Aut}(G)$ is obtained by letting $G$ act on $G$ by inner automorphisms; $\mathrm{Out}(G)$ is defined so as to render the sequence exact. If the profinite group $G$ is topologically finitely generated, then the groups $\mathrm{Aut}(G)$, $\mathrm{Out}(G)$ are naturally endowed with a profinite topology, and the above sequence may be regarded as an exact sequence of profinite groups.

If $J \to \mathrm{Out}(G)$ is a homomorphism of groups, then we shall write

$$G \overset{\mathrm{out}}{\rtimes} J := \mathrm{Aut}(G) \times_{\mathrm{Out}(G)} J$$

for the "*outer semi-direct product of $J$ with $G$*". Thus, we have a natural exact sequence

$$1 \longrightarrow G \longrightarrow G \overset{\mathrm{out}}{\rtimes} J \longrightarrow J \longrightarrow 1.$$

It is verified (cf. [4], Lemma 4.10) that if an automorphism $\phi$ of $G \overset{\mathrm{out}}{\rtimes} J$ preserves the subgroup $G \subseteq G \overset{\mathrm{out}}{\rtimes} J$ and induces the identity morphism on $G$ and the quotient $J$, then $\phi$ is the identity morphism of $G \overset{\mathrm{out}}{\rtimes} J$.

**Log schemes:**

Basic references for the notion of *log scheme* are [7] and [6]. In this paper, log structures are always considered on the étale sites of schemes. For a log scheme $X^{\log}$, we shall denote by $X$ (resp., $\mathcal{M}_X$) the underlying scheme of $X^{\log}$ (resp., the sheaf of monoids defining the log structure of $X^{\log}$). Let $X^{\log}$ and $Y^{\log}$ be log schemes, and $f^{\log} : X^{\log} \to Y^{\log}$ a morphism of log schemes. Then we shall refer to the quotient of $\mathcal{M}_X$ by the image of the morphism $f^*\mathcal{M}_Y \to \mathcal{M}_X$ induced by $f^{\log}$ as the *relative characteristic sheaf* of $f^{\log}$. Moreover, we shall refer to the relative characteristic sheaf of the morphism $X^{\log} \to X$ (where, by abuse of notation, we write $X$ for the log scheme obtained by equipping $X$ with the trivial log structure) induced by the natural inclusion $\mathcal{O}^* \hookrightarrow \mathcal{M}_X$ as the *characteristic sheaf* of $X^{\log}$.

We shall say that a log scheme $X^{\log}$ is *fs* if $\mathcal{M}_X$ is a sheaf of integral monoids, and locally for the étale topology, has a chart modeled on a finitely generated and saturated monoid. If $X^{\log}$ is *fs*, then, for $n$ a nonnegative integer, we shall refer to as the *$n$-interior* of $X^{\log}$ the open subset of $X$ on which the associated sheaf of groupifications of characteristic sheaf of $X^{\log}$ is of *rank* $\leq n$. Thus, the 0-interior of $X^{\log}$ is often referred to simply as the *interior* of $X^{\log}$.

**Curves:**

Let $f : X \to S$ be a morphism of schemes. Then we shall say that $f$ is a *family of curves of type (g,r)* if it factors $X \hookrightarrow \overline{X} \to S$ as the composite of an open immersion $X \hookrightarrow \overline{X}$ whose image is the complement $\overline{X} \setminus D$ of a relative divisor $D \subseteq \overline{X}$ which is finite étale over $S$ of relative degree $r$, and a morphism $\overline{X} \to S$ which is proper, smooth, and geometrically connected, and whose geometric fibers are one-dimensional of genus $g$. We shall refer to $\overline{X}$ as the *compactification* of $X$.

We shall say that $f$ is a *family of hyperbolic curves (resp., tripod)* if $f$ is a family of curves of type $(g,r)$ such that $(g,r)$ satisfies $2g-2+r > 0$ (resp., $(g,r) = (0,3)$ and the relative divisor $D$ is split over $S$).

We shall denote by

$$\overline{\mathcal{M}}_{g,[r]+s}$$

the moduli stack of $r+s$-pointed stable curves of genus $g$ for which $s$ sections are equipped with an ordering. This moduli stack may be obtained as the quotient of the moduli stack of ordered $(r+s)$-pointed stable curves of genus $g$ (cf. [8] for an exposition of the theory of such curves) by a suitable symmetric group action on $r$ letters. We shall denote by $\overline{\mathcal{M}}_{g,[r]+s}^{\log}$ the log stack obtained by equipping $\overline{\mathcal{M}}_{g,[r]+s}$ with the log structure associated to the divisor with normal crossings which parametrizes singular curves.

**Fundamental Groups:**

A basic reference for the notion of *Kummer étale covering* is [6]. For a locally Noetherian, connected scheme $X$ (resp., a locally Noetherian, connected, fs log scheme $X^{\log}$) equipped with a geometric point $\overline{x} \to X$ (resp., log geometric point $\tilde{x}^{\log} \to X^{\log}$), we shall denote by $\pi_1(X, \overline{x})$ (resp., $\pi_1(X^{\log}, \tilde{x}^{\log})$) the étale fundamental group of $X$ (resp., logarithmic fundamental group of $X^{\log}$). Since one knows that the étale and logarithmic fundamental groups are determined up to inner automorphisms independently of the choice of basepoint, we shall omit the basepoint, and write $\pi_1(X)$ (resp., $\pi_1(X^{\log})$ ).

For a scheme $X$ (resp., fs log scheme $X^{\log}$) which is geometrically connected and of finite type over a field $K$ in which a prime number $l$ is invertible, we shall refer to the quotient $\Pi_X$ of $\pi_1(X)$ (resp., the quotient $\Pi_{X^{\log}}$ of $\pi_1(X^{\log})$) by the closed normal subgroup obtained as the kernel of the natural projection from $\pi_1(X \times_K \overline{K})$ (resp., $\pi_1(X^{\log} \times_K \overline{K})$) (where $\overline{K}$ is a separable closure of $K$) to its maximal pro-$l$ quotient $\Delta_X$ (resp., $\Delta_{X^{\log}}$) as the *geometrically pro-$l$ étale fundamental group* of $X$ (resp., *geometrically pro-$l$ logarithmic fundamental group* of $X^{\log}$). Thus, (if we write $G_K$ for the Galois group of a separable closure of $K$ over $K$, then) we have a natural exct sequence

$$1 \longrightarrow \Delta_X \longrightarrow \Pi_X \longrightarrow G_K \longrightarrow 1$$
$$(\text{resp.}, 1 \longrightarrow \Delta_{X^{\log}} \longrightarrow \Pi_{X^{\log}} \longrightarrow G_K \longrightarrow 1).$$

Note that if the log structure of $X^{\log}$ is trivial, then we have natural isomorphisms $\Delta_{X^{\log}} \xrightarrow{\sim} \Delta_X, \Pi_{X^{\log}} \xrightarrow{\sim} \Pi_X$.

If $K$ is finite, then write $G_K^{\dagger} \subseteq G_K$ for the (unique) *maximal pro-$l$ subgroup of* $G_K$ (so $G_K^{\dagger} \cong \mathbb{Z}_l$). Also, for a profinite group $\Pi$ over $G_K$, we shall use the notation

$$\Pi^{\dagger} := \Pi \times_{G_K} G_K^{\dagger} \subseteq \Pi.$$

and refer to it as the *restricted pro-$l$ group* of $\Pi$.

## 1. Fundamental groups of (log) configuration spaces

The purpose of this section is to recall the notion of the (log) configuration space associated to a curve and review group-theoretic properties of the various fundamental groups associated to such spaces.

Let $l$ be a prime number, $K$ a field in which $l$ is invertible, $\overline{K}$ a separable closure of $K$, where we shall denote by $G_K$ the Galois group of $\overline{K}$ over $K$, and $X$ a hyperbolic curve over $K$ of type $(g, r)$.

**Definition 1.1.**

(i) For $n \in \mathbb{Z}_{\geq 1}$, write $X^{\times n}$ for the fiber product of $n$ copies of $X$ over $K$. We shall denote by

$$X_n (\subseteq X^{\times n})$$

the *$n$-th configuration space associated to* $X$, i.e., the scheme which represents the open subfunctor

$$S \mapsto \big\{(f_1, \cdots, f_n) \in X^{\times n}(S) \bigm| f_i \neq f_j \text{ if } i \neq j\big\}$$

of the functor represented by $X^{\times n}$.

(ii) Let us denote by $\overline{X}_n^{\log}$ the *$n$-th log configuration space associated to* $X$ (cf. [15]), i.e.,

$$\overline{X}_n^{\log} := \operatorname{Spec} K \times_{\overline{\mathcal{M}}_{g,[r]}^{\log}} \overline{\mathcal{M}}_{g,[r]+n}^{\log}$$

— where the (1-)morphism $\operatorname{Spec} K \to \overline{\mathcal{M}}_{g,[r]}^{\log}$ is the classifying morphism determined by the curve $X \to \operatorname{Spec} K$, and the (1-)morphism $\overline{\mathcal{M}}_{g,[r]+n}^{\log} \to \overline{\mathcal{M}}_{g,[r]}^{\log}$ is obtained by forgetting the ordered $n$ marked points of the tautological family of curves over $\overline{\mathcal{M}}_{g,[r]+n}^{\log}$. In the following, for simplicity, we shall write $\overline{X}^{\log}$ for $\overline{X}_1^{\log}$.

**Proposition 1.2.**

(i) *The* 0*-interior (cf.* § 0*) of the log scheme* $\overline{X}_n^{\log}$ *is naturally isomorphic to the $n$-th configuration space* $X_n$ *associated to* $X$.

(ii) *The log scheme* $\overline{X}_n^{\log}$ *is log regular and its underlying scheme is connected and regular.*

(iii) *The projection* $p_k^{\log} : \overline{X}_n^{\log} \to \overline{X}_{n-1}^{\log}$*, induced from the (1-)morphism* $\overline{\mathcal{M}}_{g,[r]+n}^{\log} \to \overline{\mathcal{M}}_{g,[r]+n-1}^{\log}$ *obtained by forgetting the $k$-th ($k = 1, \cdots, n$) ordered points of the tautological family of curves over* $\overline{\mathcal{M}}_{g,[r]+n}^{\log}$*, is log smooth (cf.* § 0*) and its underlying morphism of schemes is the natural projection* $p_k : X_n \twoheadrightarrow X_{n-1}$ *obtained by forgetting the $k$-th factor, and hence, is flat, geometrically connected, and geometrically reduced.*

*Proof.* See, for example, [4], Proposition 2.2. □

**Definition 1.3.**

We shall denote (cf. § 0) by

$$\Pi_{X_n} \quad (\text{resp.}, \Delta_{X_n})$$

the geometrically pro-$l$ étale fundamental group of $X_n$ (resp., $X_n \times_K \overline{K}$), and

$$\Pi_{\overline{X}_n^{\log}} \quad (\text{resp.}, \Pi_{\overline{X}^{\log \times n}})$$

the geometrically pro-$l$ log fundamental group of $\overline{X}_n^{\log}$ (resp., the fiber product $\overline{X}^{\log\times n}$ of $n$ copies of $\overline{X}^{\log}$ over $K$). Moreover, we shall denote (cf. § 0) by

$$\Pi^{\dagger}_{X_n},\quad \Delta^{\dagger}_{X_n}(\cong \Delta_{X_n}),\quad \Pi^{\dagger}_{\overline{X}_n^{\log}},\quad \Pi^{\dagger}_{\overline{X}^{\log\times n}}$$

respective restricted pro-$l$ groups.

Also we shall write

$$p_k^{\Delta}:\Delta_{X_n}\twoheadrightarrow \Delta_{X_{n-1}},\quad p_k^{\Pi}:\Pi_{X_n}\twoheadrightarrow \Pi_{X_{n-1}}$$

for the morphisms induced by the projection $p_k\times_K\overline{K}: X_n\times_K\overline{K}\twoheadrightarrow X_{n-1}\times_K\overline{K}$, $p_k: X_n\twoheadrightarrow X_{n-1}$ obtained by forgetting the $k$-th factor (these morphisms of groups are only defined up to conjugacy in the absence of appropriate choices of basepoints of respective schemes) and write

$$i_k^{\Delta}:\Delta^k_{X_{n/n-1}}\hookrightarrow \Delta_{X_n},\quad i_k^{\Delta'}:\Delta^k_{X_{n/n-1}}\hookrightarrow \Pi_{X_n}$$

for the kernels of the surjections $p_k^{\Delta}:\Delta_{X_n}\twoheadrightarrow\Delta_{X_{n-1}}$, $p_k^{\Pi}:\Pi_{X_n}\twoheadrightarrow\Pi_{X_{n-1}}$. Then we have exact sequences

$$1\longrightarrow \Delta_{X_n}\longrightarrow \Pi^{(-)}_{X_n}\longrightarrow G_K^{(-)}\longrightarrow 1$$

$$1\longrightarrow \Delta^k_{X_{n/n-1}}\xrightarrow{i_k^{\Delta}}\Delta_{X_n}\xrightarrow{p_k^{\Delta}}\Delta_{X_{n-1}}\longrightarrow 1$$

$$1\longrightarrow \Delta^k_{X_{n/n-1}}\xrightarrow{i_k^{\Delta'}}\Pi^{(-)}_{X_n}\xrightarrow{p_k^{\Pi^{(-)}}}\Pi^{(-)}_{X_{n-1}}\longrightarrow 1$$

— where the symbol $(-)$ denotes either the presence or absence of "†".

Also, we have a square diagram

$$\begin{tikzcd}
\Pi^{(-)}_{X_{n-1}} \arrow[d] & \Pi^{(-)}_{X_n} \arrow[l, "p_k^{\Pi^{(-)}}"'] \arrow[r] \arrow[d] & \overbrace{\Pi_X^{(-)}\times_{G_K^{(-)}}\cdots\times_{G_K^{(-)}}\Pi_X^{(-)}}^{n} \arrow[d] \\
\Pi^{(-)}_{\overline{X}_{n-1}^{\log}} & \Pi^{(-)}_{\overline{X}_n^{\log}} \arrow[l] \arrow[r] & \Pi^{(-)}_{\overline{X}^{\log\times n}},
\end{tikzcd}$$

— which can be made commutative without conjugate-indeterminacy by choosing compatible base points — arising from a natural commutative diagram

$$\begin{tikzcd}
X_{n-1} \arrow[d] & X_n \arrow[l, "p_k"'] \arrow[r] \arrow[d] & X^{n\times} \arrow[d] \\
\overline{X}_n^{\log} & \overline{X}_n^{\log} \arrow[l, "p_k^{\log}"'] \arrow[r] & \overline{X}^{\log\times n}.
\end{tikzcd}$$

Then, it follows from Proposition 1.2 (i), (ii) together with the log purity theorem (cf. [6], [9]) that the three vertical homomorphisms are isomorphisms. In the following, we shall identify $\Pi^{(-)}_{X_n}$ with $\Pi^{(-)}_{\overline{X}_n^{\log}}$, $\Pi^{(-)}_{\overline{X}^{\log\times n}}$ with $\overbrace{\Pi_X^{(-i)}\times_{G_K^{(-)}}\cdots\times_{G_K^{(-)}}\Pi_X^{(-)}}^{n}$

and the surjection $p_k^{\Pi} : \Pi_{X_n} \to \Pi_{X_{n-1}}$ with the surjection $\Pi^{(-)}_{\overline{X}^{\log}_n} \to \Pi^{(-)}_{\overline{X}^{\log}_{n-1}}$ by means of these specific isomorphisms.

**Proposition 1.4.**

(i) $\Delta^k_{X_{n/n-1}}$ *may be naturally identified with the maximal pro-$l$ quotient of the étale fundamental group of a geometric fiber of the projection morphism* $p_k : X_n \to X_{n-1}$.

(ii) *The images of the* $i_k^{\Delta} : \Delta^k_{X_{n/n-1}} \to \Delta_{X_n}$, *where* $k = 1, \cdots, n$, *generate* $\Delta_{X_n}$.

(iii) *The profinite groups* $\Delta_{X_n}$, $\Delta^k_{X_{n/n-1}}$, $\Pi^{\dagger}_{X_n}$, $\Pi^{\dagger}_{X^{\times n}}$ *are slim (i.e., every open subgroup of each profinite group is center-free).*

*Proof.* Assertion (i) follows from [15], Proposition 2.2, or [19], Proposition 2.3. Assertions (ii) and (iii) follow from induction on $n$, together with the exact sequence

$$1 \longrightarrow \Delta^n_{X_{n/n-1}} \xrightarrow{i_n^{\Delta}} \Delta_{X_n} \xrightarrow{p_n^{\Delta}} \Delta_{X_{n-1}} \longrightarrow 1$$

displayed in Definition 1.3. Indeed, with regard to (ii), $\Delta^k_{X_{n/n-1}}$ maps to $\Delta^k_{X_{n-1/n-2}}$ (for $k = 1, \cdots n-1$) via $p_n^{\Delta} : \Delta_{X_n} \to \Delta_{X_{n-1}}$, and it is verified that this map $\Delta^k_{X_{n/n-1}} \to \Delta^k_{X_{n-1/n-2}}$ is surjective by regarding it as the morphism induced by an open immersion between the hyperbolic curves that arise as geometric fibers of the projection morphisms involved. With regard to (iii), the slimness of $\Delta_X$ is well-known (cf., e.g., [10], Lemma 1.3.10); the slimness of $\Pi^{\dagger}_X$ follows from the fact that the character of $G^{\dagger}_K$ arising from the determinant of $\Delta^{\mathrm{ab}}_X$ coincides with some positive power of the cyclotomic character; the other statements follow from the fact that an extension of slim profinite groups is itself slim. □

Next, we recall from [12], § 3, the theory of the weight filtration of fundamental groups and the associated graded Lie algebra.

**Definition 1.5.**

Let $l$ be a prime number; $G$, $H$, $A$ topologically finitely generated pro-$l$ groups; $\phi : H \twoheadrightarrow A$ a (continuous) surjective homomorphism. Suppose further that $A$ is abelian, and that $G$ is an $l$-adic Lie group.

(i) We shall refer to as the *central filtration* $\{H(n)\}_{n \geq 1}$ *on* $H$ *with respect to the homorphism* $\phi$ the filtration defined as follows:

$$H(1) := H$$
$$H(2) := \mathrm{Ker}(\phi)$$
$$H(m) := \big\langle [H(m_1), H(m_2)] \bigm| m_1 + m_2 = m \big\rangle \text{ for } m \geq 3$$

— where $\langle N_i \mid i \in I \rangle$ is the group topologically generated by the $N_i$'s. In the following, for $a, b, n \in \mathbb{Z}$ such that $1 \leq a \leq b, n \geq 1$, we shall write

$$H(a/b) := H(a)/H(b)$$

$$\mathrm{Gr}(H) := \bigoplus_{m \geq 1} H(m/m+1)$$

$$\mathrm{Gr}(H)(a/b) := \bigoplus_{b > m \geq a} H(m/m+1)$$

$$\mathrm{Gr}_{\mathbb{Q}_l}(H) := \mathrm{Gr}(H) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

$$\mathrm{Gr}_{\mathbb{Q}_l}(a/b) := \mathrm{Gr}(H)(a/b) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

$$H(a/\infty) := \varprojlim_{b > a} H(a/b) \ .$$

(ii) We shall denote by $\mathrm{Lie}(G)$ the Lie algebra over $\mathbb{Q}_l$ determined by the $l$-adic Lie group $G$. We shall say that $G$ is *nilpotent* if there exists a positive integer $m$ such that if we denote by $\{G(n)\}$ the central filtration with respect to the natural surjection $G \twoheadrightarrow G^{\mathrm{ab}}$ (cf. (i)), then $G(m)=\{1\}$. If $G$ is nilpotent, then $\mathrm{Lie}(G)$ is a nilpotent Lie algebra over $\mathbb{Q}_l$, hence determines a connected, unipotent linear algebraic group $\mathrm{Lin}(G)$, which we shall refer to as the *linear algebraic group associated to* $G$. In this situation, there exists a natural (continuous) homomorphism (with open image)

$$G \longrightarrow \mathrm{Lin}(G)(\mathbb{Q}_l)$$

(from $G$ to the $l$-adic Lie group determined by the $\mathbb{Q}_l$-valued points of $\mathrm{Lin}(G)$) which is uniquely determined (since $\mathrm{Lin}(G)$ is connected and unipotent) by the condition that it induce the identity morphism on the associated Lie algebras.

In the situation of (i), if $1 \leq a \in \mathbb{Z}$, then we shall write

$$\mathrm{Lie}(H(a/\infty)) := \varprojlim_{b > a} \mathrm{Lie}(H(a/b))$$

$$\mathrm{Lin}(H(a/\infty)) := \varprojlim_{b > a} \mathrm{Lin}(H(a/b))$$

— where we note that each $H(a/b)$ is a nilpotent $l$-adic Lie group.

**Definition 1.6.**

For $n \in \mathbb{Z}_{\geq 1}$, we shall denote by

$$\{\Delta_{X_n}(m)\}$$

the central filtration of $\Delta_{X_n}$ with respect to the natural surjection $\Delta_{X_n} \twoheadrightarrow \Delta^{\mathrm{ab}}_{\overline{X}^{\times n}}$ (where $\overline{X}$ denotes the smooth compactification of $X$ (cf. § 0)), and refer to it as the *weight filtration* on $\Delta_{X_n}$.

**Proposition 1.7.**

*If we equip $\Delta^k_{X_{n/n-1}}$ with the central filtration induced from the identification given by Proposition 1.4 (i) and its weight filtration, then the sequence of morphisms of graded Lie algebras*

$$1 \longrightarrow \mathrm{Gr}(\Delta^k_{X_{n/n-1}}) \xrightarrow{\mathrm{Gr}(i^{\Delta}_k)} \mathrm{Gr}(\Delta_{X_n}) \xrightarrow{\mathrm{Gr}(p^{\Delta}_k)} \mathrm{Gr}(\Delta_{X_{n-1}}) \longrightarrow 1$$

*induced by the second displayed exact sequence of Definition 1.3 is exact.*

*Proof.* See [4], Proposition 4.1. □

Next, let us fix a section $\sigma : G_K \to \Pi_{X_n}$ of the surjection $\Pi_{X_n} \twoheadrightarrow G_K$ arising from the structure morphism of $X_n$. This section $\sigma$ determines a natural conjugate action of $G_K$ on $\Delta_{X_n}$, hence also on

$$\mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_n})(a/b), \quad \mathrm{Lie}(\Delta_{X_n}(a/b)), \quad \mathrm{Lin}(\Delta_{X_n}(a/b))(\mathbb{Q}_l)$$

or $a, b \in \mathbb{Z}$ such that $1 \leq a \leq b$.

**Proposition 1.8.**

*Let us assume that $K$ is a finite field whose cardinality we denote by $q_K$, and write $Fr \in G_K$ for the Frobenius element of $G_K$. Then, relative to the natural conjugate actions determined by $\sigma$:*

(i) *The eigenvalues of the action of $Fr$ on $\mathrm{Lie}_{X_n}(a/a+1)$ are algebraic numbers all of whose complex absolute values are equal to $q_K^{a/2}$ (i.e., weight $a$).*

(ii) *There is a unique $G_K$-equivariant isomorphism of Lie algebras*

$$\mathrm{Lie}(\Delta_{X_n}(a/b)) \xrightarrow{\sim} \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_n})(a/b)$$

*which induces the identity isomorphism*

$$\mathrm{Lie}(\Delta_{X_n}(c/c+1)) \xrightarrow{\sim} \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_n})(c/c+1)$$

*for all $c \in \mathbb{Z}_{\geq 1}$ such that $a \leq c < b$.*

*Proof.* Assertion (i) follows from the "Riemann hypothesis for abelian varieties over finite fields" (cf., e.g., [16], p. 206). Assertion (ii) follows formally from assertion (i) by considering the eigenspaces with respect to the action of Fr. □

The following proposition is a special case of a result proven previously (cf. [18]). For simplicity, we discuss only the case used in the proofs of the present paper.

**Proposition 1.9.**

*For $n = 1, 2$, the graded Lie algebra $\mathrm{Gr}(\Delta_{X_n})$ has the following presentation.*

(i) *The case $n = 1$ (i.e., $X_n = X$):*

*generators* $(1 \leq j \leq r,\ 1 \leq i \leq g)$

$\bullet_1$ $\zeta_j \in \Delta_X(2/3)$

$\bullet_2$ $\alpha_i,\ \beta_i \in \Delta_X(1/2)$

*relation*

$\bullet_1$ $\sum_{j=1}^{r} \zeta_j + \sum_{i=1}^{g} [\alpha_i, \beta_i] = 0$

— *where* $\zeta_j$ $(j = 1, 2, \cdots, r)$ *topologically generates the inertia subgroup in* $\Delta_X$ *(well-defined up to conjugacy) associated to the* $j$*-th cusp [relative to some ordering of the cusps of* $X \times_K \overline{K}$*].*

(ii) *The case* $n = 2$*:*

*generators* $(1 \leq j \leq r,\ 1 \leq i \leq g,\ k = 1, 2)$

$\bullet_1$ $\zeta \in \Delta_{X_2}(2/3)$

$\bullet_2$ $\zeta_j^k \in \Delta_{X_{2/1}}^k(2/3)$

$\bullet_3$ $\alpha_i^k,\ \beta_i^k \in \Delta_{X_{2/1}}^k(1/2)$

*relations* $(1 \leq j, j' \leq r,\ j \neq j',\ 1 \leq i, i' \leq g,\ \{k, k'\} = \{1, 2\})$

$\bullet_1$ $\zeta + \sum_{j=1}^{r} \zeta_j^k + \sum_{i=1}^{g} [\alpha_i^k, \beta_i^k] = 0$

$\bullet_2$ $[\alpha_i^k, \zeta_j^{k'}] = [\beta_i^k, \zeta_j^{k'}] = 0$

$\bullet_3$ $[\zeta_j^k, \zeta_{j'}^{k'}] = 0$

$\bullet_4$ $[\alpha_i^k, \alpha_{i'}^{k'}] = [\beta_i^k, \beta_{i'}^{k'}] = 0$

$\bullet_5$ $[\alpha_i^k, \beta_{i'}^{k'}] = \begin{cases} \zeta & (\textit{ if } i = i') \\ 0 & (\textit{ if } i \neq i') \end{cases}$

— *where* $\zeta$ *topologically generates the image in* $\Delta_{X_2}(2/3)$ *of the inertia subgroup in* $\Delta_{X_2}$ *(well-defined up to conjugacy) associated to the diagonal divisor of* $X \times_K X$*, and* $\zeta_j^k$ *generates the image in* $\Delta_{X_{2/1}}^k(2/3)$ *of the inertia subgroup in* $\Delta_{X_{2/1}}^k$ *asssociated to the* $j$*-th cusp [relative to some ordering of the cusps of* $X \times_K \overline{K}$*] of the* $k$*-th factor of* $X_2$*.*

## 2. Switching morphism on configuration spaces

We continue to use the notation of Section 1. In this section, we shall introduce certain closed subschemes of $\overline{X}_2^{\log}$ equipped with induced log structures — denoted by $\mathbb{D}^{\log}$ and $\overline{X}_x^{\log}$ — and consider various automorphisms induced by the automorphism of $\overline{X}_2^{\log}$ determined by switching the two factors of $X$. The geometry of such log schemes allows us to prove the uniqueness of certain specific conjugates of induced switching morphisms between fundamental groups that satisfy certain conditions. This uniqueness (Proposition 2.5) plays a key role in the proof of Theorem A.

First, we define a log scheme

$$\mathbb{D}^{\log}$$

to be the log scheme obtained by equipping the diagonal divisor $\overline{X} \subseteq \overline{X}_2$ (which is the restriction of the (1-)morphism $\overline{\mathcal{M}}_{g,[r]+1} \to \overline{\mathcal{M}}_{g,[r]+2}$ obtained by gluing the tautological family of curves over $\overline{\mathcal{M}}_{g,[r]+1}^{\log}$ to a trivial family of tripods along the final ordered marked section) with the log structure pulled back from $\overline{X}_2^{\log}$. Thus, if we write $d : \mathbb{D}^{\log} \to \overline{X}_2^{\log}$ for the natural diagonal embedding, then it follows immediately from the definitions that $p_1 \circ d = p_2 \circ d : \mathbb{D}^{\log} \to \overline{X}^{\log}$ is a morphism of type $\mathbb{N}$ (cf. [2]), i.e., the underlying morphism of schemes is an isomorphism, and the relative characteristic sheaf (cf. § 0 ) is locally constant with stalk isomorphic to $\mathbb{N}$.

Observe that the (1-)automorphism on $\overline{\mathcal{M}}_{g,[r]+2}^{\log}$ over $\overline{\mathcal{M}}_{g,[r]}^{\log}$ given by switching the two ordered marked points of the tautological family of curves over $\overline{\mathcal{M}}_{g,[r]+2}^{\log}$ induces automorphisms $s$, $\overline{s}$, and $s_{\mathbb{D}}$, which fit into a commutative diagram as follows:

$$\begin{array}{ccccc}
\mathbb{D}^{\log} & \xrightarrow{d} & \overline{X}_2^{\log} & \xrightarrow{p:=(p_1,p_2)} & \overline{X}^{\log} \times_K \overline{X}^{\log} \\
{\scriptstyle s_{\mathbb{D}}}\downarrow & & {\scriptstyle s}\downarrow & & {\scriptstyle \overline{s}}\downarrow \\
\mathbb{D}^{\log} & \xrightarrow{d} & \overline{X}_2^{\log} & \xrightarrow{p:=(p_1,p_2)} & \overline{X}^{\log} \times_K \overline{X}^{\log}.
\end{array} \qquad (*)^X$$

**Lemma 2.1.**
*In the notation of the above situation,*

(i) $\overline{s}$ *is the morphism determined by switching the two factors.*

(ii) $s_{\mathbb{D}}$ *is the identity morphism on the underlying scheme; on the sheaf of monoids defining the log structure of* $\mathbb{D}^{\log}$*, for any étale local section* $s$ *of* $\mathcal{M}_{\mathbb{D}}$ *such that "*$s = 0$*" defines the diagonal divisor* $\overline{X} \subseteq \overline{X}_2$*,*

$$s_{\mathbb{D}}(s) = -s \ .$$

*Proof.* Recall that $\overline{X}_2$ is obtained by blowing-up $\overline{X} \times_K \overline{X}$ along the intersection of the diagonal divisor and the pull-backs of the cusps via $p_1, p_2 : \overline{X}_2 \to \overline{X}$. Thus, one verifies easily that assertions (i) and (ii) follow immediately from the fact that the ring homomorphism corresponding to $\overline{s}$ in an affine neighborhood of any diagonal point may be expressed as

$$A \otimes_K A \longrightarrow A \otimes_K A$$

$$\sum_j a_j \otimes a'_j \mapsto \sum_j a'_j \otimes a_j \ ,$$

hence maps $s$ to $-s$ for any local section $s$ such that "$s = 0$" defines the diagonal divisor $\overline{X} \subseteq \overline{X} \times_K \overline{X}$. □

**Remark 2.1.1.**

Lemma 2.1 (ii) can be interpreted as the assertion that the automorphism induced by $s_{\mathbb{D}}$ on the sheaf of monoids $\mathcal{M}_{\mathbb{D}}$ defining the log structure of $\mathbb{D}^{\log}$ may be expressed, relative to the étale local splitting of $\mathcal{M}_{\mathbb{D}} \twoheadrightarrow \mathcal{M}_{\mathbb{D}}/\mathcal{O}^*_X \cong \mathbb{N}$ corresponding to $s$, as

$$\begin{aligned} \mathbb{N} \ \oplus \ \mathcal{O}^*_X &\xrightarrow{\sim} \mathbb{N} \ \oplus \ \mathcal{O}^*_X \\ (m, v) &\longmapsto (m, (-1)^m v) \ . \end{aligned}$$

Next, we introduce the log scheme $\overline{X}^{\log}_x$ that appears in the discussion at the beginning of this section. Let $x^{\log} \to \overline{X}^{\log}$ be a strict morphism (cf. [6], 1.2) such that the underlying scheme of $x^{\log}$ is $K$-isomorphic to Spec$(K)$. We shall write

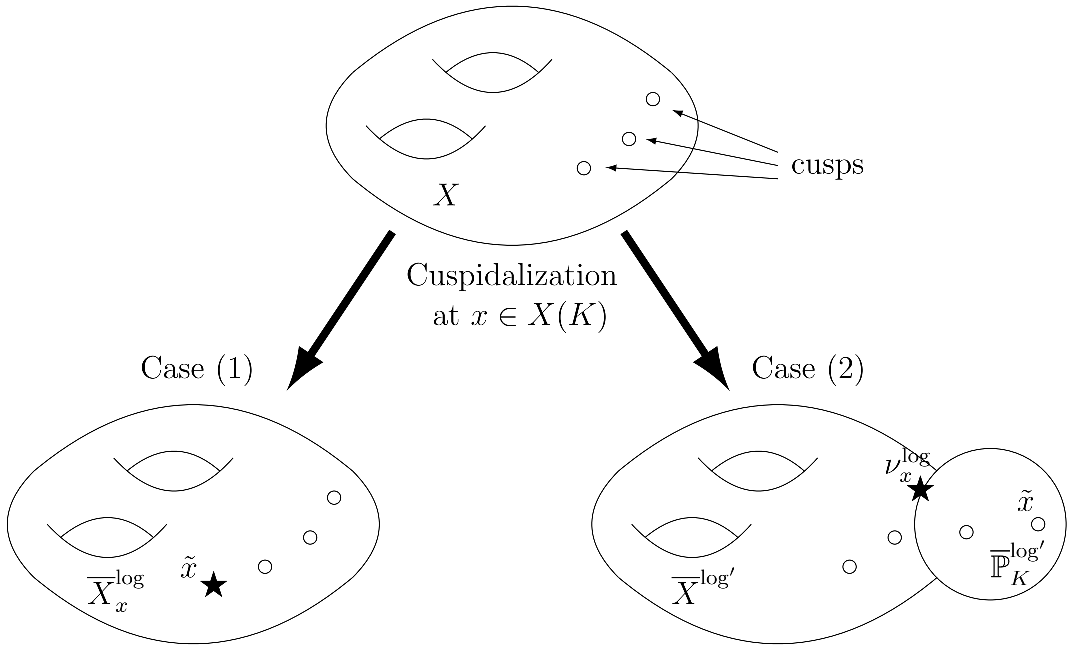
$$\begin{aligned} \overline{X}^{\log}_x &:= x^{\log} \times_{\overline{X}^{\log}} \overline{X}^{\log}_2, \\ \tilde{x}^{\log} &:= x^{\log} \times_{\overline{X}^{\log}} \mathbb{D}^{\log}, \end{aligned}$$

— where the morphism $\overline{X}^{\log}_2 \to \overline{X}^{\log}$ (resp., $\mathbb{D}^{\log} \to \overline{X}^{\log}$) in the fiber product defining $\overline{X}^{\log}_x$ (resp., $\tilde{x}^{\log}$) is $p_1$ (resp., $p_1 \circ d = p_2 \circ d$) — and refer to $\overline{X}^{\log}_x$ (resp., $\tilde{x}^{\log}$) as the *cuspidalization of* $X$ *at* $x$ (resp., *diagonal cusp of* $\overline{X}^{\log}_x$). We note that both the log structure of $x^{\log}$ and the underlying scheme of $\overline{X}^{\log}_x$ depend on the choice of $x \in \overline{X}$:

(1) *The Case* $x \in X$:
In this case, $x = x^{\log}$, i.e., the log structure of $x^{\log}$ is trivial. As we discussed in Section 1, the underlying scheme of $\overline{X}^{\log}_x$ is naturally isomorphic to $\overline{X}$; this isomorphism maps $\tilde{x}$ to $x$ and the interior of $\overline{X}^{\log}_x$ onto $X \setminus \{x\}$.

(2) *The Case* $x \in \overline{X} \setminus X$:
In this case, the log structure of $x^{\log}$ has a chart modeled on $\mathbb{N}$, which determines a local uniformizer of $X$ at $x$. The scheme $\overline{X}_x$ consists of precisely two irreducible components, one of which maps to the point $x \in X$ (resp., maps isomorphically to $\overline{X}$) via $\overline{X}^{\log}_x \xrightarrow{p_2 \circ i_1} \overline{X}^{\log}$; denote this irreducible component by $\overline{\mathbb{P}}_K$ (resp., $\overline{X}$, via a slight abuse of notation). Thus, $\overline{X}$, $\overline{\mathbb{P}}_K$ are joined at a single *node* $\nu_x$. Let us refer to $\overline{X}$ (resp., $\overline{\mathbb{P}}_K$, $\nu_x$) as the *major cuspidal component* (resp., the *minor cuspidal component*, the *nexus*) at $x$, and denote by $\overline{X}^{\log'}$, $\overline{\mathbb{P}}^{\log'}_K$, $\nu^{\log}_x$ the log schemes obtained by equipping $\overline{X}$, $\overline{\mathbb{P}}_K$, $\nu_x$ with the respective log structures pulled back from $\overline{X}^{\log}_x$ (cf. [14], Definition 1.4). Note that the

1-interior of $\overline{X}^{\log'}$ (resp., $\overline{\mathbb{P}}_K^{\log'}$) is naturally isomorphic to $X$ (resp., is a *tripod*).




(the two thick arrows in the picture do not represent morphisms of log schemes)

Now, if we denote by

$$\Pi_{\mathbb{D}^{\log}}, \quad \Pi_{\overline{X}_x^{\log}}$$

the geometrically pro-$l$ log fundamental groups of $\mathbb{D}^{\log}$, $\overline{X}_x^{\log}$ respectively, then the map $i_1 : \overline{X}_x^{\log} \rightarrow \overline{X}_2^{\log}$ of log schemes induces an outer homomorphism $[i_1^{\Pi}] : \Pi_{\overline{X}_x^{\log}} \rightarrow \Pi_{X_2}$ of profinite groups, and the above diagram $(*)^X$ induces a diagram of outer homomorphisms of profinite groups as follows:

$$\begin{array}{ccccc} \Pi_{\mathbb{D}^{\log}} & \xrightarrow{[d^{\Pi}]} & \Pi_{X_2} & \xrightarrow{[p^{\Pi}]} & \Pi_X \times_{G_K} \Pi_X \\ {\scriptstyle [s_{\mathbb{D}}^{\Pi}]}\downarrow \wr & & {\scriptstyle [s^{\Pi}]}\downarrow \wr & & {\scriptstyle [\overline{s}^{\Pi}]}\downarrow \wr \\ \Pi_{\mathbb{D}^{\log}} & \xrightarrow{[d^{\Pi}]} & \Pi_{X_2} & \xrightarrow{[p^{\Pi}]} & \Pi_X \times_{G_K} \Pi_X \ . \end{array} \qquad (*)^{\Pi}$$

Note that the homomorphisms corresponding to the arrow $[i_1^{\Pi}]$ and the arrows in the diagram $(*)^{\Pi}$ are only defined (i.e., in the absence of appropriate choices of basepoints of respective log schemes) up to conjugacy, and that $[\overline{s}^{\Pi}]$ coincides with the morphism obtained by switching the two factors. The main purpose of this section is to give characterizations of certain specific choices within these conjugacy classes of homomorphisms.

**Definition 2.2.**

(i) We shall denote by

$$(C^X_{i_1})\ \big(\text{resp., } (C^X_{p_1}), (C^X_{p_2}), (C^X_d)\big)$$

a *choice of a specific homomorphism* [i.e., in the sense that it is *not* subject to conjugacy indeterminacy!]

$$i_1^\Pi : \Pi_{\overline{X}_x^{\log}} \longrightarrow \Pi_{X_2}$$

$$\big(\text{resp., } p_1^\Pi : \Pi_{X_2} \longrightarrow \Pi_X,\ p_2^\Pi : \Pi_{X_2} \longrightarrow \Pi_X,\ d^\Pi : \Pi_{\mathbb{D}^{\log}} \longrightarrow \Pi_{X_2}\big)$$

induced by the morphism of log schemes $i_1 : \overline{X}_x^{\log} \to \overline{X}_2^{\log}$ (resp., $p_1 : \overline{X}_2^{\log} \to \overline{X}^{\log}$, $p_1 : \overline{X}_2^{\log} \to \overline{X}^{\log}$, $\mathbb{D}^{\log} \to \overline{X}_2^{\log}$).

(ii) We shall denote by

$$(C^X_{\tilde{x}})\ \big(\text{resp., } (C^X_D)\big)$$

a *choice of a specific subgroup* — i.e., of a *specific decomposition group* —

$$D_{\tilde{x}} \subseteq \Pi_{\overline{X}_x^{\log}}\ \big(\text{resp., } D_X \subseteq \Pi_{X_2}\big)$$

associated to $\tilde{x}^{\log}$ of $\overline{X}_x^{\log}$ (resp., the diagonal divisor of $\overline{X}_2$), among the various conjugates of this subgroup. Note that such a choice determines a *choice of a specific subgroup* — i.e., of a *specific inertia group* —

$$I_{\tilde{x}} := D_{\tilde{x}} \cap \Delta_{X^1_{2/1}} \subseteq \Pi_{\overline{X}_x^{\log}}\ \big(\text{resp., } I_X := D_X \cap \Delta_{X^1_{2/1}} \subseteq \Pi_{X_2}\big)$$

among the various conjugates of this subgroup.

(iii) Assume that we have fixed a choice $(C^X_{\tilde{x}})$ of a specific decomposition group $D_{\tilde{x}} \subseteq \Pi_{\overline{X}_x^{\log}}$ (hence also of a specific inertia group $I_{\tilde{x}} \subseteq \Pi_{\overline{X}_x^{\log}}$). Then we shall denote by

$$(C^{(-)X}_\sigma)$$

a *choice of a specific section*

$$\sigma^{(-)} : G_K^{(-)} \longrightarrow D_{\tilde{x}}^{(-)}$$

— where the symbol $(-)$ denotes either the presence or absence of "$\dagger$" [thus, a choice $(C^X_\sigma)$ determines a unique choice $(C^{\dagger X}_\sigma)$ by restriction] — of the natural surjection $D_{\tilde{x}}^{(-)} \to G_K^{(-)}$ (cf. Remark 2.2.1) and by

$$(C^X_\delta)$$

a *choice of a specific 1-cocycle map*

$$\delta : G_K^\dagger \longrightarrow I_{\tilde{x}}$$

representing the Kummer class $-1 \in (K^\times)^\wedge$ (cf. Remark 2.2.1).

Before proceeding, we pause to make a remark concerning Definition 2.2.

**Remark 2.2.1.**

(i) Recall that the natural surjection $D_{\tilde{x}} \twoheadrightarrow G_K$ (which, since $G_K$ is *abelian*, is *uniquely determined* without any conjugacy indeterminacies) has a section. Indeed, when $x \in X$ (resp., $x \in \overline{X} \setminus X$), fixing a choice of such a section is equivalent to extracting roots of any local uniformizer(s) of the divisor(s) $\mathbb{D} \subseteq \overline{X}_2$ (resp., $\mathbb{D} \subseteq \overline{X}_2$ and $\overline{X}_x \subseteq \overline{X}_2$) at $\tilde{x}$.

(ii) We shall consider the restriction map $H^1(G_K, I_{\tilde{x}}) \to H^1(G_K^\dagger, I_{\tilde{x}})$ of cohomology groups induced by the natural inclusion $G_K^\dagger \hookrightarrow G_K$. Since $G_K^\dagger$ is the maximal pro-$l$ subgroup of $G_K$ and $I_{\tilde{x}}$ is isomorphic to $\mathbb{Z}_l(1)$ as a $G_K$-module, this restriction map determines an isomorphism of $H^1(G_K, I_{\tilde{x}})$ with $H^1(G_K^\dagger, I_{\tilde{x}})$, hence also with the maximal pro-$l$ completion $(K^\times)^\wedge$ of the multiplicative group $K^\times$ of $K$. Therefore, if we denote by $Z^1(G_K^\dagger, I_{\tilde{x}})$ (resp., $Z^1(G_K, I_{\tilde{x}})$) the set of (continuous) 1-cocycles of $G_K^\dagger$ (resp., $G_K$) with coefficients in $I_{\tilde{x}}$, then it makes sense to refer to any element of $Z^1(G_K^\dagger, I_{\tilde{x}})$ (resp., $Z^1(G_K, I_{\tilde{x}})$) belonging to the inverse image of $a \in (K^\times)^\wedge \cong H^1(G_K^\dagger, I_{\tilde{x}})$ (resp., $\cong H^1(G_K^\dagger, I_{\tilde{x}})$), via the natural surjection, as a *(continuous) 1-cocycle representing the Kummer class* $a$.

**Lemma 2.3.**

*For any choice* $(C_{\tilde{x}}^X)$ *(resp.,* $(C_D^X)$*) of a specific decomposition group* $D_{\tilde{x}} \subseteq \Pi_{\overline{X}_x^{\log}}$ *(resp.,* $D_X \subseteq \Pi_{X_2}$*),* $I_{\tilde{x}}$ *(resp.,* $I_X$*) is normally terminal in* $\Delta^1_{X_{2/1}}$ *(cf. §0), and* $D_{\tilde{x}}^{(-)}$ *(resp.,* $D_X^{(-)}$*) — where the symbol* $(-)$ *denotes either the presence or absence of "†" — coincides with* $N_{\Pi_{\overline{X}_x^{\log}}}(I_{\tilde{x}})^{(-)}$ *(resp.,* $N_{\Pi_{X_2}}(I_X)^{(-)}$*)(cf. §0).*

*Proof.* Recall that, by definition, we have $I_{\tilde{x}} = D_{\tilde{x}} \cap \Delta^1_{X_{2/1}} \subseteq \Pi_{X_2}$ and $I_X = D_X \cap \Delta^1_{X_{2/1}} \subseteq \Pi_{X_2}$. Next, let us recall the well-known fact (cf., e.g., [17], (2.3.1)) that $I_{\tilde{x}}$ and $I_X$ are normally terminal (cf. §0) in $\Delta^1_{X_{2/1}}$. Thus, the *resp'd* assertion follows immediately from the fact that $p_1^\Pi$ maps $D_X$ onto $\Pi_X$. On the other hand, the *non-resp'd* assertion follows immediately from the observation that the images of $D_{\tilde{x}}$ and $\Pi_{\overline{X}_x^{\log}}$ via $p_1^\Pi \circ i_1^\Pi$ *coincide*. This observation is a consequence of the geometry of the corresponding morphisms of log schemes, which implies that both of these images coincide with a decomposition group $\subseteq \Pi_X$ associated to the point $x$. $\square$

**Lemma 2.4.**

(i) *If we fix a choice* $(C_d^X)$ *of* $d^\Pi : \Pi_{\mathbb{D}^{\log}} \to \Pi_{X_2}$*, then there exists a unique choice* $(C_D^X)$ *of* $D_X \subseteq \Pi_{X_2}$ *such that the image of* $d^\Pi$ *coincides with* $D_X$*. By contrast, if we fix a choice* $(C_D^X)$ *of* $D_X \subseteq \Pi_{X_2}$*, then there exists a (not necessarily unique!) choice* $(C_d^X)$ *of* $d^\Pi : \Pi_{\mathbb{D}^{\log}} \to \Pi_{X_2}$ *such that the image of* $d^\Pi$ *coincides with* $D_X$*.*

(ii) *If we fix a triple of choices* $(C_{i_1}^X)$*,* $(C_{p_1}^X)$ *and* $(C_{\tilde{x}}^X)$*, then there exists a unique pair consisting of a choice* $(C_{p_2}^X)$ *of* $p_2^\Pi : \Pi_{X_2} \to \Pi_X$ *and a choice* $(C_D^X)$ *of* $D_X \subseteq \Pi_{X_2}$ *that satisfy the following conditions:*

(1) *The image of the inertia group* $I_X \subseteq D_X$ *in* $\Pi_{X_2}$ *coincides with the image of* $I_{\tilde{x}}$ *via* $i_1^{\Pi}$.

(2) *The homomorphism* $(p_1^{\Pi}, p_2^{\Pi}) : \Pi_{X_2} \to \Pi_X \times_{G_K} \Pi_X$ *maps* $D_X$ *onto the image of the diagonal embedding* $\Pi_X \hookrightarrow \Pi_X \times_{G_K} \Pi_X$.

*Proof.* Assertion (i) follows immediately from the definitions of $\Pi_{\mathbb{D}^{\log}}$ and $D_X$. Next, we consider assertion (ii). First, let us observe that it follows immediately from the various definitions involved that $I_X$ and $I_{\tilde{x}}$ are $\Pi_{X_2}$-conjugate. Since, by Lemma 2.3, $D_X$ coincides with the normalizer of $I_X$ in $\Pi_{X_2}$, it suffices to take $D_X$ to be the normalizer of $I_{\tilde{x}}$ in $\Pi_{X_2}$ and $p_2^{\Pi}$ to be such that the condition $p_1^{\Pi} = p_2^{\Pi}$ is satisfied on $D_X$. Uniqueness follows immediately from the conditions (1), (2) and the surjectivity of the restriction of $p_2^{\Pi}$ to $D_X$. $\square$

**Proposition 2.5.**
*If we fix arbitrary choices* $(C_{i_1}^X)$, $(C_{p_1}^X)$, $(C_{\tilde{x}}^X)$, $(C_{\sigma}^{\dagger X})$, *and* $(C_{\delta}^X)$, *then there exists a unique triple of choices consisting of* $(C_{p_2}^X)$, $(C_D^X)$, *and a* **choice of a specific automorphism** *induced by* $s : \overline{X}_2^{\log} \xrightarrow{\sim} \overline{X}_2^{\log}$

$$s^{\dagger} : \Pi_{X_2}^{\dagger} \xrightarrow{\sim} \Pi_{X_2}^{\dagger}$$

*— which we shall denote by* $(C_s^{\dagger X})$ *— satisfying the two conditions (1), (2) stated in Lemma 2.4, (ii), as well as the following conditions:*

(1) *the morphism* $\overline{s}^{\dagger} : \Pi_X^{\dagger} \times_{G_K^{\dagger}} \Pi_X^{\dagger} \xrightarrow{\sim} \Pi_X^{\dagger} \times_{G_K^{\dagger}} \Pi_X^{\dagger}$ *induced by passing to the quotient* $\Pi_{X_2}^{\dagger} \overset{p^{\dagger}}{\twoheadrightarrow} \Pi_X^{\dagger} \times_{G_K^{\dagger}} \Pi_X^{\dagger}$ *determined by* $p_1^{\Pi}, p_2^{\Pi}$ *coincides with the morphism obtained by switching the two factors.*

(2) $s^{\dagger}$ *preserves* $D_X^{\dagger} \subseteq \Pi_{X_2}^{\dagger}$, *and the restriction* $s^{\dagger}|_{D_X^{\dagger}} : D_X^{\dagger} \xrightarrow{\sim} D_X^{\dagger}$ *corresponds to an automorphism induced by* $s_{\mathbb{D}} : \mathbb{D}^{\log} \xrightarrow{\sim} \mathbb{D}^{\log}$ *via the identification* $\Pi_{\mathbb{D}^{\log}}^{\dagger} \xrightarrow{\sim} D_X^{\dagger}$ *determined by some choice of a specific homomorphism* $d^{\Pi} : \Pi_{\mathbb{D}^{\log}} \to \Pi_{X_2}$ *whose image coincides with* $D_X$ *(cf. Lemma 2.4, (i)).*

(3) *The continuous function* $G_K^{\dagger} \to \Pi_{X_2}^{\dagger}$ *defined by*

$$g \mapsto (s^{\dagger} \circ \sigma^{\dagger})(g) \cdot \sigma^{\dagger}(g)^{-1}$$

*is valued in* $I_{\tilde{x}} \subseteq \Pi_{X_2}^{\dagger}$ *and coincides with the "*$\delta$*" determined by* $(C_{\delta}^X)$.

*In particular,* $s^{\dagger}$ *induces the identity morphism on* $I_X \subseteq \Pi_{X_2}^{\dagger}$.

*Proof.* We begin by proving the existence portion. Let us consider the following (not necessarily commutative) diagram

$$\begin{array}{ccccc} \Pi^\dagger_{\mathbb{D}^{\log}} & \xrightarrow{d^\dagger} & \Pi^\dagger_{X_2} & \xrightarrow{p^\dagger} & \Pi^\dagger_X \times_{G^\dagger_K} \Pi^\dagger_X \\ {\scriptstyle \underline{s}^\dagger}\downarrow & & {\scriptstyle s^\dagger}\downarrow & & {\scriptstyle \overline{s}^\dagger}\downarrow \\ \Pi^\dagger_{\mathbb{D}^{\log}} & \xrightarrow{d^\dagger} & \Pi^\dagger_{X_2} & \xrightarrow{p^\dagger} & \Pi^\dagger_X \times_{G^\dagger_K} \Pi^\dagger_X \end{array} \qquad (*)^\dagger$$

induced by $(*)^\Pi$ consisting of the horizontal arrows arising from the choice $(C^X_{p_1})$ fixed in advance and the pair of choices $(C^X_{p_2})$, $(C^X_d)$ obtained by applying Lemma 2.4 (i), (ii), and arbitrary choices of the vertical arrows. By the surjectivity of $p^\dagger$, we can take $s^\dagger, \overline{s}^\dagger$ such that the right-hand square of the diagram $(*)^\dagger$ commutes, and condition (1) is satisfied. The commutativity of the rectangle in $(*)^\dagger$ up to conjugacy implies that there exists $\lambda \in \Pi^\dagger_X \times_{G^\dagger_K} \Pi^\dagger_X$ such that $\overline{s}^\dagger \circ (p^\dagger \circ d^\dagger) = \mathrm{Inn}(\lambda) \circ (p^\dagger \circ d^\dagger) \circ \underline{s}^\dagger$ (where $\mathrm{Inn}(\lambda)$ denotes the inner automorphism obtained by conjugating by $\lambda$). By the construction of the choice $(C^X_d)$ (cf. condition (2) of Lemma 2.4 (ii)), $p^\dagger \circ d^\dagger$ maps $\Pi^\dagger_{\mathbb{D}^{\log}}$ onto the subgroup of diagonal elements of $\Pi^\dagger_X \times_{G^\dagger_K} \Pi^\dagger_X$; thus, $\mathrm{Inn}(\lambda)$ preserves this diagonal subgroup. Since $\Pi^\dagger_X$ is center-free (by Proposition 1.4 (iii)), we thus conclude that $\lambda$ is a diagonal element. Thus, by taking a lifting $\tilde{\lambda} \in \Pi^\dagger_{\mathbb{D}^{\log}}$ of $\lambda$ and replacing $\underline{s}^\dagger$ by $\mathrm{Inn}(\tilde{\lambda}^{-1}) \circ \underline{s}^\dagger$, we can make the rectangle in $(*)^\dagger$ commute in the strict sense. Next, we observe (by applying again the commutativity of the rectangle in $(*)^\dagger$ up to conjugacy) that $s^\dagger \circ d^\dagger = \mathrm{Inn}(\mu) \circ d^\dagger \circ \underline{s}^\dagger$ for some $\mu \in \Pi^\dagger_{X_2}$. By the commutativity of the rectangle in $(*)^\dagger$, $\mu$ projects via $p^\dagger$ into the center of $\Pi^\dagger_X \times_{G^\dagger_K} \Pi^\dagger_X$, hence (by Proposition 1.4, (iii)), to the unit element. Therefore, by replacing $s^\dagger$ by $\mathrm{Inn}(\mu^{-1}) \circ s^\dagger$, we conclude that we may choose $\underline{s}^\dagger$, $s^\dagger$, and $\overline{s}^\dagger$ so that the diagram $(*)^\dagger$ *commutes*, and, moreover, conditions (1) and (2) are satisfied.

Next, observe that by restricting $s^\dagger$ to $D^\dagger_X$, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_X & \longrightarrow & D^\dagger_X & \xrightarrow{p^\dagger|_{D^\dagger_X}} & \Pi^\dagger_X & \longrightarrow & 1 \\ & & {\scriptstyle s^\dagger|_{I_X}}\downarrow\wr & & {\scriptstyle s^\dagger|_{D^\dagger_X}}\downarrow\wr & & {\scriptstyle \mathrm{id}}\downarrow\wr & & \\ 1 & \longrightarrow & I_X & \longrightarrow & D^\dagger_X & \xrightarrow{p^\dagger|_{D^\dagger_X}} & \Pi^\dagger_X & \longrightarrow & 1 \end{array}$$

in which the right-hand vertical arrow is the identity automorphism of $\Pi^\dagger_X$. Write $\mathbb{M} \subseteq \mathbb{Q}$ for the monoid of positive rational numbers with $l$-power denominators, and $\mathcal{N}$ for the monoid of global sections of the sheaf of monoids defining the log structure on a universal geometrically pro-$l$ két covering of $x^{\log} \times_{\overline{X}^{\log}} \mathbb{D}^{\log}$. When $x \in X$ (resp., $x \in \overline{X} \setminus X$), $\mathcal{N}$ admits a direct sum decomposition $\mathcal{N} \cong \mathbb{M} \oplus \overline{K}^\times$ (resp., $\mathcal{N} \cong \mathbb{M} \oplus \mathbb{M} \oplus \overline{K}^\times$), where (cf. Remark 2.2.1 (i)) the first factor (resp., first two factors) of the direct sum arise(s) from extracting roots of a local uniformizer of the divisor $\mathbb{D} \subseteq \overline{X}_2$ at $\tilde{x}$ (resp., of local uniformizers of the two irreducible divisors defining the log structure of $\overline{X}_2^{\log}$ at $\tilde{x}$) in a fashion compatible with the choice ($C_{\sigma}^{\dagger X}$) of $\sigma$. Here, in the resp'd case, we assume that the first factor "$\mathbb{M}$" corresponds to the divisor $\mathbb{D} \subseteq \overline{X}_2$. Next, observe that it follows from Lemma 2.1 (ii), together with the well-known local structure of $\overline{X}_2$ in a neighborhood of $\tilde{x}$, that the automorphism of $\mathcal{N} \cong \mathbb{M} \oplus \overline{K}^{\times}$ (resp., $\mathcal{N} \cong \mathbb{M} \oplus \mathbb{M} \oplus \overline{K}^{\times}$) induced by the automorphism $\underline{s}^{\dagger}$ of $\Pi_{\mathbb{D}^{\log}}^{\dagger}$ may be expressed in the form

$$(\frac{a}{l^m}, k) \mapsto (\frac{a}{l^m}, (-1)^{\frac{a}{l^m}} \cdot k)$$

$$\big(\text{resp.}, (\frac{a_1}{l^{m_1}}, \frac{a_2}{l^{m_2}}, k) \mapsto (\frac{a_1}{l^{m_1}}, \frac{a_2}{l^{m_2}}, (-1)^{\frac{a_1}{l^{m_1}}} \cdot k)\big)$$

for a suitable choice of a projective system $\{(-1)^{\frac{1}{l^m}}\}_{m \in \mathbb{Z}_{\geq 0}}$ of $l$-power roots of $-1$. In particular, we conclude that the restriction $s^{\dagger}|_{I_X}$ is the identity morphism, and that the 1-cocycle $G_K^{\dagger} \ni g \mapsto (s^{\dagger} \circ \sigma^{\dagger})(g) \cdot \sigma^{\dagger}(g)^{-1}$ is valued in $I_X = I_{\tilde{x}}$ (cf. condition (1) of Lemma 2.4 (ii)). Therefore, by replacing $\underline{s}^{\dagger}$, $s^{\dagger}$ by their composites with a suitable $I_X$-inner automorphism, we may assume that condition (3) is satisfied. This completes the proof of the existence assertion.

Next we prove the uniqueness portion. If $s_1^{\dagger}$, $s_2^{\dagger}$ are two maps that satisfy conditions (1), (2) and (3), then $s_1^{\dagger} \circ (s_2^{\dagger})^{-1} = \mathrm{Inn}(\eta) \in \mathrm{Aut}(\Pi_{X_2}^{\dagger})$ for some $\eta \in \Pi_{X_2}^{\dagger}$, and it follows from condition (2) that $\mathrm{Inn}(\eta)$ preserves the subgroup $D_X^{\dagger} \subseteq \Pi_{X_2}^{\dagger}$. Since $D_X^{\dagger}$ is normally terminal in $\Pi_{X_2}^{\dagger}$ (cf. Lemma 2.3), we thus conclude that $\eta$ is in $D_X^{\dagger}$. Moreover, it follows from condition (1) and the fact that $\Pi_X^{\dagger}$ is center-free (cf. Proposition 1.4, (iii)), that $\eta$ lies in $\mathrm{Ker}\big(D_X^{\dagger} \xrightarrow{p^{\dagger}|_{D_X^{\dagger}}} \Pi_X^{\dagger} \times_{G_K^{\dagger}} \Pi_X^{\dagger}\big)$, i.e., $\eta \in I_X$. On the other hand, since the section $\sigma^{\dagger}$ acts *faithfully* on $I_X$ via the cyclotomic character, condition (3) implies that $\eta$ is the unit element, i.e., that $s_1^{\dagger} = s_2^{\dagger}$. □

**Remark 2.5.1.**

In the case $l \neq 2$, $-1$ coincides with the unit element 1 in $(K^{\times})^{\wedge}$. Therefore, in the statement of Proposition 2.5, by taking the choice $(C_{\delta}^{\dagger})$ to be such that 1-cocycle map $\delta$ is trivial, we may obtain an "$s^{\dagger}$" satisfying $s^{\dagger} \circ \sigma^{\dagger} = \sigma^{\dagger}$.

## 3. The proof of Theorem A

This section is devoted to proving Theorem A. We begin with a review of the notation and setup. Let $l$ be a prime number, $K$ a finite field in which $l$ is invertible, and $\overline{K}$ a separable closure of $K$. We shall denote by $G_K$ the Galois group of $\overline{K}$ over $K$. Next, let $X$ be a hyperbolic curve over $K$ of type

$(g_X, r_X)$ and $x^{\log}$ a strict $K$-rational log point of $\overline{X}^{\log} := \overline{X}_1^{\log}$; write $\overline{X}_x^{\log} := x^{\log} \times_{\overline{X}^{\log}, p_1} \overline{X}_2^{\log}$, $\tilde{x}^{\log} := x^{\log} \times_{\overline{X}^{\log}} \mathbb{D}^{\log}$. In addition, we assume that we have fixed choices $(C^X_{i_1})$, $(C^X_{p_1})$, $(C^X_{\tilde{x}})$, $(C^X_{\sigma})$, $(C^X_{\delta})$ [i.e., in the sense that they are *not* subject to conjugacy indeterminacy].

As a first step, we define two actions of $G_K$ on various topological groups, graded Lie algebras, and linear algebraic groups associated to the fundamental groups of $\overline{X}_x^{\log}$ and $X_2$. As we shall discuss in the following, these two actions are mapped to one another via the morphisms induced by the switching morphism obtained in Section 2.

**Definition 3.1.**

(i) The choice $(C_\sigma)$ of a section $\sigma : G_K \to D_{\tilde{x}}$ determines, by composing with the natural morphism $D_{\tilde{x}} \to \Pi_{\overline{X}_x^{\log}}$ (resp., $D_{\tilde{x}} \to \Pi_{X_2}$, $D_{\tilde{x}} \to \Pi_{X^{\times 2}}$), a natural action of $G_K$ by conjugation on $\Delta^1_{X_{2/1}} \cong \mathrm{Ker}(\Pi_{\overline{X}_x^{\log}} \overset{i_1^{\Pi} \circ p_1^{\Pi}}{\to} \Pi_X)$ (resp., $\Delta_{X_2}$, $\Delta_{X^{\times 2}}$), hence also on

$$\mathrm{Gr}^1_{X_{2/1}} := \mathrm{Gr}_{\mathbb{Q}_l}(\Delta^1_{X_{2/1}}),$$

$$\big(\text{resp., } \mathrm{Gr}_{X_2} := \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_2}),\ \mathrm{Gr}_{X^{\times 2}} := \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X^{\times 2}})\big),$$

$$\mathrm{Lie}^1_{X_{2/1}} := \mathrm{Lie}(\Delta^1_{X_{2/1}}(1/\infty)),$$

$$\big(\text{resp., } \mathrm{Lie}_{X_2} := \mathrm{Lie}(\Delta_{X_2}(1/\infty)),\ \mathrm{Lie}_{X^{\times 2}} := \mathrm{Lie}(\Delta_{X^{\times 2}}(1/\infty))\big),$$

$$\mathrm{Lin}^1_{X_{2/1}} := \mathrm{Lin}(\Delta^1_{X_{2/1}}(1/\infty))(\mathbb{Q}_l).$$

$$\big(\text{resp., } \mathrm{Lin}_{X_2} := \mathrm{Lin}(\Delta_{X_2}(1/\infty))(\mathbb{Q}_l),\ \mathrm{Lin}_{X^{\times 2}} := \mathrm{Lin}(\Delta_{X^{\times 2}}(1/\infty))(\mathbb{Q}_l)\big).$$

In the following, we regard these objects as being equipped with these $G_K$-actions. From the discussion in Definition 1.5 (ii), we have the following commutative diagram consisting of $G_K$-equivariant morphisms

$$\begin{array}{ccccc}
\Delta^1_{X_{2/1}} & \xrightarrow{i_1} & \Delta_{X_2} & \xrightarrow{p} & \Delta_{X^{\times 2}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Lin}^1_{X_{2/1}} & \xrightarrow{i_1^{\mathrm{Lin}}} & \mathrm{Lin}_{X_2} & \xrightarrow{p^{\mathrm{Lin}}} & \mathrm{Lin}_{X^{\times n}}
\end{array}$$

and topological groups equipped with $G_K$-actions

$$\Delta^{\mathrm{Lie}}_{X_2} := \Delta_{X^{\times 2}} \times_{\mathrm{Lin}_{X^{\times 2}}} \mathrm{Lin}_{X_2}, \quad \Pi^{\mathrm{Lie}}_{X_2} := \Delta^{\mathrm{Lie}}_{X_2} \rtimes G_K$$

as well as $G_K$-equivariant homomorphisms of topological groups

$$\mathrm{Int}^{\Delta}_X : \Delta_{X_2} \to \Delta^{\mathrm{Lie}}_{X_2}, \quad \mathrm{Int}^{\Pi}_X : \Pi_{X_2} \to \Pi^{\mathrm{Lie}}_{X_2}.$$

(ii) Next, the choice $(C_\sigma)$, $(C_\delta)$ yields a new section of the surjective homomorphism $D_{\tilde{x}} \twoheadrightarrow G_K$

$$\sigma_\delta : G_K \longrightarrow D_{\tilde{x}}$$
$$g \mapsto \delta(g) \cdot \sigma(g)$$

— which is a homomorphism of topological groups. Then the section $\sigma_\delta$ determines, in a similar way to (i), a natural action of $G_K$ by conjugation on

$$\breve{\mathrm{Gr}}^1_{X_{2/1}} := \mathrm{Gr}_{\mathbb{Q}_l}(\Delta^1_{X_{2/1}}),$$
$$\big(\text{resp., } \breve{\mathrm{Gr}}_{X_2} := \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_2}),\ \breve{\mathrm{Gr}}_{X^{\times 2}} := \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X^{\times 2}})\big),$$

$$\breve{\mathrm{Lie}}^1_{X_{2/1}} := \mathrm{Lie}(\Delta^1_{X_{2/1}}(1/\infty)),$$
$$\big(\text{resp., } \breve{\mathrm{Lie}}_{X_2} := \mathrm{Lie}(\Delta_{X_2}(1/\infty)),\ \breve{\mathrm{Lie}}_{X^{\times 2}} := \mathrm{Lie}(\Delta_{X^{\times 2}}(1/\infty))\big),$$

$$\breve{\mathrm{Lin}}^1_{X_{2/1}} := \mathrm{Lin}(\Delta^1_{X_{2/1}}(1/\infty))(\mathbb{Q}_l).$$
$$\big(\text{resp., } \breve{\mathrm{Lin}}_{X_2} := \mathrm{Lin}(\Delta_{X_2}(1/\infty))(\mathbb{Q}_l),\ \breve{\mathrm{Lin}}_{X^{\times 2}} := \mathrm{Lin}(\Delta_{X^{\times 2}}(1/\infty))(\mathbb{Q}_l)\big).$$

— where, in the following, we regard these objects as being equipped with the $G_K$-actions just defined — as well as topological groups equipped with $G_K$-actions

$$\breve{\Delta}^{\mathrm{Lie}}_{X_2} := \Delta_{X^{\times 2}} \times_{\breve{\mathrm{Lin}}_{X^{\times 2}}} \breve{\mathrm{Lin}}_{X_2}, \quad \breve{\Pi}^{\mathrm{Lie}}_{X_2} := \breve{\Delta}^{\mathrm{Lie}}_{X_2} \rtimes G_K.$$

Next, let us recall that by applying Proposition 2.5, together with the choices $(C_{i_1})$, $(C_{p_1})$, $(C_{\tilde{x}})$, $(C_\delta)$ and the choice $(C^\dagger_\sigma)$ determined naturally by $(C_\sigma)$, we obtain a choice $(C^\dagger_s)$ of a specific automorphism $s^\dagger : \Pi^\dagger_{X_2} \to \Pi^\dagger_{X_2}$. Let $s^\Pi : \Pi_{X_2} \xrightarrow{\sim} \Pi_{X_2}$ be an automorphism that induces the outer automorphism determined by the switching morphism $s_X : \overline{X}^{\log}_2 \to \overline{X}^{\log}_2$ and is compatible with $s^\dagger : \Pi^\dagger_{X_2} \to \Pi^\dagger_{X_2}$. Then, by Lemma 3.2 below, we obtain $G_K$-equivariant isomorphisms of topological groups

$$s^{\Delta^{\mathrm{Lie}}} : \Delta^{\mathrm{Lie}}_{X_2} \xrightarrow{\sim} \breve{\Delta}^{\mathrm{Lie}}_{X_2}, \quad s^{\Pi^{\mathrm{Lie}}} : \Pi^{\mathrm{Lie}}_{X_2} \xrightarrow{\sim} \breve{\Pi}^{\mathrm{Lie}}_{X_2}$$

induced by $s^\Pi$ and a (non-$G_K$-equivariant) commutative diagram as follows:

$$\begin{array}{ccc} \Delta_{X_2} & \xrightarrow{s^\Delta} & \Delta_{X_2} \\ {\scriptstyle \mathrm{Int}^\Delta_X}\downarrow & & \downarrow{\scriptstyle \mathrm{Int}^\Delta_X} \\ \Delta^{\mathrm{Lie}}_{X_2} & \xrightarrow{s^{\Delta^{\mathrm{Lie}}}} & \breve{\Delta}^{\mathrm{Lie}}_{X_2} \end{array} \qquad \begin{array}{ccc} \Pi_{X_2} & \xrightarrow{s^\Pi} & \Pi_{X_2} \\ {\scriptstyle \mathrm{Int}^\Pi_X}\downarrow & & \downarrow{\scriptstyle \mathrm{Int}^\Pi_X} \\ \Pi^{\mathrm{Lie}}_{X_2} & \xrightarrow{s^{\Pi^{\mathrm{Lie}}}} & \breve{\Pi}^{\mathrm{Lie}}_{X_2}. \end{array}$$

**Lemma 3.2.**
*The $G_K$-action induced by $\sigma_\delta$ (cf. Definition 3.1 (ii)) on $\Delta_{X_2}$ (hence also on $\breve{\mathrm{Gr}}_{X_2}$, $\breve{\mathrm{Lie}}_{X_2}$, $\breve{\mathrm{Lin}}_{X_2}$ and $\breve{\Delta}^{\mathrm{Lie}}_{X_2}$) coincides with the action*

$$G_K \longrightarrow \mathrm{Aut}(\Delta_{X_2})$$
$$g \mapsto \mathrm{Inn}(s^\Pi \circ i_1^\Pi \circ \sigma(g)).$$

*Proof.* This follows immediately from condition (3) of Proposition 2.5, together with the definition of the $G_K$-action induced by $\sigma_\delta$. □

**Lemma 3.3.**
$\mathrm{Int}^\Delta_X$ *and* $\mathrm{Int}^\Pi_X$ *are injective (cf.* [4]*, Lemma 4.3 in the case where $X$ is proper).*

*Proof.* It suffices to verify that $\Delta_{X_2} \to \mathrm{Lin}_{X_2}$ is injective. But this follows from the discussion in Definition 1.5 (ii) and the fact that $\bigcap_{m\geq 1} \Delta_X(m) = 1$ (cf. [18], Corollary 2.6). □

Next, we shall construct certain graded Lie algebras equipped with a $G_K$-action — which we shall denote by $\mathcal{L}^1_X$ and $\mathcal{L}^2_X$ — by using various subgroups of $\Pi_{\overline{X}^{\log}_x}$. Comparing these graded Lie algebras to the graded Lie algebras discussed above (cf. Lemma 3.5, 3.6) will allow us to reconstruct various groups associated to $\Pi_{X_2}$ from those associated to $\Pi_{\overline{X}^{\log}_x}$ (cf. Proposition 3.8). This will play an important role in the proof of Theorem A.

**Definition 3.4.**

(i) For each $j = 1, 2, \cdots, r$, let us fix a choice of the inertia subgroup $I_j \subseteq \Delta^1_{X_{2/1}} \cong \mathrm{Ker}(\Pi_{\overline{X}^{\log}_x} \stackrel{p_1^\Pi \circ i_1^\Pi}{\to} \Pi_X)$ associated to the $j$-th cusp (relative to some ordering of the cusps of $X \times_K \overline{K}$) among the various $\Delta^1_{X_{2/1}}$-conjugates of these subgroups. Then, we have canonical isomorphisms

$$\eta_j : I_{\tilde{x}} \stackrel{\sim}{\longrightarrow} I_j \quad (j = 1, 2, \cdots, r)$$

Indeed, recall that the *kernel* of the natural quotient $(\Delta^1_{X_{2/1}})^{\mathrm{ab}} \twoheadrightarrow \Delta^{\mathrm{ab}}_{\overline{X}}$ coincides with the submodule $\bigoplus_{j'=1}^r I_{j'} \subseteq (\Delta^1_{X_{2/1}})^{\mathrm{ab}}$; thus, since the subgroup $I_{\tilde{x}}$ of $(\Delta^1_{X_{2/1}})^{\mathrm{ab}}$ is contained in this kernel, it follows that the composite $I_{\tilde{x}} \hookrightarrow \bigoplus_{j'=1}^r I_{j'} \twoheadrightarrow I_j \stackrel{(-1)}{\to} I_j$ of this inclusion with the natural projection to $j$-th factor multiplied by $-1$ yields the required isomorphism.

For $n = 1, 2$ we shall denote by $\mathcal{V}^n$ the completion with respect to the filtration topology of the free Lie algebra generated by

$$V^n := I_{\tilde{x}} \oplus \big(\bigoplus_{j=1}^r I_j \oplus \Delta^{\mathrm{ab}}_{\overline{X}}\big)^{\oplus n}$$

equipped with a natural grading (hence also a filtration) by taking $I_{\tilde{x}}, I_j$ to be of weight 2, $\Delta^{\mathrm{ab}}_{\overline{X}}$ to be of weight 1.

(ii) If $X$ has *genus* $\geq 1$, then we shall write

$$M_X := \mathrm{Hom}_{\mathbb{Z}_l}(H^2(\Delta_{\overline{X}}, \mathbb{Z}_l), \mathbb{Z}_l).$$

Note that $M_X$ is canonically isomorphic to $I_{\tilde{x}}$ as a $G_K$-module. Indeed, recall the natural quotient $(\Delta^1_{X_{2/1}}/ < I_j >_{j=1,\cdots,r}) \twoheadrightarrow \Delta_{\overline{X}}$; the associated maximal cuspidally central quotient (cf. [12], Definition 1.1 (i)) yields an extension of $\Delta_{\overline{X}}$ by $I_{\tilde{x}}$; this extension determines a generator of the rank one free $\mathbb{Z}_l$-module $H^2(\Delta_{\overline{X}}, I_{\tilde{x}}) \cong \mathrm{Hom}_{\mathbb{Z}_l}(M_X, I_{\tilde{x}})$ (cf., e.g., [11], Lemma 4.2, (i), (ii), (iii) ), hence an isomorphism $M_X \xrightarrow{\sim} I_{\tilde{x}}$, as desired.

The cup product on the group cohomology of $\Delta_{\overline{X}}$

$$\bigwedge^2 H^1(\Delta_{\overline{X}}, M_X) \longrightarrow H^2(\Delta_{\overline{X}}, M_X \otimes_{\mathbb{Z}_l} M_X) \cong M_X$$

determines an isomophism

$$\big(H^1(\Delta_{\overline{X}}, M_X) \cong\big) \ \mathrm{Hom}(\Delta^{\mathrm{ab}}_{\overline{X}}, M_X) \ \stackrel{\sim}{\longrightarrow} \ \Delta^{\mathrm{ab}}_{\overline{X}} \ \big(\cong \mathrm{Hom}(H^1(\Delta_{\overline{X}}, M_X), M_X)\big),$$

hence composites of natural homomorphisms

$$\phi : I_{\tilde{x}} \stackrel{\sim}{\longrightarrow} M_X \longrightarrow \bigwedge^2 \Delta^{\mathrm{ab}}_{\overline{X}}, \quad \psi : \bigwedge^2 \Delta^{\mathrm{ab}}_{\overline{X}} \longrightarrow M_X \stackrel{\sim}{\longrightarrow} I_{\tilde{x}}.$$

If $X$ has *genus* 0, then we take $\phi$, $\psi$ to be the zero maps.

(iii) We define $\mathcal{L}^n_X$ to be the quotient of $\mathcal{V}^n$ by the relations determined by the images of the following morphisms (which are patterned after the presentations given in Proposition 1.9):

(1) When $n = 1$,

$\bullet_1$ $I_{\tilde{x}} \longrightarrow \mathcal{V}^1(2/3)$ ; $m \mapsto (\mathrm{id}_{I_{\tilde{x}}} + \sum \eta_j + \phi)(m)$

(2) When $n = 2$ $(1 \leq i \leq g,\ 1 \leq j, j' \leq r,\ j \neq j',\ \{k, k'\} = \{1, 2\})$,

$\bullet_1$ $I_{\tilde{x}} \longrightarrow \mathcal{V}^2(2/3)$ ; $m \mapsto m + i_k(\sum \eta_j + \phi)(m)$

$\bullet_2$ $I_{\tilde{x}} \otimes_{\mathbb{Z}_l} \Delta^{\mathrm{ab}}_{\overline{X}} \longrightarrow \mathcal{V}^2(3/4)$ ; $m \otimes a \mapsto [i_k \circ \eta_j(m), i_{k'}(a)]$

$\bullet_3$ $I_{\tilde{x}} \longrightarrow \mathcal{V}^2(4/5)$ ; $m \mapsto [i_k \circ \eta_j(m), i_{k'} \circ \eta_{j'}(m)]$

$\bullet_{4,5}$ $\bigwedge^2 \Delta^{\mathrm{ab}}_{\overline{X}} \longrightarrow \mathcal{V}^2(2/3)$ ; $a \wedge a' \mapsto [i_k(a), i_{k'}(a')] - \psi(a \wedge a')$

— where "[ , ]" denotes the Lie bracket, and for $k = 1, 2$, $i_k : (\bigoplus I_j \oplus \bigwedge^2 \Delta^{\mathrm{ab}}_{\overline{X}}) \hookrightarrow (\bigoplus I_j \oplus \bigwedge^2 \Delta^{\mathrm{ab}}_{\overline{X}})^{\oplus 2}$ denotes the inclusion into the $k$-th factor.

(iv) The natural $G_K$-action on each direct summand in $\mathcal{V}^n$ determines a natural $G_K$-action on $\mathcal{V}^n$. One verifies immediately that the ideal generated by the relations defined in (iii) is *preserved* by this $G_K$-action. Thus, we obtain a natural $G_K$-action on the graded Lie algebra

$$\mathcal{L}^1_X \quad (\text{resp.}, \mathcal{L}^2_X)$$

and a $G_K$-equivariant homomorphism

$$i_1^{\mathcal{L}} : \mathcal{L}_X^1 \longrightarrow \mathcal{L}_X^2$$

of graded Lie algebras determined by the map on generators given by

$$\begin{array}{ccc} I_{\tilde{x}} \oplus \big(\bigoplus_{j=1}^{r} I_j \oplus \Delta_{\overline{X}}^{\mathrm{ab}}\big) & \longrightarrow & I_{\tilde{x}} \oplus \big(\bigoplus_{j=1}^{r} I_j \oplus \Delta_{\overline{X}}^{\mathrm{ab}}\big)^{\oplus 2} \\ (a,b) & \mapsto & (a, i_1(b)), \end{array}$$

as well as a $G_K$-equivariant isomorphism

$$s_X^{\mathcal{L}} : \mathcal{L}_X^2 \xrightarrow{\sim} \mathcal{L}_X^2$$

of graded Lie algebras determined by the map on generators given by

$$\begin{array}{ccc} I_{\tilde{x}} \oplus \big(\bigoplus_{j=1}^{r} I_j \oplus \Delta_{\overline{X}}^{\mathrm{ab}}\big)^{\oplus 2} & \longrightarrow & I_{\tilde{x}} \oplus \big(\bigoplus_{j=1}^{r} I_j \oplus \Delta_{\overline{X}}^{\mathrm{ab}}\big)^{\oplus 2} \\ (a,b_1,b_2) & \mapsto & (a, b_2, b_1). \end{array}$$

**Lemma 3.5.**
*Consider the homomorphism of graded Lie algebras $\mathcal{V}^1 \to \mathrm{Gr}_{\mathbb{Q}_l}(\Delta^1_{X_{2/1}})$ determined by the natural inclusions $\Delta_{\overline{X}}^{\mathrm{ab}} \hookrightarrow \mathrm{Gr}_{\mathbb{Q}_l}(\Delta^1_{X_{2/1}})(1/2)$, $I_{\tilde{x}} \hookrightarrow \mathrm{Gr}_{\mathbb{Q}_l}(\Delta^1_{X_{2/1}})(2/3)$ and $I_j \hookrightarrow \mathrm{Gr}_{\mathbb{Q}_l}(\Delta^1_{X_{2/1}})(2/3)$. This homomorphism of graded Lie algebras factors through $\mathcal{L}_X^1$, and the resulting homomorphism $\mathfrak{h}^1 : \mathcal{L}_X^1 \to \mathrm{Gr}_{\mathbb{Q}_l}(\Delta^1_{X_{2/1}})$ is a $G_K$-equivariant isomorphism of graded Lie algebras, whether we regard $\mathrm{Gr}_{\mathbb{Q}_l}(\Delta^1_{X_{2/1}})$ as the underlying graded Lie algebra (i.e., without $G_K$-action) of $\mathrm{Gr}^1_{X_{2/1}}$ or as the underlying graded Lie algebra of $\breve{\mathrm{Gr}}^1_{X_{2/1}}$.*

*Proof.* The asserted *$G_K$-equivariance* follows immediately from the definitions. Thus, it suffices to verify that $\mathfrak{h}^1$ is an *isomorphism.* When $x$ is *not* a cusp of $X$, this follows immediately from Proposition 1.9 (i), applied to $X_x$. Thus, it suffices to verify that $\mathfrak{h}^1$ is an isomorphism in the case where $x$ is a cusp of $X$. Let $S$ be a mixed characteristic trait $S$ whose residue field is isomorphic to the residue field of $x$, and write $S^{\log}$ for the log scheme obtained by equipping $S$ with the log structure determined by the closed point of $S$. Next, let us take a stable log curve $\overline{X}_S^{\log} \to S^{\log}$ whose special fiber is isomorphic to $\overline{X}_x^{\log} \to x^{\log}$ and such that the interior $U$ of $X_S^{\log}$ is a hyperbolic curve over the fraction field of $S$. Then (cf. the discussion of [14], §0, in the characteristic zero case) we obtain a natural isomorphism $\Delta_{\overline{X}_x^{\log}} \xrightarrow{\sim} \Delta_U$ by composing a certain "*specialization isomorphism*" $\Delta_{\overline{X}_x^{\log}} \xrightarrow{\sim} \Delta_{\overline{X}_S^{\log}}$ with an isomorphism $\Delta_{\overline{X}_S^{\log}} \xrightarrow{\sim} \Delta_U$ arising from the "*log purity theorem*". Hence, the fact that $\mathfrak{h}^1$ is an isomorphism follows immediately by applying this isomorphism $\Delta_{\overline{X}_x^{\log}} \xrightarrow{\sim} \Delta_U$, together with Proposition 1.9 (i). $\square$

**Lemma 3.6.**
*Let*

$$i_1^{\mathrm{Lie}} : \mathrm{Lie}^1_{X_{2/1}} \longrightarrow \mathrm{Lie}_{X_2}, \quad \breve{i}_1^{\mathrm{Lie}} : \breve{\mathrm{Lie}}^1_{X_{2/1}} \longrightarrow \breve{\mathrm{Lie}}_{X_2}, \quad s_X^{\mathrm{Lie}} : \mathrm{Lie}_{X_2} \xrightarrow{\sim} \breve{\mathrm{Lie}}_{X_2}$$

*be the $G_K$-equivariant homomorphisms of graded Lie algebras induced by $i_1^{\Pi} : \Pi_{\overline{X}_x^{\log}} \to \Pi_{X_2}$, $i_1^{\Pi} : \Pi_{\overline{X}_x^{\log}} \to \Pi_{X_2}$ and $s_X^{\Pi} : \Pi_{X_2} \xrightarrow{\sim} \Pi_{X_2}$ respectively.*
*Then there exist $G_K$-equivariant isomorphisms of graded Lie algebras*

$$h_X^1 : \mathcal{L}_X^1 \xrightarrow{\sim} \mathrm{Lie}^1_{X_{2/1}}, \quad \breve{h}_X^1 : \mathcal{L}_X^1 \xrightarrow{\sim} \breve{\mathrm{Lie}}^1_{X_{2/1}},$$

$$h_X^2 : \mathcal{L}_X^2 \xrightarrow{\sim} \mathrm{Lie}_{X_2}, \quad \breve{h}_X^2 : \mathcal{L}_X^2 \xrightarrow{\sim} \breve{\mathrm{Lie}}_{X_2}$$

*which fit into the following commutative diagrams consisting of $G_K$-equivariant mophisms*

$$\begin{array}{ccc} \mathcal{L}_X^1 & \xrightarrow{i_1^{\mathcal{L}}} & \mathcal{L}_X^2 \\ {\scriptstyle h_X^1}\downarrow\wr & & {\scriptstyle h_X^2}\downarrow\wr \\ \mathrm{Lie}^1_{X_{2/1}} & \xrightarrow{i_1^{\mathrm{Lie}}} & \mathrm{Lie}_{X_2} \end{array} \qquad \begin{array}{ccc} \mathcal{L}_X^1 & \xrightarrow{i_1^{\mathcal{L}}} & \mathcal{L}_X^2 \\ {\scriptstyle \breve{h}_X^1}\downarrow\wr & & {\scriptstyle \breve{h}_X^2}\downarrow\wr \\ \breve{\mathrm{Lie}}^1_{X_{2/1}} & \xrightarrow{\breve{i}_1^{\mathrm{Lie}}} & \breve{\mathrm{Lie}}_{X_2} \end{array} \qquad \begin{array}{ccc} \mathcal{L}_X^2 & \xrightarrow{s_X^{\mathcal{L}}} & \mathcal{L}_X^2 \\ {\scriptstyle h_X^2}\downarrow\wr & & {\scriptstyle \breve{h}_X^2}\downarrow\wr \\ \mathrm{Lie}_{X_2} & \xrightarrow{s_X^{\mathrm{Lie}}} & \breve{\mathrm{Lie}}_{X_2}. \end{array}$$

*Proof.* Consider the homomorphism of graded Lie algebras $\mathcal{V}^2 \to \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_2})$ determined by

$$I_{\tilde{x}} \oplus \Big(\bigoplus_{j=1}^{r} I_j \oplus \Delta_X^{\mathrm{ab}}\Big)^{\oplus 2} \longrightarrow \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_2})$$

$$(a, b_1, b_2) \quad \mapsto \quad i_1^{\Pi}(a + b_1) + s_X^{\Pi} \circ i_1^{\Pi}(b_2).$$

Then it follows from Proposition 1.9 (ii) that this homomorphism of graded Lie algebras factors through $\mathcal{L}_X^2$, and that the resulting homomorphism $\mathfrak{h}^2 : \mathcal{L}_X^2 \to \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_2})$ is a $G_K$-equivariant isomorphism of graded Lie algebras, whether we regard $\mathfrak{h}^2$ as the morphism of underlying graded Lie algebras (i.e., without $G_K$-actions) $h^2 : \mathcal{L}_X^2 \to \mathrm{Gr}_{X_2}$ or as the morphism of underlying graded Lie algebras $\breve{h}^2 : \mathcal{L}_X^2 \to \breve{\mathrm{Gr}}_{X_2}$. If we denote by $i_1^{\mathrm{Gr}} : \mathrm{Gr}^1_{X_{2/1}} \to \mathrm{Gr}_{X_2}$, $\breve{i}_1^{\mathrm{Gr}} : \breve{\mathrm{Gr}}^1_{X_{2/1}} \to \breve{\mathrm{Gr}}_{X_2}$, $s_X^{\mathrm{Gr}} : \mathrm{Gr}_{X_2} \xrightarrow{\sim} \breve{\mathrm{Gr}}_{X_2}$ the $G_K$-equivariant homomorphisms of graded Lie algebras induced by $i_1^{\Pi} : \Pi_{\overline{X}_x^{\log}} \to \Pi_{X_2}$, $i_1^{\Pi} : \Pi_{\overline{X}_x^{\log}} \to \Pi_{X_2}$, and $s_X^{\Pi} : \Pi_{X_2} \xrightarrow{\sim} \Pi_{X_2}$, respectively, then we obtain $G_K$-equivariant commutative diagrams as follows:

$$\begin{array}{ccc} \mathcal{L}_X^1 & \xrightarrow{i_1^{\mathcal{L}}} & \mathcal{L}_X^2 \\ {\scriptstyle h^1}\downarrow\wr & & {\scriptstyle h^2}\downarrow\wr \\ \mathrm{Gr}^1_{X_{2/1}} & \xrightarrow{i_1^{\mathrm{Gr}}} & \mathrm{Gr}_{X_2} \end{array} \qquad \begin{array}{ccc} \mathcal{L}_X^1 & \xrightarrow{i_1^{\mathcal{L}}} & \mathcal{L}_X^2 \\ {\scriptstyle \breve{h}^1}\downarrow\wr & & {\scriptstyle \breve{h}^2}\downarrow\wr \\ \breve{\mathrm{Gr}}^1_{X_{2/1}} & \xrightarrow{\breve{i}_1^{\mathrm{Gr}}} & \breve{\mathrm{Gr}}_{X_2} \end{array} \qquad \begin{array}{ccc} \mathcal{L}_X^2 & \xrightarrow{s_X^{\mathcal{L}}} & \mathcal{L}_X^2 \\ {\scriptstyle h^2}\downarrow\wr & & {\scriptstyle \breve{h}^2}\downarrow\wr \\ \mathrm{Gr}_{X_2} & \xrightarrow{s_X^{\mathrm{Gr}}} & \breve{\mathrm{Gr}}_{X_2}. \end{array}$$

On the other hand, it follows from Proposition 1.8 that we have $G_K$-equivariant commutative diagrams as follows:

$$\begin{array}{ccc} \mathrm{Gr}^1_{X_{2/1}} & \xrightarrow{i_1^{\mathrm{Gr}}} & \mathrm{Gr}_{X_2} \\ \downarrow\wr & & \downarrow\wr \\ \mathrm{Lie}^1_{X_{2/1}} & \xrightarrow{i_1^{\mathrm{Lie}}} & \mathrm{Lie}_{X_2} \end{array} \quad \begin{array}{ccc} \breve{\mathrm{Gr}}^1_{X_{2/1}} & \xrightarrow{\breve{i}_1^{\mathrm{Gr}}} & \breve{\mathrm{Gr}}_{X_2} \\ \downarrow\wr & & \downarrow\wr \\ \breve{\mathrm{Lie}}^1_{X_{2/1}} & \xrightarrow{\breve{i}_1^{\mathrm{Lie}}} & \breve{\mathrm{Lie}}_{X_2} \end{array} \quad \begin{array}{ccc} \mathrm{Gr}_{X_2} & \xrightarrow{s_X^{\mathrm{Gr}}} & \breve{\mathrm{Gr}}_{X_2}. \\ \downarrow\wr & & \downarrow\wr \\ \mathrm{Lie}_{X_2} & \xrightarrow{s_X^{\mathrm{Lie}}} & \breve{\mathrm{Lie}}_{X_2}. \end{array}$$

By composing the vertical arrows in these commutative diagrams, we obtain the required isomorphisms. $\square$

Now, let $L$ be a finite field of cardinality prime to $l$, $Y$ a hyperbolic curve over $L$ of type $(g_Y, r_Y)$, $y^{\log}$ a strict $L$-rational log point of $\overline{Y}^{\log} := \overline{Y}_1^{\log}$; we shall use similar notation for objects obtained from $Y$ (e.g., $Y_2$, $\overline{Y}_y^{\log}$, $\Pi_{Y_2}$, $\Pi_{\overline{Y}_y^{\log}}$, etc.) to the notation used for objects obtained from $X$.

**Definition 3.7.**

(i) Let us take an isomorphism of profinite groups $\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$ (resp., $\Pi_{\overline{X}_x^{\log}} \xrightarrow{\sim} \Pi_{\overline{Y}_y^{\log}}$). Then the natural surjections $\Pi_X \twoheadrightarrow G_K$, $\Pi_Y \twoheadrightarrow G_L$ (resp., $\Pi_{\overline{X}_x^{\log}} \twoheadrightarrow G_K$, $\Pi_{\overline{Y}_x^{\log}} \twoheadrightarrow G_L$ ) arising from the structure morphisms over finite fields may be characterized group-theoretically (cf. [20], Proposition 3.3) as the (unique) maximal ($\hat{\mathbb{Z}}$-)free abelian quotient. Thus, $\alpha$ induces an isomorphism $G_K \xrightarrow{\sim} G_L$.

We shall say that $\alpha$ is *Frobenius-preserving* if the isomorphism $G_K \xrightarrow{\sim} G_L$ obtained as above preserves the Frobenius elements.

(ii) We shall denote by

$$(C_{x,y}^{X \xrightarrow{\sim} Y}) \text{ (resp., } (C_{x,y}^{\overline{X}_x^{\log} \xrightarrow{\sim} \overline{Y}_y^{\log}}))$$

a choice of a specific Frobenius-preserving isomorphism $\Pi_X \xrightarrow{\sim} \Pi_Y$ (resp., $\Pi_{\overline{X}_x^{\log}} \xrightarrow{\sim} \Pi_{\overline{Y}_y^{\log}}$) which maps the decomposition group of $x$ (resp., the diagonal cusp $\tilde{x}$) onto the decomposition group of $y$ (resp., the diagonal cusp $\tilde{y}$) up to conjugation.

**Proposition 3.8.**

*Let us fix specific choices of* $(C_{i_1}^Y)$, $(C_{p_1}^Y)$, $(C_{\tilde{y}}^Y)$ *and* $(C_{x,y}^{\overline{X}_x^{\log} \xrightarrow{\sim} \overline{Y}_y^{\log}})$. *Denote by* $\alpha : \Pi_{\overline{X}_x^{\log}} \xrightarrow{\sim} \Pi_{\overline{Y}_y^{\log}}$ *the isomorphism determined by* $(C_{x,y}^{\overline{X}_x^{\log} \xrightarrow{\sim} \overline{Y}_y^{\log}})$. *Let us assume that the decomposition subgroups determined by* $(C_{\tilde{x}}^X)$, $(C_{\tilde{y}}^Y)$ *are compatible with respect to* $\alpha$.

(i) *There exists a unique pair consisting of a choice* $(C_\theta^Y)$ *of a specific 1-cocycle* $\theta : G_L \to I_{\tilde{y}} := \mathrm{Ker}(D_{\tilde{y}} \twoheadrightarrow \Pi_Y)$ *and a choice* $(C_\tau^Y)$ *of a specific section* $\tau : G_L \to D_{\tilde{y}}$ *which are compatible with* $(C_\delta^X)$ *and* $(C_\sigma^X)$, *respectively, via* $(C_{x,y}^{X \xrightarrow{\sim} Y})$ *in an evident fashion.*

(ii) *There exists a $G_K$-equivariant isomorphism $\alpha_2^{\Delta^{\mathrm{Lie}}} : \Delta_{X_2}^{\mathrm{Lie}} \xrightarrow{\sim} \Delta_{Y_2}^{\mathrm{Lie}}$ of topological groups satisfying the following conditions:*

(1) *$\alpha_2^{\Delta^{\mathrm{Lie}}}$ is also $G_K$-equivariant when we regard it as a map $\breve{\Delta}_{X_2}^{\mathrm{Lie}} \xrightarrow{\sim} \breve{\Delta}_{Y_2}^{\mathrm{Lie}}$ under the natural identifications $\Delta_{X_2}^{\mathrm{Lie}} \xrightarrow{\sim} \breve{\Delta}_{X_2}^{\mathrm{Lie}}$, $\Delta_{Y_2}^{\mathrm{Lie}} \xrightarrow{\sim} \breve{\Delta}_{Y_2}^{\mathrm{Lie}}$ without $G_K$-actions.*

(2) *If we denote by $\alpha_2^{\Pi^{\mathrm{Lie}}} : \Pi_{X_2}^{\mathrm{Lie}} \xrightarrow{\sim} \Pi_{Y_2}^{\mathrm{Lie}}$, $\breve{\alpha}_2^{\Pi^{\mathrm{Lie}}} : \breve{\Pi}_{X_2}^{\mathrm{Lie}} \xrightarrow{\sim} \breve{\Pi}_{Y_2}^{\mathrm{Lie}}$ the semi-direct products of $\alpha_2^{\Delta^{\mathrm{Lie}}}$ with the isomorphism $G_K \xrightarrow{\sim} G_L$ (i.e., determined by $\alpha$) relative to the respective actions of $G_K$ and $G_L$, then these morphisms make the following diagrams commute*

$$\begin{array}{ccccc} \Pi_{\overline{X}_x^{\log}} & \xrightarrow{\mathrm{Int}_X^{\Pi}\circ i_1^{\Pi}} & \Pi_{X_2}^{\mathrm{Lie}} & \longrightarrow & \Pi_{X^{\times 2}} \\ {\scriptstyle\alpha}\downarrow\wr & & {\scriptstyle\alpha_2^{\Pi^{\mathrm{Lie}}}}\downarrow\wr & & {\scriptstyle\overline{\alpha}\times\overline{\alpha}}\downarrow\wr \\ \Pi_{\overline{Y}_y^{\log}} & \xrightarrow{\mathrm{Int}_Y^{\Pi}\circ i_1^{\Pi}} & \Pi_{Y_2}^{\mathrm{Lie}} & \longrightarrow & \Pi_{Y^{\times 2}} \end{array} \qquad \begin{array}{ccc} \Pi_{X_2}^{\mathrm{Lie}} & \xrightarrow{s_X^{\Pi^{\mathrm{Lie}}}} & \breve{\Pi}_{X_2}^{\mathrm{Lie}} \\ {\scriptstyle\alpha_2^{\Pi^{\mathrm{Lie}}}}\downarrow & & {\scriptstyle\breve{\alpha}_2^{\Pi^{\mathrm{Lie}}}}\downarrow \\ \Pi_{Y_2}^{\mathrm{Lie}} & \xrightarrow{s_Y^{\Pi^{\mathrm{Lie}}}} & \breve{\Pi}_{Y_2}^{\mathrm{Lie}}. \end{array}$$

*Proof.* Assertion (i) follows immediately by "transport of structure". Next, we consider assertion (ii). Since $\alpha$ is assumed to be Frobenius-preserving, it follows from [13], Corollary 2.7 (i) that $(g_X, r_X) = (g_Y, r_Y)$, and that $\alpha$ induces an isomorphism $\alpha^{\mathrm{cpt}} : \Delta_{\overline{X}}^{\mathrm{ab}} \xrightarrow{\sim} \Delta_{\overline{Y}}^{\mathrm{ab}}$ and a bijective correspondence between the respective sets of cusps of $\overline{X}_x^{\log}, \overline{Y}_y^{\log}$ as well as isomorphisms of the inertia subgroups of cusps corresponding via this bijection. By applying these isomorphisms (together with the construction of $\mathcal{L}_X^1$, $\mathcal{L}_X^2$, $\mathcal{L}_Y^1$, $\mathcal{L}_Y^2$), Lemma 3.6 yields $G_K$-equivariant isomorphisms $\alpha^{\mathrm{Lie}} : \mathrm{Lie}^1_{X_{2/1}} \cong \mathrm{Lie}^1_{Y_{2/1}}$, $\breve{\alpha}^{\mathrm{Lie}} : \breve{\mathrm{Lie}}^1_{X_{2/1}} \cong \breve{\mathrm{Lie}}^1_{Y_{2/1}}$, $\alpha_2^{\mathrm{Lie}} : \mathrm{Lie}_{X_2} \cong \mathrm{Lie}_{Y_2}$ and $\breve{\alpha}_2^{\mathrm{Lie}} : \breve{\mathrm{Lie}}_{X_2} \cong \breve{\mathrm{Lie}}_{Y_2}$. These morphisms give rise to a $G_K$-equivariant commutative diagram as follows:

$$\begin{array}{ccccccc} \mathrm{Lie}^1_{X_{2/1}} & \xrightarrow{i_1^{\mathrm{Lie}}} & \mathrm{Lie}_{X_2} & \xrightarrow{s_X^{\mathrm{Lie}}} & \breve{\mathrm{Lie}}_{X_2} & \xleftarrow{\breve{i}_1^{\mathrm{Lie}}} & \breve{\mathrm{Lie}}^1_{X_{2/1}} \\ {\scriptstyle\alpha^{\mathrm{Lie}}}\downarrow\wr & & {\scriptstyle\alpha_2^{\mathrm{Lie}}}\downarrow\wr & & {\scriptstyle\breve{\alpha}_2^{\mathrm{Lie}}}\downarrow\wr & & {\scriptstyle\breve{\alpha}^{\mathrm{lie}}}\downarrow\wr \\ \mathrm{Lie}^1_{Y_{2/1}} & \xrightarrow{i_1^{\mathrm{Lie}}} & \mathrm{Lie}_{Y_2} & \xrightarrow{s_Y^{\mathrm{Lie}}} & \breve{\mathrm{Lie}}_{Y_2} & \xleftarrow{\breve{i}_1^{\mathrm{Lie}}} & \breve{\mathrm{Lie}}^1_{Y_{2/1}}. \end{array}$$

Then it follows from the *functoriality* of $\mathrm{Lin}(-)$ that we obtain a $G_K$-equivariant commutative diagram as follows:

$$\begin{array}{ccccccc} \mathrm{Lin}^1_{X_{2/1}} & \xrightarrow{i_1^{\mathrm{Lin}}} & \mathrm{Lin}_{X_2} & \xrightarrow{s_X^{\mathrm{Lin}}} & \breve{\mathrm{Lin}}_{X_2} & \xleftarrow{\breve{i}_1^{\mathrm{Lin}}} & \breve{\mathrm{Lin}}^1_{X_{2/1}} \\ {\scriptstyle\alpha^{\mathrm{Lin}}}\downarrow\wr & & {\scriptstyle\alpha_2^{\mathrm{Lin}}}\downarrow\wr & & {\scriptstyle\breve{\alpha}_2^{\mathrm{Lin}}}\downarrow\wr & & {\scriptstyle\breve{\alpha}^{\mathrm{Lin}}}\downarrow\wr \\ \mathrm{Lin}^1_{Y_{2/1}} & \xrightarrow{i_1^{\mathrm{Lin}}} & \mathrm{Lin}_{Y_2} & \xrightarrow{s_Y^{\mathrm{Lin}}} & \breve{\mathrm{Lin}}_{Y_2} & \xleftarrow{\breve{i}_1^{\mathrm{Lin}}} & \breve{\mathrm{Lin}}^1_{Y_{2/1}}. \end{array}$$

Note (cf. [12], Remark 35) that modifying the choice $(C_\sigma^X)$ of a specific section $G_K \to D_{\tilde{x}}$ by a cocycle $G_K \to I_{\tilde{x}}$ determined by the choice $(C_\delta^X)$ affects the Galois invariant isomorphisms of Proposition 1.8, (ii), by conjugation by an element $c_X$ of the subgroup obtained by tensoring $I_{\tilde{x}}$ with $\mathbb{Q}_l$; a similar statement holds, with respect to some "$c_Y$", for objects associated to $Y$ when we modify $(C^Y_\tau)$ by $(C^Y_\theta)$. One may verify easily that $\alpha$ maps $c_X$ to $c_Y$, hence that $\alpha^{\mathrm{Lin}} = \breve{\alpha}^{\mathrm{Lin}}$ as a morphism of underlying topological groups (i.e., without $G_K$-actions). Next, recall that the morphisms $i_1^{\mathrm{Lin}}$ and $s_X^{\mathrm{Lin}} \circ i_1^{\mathrm{Lin}}$ are compatible with the corresponding morphisms "$i_1^{\mathrm{Lin}}$" and "$s_Y^{\mathrm{Lin}} \circ i_1^{\mathrm{Lin}}$" associated to $Y$ via the natural identification of $\mathrm{Lin}_{X_2}$ with $\breve{\mathrm{Lin}}_{X_2}$ (i.e., without $G_K$-actions). Also, let us recall that $\mathrm{Lin}_{X_2}$ (resp., $\breve{\mathrm{Lin}}_{X_2}$) is generated by the *images* of $\mathrm{Lin}^1_{X_{2/1}} \xrightarrow{i_1^{\mathrm{Lin}}} \mathrm{Lin}_{X_2}$ (resp., $\breve{\mathrm{Lin}}^1_{X_{2/1}} \xrightarrow{\breve{i}_1^{\mathrm{Lin}}} \breve{\mathrm{Lin}}_{X_2}$) and the composite $\mathrm{Lin}^1_{X_{2/1}} \xrightarrow{i_1^{\mathrm{Lin}}} \mathrm{Lin}_{X_2} \xrightarrow{s_X^{\mathrm{Lin}}} \breve{\mathrm{Lin}}_{X_2} = \mathrm{Lin}_{X_2}$ (resp., $\breve{\mathrm{Lin}}^1_{X_{2/1}} \xrightarrow{\breve{i}_1^{\mathrm{Lin}}} \breve{\mathrm{Lin}}_{X_2} = \mathrm{Lin}_{X_2} \xrightarrow{s_X^{\mathrm{Lin}}} \breve{\mathrm{Lin}}_{X_2}$). Since the restrictions of $\alpha_2^{\mathrm{Lin}}$ and $\breve{\alpha}_2^{\mathrm{Lin}}$ to these *image* subgroups coincide by virtue of the equality $\alpha^{\mathrm{Lin}} = \breve{\alpha}^{\mathrm{Lin}}$, we obtain that $\alpha_2^{\mathrm{Lin}} = \breve{\alpha}_2^{\mathrm{Lin}}$. Therefore, by construction, $\alpha_2^{\mathrm{Lin}} (= \breve{\alpha}_2^{\mathrm{Lin}})$ induces the required $G_K$-equivariant isomorphism $\alpha_2^{\Delta^{\mathrm{Lie}}} : \Delta^{\mathrm{Lie}}_{X_2} \xrightarrow{\sim} \Delta^{\mathrm{Lie}}_{Y_2}$ of topological groups satisfying conditions (1), (2). This completes the proof of assertion (ii). $\square$

One of main results of this paper, i.e., (a slightly generalized version of) Theorem A, is the following:

**Theorem 3.9.**
*Let $X$ (resp., $Y$) be a hyperbolic curve over a finite field $K$ (resp., $L$), $x$ a $K$-rational point of $\overline{X}$ (resp., $y$ an $L$-rational point of $\overline{Y}$), $X_2$ (resp., $Y_2$) the second configuration space associated to $X$ (resp., $Y$), $\overline{X}^{\log}_x$ (resp., $\overline{Y}^{\log}_y$) the cuspidalization of $X$ at $x$ (resp., of $Y$ at $y$) [cf. Definition 2.2], $D_{\tilde{x}} \subseteq \Pi_{\overline{X}^{\log}_x}$ (resp., $D_{\tilde{y}} \subseteq \Pi_{\overline{Y}^{\log}_y}$) a specific decomposition group of the diagonal cusp $\tilde{x}^{\log}$ (resp., $\tilde{y}^{\log}$) [cf. the discussion following Remark 2.1.1].*
*Let*

$$\alpha : \Pi_{\overline{X}^{\log}_x} \xrightarrow{\sim} \Pi_{\overline{Y}^{\log}_y}$$

*be a Frobenius-preserving isomorphism [cf. Definition 3.7 (i)] which maps $D_{\tilde{x}}$ onto $D_{\tilde{y}}$. Let us denote by $\overline{\alpha} : \Pi_X \xrightarrow{\sim} \Pi_Y$ the isomorphism obtained by passing to the quotients $\Pi_{\overline{X}^{\log}_x} \twoheadrightarrow \Pi_X$, $\Pi_{\overline{Y}^{\log}_y} \twoheadrightarrow \Pi_Y$. Let us denote by $D_x \subseteq \Pi_X$ (resp., $D_y \subseteq \Pi_Y$) the decomposition group of $x$ (resp., the decomposition group of $y$) determined by the image of $D_{\tilde{x}}$ (resp., as the image of $D_{\tilde{y}}$) via the quotient $\Pi_{\overline{X}^{\log}_x} \twoheadrightarrow \Pi_X$ (resp., $\Pi_{\overline{Y}^{\log}_y} \twoheadrightarrow \Pi_Y$).*
*Then there exists an isomorphism*

$$\alpha_2 : \Pi_{X_2} \xrightarrow{\sim} \Pi_{Y_2}$$

*which is uniquely determined up to composition with an inner automorphism by the condition that it be compatible with the natural switching automorphisms [cf. the discussion following Remark 2.1.1] and with the specific decomposition groups associated to the respective diagonal divisors determined by $D_{\tilde{x}}$, $D_{\tilde{y}}$ [cf.*

*Lemma 2.4 (ii)], fit into the following commutative square*

$$\begin{array}{ccc} \Pi_{X_2} & \xrightarrow{\alpha_2} & \Pi_{Y_2} \\ {\scriptstyle p_1^{\Pi}}\downarrow & & \downarrow{\scriptstyle p_1^{\Pi}} \\ \Pi_X & \xrightarrow{\overline{\alpha}} & \Pi_Y, \end{array} \qquad (**)$$

*and induce $\alpha$ upon restriction to the inverse images (via the vertical arrows of (**)) of $D_x \subseteq \Pi_X$ and $D_y \subseteq \Pi_Y$.*

*Proof.* Let us fix specific choices of $(C^X_{i_1})$, $(C^X_{p_1})$, $(C^Y_{i_1})$, $(C^Y_{p_1})$. By applying Proposition 3.8 to these choices and the choices of $(C^X_{\tilde{x}})$, $(C^Y_{\tilde{y}})$, $(C^{\overline{X}^{\log}_x \xrightarrow{\sim} \overline{Y}^{\log}_y}_{x,y})$ given by hypothesis, we obtain a commutative diagram as follows:

$$\begin{array}{ccccc} \Pi_{\overline{X}^{\log}_x} & \xrightarrow{\mathrm{Int}^{\Pi}_X \circ i^{\Pi}_1} & \Pi^{\mathrm{Lie}}_{X_2} & \xrightarrow{s^{\Pi^{\mathrm{Lie}}}_X} & \Pi^{\mathrm{Lie}}_{X_2} \\ {\scriptstyle \alpha}\downarrow & & {\scriptstyle \alpha_2^{\Pi^{\mathrm{Lie}}}}\downarrow & & {\scriptstyle \breve{\alpha}_2^{\Pi^{\mathrm{Lie}}}}\downarrow \\ \Pi_{\overline{Y}^{\log}_y} & \xrightarrow{\mathrm{Int}^{\Pi}_Y \circ i^{\Pi}_1} & \Pi^{\mathrm{Lie}}_{Y_2} & \xrightarrow{s^{\Pi^{\mathrm{Lie}}}_Y} & \Pi^{\mathrm{Lie}}_{Y_2} \end{array}$$

Now observe that by the various constructions involved, $s^{\Pi^{\mathrm{Lie}}}_X \circ s^{\Pi^{\mathrm{Lie}}}_X = \mathrm{id}_{\Pi^{\mathrm{Lie}}_{X_2}}$, and $s^{\Pi^{\mathrm{Lie}}}_X \circ \mathrm{Int}^{\Pi}_X \circ i^{\Pi}_1$ coincides with $\mathrm{Int}^{\Pi}_X \circ i^{\Pi}_2$ for some $i^{\Pi}_2 : \Pi_{\overline{X}^{\log}_x} \to \Pi_{\overline{X}^{\log}_2}$ (within the conjugacy class of homomorphisms determined by $i^{\Pi}_2$) induced by $i_2 : \overline{X}^{\log}_x \to \overline{X}^{\log}_2$. Thus, it follows from Proposition 1.4 (ii) that $(\mathrm{Int}^{\Pi}_X \circ i^{\Pi}_1)(\Pi_{\overline{X}^{\log}_x})$ and $(s^{\Pi^{\mathrm{Lie}}}_X \circ \mathrm{Int}^{\Pi}_X \circ i^{\Pi}_1)(\Delta^1_{X_{2/1}})$ generate $\Pi_{X_2}$, and that $\Pi_{X_2}$ is preserved by the action of $s^{\Pi^{\mathrm{Lie}}}_X$. Similarly, $\Pi_{Y_2}$ is generated by $(\mathrm{Int}^{\Pi}_Y \circ i^{\Pi}_1)(\Pi_{\overline{Y}^{\log}_y})$ and $(s^{\Pi^{\mathrm{Lie}}}_Y \circ \mathrm{Int}^{\Pi}_Y \circ i^{\Pi}_1)(\Delta^1_{Y_{2/1}})$, and $\Pi_{Y_2}$ is preserved by the action of $s^{\Pi^{\mathrm{Lie}}}_Y$. Therefore, since the above diagram is commutative, $\alpha_2^{\Pi^{\mathrm{Lie}}}$ maps $\Pi_{X_2}$ onto $\Pi_{Y_2}$. Thus, the restriction $\alpha_2$ of $\alpha_2^{\Pi^{\mathrm{Lie}}}$ to $\Pi_{X_2}$ makes the diagram $(**)$ commute and is compatible with the switching automorphisms. Since the specific inertia subgroup of $\Pi_{X_2}$ associated to the diagonal divisor determined by $D_{\tilde{x}}$ is the image of $I_{\tilde{x}} \subseteq \Pi_{\overline{X}^{\log}_x}$ via $\mathrm{Int}^{\Pi}_X \circ i^{\Pi}_1$ (cf. Lemma 2.4 (ii)), the isomorphism $\alpha_2$, which is an extension of the isomorphism $\alpha$, is compatible with the corresponding specific decomposition groups associated to the respective diagonal divisors. This completes the proof of the existence assertion.

Next, we consider uniqueness. Let $\dot{\alpha}_2, \ddot{\alpha}_2 : \Pi_{X_2} \xrightarrow{\sim} \Pi_{Y_2}$ be isomorphisms both of which make the diagram $(**)$ commute and induce $\alpha|_{\Delta^1_{X_{2/1}}}$ (i.e., the restriction of $\alpha$ to $\Delta^1_{X_{2/1}}$) upon restriction to the kernels of the vertical arrows of $(**)$. Then $\dot{\alpha}_2^{-1} \circ \ddot{\alpha}_2$ determines an automorphism of the exact sequence

$$1 \longrightarrow \Delta^1_{X_{2/1}} \xrightarrow{i^{\Pi}_1} \Pi_{X_2} \xrightarrow{p^{\Pi}_1} \Pi_X \longrightarrow 1$$

which induces the identity automorphism on $\Delta^1_{X_{2/1}}$ and $\Pi_X$. This implies that $\dot{\alpha}_2^{-1} \circ \ddot{\alpha}_2$ is the identity morphism (cf. § 0). □

**Corollary 3.10.**
*Let $X$ (resp., $Y$) be a hyperbolic curve over a finite field $K$ (resp., $L$), $x, x'$ $K$-rational points of $\overline{X}$ (resp., $y, y'$ $L$-rational points of $\overline{Y}$). Let*

$$\alpha : \Pi_{\overline{X}^{\log}_x} \longrightarrow \Pi_{\overline{Y}^{\log}_y}$$

*be a Frobenius-preserving isomorphism such that the decomposition groups of $\tilde{x}$ and $\tilde{y}$ (which are well-defined up to conjugacy) correspond via $\alpha$. Suppose that the isomorphism $\overline{\alpha} : \Pi_X \xrightarrow{\sim} \Pi_Y$ induced by passing to the quotients $\Pi_{\overline{X}^{\log}_x} \twoheadrightarrow \Pi_X$, $\Pi_{\overline{Y}^{\log}_y} \twoheadrightarrow \Pi_Y$ maps the conjugacy class of the decomposition group of $x'$ to the conjugacy class of the decomposition group of $y'$.*

*Then there exists a Frobenius-preserving isomorphism*

$$\alpha' : \Pi_{\overline{X}^{\log}_{x'}} \longrightarrow \Pi_{\overline{Y}^{\log}_{y'}}$$

*which is uniquely determined up to composition with an inner automorphism by the condition that it induce $\overline{\alpha}$ upon passing to the respective quotients and map the conjugacy class of the decomposition group of the diagonal cusp $\tilde{x}'$ to the conjugacy class of the decomposition group of the diagonal cusp $\tilde{y}'$.*

*Proof.* The existence assertion follows from Theorem 3.9 and the fact that if $D_{x'} \subseteq \Pi_X$, $D_{y'} \subseteq \Pi_Y$ denote the decomposition groups of $x'$, $y'$ respectively, then we have natural isomorphisms $\Pi_{\overline{X}^{\log}_{x'}} \cong D_{x'} \times_{\Pi_X} \Pi_{X_2}$, $\Pi_{\overline{Y}^{\log}_{y'}} \cong D_{y'} \times_{\Pi_Y} \Pi_{Y_2}$.

Next, we consider the uniqueness assertion. Let $\dot{\alpha}', \ddot{\alpha}' : \Pi_{\overline{X}^{\log}_{x'}} \xrightarrow{\sim} \Pi_{\overline{Y}^{\log}_{y'}}$ be Frobenius-preserving isomorphisms both of which induce $\overline{\alpha}$ upon passing to the respective quotients and map some *specific* decomposition group of the diagonal cusp $\tilde{x}'$ to the *same* decomposition group of the diagonal cusp $\tilde{y}'$. Write $\beta := (\dot{\alpha}')^{-1} \circ \ddot{\alpha}' \in \mathrm{Aut}(\Pi_{\overline{X}^{\log}_{x'}})$. Then it follows from the existence portion of Theorem 3.9 that $\beta$ induces an element $\beta_2 \in \mathrm{Aut}(\Pi_{X_2})$ which induces the identity morphism of $\Pi_{X^{\times 2}}$ upon passing to the natural quotient $\Pi_{X_2} \twoheadrightarrow \Pi_{X^{\times 2}}$. Note that $\beta_2$ defines an element $[\beta_2] \in \mathrm{Out}^{\mathrm{FC}}(\Delta_{X_2})$. Moreover, since $\beta_2$ induces the identity morphism of $\Pi_{X^{\times 2}}$, it follows that $[\beta_2]$ maps to the identity element of $\mathrm{Out}(\Delta_X)$ (cf. [5] for the definition of and results concerning to "$\mathrm{Out}^{\mathrm{FC}}$"). But $\mathrm{Out}^{\mathrm{FC}}(\Delta_{X_2}) \to \mathrm{Out}(\Delta_X)$ is injective (cf., e.g., [5], Theorem A), so we have $[\beta_2] = 1$ i.e., the restriction of $\beta_2$ to $\Delta_{X_2}$ coincides with an inner automorphism $\mathrm{Inn}(b)$ determined by an element $b$ of $\Delta_{X_2}$. By the construction of $\beta_2$, $\mathrm{Inn}(b)$ (preserves the subgroup $\Delta^1_{X_{2/1}}$ of $\Delta_{X_2}$ and) induces the identity morphism of $\Delta_X$ upon passing to the quotient $\Delta_{X_2} \twoheadrightarrow \Delta_{X_2}/\Delta^1_{X_{2/1}} \cong \Delta_X$. Since $\Delta_X$ is center-free (cf. Proposition 1.4 (iii)), we thus conclude that $b$ maps to the identity element of $\Delta_X$ via $\Delta_{X_2} \twoheadrightarrow \Delta_X$. In particular, $b$ is an element of $\Pi_{\overline{X}^{\log}_{x'}}$. Thus we have two automorphisms $\beta$, $\mathrm{Inn}(b)$ on $\Pi_{\overline{X}^{\log}_{x'}}$ which coincide upon passing to the quotient

$\Pi_{\overline{X}_{x'}^{\log}} \twoheadrightarrow D_{x'} \subseteq \Pi_X$ as well as upon the restriction to $\Delta^1_{X_{2/1}} \subseteq \Pi_{\overline{X}_{x'}^{\log}}$. This implies that $\beta = \mathrm{Inn}(b)$ (cf. §0), hence completes the proof of the uniqueness assertion. $\square$

**Remark 3.10.1.**

Any Frobenius-preserving isomorphism is quasi-point-theoretic (cf. [20], Corollary 2.10, Proposition 3.8; [13], Remark 10, (iii)), i.e., induces a bijection between the sets of decomposition groups of the points of $\overline{X}, \overline{Y}$. Therefore, in the statement of Corollary 3.10, given a closed point $x''$ of $\overline{X}$, there always *exists* a closed point $y''$ of $\overline{Y}$ which corresponds, at the level of conjugacy classes of decomposition groups, to $x''$ via $\overline{\alpha}$ (but this choice is *not necessarily unique*!).

## 4. Cuspidalization Problems for hyperbolic curves

In this last section, we apply Theorem 3.9 to obtain group-theoretic constructions of the cuspidalization of a hyperbolic curve at a point infinitesimally close to a cusp (cf. Theorem 4.3), as well as of arithmetic fundamental groups of configuration spaces of arbitrary dimension (cf. Theorem 4.4).

We maintain the notation and set-up of the discussion at the beginning of Section 3. Moreover, until the end of Theorem 4.3, we shall assume that $X$ is affine (i.e., $r > 0$), and that $x$ is split cusp of $X$, i.e., $x \in \overline{X}(K) \setminus X(K)$. As discussed following Remark 2.1.1, the major and minor cuspidal components $\overline{X}^{\log'}$, $\overline{\mathbb{P}}_X^{\log'}$ at $x$, together with the nexus $\nu_x^{\log}$ at $x$, determine *strict* (cf. [6], 1.2) closed sub-log schemes of $\overline{X}_x^{\log}$. These closed sub-log schemes determine subgroups well-defined up to conjugacy

$$\Pi_{\overline{X}^{\log'}},\ \Pi_{\overline{\mathbb{P}}_X^{\log'}},\ \Pi_{\nu_x^{\log}} \subseteq \Pi_{\overline{X}_x^{\log}}$$

— which we shall refer to, respectively, as the *major verticial*, *minor verticial*, and *nexus subgroups* (cf. [14], Definition 1.4).

**Lemma 4.1.**

*Write*

$$D_x := \mathrm{Im}(\Pi_{\overline{X}_x^{\log}} \stackrel{p_1^{\Pi}\circ i_1^{\Pi}}{\longrightarrow} \Pi_X).$$

*(Thus, $D_x \subseteq \Pi_X$ is a specific decomposition group of $x$, i.e., well-defined without any conjugacy indeterminacies.) Then:*

(i) *For any choice of a specific major verticial subgroup $\Pi_{\overline{X}^{\log'}} \subseteq \Pi_{\overline{X}_x^{\log}}$, the composite morphism*

$$\Pi_{\overline{X}^{\log'}} \longrightarrow \Pi_{\overline{X}_x^{\log}} \stackrel{(p_1^{\Pi}\circ i_1^{\Pi}, p_2^{\Pi}\circ i_1^{\Pi})}{\longrightarrow} D_x \times_{G_K} \Pi_X$$

*is an isomorphism. (In particular, the major verticial subgroups may be thought of as defining sections of the natural surjection* $\Pi_{\overline{X}_x^{\log}} \twoheadrightarrow \Pi_X \times_{G_K} D_x$*.) Moreover, the inverse of this isomorphism maps the subgroup* $D_x \times_{G_K} D_x \subseteq D_x \times_{G_K} \Pi_X$ *to the nexus subgroup* $\Pi_{\nu_x^{\log}} \subseteq \Pi_{\overline{X}^{\log\prime}}$*.*

(ii) *In a similar vein, let* $\overline{\mathbb{P}}_K^{\log}$ *be the 1-st log configuration space associated to a tripod* $\mathbb{P}_K$ *over* $K$ *(cf. Definition 1.1 (ii)). Then for any choice of a specific minor verticial subgroup* $\Pi_{\overline{\mathbb{P}}_X^{\log\prime}} \subseteq \Pi_{\overline{X}_x^{\log}}$*, the composite morphism*

$$\Pi_{\overline{\mathbb{P}}_X^{\log\prime}} \longrightarrow \Pi_{\overline{X}_x^{\log}} \stackrel{(p_{\mathbb{P}}^{\Pi}, p_1^{\Pi} \circ i_1^{\Pi})}{\longrightarrow} \Pi_{\overline{\mathbb{P}}_K^{\log}} \times_{G_K} D_x$$

*— where* $p_{\mathbb{P}}^{\Pi}$ *denotes the homomorphism* $\Pi_{\overline{X}_x^{\log}} \to \Pi_{\overline{\mathbb{P}}_K^{\log}}$ *(well-defined up to conjugation) induced by the natural morphism* $\overline{X}_x^{\log} \to \overline{\mathbb{P}}_K^{\log}$ *given by contracting* $\overline{X}$ *(*$\subseteq \overline{X}_x$*) to* $\nu_x$ *— is an isomorphism.*

*Proof.* We shall only consider assertion (i) since assertion (ii) follows from a similar argument. Let us consider the commutative diagram of natural morphisms of log schemes

$$\begin{array}{ccc} \nu_x^{\log} & \longrightarrow & x^{\log} \times_K x^{\log} \\ \downarrow & & \downarrow \\ \overline{X}^{\log\prime} & \longrightarrow & \overline{X}^{\log} \times_K x^{\log} \end{array}$$

— where the horizontal arrows are the strict closed immersions. Now recall that: (a) két coverings may be constructed by means of descent with respect to (non-logarithmic!) étale morphisms; (b) restriction from a henselian trait to its closed point induces an equivalence between the respective categories of két coverings (cf. [6]). Since the bottom horizontal arrow $\overline{X}^{\log\prime} \to \overline{X}^{\log} \times_K x^{\log}$ in the above diagram is an isomorphism on the respective complements of the images of the horizontal arrows in the above diagram, it suffices (by (a), (b)) to verify that the induced morphism between the log inertia groups of $\nu_x^{\log}$ and $x^{\log} \times_K x^{\log}$ (i.e., $\mathrm{Ker}(\Pi_{\nu_x^{\log}} \twoheadrightarrow G_K)$ and $\mathrm{Ker}(\Pi_{x^{\log} \times_K x^{\log}} \twoheadrightarrow G_K)$) is an isomorphism (cf. [6], 4.7 for the terminology "log inertia subgroup"). Fix a chart, modeled on $\mathbb{N}$, of $x^{\log}$ (i.e., roots of a local uniformizer at $x$ in $\overline{X}$). Then such a chart determines charts, modeled on $\mathbb{N} \oplus \mathbb{N}$, of $x^{\log} \times_K x^{\log}$ and $\nu_x^{\log}$. By using these charts, one verifies easily that the homomorphism of monoids induced by the morphism $\nu_x^{\log} \to x^{\log} \times_K x^{\log}$ may be expressed as follows:

$$\mathbb{N} \oplus \mathbb{N} \longrightarrow \mathbb{N} \oplus \mathbb{N}$$

$$(a, b) \ \mapsto \ (a + b, b).$$

Then, by applying the functor $\mathrm{Hom}((\ \_\ )^{\mathrm{gp}}, \mathbb{Z}_l(1))$ to this morphism of monoids, one verifies immediately that the induced morphism of log inertia groups between $\nu_x^{\log}$ and $x^{\log} \times_K x^{\log}$ is an isomorphism. $\square$

**Lemma 4.2.**

*Suppose that we fix a choice of a nexus subgroup* $\Pi_{\nu_x^{\log}} \subseteq \Pi_{\overline{X}_x^{\log}}$ *among its various* $\Pi_{\overline{X}_x^{\log}}$*-conjugates. Then:*

(i) *There exists a unique pair of inclusions*

$$\Pi_{\overline{X}^{\log\prime}} \subseteq \Pi_{\overline{X}_x^{\log}}, \quad \Pi_{\overline{\mathbb{P}}_X^{\log\prime}} \subseteq \Pi_{\overline{X}_x^{\log}}$$

*(among their various* $\Pi_{\overline{X}_x^{\log}}$*-conjugates) both of which contain* $\Pi_{\nu_x^{\log}} \subseteq \Pi_{\overline{X}_x^{\log}}$ *.*

(ii) *The inclusions* $\Pi_{\nu_x^{\log}} \subseteq \Pi_{\overline{X}^{\log\prime}} \subseteq \Pi_{\overline{X}_x^{\log}}$, $\Pi_{\nu_x^{\log}} \subseteq \Pi_{\overline{\mathbb{P}}_X^{\log\prime}} \subseteq \Pi_{\overline{X}_x^{\log}}$ *obtained in (i) make the diagram*

$$\begin{array}{ccc} \Pi_{\nu_x^{\log}} & \longrightarrow & \Pi_{\overline{\mathbb{P}}_X^{\log\prime}} \\ \big\downarrow & & \big\downarrow \\ \Pi_{\overline{X}^{\log\prime}} & \longrightarrow & \Pi_{\overline{X}_x^{\log}} \end{array}$$

*commute and co-cartesian in the category of profinite groups equipped with an augmentation to* $G_K$ *whose kernel is pro-l.*

*Proof.* Assertion (i) (respectively, (ii)) follows immediately from [14], Proposition 1.5, (ii) (respectively 1.5, (iii)). □

Next, we turn to the proof of Theorem B. Theorem 4.3 given below may be regarded as a slightly weakened version of Theorem B (as stated in the Introduction). This weakened version, however, will be sufficient to prove Theorem 4.4 below (which corresponds *precisely* to Theorem C in the Introduction). Moreover, one may conclude Theorem B (as stated in the Introduction) from Theorem C (cf. Remark 4.4.1). On the other hand, if we did *not* restrict our attention, in the statement of Theorem 4.3, to this slightly weakened version of Theorem B, then it would have been necessary to (essentially) *repeat*, in our proof of Theorem 4.4 below, arguments already applied in the proof of Theorem 4.3.

**Theorem 4.3.**

*Let* $X$ *(resp.,* $Y$*) be an affine hyperbolic curve over a finite field* $K$ *(resp.,* $L$*),* $x$ *(resp.,* $y$*) a* $K$*-(resp.,* $L$*-)rational point of* $\overline{X} \setminus X$ *(resp.,* $\overline{Y} \setminus Y$*). Let*

$$\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$$

*be a Frobenius-preserving isomorphism such that the decomposition groups of* $x$ *and* $y$ *(which are well-defined up to conjugacy) correspond via* $\alpha$*. In the following, we shall apply the notational conventions introduced in the discussion following Remark 2.1.1.*

*Then there exist finite extensions* $\dot{K}$ *of* $K$ *and* $\dot{L}$ *of* $L$ *and an isomorphism*

$$\dot{\alpha}_{x,y} : \Pi_{\overline{X}_x^{\log}} \times_{G_K} G_{\dot{K}} \xrightarrow{\sim} \Pi_{\overline{Y}_y^{\log}} \times_{G_L} G_{\dot{L}}$$

*which is uniquely determined up to composition with an inner automorphism by the condition that it map the conjugacy class of the decomposition group of* $\tilde{x}$ *to the conjugacy class of the decomposition group of* $\tilde{y}$ *and induce* $\alpha|_{\Pi_{X\times_K\dot{K}}}$ : $\Pi_{X\times_K\dot{K}} \xrightarrow{\sim} \Pi_{Y\times_L\dot{L}}$ *upon passing to the quotients* $\Pi_{\overline{X}_x^{\log}} \times_{G_K} G_{\dot{K}} \twoheadrightarrow \Pi_{X\times_K\dot{K}}$, $\Pi_{\overline{Y}_y^{\log}} \times_{G_L} G_{\dot{L}} \twoheadrightarrow \Pi_{Y\times_L\dot{L}}$.

*Proof.* The asserted uniqueness follows immediately from the uniqueness portion of Corollary 3.10. Next, we shall consider the existence assertion. First, observe that there exists a connected finite étale covering $f : \dot{Z} \to X$, where $\dot{Z}$ is a hyperbolic curve over a finite extension field $\dot{K}$ of $K$ whose (smooth) compactification admits at least two distinct $\dot{K}$-rational points $z$, $z'$ lying over $x$ at which $f$ is *unramified*. Indeed, this follows immediately from the well-known structure of $\Delta_X$. In the following, we shall, for simplicity, replace $\dot{K}$ by $K$ (i.e., assume that the base fields of $X$ and $\dot{Z}$ coincide).

Write $Z$ for the partial (smooth) compactification of $\dot{Z}$ at $z'$ and $\overline{Z}_z^{\log}$ for the cuspidalization of $Z$ at $z$. Thus, the underlying scheme $\overline{Z}_z$ of $\overline{Z}_z^{\log}$ is *proper*. Denote by

$$\overline{Z}^{\log'}, \quad \overline{\mathbb{P}}_Z^{\log'}, \quad \nu_z^{\log}$$

the major and minor cuspidal components and the nexus of $\overline{Z}_z^{\log}$ at $z$, respectively (cf. the discussion at the beginning of the present §4). Let us fix specific choices of the decomposition groups $\dot{D}_z \subseteq \Pi_{\dot{Z}}$ of $z$ and $D_x \subseteq \Pi_X$ of $x$ such that $D_x \cap \Pi_{\dot{Z}} = \dot{D}_z$. Denote by $D_z$ the image of $\dot{D}_z$ via the quotient $\Pi_{\dot{Z}} \twoheadrightarrow \Pi_Z$ (which may be considered as the decomposition group of $z$ in $\Pi_Z$). Thus, the natural inclusion $\dot{D}_z \subseteq D_x$ is in fact an equality $\dot{D}_z = D_x$, and we have a natural isomorphism $\dot{D}_z \xrightarrow{\sim} D_z$. By applying Corollary 3.10 (cf. also Theorem 3.9) to the hyperbolic curve $\dot{Z} = Z \setminus \{z'\}$ together with the $K$-rational points $z$ and $z'$, we may reconstruct, group-theoretically from $\Pi_{\dot{Z}}$, the profinite group $\Pi_{\overline{Z}_z^{\log}}$ together with its natural augmentation to $D_z$. Also, by [13], Corollary 2.7 (iii), we may reconstruct, group-theoretically from the natural augmentation $\Pi_{\overline{Z}_z^{\log}} \twoheadrightarrow D_z$, the conjugacy classes of the major verticial, minor verticial and nexus subgroups of $\Pi_{\overline{Z}_z^{\log}}$ associated to the cuspidalization at $z$. Now let us fix *specific choices* of the major verticial, minor verticial and nexus subgroups of $\Pi_{\overline{Z}_z^{\log}}$

$$\Pi_{\overline{Z}^{\log'}}, \quad \Pi_{\overline{\mathbb{P}}_Z^{\log'}}, \quad \Pi_{\nu_z^{\log}}$$

such that: (a) the subgroup $\Pi_{\nu_z^{\log}} \subseteq \Pi_{\overline{Z}_z^{\log}}$ maps, via the natural morphism $\Pi_{\overline{Z}_z^{\log}} \twoheadrightarrow \Pi_Z$, onto the subgroup $D_z$; (b) $\Pi_{\nu_z^{\log}} \subseteq \Pi_{\overline{Z}^{\log'}} \subseteq \Pi_{\overline{Z}_z^{\log}}$ and $\Pi_{\nu_z^{\log}} \subseteq \Pi_{\overline{\mathbb{P}}_Z^{\log'}} \subseteq \Pi_{\overline{Z}_z^{\log}}$. (These choices are possible by virtue of Lemmas 4.1(i), 4.2.) If we denote by $\overline{\mathbb{P}}_K^{\log}$ the 1-st log configuration space associated to a tripod $\mathbb{P}_K$ over $K$, then we obtain (cf. Lemma 4.1 (ii)) a composite

$$\Pi_{\nu_z^{\log}} \longrightarrow \Pi_{\overline{\mathbb{P}}_Z^{\log'}} \xrightarrow{\sim} \Pi_{\overline{\mathbb{P}}_K^{\log}} \times_{G_K} D_x.$$

Here, we may regard $\Pi_{\overline{\mathbb{P}}_K^{\log}}$ as an object group-theoretically reconstructed from $\Pi_{\overline{\mathbb{P}}_Z^{\log\prime}}$ by thinking of $\Pi_{\overline{\mathbb{P}}_K^{\log}}$ as the quotient of the kernel of the natural composite augmentation $\Pi_{\overline{\mathbb{P}}_Z^{\log\prime}} \twoheadrightarrow D_z \twoheadrightarrow G_K$ (i.e., which is naturally isomorphic to $\Delta_{\mathbb{P}_K} \times \mathbb{Z}_l(1)$) by its *center* (i.e., $\mathbb{Z}_l(1)$ — cf. Proposition 1.4 (iii)). Also, we obtain (cf. Lemma 4.1 (i)) a diagram of natural morphisms

$$D_x \times_{G_K} \Pi_X \longleftarrow \dot{D}_z \times_{G_K} \dot{D}_z \xrightarrow{\sim} D_z \times_{G_K} D_z \xleftarrow{\sim} \Pi_{\nu_z^{\log}}$$

induced, by restriction, from a diagram of natural morphisms

$$D_x \times_{G_K} \Pi_X \quad \hookleftarrow \quad \dot{D}_z \times_{G_K} \Pi_{\dot{Z}} \quad \twoheadrightarrow \quad D_z \times_{G_K} \Pi_Z \quad \xleftarrow{\sim} \quad \Pi_{\overline{Z}^{\log\prime}}.$$

Thus, for suitable choices of the subgroups $\Pi_{\overline{X}^{\log\prime}}$, $\Pi_{\overline{\mathbb{P}}_X^{\log\prime}}$, $\Pi_{\nu_x^{\log}} \subseteq \Pi_{\overline{X}_x^{\log}}$ (cf. Lemma 4.2 (i)), we obtain a natural commutative diagram:

$$\begin{tikzcd}
\Pi_{\overline{X}^{\log\prime}} \arrow[d, "\wr"'] & \Pi_{\nu_x^{\log}} \arrow[l] \arrow[r] \arrow[d, "\wr"'] & \Pi_{\overline{\mathbb{P}}_X^{\log\prime}} \arrow[d, "\wr"'] \\
D_x \times_{G_K} \Pi_X & \Pi_{\nu_z^{\log}} \arrow[l] \arrow[r] & \Pi_{\overline{\mathbb{P}}_K^{\log}} \times_{G_K} D_x,
\end{tikzcd}$$

where the vertical arrows are all isomorphisms by Lemma 4.1(i), (ii). In particular, it follows from Lemma 4.2 (ii) that $\Pi_{\overline{X}_x^{\log}}$ may be identified with the colimit of the lower horizontal sequence — which, by the above discussion, may be *reconstructed group-theoretically* from the data $(\Pi_X, D_x \subseteq \Pi_X)$! — in the above diagram. Therefore, by comparing this diagram to the corresponding diagram for $Y$, the proof is completed. □

Next, we consider Theorem C, i.e., the cuspidalization problem for geometrically pro-$l$ fundamental groups of configuration spaces of (not necessarily proper) hyperbolic curves over finite fields.

**Theorem 4.4.** *(cf.* [12]*, Theorem 3.1;* [4]*, Theorem 4.1)*
*Let $X$ (resp., $Y$) be a hyperbolic curve over a finite field $K$ (resp., $L$). Let*

$$\alpha_1 : \Pi_X \xrightarrow{\sim} \Pi_Y$$

*be a Frobenius-preserving isomorphism. Then for any $n \in \mathbb{Z}_{\geq 0}$, there exists an isomorphism*

$$\alpha_n : \Pi_{X_n} \xrightarrow{\sim} \Pi_{Y_n}$$

*which is uniquely determined up to composition with an inner automorphism by the condition that it be compatible with the natural respective outer actions of the symmetric group on $n$ letters and make the diagram*

$$\begin{tikzcd}
\Pi_{X_{n+1}} \arrow[r, "\alpha_{n+1}"] \arrow[d, "p_i"'] & \Pi_{Y_{n+1}} \arrow[d, "p_i"] \\
\Pi_{X_n} \arrow[r, "\alpha_n"] & \Pi_{Y_n}
\end{tikzcd}$$

*$(i = 1, \cdots, n+1)$ commute.*

*Proof.* First, we recall that the case where $n = 2$ and $X$ is proper follows from [12], Theorem 3.1. Next, we consider the case where $n = 2$ and $X$ is affine. As we noted in Definition 3.7 (i), $\alpha_1$ induces an isomorphism

$$\alpha_0 : G_K \xrightarrow{\sim} G_L$$

of profinite groups. Now, by combining Theorem 3.9 and Theorem 4.3 together with the fact that $\alpha_1$ is quasi-point-theoretic (cf. Remark 3.10.1), we conclude that $\alpha_1$ induces an isomorphism

$$\dot{\alpha}_2 : \Pi_{X_2} \times_{G_K} G_{\dot{K}} \xrightarrow{\sim} \Pi_{Y_2} \times_{G_L} G_{\dot{L}}$$

— where $G_{\dot{K}} \subseteq G_K$, $G_{\dot{L}} \subseteq G_L$ denote open subgroups corresponding to certain finite extensions $\dot{K}$ of $K$ and $\dot{L}$ of $L$, respectively. If we denote by $\alpha_2^{\Delta}$ the restriction of $\dot{\alpha}_2$ to $\Delta_{X_2}$, then (cf. Theorem 3.9) $\alpha_2^{\Delta}$ maps onto $\Delta_{Y_2}$, i.e., determines an isomorphism

$$\alpha_2^{\Delta} : \Delta_{X_2} \xrightarrow{\sim} \Delta_{Y_2}.$$

Let

$$\gamma_X : G_K \to \mathrm{Out}^{\mathrm{FC}}(\Delta_{X_2}) \quad (\text{resp.}, \gamma_Y : G_L \to \mathrm{Out}^{\mathrm{FC}}(\Delta_{Y_2}))$$

(cf. [5] for the definition and results concerning to "$\mathrm{Out}^{\mathrm{FC}}$") be the morphism obtained by lifting elements of $G_K$ (resp., $G_L$) via the surjection $\Pi_{X_2} \twoheadrightarrow G_K$ (resp., $\Pi_{Y_2} \twoheadrightarrow G_L$) and considering the action of these elements by conjugation. Then $\alpha_2^{\Delta}$, $\alpha_0$ give rise to two composites $\gamma_Y \circ \alpha_0$ and $[\alpha_2^{\Delta}] \circ \gamma_X$

$$\gamma_Y \circ \alpha_0,\ [\alpha_2^{\Delta}] \circ \gamma_X : G_K \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\Delta_{Y_2})$$

— where $[\alpha_2^{\Delta}]$ denotes the isomorphism $\mathrm{Out}^{\mathrm{FC}}(\Delta_{X_2}) \xrightarrow{\sim} \mathrm{Out}^{\mathrm{FC}}(\Delta_{Y_2})$ that sends an element $g \in \mathrm{Aut}(\Delta_{X_2})$ to $\alpha_2^{\Delta} \circ g \circ (\alpha_2^{\Delta})^{-1} \in \mathrm{Aut}(\Delta_{Y_2})$. It follows from the constructions of $\alpha_0$, $\alpha_2^{\Delta}$ that $\gamma_Y \circ \alpha_0$ and $[\alpha_2^{\Delta}] \circ \gamma_X$ coincide after composing with the natural morphism $\mathrm{Out}^{\mathrm{FC}}(\Delta_{Y_2}) \to \mathrm{Out}(\Delta_Y)$. On the other hand, since $\mathrm{Out}^{\mathrm{FC}}(\Delta_{Y_2}) \to \mathrm{Out}(\Delta_Y)$ is injective (cf., e.g., [5], Theorem A), we conclude that $\gamma_Y \circ \alpha_0 = [\alpha_2^{\Delta}] \circ \gamma_X$. Therefore, by applying the natural isomorphisms $\Pi_{X_2} \cong \Delta_{X_2} \overset{\mathrm{out}}{\rtimes} G_K$ and $\Pi_{Y_2} \cong \Delta_{Y_2} \overset{\mathrm{out}}{\rtimes} G_L$, we obtain an isomorphism $\Pi_{X_2} \cong \Pi_{Y_2}$, which satisfies the required uniqueness and compatibility properties (cf. the construction of $\dot{\alpha}_2$; Theorem 3.9). This completes the proof of the assertion in the case where $n = 2$ and $X$ is affine.

Finally, the assertion in the case $n \geq 3$ follows from an inductive argument on $n$ applied to an argument similar to the argument of the above discussion. Indeed, consider the natural exact sequence

$$1 \longrightarrow \Delta_{(X \times_K \overline{K} \setminus \{x\})_{n-1}} \longrightarrow \Pi_{X_n} \xrightarrow{q_j^{\Pi}} \Pi_X \longrightarrow 1$$

(which induces an isomorphism $\Pi_{X_n} \cong \Delta_{(X \times_K \overline{K} \setminus \{x\})_{n-1}} \overset{\mathrm{out}}{\rtimes} \Pi_X$), where $x$ denotes a $\overline{K}$-rational point of $X$, and $q_j^{\Pi}$ denotes the morphism induced by the projection $X_n \to X$ to the $j$-th factor. Since the natural morphism $\mathrm{Out}^{\mathrm{FC}}(\Delta_{(X \times_K \overline{K} \setminus \{x\})_{n-1}}) \to \mathrm{Out}^{\mathrm{FC}}(\Delta_{(X \times_K \overline{K} \setminus \{x\})_{n-2}})$ is *injective* (cf. [5]), we may carry out a similar argument to the above discussion by replacing $G_K$ by $\Pi_X$ and $\Delta_{X_2}$ by $\Delta_{(X \times_K \overline{K} \setminus \{x\})_{n-1}}$.

Hence, for $j = 1, \cdots, n$, we obtain an isomorphism $\alpha_n^j : \Pi_{X_n} \xrightarrow{\sim} \Pi_{Y_n}$ that fits into a commutative diagram

$$\begin{array}{ccc} \Pi_{X_n} & \xrightarrow{\alpha_n^j} & \Pi_{Y_n} \\ p_i^\Pi \big\downarrow & & \big\downarrow p_i^\Pi \\ \Pi_{X_{n-1}} & \xrightarrow{\alpha_{n-1}} & \Pi_{Y_{n-1}}. \end{array}$$

for $i = 1, \cdots, n-1$. But it follows from the induction hypothesis (concerning to the asserted uniqueness), together with the *injectivity* applied above, that the $\alpha_n^j$'s coincide, for $j = 1, \cdots, n$, up to composition with an inner automorphism, and that the asserted uniqueness and compatibility with symmetric group actions for $n$ are satisfied. $\square$

**Remark 4.4.1.**

As explained in the discussion preceding Theorem 4.3, one may conclude Theorem B (as stated in Introduction) directly from Theorem 4.4 as follows. Let $X$, $Y$, $x$, $y$ and $\alpha$ be as in the statement of Theorem B. Then, by applying Theorem 4.4 in the case $n = 2$, we obtain a unique isomorphism

$$\alpha_2 : \Pi_{X_2} \xrightarrow{\sim} \Pi_{Y_2}$$

well-defined up to composition with an inner automorphism, which fits into two commutative diagrams as follows:

$$\begin{array}{ccc} \Pi_{X_2} & \xrightarrow{\alpha_2} & \Pi_{Y_2} \\ p_1^\Pi \big\downarrow & & \big\downarrow p_1^\Pi \\ \Pi_X & \xrightarrow{\alpha} & \Pi_Y, \end{array} \qquad \begin{array}{ccc} \Pi_{X_2} & \xrightarrow{\alpha_2} & \Pi_{Y_2} \\ p_2^\Pi \big\downarrow & & \big\downarrow p_2^\Pi \\ \Pi_X & \xrightarrow{\alpha} & \Pi_Y. \end{array}$$

On the other hand, we may have natural identifications $\Pi_{\overline{X}_x^{\log}} \xrightarrow{\sim} (p_2^\Pi)^{-1}(D_x)$, $\Pi_{\overline{Y}_y^{\log}} \xrightarrow{\sim} (p_2^\Pi)^{-1}(D_y)$. Hence the right-hand diagram above induces (since $\alpha(D_x) = D_y$) an isomorphism

$$\alpha_{x,y} : \Pi_{\overline{X}_x^{\log}} \xrightarrow{\sim} \Pi_{\overline{Y}_y^{\log}}$$

by restricting $\alpha_2$ to the inverse images (via the vertical arrows) of $D_x \subseteq \Pi_X$ and $D_y \subseteq \Pi_Y$. On the other hand, it follows from [13], Corollary 2.7 (i) that $\alpha_{x,y}$ maps the conjugacy class of the decomposition group of $\tilde{x}$ to the conjugacy class of the decomposition group of $\tilde{y}$. Thus, the left-hand commutative diagram above induces, by restricting the upper horizontal arrow of the diagram to the domain and codomain of $\alpha_{x,y}$, a commutative diagram

$$\begin{array}{ccc} \Pi_{\overline{X}_x^{\log}} & \xrightarrow{\alpha_{x,y}} & \Pi_{\overline{Y}_y^{\log}} \\ p_1^\Pi \big\downarrow & & \big\downarrow p_1^\Pi \\ \Pi_X & \xrightarrow{\overline{\alpha}} & \Pi_Y, \end{array}$$

which completes the proof of Theorem B. (The proof of uniqueness is similar to the proof of the asserted uniqueness in Corollary 3.10.)

Finally, we shall conclude the paper with the following corollary.

**Corollary 4.5.** *(cf.* [4], *Corollary 4.1)*
*Let $X$ (resp., $Y$) be a hyperbolic curve over a finite field $K$ (resp., $L$), and $n \in \mathbb{Z}_{\geq 0}$. Let*

$$\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$$

*be a Frobenius-preserving isomorphism, and $x_\bullet := \{x_1, \cdots, x_n\}$ an ordered set of distinct $K$-rational points of $X$. Then there exist an ordered set $y_\bullet := \{y_1, \cdots, y_n\}$ of distinct $L$-rational points of $Y$ and an isomorphism*

$$\tilde{\alpha} : \Pi_{X \setminus \{x_1, \cdots, x_n\}} \xrightarrow{\sim} \Pi_{Y \setminus \{y_1, \cdots, y_n\}}$$

*which is uniquely determined up to composition with an inner automorphism by the condition that it induce $\alpha$ upon passing to quotients $\Pi_{X \setminus \{x_1, \cdots, x_n\}} \twoheadrightarrow \Pi_X$, $\Pi_{Y \setminus \{y_1, \cdots, y_n\}} \twoheadrightarrow \Pi_Y$ and map the conjugacy classes of the decomposition groups of the points in $x_\bullet$ to the conjugacy classes of the decomposition groups of the points in $y_\bullet$ in the order of numbering.*

*Proof.* The existence assertion follows, by induction on $n$, from Theorem 4.4 together with the fact that any Frobenius-preserving isomorphism between hyperbolic curves over finite fields preserves the set of decomposition groups of closed points (cf. Remark 3.10.1). The asserted uniqueness follows from the uniqueness asserted in Corollary 3.10, applied successively to the cuspidalizations at corresponding points of $x_\bullet$, $y_\bullet$. $\square$

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