RIMS-1759

INTER-UNIVERSAL TEICHMÜLLER THEORY IV: LOG-VOLUME COMPUTATIONS AND SET-THEORETIC FOUNDATIONS

By

Shinichi MOCHIZUKI

August 2012

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

INTER-UNIVERSAL TEICHMÜLLER THEORY IV: LOG-VOLUME COMPUTATIONS AND SET-THEORETIC FOUNDATIONS

Shinichi Mochizuki

August 2012

Abstract. The present paper forms the fourth and final paper in a series of papers concerning **"inter-universal Teichmüller theory"**. In the first three papers of the series, we introduced and studied the theory surrounding the **log-theta-lattice**, a *highly non-commutative* two-dimensional diagram of *"miniature models of conventional scheme theory"*, called $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters*, that were associated, in the first paper of the series, to certain data, called *initial* Θ*-data*. This data includes an *elliptic curve* E_F over a *number field* F, together with a *prime number* $l \geq 5$. Consideration of various properties of the log-theta-lattice led naturally to the establishment, in the third paper of the series, of **multiradial algorithms** for constructing **"splitting monoids of LGP-monoids"**. Here, we recall that "multiradial algorithms" are algorithms that make sense from the point of view of an **"alien arithmetic holomorphic structure"**, i.e., the ring/scheme structure of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater related to a given $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater by means of a *non-ring/scheme-theoretic* horizontal arrow of the log-theta-lattice. In the present paper, estimates arising from these multiradial algorithms for splitting monoids of LGP-monoids are applied to verify various **diophantine results** which imply, for instance, the so-called **Vojta Conjecture** for hyperbolic curves, the **ABC Conjecture**, and the **Szpiro Conjecture** for elliptic curves. Finally, we examine the *foundational/set-theoretic* issues surrounding the *vertical* and *horizontal arrows* of the log-theta-lattice by introducing and studying the basic properties of the notion of a **"species"**, which may be thought of as a sort of formalization, via set-theoretic formulas, of the intuitive notion of a *"type of mathematical object"*. These foundational issues are closely related to the central role played in the present series of papers by various results from **absolute anabelian geometry**, as well as to the idea of **gluing together** *distinct models of conventional scheme theory*, i.e., in a fashion that lies outside the framework of conventional scheme theory. Moreover, it is precisely these foundational issues surrounding the vertical and horizontal arrows of the log-theta-lattice that led naturally to the introduction of the term **"inter-universal"**.

Contents:

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Introduction

The present paper forms the fourth and final paper in a series of papers concerning **"inter-universal Teichmüller theory"**. In the first three papers, [IUTchI], [IUTchII], and [IUTchIII], of the series, we introduced and studied the theory surrounding the **log-theta-lattice** [cf. the discussion of [IUTchIII], Introduction], a *highly non-commutative* two-dimensional diagram of *"miniature models of conventional scheme theory"*, called $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters*, that were associated, in the first paper [IUTchI] of the series, to certain data, called *initial* Θ*-data*. This data includes an *elliptic curve* E_F over a *number field* F, together with a *prime number* $l \geq 5$ [cf. [IUTchI], §I1]. Consideration of various properties of the log-theta-lattice leads naturally to the establishment of **multiradial algorithms** for constructing **"splitting monoids of LGP-monoids"** [cf. [IUTchIII], Theorem A]. Here, we recall that "multiradial algorithms" [cf. the discussion of [IUTchIII], Introduction] are algorithms that make sense from the point of view of an **"alien arithmetic holomorphic structure"**, i.e., the ring/scheme structure of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater related to a given $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater by means of a *non-ring/scheme-theoretic* horizontal arrow of the log-theta-lattice. In the final portion of [IUTchIII], by applying these multiradial algorithms for splitting monoids of LGP-monoids, we obtained *estimates* for the *log-volume* of these LGP-monoids [cf. [IUTchIII], Theorem B]. In the present paper, these estimates will be applied to verify various **diophantine results**.

In §1 of the present paper, we start by discussing various *elementary estimates* for the *log-volume* of various *tensor products* of the modules obtained by applying the *p-adic logarithm* to the *local units* — i.e., in the terminology of [IUTchIII], *"tensor packets of log-shells"* [cf. the discussion of [IUTchIII], Introduction] — in terms of various well-known invariants, such as *differents*, associated to a *mixed-characteristic nonarchimedean local field* [cf. Propositions 1.1, 1.2, 1.3, 1.4]. We then discuss similar — but technically much simpler! — log-volume estimates in the case of *complex archimedean local fields* [cf. Proposition 1.5]. After reviewing a certain classical estimate concerning the *distribution of prime numbers* [cf. Proposition 1.6], as well as some elementary general nonsense concerning *weighted averages* [cf. Proposition 1.7] and well-known elementary facts concerning *elliptic curves* [cf. Proposition 1.8], we then proceed to *compute explicitly*, in more elementary language, the quantity that was estimated in [IUTchIII], Theorem B. These computations yield a *quite strong/explicit* **diophantine inequality** [cf. Theorem 1.10] concerning elliptic curves that are in **"sufficiently general position"**, so that one may apply the general theory developed in the first three papers of the series.

In §2 of the present paper, after reviewing another classical estimate concerning the *distribution of prime numbers* [cf. Proposition 2.1, (ii)], we then proceed to apply the theory of [GenEll] to **reduce** various diophantine results concerning an **arbitrary** *elliptic curve over a number field* to results of the type obtained in Theorem 1.10 concerning elliptic curves that are in **"sufficiently general position"** [cf. Corollary 2.2; the discussion of Remark 2.3.2, (ii)]. This reduction allows us to derive the following result [cf. Corollary 2.3], which constitutes the **main application** of the *"inter-universal Teichmüller theory"* developed in the present series of papers.

Theorem A. **(Diophantine Inequalities)** *Let* X *be a smooth, proper, geometrically connected curve over a number field;* $D \subseteq X$ *a reduced divisor;* $U_X \overset{\text{def}}{=} X \backslash D$*;* d *a positive integer;* $\epsilon \in \mathbb{R}_{>0}$ *a positive real number. Write* ω_X *for the canonical sheaf on* X*. Suppose that* U_X *is a* **hyperbolic curve***, i.e., that the degree of the line bundle* $\omega_X(D)$ *is* **positive***. Then, relative to the notation of [GenEll] [reviewed in the discussion preceding Corollary 2.2 of the present paper], one has an* **inequality of "bounded discrepancy classes"**

$$\mathrm{ht}_{\omega_X(D)} \ \lesssim \ (1+\epsilon)(\text{log-diff}_X + \text{log-cond}_D)$$

of functions on $U_X(\overline{\mathbb{Q}})^{\leq d}$ *— i.e., the function* $(1+\epsilon)(\text{log-diff}_X + \text{log-cond}_D) - \mathrm{ht}_{\omega_X(D)}$ *is bounded below by a* **constant** *on* $U_X(\overline{\mathbb{Q}})^{\leq d}$ *[cf. [GenEll], Definition 1.2, (ii), as well as Remark 2.3.1, (ii), of the present paper].*

Thus, Theorem A asserts an *inequality* concerning the *canonical height* [i.e., "$\mathrm{ht}_{\omega_X(D)}$"], the *logarithmic different* [i.e., "log-diff_X"], and the *logarithmic conductor* [i.e., "log-cond_D"] of points of the curve U_X valued in number fields whose extension degree over $\mathbb{Q}$ is $\leq d$. In particular, the so-called **Vojta Conjecture** for hyperbolic curves, the **ABC Conjecture**, and the **Szpiro Conjecture** for elliptic curves all follow as special cases of Theorem A. We refer to [Vjt] for a detailed exposition of these conjectures.

Finally, in §3, we examine certain **foundational issues** underlying the theory of the present series of papers. Typically in mathematical discussions — such as, for instance, the theory developed in the present series of papers! — one defines various **"types of mathematical objects"** [i.e., such as groups, topological spaces, or schemes], together with a notion of *"morphisms"* between two particular examples of a specific type of mathematical object [i.e., morphisms between groups, between topological spaces, or between schemes]. Such objects and morphisms [typically] determine a *category.* On the other hand, if one restricts one's attention to such a category, then one must keep in mind the fact that the structure of the category — i.e., which consists *only of a collection of objects and morphisms satisfying certain properties!* — does not include any mention of the various sets and conditions satisfied by those sets that give rise to the "type of mathematical object" under consideration. For instance, the data consisting of the underlying set of a group, the group multiplication law on the group, and the properties satisfied by this group multiplication law *cannot be recovered* [at least in an *a priori* sense!] from the structure of the "*category* of groups". Put another way, although the notion of a "type of mathematical object" may give rise to a "category of such objects", the notion of a "type of mathematical object" is much *stronger* — in the sense that it involves much more *mathematical structure* — than the notion of a category. Indeed, a given "type of mathematical object" may have a *very complicated internal structure*, but may give rise to a category equivalent to a *one-morphism category* [i.e., a category with precisely one morphism]; in particular, in such cases, the structure of the associated category does not retain any information of interest concerning the internal structure of the "type of mathematical object" under consideration.

In Definition 3.1, (iii), we formalize this intuitive notion of a "type of mathematical object" by defining the notion of a **species** as, roughly speaking, a *collection* *of set-theoretic formulas* that gives rise to a category in any given *model of set theory* [cf. Definition 3.1, (iv)], but, unlike any *specific* category [e.g., of groups, etc.] is **not confined** to any **specific model of set theory**. In a similar vein, by working with *collections of set-theoretic formulas*, one may define a species-theoretic analogue of the notion of a *functor*, which we refer to as a **mutation** [cf. Definition 3.3, (i)]. Given a diagram of mutations, one may then define the notion of a "mutation that extracts, from the diagram, a certain portion of the types of mathematical objects that appear in the diagram that is *invariant* with respect to the mutations in the diagram"; we refer to such a mutation as a **core** [cf. Definition 3.3, (v)].

One fundamental example, in the context of the present series of papers, of a diagram of mutations is the usual set-up of **[absolute] anabelian geometry** [cf. Example 3.5 for more details]. That is to say, one begins with the *species* constituted by schemes satisfying certain conditions. One then considers the *mutation*

$$X \quad \rightsquigarrow \quad \Pi_X$$

that associates to such a scheme X its étale fundamental group Π_X [say, considered up to inner automorphisms]. Here, it is important to note that the codomain of this mutation is the *species* constituted by topological groups [say, considered up to inner automorphisms] that satisfy certain conditions which *do not include* any information concerning *how the group is related* [for instance, via some sort of étale fundamental group mutation] *to a scheme*. The notion of an **anabelian reconstruction algorithm** may then be formalized as a *mutation* that forms a *"mutation-quasi-inverse"* to the fundamental group mutation.

Another fundamental example, in the context of the present series of papers, of a diagram of mutations arises from the *Frobenius morphism* in positive characteristic scheme theory [cf. Example 3.6 for more details]. That is to say, one fixes a prime number p and considers the *species* constituted by reduced schemes of characteristic p. One then considers the *mutation* that associates

$$S \quad \rightsquigarrow \quad S^{(p)}$$

to such a scheme S the scheme $S^{(p)}$ with the same topological space, but whose regular functions are given by the p-th powers of the regular functions on the original scheme. Thus, the domain and codomain of this mutation are given by the same species. One may also consider a *log scheme* version of this example, which, at the level of monoids, corresponds, in essence, to assigning

$$M \quad \rightsquigarrow \quad p \cdot M$$

to a torsion-free abelian monoid M the submonoid $p \cdot M \subseteq M$ determined by the image of multiplication by p. Returning to the case of schemes, one may then observe that the well-known constructions of the **perfection** and the **étale site**

$$S \quad \rightsquigarrow \quad S^{\mathrm{pf}}; \quad S \quad \rightsquigarrow \quad S_{\text{ét}}$$

associated to a reduced scheme S of characteristic p give rise to **cores** of the diagram obtained by considering iterates of the **"Frobenius mutation"** just discussed.

This last example of the *Frobenius mutation* and the associated core constituted by the *étale site* is of particular importance in the context of the present series of papers in that it forms the *"intuitive prototype"* that underlies the theory of the **vertical** and **horizontal** lines of the **log-theta-lattice** [cf. the discussion of Remark 3.6.1, (i)]. One notable aspect of this example is the [evident!] fact that the *domain* and *codomain* of the Frobenius mutation are given by the *same species*. That is to say, despite the fact that in the *construction* of the scheme $S^{(p)}$ [cf. the notation of the preceding paragraph] from the scheme S, the scheme $S^{(p)}$ is *"subordinate"* to the scheme S, the domain and codomain species of the resulting Frobenius mutation *coincide*, hence, in particular, are *on a par with one another*. This sort of situation served, for the author, as a sort of model for the $\mathfrak{log}$**-** and $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$**-links** of the log-theta-lattice, which may be formulated as *mutations* between the *species* constituted by the notion of a $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theater*. That is to say, although in the *construction* of either the $\mathfrak{log}$- or the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link, the domain and codomain $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters are by no means on a "par" with one another, the domain and codomain $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters of the resulting $\mathfrak{log}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-links are regarded as objects of the *same species*, hence, in particular, completely *on a par with one another*. This sort of **"relativization"** of **distinct models** *of conventional scheme theory over* $\mathbb{Z}$ via the notion of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater [cf. Fig. I.1 below; the discussion of "gluing together" such models of conventional scheme theory in [IUTchI], §I2] is one of the *most characteristic features* of the theory developed in the present series of papers and, in particular, lies [tautologically!] *outside the framework of conventional scheme theory over* $\mathbb{Z}$. That is to say, in the framework of conventional scheme theory over $\mathbb{Z}$, if one starts out with schemes over $\mathbb{Z}$ and constructs from them, say, by means of geometric objects such as the *theta function* on a Tate curve, some sort of Frobenioid that is isomorphic to a Frobenioid associated to $\mathbb{Z}$, then — unlike, for instance, the case of the *Frobenius morphism* in positive characteristic scheme theory —

> there is no way, within the framework of conventional scheme theory, to treat the newly constructed Frobenioid *"as if it is the Frobenioid associated to* $\mathbb{Z}$*, relative to some* **new** *version/model of conventional scheme theory"*.

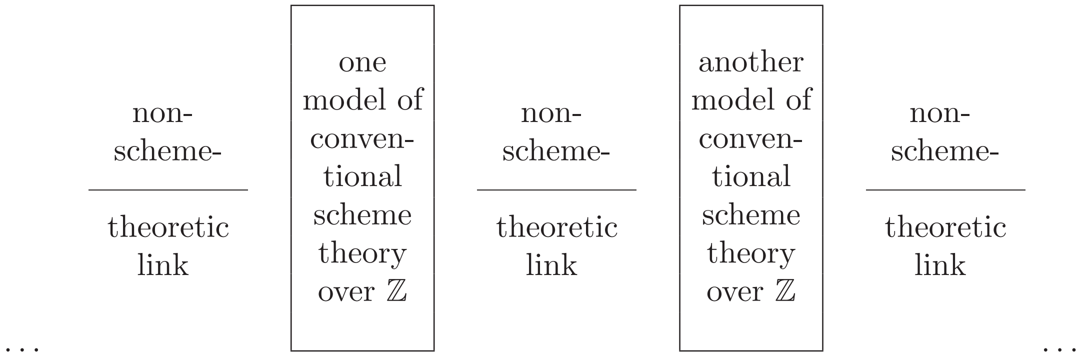

Fig. I.1: Relativized models of conventional scheme theory over $\mathbb{Z}$

If, moreover, one thinks of $\mathbb{Z}$ as being constructed, in the usual way, via *axiomatic set theory*, then one may interpret the *"absolute"* — i.e., *"tautologically* *unrelativizable"* — nature of conventional scheme theory over $\mathbb{Z}$ at a *purely set-theoretic level.* Indeed, from the point of view of the *"$\in$-structure"* of axiomatic set theory, there is *no way to treat* sets constructed at *distinct levels* of this $\in$-structure as being on a par with one another. On the other hand, if one focuses not on the level of the $\in$-structure to which a set belongs, but rather on *species*, then the notion of a species allows one to relate — i.e., to treat on a par with one another — objects belonging to the species that arise from sets constructed at distinct levels of the $\in$-structure. That is to say,

> the notion of a **species** allows one to **"simulate $\in$-loops"** *without violating the axiom of foundation of axiomatic set theory*

— cf. the discussion of Remark 3.3.1, (i).

As one constructs sets at higher and higher levels of the $\in$-structure of some model of axiomatic set theory — e.g., as one travels along *vertical* or *horizontal lines* of the *log-theta-lattice*! — one typically encounters *new schemes*, which give rise to *new Galois categories*, hence to *new Galois or étale fundamental groups*, which may only be constructed if one allows oneself to consider *new basepoints*, relative to *new universes.* In particular, one must continue to *extend the universe*, i.e., to *modify the model of set theory*, relative to which one works. Here, we recall in passing that such "extensions of universe" are possible on account of an **existence axiom** concerning **universes**, which is apparently attributed to the *"Grothendieck school"* and, moreover, cannot, apparently, be obtained as a consequence of the conventional ZFC axioms of axiomatic set theory [cf. the discussion at the beginning of §3 for more details]. On the other hand, ultimately in the present series of papers [cf. the discussion of [IUTchIII], Introduction], we wish to obtain **algorithms** for constructing various objects that arise in the context of the *new schemes/universes* discussed above — i.e., at *distant* $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters* of the log-theta-lattice — that *make sense* from the point of view the *original schemes/universes* that occurred at the outset of the discussion. Again, the fundamental tool that makes this possible, i.e., that allows one to express constructions in the new universes in terms that makes sense in the original universe is precisely

> the **species-theoretic formulation** — i.e., the formulation via **set-theoretic formulas** that *do not depend on* **particular choices** *invoked in* **particular universes** — of the constructions of interest

— cf. the discussion of Remarks 3.1.2, 3.1.3, 3.1.4, 3.1.5, 3.6.2, 3.6.3. This is the point of view that gave rise to the term **"inter-universal"**. At a more concrete level, this "inter-universal" contact between constructions in distant models of conventional scheme theory in the log-theta-lattice is realized by considering [the *étale-like structures* given by] the various Galois or étale fundamental groups that occur as [the "type of mathematical object", i.e., *species* constituted by] **abstract topological groups** [cf. the discussion of Remark 3.6.3, (i); [IUTchI], §I3]. These abstract topological groups give rise to **vertical** or **horizontal cores** of the log-theta-lattice. Moreover, once one obtains cores that are sufficiently *"non-degenerate"*, or *"rich in structure"*, so as to serve as *containers* for the *non-coric* portions of the various mutations [e.g., vertical and horizontal arrows of the log-theta-lattice] under consideration, then one may construct the desired algorithms, or **descriptions**, of these **non-coric portions** in terms of **coric containers**, up to certain *relatively mild* **indeterminacies** [i.e., which reflect the non-coric nature of these non-coric portions!] — cf. the illustration of this sort of situation given in Fig. I.2 below; Remark 3.3.1, (iii); Remark 3.6.1, (ii). In the context of the log-theta-lattice, this is precisely the sort of situation that was achieved in [IUTchIII], Theorem A [cf. the discussion of [IUTchIII], Introduction].

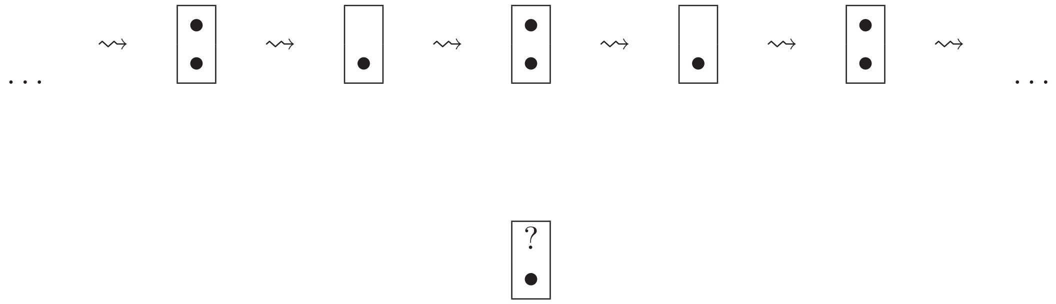

Fig. I.2: A coric container underlying a sequence of mutations

In the context of the above discussion of set-theoretic aspects of the theory developed in the present series of papers, it is of interest to note the following observation, relative to the analogy between the theory of the present series of papers and **p-adic Teichmüller theory** [cf. the discussion of [IUTchI], §I4]. If, instead of working *species-theoretically*, one attempts to document *all of the possible choices* that occur in various *newly introduced universes* that occur in a construction, then one finds that one is obliged to work with sets, such as sets obtained via **set-theoretic exponentiation**, of **very large cardinality**. Such sets of large cardinality are reminiscent of the **exponentially large** denominators that occur if one attempts to *p-adically formally integrate an arbitrary connection* as opposed to a **canonical crystalline connection** of the sort that occurs in the context of the **canonical liftings** of p-adic Teichmüller theory [cf. the discussion of Remark 3.6.2, (iii)]. In this context, it is of interest to recall the computations of [Finot], which assert, roughly speaking, that the canonical liftings of p-adic Teichmüller theory may, in certain cases, be characterized as liftings *"of minimal complexity"* in the sense that their Witt vector coordinates are given by *polynomials of* **minimal degree**.

Finally, we observe that although, in the above discussion, we concentrated on the *similarities*, from an *"inter-universal"* point of view, between the *vertical* and *horizontal* arrows of the log-theta-lattice, there is one important *difference* between these vertical and horizontal arrows: namely,

- whereas the copies of the *full arithmetic fundamental group* — i.e., in particular, the copies of the **geometric fundamental group** — on either side of a **vertical** arrow are **identified** with one another,

- in the case of a **horizontal** arrow, only the **Galois groups** *of the local base fields* on either side of the arrow are identified with one another

— cf. the discussion of Remark 3.6.3, (ii). One way to understand the reason for this difference is as follows. In the case of the *vertical* arrows — i.e., the $\mathfrak{log}$-*links*, which, in essence, amount to the various *local p-adic logarithms* — in order to *construct* the $\mathfrak{log}$-link, it is necessary to make use, in an essential way, of the **local ring structures** at $\underline{v} \in \underline{\mathbb{V}}$ [cf. the discussion of [IUTchIII], Definition 1.1, (i), (ii)], which may only be reconstructed from the *full arithmetic fundamental group*. By contrast, in order to construct the horizontal arrows — i.e., the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-*links* — this local ring structure is *unnecessary*. On the other hand, in order to construct the horizontal arrows, it is necessary to work with structures that, up to isomorphism, are *common* to both the *domain* and the *codomain* of the arrow. Since the construction of the domain of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link **depends**, in an essential way, on the *Gaussian monoids*, i.e., on the **labels** $\in \mathbb{F}^{\circledast}_{l}$ for the **theta values**, which are constructed from the *geometric fundamental group*, while the codomain only involves monoids arising from the local *q-parameters* "$\underline{\underline{q}}_{\underline{v}}$" [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$], which are constructed in a fashion that is **independent** of these **labels**, in order to obtain an isomorphism between structures arising from the domain and codomain, it is necessary to restrict one's attention to the *Galois groups of the local base fields*, which are *free of any dependence on these labels.*

Acknowledgements:

I would like to thank *Fumiharu Kato* and *Akio Tamagawa* for many helpful discussions concerning the material presented in this paper. Also, I would like to thank *Kentaro Sato* for informing me of [Ffmn].

Notations and Conventions:

We shall continue to use the "Notations and Conventions" of [IUTchI], §0.

Section 1: Log-volume Estimates

In the present §1, we perform various *elementary* **local computations** concerning nonarchimedean and archimedean local fields which allow us to obtain **more explicit versions** [cf. Theorem 1.10 below] of the **log-volume estimates** for **Θ-pilot objects** obtained in [IUTchIII], Corollary 3.12.

In the following, if $\lambda \in \mathbb{R}$, then we shall write

$$\lceil \lambda \rceil \text{ (respectively, } \lfloor \lambda \rfloor\text{)}$$

for the *smallest* (respectively, *largest*) $n \in \mathbb{Z}$ such that $n \geq \lambda$ (respectively, $n \leq \lambda$).

Proposition 1.1. **(Multiple Tensor Products and Differents)** *Let p be a prime number, I a nonempty finite set, $\overline{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$. Write $\overline{R} \subseteq \overline{\mathbb{Q}}_p$ for the ring of integers of $\overline{\mathbb{Q}}_p$ and* $\mathrm{ord} : \overline{\mathbb{Q}}_p^{\times} \to \mathbb{Q}$ *for the natural p-adic valuation on $\overline{\mathbb{Q}}_p$, normalized so that* $\mathrm{ord}(p) = 1$. *For $i \in I$, let $k_i \subseteq \overline{\mathbb{Q}}_p$ be a finite extension of $\mathbb{Q}_p$; write $R_i \overset{\mathrm{def}}{=} \mathcal{O}_{k_i} = \overline{R} \bigcap k_i$ for the ring of integers of k_i and $\mathfrak{d}_i \in \mathbb{Q}_{\geq 0}$ for the order [i.e., "*$\mathrm{ord}(-)$*"] of any generator of the* **different ideal** *of R_i over $\mathbb{Z}_p$. Also, for any nonempty subset $E \subseteq I$, let us write*

$$R_E \quad \overset{\mathrm{def}}{=} \quad \bigotimes_{i \in E} R_i; \qquad \mathfrak{d}_E \quad \overset{\mathrm{def}}{=} \quad \sum_{i \in E} \mathfrak{d}_i$$

— where the tensor product is over $\mathbb{Z}_p$. Fix an element $* \in I$; write $I^* \overset{\mathrm{def}}{=} I \setminus \{*\}$. Then

$$R_I \quad \subseteq \quad (R_I)^{\sim}; \qquad p^{\lceil \mathfrak{d}_{I^*} \rceil} \cdot (R_I)^{\sim} \quad \subseteq \quad R_I$$

— where we write "$(-)^{\sim}$" for the **normalization** *of the ring in parentheses.*

Proof. Let us regard R_I as an R_**-algebra* in the evident fashion. It is immediate from the definitions that $R_I \subseteq (R_I)^{\sim}$. Now observe that

$$\overline{R} \otimes_{R_*} R_I \quad \subseteq \quad \overline{R} \otimes_{R_*} (R_I)^{\sim} \quad \subseteq \quad (\overline{R} \otimes_{R_*} R_I)^{\sim}$$

— where $(\overline{R} \otimes_{R_*} R_I)^{\sim}$ decomposes as a *direct sum* of finitely many copies of $\overline{R}$. In particular, one verifies immediately, in light of the fact the $\overline{R}$ is *faithfully flat* over R_*, that to complete the proof of Proposition 1.1, it suffices to verify that

$$p^{\lceil \mathfrak{d}_{I^*} \rceil} \cdot (\overline{R} \otimes_{R_*} R_I)^{\sim} \quad \subseteq \quad \overline{R} \otimes_{R_*} R_I$$

— or, indeed, that

$$p^{\mathfrak{d}_{I^*}} \cdot (\overline{R} \otimes_{R_*} R_I)^{\sim} \quad \subseteq \quad \overline{R} \otimes_{R_*} R_I$$

— where, for $\lambda \in \mathbb{Q}$, we write p^{λ} for any element of $\overline{\mathbb{Q}}_p$ such that $\mathrm{ord}(p^{\lambda}) = \lambda$. On other hand, it follows immediately from *induction on the cardinality of I* that to verify this last inclusion, it suffices to verify the inclusion in the case where I is of *cardinality two*. But in this case, the desired inclusion follows immediately from the *definition of the different ideal*. This completes the proof of Proposition 1.1. $\bigcirc$

Proposition 1.2. **(Differents and Logarithms)** *We continue to use the notation of Proposition 1.1. For $i \in I$, write e_i for the* **ramification index** *of k_i over $\mathbb{Q}_p$;*

$$a_i \stackrel{\mathrm{def}}{=} \frac{1}{e_i} \cdot \lceil \frac{e_i}{p-2} \rceil \ \text{if}\ p > 2, \qquad a_i \stackrel{\mathrm{def}}{=} 2 \ \text{if}\ p = 2.$$

Here, "$\log(-)$" denotes the natural logarithm. Thus,

$$\text{if}\ p > 2\ \text{and}\ e_i \leq p-2,\ \text{then}\ a_i = \frac{1}{e_i}.$$

For any nonempty subset $E \subseteq I$, let us write

$$\log_p(R_E^\times) \quad \stackrel{\mathrm{def}}{=} \quad \bigotimes_{i \in E} \log_p(R_i^\times); \qquad a_E \quad \stackrel{\mathrm{def}}{=} \quad \sum_{i \in E} a_i$$

— where the tensor product is over $\mathbb{Z}_p$; we write "$\log_p(-)$" for the p-adic logarithm. For $\lambda \in \frac{1}{e_i} \cdot \mathbb{Z}$, we shall write $p^\lambda \cdot R_i$ for the fractional ideal of R_i generated by any element "p^λ" of k_i such that $\mathrm{ord}(p^\lambda) = \lambda$. *Let*

$$\phi: \ \log_p(R_I^\times) \otimes \mathbb{Q}_p \ \stackrel{\sim}{\to} \ \log_p(R_I^\times) \otimes \mathbb{Q}_p$$

be an **automorphism** *of the finite dimensional $\mathbb{Q}_p$-vector space* $\log_p(R_I^\times) \otimes \mathbb{Q}_p$ *that induces an automorphism of the submodule* $\log_p(R_I^\times)$. *Then:*

(i) We have:

$$p^{a_i} \cdot R_i \quad \subseteq \quad \log_p(R_i^\times)$$

— where the "$\subseteq$" is an **equality** *when $p > 2$ and $e_i \leq p-2$.*

(ii) We have:

$$\phi(p^\lambda \cdot R_i \otimes_{R_i} (R_I)^\sim) \quad \subseteq \quad p^{\lfloor \lambda \rfloor - \lceil \mathfrak{d}_I \rceil - \lceil a_I \rceil} \cdot \log_p(R_I^\times)$$

for any $\lambda \in \frac{1}{e_i} \cdot \mathbb{Z}$, $i \in I$. In particular, $\phi((R_I)^\sim) \ \subseteq \ p^{-\lceil \mathfrak{d}_I \rceil - \lceil a_I \rceil} \cdot \log_p(R_I^\times)$.

(iii) Suppose that $p > 2$, and that $e_i \leq p-2$ for all $i \in I$. Then we have:

$$\phi(p^\lambda \cdot R_i \otimes_{R_i} (R_I)^\sim) \quad \subseteq \quad p^{\lfloor \lambda \rfloor - \lceil \mathfrak{d}_I \rceil - 1} \cdot (R_I)^\sim$$

for any $\lambda \in \frac{1}{e_i} \cdot \mathbb{Z}$, $i \in I$. In particular, $\phi((R_I)^\sim) \ \subseteq \ p^{-\lceil \mathfrak{d}_I \rceil - 1} \cdot (R_I)^\sim$.

(iv) If $p > 2$ and $e_i = 1$ for all $i \in I$, then $\phi((R_I)^\sim) \ \subseteq \ (R_I)^\sim$.

Proof. Assertion (i) follows immediately from the well-known theory of the *p-adic logarithm* and *exponential* maps [cf., e.g., [Kobl], p. 81]. Next, let us observe that to verify assertions (ii) and (iii), it suffices to consider the case where $\lambda = 0$. Now it follows from the second displayed inclusion of Proposition 1.1 that

$$p^{\lceil \mathfrak{d}_I \rceil} \cdot (R_I)^{\sim} \quad \subseteq \quad R_I \quad = \quad \bigotimes_{i \in I} R_i$$

and hence that

$$p^{\lceil \mathfrak{d}_I \rceil + \lceil a_I \rceil} \cdot (R_I)^{\sim} \quad \subseteq \quad \bigotimes_{i \in I} p^{a_i} \cdot R_i \quad \subseteq \quad \bigotimes_{i \in I} \log_p(R_i^{\times}) \quad = \quad \log_p(R_I^{\times})$$

— where the first inclusion follows immediately from the fact that $(R_I)^{\sim}$ decomposes as a *direct sum* of rings of integers of finite extensions of $\mathbb{Q}_p$, and the second inclusion follows from assertion (i). Thus, assertion (ii) follows immediately from the fact that ϕ induces an automorphism of the submodule $\log_p(R_I^{\times})$. When $p > 2$ and $e_i \leq p - 2$ for all $i \in I$, we thus obtain that

$$p^{\lceil \mathfrak{d}_I \rceil + \lceil a_I \rceil} \cdot \phi((R_I)^{\sim}) \quad \subseteq \quad \log_p(R_I^{\times}) \quad = \quad \bigotimes_{i \in I} p^{a_i} \cdot R_i \quad \subseteq \quad p^{\lfloor a_I \rfloor} \cdot (R_I)^{\sim}$$

— where the equality follows from assertion (i), and the final inclusion follows immediately from the fact that $(R_I)^{\sim}$ decomposes as a *direct sum* of rings of integers of finite extensions of $\mathbb{Q}_p$. Thus, assertions (iii) and (iv) follow immediately from the fact that $\lfloor a_I \rfloor - \lceil a_I \rceil \geq -1$, together with the fact that $a_i = 1$, $\mathfrak{d}_i = 0$ whenever $e_i = 1$. This completes the proof of Proposition 1.2. ○

Proposition 1.3. **(Estimates of Differents)** *We continue to use the notation of Proposition 1.2. Suppose that $k_0 \subseteq k_i$ is a subfield that contains $\mathbb{Q}_p$. Write $R_0 \overset{\mathrm{def}}{=} \mathcal{O}_{k_0}$ for the ring of integers of k_0, $\mathfrak{d}_0$ for the order [i.e., "$\mathrm{ord}(-)$"] of any generator of the different ideal of R_0 over $\mathbb{Z}_p$, e_0 for the ramification index of k_0 over $\mathbb{Q}_p$, $e_{i/0} \overset{\mathrm{def}}{=} e_i/e_0$ ($\in \mathbb{Z}$), $[k_i : k_0]$ for the degree of the extension k_i/k_0, n_i for the unique nonnegative integer such that $[k_i : k_0]/p^{n_i}$ is an integer prime to p. Then:*

(i) We have:

$$\mathfrak{d}_i \geq \mathfrak{d}_0 + (e_{i/0} - 1)/(e_{i/0} \cdot e_0) = \mathfrak{d}_0 + (e_{i/0} - 1)/e_i$$

— where the "$\geq$" is an **equality** *when k_i is tamely ramified over k_0.*

(ii) Suppose that k_i is a finite Galois extension of a subfield $k_1 \subseteq k_i$ such that $k_0 \subseteq k_1$, and k_1 is tamely ramified over k_0. Then we have: $\mathfrak{d}_i \leq \mathfrak{d}_0 + n_i + 1/e_0$.

Proof. First, we consider assertion (i). By replacing k_0 by an *unramified* extension of k_0 contained in k_i, we may assume without loss of generality that k_i is a *totally ramified* extension of k_0. Let π_0 be a uniformizer of R_0. Then there exists an isomorphism R_0-algebras $R_0[x]/(f(x)) \overset{\sim}{\to} R_i$, where $f(x) \in R_0[x]$ is a monic polynomial which is $\equiv x^{e_{i/0}} \pmod{\pi_0}$, that maps $x \mapsto \pi_i$ for some uniformizer π_i of R_i. Thus, the different $\mathfrak{d}_i$ may be computed as follows:

$$\begin{aligned} \mathfrak{d}_i - \mathfrak{d}_0 \ &= \ \mathrm{ord}(f'(\pi_i)) \ \geq \ \min(\mathrm{ord}(\pi_0), \mathrm{ord}(e_{i/0} \cdot \pi_i^{e_{i/0}-1})) \\ &\geq \ \min\left(\frac{1}{e_0}, \mathrm{ord}(\pi_i^{e_{i/0}-1})\right) \ = \ \min\left(\frac{1}{e_0}, \frac{e_{i/0}-1}{e_{i/0} \cdot e_0}\right) \ = \ \frac{e_{i/0}-1}{e_i} \end{aligned}$$

— where, for $\lambda, \mu \in \mathbb{R}$ such that $\lambda \geq \mu$, we define $\min(\lambda, \mu) \overset{\mathrm{def}}{=} \mu$. When k_i is *tamely ramified* over k_0, one verifies immediately that the inequalities of the above display are, in fact, equalities. This completes the proof of assertion (i).

Next, we consider assertion (ii). We apply *induction* on n_i. Since assertion (ii) follows immediately from assertion (i) when $n_i = 0$, we may assume that $n_i \geq 1$, and that assertion (ii) has been verified for smaller "n_i". By replacing k_1 by some tamely ramified extension of k_1 contained in k_i, we may assume without loss of generality that $\mathrm{Gal}(k_i/k_1)$ is a p-group. Since p-groups are solvable, it follows that there exists a subextension $k_1 \subseteq k_* \subseteq k_i$ such that k_i/k_* and k_*/k_1 are Galois extensions of degree p and p^{n_i-1}, respectively. Write $R_* \overset{\mathrm{def}}{=} \mathcal{O}_{k_*}$ for the ring of integers of k_*, $\mathfrak{d}_*$ for the order [i.e., "$\mathrm{ord}(-)$"] of any generator of the different ideal of R_* over $\mathbb{Z}_p$, and e_* for the ramification index of k_* over $\mathbb{Q}_p$. Thus, by the induction hypothesis, it follows that $\mathfrak{d}_* \leq \mathfrak{d}_0 + n_i - 1 + 1/e_0$. To verify that $\mathfrak{d}_i \leq \mathfrak{d}_0 + n_i + 1/e_0$, it suffices to verify that $\mathfrak{d}_i \leq \mathfrak{d}_0 + n_i + 1/e_0 + \epsilon$ for any positive real number ϵ. By possibly enlarging k_i and k_1, we may also assume without loss of generality that the tamely ramified extension k_1 of k_0 contains a *primitive p-th root of unity*, and, moreover, that the ramification index e_1 of k_1 over $\mathbb{Q}_p$ satisfies $e_1 \geq p/\epsilon$ [so $e_* \geq e_1 \geq p/\epsilon$]. Thus, k_i is a *Kummer extension* of k_*. In particular, there exists an *inclusion* of R_*-algebras $R_*[x]/(f(x)) \hookrightarrow R_i$, where $f(x) \in R_*[x]$ is a monic polynomial which is of the form $f(x) = x^p - \varpi_*$ for some element ϖ_* of R_* satisfying $0 \leq \mathrm{ord}(\varpi_*) \leq \frac{p-1}{e_*}$, that maps $x \mapsto \varpi_i$ for some element ϖ_i of R_i satisfying $0 \leq \mathrm{ord}(\varpi_i) \leq \frac{p-1}{p \cdot e_*}$. Now we compute:

$$\begin{aligned} \mathfrak{d}_i \ &\leq \ \mathrm{ord}(f'(\varpi_i)) + \mathfrak{d}_* \ \leq \ \mathrm{ord}(p \cdot \varpi_i^{p-1}) + \mathfrak{d}_0 + n_i - 1 + 1/e_0 \\ &= \ (p-1) \cdot \mathrm{ord}(\varpi_i) + \mathfrak{d}_0 + n_i + 1/e_0 \ \leq \ \frac{(p-1)^2}{p \cdot e_*} + \mathfrak{d}_0 + n_i + 1/e_0 \\ &\leq \ \frac{p}{e_*} + \mathfrak{d}_0 + n_i + 1/e_0 \ \leq \ \mathfrak{d}_0 + n_i + 1/e_0 + \epsilon \end{aligned}$$

— thus completing the proof of assertion (ii). $\bigcirc$

Proposition 1.4. **(Nonarchimedean Normalized Log-volume Estimates)** *We continue to use the notation of Proposition 1.2. Also, for* $i \in I$, *write* $R_i^{\boldsymbol{\mu}} \subseteq R_i^{\times}$ *for the torsion subgroup of* $R_i^{\times}$, $R_i^{\times\boldsymbol{\mu}} \overset{\mathrm{def}}{=} R_i^{\times}/R_i^{\boldsymbol{\mu}}$, p^{f_i} *for the cardinality of the residue field of* k_i, *and* p^{m_i} *for the order of the p-primary component of* $R_i^{\boldsymbol{\mu}}$. *Thus, the order of* $R_i^{\boldsymbol{\mu}}$ *is equal to* $p^{m_i} \cdot (p^{f_i} - 1)$. *Then:*

(i) The **log-volumes** *constructed in [AbsTopIII], Proposition 5.7, (i), on the various finite extensions of* $\mathbb{Q}_p$ *contained in* $\overline{\mathbb{Q}}_p$ *may be suitably* **normalized** *[i.e., by dividing by the degree of the finite extension] so as to yield a notion of log-volume*

$$\mu^{\log}(-)$$

defined on compact open subsets of finite extensions of $\mathbb{Q}_p$ *contained in* $\overline{\mathbb{Q}}_p$*, valued in* $\mathbb{R}$*, and normalized so that* $\mu^{\log}(R_i) = 0$*,* $\mu^{\log}(p \cdot R_i) = -\log(p)$*, for each* $i \in I$*. Moreover, by applying the fact that tensor products of finitely many copies of* $\overline{\mathbb{Q}}_p$ *over* $\mathbb{Z}_p$ *decompose, naturally, as direct sums of finitely many copies of* $\overline{\mathbb{Q}}_p$*, we obtain a notion of log-volume — which, by abuse of notation, we shall also denote by "*$\mu^{\log}(-)$*" — defined on compact open subsets of finitely generated* $\mathbb{Z}_p$*-submodules of such tensor products, valued in* $\mathbb{R}$*, and normalized so that* $\mu^{\log}((R_E)^\sim) = 0$*,* $\mu^{\log}(p \cdot (R_E)^\sim) = -\log(p)$*, for any nonempty set* $E \subseteq I$*.*

(ii) We have:

$$\mu^{\log}(\log_p(R_i^\times)) = -\Big(\frac{1}{e_i} + \frac{m_i}{e_i f_i}\Big) \cdot \log(p)$$

[cf. [AbsTopIII], Proposition 5.8, (iii)].

(iii) Let $I^* \subseteq I$ *be a subset such that for each* $i \in I \setminus I^*$*, it holds that* $p - 2 \geq e_i$ (≥ 1)*. Then for any* $\lambda \in \frac{1}{e_{i^\dagger}} \cdot \mathbb{Z}$*,* $i^\dagger \in I$*, we have* $\phi(p^\lambda \cdot R_{i^\dagger} \otimes_{R_{i^\dagger}} (R_I)^\sim) \subseteq p^{\lfloor \lambda \rfloor - \lceil \mathfrak{d}_I \rceil - \lceil a_I \rceil} \cdot \log_p(R_I^\times)$*, and*

$$\mu^{\log}(p^{\lfloor \lambda \rfloor - \lceil \mathfrak{d}_I \rceil - \lceil a_I \rceil} \cdot \log_p(R_I^\times)) \;\leq\; \Big(-\lambda + \mathfrak{d}_I + 3 + 4 \cdot |I^*|/p\Big) \cdot \log(p)$$

— where we write $|I^*|$ *for the cardinality of* I^**. Moreover,* $\lceil \mathfrak{d}_I \rceil + \lceil a_I \rceil \;\geq\; |I|$ *if* $p > 2$*;* $\lceil \mathfrak{d}_I \rceil + \lceil a_I \rceil \;\geq\; 2 \cdot |I|$ *if* $p = 2$*.*

(iv) If $p > 2$ *and* $e_i = 1$ *for all* $i \in I$*, then* $\phi((R_I)^\sim) \subseteq (R_I)^\sim$*, and* $\mu^{\log}((R_I)^\sim) = 0$*.*

Proof. Assertion (i) follows immediately from the definitions. Next, we consider assertion (ii). Note that the log-volume on $R_i^\times$ determines, in a natural way, a log-volume on the quotient $R_i^\times \twoheadrightarrow R_i^{\times\boldsymbol{\mu}}$. Moreover, in light of the compatibility of the log-volume with "$\log_p(-)$" [cf. [AbsTopIII], Proposition 5.7, (i), (c)], it follows immediately that $\mu^{\log}(\log_p(R_i^\times)) = \mu^{\log}(R_i^{\times\boldsymbol{\mu}})$. Thus, it suffices to compute $e_i \cdot f_i \cdot \mu^{\log}(R_i^{\times\boldsymbol{\mu}}) = e_i \cdot f_i \cdot \mu^{\log}(R_i^\times) - \log(p^{m_i} \cdot (p^{f_i} - 1))$. On the other hand, it follows immediately from the basic properties of the log-volume [cf. [AbsTopIII], Proposition 5.7, (i), (a)] that $e_i \cdot f_i \cdot \mu^{\log}(R_i^\times) = \log(1 - p^{-f_i})$, so $e_i \cdot f_i \cdot \mu^{\log}(R_i^{\times\boldsymbol{\mu}}) = -(f_i + m_i) \cdot \log(p)$, as desired. This completes the proof of assertion (ii).

The inclusion of assertion (iii) follows immediately from Proposition 1.2, (ii). When $p = 2$, the fact that $\lceil \mathfrak{d}_I \rceil + \lceil a_I \rceil \;\geq\; 2 \cdot |I|$ follows immediately from the definition of "a_i" in Proposition 1.2. When $p > 2$, it follows immediately from the definition of "a_i" in Proposition 1.2 that $a_i \geq 1/e_i$, for all $i \in I$; thus, since $\mathfrak{d}_i \geq (e_i - 1)/e_i$ for all $i \in I$ [cf. Proposition 1.3, (i)], we conclude that $\mathfrak{d}_i + a_i \geq 1$ for all $i \in I$, and hence that $\lceil \mathfrak{d}_I \rceil + \lceil a_I \rceil \;\geq\; \mathfrak{d}_I + a_I \;\geq\; |I|$, as asserted in the stament of assertion (iii). Next, let us observe that $\frac{1}{p-2} \leq \frac{4}{p}$ for $p \geq 3$. Thus, it follows immediately from the definition of a_i in Proposition 1.2 that $a_i \leq \frac{4}{p} + \frac{1}{e_i}$ for $i \in I$, $a_i = \frac{1}{e_i}$ for $i \in I \setminus I^*$. On the other hand, by assertion (i), we have

$\mu^{\log}(R_I) \leq \mu^{\log}((R_I)^{\sim}) = 0$; by assertion (ii), we have $\mu^{\log}(\log_p(R_i^{\times})) \leq -\frac{1}{e_i} \cdot \log(p)$. Now we compute:

$$\begin{aligned}
\mu^{\log}(p^{\lfloor \lambda \rfloor - \lceil \mathfrak{d}_I \rceil - \lceil a_I \rceil} \cdot \log_p(R_I^{\times})) \ &\leq \ \Big(- \lambda + \mathfrak{d}_I + a_I + 3 \Big) \cdot \log(p) \ + \ \mu^{\log}(\log_p(R_I^{\times})) \\
&\leq \ \Big(- \lambda + \mathfrak{d}_I + a_I + 3 \Big) \cdot \log(p) \\
&\quad + \ \Big\{ \sum_{i \in I} \ \mu^{\log}(\log_p(R_i^{\times})) \Big\} \ + \ \mu^{\log}(R_I) \\
&\leq \ \Big\{ - \lambda + \mathfrak{d}_I + 3 + \sum_{i \in I} \ (a_i - \frac{1}{e_i}) \Big\} \cdot \log(p) \\
&\leq \ \Big(- \lambda + \mathfrak{d}_I + 3 + 4 \cdot |I^*|/p \Big) \cdot \log(p)
\end{aligned}$$

— thus completing the proof of assertion (iii). Assertion (iv) follows immediately from assertion (i) and Proposition 1.2, (iv). ○

Proposition 1.5. **(Archimedean Metric Estimates)** *In the following, we shall regard the* **complex archimedean field** $\mathbb{C}$ *as being equipped its standard* **Hermitian metric***, i.e., the metric determined by the complex norm. Let us refer to as the* **primitive automorphisms** *of* $\mathbb{C}$ *the group of automorphisms [of order* 8*] of the underlying metrized real vector space of* $\mathbb{C}$ *generated by the operations of* **complex conjugation** *and* **multiplication** *by* ± 1 *or* $\pm\sqrt{-1}$.

(i) **(Direct Sum vs. Tensor Product Metrics)** *The metric on* $\mathbb{C}$ *determines a tensor product metric on* $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$*, as well as a direct sum metric on* $\mathbb{C} \oplus \mathbb{C}$*. Then, relative to these metrics, any* **isomorphism of topological rings** *[i.e., arising from the Chinese remainder theorem]*

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \ \stackrel{\sim}{\to} \ \mathbb{C} \oplus \mathbb{C}$$

is **compatible** *with these* **metrics***, up a factor of* 2*, i.e., the metric on the right-hand side corresponds to* 2 *times the metric on the left-hand side. [Thus, lengths differ by a factor of* $\sqrt{2}$*.]*

(ii) **(Direct Sum vs. Tensor Product Automorphisms)** *Relative to the notation of (i), the* **direct sum decomposition** $\mathbb{C} \oplus \mathbb{C}$*, together with its Hermitian metric, is* **preserved***, relative to the displayed isomorphism of (i), by the automorphisms of* $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ *induced by the various* **primitive automorphisms** *of the two copies of "*$\mathbb{C}$*" that appear in the tensor product* $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$*.*

(iii) **(Direct Sums and Tensor Products of Multiple Copies)** *Let* I*,* V *be nonempty finite sets, whose cardinalities we denote by* $|I|$*,* $|V|$*, respectively. Write*

$$M \ \stackrel{\text{def}}{=} \ \bigoplus_{v \in V} \ \mathbb{C}_v$$

for the direct sum of copies $\mathbb{C}_v \stackrel{\text{def}}{=} \mathbb{C}$ *of* $\mathbb{C}$ *labeled by* $v \in V$*, which we regard as equipped with the direct sum metric, and*

$$M_I \ \stackrel{\text{def}}{=} \ \bigotimes_{i \in I} \ M_i$$

for the tensor product over $\mathbb{R}$ *of copies* $M_i \stackrel{\mathrm{def}}{=} M$ *of* M *labeled by* $i \in I$*, which we regard as equipped with the* **tensor product metric** *[cf. the constructions of [IUTchIII], Proposition 3.2, (ii)]. Then the topological ring structure on each* $\mathbb{C}_v$ *determines a* **topological ring structure** *on* M_I *with respect to which* M_I *admits a unique* **direct sum decomposition** *as a direct sum of*

$$2^{|I|-1} \cdot |V|^{|I|}$$

copies of $\mathbb{C}$ *[cf. [IUTchIII], Proposition 3.1, (i)]. The* **direct sum metric** *on* M_I *— i.e., the metric determined by the natural metrics on these copies of* $\mathbb{C}$ *— is equal to*

$$2^{|I|-1}$$

times the original tensor product metric on M_I*. Write*

$$B_I \quad \subseteq \quad M_I$$

for the **"integral structure"** *[cf. the constructions of [IUTchIII], Proposition 3.1, (ii)] given by the direct product of the* **unit balls** *of the copies of* $\mathbb{C}$ *that occur in the direct sum decomposition of* M_I*. Then the tensor product metric on* M_I*, the direct sum decomposition of* M_I*, the direct sum metric on* M_I*, and the integral structure* $B_I \subseteq M_I$ *are* **preserved** *by the automorphisms of* M_I *induced by the various* **primitive automorphisms** *of the direct summands "*$\mathbb{C}_v$*" that appear in the factors "*M_i*" of the tensor product* M_I*.*

(iv) **(Tensor Product of Vectors of a Given Length)** *Suppose that we are in the situation of (iii). Fix* $\lambda \in \mathbb{R}_{>0}$*. Then*

$$M_I \quad \ni \quad \bigotimes_{i \in I} m_i \quad \in \quad \lambda^{|I|} \cdot B_I$$

for any collection of elements $\{m_i \in M_i\}_{i \in I}$ *such that the component of* m_i *in each direct summand "*$\mathbb{C}_v$*" of* M_i *is of* **length** λ*.*

Proof. Assertions (i) and (ii) are discussed in [IUTchIII], Remark 3.9.1, (ii), and may be verified by means of routine and elementary arguments. Assertion (iii) follows immediately from assertions (i) and (ii). Assertion (iv) follows immediately from the various definitions involved. ○

Proposition 1.6. **(The Prime Number Theorem)** *If* n *is a positive integer, then let us write* p_n *for the* n*-th largest prime number. [Thus,* $p_1 = 2$*,* $p_2 = 3$*, and so on.] Then there exists an integer* n_0 *such that holds that*

$$\sum_{m=1}^{n} \frac{\log(p_m)}{p_m} \quad \leq \quad 2 \cdot \log(n) \quad (\leq \quad 2 \cdot \log(p_n))$$

for all $n \geq n_0$*. In particular, there exists a positive real number* η_{prm} *such that*

$$\sum_{\eta \leq p^{-1}} \frac{\log(p)}{p} \quad \leq \quad -\, 2 \cdot \log(\eta)$$

— where the sum ranges over the prime numbers p *such that* $\eta \leq p^{-1}$ *— for all positive real* $\eta < \eta_{\mathrm{prm}}$*.*

Proof. Relative to our notation, the *Prime Number Theorem* [cf., e.g., [DmMn], §3.10] implies that

$$\lim_{m \to \infty} \frac{m \cdot \log(p_m)}{p_m} = 1$$

— i.e., in particular, that for some positive integer m_0, it holds that

$$\frac{\log(p_m)}{p_m} \leq \frac{4}{3} \cdot \frac{1}{m}$$

for all $m \geq m_0$. On the other hand, one verifies immediately [i.e., by estimating the integral of the function $\mathbb{R}_{>0} \ni x \mapsto \frac{1}{x} \in \mathbb{R}_{>0}$] that m_0 may be chosen so that

$$\sum_{m=1}^{n} \frac{1}{m} \leq \frac{4}{3} \cdot \log(n)$$

for all $n \geq m_0$. Thus, we conclude that for some $n_0 \geq m_0$, it holds that

$$\begin{aligned} \sum_{m=1}^{n} \frac{\log(p_m)}{p_m} &\leq \sum_{m=1}^{m_0-1} \frac{\log(p_m)}{p_m} + \sum_{m=m_0}^{n} \frac{\log(p_m)}{p_m} \\ &\leq \frac{1}{9} \cdot \log(n_0) + \sum_{m=m_0}^{n} \frac{4}{3} \cdot \frac{1}{m} \leq \frac{1}{9} \cdot \log(n_0) + \frac{16}{9} \cdot \log(n) \\ &\leq 2 \cdot \log(n) \end{aligned}$$

for all $n \geq n_0$, as desired. The final portion of Proposition 1.6 follows formally. ○

Proposition 1.7. (Weighted Averages) *Let* E *be a nonempty finite set,* n *a positive integer. For* $e \in E$*, let* $\lambda_e \in \mathbb{R}_{>0}$*,* $\beta_e \in \mathbb{R}$*. Then, for any* $i = 1, \dots, n$*, we have:*

$$\frac{\sum_{\vec{e} \in E^n} \beta_{\vec{e}} \cdot \lambda_{\Pi \vec{e}}}{\sum_{\vec{e} \in E^n} \lambda_{\Pi \vec{e}}} = \frac{\sum_{\vec{e} \in E^n} n \cdot \beta_{e_i} \cdot \lambda_{\Pi \vec{e}}}{\sum_{\vec{e} \in E^n} \lambda_{\Pi \vec{e}}} = n \cdot \beta_{\mathrm{avg}}$$

— where we write $\beta_{\mathrm{avg}} \stackrel{\mathrm{def}}{=} \beta_E / \lambda_E$*,* $\beta_E \stackrel{\mathrm{def}}{=} \sum_{e \in E} \beta_e \cdot \lambda_e$*,* $\lambda_E \stackrel{\mathrm{def}}{=} \sum_{e \in E} \lambda_e$*,*

$$\beta_{\vec{e}} \stackrel{\mathrm{def}}{=} \sum_{j=1}^{n} \beta_{e_j}; \qquad \lambda_{\Pi \vec{e}} \stackrel{\mathrm{def}}{=} \prod_{j=1}^{n} \lambda_{e_j}$$

for any n*-tuple* $\vec{e} = (e_1, \dots, e_n) \in E^n$ *of elements of* E*.*

Proof. We begin by observing that

$$\lambda_E^n = \sum_{\vec{e} \in E^n} \lambda_{\Pi \vec{e}} \, ; \qquad \beta_E \cdot \lambda_E^{n-1} = \sum_{\vec{e} \in E^n} \beta_{e_i} \cdot \lambda_{\Pi \vec{e}}$$

for any $i = 1, \dots, n$. Thus, summing over i, we obtain that

$$n \cdot \beta_E \cdot \lambda_E^{n-1} \quad = \quad \sum_{\vec{e} \in E^n} \beta_{\vec{e}} \cdot \lambda_{\Pi \vec{e}} \quad = \quad \sum_{\vec{e} \in E^n} n \cdot \beta_{e_i} \cdot \lambda_{\Pi \vec{e}}$$

and hence that

$$\begin{aligned} n \cdot \beta_{\mathrm{avg}} \; = \; n \cdot \beta_E \cdot \lambda_E^{n-1} / \lambda_E^n \; &= \; \Big(\sum_{\vec{e} \in E^n} \beta_{\vec{e}} \cdot \lambda_{\Pi \vec{e}} \Big) \cdot \Big(\sum_{\vec{e} \in E^n} \lambda_{\Pi \vec{e}} \Big)^{-1} \\ &= \; \Big(\sum_{\vec{e} \in E^n} n \cdot \beta_{e_i} \cdot \lambda_{\Pi \vec{e}} \Big) \cdot \Big(\sum_{\vec{e} \in E^n} \lambda_{\Pi \vec{e}} \Big)^{-1} \end{aligned}$$

as desired. $\bigcirc$

Remark 1.7.1. In Theorem 1.10 below, we shall apply Proposition 1.7 to compute various **packet-normalized log-volumes** of the sort discussed in [IUTchIII], Proposition 3.9, (i) — i.e., log-volumes normalized by means of the **normalized weights** discussed in [IUTchIII], Remark 3.1.1, (ii). Here, we recall that the normalized weights discussed in [IUTchIII], Remark 3.1.1, (ii), were computed relative to the *non-normalized log-volumes* of [AbsTopIII], Proposition 5.8, (iii), (vi) [cf. the discussion of [IUTchIII], Remark 3.1.1, (ii); [IUTchI], Example 3.5, (iii)]. By contrast, in the discussion of the present §1, our computations are performed relative to *normalized log-volumes* as discussed in Proposition 1.4, (i). In particular, it follows that the *weights* $[K_{\underline{v}} : (F_{\mathrm{mod}})_v]^{-1}$, where $\underline{\mathbb{V}} \ni \underline{v} \mid v \in \mathbb{V}_{\mathrm{mod}}$, of the discussion of [IUTchIII], Remark 3.1.1, (ii), must be *replaced* — i.e., when one works with *normalized log-volumes* as in Proposition 1.4, (i) — by the **weights**

$$[K_{\underline{v}} : \mathbb{Q}_{v_{\mathbb{Q}}}] \cdot [K_{\underline{v}} : (F_{\mathrm{mod}})_v]^{-1} \; = \; [(F_{\mathrm{mod}})_v : \mathbb{Q}_{v_{\mathbb{Q}}}]$$

— where $\mathbb{V}_{\mathrm{mod}} \ni v \mid v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$. This means that the *normalized weights* of the final display of [IUTchIII], Remark 3.1.1, (ii), must be replaced, when one works with *normalized log-volumes* as in Proposition 1.4, (i), by the **normalized weights**

$$\frac{\Big(\prod_{\alpha \in A} [(F_{\mathrm{mod}})_{v_\alpha} : \mathbb{Q}_{v_{\mathbb{Q}}}] \Big)}{\sum_{\{w_\alpha\}_{\alpha \in A}} \Big(\prod_{\alpha \in A} [(F_{\mathrm{mod}})_{w_\alpha} : \mathbb{Q}_{v_{\mathbb{Q}}}] \Big)}$$

— where the sum is over all collections $\{w_\alpha\}_{\alpha \in A}$ of [not necessarily distinct!] elements $w_\alpha \in \mathbb{V}_{\mathrm{mod}}$ lying over $v_{\mathbb{Q}}$ and indexed by $\alpha \in A$. Thus, in summary, when one works with *normalized log-volumes* as in Proposition 1.4, (i), the appropriate *normalized weights* are given by the expressions

$$\frac{\lambda_{\Pi \vec{e}^{\dagger}}}{\sum_{\vec{e} \in E^n} \lambda_{\Pi \vec{e}}}$$

[where $\vec{e}^{\dagger} \in E^n$] that appear in Proposition 1.7. Here, one takes "E" to be the set of elements of $\underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$ lying over a fixed $v_{\mathbb{Q}}$; one takes "n" to be the cardinality of A, so that one can write $A = \{\alpha_1, \dots, \alpha_n\}$ [where the α_i are *distinct*]; if $e \in E$ corresponds to $\underline{v} \in \underline{\mathbb{V}}$, $v \in \mathbb{V}_{\mathrm{mod}}$, then one takes

$$\text{“}\lambda_e\text{”} \overset{\mathrm{def}}{=} [(F_{\mathrm{mod}})_v : \mathbb{Q}_{v_{\mathbb{Q}}}] \in \mathbb{R}_{>0}$$

and “β_e” to be a normalized log-volume of some compact open subset of $K_{\underline{v}}$.

Before proceeding, we recall some well-known elementary facts concerning *elliptic curves*. In the following, we shall write $\mathcal{M}_{\mathrm{ell}}$ for the *moduli stack of elliptic curves* over $\mathbb{Z}$ and

$$\mathcal{M}_{\mathrm{ell}} \subseteq \overline{\mathcal{M}}_{\mathrm{ell}}$$

for the *natural compactification* of $\mathcal{M}_{\mathrm{ell}}$, i.e., the moduli stack of one-dimensional semi-abelian schemes over $\mathbb{Z}$. Also, if R is a $\mathbb{Z}$-algebra, then we shall write $(\mathcal{M}_{\mathrm{ell}})_R \overset{\mathrm{def}}{=} \mathcal{M}_{\mathrm{ell}} \times_{\mathbb{Z}} R$, $(\overline{\mathcal{M}}_{\mathrm{ell}})_R \overset{\mathrm{def}}{=} \overline{\mathcal{M}}_{\mathrm{ell}} \times_{\mathbb{Z}} R$.

Proposition 1.8. **(Torsion Points of Elliptic Curves)** *Let k be a perfect field k, $\overline{k}$ an algebraic closure of k. Write $G_k \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$.*

(i) Let $l \geq 3$ be a **prime number** *that is* **invertible** *in k; suppose that $\overline{k} = k$. Let A be an* **abelian variety** *over k, equipped with a polarization λ. Write $A[l] \subseteq A(k)$ for the group of l-torsion points of $A(k)$. Then the natural map*

$$\phi : \quad \mathrm{Aut}_k(A, \lambda) \quad \to \quad \mathrm{Aut}(A[l])$$

from the group of automorphisms of the polarized abelian variety (A, λ) over k to the group of automorphisms of the abelian group $A[l]$ is **injective**.

(ii) Let $E_{\overline{k}}$ be an **elliptic curve** *over $\overline{k}$ with origin $\epsilon_E \in E(\overline{k})$. For n a positive integer, write $E_{\overline{k}}[n] \subseteq E_{\overline{k}}(\overline{k})$ for the G_k-module of n-torsion points of $E_{\overline{k}}(\overline{k})$ and*

$$\mathrm{Aut}_{\overline{k}}(E_{\overline{k}}) \quad \subseteq \quad \mathrm{Aut}_k(E_{\overline{k}})$$

for the respective groups of ϵ_E-preserving automorphisms of the $\overline{k}$-scheme $E_{\overline{k}}$ and the k-scheme $E_{\overline{k}}$. Then we have a **natural exact sequence**

$$1 \longrightarrow \mathrm{Aut}_{\overline{k}}(E_{\overline{k}}) \longrightarrow \mathrm{Aut}_k(E_{\overline{k}}) \longrightarrow G_k$$

— where the image $G_E \subseteq G_k$ of the homomorphism $\mathrm{Aut}_k(E_{\overline{k}}) \to G_k$ is **open** *— and a* **natural representation**

$$\rho_n : \ \mathrm{Aut}_k(E_{\overline{k}}) \ \to \ \mathrm{Aut}(E_{\overline{k}}[n])$$

on the n-torsion points of $E_{\overline{k}}$. The finite extension k_E of k determined by G_E is the **minimal field of definition** *of $E_{\overline{k}}$, i.e., the field generated over k by the* **j-invariant** *of $E_{\overline{k}}$. Finally, if $H \subseteq G_k$ is any closed subgroup, which corresponds to an extension k_H of k, then the datum of a* **model** *of $E_{\overline{k}}$ over k_H [i.e., descent data for $E_{\overline{k}}$ from $\overline{k}$ to k_H] is equivalent to the datum of a* **section** *of the homomorphism*

$\mathrm{Aut}_k(E_{\overline{k}}) \to G_k$ *over* H. *In particular, the homomorphism* $\mathrm{Aut}_k(E_{\overline{k}}) \to G_k$ *admits a section over* G_E.

(iii) In the situation of (ii), suppose further that $\mathrm{Aut}_{\overline{k}}(E_{\overline{k}}) = \{\pm 1\}$. *Then the representation* ρ_2 **factors** *through* G_E *and hence defines a* **natural representation** $G_E \to \mathrm{Aut}(E_{\overline{k}}[2])$.

(iv) In the situation of (ii), suppose further that $l \geq 3$ *is a* **prime number** *that is* **invertible** *in* k, *and that* $E_{\overline{k}}$ *descends to elliptic curves* E'_k *and* E''_k *over* k, *all of whose* l*-torsion points are* **rational** *over* k. *Then* E'_k *is* **isomorphic** *to* E''_k *over* k.

(v) In the situation of (ii), suppose further that k *is a* **complete discrete valuation field** *with ring of integers* $\mathcal{O}_k$, *that* $l \geq 3$ *is a* **prime number** *that is* **invertible** *in* $\mathcal{O}_k$, *and that* $E_{\overline{k}}$ *descends to an elliptic curve* E_k *over* k, *all of whose* l*-torsion points are* **rational** *over* k. *Then* E_k *has* **semi-stable reduction** *over* $\mathcal{O}_k$ *[i.e., extends to a semi-abelian scheme over* $\mathcal{O}_k$*].*

(vi) In the situation of (iii), suppose further that 2 *is* **invertible** *in* k, *that* $G_E = G_k$, *and that the representation* $G_E \to \mathrm{Aut}(E_{\overline{k}}[2])$ *is* **trivial**. *Then* $E_{\overline{k}}$ *descends to an elliptic curve* E_k *over* k *which is defined by means of the* **Legendre form** *of the Weierstrass equation [cf., e.g., the statement of Corollary 2.2, below]. If, moreover,* k *is a* **complete discrete valuation field** *with ring of integers* $\mathcal{O}_k$ *such that* 2 *is* **invertible** *in* $\mathcal{O}_k$, *then* E_k *has* **semi-stable reduction** *over* $\mathcal{O}_{k'}$ *[i.e., extends to a semi-abelian scheme over* $\mathcal{O}_{k'}$*] for some finite extension* $k' \subseteq \overline{k}$ *of* k *such that* $[k' : k] \leq 2$*; if* E_k *has* **good reduction** *over* $\mathcal{O}_{k'}$ *[i.e., extends to an abelian scheme over* $\mathcal{O}_{k'}$*], then one may in fact take* k' *to be* k.

(vii) In the situation of (ii), suppose further that k *is a* **complete discrete valuation field** *with ring of integers* $\mathcal{O}_k$, *that* $E_{\overline{k}}$ *descends to an elliptic curve* E_k *over* k, *and that* n *is* **invertible** *in* $\mathcal{O}_k$. *If* E_k *has* **good reduction** *over* $\mathcal{O}_k$ *[i.e., extends to an abelian scheme over* $\mathcal{O}_k$*], then the action of* G_k *on* $E_{\overline{k}}[n]$ *is* **unramified**. *If* E_k *has* **bad multiplicative reduction** *over* $\mathcal{O}_k$ *[i.e., extends to a non-proper semi-abelian scheme over* $\mathcal{O}_k$*], then the action of* G_k *on* $E_{\overline{k}}[n]$ *is* **tamely ramified**.

Proof. First, we consider assertion (i). Suppose that ϕ is *not injective*. Since $\mathrm{Aut}_k(A, \lambda)$ is well-known to be *finite* [cf., e.g., [Milne], Proposition 17.5, (i)], we thus conclude that there exists an $\alpha \in \mathrm{Ker}(\phi)$ of order $n \neq 1$. We may assume without loss of generality that n is *prime*. Now we follow the argument of [Milne], Proposition 17.5, (ii). Since α acts trivially on $A[l]$, it follows immediately that the endomorphism of A given by $\alpha - \mathrm{id}_A$ [where id_A denotes the identity automorphism of A] may be written in the form $l \cdot \beta$, for β an endomorphism of A over k. Write $T_l(A)$ for the *l-adic Tate module* of A. Since $\alpha^n = \mathrm{id}_A$, it follows that the eigenvalues of the action of α on $T_l(A)$ are *n-th roots of unity*. On the other hand, the eigenvalues of the action of β on $T_l(A)$ are *algebraic integers* [cf. [Milne], Theorem 12.5]. We thus conclude that each eigenvalue ζ of the action of α on $T_l(A)$ is an n-th root of unity which, as an algebraic integer, is $\equiv 1 \pmod{l}$ [where $l \geq 3$], hence $= 1$. Since $\alpha^n = \mathrm{id}_A$, it follows that α acts on $T_l(A)$ as a *semi-simple matrix* which is also *unipotent*, hence equal to the *identity matrix*. But this implies that

$\alpha = \mathrm{id}_A$ [cf. [Milne], Theorem 12.5]. This contradiction completes the proof of assertion (i).

Next, we consider assertion (ii). Since $E_{\overline{k}}$ is *proper* over $\overline{k}$, it follows [by considering the space of global sections of the structure sheaf of $E_{\overline{k}}$] that any automorphism of the scheme $E_{\overline{k}}$ lies over an automorphism of $\overline{k}$. This implies the existence of a *natural exact sequence* and *natural representation* as in the statement of assertion (ii). The relationship between k_E and the *j-invariant* of $E_{\overline{k}}$ follows immediately from the well-known theory of the j-invariant of an elliptic curve [cf., e.g., [Silv], Chapter III, Proposition 1.4, (b), (c)]. The final portion of assertion (ii) concerning *models* of $E_{\overline{k}}$ follows immediately from the definitions. This completes the proof of assertion (ii). Assertion (iii) follows immediately from the fact that $\{\pm 1\}$ acts *trivially* on $E_{\overline{k}}[2]$.

Next, we consider assertion (iv). First, let us observe that it follows immediately from the final portion of assertion (ii) that a model E^*_k of $E_{\overline{k}}$ over k all of whose l-torsion points are *rational* over k corresponds to a closed subgroup $H^* \subseteq \mathrm{Aut}_k(E_{\overline{k}})$ that lies in the kernel of ρ_l and, moreover, maps isomorphically to G_k. On the other hand, it follows from assertion (i) that the restriction of ρ_l to $\mathrm{Aut}_{\overline{k}}(E_{\overline{k}}) \subseteq \mathrm{Aut}_k(E_{\overline{k}})$ is *trivial.* Thus, a closed subgroup $H^* \subseteq \mathrm{Aut}_k(E_{\overline{k}})$ is *uniquely determined* by the condition that it lie in the kernel of ρ_l and, moreover, map isomorphically to G_k. This completes the proof of assertion (iv).

Next, we consider assertion (v). First, let us observe that, by considering *l-level structures*, we obtain a *finite covering* of $S \to (\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{Z}[\frac{1}{l}]}$ which is *étale* over $(\mathcal{M}_{\mathrm{ell}})_{\mathbb{Z}[\frac{1}{l}]}$ and *tamely ramified* over the divisor at infinity. Then it follows from assertion (i) that the algebraic stack S is in fact a *scheme*, which is, moreover, *proper* over $\mathbb{Z}[\frac{1}{l}]$. Thus, it follows from the valuative criterion for properness that any k-valued point of S determined by E_k — where we observe that such a point necessarily exists, in light of our assumption that the l-torsion points of E_k are *rational* over k — extends to an $\mathcal{O}_k$-valued point of S, hence also of $\overline{\mathcal{M}}_{\mathrm{ell}}$, as desired. This completes the proof of assertion (v).

Next, we consider assertion (vi). Since $G_E = G_k$, it follows from assertion (ii) that $E_{\overline{k}}$ descends to an elliptic curve E_k over k. Our assumption that the representation $G_k = G_E \to \mathrm{Aut}(E_{\overline{k}}[2])$ of assertion (iii) is *trivial* implies that the 2-torsion points of E_k are *rational* over k. Thus, by considering suitable *global sections* of tensor powers of the line bundle on E_k determined by the origin *on which the automorphism "-1" of E_k acts via multiplication by* ± 1 [cf., e.g., [Harts], Chapter IV, the proof of Proposition 4.6], one concludes immediately that a suitable [possibly trivial] *twist* E'_k of E_k over k [i.e., such that E'_k and E_k are isomorphic over some quadratic extension k' of k] may be defined by means of the *Legendre form* of the Weierstrass equation. Now suppose that k is a *complete discrete valuation field* with ring of integers $\mathcal{O}_k$ such that 2 is *invertible* in $\mathcal{O}_k$, and that E_k is defined by means of the *Legendre form* of the Weierstrass equation. Then the fact that E_k has *semi-stable reduction* over $\mathcal{O}_{k'}$ for some finite extension $k' \subseteq \overline{k}$ of k such that $[k' : k] \leq 2$ follows from the explicit computations of the proof of [Silv], Chapter VII, Proposition 5.4, (c). These explicit computations also imply that if E_k has *good reduction* over $\mathcal{O}_{k'}$, then one may in fact take k' to be k. This completes the proof of assertion (vi).

Assertion (vii) follows immediately from [NerMod], §7.4, Theorem 5, in the case of *good reduction* and from [NerMod], §7.4, Theorem 6, in the case of *bad multiplicative reduction.* ○

We are now ready to apply the elementary computations discussed above to give *more explicit log-volume estimates for* Θ*-pilot objects.* We begin by recalling some notation and terminology from [GenEll], §1.

Definition 1.9. Let F be a *number field* [i.e., a finite extension of the rational number field $\mathbb{Q}$], whose *set of valuations* we denote by $\mathbb{V}(F)$. Thus, $\mathbb{V}(F)$ decomposes as a disjoint union $\mathbb{V}(F) = \mathbb{V}(F)^{\mathrm{non}} \bigcup \mathbb{V}(F)^{\mathrm{arc}}$ of *nonarchimedean* and *archimedean* valuations. If $v \in \mathbb{V}(F)$, then we shall write F_v for the *completion* of F at v; if $v \in \mathbb{V}(F)^{\mathrm{non}}$, then we shall write $\mathrm{ord}_v(-) : F_v^\times \twoheadrightarrow \mathbb{Z}$ for the *order* defined by v, e_v for the *ramification index* of F_v over $\mathbb{Q}_{p_v}$, and q_v for the cardinality of the *residue field* of F_v.

(i) A(n) *[*$\mathbb{Q}$*-]arithmetic divisor* $\mathfrak{a}$ on F is defined to be a finite formal sum

$$\sum_{v \in \mathbb{V}(F)} c_v \cdot v$$

— where $c_v \in \mathbb{Q}$ if $v \in \mathbb{V}(F)^{\mathrm{non}}$ and $c_v \in \mathbb{R}$ if $v \in \mathbb{V}(F)^{\mathrm{arc}}$. Here, we shall refer to the set

$$\mathrm{Supp}(\mathfrak{a})$$

of $v \in \mathbb{V}(F)$ such that $c_v \neq 0$ as the *support of* $\mathfrak{a}$; if all of the c_v are ≥ 0, then we shall say that the arithmetic divisor is *effective*. Thus, the [$\mathbb{Q}$-]arithmetic divisors on F naturally form a group $\mathrm{ADiv}_{\mathbb{Q}}(F)$. The assignment

$$\mathbb{V}(F)^{\mathrm{non}} \ni v \mapsto \log(q_v); \quad \mathbb{V}(F)^{\mathrm{arc}} \ni v \mapsto 1$$

determines a homomorphism

$$\deg_F : \mathrm{ADiv}_{\mathbb{Q}}(F) \to \mathbb{R}$$

which we shall refer to as the *degree* map. If $\mathfrak{a} \in \mathrm{ADiv}_{\mathbb{Q}}(F)$, then we shall refer to

$$\underline{\deg}(\mathfrak{a}) \quad \overset{\mathrm{def}}{=} \quad \frac{1}{[F:\mathbb{Q}]} \cdot \deg_F(\mathfrak{a})$$

as the *normalized degree* of $\mathfrak{a}$. Thus, for any finite extension K of F, we have

$$\underline{\deg}(\mathfrak{a}|_K) = \underline{\deg}(\mathfrak{a})$$

— where we write $\underline{\deg}(\mathfrak{a}|_K)$ for the normalized degree of the pull-back $\mathfrak{a}|_K \in \mathrm{ADiv}_{\mathbb{Q}}(K)$ [defined in the evident fashion] of $\mathfrak{a}$ to K.

(ii) Let $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}} \overset{\mathrm{def}}{=} \mathbb{V}(\mathbb{Q})$, $E \subseteq \mathbb{V}(F)$ a nonempty set of elements lying over $v_{\mathbb{Q}}$. If $\mathfrak{a} = \sum_{v \in \mathbb{V}(F)} c_v \cdot v \in \mathrm{ADiv}_{\mathbb{Q}}(F)$, then we shall write

$$\mathfrak{a}_E \overset{\mathrm{def}}{=} \sum_{v \in E} c_v \cdot v \; \in \; \mathrm{ADiv}_{\mathbb{Q}}(F); \quad \underline{\deg}_E(\mathfrak{a}) \overset{\mathrm{def}}{=} \frac{\deg(\mathfrak{a}_E)}{\sum_{v \in E} [F_v : \mathbb{Q}_{v_{\mathbb{Q}}}]}$$

for the portion of $\mathfrak{a}$ *supported in* E and the *"normalized* E*-degree"* of $\mathfrak{a}$, respectively. Thus, for any finite extension K of F, we have

$$\underline{\deg}_{E|_K}(\mathfrak{a}|_K) = \underline{\deg}_E(\mathfrak{a})$$

— where we write $E|_K \subseteq \mathbb{V}(K)$ for the set of valuations lying over valuations $\in E$.

Theorem 1.10. (Log-volume Estimates for Θ-Pilot Objects) *Set*

$$\epsilon_\Theta \quad \overset{\mathrm{def}}{=} \quad \min\left(\tfrac{40\eta_{\mathrm{prm}}}{3}, 3^{-210}\right) \ \in \ \mathbb{R}_{>0}$$

— where the constant $\eta_{\mathrm{prm}} \in \mathbb{R}_{>0}$ *is as in Proposition 1.6. Then the constant* $\epsilon_\Theta \in \mathbb{R}_{>0}$ *satisfies the following property:*

Fix a collection of **initial Θ-data** *as in [IUTchI], Definition 3.1. Suppose that we are in the situation of [IUTchIII], Corollary 3.12. Also, in the notation of [IUTchI], Definition 3.1, let us write* $d_{\mathrm{mod}} \overset{\mathrm{def}}{=} [F_{\mathrm{mod}} : \mathbb{Q}]$ *and*

$$F_{\mathrm{mod}} \quad \subseteq \quad F_{\mathrm{tpd}} \overset{\mathrm{def}}{=} F_{\mathrm{mod}}(\ E_{F_{\mathrm{mod}}}[2]\ ) \quad \subseteq \quad F$$

for the **"tripodal"** *intermediate field obtained from* F_{mod} *by adjoining the fields of definition of the* 2*-torsion points of any model of* E_F *over* F_{mod} *[cf. Proposition 1.8, (ii), (iii)]. Moreover, we assume that the* $(3 \cdot 5)$*-torsion points of* E_F *are defined over* F*, and that*

$$F \quad = \quad F_{\mathrm{mod}}(\ \sqrt{-1},\ E_{F_{\mathrm{mod}}}[2\cdot 3\cdot 5]\ ) \quad \overset{\mathrm{def}}{=} \quad F_{\mathrm{tpd}}(\ \sqrt{-1},\ E_{F_{\mathrm{tpd}}}[3\cdot 5]\ )$$

— i.e., that F *is obtained from* F_{tpd} *by adjoining* $\sqrt{-1}$*, together with the fields of definition of the* $(3\cdot 5)$*-torsion points of a model* $E_{F_{\mathrm{tpd}}}$ *of the elliptic curve* E_F *over* F_{tpd} *determined by the* **Legendre form** *of the Weierstrass equation [cf., e.g., the statement of Corollary 2.2, below; Proposition 1.8, (vi)]. [Thus, it follows from Proposition 1.8, (iv), that* $E_F \cong E_{F_{\mathrm{tpd}}} \times_{F_{\mathrm{tpd}}} F$ *over* F*.] If* $F_{\mathrm{mod}} \subseteq F_{\Box} \subseteq K$ *is any intermediate extension which is Galois over* F_{mod}*, then we shall write*

$$\mathfrak{d}^{F_\Box}_{\mathrm{ADiv}} \ \in \ \mathrm{ADiv}_{\mathbb{Q}}(F_\Box)$$

for the effective divisor determined by the **different ideal** *of* $F_\Box$ *over* $\mathbb{Q}$*,*

$$\mathfrak{q}^{F_\Box}_{\mathrm{ADiv}} \in \mathrm{ADiv}_{\mathbb{Q}}(F_\Box)$$

for the effective arithmetic divisor determined by the ***q*-parameters** *of the elliptic curve* E_F *at the elements of* $\mathbb{V}(F_\Box)^{\mathrm{bad}} \overset{\mathrm{def}}{=} \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}} \times_{\mathbb{V}_{\mathrm{mod}}} \mathbb{V}(F_\Box)$ $(\neq \emptyset)$*,*

$$\mathfrak{f}^{F_\Box}_{\mathrm{ADiv}} \in \mathrm{ADiv}_{\mathbb{Q}}(F_\Box)$$

for the effective arithmetic divisor whose support coincides with $\mathrm{Supp}(\mathfrak{q}^{F_\Box}_{\mathrm{ADiv}})$*, but all of whose coefficients are equal to* 1 *— i.e., the* **conductor** *— and*

$$\log(\mathfrak{d}^{F_\Box}_v) \ \overset{\mathrm{def}}{=} \ \underline{\deg}_{\mathbb{V}(F_\Box)_v}(\mathfrak{d}^{F_\Box}_{\mathrm{ADiv}}) \ \in \ \mathbb{R}_{\geq 0}; \quad \log(\mathfrak{d}^{F_\Box}_{v_{\mathbb{Q}}}) \ \overset{\mathrm{def}}{=} \ \underline{\deg}_{\mathbb{V}(F_\Box)_{v_{\mathbb{Q}}}}(\mathfrak{d}^{F_\Box}_{\mathrm{ADiv}}) \ \in \ \mathbb{R}_{\geq 0}$$

$$\log(\mathfrak{d}^{F_\Box}) \ \overset{\mathrm{def}}{=} \ \underline{\deg}(\mathfrak{d}^{F_\Box}_{\mathrm{ADiv}}) \ \in \ \mathbb{R}_{\geq 0}$$

$$\log(\mathfrak{q}_v) \ \stackrel{\mathrm{def}}{=}\ \underline{\deg}_{\mathbb{V}(F_{\Box})_v}(\mathfrak{q}^{F_{\Box}}_{\mathrm{ADiv}}) \ \in\ \mathbb{R}_{\geq 0}; \quad \log(\mathfrak{q}_{v_{\mathbb{Q}}}) \ \stackrel{\mathrm{def}}{=}\ \underline{\deg}_{\mathbb{V}(F_{\Box})_{v_{\mathbb{Q}}}}(\mathfrak{q}^{F_{\Box}}_{\mathrm{ADiv}}) \ \in\ \mathbb{R}_{\geq 0}$$

$$\log(\mathfrak{q}) \ \stackrel{\mathrm{def}}{=}\ \underline{\deg}(\mathfrak{q}^{F_{\Box}}_{\mathrm{ADiv}}) \ \in\ \mathbb{R}_{\geq 0}$$

$$\log(\mathfrak{f}^{F_{\Box}}_v) \ \stackrel{\mathrm{def}}{=}\ \underline{\deg}_{\mathbb{V}(F_{\Box})_v}(\mathfrak{f}^{F_{\Box}}_{\mathrm{ADiv}}) \ \in\ \mathbb{R}_{\geq 0}; \quad \log(\mathfrak{f}^{F_{\Box}}_{v_{\mathbb{Q}}}) \ \stackrel{\mathrm{def}}{=}\ \underline{\deg}_{\mathbb{V}(F_{\Box})_{v_{\mathbb{Q}}}}(\mathfrak{f}^{F_{\Box}}_{\mathrm{ADiv}}) \ \in\ \mathbb{R}_{\geq 0}$$

$$\log(\mathfrak{f}^{F_{\Box}}) \ \stackrel{\mathrm{def}}{=}\ \underline{\deg}(\mathfrak{f}^{F_{\Box}}_{\mathrm{ADiv}}) \ \in\ \mathbb{R}_{\geq 0}$$

— *where* $v \in \mathbb{V}_{\mathrm{mod}} = \mathbb{V}(F_{\mathrm{mod}})$, $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}} = \mathbb{V}(\mathbb{Q})$, $\mathbb{V}(F_{\Box})_v \stackrel{\mathrm{def}}{=} \mathbb{V}(F_{\Box}) \times_{\mathbb{V}_{\mathrm{mod}}} \{v\}$, $\mathbb{V}(F_{\Box})_{v_{\mathbb{Q}}} \stackrel{\mathrm{def}}{=} \mathbb{V}(F_{\Box}) \times_{\mathbb{V}_{\mathbb{Q}}} \{v_{\mathbb{Q}}\}$. *Here, we observe that the various "*$\log(\mathfrak{q}_{(-)})$*'s" are independent of the choice of* $F_{\Box}$, *and that the quantity "*$|\log(\underline{\underline{q}})| \in \mathbb{R}_{>0}$*" defined in [IUTchIII], Corollary 3.12, is equal to* $\frac{1}{2l} \cdot \log(\mathfrak{q}) \in \mathbb{R}$ *[cf. the definition of "*$\underline{\underline{q}}_{v}$*" in [IUTchI], Example 3.2, (iv)]. Then one may take the constant "*$C_{\Theta} \in \mathbb{R}$*" of [IUTchIII], Corollary 3.12, to be*

$$\tfrac{l+1}{4} \cdot \Big\{(1 + \epsilon + \tfrac{28 \cdot d_{\mathrm{mod}}}{l}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 2 \cdot \log(l \cdot \epsilon^{-7}) - \tfrac{1}{6} \cdot \log(\mathfrak{q})\Big\} - 1$$

and hence, by applying [IUTchIII], Corollary 3.12, conclude that

$$\begin{aligned} \tfrac{1}{6} \cdot \log(\mathfrak{q}) \ &\leq\ (1 + \epsilon + \tfrac{28 \cdot d_{\mathrm{mod}}}{l}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 2 \cdot \log(l \cdot \epsilon^{-7}) \\ &\leq\ (1 + \epsilon + \tfrac{28 \cdot d_{\mathrm{mod}}}{l}) \cdot (\log(\mathfrak{d}^{F}) + \log(\mathfrak{f}^{F})) + 2 \cdot \log(l \cdot \epsilon^{-7}) \end{aligned}$$

for any $\epsilon \in \mathbb{R}_{>0}$ *satisfying* $\epsilon < \epsilon_{\Theta}$.

Proof. For ease of reference, we divide our discussion into *steps*, as follows.

(i) We begin by recalling the following *elementary identities* for $n \in \mathbb{N}_{\geq 1}$:

(E1) $\frac{1}{n} \sum\limits_{m=1}^{n} m \ = \ \frac{1}{2}(n+1)$;

(E2) $\frac{1}{n} \sum\limits_{m=1}^{n} m^2 \ = \ \frac{1}{6}(2n+1)(n+1)$.

Also, we recall the following *elementary facts*:

(E3) For p a prime number, the cardinality $|GL_2(\mathbb{F}_p)|$ of $GL_2(\mathbb{F}_p)$ is given by $|GL_2(\mathbb{F}_p)| = p(p+1)(p-1)^2$.

(E4) For $p = 2, 3, 5$, the expression of (E3) may be computed as follows: $2(2+1)(2-1)^2 = 2 \cdot 3$; $3(3+1)(3-1)^2 = 3 \cdot 2^4$; $5(5+1)(5-1)^2 = 5 \cdot 2^5 \cdot 3$.

(E5) The degree of the extension $F_{\mathrm{mod}}(\sqrt{-1}\ )/F_{\mathrm{mod}}$ is ≤ 2.

(E6) We have: $0 \leq \log(2) \leq 1$, $\quad 1 \leq \log(3) \leq \log(\pi) \leq \log(5) \leq 2$.

(ii) Next, let us observe that the *inequality*

$$\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}) \ \leq\ \log(\mathfrak{d}^{F}) + \log(\mathfrak{f}^{F})$$

follows immediately from Proposition 1.3, (i), and the various definitions involved. On the other hand, the *inequality*

$$\begin{aligned}\log(\mathfrak{d}^{F}) + \log(\mathfrak{f}^{F}) \;&\leq\; \log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}) + \log(2^{11} \cdot 3^{3} \cdot 5^{2}) \\ &\leq\; \log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}) + 21\end{aligned}$$

follows by applying Proposition 1.3, (i), at the primes that do *not* divide $2 \cdot 3 \cdot 5$ [over which the extension F/F_{tpd} is *tamely ramified* — cf. Proposition 1.8, (vi), (vii)] and applying Proposition 1.3, (ii), together with (E3), (E4), (E5), (E6), and the fact that we have a natural inclusion $\mathrm{Gal}(F/F_{\mathrm{tpd}}) \hookrightarrow GL_2(\mathbb{F}_3) \times GL_2(\mathbb{F}_5) \times \mathbb{Z}/2\mathbb{Z}$, at the primes that *divide* $2 \cdot 3 \cdot 5$. In a similar vein, since the extension K/F is *tamely ramified* at the primes that do *not* divide l, and we have a natural isomorphism $\mathrm{Gal}(K/F) \overset{\sim}{\to} GL_2(\mathbb{F}_l)$, the *inequality*

$$\begin{aligned}\log(\mathfrak{d}^{K}) \;\leq\; \log(\mathfrak{d}^{K}) + \log(\mathfrak{f}^{K}) \;&\leq\; \log(\mathfrak{d}^{F}) + \log(\mathfrak{f}^{F}) + 2 \cdot \log(l) \\ &\leq\; \log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}) + 2 \cdot \log(l) + 21\end{aligned}$$

follows immediately from Proposition 1.3, (i), (ii). Finally, for later reference, we observe that

$$(1 + \tfrac{4}{l}) \cdot \log(\mathfrak{d}^{K}) \;\leq\; (1 + \tfrac{4}{l}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 2 \cdot \log(l) + 46$$

— where we apply the estimates $\frac{\log(l)}{l} \leq \frac{1}{2}$ and $1 + \frac{4}{l} \leq 2$, both of which are consequences of the fact that $l \geq 5$ [cf. also (E6)].

(iii) If $F_{\mathrm{tpd}} \subseteq F_{\square} \subseteq K$ is any intermediate extension which is *Galois* over F_{mod}, then we shall write

$$\mathbb{V}(F_{\square})^{\mathrm{dst}} \;\subseteq\; \mathbb{V}(F_{\square})^{\mathrm{non}}$$

for the set of **"distinguished"** nonarchimedean valuations $v \in \mathbb{V}(F_{\square})^{\mathrm{non}}$, i.e., v that extend to a valuation $\in \mathbb{V}(K)^{\mathrm{non}}$ that *ramifies* over $\mathbb{Q}$. Now observe that it follows immediately from Proposition 1.8, (vi), (vii), that

(D0) if $v \in \mathbb{V}(F_{\mathrm{tpd}})^{\mathrm{non}}$ does *not divide* $2 \cdot 3 \cdot 5 \cdot l$ and, moreover, is *not contained* in $\mathrm{Supp}(\mathfrak{q}^{F_{\mathrm{tpd}}}_{\mathrm{ADiv}})$, then the extension K/F_{tpd} is *unramified* over v.

Also, let us recall [cf. the various definitions involved] that K contains a *primitive* $4 \cdot 3 \cdot 5 \cdot l$*-th root of unity*, hence is ramified over $\mathbb{Q}$ at any valuation $\in \mathbb{V}(K)^{\mathrm{non}}$ that divides $2 \cdot 3 \cdot 5 \cdot l$. Thus, one verifies immediately [i.e., by applying (D0); cf. also [IUTchI], Definition 3.1, (c)] that the following *conditions* on a valuation $v \in \mathbb{V}(F_{\square})^{\mathrm{non}}$ are *equivalent*:

(D1) $v \in \mathbb{V}(F_{\square})^{\mathrm{dst}}$.
(D2) The valuation v either divides $2 \cdot 3 \cdot 5 \cdot l$ or lies in $\mathrm{Supp}(\mathfrak{q}^{F_{\square}}_{\mathrm{ADiv}} + \mathfrak{d}^{F_{\square}}_{\mathrm{ADiv}})$.
(D3) The image of v in $\mathbb{V}(F_{\mathrm{tpd}})$ lies in $\mathbb{V}(F_{\mathrm{tpd}})^{\mathrm{dst}}$.

Let us write

$$\mathbb{V}^{\mathrm{dst}}_{\mathrm{mod}} \;\subseteq\; \mathbb{V}^{\mathrm{non}}_{\mathrm{mod}}; \quad \mathbb{V}^{\mathrm{dst}}_{\mathbb{Q}} \;\subseteq\; \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$$

for the respective *images* of $\mathbb{V}(F_{\mathrm{tpd}})^{\mathrm{dst}}$ in $\mathbb{V}_{\mathrm{mod}}$, $\mathbb{V}_{\mathbb{Q}}$ and, for $F_* \in \{F_{\square}, F_{\mathrm{mod}}, \mathbb{Q}\}$ and $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$,

$$\mathfrak{s}^{F_*}_{\mathrm{ADiv}} \stackrel{\mathrm{def}}{=} \sum_{v \in \mathbb{V}(F_*)^{\mathrm{dst}}} e_v \cdot v \ \in \ \mathrm{ADiv}_{\mathbb{Q}}(F_*)$$

$$\log(\mathfrak{s}^{F_*}_{v_{\mathbb{Q}}}) \stackrel{\mathrm{def}}{=} \underline{\deg}_{\mathbb{V}(F_*)_{v_{\mathbb{Q}}}}(\mathfrak{s}^{F_*}_{\mathrm{ADiv}}) \ \in \ \mathbb{R}_{\geq 0}; \quad \log(\mathfrak{s}^{F_*}) \stackrel{\mathrm{def}}{=} \underline{\deg}(\mathfrak{s}^{F_*}_{\mathrm{ADiv}}) \ \in \ \mathbb{R}_{\geq 0}$$

— where we write $\mathbb{V}(F_*)_{v_{\mathbb{Q}}} \stackrel{\mathrm{def}}{=} \mathbb{V}(F_*) \times_{\mathbb{V}_{\mathbb{Q}}} \{v_{\mathbb{Q}}\}$. One verifies immediately [again, by applying (D0); cf. also [IUTchI], Definition 3.1, (c)] that the following *conditions* on a valuation $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$ are *equivalent*:

(D4) $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{dst}}_{\mathbb{Q}}$.
(D5) The valuation $v_{\mathbb{Q}}$ *ramifies* in K.
(D6) Either $p_{v_{\mathbb{Q}}} \mid 2 \cdot 3 \cdot 5 \cdot l$ or $v_{\mathbb{Q}}$ lies in the image of $\mathrm{Supp}(\mathfrak{q}^{F_{\mathrm{tpd}}}_{\mathrm{ADiv}} + \mathfrak{d}^{F_{\mathrm{tpd}}}_{\mathrm{ADiv}})$.
(D7) Either $p_{v_{\mathbb{Q}}} \mid 2 \cdot 3 \cdot 5 \cdot l$ or $v_{\mathbb{Q}}$ lies in the image of $\mathrm{Supp}(\mathfrak{q}^{F}_{\mathrm{ADiv}} + \mathfrak{d}^{F}_{\mathrm{ADiv}})$.

Next, for later reference, we observe that the *inequality*

$$\tfrac{1}{p_{v_{\mathbb{Q}}}} \cdot \log(\mathfrak{s}^{F_{\mathrm{mod}}}_{v_{\mathbb{Q}}}) \ \leq \ \tfrac{1}{p_{v_{\mathbb{Q}}}} \cdot \log(p_{v_{\mathbb{Q}}})$$

holds for *any* $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$; in particular, when $p_{v_{\mathbb{Q}}} = l$ (≥ 5), it holds that

$$\tfrac{1}{p_{v_{\mathbb{Q}}}} \cdot \log(\mathfrak{s}^{F_{\mathrm{mod}}}_{v_{\mathbb{Q}}}) \ \leq \ \tfrac{1}{p_{v_{\mathbb{Q}}}} \cdot \log(p_{v_{\mathbb{Q}}}) \ \leq \ \tfrac{1}{2}$$

— cf. (E6). On the other hand, it follows immediately from Proposition 1.3, (i), by considering the various possibilities for elements $\in \mathrm{Supp}(\mathfrak{s}^{F_{\mathrm{mod}}}_{\mathrm{ADiv}})$, that

$$\log(\mathfrak{s}^{F_{\mathrm{mod}}}_{v_{\mathbb{Q}}}) \ \leq \ 2 \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}_{v_{\mathbb{Q}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}_{v_{\mathbb{Q}}}))$$

— and hence that

$$\tfrac{1}{p_{v_{\mathbb{Q}}}} \cdot \log(\mathfrak{s}^{F_{\mathrm{mod}}}_{v_{\mathbb{Q}}}) \ \leq \ \tfrac{2}{p_{v_{\mathbb{Q}}}} \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}_{v_{\mathbb{Q}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}_{v_{\mathbb{Q}}}))$$

— for any $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ such that $p_{v_{\mathbb{Q}}} \notin \{2, 3, 5, l\}$. In a similar vein, we conclude that

$$\begin{aligned}\log(\mathfrak{s}^{\mathbb{Q}}) \ &\leq \ 2 \cdot d_{\mathrm{mod}} \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + \log(2 \cdot 3 \cdot 5 \cdot l) \\ &\leq \ 2 \cdot d_{\mathrm{mod}} \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 5 + \log(l)\end{aligned}$$

and hence that

$$\tfrac{12}{l} \cdot \log(\mathfrak{s}^{\mathbb{Q}}) \ \leq \ \tfrac{24 \cdot d_{\mathrm{mod}}}{l} \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 18$$

— cf. (E6); the fact that $l \geq 5$. Combining this last inequality with the inequality of the final display of Step (ii) yields the *inequality*

$$(1 + \tfrac{4}{l}) \cdot \log(\mathfrak{d}^{K}) + \tfrac{12}{l} \cdot \log(\mathfrak{s}^{\mathbb{Q}}) \ \leq \ (1 + \tfrac{28 \cdot d_{\mathrm{mod}}}{l}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 2 \cdot \log(l) + 64$$

— where we apply the estimate $d_{\mathrm{mod}} \geq 1$.

(iv) In order to estimate the constant "C_Θ" of [IUTchIII], Corollary 3.12, we must, according to the various definitions given in the statement of [IUTchIII], Corollary 3.12, compute an *upper bound* for the

> *procession-normalized mono-analytic log-volume* of the *union* of the *possible images* of a *Θ-pilot object*, relative to the relevant *Kummer isomorphisms* [cf. [IUTchIII], Theorem 3.11, (ii)], in the *multiradial representation* of [IUTchIII], Theorem 3.11, (i), which we regard as *subject* to the *indeterminacies* (Ind1), (Ind2), (Ind3) described in [IUTchIII], Theorem 3.11, (i), (ii).

Thus, we proceed to estimate this log-volume at *each* $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$. Once one fixes $v_{\mathbb{Q}}$, this amounts to estimating the component of this log-volume in

$$\text{“}\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})\text{”}$$

[cf. the notation of [IUTchIII], Theorem 3.11, (i), (a)], for *each* $j \in \{1, \dots, l^*\}$, which we shall also regard as an element of $\mathbb{F}_l^*$, and then computing the *average*, over $j \in \{1, \dots, l^*\}$, of these estimates. Here, we recall [cf. [IUTchI], Proposition 6.9, (i); [IUTchIII], Proposition 3.4, (ii)] that $\mathbb{S}^{\pm}_{j+1} = \{0, 1, \dots, j\}$. Also, we recall from [IUTchIII], Proposition 3.2, that "$\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$" is, by definition, a *tensor product* of $j+1$ copies, indexed by the elements of $\mathbb{S}^{\pm}_{j+1}$, of the *direct sum* of the $\mathbb{Q}$-spans of the *log-shells* associated to each of the elements of $\mathbb{V}(F_{\text{mod}})_{v_{\mathbb{Q}}}$ [cf., especially, the second and third displays of [IUTchIII], Proposition 3.2]. In particular, for *each collection*

$$\{v_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$$

of [not necessarily distinct!] elements of $\mathbb{V}(F_{\text{mod}})_{v_{\mathbb{Q}}}$, we must estimate the component of the log-volume in question corresponding to the *tensor product* of the $\mathbb{Q}$-spans of the *log-shells* associated to this collection $\{v_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$ and then compute the *weighted average* [cf. the discussion of Remark 1.7.1], over possible collections $\{v_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$, of these estimates.

(v) Let $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\text{dst}}$. Fix j, $\{v_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$ as in Step (iv). Write $\underline{v}_i \in \underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\text{mod}} = \mathbb{V}(F_{\text{mod}})$ for the element corresponding to v_i. We would like to apply Proposition 1.4, (iii), to the present situation, by taking

- "I" to be $\mathbb{S}^{\pm}_{j+1}$;
- "$I^* \subseteq I$" to be the set of $i \in I$ such that $\underline{v}_i \in \mathbb{V}(K)^{\text{dst}}$ [i.e., such that $\underline{v}_i$ is *ramified* over $\mathbb{Q}$];
- "k_i" to be $K_{\underline{v}_i}$ [so "R_i" will be the ring of integers $\mathcal{O}_{K_{\underline{v}_i}}$ of $K_{\underline{v}_i}$];
- "$i^\dagger$" to be $j \in \mathbb{S}^{\pm}_{j+1}$;
- "λ" to be 0 if $\underline{v}_j \in \underline{\mathbb{V}}^{\text{good}}$;
- "λ" to be "ord$(-)$" of the element $\underline{\underline{q}}_{\underline{v}_j}^{j^2}$ [cf. the definition of "$\underline{\underline{q}}_{\underline{v}}$" in [IUTchI], Example 3.2, (iv)] if $\underline{v}_j \in \underline{\mathbb{V}}^{\text{bad}}$.

Thus, the *inclusion* "$\phi(p^{\lambda} \cdot R_{i^\dagger} \otimes_{R_{i^\dagger}} (R_I)^{\sim}) \subseteq p^{\lfloor \lambda \rfloor - \lceil \mathfrak{d}_I \rceil - \lceil a_I \rceil} \cdot \log_p(R_I^{\times})$" of Proposition 1.4, (iii), implies that the result of multiplying the tensor product of log-shells under consideration by a suitable *nonpositive* [cf. the final portion of Proposition 1.4, (iii)] integer power of $p_{v_{\mathbb{Q}}}$ *contains* the "union of possible images of a Θ-pilot object" discussed in Step (iv). That is to say, the *indeterminacies* (Ind1) and (Ind2) are taken into account by the *arbitrary* nature of the automorphism "ϕ" [cf. Proposition 1.2], while the *indeterminacy* (Ind3) is taken into account by the fact that we are considering *upper bounds* [cf. the discussion of Step (x) of the proof of [IUTchIII], Corollary 3.12], together with the fact that the above-mentioned integer power of $p_{v_{\mathbb{Q}}}$ is *nonpositive*, which implies that the module obtained by multiplying by this power of $p_{v_{\mathbb{Q}}}$ *contains* the tensor product of log-shells under consideration. Thus, an upper bound on the component of the log-volume under consideration may be obtained by computing an upper bound for the log-volume of the right-hand side "$p^{\lfloor \lambda \rfloor - \lceil \mathfrak{d}_I \rceil - \lceil a_I \rceil} \cdot \log_p(R_I^{\times})$" of the above inclusion. Such an *upper bound*

$$\text{“}\Big(-\lambda + \mathfrak{d}_I + 3 + 4 \cdot |I^*|/p \Big) \cdot \log(p)\text{”}$$

is given in the displayed inequality of Proposition 1.4, (iii). Here, we note that, unlike the other terms that appear in this *upper bound*, "λ" is *asymmetric* with respect to the choice of "$i^\dagger \in I$" in $\mathbb{S}^{\pm}_{j+1}$. Since we would like to compute *weighted averages* [cf. the discussion of Remark 1.7.1], we thus observe that, after *symmetrizing* with respect to the choice of "$i^\dagger \in I$" in $\mathbb{S}^{\pm}_{j+1}$, this *upper bound* may be written in the form

$$\text{“}\beta_{\vec{e}}\text{”}$$

[cf. the notation of Proposition 1.7] if, in the situation of Proposition 1.7, one takes

- "E" to be $\mathbb{V}(F_{\mathrm{mod}})_{v_{\mathbb{Q}}}$;
- "n" to be $j+1$, so an element "$\vec{e} \in E^n$" corresponds precisely to a collection $\{v_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$;
- "λ_e", for an element $e \in E$ corresponding to $v \in \mathbb{V}(F_{\mathrm{mod}}) = \mathbb{V}_{\mathrm{mod}}$, to be $[(F_{\mathrm{mod}})_v : \mathbb{Q}_{v_{\mathbb{Q}}}] \in \mathbb{R}_{>0}$;
- "β_e", for an element $e \in E$ corresponding to $\underline{v} \in \underline{\mathbb{V}}$, $v \in \mathbb{V}(F_{\mathrm{mod}}) = \mathbb{V}_{\mathrm{mod}}$, to be

$$\log(\mathfrak{d}^K_v) - \tfrac{j^2}{2l(j+1)} \cdot \log(\mathfrak{q}_v) + \tfrac{3}{j+1} \cdot \log(p_{v_{\mathbb{Q}}}) + \tfrac{4 \cdot \iota_v}{p_{v_{\mathbb{Q}}}} \cdot \log(p_{v_{\mathbb{Q}}})$$

— where we set $\iota_v \overset{\mathrm{def}}{=} 1$ if $\underline{v} \in \mathbb{V}(K)^{\mathrm{dst}}$, $\iota_v \overset{\mathrm{def}}{=} 0$ if $\underline{v} \in \mathbb{V}(K)^{\mathrm{dst}}$.

Here, we note that it follows immediately from the first equality of the first display of Proposition 1.7 that, after passing to *weighted averages*, the operation of *symmetrizing* with respect to the choice of "$i^\dagger \in I$" in $\mathbb{S}^{\pm}_{j+1}$ does *not affect* the computation of the *upper bound* under consideration. Thus, by applying Proposition 1.7, we obtain that the resulting **"weighted average upper bound"** is given by

$$(j+1) \cdot \log(\mathfrak{d}^K_{v_{\mathbb{Q}}}) - \tfrac{j^2}{2l} \cdot \log(\mathfrak{q}_{v_{\mathbb{Q}}}) + 3 \cdot \log(\mathfrak{s}^{\mathbb{Q}}_{v_{\mathbb{Q}}}) + \tfrac{4(j+1)}{p_{v_{\mathbb{Q}}}} \cdot \log(\mathfrak{s}^{F_{\mathrm{mod}}}_{v_{\mathbb{Q}}})$$

— where we recall the notational conventions introduced in Step (iii). Thus, it remains to compute the *average over* $j \in \mathbb{F}^{\circledast}_l$. By *averaging* over $j \in \{1, \dots, l^{\circledast} =$

$\frac{l-1}{2}\}$ and applying (E1), (E2), we obtain the **"procession-normalized upper bound"**

$$\begin{aligned}
&\tfrac{(l^*+3)}{2} \cdot \log(\mathfrak{d}^K_{v_{\mathbb{Q}}}) - \tfrac{(2l^*+1)(l^*+1)}{12l} \cdot \log(\mathfrak{q}_{v_{\mathbb{Q}}}) + 3 \cdot \log(\mathfrak{s}^{\mathbb{Q}}_{v_{\mathbb{Q}}}) + \tfrac{2(l^*+3)}{p_{v_{\mathbb{Q}}}} \cdot \log(\mathfrak{s}^{F_{\mathrm{mod}}}_{v_{\mathbb{Q}}}) \\
&\quad = \tfrac{l+5}{4} \cdot \log(\mathfrak{d}^K_{v_{\mathbb{Q}}}) - \tfrac{l+1}{24} \cdot \log(\mathfrak{q}_{v_{\mathbb{Q}}}) + 3 \cdot \log(\mathfrak{s}^{\mathbb{Q}}_{v_{\mathbb{Q}}}) + \tfrac{(l+5)}{p_{v_{\mathbb{Q}}}} \cdot \log(\mathfrak{s}^{F_{\mathrm{mod}}}_{v_{\mathbb{Q}}}) \\
&\quad \leq \tfrac{l+1}{4} \cdot \left\{ (1 + \tfrac{4}{l}) \cdot \log(\mathfrak{d}^K_{v_{\mathbb{Q}}}) - \tfrac{1}{6} \cdot \log(\mathfrak{q}_{v_{\mathbb{Q}}}) + \tfrac{12}{l} \cdot \log(\mathfrak{s}^{\mathbb{Q}}_{v_{\mathbb{Q}}}) + \tfrac{20}{3p_{v_{\mathbb{Q}}}} \cdot \log(\mathfrak{s}^{F_{\mathrm{mod}}}_{v_{\mathbb{Q}}}) \right\}
\end{aligned}$$

— where, in the passage to the final displayed inequality, we apply the estimates $\frac{1}{l+1} \leq \frac{1}{l}$ and $\frac{4(l+5)}{l+1} \leq \frac{20}{3}$, both of which are consequences of the fact that $l \geq 5$.

(vi) Next, let $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}} \setminus \mathbb{V}^{\mathrm{dst}}_{\mathbb{Q}}$. Fix j, $\{v_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$ as in Step (iv). Write $\underline{v}_i \in \underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}} = \mathbb{V}(F_{\mathrm{mod}})$ for the element corresponding to v_i. We would like to apply Proposition 1.4, (iv), to the present situation, by taking

- · "I" to be $\mathbb{S}^{\pm}_{j+1}$;
- · "k_i" to be $K_{\underline{v}_i}$ [so "R_i" will be the ring of integers $\mathcal{O}_{K_{\underline{v}_i}}$ of $K_{\underline{v}_i}$].

Here, we note that our assumption that $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}} \setminus \mathbb{V}^{\mathrm{dst}}_{\mathbb{Q}}$ implies that the hypotheses of Proposition 1.4, (iv), are satisfied. Thus, the *inclusion* "$\phi((R_I)^{\sim}) \subseteq (R_I)^{\sim}$" of Proposition 1.4, (iv), implies that the tensor product of log-shells under consideration *contains* the "union of possible images of a Θ-pilot object" discussed in Step (iv). That is to say, the *indeterminacies* (Ind1) and (Ind2) are taken into account by the *arbitrary* nature of the automorphism "ϕ" [cf. Proposition 1.2], while the *indeterminacy* (Ind3) is taken into account by the fact that we are considering *upper bounds* [cf. the discussion of Step (x) of the proof of [IUTchIII], Corollary 3.12], together with the fact that the "container of possible images" is *precisely equal* to the tensor product of log-shells under consideration. Thus, an upper bound on the component of the log-volume under consideration may be obtained by computing an upper bound for the log-volume of the right-hand side "$(R_I)^{\sim}$" of the above inclusion. Such an *upper bound*

$$\text{“}0\text{”}$$

is given in the final equality of Proposition 1.4, (iv). One may then compute a **"weighted average upper bound"** and then a *"procession-normalized upper bound"*, as was done in Step (v). The resulting **"procession-normalized upper bound"** is clearly equal to 0.

(vii) Next, let $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{arc}}_{\mathbb{Q}}$. Fix j, $\{v_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$ as in Step (iv). Write $\underline{v}_i \in \underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}} = \mathbb{V}(F_{\mathrm{mod}})$ for the element corresponding to v_i. We would like to apply Proposition 1.5, (iii), (iv), to the present situation, by taking

- · "I" to be $\mathbb{S}^{\pm}_{j+1}$ [so $|I| = j + 1$];
- · "V" to be $\mathbb{V}(F_{\mathrm{mod}})_{v_{\mathbb{Q}}}$;
- · "$\mathbb{C}_v$" to be $K_{\underline{v}}$, where we write $\underline{v} \in \underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$ for the element determined by $v \in V$.

Then it follows from Proposition 1.5, (iii), (iv), that

$$\pi^{j+1} \cdot B_I$$

serves as a *container* for the "union of possible images of a Θ-pilot object" discussed in Step (iv). That is to say, the *indeterminacies* (Ind1) and (Ind2) are taken into account by the fact that $B_I \subseteq M_I$ is *preserved* by *arbitrary automorphisms* of the type discussed in Proposition 1.5, (iii), while the *indeterminacy* (Ind3) is taken into account by the fact that we are considering *upper bounds* [cf. the discussion of Step (x) of the proof of [IUTchIII], Corollary 3.12], together with the fact that, by Proposition 1.5, (iv), together with our choice of the factor π^{j+1}, this "container of possible images" contains the elements of M_I obtained by forming the tensor product of elements of the log-shells under consideration. Thus, an upper bound on the component of the log-volume under consideration may be obtained by computing an upper bound for the log-volume of this container. Such an *upper bound*

$$(j+1) \cdot \log(\pi)$$

follows immediately from the fact that [in order to ensure compatibility with *arithmetic degrees* of arithmetic line bundles — cf. [IUTchIII], Proposition 3.9, (iii) — one is obliged to adopt normalizations which imply that] *the log-volume of B_I is equal to* 0. One may then compute a **"weighted average upper bound"** and then a *"procession-normalized upper bound"*, as was done in Step (v). The resulting **"procession-normalized upper bound"** is given by

$$\tfrac{l+5}{4} \cdot \log(\pi) \;\; \leq \;\; \tfrac{l+1}{4} \cdot 4$$

— cf. (E1), (E6); the fact that $l \geq 5$.

(viii) Now we return to the discussion of Step (iv). In order to compute the desired *upper bound* for "C_Θ", it suffices to **sum over** $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ the various local **"procession-normalized upper bounds"** obtained in Steps (v), (vi), (vii) for $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$. By applying the *inequality* of the final display of Step (iii), we thus obtain the following *upper bound* for "C_Θ":

$$\tfrac{l+1}{4} \cdot \Bigg\{ (1 + \tfrac{28 \cdot d_{\mathrm{mod}}}{l}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 2 \cdot \log(l) + 69 - \tfrac{1}{6} \cdot \log(\mathfrak{q}) \\ + \tfrac{20}{3} \cdot \sum_{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{dst}}} \tfrac{1}{p_{v_{\mathbb{Q}}}} \cdot \log(\mathfrak{s}_{v_{\mathbb{Q}}}^{F_{\mathrm{mod}}}) \Bigg\} - 1$$

— where we apply the estimate $\frac{l+1}{4} \geq 1$ [a consequence of the fact that $l \geq 5$]. Thus, it remains to evaluate the *sum over* $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{dst}}$ that appears in the last display. Recall the constant "η_{prm}" of Proposition 1.6. Now we define

$$\epsilon_\Theta \;\; \stackrel{\mathrm{def}}{=} \;\; \min\left(\tfrac{40\eta_{\mathrm{prm}}}{3}, 3^{-210}\right)$$

and suppose that $\epsilon \in \mathbb{R}_{>0}$ satisfies $\epsilon < \epsilon_\Theta$. Note that this implies that $\frac{40}{3\epsilon} > 5$ and $\frac{2}{3} \cdot \log(\epsilon^{-1}) \geq 140$ [cf. (E6)]. Write

- $\mathbb{V}_{\mathbb{Q}}^{\prec \epsilon^{-1}} \stackrel{\mathrm{def}}{=} \{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}} \mid p_{v_{\mathbb{Q}}} \leq \frac{40}{3\epsilon}\}$;
- $\mathbb{V}_{\mathbb{Q}}^{\succ \epsilon^{-1}} \stackrel{\mathrm{def}}{=} \{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{dst}} \mid l \neq p_{v_{\mathbb{Q}}} \geq \frac{40}{3\epsilon}\ (> 5)\}$.

Then by applying Proposition 1.6, where we take "η" to be $\frac{3\epsilon}{40} < \eta_{\mathrm{prm}}$, together with the discussion of Step (iii), we conclude that

$$\begin{aligned}
&\tfrac{20}{3} \cdot \sum_{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{dst}}} \tfrac{1}{p_{v_{\mathbb{Q}}}} \cdot \log(\mathfrak{s}_{v_{\mathbb{Q}}}^{F_{\mathrm{mod}}}) \\
&\leq\ \tfrac{20}{3} \cdot \sum_{l = p_{v_{\mathbb{Q}}}} \frac{\log(\mathfrak{s}_{v_{\mathbb{Q}}}^{F_{\mathrm{mod}}})}{p_{v_{\mathbb{Q}}}} + \tfrac{20}{3} \cdot \sum_{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\succ \epsilon^{-1}}} \frac{\log(\mathfrak{s}_{v_{\mathbb{Q}}}^{F_{\mathrm{mod}}})}{p_{v_{\mathbb{Q}}}} + \tfrac{20}{3} \cdot \sum_{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\prec \epsilon^{-1}}} \frac{\log(\mathfrak{s}_{v_{\mathbb{Q}}}^{F_{\mathrm{mod}}})}{p_{v_{\mathbb{Q}}}} \\
&\leq\ 4 + \tfrac{40}{3} \cdot \sum_{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\succ \epsilon^{-1}}} \tfrac{1}{p_{v_{\mathbb{Q}}}} \cdot (\log(\mathfrak{d}_{v_{\mathbb{Q}}}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}_{v_{\mathbb{Q}}}^{F_{\mathrm{tpd}}})) + \tfrac{20}{3} \cdot \sum_{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\prec \epsilon^{-1}}} \frac{\log(p_{v_{\mathbb{Q}}})}{p_{v_{\mathbb{Q}}}} \\
&\leq\ 4 + \epsilon \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) - \tfrac{40}{3} \cdot \log(\epsilon) + \tfrac{40}{3} \cdot \log(\tfrac{40}{3}) \\
&\leq\ \epsilon \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 2 \cdot \log(\epsilon^{-7}) - \tfrac{2}{3} \cdot \log(\epsilon^{-1}) + 60 \\
&\leq\ \epsilon \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 2 \cdot \log(\epsilon^{-7}) - 80
\end{aligned}$$

— where we apply the estimates $\frac{10}{3} \leq 4$ and $\frac{40}{3} \cdot \log(\frac{40}{3}) \leq 14 \cdot \log(15) \leq 56$ [cf. (E6)]. Thus, substituting back into our *original upper bound* for "C_{Θ}", we obtain the following *upper bound* for "C_{Θ}":

$$\tfrac{l+1}{4} \cdot \Big\{(1 + \epsilon + \tfrac{28 \cdot d_{\mathrm{mod}}}{l}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 2 \cdot \log(l \cdot \epsilon^{-7}) - \tfrac{1}{6} \cdot \log(\mathfrak{q})\Big\} - 1$$

— where we apply the estimate $69 - 80 \leq 0$ — i.e., as asserted in the statement of Theorem 1.10. The final portion of Theorem 1.10 follows immediately from [IUTchIII], Corollary 3.12, together with the inequality of the first display of Step (ii). ○

Remark 1.10.1. One of the main original motivations for the development of the theory discussed in the present series of papers was to create a *framework*, or *geometry*, within which a suitable analogue of the *scheme-theoretic Hodge-Arakelov theory* of [HASurI], [HASurII] could be realized in such a way that the *obstructions to diophantine applications* that arose in the scheme-theoretic formulation of [HASurI], [HASurII] [cf. the discussion of [HASurI], §1.5.1; [HASurII], Remark 3.7] could be *avoided*. From this point of view, it is of interest to observe that the computation of the *"leading term"* of the inequality of the final display of the statement of Theorem 1.10 — i.e., of the term

$$\tfrac{(l^{*}+3)}{2} \cdot \log(\mathfrak{d}_{v_{\mathbb{Q}}}^{K}) - \tfrac{(2l^{*}+1)(l^{*}+1)}{12l} \cdot \log(\mathfrak{q}_{v_{\mathbb{Q}}})$$

that occurs in the final display of Step (v) of the proof of Theorem 1.10 — via the identities (E1), (E2) is *essentially identical* to the computation of the leading term that occurs in the proof of [HASurI], Theorem A [cf. the discussion following the statement of Theorem A in [HASurI], §1.1]. That is to say, in some sense,

> the **computations** performed in the proof of Theorem 1.10 were already essentially known to the author around the year 2000; the problem then was to construct an appropriate **framework**, or **geometry**, in which these computations could be performed!

This sort of situation may be compared to the computations underlying the **Weil Conjectures** priori to the construction of a "Weil cohomology" in which those computations could be performed, or, alternatively, to various computations of invariants in topology or differential geometry that were motivated by computations in **physics**, again prior to the construction of a suitable mathematical framework in which those computations could be performed.

Remark 1.10.2. The computation performed in the proof of Theorem 1.10 may be thought of as the computation of a sort of **derivative** in the $\mathbb{F}_l^{\circledast}$**-direction**, which, relative to the analogy between the theory of the present series of papers and the p-adic Teichmüller theory of [pOrd], [pTeich], corresponds to the *derivative* of the *canonical Frobenius lifting* — cf. the discussion of [IUTchIII], Remark 3.12.4, (iii). In this context, it is useful to recall the *arithmetic Kodaira-Spencer morphism* that occurs in scheme-theoretic Hodge-Arakelov theory [cf. [HASurII], §3]. In particular, in [HASurII], Corollary 3.6, it is shown that, when suitably formulated, a *"certain portion"* of this arithmetic Kodaira-Spencer morphism *coincides* with the usual *geometric Kodaira-Spencer morphism.* From the point of view of the *action* of $GL_2(\mathbb{F}_l)$ on the l-torsion points involved, this "certain portion" consists of the *unipotent matrices*

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

of $GL_2(\mathbb{F}_l)$. By contrast, the $\mathbb{F}_l^{\circledast}$*-symmetries* that occur in the present series of papers correspond to the *toral matrices*

$$\begin{pmatrix} 1 & 0 \\ 0 & \pm * \end{pmatrix}$$

of $GL_2(\mathbb{F}_l)$ — cf. the discussion of [IUTchI], Example 4.3, (i). As we shall see in §2 below, in the present series of papers, we shall ultimately take l to be *"large"*. When l is "sufficiently large", $GL_2(\mathbb{F}_l)$ may be thought of as a *"good approximation"* for $GL_2(\mathbb{Z})$ or $GL_2(\mathbb{R})$ — cf. the discussion of [IUTchI], Remark 6.12.3, (i), (iii). In the case of $GL_2(\mathbb{R})$, "toral subgroups" may thought of as corresponding to the *isotropy subgroups* [isomorphic to $\mathbb{S}^1$] of points that arise from the action of $GL_2(\mathbb{R})$ on the *upper half-plane*, i.e., subgroups which may be thought of as a sort of *geometric, group-theoretic representation* of **tangent vectors** at a point.

Remark 1.10.3. The *"*$\log(l)$ *term"* that occurs in the inequality of the final display of Theorem 1.10 may be thought of as an *inevitable consequence* of the *fundamental role* played in the theory of the present series of papers by the l*-torsion points* of the elliptic curve under consideration. Here, we note that it is of crucial importance to work over the *field of rationality* of the l-torsion points [i.e., "K" as opposed to "F"] *not only* when considering the global portions of the various ΘNF-

and $\Theta^{\pm \mathrm{ell}}$*-Hodge-theaters* involved, but also when considering the local portions — i.e., the *prime-strips* — of these ΘNF- and $\Theta^{\pm \mathrm{ell}}$-Hodge-theaters. That is to say, these local portions are necessary, for instance, in order to *glue together* the ΘNF- and $\Theta^{\pm \mathrm{ell}}$-Hodge-theaters that appear so as to form a $\Theta^{\pm \mathrm{ell}}$*NF-Hodge-theater* [cf. the discussion of [IUTchI], Remark 6.12.2]. In particular, to allow, within these local portions, any sort of *"Galois indeterminacy"* with respect to the l-torsion points — even, for instance, at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$, which, at first glance, might appear *irrelevant* to the theory of *Hodge-Arakelov-theoretic evaluation at* l*-torsion points* developed in [IUTchII] — would have the effect of *invalidating* the various *delicate manipulations involving* l*-torsion points* discussed in [IUTchI], §4, §6 [cf., e.g., [IUTchI], Propositions 4.7, 6.5].

Remark 1.10.4. The various *fluctuations in log-volume* — i.e., whose computation is the subject of Theorem 1.10! — that arise from the *multiradial representation* of [IUTchIII], Theorem 3.11, (i), may be thought of as a sort of **"inter-universal analytic torsion"**. Indeed,

> in general, **"analytic torsion"** may be understood as a sort of *measure* — in **"metrized"** [e.g., log-volume!] terms — of the degree of **deviation** of the *"holomorphic functions"* [such as sections of a line bundle] on a variety — i.e., which depend, in an essential way, on the **holomorphic moduli** of the variety! — from the *"real analytic functions"* — i.e., which are **invariant** with respect to **deformations of the holomorphic moduli** of the variety.

For instance:

(a) In *"classical" Arakelov theory*, analytic torsion typically arises as [the logarithm of] a sort of **normalized determinant** of the **Laplacian** acting on some space of real analytic [or L^2-] sections of a line bundle on a complex variety equipped with a real analytic Kähler metric [cf., e.g., [Arak], Chapters V, VI]. Here, we recall that in this sort of situation, the space of *holomorphic* sections of the line bundle is given by the kernel of the Laplacian; the definition of the *Laplacian* depends, in an essential way, on the *Kähler metric*, hence, in particular, on the *holomorphic moduli* of the variety under consideration [cf., e.g., the case of the Poincaré metric on a hyperbolic Riemann surface!].

(b) In the *scheme-theoretic Hodge-Arakelov theory* discussed in [HASurI], [HASurII], the main theorem consists of a sort of *comparison isomorphism* [cf. [HASurI], Theorem A] between the space of *sections of an ample line bundle on the universal vectorial extension* of an elliptic curve and the space of *set-theoretic functions on the torsion points* of the elliptic curve. That is to say, the former space of sections contains, in a natural way, the space of *holomorphic* sections of the ample line on the elliptic curve, while the latter space of functions may be thought of as a sort of "discrete approximation" of the space of *real analytic* functions on the elliptic curve [cf. the discussion of [HASurI], §1.3.2, §1.3.4]. In this context, the **"Gaussian poles"** [cf. the discussion of [HASurI], §1.1] arise as a measure of the *discrepancy of integral structures* between these two spaces in a neighborhood of the divisor at infinity of the moduli stack of elliptic curves, hence may be thought of as a sort of *"analytic torsion at the divisor at infinity"* [cf. the discussion of [HASurI], §1.2].

(c) In the case of the *multiradial representation* of [IUTchIII], Theorem 3.11, (i), the fluctuations of log-volume computed in Theorem 1.10 arise precisely as a result of the execution of a *comparison of an* **"alien" arithmetic holomorphic structure** to this **multiradial representation**, which is compatible with the *permutation symmetries of the étale-picture*, i.e., which is "invariant with respect to deformations of the arithmetic holomorphic moduli of the number field under consideration" in the sense that it makes sense *simultaneously with respect to distinct arithmetic holomorphic structures* [cf. [IUTchIII], Remark 3.11.1; [IUTchIII], Remark 3.12.3, (ii)]. Here, it is of interest to observe that the object of this comparison consists of the *values of the theta function*, i.e., in essence, a "holomorphic section of an ample line bundle". In particular, the resulting fluctuations of log-volume may be thought as a sort of *"analytic torsion"*. By analogy to the terminology *"Gaussian poles"* discussed in (b) above, it is natural to think of the terms involving the *different* $\mathfrak{d}^K_{(-)}$ that appear in the computation underlying Theorem 1.10 [cf., e.g., the final display of Step (v) of the proof of Theorem 1.10] as **"differential poles"** [cf. the discussion of Remarks 1.10.1, 1.10.2]. Finally, in the context of the *normalized determinants* that appear in (a), it is interesting to note the role played by the **prime number theorem** — i.e., in essence, the **Riemann zeta function** [cf. Proposition 1.6 and its proof] — in the computation of **"inter-universal analytic torsion"** given in the proof of Theorem 1.10.

Section 2: Diophantine Inequalities

In the present §2, we combine Theorem 1.10 with the theory of [GenEll] to give a proof of the **ABC Conjecture**, or, equivalently, **Vojta's Conjecture for hyperbolic curves** [cf. Corollary 2.3 below].

We begin by reviewing some well-known estimates.

Proposition 2.1. (Well-known Estimates)

(i) **(Linearization of Logarithms)** *Let* $C, \epsilon \in \mathbb{R}_{>0}$. *Then there exists an* $x_0 \in \mathbb{R}_{>0}$ *such that* $\log(x + C) \leq \epsilon \cdot x$ *for all* $(\mathbb{R} \ni)\ x \geq x_0$.

(ii) **(The Prime Number Theorem)** *There exists a* $\xi_{\mathrm{prm}} \in \mathbb{R}_{>0}$ *such that*

$$\sum_{p \leq x} \log(p) \quad \leq \quad 2x$$

— where the sum ranges over the prime numbers p *such that* $p \leq x$ *— for all* $(\mathbb{R} \ni)\ x \geq \xi_{\mathrm{prm}}$.

Proof. Assertion (i) is well-known and entirely elementary. Assertion (ii) is a well-known consequence of the *Prime Number Theorem* [cf., e.g., [Edw], p. 76]. ○

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. In the following discussion, we shall apply the *notation* and *terminology* of [GenEll]. Let X be a smooth, proper, geometrically connected curve over a number field; $D \subseteq X$ a reduced divisor; $U_X \stackrel{\mathrm{def}}{=} X \backslash D$; d a positive integer. Write ω_X for the canonical sheaf on X. Suppose that U_X is a *hyperbolic curve*, i.e., that the degree of the line bundle $\omega_X(D)$ is *positive*. Then we recall the following notation:

- $U_X(\overline{\mathbb{Q}})^{\leq d} \subseteq U_X(\overline{\mathbb{Q}})$ denotes the subset of $\overline{\mathbb{Q}}$-rational points *defined over a finite extension field* of $\mathbb{Q}$ of *degree* $\leq d$ [cf. [GenEll], Example 1.3, (i)].
- log-diff_X denotes the *(normalized) log-different* function on $U_X(\overline{\mathbb{Q}})^{\leq d}$ [cf. [GenEll], Definition 1.5, (iii)].
- log-cond_D denotes the *(normalized) log-conductor* function on $U_X(\overline{\mathbb{Q}})^{\leq d}$ [cf. [GenEll], Definition 1.5, (iv)].
- $\mathrm{ht}_{\omega_X(D)}$ denotes the *(normalized) height* function on $U_X(\overline{\mathbb{Q}})^{\leq d}$ associated to $\omega_X(D)$, which is well-defined up to a *"bounded discrepancy"* [cf. [GenEll], Proposition 1.4, (iii)];

In order to apply the theory of the present series of papers, it is neceesary to *construct suitable initial* Θ*-data*, as follows.

Corollary 2.2. **(Construction of Suitable Initial Θ-Data)** *Suppose that* $X = \mathbb{P}^1_{\mathbb{Q}}$ *is the* **projective line** *over* $\mathbb{Q}$*, and that* $D \subseteq X$ *is the divisor consisting of the* **three points** *"0", "1", and "∞". We shall regard X as the* **"λ-line"** *— i.e., we shall regard the standard coordinate on* $X = \mathbb{P}^1_{\mathbb{Q}}$ *as the "λ" in the* **Legendre form** *"$y^2 = x(x-1)(x-\lambda)$" of the Weierstrass equation defining an elliptic curve — and hence as being equipped with a natural* **classifying morphism** $U_X \to (\mathcal{M}_{\mathrm{ell}})_{\mathbb{Q}}$ *[cf. the discussion preceding Proposition 1.8]. Let*

$$\mathcal{K}_V \subseteq U_X(\overline{\mathbb{Q}})$$

be a **compactly bounded subset** *[i.e., regarded as a subset of $X(\overline{\mathbb{Q}})$ — cf. [GenEll], Example 1.3, (ii)] whose* **support** *contains the nonarchimedean prime "2". Then:*

(i) The normalized degree "$\underline{\deg}(-)$" of the effective arithmetic divisor determined by the **q-parameters** *of an elliptic curve over a number field at the nonarchimedean primes that do not divide* 2 *[cf. the invariant "$\log(\mathfrak{q})$" associated, in the statement of Theorem 1.10, to the elliptic curve E_F] determines an $\mathbb{R}$-valued function on $\mathcal{M}_{\mathrm{ell}}(\overline{\mathbb{Q}})$, hence also on $U_X(\overline{\mathbb{Q}})$. If we denote this function by the notation "$\log(\mathfrak{q}_{(-)})$", then we have an* **equality of "bounded discrepancy classes"** *[cf. [GenEll], Definition 1.2, (ii), as well as Remark 2.3.1, (ii), below]*

$$\tfrac{1}{6} \cdot \log(\mathfrak{q}_{(-)}) \;\approx\; \mathrm{ht}_{\omega_X(D)}$$

of functions on $\mathcal{K}_V \subseteq U_X(\overline{\mathbb{Q}})$.

(ii) Let

$$\mathbb{R}_{>0} \;\ni\; \epsilon \;<\; \min(\xi_{\mathrm{prm}}^{-1}, 5 \cdot \epsilon_\Theta)$$

— where ϵ_Θ is as in Theorem 1.10; ξ_{prm} is as in Proposition 2.1, (ii). Then there exists a **Galois-finite** *[cf. [GenEll], Example 1.3, (i)] subset $\mathfrak{Exc}_\epsilon \subseteq U_X(\overline{\mathbb{Q}})$ which contains all points corresponding to elliptic curves with automorphisms of order* > 2 *and, moreover, satisfies the following property:*

Let E_F be an **elliptic curve** *over a number field $F \subseteq \overline{\mathbb{Q}}$ that lifts [not necessarily uniquely!] to a point $x_E \in U_X(F)$ such that*

$$[x_E] \in \mathcal{K}_V, \quad [x_E] \notin \mathfrak{Exc}_\epsilon.$$

Write F_{mod} for the **minimal field of definition** *of the corresponding point* $\in \mathcal{M}_{\mathrm{ell}}(\overline{\mathbb{Q}})$ *and*

$$F_{\mathrm{mod}} \quad \subseteq \quad F_{\mathrm{tpd}} \overset{\mathrm{def}}{=} F_{\mathrm{mod}}(\ E_{F_{\mathrm{mod}}}[2]\ ) \quad \subseteq \quad F$$

for the **"tripodal"** *intermediate field obtained from F_{mod} by adjoining the fields of definition of the* 2*-torsion points of any model of E_F over F_{mod} [cf. Proposition 1.8, (ii), (iii)]. Moreover, we assume that the $(3\cdot5)$-torsion points of E_F are defined over F, and that*

$$F \quad = \quad F_{\mathrm{mod}}(\ \sqrt{-1},\ E_{F_{\mathrm{mod}}}[2 \cdot 3 \cdot 5]\ ) \quad \overset{\mathrm{def}}{=} \quad F_{\mathrm{tpd}}(\ \sqrt{-1},\ E_{F_{\mathrm{tpd}}}[3 \cdot 5]\ )$$

— i.e., that F *is obtained from* F_{tpd} *by adjoining* $\sqrt{-1}$*, together with the fields of definition of the* $(3 \cdot 5)$*-torsion points of a model* $E_{F_{\mathrm{tpd}}}$ *of the elliptic curve* E_F *over* F_{tpd} *determined by the* **Legendre form** *of the Weierstrass equation discussed above [cf. Proposition 1.8, (vi)]. [Thus, it follows from Proposition 1.8, (iv), that* $E_F \cong E_{F_{\mathrm{tpd}}} \times_{F_{\mathrm{tpd}}} F$ *over* F*, so* $x_E \in U_X(F_{\mathrm{tpd}}) \subseteq U_X(F)$*.] Then* E_F *and* F_{mod} *arise as the "*E_F*" and "*F_{mod}*" for a collection of* **initial Θ-data** *as in Theorem 1.10 such that, in the notation of Theorem 1.10, the* **prime number** $\boldsymbol{l}$ *satisfies the following conditions:*

(P1) $2 \cdot \log(l) \ \leq \ \frac{1}{5} \cdot \epsilon \cdot \left\{ \frac{1}{6} \cdot \log(\mathfrak{q}) + \log(\mathfrak{d}^{F_{\mathrm{tpd}}}) \right\} + 2 \cdot \log(d_{\mathrm{mod}})$*;*
(P2) $l \geq 5 \cdot 28 \cdot d_{\mathrm{mod}} \cdot \epsilon^{-1}$*.*

In particular, by applying Theorem 1.10, we conclude that

$$\frac{1}{6} \cdot \log(\mathfrak{q}) \ \leq \ (1+\epsilon) \cdot (\text{log-diff}_X(x_E) + \text{log-cond}_X(x_E)) + 28 \cdot \log(5 \cdot \epsilon^{-1}) + 4 \cdot \log(d_{\mathrm{mod}})$$

— where we observe that [it follows tautologically from the definitions that] we have: $\text{log-diff}_X(x_E) = \log(\mathfrak{d}^{F_{\mathrm{tpd}}})$*,* $\text{log-cond}_D(x_E) = \log(\mathfrak{f}^{F_{\mathrm{tpd}}})$*.*

Proof. First, we consider assertion (i). We begin by observing that since the *support* of $\mathcal{K}_V$ contains the nonarchimedean prime "2", it follows immediately from the various definitions involved that

$$\log(\mathfrak{q}_{(-)}) \ \approx \ \underline{\deg}_\infty$$

— where "$\underline{\deg}_\infty$" is as in the discussion preceding [GenEll], Proposition 3.4 — on $\mathcal{K}_V \subseteq U_X(\overline{\mathbb{Q}})$. In a similar vein, since the *support* of $\mathcal{K}_V$ contains the unique archimedean prime of $\mathbb{Q}$, it follows immediately from the various definitions involved that

$$\underline{\deg}_\infty \ \approx \ \mathrm{ht}_\infty$$

— where "ht_∞" is as in the discussion preceding [GenEll], Proposition 3.4 — on $\mathcal{K}_V \subseteq U_X(\overline{\mathbb{Q}})$ [cf. the argument of the final paragraph of the proof of [GenEll], Lemma 3.7]. Thus, we conclude that $\log(\mathfrak{q}_{(-)}) \ \approx \ \mathrm{ht}_\infty$. Since [as is well-known] the pull-back to X of the divisor at infinity of the natural compatification $(\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{Q}}$ of $(\mathcal{M}_{\mathrm{ell}})_{\mathbb{Q}}$ is of *degree* 6, while the line bundle $\omega_X(D)$ is of *degree* 1, the desired equality of BD-classes $\frac{1}{6} \cdot \log(\mathfrak{q}_{(-)}) \ \approx \ \mathrm{ht}_{\omega_X(D)}$ follows immediately from [GenEll], Proposition 1.4, (i), (iii). This completes the proof of assertion (i).

Next, we consider assertion (ii). First, we observe that [one verifies easily that] the *image* in $\mathcal{M}_{\mathrm{ell}}(\overline{\mathbb{Q}})$ of $\mathcal{K}_V$ determines a *compactly bounded subset* of $\mathcal{M}_{\mathrm{ell}}(\overline{\mathbb{Q}})$. Thus, by applying [GenEll], Corollary 4.4, to this compactly bounded subset of $\mathcal{M}_{\mathrm{ell}}(\overline{\mathbb{Q}})$, we obtain a *Galois-finite* subset "$\mathfrak{Exc}$" of $\mathcal{M}_{\mathrm{ell}}(\overline{\mathbb{Q}})$, together with a constant "$C \in \mathbb{R}$", that satisfy a *certain property* [cf. the statement of [GenEll], Corollary 4.4], which we shall discuss below in detail. Let us write

$$\mathfrak{Exc}_\epsilon \ \subseteq \ U_X(\overline{\mathbb{Q}})$$

for the inverse image of the subset "$\mathfrak{Exc}$" of [GenEll], Corollary 4.4, and C_1 for the constant "C". One verifies immediately that this subset $\mathfrak{Exc}_\epsilon \ \subseteq \ U_X(\overline{\mathbb{Q}})$ is

Galois-finite. Although $\mathfrak{Exc}_\epsilon$, defined in this way, does not depend on ϵ, we shall, in the argument to follow, *enlarge* $\mathfrak{Exc}_\epsilon$ several times — i.e., by an *abuse of notation*, for the purpose of simplifying the notation! — in such a way that the resulting enlargement does in fact *depend on* ϵ.

Next, let us recall that if the once-punctured elliptic curve associated to E_F *fails to admit an* F*-core*, then there are only *four possibilities* for the j-invariant of E_F [cf. [CanLift], Proposition 2.7]. Thus, by *possibly enlarging* $\mathfrak{Exc}_\epsilon$ [in a fashion that is still *independent* of ϵ!], which is possible in light of [GenEll], Proposition 1.4, (iv), we may assume that the once-punctured elliptic curve associated to E_F *admits an* F*-core*, hence, in particular, does not have any automorphisms of order > 2 over $\overline{\mathbb{Q}}$.

Now before proceeding, let us observe [cf., Proposition 1.8, (v)] that our assumptions concerning the extension F/F_{mod} imply that E_F has *stable reduction* at all of the nonarchimedean primes of F [cf. the proof of [GenEll], Theorem 3.8]. Next, let us observe that it follows from assertion (i) that the function "$\mathrm{ht}_{\omega_X(D)}$" is *bounded* on the set of $[x_E]$ corresponding to E_F with *good reduction* at all nonarchimedean primes that do not divide 2. In particular, by *possibly enlarging* $\mathfrak{Exc}_\epsilon$ [in a fashion that is still *independent* of ϵ!], which is possible in light of [GenEll], Proposition 1.4, (iv), we may assume that E_F has *bad [but stable!] reduction* at some nonarchimedean prime that does not divide 2. Thus, in summary, one verifies immediately [cf., especially, our assumptions concerning the extension F/F_{mod}] that *all of the conditions of [IUTchI], Definition 3.1, (a), (b), (d), (e), (f), are satisfied.* That is to say, in order to obtain a collection of *initial* Θ*-data* as in the statement of assertion (ii), it suffices to show the existence of a *prime number* l that satisfies the *conditions of [IUTchI], Definition 3.1, (c),* as well as the *conditions (P1), (P2) of the statement of assertion (ii).*

Next, we would like to apply the *property* satisfied by the subset "$\mathfrak{Exc}$" of [GenEll], Corollary 4.4. We take the set "$\mathcal{S}$" of *loc. cit.* to be the set

$$\mathcal{S} \ \stackrel{\mathrm{def}}{=} \ \{p \mid p \text{ is a prime number} \leq 5 \cdot 28 \cdot d_{\mathrm{mod}} \cdot \epsilon^{-1} \ (> 5)\}$$

— cf. condition (P2). Thus, we obtain the estimate

$$x_{\mathcal{S}} \ \stackrel{\mathrm{def}}{=} \ \sum_{p \in \mathcal{S}} \log(p) \ \leq \ 10 \cdot 28 \cdot d_{\mathrm{mod}} \cdot \epsilon^{-1}$$

— cf. our assumption that $\epsilon^{-1} \geq \xi_{\mathrm{prm}}$; Proposition 2.1, (ii). Note that the quantity "d" of *loc. cit.* corresponds to the quantity d_{mod} of the present discussion. Now we take the *prime number* l to be the prime number "$l_\bullet$" of [GenEll], Corollary 4.4. Thus, $l \notin \mathcal{S}$, so the *condition (P2) is satisfied.* Moreover, since $2, 3, 5 \in \mathcal{S}$, it follows from conditions (a), (b) of [GenEll], Corollary 4.4, that the *conditions of [IUTchI], Definition 3.1, (c), are satisfied.*

Next, let us observe that it follows from the argument applied in the proof of assertion (i), together with [GenEll], Proposition 3.4, that we have *equalities/inequalities of BD-classes*

$$\log(\mathfrak{q}_{(-)}) \ \approx \ \mathrm{ht}_\infty \ \gtrsim \ 12 \cdot \mathrm{ht}^{\mathrm{Falt}}$$

on $\mathcal{K}_V \subseteq U_X(\overline{\mathbb{Q}})$. Thus, it follows from condition (c) of [GenEll], Corollary 4.4, that

$$\begin{aligned} l \ &\leq \ d_{\mathrm{mod}} \cdot \left\{ 23040 \cdot 50 \cdot \tfrac{1}{6} \cdot \log(\mathfrak{q}) + 6 \cdot \log(\mathfrak{d}^{F_{\mathrm{mod}}}) + 2x_{\mathcal{S}} \cdot d_{\mathrm{mod}}^{-1} + C_2 \right\} \\ &\leq \ d_{\mathrm{mod}} \cdot \left\{ 23040 \cdot 50 \cdot \tfrac{1}{6} \cdot \log(\mathfrak{q}) + 6 \cdot \log(\mathfrak{d}^{F_{\mathrm{mod}}}) + 20 \cdot 28 \cdot \epsilon^{-1} + C_2 \right\} \\ &\leq \ d_{\mathrm{mod}} \cdot \left\{ 23040 \cdot 50 \cdot \tfrac{1}{6} \cdot \log(\mathfrak{q}) + 6 \cdot \log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + C_{\epsilon} \right\} \end{aligned}$$

— where, in the first inequality, we replace the constant C_1 by a new constant C_2, so as to take into account the inequality of BD-classes discussed above; in the third inequality, we take $C_{\epsilon} \overset{\mathrm{def}}{=} 20 \cdot 28 \cdot \epsilon^{-1} + C_2$; we observe that the quantity "log-diff$_{\overline{\mathcal{M}}_{\mathrm{ell}}}([E_L])$" of *loc. cit.* [cf. Remark 2.3.1, (iv), below] corresponds to the quantity $\log(\mathfrak{d}^{F_{\mathrm{mod}}})$ $(\leq \log(\mathfrak{d}^{F_{\mathrm{tpd}}}))$ of the present discussion. Next, let us observe that since $\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) \geq 0$ and $C_{\epsilon} = 20 \cdot 28 \cdot \epsilon^{-1} + C_2 \geq C_2$, it follows that any *upper bound* on the quantity

$$\left\{ 23040 \cdot 50 \cdot \tfrac{1}{6} \cdot \log(\mathfrak{q}) + 6 \cdot \log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + C_{\epsilon} \right\}$$

of the final line of the preceding display *implies an upper bound* on the quantity $\log(\mathfrak{q})$, i.e., [by applying the *equalities of BD-classes* discussed above] an *upper bound* on the quantity "ht$_{\infty}$", which [cf., e.g., [GenEll], Proposition 1.4, (iv)] can only be satisfied by *finitely many elements* of $\mathcal{M}_{\mathrm{ell}}(\overline{\mathbb{Q}})^{\leq n}$, for a given integer n. Thus, by *possibly enlarging* $\mathfrak{Exc}_{\epsilon}$ [this time in a way that *depends* on ϵ!], we may assume, by applying Proposition 2.1, (i), that

$$2 \cdot \log(l) \ \leq \ \tfrac{1}{5} \cdot \epsilon \cdot \left\{ \tfrac{1}{6} \cdot \log(\mathfrak{q}) + \log(\mathfrak{d}^{F_{\mathrm{tpd}}}) \right\} + 2 \cdot \log(d_{\mathrm{mod}})$$

— i.e., that the *condition (P1) is satisfied.*

Finally, we observe that since, by assumption, $\frac{1}{5} \cdot \epsilon < \epsilon_{\Theta}$, it follows from the final portion of Theorem 1.10 that

$$\begin{aligned} \tfrac{1}{6} \cdot \log(\mathfrak{q}) \ &\leq \ (1 + \tfrac{1}{5} \cdot \epsilon + \tfrac{28 \cdot d_{\mathrm{mod}}}{l}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) \\ &\qquad + 2 \cdot \log(l) + 14 \cdot \log(5 \cdot \epsilon^{-1}) \\ &\leq \ (1 + \tfrac{1}{5} \cdot \epsilon + \tfrac{1}{5} \cdot \epsilon) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) \\ &\qquad + \tfrac{1}{5} \cdot \epsilon \cdot \left\{ \tfrac{1}{6} \cdot \log(\mathfrak{q}) + \log(\mathfrak{d}^{F_{\mathrm{tpd}}}) \right\} + 14 \cdot \log(5 \cdot \epsilon^{-1}) + 2 \cdot \log(d_{\mathrm{mod}}) \\ &\leq \ \tfrac{1}{5} \cdot \epsilon \cdot \tfrac{1}{6} \cdot \log(\mathfrak{q}) + (1 + \tfrac{3}{5} \cdot \epsilon) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) \\ &\qquad + 14 \cdot \log(5 \cdot \epsilon^{-1}) + 2 \cdot \log(d_{\mathrm{mod}}) \end{aligned}$$

— where we apply the inequalities of (P1), (P2), as well as the inequality $\log(\mathfrak{f}^{F_{\mathrm{tpd}}}) \geq 0$. The inequality

$$\tfrac{1}{6} \cdot \log(\mathfrak{q}) \ \leq \ (1+\epsilon) \cdot (\text{log-diff}_X(x_E) + \text{log-cond}_D(x_E)) + 28 \cdot \log(5 \cdot \epsilon^{-1}) + 4 \cdot \log(d_{\mathrm{mod}})$$

[cf. the final display of the statement of assertion (ii)] thus follows by applying the estimates

$$\frac{1 + \frac{3}{5} \cdot \epsilon}{1 - \frac{1}{5} \cdot \epsilon} \quad \leq \quad 1 + \epsilon; \qquad 1 - \tfrac{1}{5} \cdot \epsilon \quad \geq \quad \tfrac{1}{2}$$

— both of which are consequences of the fact that $0 < \epsilon \leq 1$ — together with the observation that it follows immediately from the definitions [cf. also Proposition 1.8, (vi)] that the quantities $\text{log-diff}_X(x_E)$, $\text{log-cond}_D(x_E)$ correspond precisely to the quantities $\log(\mathfrak{d}^{F_{\text{tpd}}})$, $\log(\mathfrak{f}^{F_{\text{tpd}}})$, respectively. ○

We are now ready to state and prove the *main theorem* of the present §2, which may also be regarded as the *main application* of the theory developed in the present series of papers.

Corollary 2.3. **(Diophantine Inequalities)** *Let X be a smooth, proper, geometrically connected curve over a number field; $D \subseteq X$ a reduced divisor; $U_X \overset{\text{def}}{=} X \backslash D$; d a positive integer; $\epsilon \in \mathbb{R}_{>0}$ a positive real number. Write ω_X for the canonical sheaf on X. Suppose that U_X is a* **hyperbolic curve**, *i.e., that the degree of the line bundle $\omega_X(D)$ is* **positive**. *Then, relative to the notation reviewed above, one has an* **inequality of "bounded discrepancy classes"**

$$\text{ht}_{\omega_X(D)} \lesssim (1+\epsilon)(\text{log-diff}_X + \text{log-cond}_D)$$

of functions on $U_X(\overline{\mathbb{Q}})^{\leq d}$ — i.e., the function $(1+\epsilon)(\text{log-diff}_X + \text{log-cond}_D) - \text{ht}_{\omega_X(D)}$ is bounded below by a **constant** *on $U_X(\overline{\mathbb{Q}})^{\leq d}$ [cf. [GenEll], Definition 1.2, (ii), as well as Remark 2.3.1, (ii), below].*

Proof. One verifies immediately that the content of the statement of Corollary 2.3 coincides precisely with the content of [GenEll], Theorem 2.1, (i). Thus, it follows from the *equivalence* of [GenEll], Theorem 2.1, that, in order to complete the proof of Corollary 2.3, it suffices to verify that Theorem 2.1, (ii), holds. That is to say, we may assume without loss of generality that:

- $X = \mathbb{P}^1_{\mathbb{Q}}$ is the *projective line* over $\mathbb{Q}$;
- $D \subseteq X$ is the divisor consisting of the *three points* "0", "1", and "∞";
- $\mathcal{K}_V \subseteq U_X(\overline{\mathbb{Q}})$ is a *compactly bounded subset* whose *support* contains the nonarchimedean prime "2".

Then it suffices to show that the *inequality of BD-classes of functions* [cf. [GenEll], Definition 1.2, (ii), as well as Remark 2.3.1, (ii), below]

$$\text{ht}_{\omega_X(D)} \lesssim (1+\epsilon)(\text{log-diff}_X + \text{log-cond}_D)$$

holds on $\mathcal{K}_V \bigcap U_X(\overline{\mathbb{Q}})^{\leq d}$. But such an inequality follows immediately, in light of the *equality of BD-classes* of Corollary 2.2, (i), from the final portion of Corollary 2.2, (ii) [where we note that it follows immediately from the various definitions involved that $d_{\text{mod}} \leq d$]. This completes the proof of Corollary 2.3. ○

Remark 2.3.1. We take this opportunity to correct some *unfortunate misprints* in [GenEll].

(i) The notation "$\text{ord}_v(-) : F_v \to \mathbb{Z}$" in the final sentence of the first paragraph following [GenEll], Definition 1.1, should read "$\text{ord}_v(-) : F_v^{\times} \to \mathbb{Z}$".

(ii) In [GenEll], Definition 1.2, (ii), the *non-resp'd* and *first resp'd* items in the display should be *reversed*! That is to say, the notation "$\alpha \lesssim_{\mathcal{F}} \beta$" corresponds to "$\alpha(x) - \beta(x) \leq C$"; the notation "$\alpha \gtrsim_{\mathcal{F}} \beta$" corresponds to "$\beta(x) - \alpha(x) \leq C$".

(iii) The first portion of the first sentence of the statement of [GenEll], Corollary 4.4, should read: "Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$; ...".

(iv) The "$\text{log-diff}_{\overline{\mathcal{M}}_{\text{ell}}}([E_L]))$" in the second inequality of the final display of the statement of [GenEll], Corollary 4.4, should read "$\text{log-diff}_{\overline{\mathcal{M}}_{\text{ell}}}([E_L])$".

Remark 2.3.2.

(i) The reader will note that, by arguing with a "bit more care", it is not difficult to give **stronger estimates** for the various *"constants"* that occur in Theorem 1.10; Corollaries 2.2, 2.3 and their proofs. Such stronger estimates are, however, being the scope of the present series of papers, so we shall not pursue this topic further in the present paper.

(ii) On the other hand, we recall that the constant "1" in the inequality of the display of Corollary 2.3 *cannot be improved* — cf. the examples constructed in [Mss]. In the context of the examples constructed in [Mss], it is of interest to note that the estimates obtained in [Mss] for these examples appear, at first glance, to contradict the rather strong inequality obtained in the final display of Corollary 2.2, (ii). Indeed, fix a $\xi \in \mathbb{R}$ such that $\frac{1}{2} < \xi < 1$. Then if one *assumes* that

(1) the quantity "$\log(\mathfrak{q})$" in the final display of Corollary 2.2, (ii), is roughly equal to the *height* of the elliptic curve, i.e., the relation "$\log(\mathfrak{q}_{(-)}) \approx \text{ht}_{\infty}$" derived at the beginning of proof of Corollary 2.2 — which amounts, in essence, to the statement that one may *ignore* the contributions to the height at the *archimedean primes*, as well as at the *primes over* 2 — holds and, moreover, that

(2) the inequality in the final display of Corollary 2.2, (ii), may be applied to the elliptic curves constructed in [Mss],

then a straightforward substitution reveals that if one takes

$$\epsilon \quad \stackrel{\text{def}}{=} \quad \{\log(\mathfrak{d}^F) + \log(\mathfrak{f}^F)\}^{-\xi}$$

in the inequality in the final display of Corollary 2.2, (ii), then one obtains, at least *asymptotically*, a *contradiction* to the estimates obtained in [Mss]. In fact, it is not clear that the elliptic curves constructed in [Mss] satisfy either of the assumptions (1), (2), both of which may be thought of as assumptions to the effect that the elliptic curve in question is in **"sufficiently general position"**. That is to say, in order to obtain elliptic curves satisfying assumptions (1), (2), one must apply the theory of [GenEll] [cf. the proofs of Corollaries 2.2, 2.3!], which involves constructing various *"noncritical Belyi maps"* on finite étale coverings of the projective line minus three points. Moreover, these Belyi maps and coverings *depend*, in an essential way, on ϵ, and it is difficult to see how to *bound* the constants that arise in the construction of these Belyi maps and coverings in such a way as to assure that these constants do not affect the *delicate estimates* of [Mss]. In particular, despite the *apparently sharper and more explicit nature* [i.e., by comparison to the inequality of Corollary 2.3] of the inequality of the final display of Corollary 2.2, (ii), there is, in fact, *no contradiction* — as far as the author can see at the time of writing! — between Corollary 2.2, (ii), and the estimates obtained in [Mss].

Section 3: Inter-universal Formalism: the Language of Species

In the present §3, we develop the language of **species**. Roughly speaking, a "species" is a **"type of mathematical object"**, such as a "group", a "ring", a "scheme", etc. In some sense, this language may be thought of as an *explicit description* of certain tasks typically executed at an *implicit, intuitive level* by mathematicians via a sort of "mental arithmetic" in the course of interpreting various mathematical arguments. In the context of the theory developed in the present series of papers, however, it is useful to describe these intuitive operations explicitly.

In the following discussion, we shall work with various **models** — consisting of *"sets"* and a *relation* "$\in$" — of the standard *ZFC axioms* of axiomatic set theory [i.e., the nine axioms of *Zermelo-Fraenkel*, together with the *axiom of choice* — cf., e.g., [Drk], Chapter 1, §3]. We shall refer to such models as **ZFC-models**. Recall that a *(Grothendieck) universe* V is a set satisfying the following axioms [cf. [McLn], p. 194]:

(i) V is *transitive*, i.e., if $y \in x$, $x \in V$, then $y \in V$.

(ii) The set of *natural numbers* $\mathbb{N} \in V$.

(iii) If $x \in V$, then the *power set of* x also belongs to V.

(iv) If $x \in V$, then the *union of all members of* x also belongs to V.

(v) If $x \in V$, $y \subseteq V$, and $f : x \to y$ is a *surjection*, then $y \in V$.

We shall say that a set E is a V*-set* if $E \in V$.

The various ZFC-models that we work with may be thought of as [but are *not restricted* to be!] the ZFC-models determined by various *universes* that are sets relative to some *ambient ZFC-model* which, in addition to the standard axioms of ZFC set theory, satisfies the following **existence axiom** [attributed to the "Grothendieck school" — cf. the discussion of [McLn], p. 193]:

($\dagger^{\mathrm{G}}$) *Given any set x, there exists a universe V such that $x \in V$.*

We shall refer to a ZFC-model that also satisfies this additional axiom of the Grothendieck school as a *ZFCG-model.* This existence axiom ($\dagger^{\mathrm{G}}$) implies, in particular, that:

> *Given a set I and a collection of universes V_i (where $i \in I$), indexed by I [i.e., a 'function' $I \ni i \mapsto V_i$], there exists a [larger] universe V such that $V_i \in V$, for $i \in I$.*

Indeed, since the graph of the function $I \ni i \mapsto V_i$ is a *set*, it follows that $\{V_i\}_{i \in I}$ is a *set*. Thus, it follows from the *existence axiom* ($\dagger^{\mathrm{G}}$) that there exists a universe V such that $\{V_i\}_{i \in I} \in V$. Hence, by condition (i), we conclude that $V_i \in V$, for all $i \in I$, as desired. Note that this means, in particular, that there exist *infinite ascending chains of universes*

$$V_0 \in V_1 \in V_2 \in V_3 \in \ldots \in V_n \in \ldots \in V$$

— where n ranges over the natural numbers. On the other hand, by the *axiom of foundation*, there do not exist *infinite descending chains of universes*

$$V_0 \ni V_1 \ni V_2 \ni V_3 \ni \ldots \ni V_n \ni \ldots$$

— where n ranges over the natural numbers.

Although we shall not discuss in detail here the quite difficult issue of *whether or not there actually exist ZFCG-models*, we remark in passing that one may justify the stance of ignoring such issues — at least from the point of view of establishing the validity of various *"final results"* that may be formulated in ZFC-models — by invoking a result of *Feferman* [cf. [Ffmn], §2.3] concerning the *"conservative extensionality"* of ZFCG relative to ZFC, i.e., roughly speaking, that *"any proposition that may be formulated in a ZFC-model and, moreover, holds in a ZFCG-model in fact holds in the original ZFC-model"*.

In the following discussion, we use the phrase *"set-theoretic formula"* as it is conventionally used in discussions of *axiomatic set-theory* [cf., e.g., [Drk], Chapter 1, §2].

Definition 3.1.

(i) A 0-*species* $\mathfrak{S}_0$ is a collection of conditions given by a *set-theoretic formula*

$$\Phi_0(\mathfrak{E})$$

involving an ordered collection $\mathfrak{E} = (\mathfrak{E}_1, \ldots, \mathfrak{E}_{n_0})$ of sets $\mathfrak{E}_1, \ldots, \mathfrak{E}_{n_0}$ [which we think of as "indeterminates"], for some integer $n_0 \geq 1$; in this situation, we shall refer to $\mathfrak{E}$ as a *collection of species-data* for $\mathfrak{S}_0$. If $\mathfrak{S}_0$ is a 0-species given by a set-theoretic formula $\Phi_0(\mathfrak{E})$, then a 0-*specimen* of $\mathfrak{S}_0$ is a *specific* ordered collection of n_0 sets $E = (E_1, \ldots, E_{n_0})$ in some *specific* ZFC-model that satisfies $\Phi_0(E)$. If E is a 0-specimen of a 0-species $\mathfrak{S}_0$, then we shall write $E \in \mathfrak{S}_0$. If, moreover, it holds, in any ZFC-model, that the 0-specimens of $\mathfrak{S}_0$ form a *set*, then we shall refer to $\mathfrak{S}_0$ as 0-*small*.

(ii) Let $\mathfrak{S}_0$ be a 0-species. Then a 1-*species* $\mathfrak{S}_1$ *acting on* $\mathfrak{S}_0$ is a collection of *set-theoretic formulas* Φ_1, $\Phi_{1\circ 1}$ satisfying the following conditions:

(a) Φ_1 is a set-theoretic formula

$$\Phi_1(\mathfrak{E}, \mathfrak{E}', \mathfrak{F})$$

involving two collections of species-data $\mathfrak{E}$, $\mathfrak{E}'$ for $\mathfrak{S}_0$ [i.e., the conditions $\Phi_0(\mathfrak{E})$, $\Phi_0(\mathfrak{E}')$ hold] and an ordered collection $\mathfrak{F} = (\mathfrak{F}_1, \ldots, \mathfrak{F}_{n_1})$ of ["indeterminate"] sets $\mathfrak{F}_1, \ldots, \mathfrak{F}_{n_1}$, for some integer $n_1 \geq 1$; in this situation, we shall refer to $(\mathfrak{E}, \mathfrak{E}', \mathfrak{F})$ as a *collection of species-data* for $\mathfrak{S}_1$ and write $\mathfrak{F} : \mathfrak{E} \to \mathfrak{E}'$. If, in some ZFC-model, $E, E' \in \mathfrak{S}_0$, and F is a *specific* ordered collection of n_1 sets that satisfies the condition $\Phi_1(E, E', F)$, then we shall refer to the data (E, E', F) as a 1-*specimen* of $\mathfrak{S}_1$ and write $(E, E', F) \in \mathfrak{S}_1$; alternatively, we shall denote a 1-specimen (E, E', F) via the notation $F : E \to E'$ and refer to E (respectively, E') as the *domain* (respectively, *codomain*) of $F : E \to E'$.

(b) $\Phi_{1\circ 1}$ is a set-theoretic formula

$$\Phi_{1\circ 1}(\mathfrak{E}, \mathfrak{E}', \mathfrak{E}'', \mathfrak{F}, \mathfrak{F}', \mathfrak{F}'')$$

involving three collections of species-data $\mathfrak{F} : \mathfrak{E} \to \mathfrak{E}'$, $\mathfrak{F}' : \mathfrak{E}' \to \mathfrak{E}''$, $\mathfrak{F}'' : \mathfrak{E} \to \mathfrak{E}''$ for $\mathfrak{S}_1$ [i.e., the conditions $\Phi_0(\mathfrak{E})$; $\Phi_0(\mathfrak{E}')$; $\Phi_0(\mathfrak{E}'')$; $\Phi_1(\mathfrak{E}, \mathfrak{E}', \mathfrak{F})$; $\Phi_1(\mathfrak{E}', \mathfrak{E}'', \mathfrak{F}')$; $\Phi_1(\mathfrak{E}, \mathfrak{E}'', \mathfrak{F}'')$ hold]; in this situation, we shall refer to $\mathfrak{F}''$ as a *composite of* $\mathfrak{F}$ *with* $\mathfrak{F}'$ and write $\mathfrak{F}'' = \mathfrak{F}' \circ \mathfrak{F}$ [which is, *a priori*, an abuse of notation, since there may exist *many* composites of $\mathfrak{F}$ with $\mathfrak{F}'$ — cf. (c) below]; we shall use similar terminology and notation for 1-specimens in specific ZFC-models.

(c) Given a pair of 1-specimens $F : E \to E'$, $F' : E' \to E''$ of $\mathfrak{S}_1$ in some ZFC-model, there *exists a unique composite* $F'' : E \to E''$ of F with F' in the given ZFC-model.

(d) Composition of 1-specimens $F : E \to E'$, $F' : E' \to E''$, $F'' : E'' \to E'''$ of $\mathfrak{S}_1$ in a ZFC-model is *associative*.

(e) For any 0-specimen E of $\mathfrak{S}_0$ in a ZFC-model, there exists a [necessarily unique] 1-specimen $F : E \to E$ of $\mathfrak{S}_1$ [in the given ZFC-model] — which we shall refer to as the *identity* 1-*specimen* id_E of E — such that for any 1-specimens $F' : E' \to E$, $F'' : E \to E''$ of $\mathfrak{S}_1$ [in the given ZFC-model] we have $F \circ F' = F'$, $F'' \circ F = F''$.

If, moreover, it holds, in any ZFC-model, that for any two 0-specimens E, E' of $\mathfrak{S}_0$, the 1-specimens $F : E \to E'$ of $\mathfrak{S}_1$ [i.e., the 1-specimens of $\mathfrak{S}_1$ with domain E and codomain E'] form a *set*, then we shall refer to $\mathfrak{S}_1$ as 1-*small*.

(iii) A *species* $\mathfrak{S}$ is defined to be a pair consisting of a 0-species $\mathfrak{S}_0$ and a 1-species $\mathfrak{S}_1$ acting on $\mathfrak{S}_0$. Fix a species $\mathfrak{S} = (\mathfrak{S}_0, \mathfrak{S}_1)$. Let $i \in \{0, 1\}$. Then we shall refer to an i-specimen of $\mathfrak{S}_i$ as an *i-specimen of* $\mathfrak{S}$. We shall refer to a 0-specimen (respectively, 1-specimen) of $\mathfrak{S}$ as a *species-object* (respectively, a *species-morphism*) of $\mathfrak{S}$. We shall say that $\mathfrak{S}$ is *i-small* if $\mathfrak{S}_i$ is i-small. We shall refer to a species-morphism $F : E \to E'$ as a *species-isomorphism* if there exists a species-morphism $F' : E' \to E$ such that the composites $F \circ F'$, $F' \circ F$ are *identity* species-morphisms; in this situation, we shall say that E, E' are *species-isomorphic*. [Thus, one verifies immediately that *composites* of *species-isomorphisms* are species-isomorphisms.] We shall refer to a species-isomorphism whose domain and codomain are equal as a *species-automorphism*. We shall refer to as *model-free* [cf. Remark 3.1.1 below] an i-specimen of $\mathfrak{S}$ equipped with a description via a *set-theoretic formula* that is "*independent* of the ZFC-model in which it is given" in the sense that for any pair of universes V_1, V_2 of some ZFC-model such that $V_1 \in V_2$, the set-theoretic formula determines the *same* i-specimen of $\mathfrak{S}$, whether interpreted relative to the ZFC-model determined by V_1 or the ZFC-model determined by V_2.

(iv) We shall refer to as the *category determined by* $\mathfrak{S}$ in a ZFC-model the *category* whose objects are the *species-objects* of $\mathfrak{S}$ in the given ZFC-model and whose arrows are the *species-morphisms* of $\mathfrak{S}$ in the given ZFC-model. [One verifies immediately that this description does indeed determine a category.]

Remark 3.1.1. We observe that any of the familiar descriptions of $\mathbb{N}$ [cf., e.g., [Drk], Chapter 2, Definition 2.3], $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}_p$, or $\mathbb{R}$, for instance, yield *species* [all of whose species-morphisms are identity species-morphisms] each of which has a *unique* species-object in any given ZFC-model. Such species are *not to be confused* with such species as the species of "monoids isomorphic to $\mathbb{N}$ and monoid isomorphisms", which admits *many species-objects* [all of which are species-isomorphic] in any ZFC-model. On the other hand, the set-theoretic formula used, for instance, to define the former "species $\mathbb{N}$" may be applied to define a *"model-free species-object* $\mathbb{N}$*"* of the latter "species of monoids isomorphic to $\mathbb{N}$".

Remark 3.1.2.

(i) It is important to remember when working with species that

> the **essence** *of a species* lies *not in the specific sets* that occur as species-objects or species-morphisms of the species in various ZFC-models, but rather in the **collection of rules**, i.e., *set-theoretic formulas*, that govern the construction of such sets in an *unspecified, "indeterminate" ZFC-model.*

Put another way, the emphasis in the theory of species lies in the *programs* — i.e., **"software"** — that yield the desired output data, *not on the output data itself.* From this point of view, one way to describe the various set-theoretic formulas that constitute a species is as a *"deterministic algorithm"* [a term suggested to the author by M. Kim] for constructing the sets to be considered.

(ii) One interesting point of view that arose in discussions between the author and F. Kato is the following. The relationship between the classical *"set-theoretic"* approach to discussing mathematics — in which *specific sets* play a central role — and the *"species-theoretic"* approach considered here — in which the **rules**, given by set-theoretic formulas for constructing the sets of interest [i.e., not specific sets themselves!], play a central role — may be regarded as *analogous* to the relationship between *classical approaches to algebraic varieties* — in which specific sets of solutions of polynomial equations in an algebraically closed field play a central role — and *scheme theory* — in which the functor determined by a scheme, i.e., the polynomial equations, or "rules", that determine solutions, as opposed to specific sets of solutions themselves, play a central role. That is to say, in summary:

$$\text{[set-th. approach : species-th. approach]} \quad \longleftrightarrow \quad \text{[varieties : schemes]}$$

A similar analogy — i.e., of the form

$$\begin{aligned}&\text{[set-th. approach : species-th. approach]}\\ &\qquad\qquad\longleftrightarrow \quad \text{[groups of specific matrices : abstract groups]}\end{aligned}$$

— may be made to the notion of an "abstract group", as opposed to a "group of specific matices". That is to say, just as a "group of specific matrices may be thought of as a *specific representation* of an "abstract group", the category of objects determined by a species in a specific ZFC-model may be thought of as a *specific representation* of an "abstract species".

(iii) If, in the context of the discussion of (i), (ii), one tries to form a sort of *quotient*, in which "programs" that yield the same sets as "output data" are *identified*, then one must contend with the resulting *indeterminacy*, i.e., working with programs is only well-defined up to internal modifications of the programs in question that does not affect the final output. This leads to somewhat *intractable problems* concerning the internal structure of such programs — a topic that lies well beyond the scope of the present work.

Remark 3.1.3.

(i) Typically, in the discussion to follow, we shall not write out explicitly the various set-theoretic formulas involved in the definition of a species. Rather, it is to be understood that the set-theoretic formulas to be used are those arising from the *conventional descriptions* of the mathematical objects involved. When applying such conventional descriptions, however, it is important to check that they are *well-defined* and *do* **not** *depend* upon the use of **arbitrary choices** that are not describable via well-defined set-theoretic formulas.

(ii) The fact that the data involved in a species is given by abstract *set-theoretic formulas* imparts a certain **canonicality** to the mathematical notion constituted by the species, a canonicality that is **not shared**, for instance, by mathematical objects whose construction depends on an **invocation of the axiom of choice** in some particular ZFC-model [cf. the discussion of (i) above]. Moreover, by furnishing a stock of such "canonical notions", the theory of species allows one, in effect, to *compute the* **extent of deviation** of various *"non-canonical objects"* [i.e., whose construction depends upon the invocation of the axiom of choice!] from a sort of *"canonical norm"*.

Remark 3.1.4. Note that because the data involved in a species is given by abstract *set-theoretic formulas*, the mathematical notion constituted by the species is **immune** to, i.e., unaffected by, **extensions of the universe** — i.e., such as the ascending chain $V_0 \in V_1 \in V_2 \in V_3 \in \ldots \in V_n \in \ldots \in V$ that appears in the discussion preceding Definition 3.1 — in which one works. This is the sense in which we apply the term **"inter-universal"**. That is to say, *"inter-universal geometry"* allows one to relate the "geometries" that occur in distinct universes.

Remark 3.1.5. Similar remarks to the remarks made in Remarks 3.1.2, 3.1.3, and 3.1.4 concerning the significance of working with *set-theoretic formulas* may be made with regard to the notions of *mutations*, *morphisms of mutations*, *mutation-histories*, *observables*, and *cores* to be introduced in Definition 3.2 below.

One fundamental example of a species is the following.

Example 3.2. **Categories.** The notions of a [small] category and an isomorphism class of [covariant] functors between two given [small] categories yield an example of a *species*. That is to say, at a set-theoretic level, one may think of a [small] *category* as, for instance, a set of arrows, together with a set of composition relations, that satisfies certain properties; one may think of a [covariant] *functor* between [small] categories as the set given by the graph of the map on arrows determined by the functor [which satisfies certain properties]; one may think of an *isomorphism class of functors* as a collection of such graphs, i.e., the graphs determined by the functors in the isomorphism class, which satisfies certain properties. Then one has *"dictionaries"*

$$\text{0-species} \quad \longleftrightarrow \quad \text{the notion of a category}$$

$$\text{1-species} \quad \longleftrightarrow \quad \text{the notion of an isomorphism class of functors}$$

at the level of *notions* and

$$\text{a 0-specimen} \quad \longleftrightarrow \quad \text{a particular [small] category}$$

$$\text{a 1-specimen} \quad \longleftrightarrow \quad \text{a particular isomorphism class of functors}$$

at the level of *specific mathematical objects* in a specific ZFC-model. Moreover, one verifies easily that species-isomorphisms between 0-species correspond to isomorphism classes of equivalences of categories in the usual sense.

Remark 3.2.1. Note that in the case of Example 3.2, one could also define a notion of *"2-species"*, *"2-specimens"*, etc., via the notion of an "isomorphism of functors", and then take the 1-species under consideration to be the notion of a functor [i.e., not an isomorphism class of functors]. Indeed, more generally, one could define a notion of "n-species" for arbitrary integers $n \geq 1$. Since, however, this approach would only serve to add an *unnecessary level of complexity* to the theory, we choose here to take the approach of working with "functors considered up to isomorphism".

Definition 3.3. Let $\mathfrak{S} = (\mathfrak{S}_0, \mathfrak{S}_1)$; $\underline{\mathfrak{S}} = (\underline{\mathfrak{S}}_0, \underline{\mathfrak{S}}_1)$; $\underline{\underline{\mathfrak{S}}} = (\underline{\underline{\mathfrak{S}}}_0, \underline{\underline{\mathfrak{S}}}_1)$ be *species*.

(i) A *mutation* $\mathfrak{M} : \mathfrak{S} \rightsquigarrow \underline{\mathfrak{S}}$ is defined to be a collection of *set-theoretic formulas* Ψ_0, Ψ_1 satisfying the following properties:

(a) Ψ_0 is a set-theoretic formula

$$\Psi_0(\mathfrak{E}, \underline{\mathfrak{E}})$$

involving a collection of species-data $\mathfrak{E}$ for $\mathfrak{S}_0$ and a collection of species-data $\underline{\mathfrak{E}}$ for $\underline{\mathfrak{S}}_0$; in this situation, we shall write $\mathfrak{M}(\mathfrak{E})$ for $\underline{\mathfrak{E}}$. Moreover, if, in some ZFC-model, $E \in \mathfrak{S}_0$, then we require that there *exist a unique* $\underline{E} \in \underline{\mathfrak{S}}_0$ such that $\Psi_0(E, \underline{E})$ holds; in this situation, we shall write $\mathfrak{M}(E)$ for $\underline{E}$.

(b) Ψ_1 is a set-theoretic formula

$$\Psi_1(\mathfrak{E}, \mathfrak{E}', \mathfrak{F}, \underline{\mathfrak{F}})$$

involving a collection of species-data $\mathfrak{F} : \mathfrak{E} \to \mathfrak{E}'$ for $\mathfrak{S}_1$ and a collection of species-data $\underline{\mathfrak{F}} : \underline{\mathfrak{E}} \to \underline{\mathfrak{E}}'$ for $\underline{\mathfrak{S}}_1$, where $\underline{\mathfrak{E}} = \mathfrak{M}(\mathfrak{E})$, $\underline{\mathfrak{E}}' = \mathfrak{M}(\mathfrak{E}')$; in this situation, we shall write $\mathfrak{M}(\mathfrak{F})$ for $\underline{\mathfrak{F}}$. Moreover, if, in some ZFC-model, $(F : E \to E') \in \mathfrak{S}_1$, then we require that there *exist a unique* $(\underline{F} : \underline{E} \to \underline{E}') \in \underline{\mathfrak{S}}_1$ such that $\Psi_0(E, E', F, \underline{F})$ holds; in this situation, we shall write $\mathfrak{M}(F)$ for $\underline{F}$. Finally, we require that the assignment $F \mapsto \mathfrak{M}(F)$ be compatible with *composites* and map *identity* species-morphisms of $\mathfrak{S}$ to identity species-morphisms of $\underline{\mathfrak{S}}$. In particular, if one fixes a ZFC-model, then $\mathfrak{M}$ determines a *functor* from the category determined by $\mathfrak{S}$ in the given ZFC-model to the category determined by $\underline{\mathfrak{S}}$ in the given ZFC-model.

There are evident notions of *"composition of mutations"* and *"identity mutations"*.

(ii) Let $\mathfrak{M}, \mathfrak{M}' : \mathfrak{S} \rightsquigarrow \underline{\mathfrak{S}}$ be *mutations*. Then a *morphism of mutations* $\mathfrak{Z} : \mathfrak{M} \to \mathfrak{M}'$ is defined to be a *set-theoretic formula* Ξ satisfying the following properties:

(a) Ξ is a set-theoretic formula

$$\Xi(\mathfrak{E}, \underline{\mathfrak{F}})$$

involving a collection of species-data $\mathfrak{E}$ for $\mathfrak{S}_0$ and a collection of species-data $\underline{\mathfrak{F}} : \mathfrak{M}(\mathfrak{E}) \to \mathfrak{M}'(\mathfrak{E})$ for $\mathfrak{S}_1$; in this situation, we shall write $\mathfrak{Z}(\mathfrak{E})$ for $\underline{\mathfrak{F}}$. Moreover, if, in some ZFC-model, $E \in \mathfrak{S}_0$, then we require that there *exist a unique* $\underline{F} \in \underline{\mathfrak{S}}_1$ such that $\Xi(E, \underline{F})$ holds; in this situation, we shall write $\mathfrak{Z}(E)$ for $\underline{F}$.

(b) Suppose, in some ZFC-model, that $F : E_1 \to E_2$ is a species-morphism of $\mathfrak{S}$. Then one has an equality of composite species-morphisms $\mathfrak{M}'(F) \circ \mathfrak{Z}(E_1) = \mathfrak{Z}(E_2) \circ \mathfrak{M}(F) : \mathfrak{M}(E_1) \to \mathfrak{M}'(E_2)$. In particular, if one fixes a ZFC-model, then a morphism of mutations $\mathfrak{M} \to \mathfrak{M}'$ determines a *natural transformation* between the functors determined by $\mathfrak{M}$, $\mathfrak{M}'$ in the ZFC-model — cf. (i).

There are evident notions of *"composition of morphisms of mutations"* and *"identity morphisms of mutations"*. If it holds that for every species-object E of $\mathfrak{S}$, $\mathfrak{Z}(E)$ is a *species-isomorphism*, then we shall refer to $\mathfrak{Z}$ as an *isomorphism of mutations*. In particular, one verifies immediately that $\mathfrak{Z}$ is an isomorphism of mutations if and only if there exists a morphism of mutations $\mathfrak{Z}' : \mathfrak{M}' \to \mathfrak{M}$ such that the composite morphisms of mutations $\mathfrak{Z}' \circ \mathfrak{Z} : \mathfrak{M} \to \mathfrak{M}$, $\mathfrak{Z} \circ \mathfrak{Z}' : \mathfrak{M}' \to \mathfrak{M}'$ are the respective *identity* morphisms of the mutations $\mathfrak{M}$, $\mathfrak{M}'$.

(iii) Let $\mathfrak{M} : \mathfrak{S} \rightsquigarrow \underline{\mathfrak{S}}$ be a *mutation*. Then we shall say that $\mathfrak{M}$ is a *mutation-equivalence* if there exists a mutation $\mathfrak{M}' : \underline{\mathfrak{S}} \rightsquigarrow \mathfrak{S}$, together with isomorphisms of mutations between the composites $\mathfrak{M} \circ \mathfrak{M}'$, $\mathfrak{M}' \circ \mathfrak{M}$ and the respective identity mutations. In this situation, we shall say that $\mathfrak{M}$, $\mathfrak{M}'$ are *mutation-quasi-inverses* to one another. Note that for any two given species-objects in the domain species of a mutation-equivalence, the *mutation-equivalence* induces a *bijection* between the collection of *species-isomorphisms* between the two given species-objects [of the domain species] and the collection of species-isomorphisms between the two species-objects [of the codomain species] obtained by applying the mutation-equivalence to the two given species-objects.

(iv) Let $\vec{\Gamma}$ be an *oriented graph*, i.e., a graph Γ, which we shall refer to as the *underlying graph* of $\vec{\Gamma}$, equipped with the additional data of a total ordering, for each edge e of Γ, on the set [of cardinality 2] of *branches* of e [cf., e.g., [AbsTopIII], §0]. Then we define a *mutation-history* $\mathfrak{H} = (\vec{\Gamma}, \mathfrak{S}^*, \mathfrak{M}^*)$ *[indexed by* $\vec{\Gamma}$*]* to be a collection of data as follows:

(a) for each vertex v of $\vec{\Gamma}$, a *species* $\mathfrak{S}^v$;

(b) for each edge e of $\vec{\Gamma}$, running from a vertex v_1 to a vertex v_2, a *mutation* $\mathfrak{M}^e : \mathfrak{S}^{v_1} \rightsquigarrow \mathfrak{S}^{v_2}$.

In this situation, we shall refer to the vertices, edges, and branches of $\vec{\Gamma}$ as vertices, edges, and branches of $\mathfrak{H}$. Thus, the notion of a "mutation-history" may be thought of as a *species-theoretic* version of the notion of a "diagram of categories" given in [AbsTopIII], Definition 3.5, (i).

(v) Let $\mathfrak{H} = (\vec{\Gamma}, \mathfrak{S}^*, \mathfrak{M}^*)$ be a *mutation-history*; $\underline{\mathfrak{S}}$ a *species*. For simplicity, we assume that the underlying graph of $\vec{\Gamma}$ is *simply connected*. Then we shall refer to as a(n) *[*$\underline{\mathfrak{S}}$*-valued] covariant* (respectively, *contravariant*) *observable* $\mathfrak{V}$ of the mutation-history $\mathfrak{H}$ a collection of data as follows:

(a) for each vertex v of $\vec{\Gamma}$, a mutation $\mathfrak{V}^v : \mathfrak{S}^v \to \underline{\mathfrak{S}}$, which we shall refer to as the *observation mutation* at v;

(b) for each edge e of $\vec{\Gamma}$, running from a vertex v_1 to a vertex v_2, a *morphism of mutations* $\mathfrak{V}^e : \mathfrak{V}^{v_1} \to \mathfrak{V}^{v_2} \circ \mathfrak{M}^e$ (respectively, $\mathfrak{V}^e : \mathfrak{V}^{v_2} \circ \mathfrak{M}^e \to \mathfrak{V}^{v_1}$).

If $\mathfrak{V}$ is a covariant observable such that all of the morphisms of mutations "$\mathfrak{V}^e$" are *isomorphisms of mutations*, then we shall refer to the covariant observable $\mathfrak{V}$ as a *core*. Thus, one may think of a core $\mathfrak{C}$ of a mutation-history as lying *"under"* the entire mutation-history in a *"uniform fashion"*. Also, we shall refer to the "property [of an observable] of being a core" as the *"coricity"* of the observable. Finally, we note that the notions of an "observable" and a "core" given here may be thought of as simplified, *species-theoretic* versions of the notions of "observable" and "core" given in [AbsTopIII], Definition 3.5, (iii).

Remark 3.3.1.

(i) One well-known consequence of the *axiom of foundation* of axiomatic set-theory is the assertion that **"∈-loops"**

$$a \in b \in c \in \ldots \in a$$

can *never occur* in the set theory in which one works. On the other hand, there are many situations in mathematics in which one wishes to somehow **"identify"** mathematical objects that arise at *higher levels* of the $\in$-structure of the set theory under consideration with mathematical objects that arise at *lower levels* of this $\in$-structure. In some sense, the notions of a *"set"* and of a *"bijection of sets"* allow one to achieve such "identifications". That is to say, the mathematical objects at both higher and lower levels of the $\in$-structure constitute examples of the *same mathematical notion of a "set"*, so that one may consider "bijections of sets" between those sets without violating the axiom of foundation. In some sense, the notion of a **species** may be thought of as a natural *extension* of this observation. That is to say,

> the notion of a "species" allows one to consider, for instance, *species-isomorphisms* between species-objects that occur at *different levels* of the $\in$-structure of the set theory under consideration — i.e., roughly speaking, to **"simulate $\in$-loops"** — *without violating the axiom of foundation.*

Moreover, typically the species-objects at higher levels of the $\in$-structure occur as the result of executing the *mutations* that arise in some sort of **mutation-history**

$$\dots \rightsquigarrow \mathfrak{S} \rightsquigarrow \underline{\mathfrak{S}} \rightsquigarrow \underline{\underline{\mathfrak{S}}} \rightsquigarrow \dots \rightsquigarrow \mathfrak{S} \rightsquigarrow \dots$$

— e.g., the *"output species-objects"* of the "$\mathfrak{S}$" on the *right* that arise from applying various mutations to the *"input species-objects"* of the "$\mathfrak{S}$" on the *left.*

(ii) In the context of constructing *"loops"* in a mutation-history as in the final display of (i), we observe that

> the **simpler** the structure of the **species** involved, the **easier** it is to construct **"loops"**.

It is for this reason that species such as the species determined by the notion of a **category** [cf. Example 3.2] are easier to work with, from the point of view of constructing "loops", than more complicated species such as the species determined by the notion of a *scheme.* This is one of the *principal motivations* for the *"geometry of categories"* — of which *"absolute anabelian geometry"* is the special case that arises when the categories involved are Galois categories — i.e., for the theory of *representing scheme-theoretic geometries via categories* [cf., e.g., the Introductions of [LgSch], [ArLgSch], [SemiAnbd], [Cusp], [FrdI]]. At a more concrete level, the utility of working with categories to reconstruct objects that occurred at lower levels of some sort of "series of constructions" [cf. the mutation-history of the final display of (i)!] may be seen in the *"reconstruction of the underlying scheme"*, given in [LgSch], Corollary 2.15, from a certain category constructed from a log scheme, as well as in the theory of *"slim exponentiation"* discussed in the Appendix to [FrdI].

(iii) Again in the context of mutation-histories such as the one given in the final display of (i), although one may, on certain occasions, wish to apply various mutations that *fundamentally alter* the structure of the mathematical objects involved and hence give rise to *"output species-objects"* of the "$\mathfrak{S}$" on the *right* that are related in a *highly nontrivial fashion* to the *"input species-objects"* of the "$\mathfrak{S}$" on the *left*, it is also of interest to consider

> *"portions"* of the various mathematical objects that occur that are left **unaltered** by the various mutations that one applies.

This is precisely the reason for the introduction of the notion of a *core* of a mutation-history. One important consequence of the construction of various cores associated to a mutation-history is that often

> one may apply various cores associated to a mutation-history to **describe**, by means of **non-coric observables**, the portions of the various mathematical objects that occur which *are* **altered** by the various mutations that one applies *in terms of* the **unaltered** portions, i.e., **cores**.

Indeed, this point of view plays a *central role* in the theory of the present series of papers — cf. the discussion of Remark 3.6.1, (ii), below.

Remark 3.3.2. One somewhat *naive* point of view that constituted one of the original motivations for the author in the development of theory of the present series of papers is the following. In the classical theory of schemes, when considering **local systems** on a scheme, there is no reason to restrict oneself to considering local systems valued in, say, modules over a finite ring. If, moreover, there is no reason to make such a restriction, then one is naturally led to consider, for instance, local systems of *schemes* [cf., e.g., the theory of the "Galois mantle" in [pTeich]], or, indeed, local systems of **"entire set-theoretic mathematical theaters"**. One may then ask what happens if one tries to consider local systems on the schemes that occur as fibers of a local system of schemes. [More concretely, if X is, for instance, a connected scheme, then one may consider local systems $\mathcal{X}$ over X whose *fibers* are isomorphic to X; then one may repeat this process, by considering such local systems *over each fiber* of the local system $\mathcal{X}$ on X, etc.] In this way, one is eventually led to the consideration of **"systems of nested local systems"** — i.e., a local system over a local system over a local system, etc. It is precisely this point of view that underlies the notion of *"successive iteration of a given mutation-history"*, relative to the terminology formulated in the present §3. If, moreover, one thinks of such "successive iterates of a given mutation-history" as being a sort of abstraction of the naive idea of a "system of nested local systems", then the notion of a **core** may be thought of as a sort of mathematical object that is *invariant* with respect to the application of the operations that gave rise to the "system of nested local systems".

Example 3.4. Topological Spaces and Fundamental Groups.

(i) One verifies easily that the notions of a *topological space* and a *continuous map* between topological spaces determine an example of a *species* $\mathfrak{S}^{\mathrm{top}}$. In a similar vein, the notions of a *universal covering* $\widetilde{X} \to X$ of a pathwise connected topological space X and a *continuous map* between such universal coverings $\widetilde{X} \to X$, $\widetilde{Y} \to Y$ [i.e., a pair of compatible continuous maps $\widetilde{X} \to \widetilde{Y}$, $X \to Y$], considered up to composition with a *deck transformation* of the universal covering $\widetilde{Y} \to Y$, determine an example of a *species* $\mathfrak{S}^{\text{u-top}}$. We leave to the reader the routine task of writing out the various *set-theoretic formulas* that define the species structures of $\mathfrak{S}^{\text{top}}$, $\mathfrak{S}^{\text{u-top}}$. Here, we note that at a set-theoretic level, the species-morphisms of $\mathfrak{S}^{\text{u-top}}$ are *collections* of continuous maps [between two given universal coverings], any two of which differ from one another by composition with a deck transformation.

(ii) One verifies easily that the notions of a *group* and an *outer homomorphism* between groups [i.e., a homomorphism considered up to composition with an inner automorphism of the codomain group] determine an example of a *species* $\mathfrak{S}^{\text{gp}}$. We leave to the reader the routine task of writing out the various *set-theoretic formulas* that define the species structure of $\mathfrak{S}^{\text{gp}}$. Here, we note that at a set-theoretic level, the species-morphisms of $\mathfrak{S}^{\text{gp}}$ are *collections* of homomorphisms [between two given groups], any two of which differ from one another by composition with an inner automorphism.

(iii) Now one verifies easily that the assignment

$$(\widetilde{X} \to X) \quad \mapsto \quad \text{Aut}(\widetilde{X}/X)$$

— where $(\widetilde{X} \to X)$ is a species-object of $\mathfrak{S}^{\text{u-top}}$, and $\text{Aut}(\widetilde{X}/X)$ denotes the group of deck transformations of the universal covering $\widetilde{X} \to X$ — determines a *mutation* $\mathfrak{S}^{\text{u-top}} \rightsquigarrow \mathfrak{S}^{\text{gp}}$. That is to say, the *"fundamental group"* may be thought of as a sort of mutation.

Example 3.5. Absolute Anabelian Geometry.

(i) Let $\mathcal{S}$ be a class of connected normal schemes that is closed under isomorphism [of schemes]. Suppose that there exists a set $E_{\mathcal{S}}$ of schemes describable by a *set-theoretic formula* with the property that every scheme of $\mathcal{S}$ is isomorphic to some scheme belonging to $E_{\mathcal{S}}$. Then just as in the case of universal coverings of topological spaces discussed in Example 3.4, (i), one verifies easily, by applying the set-theoretic formula describing $E_{\mathcal{S}}$, that the *universal pro-finite étale coverings* $\widetilde{X} \to X$ of schemes X belonging to $\mathcal{S}$ and *isomorphisms* of such coverings considered up to composition with a *deck transformation* give rise to a *species* $\mathfrak{S}^{\mathcal{S}}$.

(ii) Let $\mathcal{G}$ be a class of topological groups that is closed under isomorphism [of topological groups]. Suppose that there exists a set $E_{\mathcal{G}}$ of topological groups describable by a *set-theoretic formula* with the property that every topological group of $\mathcal{G}$ is isomorphic to some topological group belonging to $E_{\mathcal{G}}$. Then just as in the case of abstract groups discussed in Example 3.4, (ii), one verifies easily, by applying the set-theoretic formula describing $E_{\mathcal{G}}$, that *topological groups* belonging to $\mathcal{G}$ and [bi-continuous] *outer isomorphisms* between such topological groups give rise to a *species* $\mathfrak{S}^{\mathcal{G}}$.

(iii) Let $\mathcal{S}$ be as in (i). Then for an appropriate choice of $\mathcal{G}$, by associating to a universal pro-finite étale covering the resulting group of deck transformations, one obtains a *mutation*

$$\Pi : \quad \mathfrak{S}^{\mathcal{S}} \quad \rightsquigarrow \quad \mathfrak{S}^{\mathcal{G}}$$

[cf. Example 3.4, (iii)]. Then one way to define the notion that the schemes belonging to the class $\mathcal{S}$ are **"[absolute] anabelian"** is to require the specification of a *mutation*

$$\mathbb{A}: \quad \mathfrak{S}^{\mathcal{G}} \quad \rightsquigarrow \quad \mathfrak{S}^{\mathcal{S}}$$

which forms a *mutation-quasi-inverse* to Π. Here, we note that the existence of the bijections [i.e., *"fully faithfulness"*] discussed in Definition 3.3, (iii), is, in essence, the condition that is usually taken as the definition of "anabelian". By contrast, the *species-theoretic* approach of the present discussion may be thought of as an explicit mathematical formulation of the **algorithmic** *approach to [absolute] anabelian geometry* discussed in the Introduction to [AbsTopI].

(iv) The framework of [absolute] anabelian geometry [cf., e.g., the framework discussed above] gives a good example of the importance of specifying *precisely* **what species** *one is working with* in a given "series of constructions" [cf., e.g., the mutation-history of the final display of Remark 3.3.1, (i)]. That is to say, there is a quite substantial difference between working with a

profinite group in its sole capacity as a profinite group

and working with the same profinite group — which may happen to arise as the *étale fundamental group* of some scheme! —

> *regarded as being equipped with various data that arise from the construction of the profinite group as the étale fundamental group of some scheme.*

It is precisely this sort of issue that constituted one of the original motivations for the author in the development of the theory of species presented here.

Example 3.6. The Étale Site and Frobenius.

(i) Let p be a *prime number*. If S is a *reduced* scheme over $\mathbb{F}_p$, then denote by $S^{(p)}$ the scheme with the same topological space as S, but whose structure sheaf is given by the *subsheaf*

$$\mathcal{O}_{S^{(p)}} \stackrel{\text{def}}{=} (\mathcal{O}_S)^p \subseteq \mathcal{O}_S$$

of p-th powers of sections of S. Thus, the natural inclusion $\mathcal{O}_{S^{(p)}} \hookrightarrow \mathcal{O}_S$ induces a morphism $\Phi_S : S \to S^{(p)}$. Moreover, "raising to the p-th power" determines a *natural isomorphism* $\alpha_S : S^{(p)} \xrightarrow{\sim} S$ such that the resulting composite $\alpha_S \circ \Phi_S : S \to S$ is the *Frobenius morphism* of S. Write

$$\mathfrak{S}^{p\text{-sch}}$$

for the *species* of reduced schemes over $\mathbb{F}_p$ and morphisms of schemes. Note that by considering, for instance, [necessarily quasi-affine!] étale morphisms of finite presentation $T \to S$ equipped with factorizations $T|_U \subseteq \mathbb{A}^N_U \to U$ for each affine open $U \subseteq S$ [where $\mathbb{A}^N_U$ denotes a "standard copy of affine N-space over U", for some integer $N \geq 1$; the "$\subseteq$" exhibits $T|_U$ as a finitely presented subscheme of $\mathbb{A}^N_U$], one may construct an *assignment*

$$S \ \mapsto \ S_{\text{ét}}$$

that maps a species-object S of $\mathfrak{S}^{p\text{-sch}}$ to the category $S_{\text{ét}}$ of such étale morphisms of finite presentation $T \to S$ and S-morphisms — i.e., "the *small étale site* of S" — in such a way that the assignment $S \mapsto S_{\text{ét}}$ is *contravariantly functorial* with respect to species-morphisms $S_1 \to S_2$ of $\mathfrak{S}^{p\text{-sch}}$, and, moreover, may be described via *set-theoretic formulas.* Thus, such an assignment determines an *"étale site mutation"*

$$\mathfrak{M}^{\text{ét}} : \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \mathfrak{S}^{\text{cat}}$$

— where we write $\mathfrak{S}^{\text{cat}}$ for the *species* of categories and isomorphism classes of contravariant functors [cf. Example 3.2]. Another natural assignment in the present context is the assignment

$$S \mapsto S^{\text{pf}}$$

which maps S to its *perfection* S^{pf}, i.e., the scheme determined by taking the inverse limit of the inverse system $\ldots \to S \to S \to S$ obtained by considering *iterates* of the *Frobenius morphism* of S. Thus, by considering the final copy of "S" in this inverse system, one obtains a natural morphism $\beta_S : S^{\text{pf}} \to S$. Finally, one obtains a *"perfection mutation"*

$$\mathfrak{M}^{\text{pf}} : \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \mathfrak{S}^{p\text{-sch}}$$

by considering the *set-theoretic formulas* underlying the assignment $S \mapsto S^{\text{pf}}$.

(ii) Write

$$\mathfrak{F}^{p\text{-sch}} : \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \mathfrak{S}^{p\text{-sch}}$$

for the **"Frobenius mutation"** obtained by considering the *set-theoretic formulas* underlying the assignment $S \mapsto S^{(p)}$. Thus, one may formulate the well-known *"invariance of the étale site under Frobenius"* [cf., e.g., [FK], Chapter I, Proposition 3.16] as the statement that the **"étale site mutation"** $\mathfrak{M}^{\text{ét}}$ exhibits $\mathfrak{S}^{\text{cat}}$ as a **core** — i.e., an *"invariant piece"* — of the **"Frobenius mutation-history"**

$$\ldots \rightsquigarrow \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \ldots$$

determined by the *"Frobenius mutation"* $\mathfrak{F}^{p\text{-sch}}$. In this context, we observe that the **"perfection mutation"** $\mathfrak{M}^{\text{pf}}$ also yields a **core** — i.e., another *"invariant piece"* — of the Frobenius mutation-history. On the other hand, the natural morphism $\Phi_S : S \to S^{(p)}$ may be interpreted as a *covariant* **observable** of this mutation-history whose *observation mutations* are the identity mutations $\text{id}_{\mathfrak{S}^{p\text{-sch}}} : \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \mathfrak{S}^{p\text{-sch}}$. Since Φ_S is not, in general, an isomorphism, it follows that this observable constitutes an example of an **non-coric** observable. Nevertheless, the natural morphism $\beta_S : S^{\text{pf}} \to S$ may be interpreted as a *morphism of mutations* $\mathfrak{M}^{\text{pf}} \to \text{id}_{\mathfrak{S}^{p\text{-sch}}}$ that serves to relate the *non-coric observable* just considered to the *coric observable* arising from $\mathfrak{M}^{\text{pf}}$.

(iii) One may also develop a version of (i), (ii) for *log schemes*; we leave the routine details to the interested reader. Here, we pause to mention that the theory of log schemes motivates the following **"combinatorial monoid-theoretic"** version of the *non-coric observable* on the *Frobenius mutation-history* of (ii). Write

$$\mathfrak{S}^{\text{mon}}$$

for the species of *torsion-free abelian monoids* and *morphisms of monoids*. If M is a species-object of $\mathfrak{S}^{\mathrm{mon}}$, then write $M^{(p)} \overset{\mathrm{def}}{=} p \cdot M \subseteq M$. Then the assignment $M \mapsto M^{(p)}$ determines a **"monoid-Frobenius mutation"**

$$\mathfrak{F}^{\mathrm{mon}} : \mathfrak{S}^{\mathrm{mon}} \rightsquigarrow \mathfrak{S}^{\mathrm{mon}}$$

and hence a **"monoid-Frobenius mutation-history"**

$$\ldots \rightsquigarrow \mathfrak{S}^{\mathrm{mon}} \rightsquigarrow \mathfrak{S}^{\mathrm{mon}} \rightsquigarrow \ldots$$

which is equipped with a **non-coric** *contravariant* **observable** determined by the natural inclusion morphism $M^{(p)} \hookrightarrow M$ and the observation mutations given by the identity mutations $\mathrm{id}_{\mathfrak{S}^{\mathrm{mon}}} : \mathfrak{S}^{\mathrm{mon}} \rightsquigarrow \mathfrak{S}^{\mathrm{mon}}$. On the other hand, the *p-perfection* M^{pf} of M, i.e., the inductive limit of the inductive system $M \hookrightarrow M \hookrightarrow M \hookrightarrow \ldots$ obtained by considering the inclusions given by *multiplying by* p, gives rise to a **"monoid-p-perfection mutation"**

$$\mathfrak{F}^{\mathrm{pf}} : \mathfrak{S}^{\mathrm{mon}} \rightsquigarrow \mathfrak{S}^{\mathrm{mon}}$$

which may be interpreted as a **core** of the monoid-Frobenius mutation-history. Finally, the natural inclusion of monoids $M \hookrightarrow M^{\mathrm{pf}}$ may be interpreted as a *morphism of mutations* $\mathrm{id}_{\mathfrak{S}^{\mathrm{mon}}} \to \mathfrak{F}^{\mathrm{pf}}$ that serves to relate the *non-coric observable* just considered to the *coric observable* arising from $\mathfrak{F}^{\mathrm{pf}}$.

Remark 3.6.1.

(i) The various constructions of Example 3.6 may be thought of as providing, in the case of the phenomena of *"invariance of the étale site under Frobenius"* and *"invariance of the perfection under Frobenius"*, a *"species-theoretic intepretation"* — i.e., via consideration of

"coric" versus **"non-coric"** observables

— of the difference between **"étale-type"** and **"Frobenius-type"** structures [cf. the discussion of [FrdI], §I4]. This sort of approach via *"combinatorial patterns"* to expressing the difference between "étale-type" and "Frobenius-type" structures plays a *central role* in the theory of the present series of papers. Indeed, the mutation-histories and cores considered in Example 3.6, (ii), (iii), may be thought of as the **underlying motivating examples** for the theory of both

- the **vertical lines**, i.e., consisting of **$\mathfrak{log}$-links**, and
- the **horizontal lines**, i.e., consisting of **$\Theta^{\times\boldsymbol{\mu}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-links**,

of the **log-theta-lattice** [cf. [IUTchIII], Definitions 1.4, 3.8]. Finally, we recall that this approach to understanding the $\mathfrak{log}$-links may be seen in the introduction of the terminology of *"observables"* and *"cores"* in [AbsTopIII], Definition 3.5, (iii).

(ii) Example 3.6 also provides a good example of the *important theme* [cf. the discussion of Remark 3.3.1, (iii)] of *describing* **non-coric** *data in terms of* **coric** *data*

— cf. the morphism $\beta_S : S^{\mathrm{pf}} \to S$ of Example 3.6, (ii); the natural inclusion $M \hookrightarrow M^{\mathrm{pf}}$ of Example 3.6, (iii). From the point of view of the *vertical* and *horizontal* lines of the **log-theta-lattice** [cf. the discussion of (i)], this theme may also be observed in the *vertically coric log-shells* that serve as a *common receptacle* for the various arrows of the **$\mathfrak{log}$-Kummer correspondences** of [IUTchIII], Corollary 3.11, (ii), as well as in the *multiradial representations* of [IUTchIII], Corollary 3.11, (i), which describe [certain aspects of] the **arithmetic holomorphic structure** on one vertical line of the log-theta-lattice in terms that may be understood relative to an **alien** *arithmetic holomorphic structure* on another vertical line — i.e., separated from the first vertical line by *horizontal arrows* — of the log-theta-lattice [cf. [IUTchIII], Remark 3.11.1; [IUTchIII], Remark 3.12.3, (ii)].

Remark 3.6.2.

(i) In the context of the theme of *"coric descriptions of non-coric data"* discussed in Remark 3.6.1, (ii), it is of interest to observe the significance of the use of **set-theoretic formulas** [cf. the discussion of Remarks 3.1.2, 3.1.3, 3.1.4] to realize such descriptions. That is to say, descriptions in terms of *arbitrary choices* that depend on a particular model of set theory [cf. Remark 3.1.3] do not allow one to **calculate** *in terms that make sense in one universe* the operations performed in an **alien universe**! This is precisely the sort of situation that one encounters when one considers the vertical and horizontal arrows of the log-theta-lattice [cf. (ii) below], where *distinct universes* arise from the *distinct* **scheme-theoretic basepoints** on either side of such an arrow that correspond to *distinct* **ring theories**, i.e., ring theories that *cannot be related* to one another by means of a *ring homomorphism* — cf. the discussion of Remark 3.6.3 below. Indeed,

> it was precisely the need to understand this sort of situation that led the author to develop the **"inter-universal"** version of **Teichmüller theory** exposed in the present series of papers.

Finally, we observe that the *algorithmic approach* [i.e., as opposed to the *"fully faithfulness/Grothendieck Conjecture-style approach"* — cf. Example 3.5, (iii)] to reconstruction issues via *set-theoretic formulas* plays an essential role in this context. That is to say, although different algorithms, or *software*, may yield the *same output data*, it is only by working with **specific algorithms** that one may understand the *delicate inter-relations* that exist between various **components** of the structures that occur as one performs various operations [i.e., the mutations of a mutation-history]. In the case of the theory developed in the present series of papers, one central example of this phenomenon is the **cyclotomic rigidity isomorphisms** that underlie the theory of **$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link compatibility** discussed in [IUTchIII], Corollary 3.11, (iii), (c), (d) [cf. also [IUTchIII], Remarks 2.2.1, 3.10.1].

(ii) The **algorithmic** *approach to reconstruction* that is taken throughout the present series of papers, as well as, for instance, in [FrdI], [EtTh], and [AbsTopIII], was conceived by the author in the spirit of the **species-theoretic** *formulation* exposed in the present §3. Nevertheless, [cf. Remark 3.1.3, (i)] we shall not explicitly write out the various set-theoretic formulas involved in the various species, mutations, etc. that are *implicit* throughout the theory of the present series of papers. Rather, it is to be understood that the set-theoretic formulas to be used are those arising from the *conventional descriptions* that are given of the mathematical objects involved. When applying such conventional descriptions, however, the reader is obliged to check that they are *well-defined* and *do* **not** *depend* upon the use of **arbitrary choices** that are not describable via well-defined set-theoretic formulas.

(iii) The *sharp contrast* between

- the **canonicality** imparted by descriptions via *set-theoretic formulas* in the context of *extensions of the universe* in which one works

[cf. Remarks 3.1.3, 3.1.4] and

- the situation that arises if one allows, in one's descriptions, the various **arbitrary choices** arising from *invocations of the axiom of choice*

may be understood somewhat explicitly if one attempts to *"catalogue the various possibilites"* corresponding to various possible choices that may occur in one's description. That is to say, such a "cataloguing operation" typically obligates one to work with *"sets of very large cardinality"*, many of which must be constructed by means of **set-theoretic exponentiation** [i.e., such as the operation of passing from a set E to the set "2^E" of all subsets of E]. Such a rapid outbreak of *"unwieldy large sets"* is reminiscent of the *rapid growth*, in the p-adic crystalline theory, of the p-adic valuations of the *denominators* that occur when one *formally integrates an* **arbitrary connection**, as opposed to a **"canonical connection"** of the sort that arises from a crystalline representation. In the p-adic theory, such "canonical connections" are typically related to **"canonical liftings"**, such as, for instance, those that occur in ***p*-adic Teichmüller theory** [cf. [pOrd], [pTeich]]. In this context, it is of interest to recall that the canonical liftings of p-adic Teichmüller theory may, under certain conditions, be thought of as liftings *"of minimal complexity"* in the sense that their Witt vector coordinates are given by *polynomials of* **minimal degree** — cf. the computations of [Finot].

Remark 3.6.3.

(i) In the context of Remark 3.6.2, it is useful to recall the *fundamental reason* for the need to pursue **"inter-universality"** in the present series of papers [cf. the discussion of [IUTchIII], Remark 1.2.4; [IUTchIII], Remark 1.4.2], namely,

> since *étale fundamental groups* — i.e., in essence, **Galois groups** — are defined as certain *automorphism groups of fields/rings*, the definition of such a Galois group as a certain automorphism group of some ring structure is **fundamentally incompatible** with the **vertical** and **horizontal** arrows of the **log-theta-lattice** [i.e., which do *not* arise from ring homomorphisms]!

In this respect, "transformations" such as the vertical and horizontal arrows of the log-theta-lattice *differ*, quite fundamentally, from "transformations" that are *compatible with the ring structures* on the domain and codomain, i.e., **morphisms of rings/schemes**, which **tautologically** give rise to **functorial morphisms** between the respective étale fundamental groups. Put another way, in the notation of [IUTchI], Definition 3.1, (e), (f), for, say, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$,

> the only *natural correspondence* that may be described by means of **set-theoretic formulas** between the isomorphs of the local base field Galois groups "$G_{\underline{v}}$" on either side of a vertical or horizontal arrow of the log-theta-lattice is the correspondence constituted by an **indeterminate isomorphism of topological groups**.

A similar statement may be made concerning the isomorphs of the *geometric fundamental group* $\Delta_{\underline{v}} \overset{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}})$ on either side of a *vertical* [but *not horizontal*! — cf. the discussion of (ii) below] arrow of the log-theta-lattice — that is to say,

> the only *natural correspondence* that may described by means of **set-theoretic formulas** between these isomorphs is the correspondence constituted by an **indeterminate isomorphism of topological groups** *equipped with some* **outer action** *by the respective isomorph of "$G_{\underline{v}}$"*

— cf. the discussion of [IUTchIII], Remark 1.2.4. Here, again we recall from the discussion of Remark 3.6.2, (i), (ii), that it is only by working with such correspondences that may be described by means of *set-theoretic formulas* that one may obtain descriptions that allow one to **calculate** the operations performed in *one universe* from the point of view of an **alien universe**.

(ii) One *fundamental difference* between the *vertical* and *horizontal* arrows of the log-theta-lattice is that whereas, for, say, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$,

(V1) one **identifies**, up to isomorphism, the isomorphs of the *full* arithmetic fundamental group "$\Pi_{\underline{v}}$" on either side of a **vertical** arrow,

(H1) one **distinguishes** the "$\Delta_{\underline{v}}$'s" on either side of a **horizontal** arrow, i.e., one *only identifies*, up to isomorphism, the local base field Galois groups "$G_{\underline{v}}$" on either side of a horizontal arrow.

— cf. the discussion of [IUTchIII], Remark 1.4.2. One way to understand the fundamental reason for this difference is as follows.

(V2) In order to construct the **$\mathfrak{log}$-link** — i.e., at a more concrete level, the power series that defines the $p_{\underline{v}}$-adic logarithm at $\underline{v}$ — it is necessary to avail oneself of the local **ring structures** at $\underline{v}$ [cf. the discussion of [IUTchIII], Definition 1.1, (i), (ii)], which may only be reconstructed from the *full* "$\Pi_{\underline{v}}$" [i.e., not from "$G_{\underline{v}}$ stripped of its structure as a *quotient* of $\Pi_{\underline{v}}$" — cf. the discussion of [IUTchIII], Remark 1.4.1, (i); [IUTchIII], Remark 2.1.1, (ii); [AbsTopIII], §I3].

(H2) In order to construct the **$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-links** — i.e., at a more concrete level, the correspondence

$$\underline{\underline{q}} \;\mapsto\; \left\{ \underline{\underline{q}}^{j^2} \right\}_{j=1,\dots,l^{\divideontimes}}$$

[cf. [IUTchII], Remark 4.11.1] — it is necessary, in effect, to construct an *"isomorphism"* between a mathematical object [i.e., the *theta values* "$\underline{\underline{q}}^{j^2}$"] that **depends**, in an essential way, on regarding the various "j" as **distinct labels** [which are constructed from "$\Delta_{\underline{v}}$"!] and a mathematical object [i.e., "$\underline{\underline{q}}$"] that is **independent** *of these labels*; it is then a **tautology** that such an "isomorphism" may only be achieved if the labels — i.e., in essence, "$\Delta_{\underline{v}}$" — on either side of the "isomorphism" are kept **distinct** from one another.

Here, we observe in passing that the "apparently *horizontal* arrow-related" issue discussed in (H2) of **simultaneous realization** of **"label-dependent"** and **"label-free"** mathematical objects is reminiscent of the *vertical* arrow portion of the **bi-coricity** theory of [IUTchIII], Theorem 1.5 — cf. the discussion of [IUTchIII], Remark 1.5.1, (i), (ii); Step (vii) of the proof of [IUTchIII], Corollary 3.12.

Bibliography

[NerMod] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron Models*, *Ergebnisse der Mathematik und ihrer Grenzgebiete* **21**, Springer-Verlag (1990).

[Drk] F. R. Drake, *Set Theory: an Introduction to Large Cardinals*, *Studies in Logic and the Foundations of Mathematics* **76**, North-Holland (1974).

[DmMn] H. Dym and H. P. McKean, *Fourier Series and Integrals*, Academic Press (1972).

[Edw] H. M. Edwards, *Riemann's Zeta Function*, Academic Press (1974).

[Ffmn] S. Feferman, Set-theoretical Foundations of Category Theory, *Reports of the Midwest Category Seminar III*, *Lecture Notes in Mathematics* **106**, Springer-Verlag (1969), pp. 201-247.

[Finot] L. R. A. Finotti, Minimal degree liftings of hyperelliptic curves, *J. Math. Sci. Univ. Tokyo* **11** (2004), pp. 1-47.

[FK] E. Freitag and R. Kiehl, *Étale Cohomology and the Weil Conjecture*, Springer Verlag (1988).

[Harts] R. Hartshorne, *Algebraic Geometry*, *Graduate Texts in Mathematics* **52**, Springer-Verlag (1977).

[Kobl] N. Koblitz, *p-adic Numbers, p-adic Analysis, and Zeta-Functions*, *Graduate Texts in Mathematics* **58**, Springer-Verlag (1984).

[McLn] S. MacLane, One Universe as a Foundation for Category Theory, *Reports of the Midwest Category Seminar III*, *Lecture Notes in Mathematics* **106**, Springer-Verlag (1969).

[Mss] D. W. Masser, *Note on a conjecture of Szpiro* in *Astérisque* **183** (1990), pp. 19-23.

[Milne] J. S. Milne, *Abelian Varieties* in *Arithmetic Geometry*, edited by G. Cornell and J. H. Silverman, Springer-Verlag (1986), pp. 103-150.

[pOrd] S. Mochizuki, A Theory of Ordinary p-adic Curves, *Publ. Res. Inst. Math. Sci.* **32** (1996), pp. 957-1151.

[pTeich] S. Mochizuki, *Foundations of p-adic Teichmüller Theory*, AMS/IP Studies in Advanced Mathematics **11**, American Mathematical Society/International Press (1999).

[HASurI] S. Mochizuki, A Survey of the Hodge-Arakelov Theory of Elliptic Curves I, *Arithmetic Fundamental Groups and Noncommutative Algebra*, *Proceedings of Symposia in Pure Mathematics* **70**, American Mathematical Society (2002), pp. 533-569.

[HASurII] S. Mochizuki, A Survey of the Hodge-Arakelov Theory of Elliptic Curves II, *Algebraic Geometry 2000, Azumino*, *Adv. Stud. Pure Math.* **36**, Math. Soc. Japan (2002), pp. 81-114.

[CanLift] S. Mochizuki, The Absolute Anabelian Geometry of Canonical Curves, *Kazuya Kato's fiftieth birthday, Doc. Math. 2003, Extra Vol.*, pp. 609-640.

[LgSch] S. Mochizuki, Categorical representation of locally noetherian log schemes, *Adv. Math.* **188** (2004), pp. 222-246.

[ArLgSch] S. Mochizuki, Categories of log schemes with Archimedean structures, *J. Math. Kyoto Univ.* **44** (2004), pp. 891-909.

[SemiAnbd] S. Mochizuki, Semi-graphs of Anabelioids, *Publ. Res. Inst. Math. Sci.* **42** (2006), pp. 221-322.

[Cusp] S. Mochizuki, Absolute anabelian cuspidalizations of proper hyperbolic curves, *J. Math. Kyoto Univ.* **47** (2007), pp. 451-539.

[FrdI] S. Mochizuki, The Geometry of Frobenioids I: The General Theory, *Kyushu J. Math.* **62** (2008), pp. 293-400.

[EtTh] S. Mochizuki, The Étale Theta Function and its Frobenioid-theoretic Manifestations, *Publ. Res. Inst. Math. Sci.* **45** (2009), pp. 227-349.

[GenEll] S. Mochizuki, Arithmetic Elliptic Curves in General Position, *Math. J. Okayama Univ.* **52** (2010), pp. 1-28.

[AbsTopIII] S. Mochizuki, *Topics in Absolute Anabelian Geometry III: Global Reconstruction Algorithms*, RIMS Preprint **1626** (March 2008).

[IUTchI] S. Mochizuki, *Inter-universal Teichmüller Theory I: Construction of Hodge Theaters*, preprint.

[IUTchII] S. Mochizuki, *Inter-universal Teichmüller Theory II: Hodge-Arakelov-theoretic Evaluation*, preprint.

[IUTchIII] S. Mochizuki, *Inter-universal Teichmüller Theory III: Canonical Splittings of the Log-theta-lattice*, preprint.

[Silv] J. H. Silverman, *The Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics **106**, Springer-Verlag (1986).

[Arak] C. Soulé, D. Abramovich, J.-F. Burnol, J. Kramer, *Lectures on Arakelov Geometry, Cambridge studies in advanced mathematics* **33**, Cambridge University Press (1994).

[Vjt] P. Vojta, *Diophantine approximations and value distribution theory*, Lecture Notes in Mathematics **1239**, Springer-Verlag (1987).