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Exact WKB analysis of second-order non-homogeneous linear ordinary differential equations

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Abstract

In this paper we consider the exact WKB analysis of a second-order non-homogeneous linear ordinary differential equation with a large parameter. We give a geometric criterion which guarantees the Borel summability of formal solutions of a non-homogeneous equation; this criterion is described in terms of exact steepest descent paths introduced in [AKT2]. An example related to the BNR equation ([BNR]) is also discussed from this viewpoint.

§ 1. Introduction and main result

In this paper we consider a second-order non-homogeneous linear ordinary differential equation with a large parameter η (> 0) of the following form:

$$\left(\frac{d^2}{dx^2} + \eta p(x)\frac{d}{dx} + \eta^2 q(x)\right)\psi = F(x). \tag{1.1}$$

Here, for the sake of simplicity, we assume the coefficients $p(x)$, $q(x)$ and the non-homogeneous term $F(x)$ are all polynomials of x. As (1.1) contains a large parameter η, we readily find that (1.1) has the following formal power series solution in η^{-1}:

$$\widehat{\psi}(x,\eta) = \sum_{n=2}^{\infty} \psi_n(x)\eta^{-n}, \tag{1.2}$$

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where the coefficients $\psi_n(x)$ $(n = 2, 3, \ldots)$ are determined by the recursion formula

$$\begin{aligned} q(x)\psi_2 = F(x), \quad q(x)\psi_3 + p(x)\frac{d\psi_2}{dx} = 0, \\ q(x)\psi_{n+2} + p(x)\frac{d\psi_{n+1}}{dx} + \frac{d^2\psi_n}{dx^2} = 0 \quad (n \geq 2). \end{aligned} \tag{1.3}$$

In general, the formal solution (1.2) does not converge. *The purpose of this paper is to give a criterion that guarantees the Borel summability of the formal solution* (1.2)*, that is, to discuss when the Borel sum*

$$\int_0^\infty e^{-y\eta}\widehat{\psi}_B(x, y)\, dy \tag{1.4}$$

of $\widehat{\psi}(x, \eta)$ *is well-defined*, where $\widehat{\psi}_B(x, y)$ denotes the Borel transform of $\widehat{\psi}(x, \eta)$, i.e.,

$$\widehat{\psi}_B(x, y) := \sum_{n=2}^{\infty} \frac{\psi_n(x)}{(n-1)!}\, y^{n-1}. \tag{1.5}$$

Let us first explain the motivations of our research. In the case of a homogeneous equation

$$\left(\frac{d^2}{dx^2} + \eta p(x)\frac{d}{dx} + \eta^2 q(x)\right)\psi = 0, \tag{1.6}$$

there exist the following formal solutions with an exponential term called *WKB solutions*:

$$\psi_\pm(x, \eta) = \exp\left(\eta \int^x \zeta_\pm(x)dx\right) \sum_{n=0}^{\infty} \psi_{\pm,n}(x)\eta^{-(n+1/2)}, \tag{1.7}$$

where $\zeta_\pm(x)$ are roots of the characteristic equation

$$\zeta^2 + p(x)\zeta + q(x) = 0. \tag{1.8}$$

A criterion for the Borel summability of WKB solutions (1.7) is now well-known (cf. [DLS], [CDK], [KS], etc.), as described in the following

Theorem 1.1. *The WKB solutions* $\psi_\pm(x, \eta)$ *of the homogeneous equation* (1.6) *are Borel summable except on Stokes curves of* (1.6)*. Here a Stokes curve of* (1.6) *is, by definition, an integral curve of the direction field*

$$\mathrm{Im}\,(\zeta_+(x) - \zeta_-(x))\, dx = 0 \tag{1.9}$$

emanating from a turning point of (1.6)*, i.e., a zero of the discriminant of* (1.8)*. To be more precise, if the steepest descent paths of*

$$\mathrm{Re}\left(\pm \int^x (\zeta_+(x) - \zeta_-(x))\, dx\right) \tag{1.10}$$

passing through x_0 can be prolonged to $x = \infty$ without flowing into any turning point, then the WKB solutions $\psi_\pm(x, \eta)$ normalized at x_0 are Borel summable in a neighborhood of x_0.

One of our motivations is to generalize Theorem 1.1 to non-homogeneous equations. As a matter of fact, since a general solution of (1.1) is given by a linear combination of the formal solution (1.2) and WKB solutions of (1.6), it suffices to consider only (1.2) to discuss the Borel summability of a general solution of (1.1) in view of Theorem 1.1.

Another motivation is concerned with generalization of Theorem 1.1 to third-order (or, more generally, higher-order) homogeneous equations. As is rigorously discussed in [KS], Theorem 1.1 can be proved by considering the Riccati equation (i.e., a first-order nonlinear ordinary differential equation) associated with (1.6) instead of dealing directly with (1.6). A crucial step in the proof is to apply the iteration method (or, equivalently, the fixed point theorem for a contraction mapping) after recursively solving first-order non-homogeneous linear ordinary differential equations which are, roughly speaking, obtained as linearized equation of the Riccati equation. Thus, in order to generalize this scheme to third-order homogeneous equations, we are compelled to solve a second-order non-homogeneous linear differential equation. As the first step toward the proof of the Borel summability of WKB solutions of higher-order equations, we investigate second-order non-homogeneous linear differential equations in this paper.

We now state our main result. Let x_0 be a point that is not located on any Stokes curve of (1.6). Defining $f_\pm(x)$ by

$$f_\pm(x) = -\int_{x_0}^{x} \zeta_\mp dx = \frac{1}{2}\int_{x_0}^{x} \{-(\zeta_+ + \zeta_-) \pm (\zeta_+ - \zeta_-)\} dx \tag{1.11}$$
$$= \frac{1}{2}\int_{x_0}^{x} \{p(x) \pm (\zeta_+ - \zeta_-)\} dx,$$

we let $\Gamma_\pm^{(0)}$ denote a steepest descent path of Re $f_\pm$ passing through x_0. If $\Gamma_+^{(0)}$ (resp., $\Gamma_-^{(0)}$) crosses a Stokes curve of (1.6) of type $+ > -$ (resp., of type $- > +$, where a Stokes curve of (1.6) is said to be of type $+ > -$ if Re $\int^x (\zeta_+(x) - \zeta_-(x))\, dx > 0$ holds on the curve in question) at some point $x = x_1$, then we also consider a steepest descent path ("*bifurcated steepest descent path*") $\Gamma_-^{(1)}$ of Re f_- (resp., $\Gamma_+^{(1)}$ of Re f_+) passing through x_1. In case these steepest descent paths $\Gamma_\pm^{(0)}$ and $\Gamma_\pm^{(1)}$ further cross a Stokes curve of (1.6), we similarly define $\Gamma_\pm^{(2)}$, $\Gamma_\pm^{(3)}$, ... by repeating the same process. We now assume that these processes terminate in finite steps, that is, there exists a finite number of steepest descent paths $\{\Gamma_\pm^{(l)}\}_{0 \le l \le L}$ so that every bifurcated steepest descent path passing through a crossing point of $\Gamma_\pm^{(l)}$ and a Stokes curve of (1.6) is contained in $\{\Gamma_\pm^{(l)}\}_{0 \le l \le L}$. The totality of the steepest descent paths $\{\Gamma_\pm^{(l)}\}_{0 \le l \le L}$ is called an "*exact steepest descent path*" of (1.1) passing through x_0. Under these situations we can prove

Theorem 1.2. *The formal solution $\widehat{\psi}(x,\eta)$ of (1.1) of the form (1.2) is Borel summable at $x = x_0$ if all the steepest descent paths belonging to an exact steepest descent path passing through x_0 can be prolonged to $x = \infty$ without flowing into any turning point.*

Note that exact steepest descent paths were first introduced in [AKT2] in the study of WKB solutions of homogeneous equations through the Laplace transformation method with respect to an independent variable of the differential equation ("*exact steepest descent method*").

The paper is organized as follows: Making use of WKB solutions of the corresponding homogeneous equation (1.6) together with the method of variation of constants, we obtain an integral representation for the Borel transform of the formal solution $\widehat{\psi}(x,\eta)$ of (1.1) in Section 2. Then in Section 3 we study the analytic continuation of the Borel transform of $\widehat{\psi}(x,\eta)$ by using this integral representation to prove Theorem 1.2. In [AKT2] an exact steepest descent path was introduced to investigate the (inverse) Laplace integral for the Laplace transformed (with respect to an independent variable of the differential equation) WKB solutions. Here the integral representation obtained in Section 2 plays the same role as the (inverse) Laplace integral in [AKT2]; this explains why an exact steepest descent path appears in describing the condition for the Borel summability of $\widehat{\psi}(x,\eta)$. Finally in Section 4 we discuss an example related to the BNR equation, a third-order homogeneous equation for which a new Stokes curve appears, as was first observed by Berk et al ([BNR]).

§ 2. Explicit formula for the Borel transform of $\widehat{\psi}(x,\eta)$

In this section, applying the method of variation of constants, we obtain an integral representation for the Borel transform of $\widehat{\psi}(x,\eta)$ in terms of the Borel transform of WKB solutions of the corresponding homogeneous equation (1.6).

Let

$$\psi_{\pm}(x,\eta) = \exp\left(\eta\int_{x_0}^{x}\zeta_{\pm}(x)dx\right)\varphi_{\pm}(x,\eta), \quad \varphi_{\pm}(x,\eta) = \sum_{n=0}^{\infty}\psi_{\pm,n}(x)\eta^{-(n+1/2)} \tag{2.1}$$

be WKB solutions of the homogeneous equation (1.6) normalized at x_0, where $\zeta_{\pm}(x)$ are roots of (1.8). Note that $\psi_{\pm}(x,\eta)$ can be constructed by using formal power series solutions

$$S_{\pm}(x,\eta) = \eta\zeta_{\pm}(x) + S_{\pm,0}(x) + \eta^{-1}S_{\pm,1}(x) + \cdots \tag{2.2}$$

of the Riccati equation

$$S^2 + \frac{dS}{dx} + \eta p(x)S + \eta^2 q(x) = 0 \tag{2.3}$$

associated with (1.6) in such a way that

$$\psi_{\pm}(x,\eta) = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(-\frac{1}{2}\eta \int_{x_0}^{x} p(x)dx \pm \int_{x_0}^{x} S_{\mathrm{odd}} dx\right), \tag{2.4}$$

where S_{odd} denotes the odd part of $S_{\pm}(x,\eta)$, i.e.,

$$S_{\pm}(x,\eta) = \pm S_{\mathrm{odd}}(x,\eta) + S_{\mathrm{even}}(x,\eta). \tag{2.5}$$

For WKB solutions $\psi_{\pm}(x,\eta)$ we have

Lemma 2.1. *Let $W = W(\psi_+,\psi_-) = \psi_+(d\psi_-/dx) - (d\psi_+/dx)\psi_-$ denote the Wronskian of ψ_+ and ψ_-. Then the following holds:*

$$W(\psi_+,\psi_-) = -2\exp\left(-\eta\int_{x_0}^{x} p(x)dx\right). \tag{2.6}$$

Proof. Substituting (2.5) into the Riccati equation (2.3) and taking its odd part, we find

$$2S_{\mathrm{odd}}S_{\mathrm{even}} + \frac{dS_{\mathrm{odd}}}{dx} + \eta p(x) S_{\mathrm{odd}} = 0, \tag{2.7}$$

that is,

$$S_{\mathrm{even}} = -\frac{1}{2}\frac{d}{dx}\log S_{\mathrm{odd}} - \frac{1}{2}\eta p(x). \tag{2.8}$$

Note that (2.8) and a well known relation $\psi_{\pm} = \exp\int^{x} S_{\pm}dx$ justify the expression (2.4). Then, using the relation

$$W(\psi_+,\psi_-) = \frac{d}{dx}\left(\frac{\psi_-}{\psi_+}\right)\cdot\psi_+^2 = -2S_{\mathrm{odd}}\exp\left(-2\int_{x_0}^{x} S_{\mathrm{odd}}dx\right)\psi_+^2 \tag{2.9}$$

and (2.4), we immediately obtain

$$W(\psi_+,\psi_-) = -2\exp\left(-\eta\int_{x_0}^{x} p(x)dx\right). \tag{2.10}$$

This completes the proof of Lemma 2.1. □

Using $\psi_{\pm}$ as a fundamental system of solutions of the corresponding homogeneous equation (1.6), we now apply the method of variation of constants to obtain a solution of the non-homogeneous equation (1.1). In view of Lemma 2.1, it is explicitly given by the following

$$\begin{aligned}&\frac{1}{2}\psi_+(x,\eta)\int^{x} F(x')\exp\Big(\eta\int_{x_0}^{x'} p(z)dz\Big)\psi_-(x',\eta)dx' \\ &\quad -\frac{1}{2}\psi_-(x,\eta)\int^{x} F(x')\exp\Big(\eta\int_{x_0}^{x'} p(z)dz\Big)\psi_+(x',\eta)dx'.\end{aligned} \tag{2.11}$$

As the behavior of the first term is similar to that of the second term, we mainly consider only the second term of (2.11) in what follows. Furthermore, since the Borel summability of $\psi_\pm$ (or, more precisely, of $\varphi_\pm$) is now known thanks to Theorem 1.1, we are going to discuss only the core part of the second term defined by the following, which will be denoted by $\Phi(x,\eta)$:

(2.12)
$$\begin{aligned}\Phi(x,\eta) &= \exp\Big(\eta\int_{x_0}^{x}\zeta_-(z)dz\Big)\int^{x}\exp\Big(\eta\int_{x_0}^{x'}(p(z)+\zeta_+(z))dz\Big)F(x')\varphi_+(x',\eta)dx' \\ &= \int^{x}\exp\Big(-\eta\int_{x}^{x'}\zeta_-(z)dz\Big)\phi_+(x',\eta)dx',\end{aligned}$$

where $\phi_+(x',\eta)=F(x')\varphi_+(x',\eta)$.

We thus obtain a solution of the non-homogeneous equation (1.1) provided by (2.11). It can be expanded into the formal power series of η^{-1} as follows:

Proposition 2.2. *The integral* (2.12) *is expanded as*

$$\Phi(x,\eta)=\sum_{n=0}^{\infty}\Phi_n(x)\eta^{-(n+3/2)}, \tag{2.13}$$

the right-hand side of which is explicitly given by the following:

$$-\sum_{n=0}^{\infty}\eta^{-(n+1)}\left(\frac{1}{\zeta_-(x')}\frac{d}{dx'}\right)^n\left(\frac{1}{\zeta_-(x')}\phi_+(x',\eta)\right)\Bigg|_{x'=x}. \tag{2.14}$$

Consequently (2.11) *provides the formal power series solution* (1.2) *of* (1.1) *under consideration.*

Proof. By a change of integration variable $X'=-\displaystyle\int_x^{x'}\zeta_-(z)dz$, we have

$$\Phi(x,\eta)=\int^{0}e^{\eta X'}\widetilde{\phi}_+(X',\eta)dX' \quad\text{with}\quad \widetilde{\phi}_+(X',\eta)=\phi_+(x'(X'),\eta)\Big(\frac{dX'}{dx'}\Big)^{-1}. \tag{2.15}$$

Then integration by parts tells us that

$$\Phi(x,\eta)=\eta^{-1}e^{\eta X'}\widetilde{\phi}_+(X',\eta)\Big|_{X'=0}-\int^{0}\eta^{-1}e^{\eta X'}\frac{d\widetilde{\phi}_+}{dX'}dX'. \tag{2.16}$$

Using integration by parts repeatedly, we thus obtain

$$\Phi(x,\eta)=\sum_{n=0}^{\infty}(-1)^n\eta^{-(n+1)}\left(\frac{d}{dX'}\right)^n\widetilde{\phi}_+\Big|_{X'=0}, \tag{2.17}$$

which verifies (2.14). □

Thanks to Proposition 2.2 the formal power series solution (1.2) in question has an integral representation (2.11). In what follows we discuss the Borel summability of (2.11), mainly that of $\Phi(x,\eta)$ given by (2.12). To this end we first seek for an integral representation of the Borel transform of $\Phi(x,\eta)$, which is given by the following:

Proposition 2.3. *The Borel transform of $\Phi(x,\eta)$, that is, the formal inverse Laplace transform of $\Phi(x,\eta)$ defined by*

$$\Phi_B(x,y)=\mathcal{L}^{-1}_{\eta\to y}\Phi=\sum_{n=0}^{\infty}\frac{\Phi_n(x)}{\Gamma(n+3/2)}y^{n+1/2}, \tag{2.18}$$

has the following integral representation

$$\Phi_B(x,y)=-\int_x^{x_*}\phi_{+,B}(x',y'(x'))dx'. \tag{2.19}$$

Here $\phi_{+,B}(x,y)$ denotes the Borel transform of $\phi_+(x,\eta)$, $y'(x')$ is a function defined by

$$y'(x')=y+f_+(x') \quad \textit{with} \quad f_+(x')=-\int_x^{x'}\zeta_-(z)dz, \tag{2.20}$$

x_* is a point determined by $f_+(x_*)=-y$ (i.e., $y'(x_*)=0$), and the integral is performed along a path $f_+(x')=-u$ $(0\le u\le y)$, i.e., along the steepest descent path of $\mathrm{Re}\, f_+$ *passing through x.*

Proof. It follows from (2.14) or, more conveniently, the equivalent expression (2.17) of $\Phi(x,\eta)$ that

$$\Phi_B(x,y)=\sum_{n=0}^{\infty}(-1)^n\frac{y^n}{\Gamma(n+1)}*\left(\frac{d}{dX'}\right)^n\widetilde{\phi}_{+,B}\Big|_{X'=0}, \tag{2.21}$$

where $*$ stands for the convolution product with respect to the variable y. Hence we have

$$\begin{aligned}\Phi_B(x,y)&=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^y(y-t)^n\left(\frac{d}{dX'}\right)^n\widetilde{\phi}_{+,B}\Big|_{X'=0}dt \\ &=\int_0^y\widetilde{\phi}_{+,B}(t-y,t)dt \\ &=\int_{-y}^0\widetilde{\phi}_{+,B}(X',y+X')dX' \\ &=\int_{x_*}^x\phi_{+,B}\left(x',y-\int_x^{x'}\zeta_-(z)dz\right)dx'.\end{aligned} \tag{2.22}$$

This implies (2.19), completing the proof of Proposition 2.3. □

Using Proposition 2.3, we investigate the analytic continuation of $\Phi_B(x,y)$ to obtain the Borel summability of $\Phi(x,\eta)$ in the subsequent section.

§ 3. Proof of the main result (Theorem 1.2)

Assume that the point x_0 in question is not located on any Stokes curve of the corresponding homogeneous equation (1.6). In what follows we show that the Borel transform $\Phi_B(x_0,y)$ of $\Phi(x_0,\eta)$ given by the integral (2.19) can be analytically continued along the positive real axis $\{y \in \mathbb{C} \,|\, y \geq 0\}$ under the condition of Theorem 1.2.

First of all, in view of the expression (2.19) and the analytic structure of $\phi_{+,B}(x', y'(x'))$ near $(x',y') = (x_0,0)$, we can readily confirm that $\Phi_B(x_0,y)$ defines an analytic function of y when $|y|$ is small. (Cf. Figure 1. Note that, replacing the integral

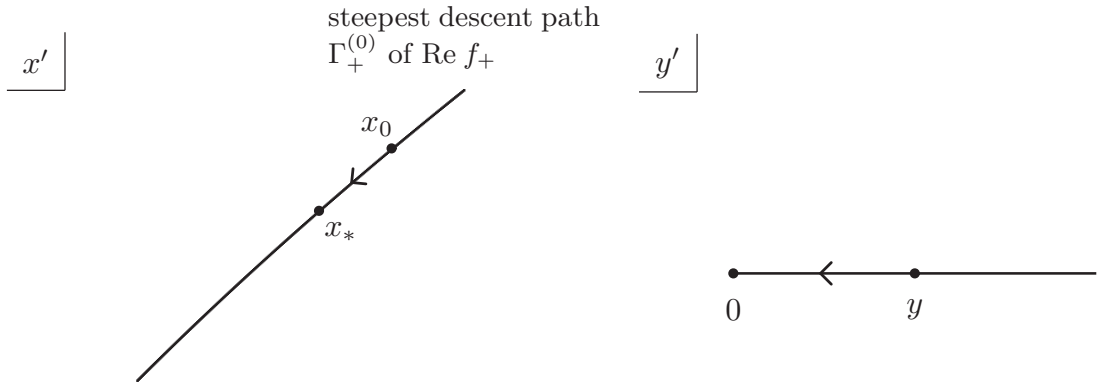

Figure 1. Integration path for $\Phi_B(x_0,y)$ when $|y|$ is small.

(2.19) by a contour integral around x_*, we find that the square-root type singularity of $\phi_{+,B}(x',y'(x'))$ at $(x',y') = (x_0,0)$ is irrelevant to the analyticity of $\Phi_B(x_0,y)$.) Thus our task is to verify the analytic continuability of $\Phi_B(x_0,y)$ when $y \geq 0$ is large. A key lemma is the following Lemma 3.1, which explicitly describes the singularity structure of the integrand $\phi_{+,B}$ of (2.19).

Lemma 3.1. (i) *The Borel transform $\phi_{+,B}(x',y') = F(x')\varphi_{+,B}(x',y')$ of $\phi_+(x',\eta) = F(x')\varphi_+(x',\eta)$ has a singular point at*

$$y' = \omega(x') := \int_a^{x'} (\zeta_+(z) - \zeta_-(z))dz, \tag{3.1}$$

where a is a turning point of (1.6).
(ii) *At $y' = \omega(x')$, $\phi_{+,B}(x', y')$ satisfies the following relation:*

$$\Delta_{y'=\omega(x')}\phi_{+,B}(x', y') = i\phi_{-,B}(x', y' - \omega(x')), \tag{3.2}$$

where $\Delta_{y'=\omega(x')}\phi_{+,B}(x', y')$ denotes the discontinuity of $\phi_{+,B}(x', y')$ at $y' = \omega(x')$, that is, the difference of the boundary value of $\phi_{+,B}(x', y')$ from the upper side of the cut $\{y' \in \mathbb{C} \,|\, \mathrm{Re}\,(y' - \omega(x')) \geq 0,\ \mathrm{Im}\,(y' - \omega(x')) = 0\}$ and the boundary value from its lower side.

It is a well known fact in the exact WKB analysis of second-order homogeneous ordinary differential equations that $\varphi_{+,B}(x', y')$ has the same singularity structure as described in Lemma 3.1 (see, e.g., [DP], [KT]). Since $F(x)$ is a polynomial, Lemma 3.1 immediately follows from this fact.

It follows from Lemma 3.1 that the integrand of (2.19) is singular at a point where

$$y'(x') = y - \int_{x_0}^{x'} \zeta_-(z)dz \quad \text{and} \quad y'(x') = \int_a^{x'} (\zeta_+(z) - \zeta_-(z))dz \tag{3.3}$$

are satisfied. Eliminating $y'(x')$ in (3.3), we readily find that at a singular point of the integrand of (2.19) we have

$$y - \int_{x_0}^{x'} \zeta_-(z)dz = \int_a^{x'} (\zeta_+(z) - \zeta_-(z))dz. \tag{3.4}$$

Since the left-hand side of (3.4) is real and positive for positive y, the integrand of (2.19) cannot have a singularity for positive y as far as the steepest descent path $\Gamma_+^{(0)}$ of $\mathrm{Re}\, f_+$ passing through x_0 is prolonged without flowing into any turning point or crossing any Stokes curve of (1.6). Hence $\Phi_B(x_0, y)$ is analytic along $\mathbb{R}_y^+ = \{y \in \mathbb{C} \,|\, y \geq 0\}$ under such a situation. On the contrary, if $\Gamma_+^{(0)}$ flows into a turning point, then $\Phi_B(x_0, y)$ has a singularity on $\mathbb{R}_y^+$ in general as a singular point (3.1) hits the endpoint of the integral (2.19) at a turning point.

Now let us consider a more intriguing situation, that is, the situation where $\Gamma_+^{(0)}$ crosses a Stokes curve of (1.6) of type $+ > -$. In this situation a singular point of the integrand $\phi_{+,B}(x', y'(x'))$ hits a path of integration of (2.19) and consequently it becomes necessary to take the effect of such a singular point into acccount for sufficiently large $y \geq 0$. For the sake of simplicity we suppose that the steepest descent path $\Gamma_+^{(0)}$ of $\mathrm{Re}\, f_+$ passing through x_0 crosses just one Stokes curve of type $+ > -$ emanating from a turning point a of (1.6) in what follows. We denote the crossing point of $\Gamma_+^{(0)}$ and the Stokes curve by $x = x_1$. Then (3.4) implies that a singular point of the integrand $\phi_{+,B}(x', y'(x'))$ hits a path of integration of (2.19) at

$$y = \hat{y} := \int_a^{x_1} (\zeta_+(z) - \zeta_-(z))dz + \int_{x_0}^{x_1} \zeta_-(z)dz > 0. \tag{3.5}$$

To analyze explicitly the effect of the singular point of $\phi_{+,B}(x', y'(x'))$ to the analytic continuation of $\Phi_B(x_0, y)$ when $y \geq \hat{y}$, we first make a change of integration variable from x' to $y' = y + f_+(x')$ in (2.19) as follows:

$$\Phi_B(x_0, y) = \int_y^0 \phi_{+,B}(g_+(y' - y), y') \, \frac{dy'}{\zeta_-(g_+(y' - y))}, \tag{3.6}$$

where $g_+(y')$ denotes an inverse function of $f_+(x')$, i.e., $y' = y + f_+(x')$ can be expressed also as $x' = g_+(y' - y)$. Rewriting (3.4) as

$$\int_{x_1}^{x'} \zeta_+(z)dz = y - \int_a^{x_1} (\zeta_+(z) - \zeta_-(z))dz - \int_{x_0}^{x_1} \zeta_-(z)dz = y - \hat{y} \tag{3.7}$$

and denoting x' that satisfies (3.7) by x_{**}, we find that the integrand of (3.6) has a singularity at

$$y_{**} := \int_a^{x_1} (\zeta_+(z) - \zeta_-(z))dz + \int_{x_1}^{x_{**}} (\zeta_+(z) - \zeta_-(z))dz \tag{3.8}$$

and the analytic continuation of $\Phi_B(x_0, y)$ to $y \geq \hat{y}$ is given by an integral of (3.6) along a path C indicated as in Figure 2. (Here we assume that $\mathrm{Im}\, y_{**} > 0$ for the

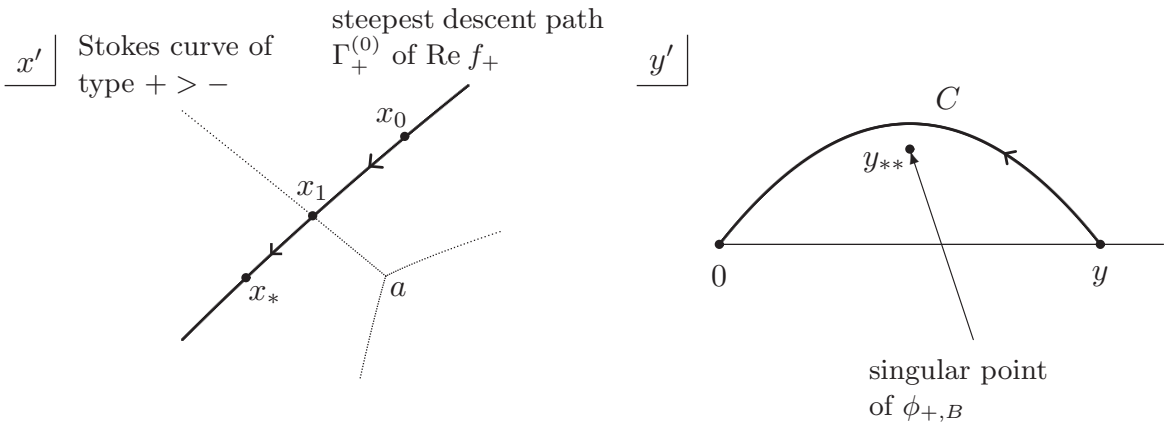

Figure 2. Integration path of (3.6) when $y \geq \hat{y}$.

sake of definiteness.) Then we deform the path C to a new integration path $\widetilde{C}$ which is homotopically equivalent to C and consists of the following three portions:

$$C \simeq \widetilde{C} = C_1 \cup C_2 \cup C_3, \tag{3.9}$$

where

C_1 : a path from y to $y_1 := y'(x_1) = y - \int_{x_0}^{x_1} \zeta_- dz$ on the real axis,
C_2 : a path starting from y_1, encircling y_{**} anticlockwise and returning to y_1,
C_3 : a path from y_1 to 0 on the real axis.

(Cf. Figure 3.) Corresponding to this decomposition of $\widetilde{C} \simeq C$, we also decompose

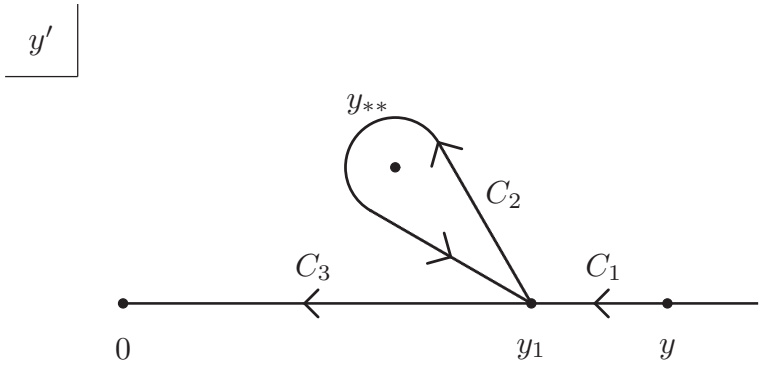

Figure 3. Integration path $\widetilde{C} = C_1 \cup C_2 \cup C_3$.

$\Phi_B(x_0, y)$ as follows:

$$\Phi_B(x_0, y) = \sum_{j=1}^{3} \Phi_B^{(j)}(x_0, y) \tag{3.10}$$

with

$$\Phi_B^{(j)}(x_0, y) = \int_{C_j} \phi_{+,B}(g_+(y' - y), y') \, \frac{dy'}{\zeta_-(g_+(y' - y))}. \tag{3.11}$$

Among the three, $\Phi_B^{(1)}(x_0, y)$ and $\Phi_B^{(3)}(x_0, y)$ can be easily rewritten in terms of the steepest descent path $\Gamma_+^{(0)}$ as

$$\Phi_B^{(1)}(x_0, y) = -\int_{x_0, \text{ along } \Gamma_+^{(0)}}^{x_1} \phi_{+,B}(x', y'(x'))dx', \tag{3.12}$$

$$\Phi_B^{(3)}(x_0, y) = -\int_{x_1, \text{ along } \Gamma_+^{(0)}}^{x_*} \phi_{+,B}(x', y'(x'))dx', \tag{3.13}$$

while $\Phi_B^{(2)}(x_0, y)$ is not directly related to $\Gamma_+^{(0)}$. However, using Lemma 3.1, we can express $\Phi_B^{(2)}(x_0, y)$ in terms of the bifurcated steepest descent path $\Gamma_-^{(1)}$ of $\operatorname{Re} f_-$ passing through x_1 in the following manner:

Proposition 3.2. *The integral $\Phi_B^{(2)}(x_0, y)$ has the following expression:*

$$\Phi_B^{(2)}(x_0, y) = -i \int_{x_1}^{x_{**}} \phi_{-,B}(x'', y''(x''))dx'', \tag{3.14}$$

where $y''(x'')$ is defined by

$$y''(x'') = (y-\hat{y}) + f_-(x'') = \left(y - \int_a^{x_1} (\zeta_+ - \zeta_-)dz - \int_{x_0}^{x_1} \zeta_- dz\right) + f_-(x'') \tag{3.15}$$

with

$$f_-(x'') = -\int_{x_1}^{x''} \zeta_+(z)dz, \tag{3.16}$$

x_{**} is a point determined by $f_-(x_{**}) = -(y-\hat{y})$ (i.e., $y''(x_{**}) = 0$, cf. (3.7)), and the integral is performed along a path $f_-(x'') = -v$ $(0 \le v \le y-\hat{y})$, i.e., along the steepest descent path of $\mathrm{Re}\, f_-$ *passing through x_1 (cf. Figure 4).*

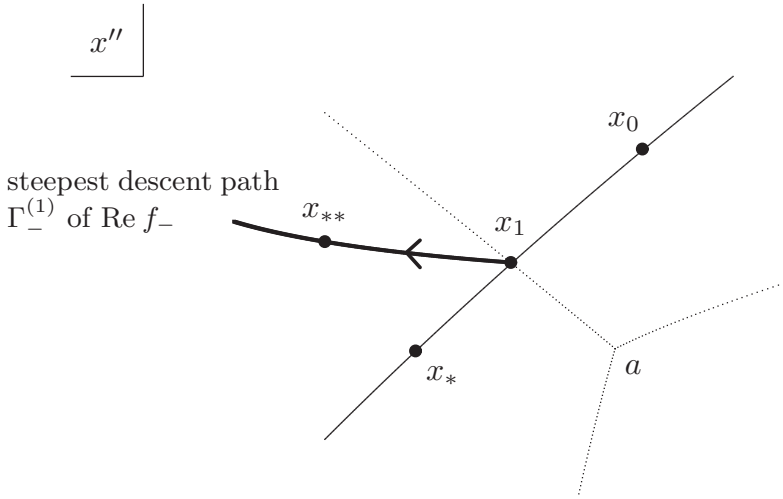

Figure 4. Integration path of (3.14).

Proof. It follows from Lemma 3.1, (ii) that

$$\begin{aligned}\Phi_B^{(2)}(x_0,y) &= \int_{C_2} \phi_{+,B}(g_+(y'-y),y')\,\frac{dy'}{\zeta_-(g_+(y'-y))} \\ &= -\int_{y_{**}}^{y_1} \Delta_{y'=y_{**}}\phi_{+,B}(g_+(y'-y),y')\,\frac{dy'}{\zeta_-(g_+(y'-y))} \\ &= -i\int_{y_{**}}^{y_1} \phi_{-,B}(g_+(y'-y),y'-\omega(g_+(y'-y)))\,\frac{dy'}{\zeta_-(g_+(y'-y))}.\end{aligned} \tag{3.17}$$

Here, letting x'' denote $g_+(y'-y)$, we employ a change of integration variable from y' to $y'' = y' - \omega(g_+(y'-y))$. Firstly, we note that $x'' = g_+(y'-y)$ means

$$y'-y = f_+(x'') = -\int_{x_0}^{x''} \zeta_-\, dz \tag{3.18}$$

and that

$$y'' - y' = -\int_a^{g_+(y'-y)} (\zeta_+ - \zeta_-)\, dz = -\int_a^{x''} (\zeta_+ - \zeta_-)\, dz \tag{3.19}$$

holds. Hence

$$\begin{aligned} y'' &= y - \int_{x_0}^{x''} \zeta_-\, dz - \int_a^{x''} (\zeta_+ - \zeta_-)\, dz \\ &= y - \left(\int_{x_0}^{x_1} \zeta_-\, dz + \int_a^{x_1} (\zeta_+ - \zeta_-)\, dz + \int_{x_1}^{x''} \zeta_+\, dz \right) \\ &= (y - \hat{y}) + f_-(x''). \end{aligned} \tag{3.20}$$

Secondly, we have

$$\begin{aligned} dy'' &= dy' - (\zeta_+ - \zeta_-)(x'')\, \frac{dg_+}{dy'}(y' - y)\, dy' \\ &= dy' - (\zeta_+ - \zeta_-)(x'')\, \frac{1}{(df_+/dx)(x'')}\, dy' \\ &= dy' + \frac{(\zeta_+ - \zeta_-)(x'')}{\zeta_-(x'')}\, dy' \\ &= \frac{\zeta_+(x'')}{\zeta_-(x'')}\, dy', \end{aligned} \tag{3.21}$$

that is,

$$\frac{dy''}{\zeta_+(x'')} = \frac{dy'}{\zeta_-(x'')}. \tag{3.22}$$

Thirdly, it follows from the definition of x_{**} and y_{**} that $x'' = g_+(y' - y) = x_{**}$ holds when $y' = y_{**}$ (cf. (3.4), (3.7) and (3.8)). Hence, by (3.8), we find that the point $y' = y_{**}$ corresponds in the new variable y'' to the point

$$y' - \int_a^{x''} (\zeta_+ - \zeta_-)\, dz \Bigg|_{y'=y_{**},\, x''=x_{**}} = 0, \tag{3.23}$$

that is, $y'' = 0$. On the other hand, since the definition of y_1, i.e., $y_1 = y'(x_1) = y + f_+(x_1)$ implies that $g_+(y_1 - y) = x_1$, the point $y' = y_1$ corresponds in the new variable y'' to the point

$$y' - \int_a^{x''} (\zeta_+ - \zeta_-)\, dz \Bigg|_{y'=y_1,\, x''=x_1} = y - \int_{x_0}^{x_1} \zeta_-\, dz - \int_a^{x_1} (\zeta_+ - \zeta_-)\, dz = y - \hat{y}. \tag{3.24}$$

We thus conclude

$$\Phi_B^{(2)}(x_0,y) = -i\int_0^{y-\hat{y}} \phi_{-,B}(x'',y'')\,\frac{dy''}{\zeta_+(x'')} \quad \text{with} \quad y'' = (y-\hat{y}) + f_-(x''). \tag{3.25}$$

Finally, if we make a change of integration variable from y'' to x'' in (3.25), we obtain the expression (3.14) for $\Phi_B^{(2)}(x_0,y)$. □

Thanks to Proposition 3.2, each $\Phi_B^{(j)}(x_0,y)$ $(j=1,2,3)$ has the same form as the original integral (2.19). Hence, if both the steepest descent paths $\Gamma_+^{(0)}$ and $\Gamma_-^{(1)}$ are prolonged to $x=\infty$ without crossing any further Stokes curves of (1.6), the above reasoning verifies that each $\Phi_B^{(j)}(x_0,y)$ (and hence $\Phi_B(x_0,y)$ as well) is analytically continued along the whole positive real axis. In case $\Gamma_+^{(0)}$ and/or $\Gamma_-^{(1)}$ may cross any other Stokes curve of (1.6), we can apply the discussion in this section again to $\Phi_B^{(j)}(x_0,y)$ as it has the same form as (2.19). Thus we obtain the analyticity of $\Phi_B(x_0,y)$ on the positive real axis $\{y\in\mathbb{C}\,|\,y\geq 0\}$ under the condition of Theorem 1.2.

Finally, the exponential growth of $\Phi_B(x,y)$ with respect to the y-variable easily follows from that of $\phi_{+,B}(x,y)$ (cf. [KS]). The proof of Theorem 1.2 is now completed.

Remark. In case $\Gamma_+^{(0)}$ and $\Gamma_-^{(1)}$ are prolonged to $x=\infty$ without crossing any further Stokes curves of (1.6), corresponding to the decomposition (3.10) of the Borel transform $\Phi_B(x_0,y)$, we can obtain the following expression of the Borel sum of $\Phi(x_0,\eta)$ in view of Proposition 3.2:

$$\begin{aligned}(\mathcal{S}\Phi)(x_0,\eta) = &\int_{x_1,\text{ along }\Gamma_+^{(0)}}^{x_0} \exp\left(-\eta\int_{x_0}^{x'}\zeta_-\,dz\right)(\mathcal{S}\phi_+)(x',\eta)\,dx' \\ &+\int_{\infty,\text{ along }\Gamma_+^{(0)}}^{x_1} \exp\left(-\eta\int_{x_0}^{x'}\zeta_-\,dz\right)(\mathcal{S}\phi_+)(x',\eta)\,dx' \\ &+ i\int_{\infty,\text{ along }\Gamma_-^{(1)}}^{x_1} \exp\left(-\eta\hat{y}-\eta\int_{x_1}^{x''}\zeta_+\,dz\right)(\mathcal{S}\phi_-)(x'',\eta)\,dx'',\end{aligned} \tag{3.26}$$

where $\mathcal{S}\Phi$ denotes the Borel sum of Φ.

§ 4. An example related to the BNR equation

In [BNR] Berk et al considered the following third-order linear differential equation

$$\left(\frac{d^3}{dx^3} + 3\eta^2\frac{d}{dx} + 2ix\eta^3\right)\psi = 0 \tag{4.1}$$

and showed that some Stokes phenomena for Borel resummed WKB solutions of (4.1) occur not only on ordinary Stokes curves but also on the so-called "new Stokes curves".

Here an ordinary Stokes curve means a Stokes curve emanating from a turning point, while a new Stokes curve is a Stokes curve passing through no turning points. (See [AKT1] for more precise definitions of new Stokes curves.)

To be more concrete, as the characteristic equation of (4.1) is given by

$$\zeta^3 + 3\zeta + 2ix = 0 \tag{4.2}$$

and a turning point is a zero of the discriminant of (4.2), i.e., a point where two characteristic roots merge, the BNR equation (4.1) has two turning points at $x = \pm 1$, from each of which three Stokes curves emanate. See Figure 5 for the figure of Stokes curves of (4.1). In Figure 5 "a turning point of type (j, k)" means a turning point where two characteristic roots labeled by ζ_j and ζ_k merge. Similarly, "a Stokes curve of type $j > k$" is a Stokes curve on which $\mathrm{Im} \int^x (\zeta_j(x) - \zeta_k(x))\, dx = 0$ and $\mathrm{Re} \int^x (\zeta_j(x) - \zeta_k(x))\, dx > 0$ hold. (To number the characteristic roots, in Figure 5 we have placed two cuts, designated by wiggly lines, which enable us to define a characteristic root as a single-valued analytic function.) In addition to ordinary Stokes curves emanating from turning points new Stokes curves are also included in Figure 5: In the case of (4.1) the imaginary axis is a new Stokes curve. Note that, as is shown in [BNR] and [AKT1], Stokes phenomena for Borel resummed WKB solutions do really occur on the solid portion of the new Stokes curve, but not on its dashed portion.

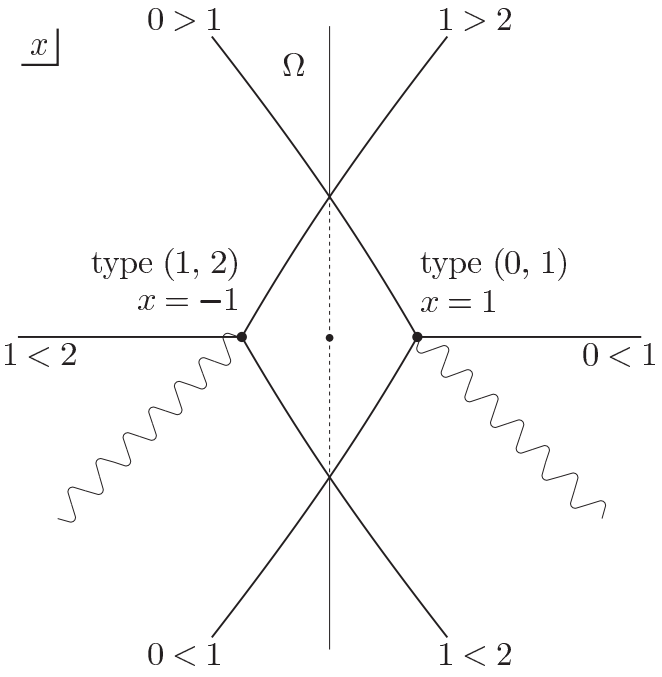

Figure 5. Stokes curves of the BNR equation (4.1).

Here let us recall the construction of WKB solutions of (4.1). We first assume that

an unknown function ψ of (4.1) has the form

$$\psi = \exp \int^x S(x,\eta)\,dx. \tag{4.3}$$

Then, substituting (4.3) into (4.1), we find that $S(x,\eta)$ should satisfy

$$S^3 + 3S\frac{dS}{dx} + \frac{d^2S}{dx^2} + 3\eta^2 S + 2ix\eta^3 = 0, \tag{4.4}$$

a higher-order analogue of the Riccati equation (2.3). Equation (4.4) has the following formal power series solution with the characteristic root ζ_j $(j = 0, 1, 2)$ of (4.2) as its top order term:

$$S^{(j)}(x,\eta) = \eta\zeta_j(x) + S_0^{(j)}(x) + \eta^{-1}S_1^{(j)}(x) + \cdots. \tag{4.5}$$

A WKB solution $\psi_j(x,\eta)$ of (4.1) is a formal solution obtained by substituting (4.5) into (4.3).

We now set $S = \eta\zeta_0(x) + T$ in (4.4). Then T should satisfy

$$\frac{d^2T}{dx^2} + 3\eta\zeta_0(x)\frac{dT}{dx} + 3\eta^2((\zeta_0(x))^2 + 1)T + R = 0 \tag{4.6}$$

with

$$R = 3\eta^2\zeta_0\frac{d\zeta_0}{dx} + \eta\frac{d^2\zeta_0}{dx^2} + 3\eta\frac{d\zeta_0}{dx}T + 3\eta\zeta_0T^2 + T^3 + 3T\frac{dT}{dx}. \tag{4.7}$$

The remainder term R consists of terms containing only ζ_0, lower order terms with respect to η, and higher order (i.e., nonlinear) terms with respect to T. In what follows, neglecting the remainder term R and regarding it as a given non-homogeneous term, we consider

$$\left(\frac{d^2}{dx^2} + 3\eta\zeta_0(x)\frac{d}{dx} + 3\eta^2((\zeta_0(x))^2 + 1)\right)T = F(x) \tag{4.8}$$

as an example of application of our main theorem (Theorem 1.2). Note that the homogeneous equation corresponding to (4.8) is nothing but (the principal part of) the linearlized equation or the Fréchet derivative of (4.4) at its formal solution $S^{(0)}(x,\eta)$. Our main interest lies in the Borel summability of the formal solution (1.2) of (4.8) and its comparison with that of WKB solutions of the BNR equation (4.1).

Equation (4.8) can be written also in terms of ζ_j as

$$\left(\frac{d^2}{dx^2} - \eta\big((\zeta_1 - \zeta_0) + (\zeta_2 - \zeta_0)\big)\frac{d}{dx} + \eta^2(\zeta_1 - \zeta_0)(\zeta_2 - \zeta_0)\right)T = F(x). \tag{4.9}$$

Hence the characteristic roots of (4.9) are given by

$$\zeta_{1-0} := \zeta_1 - \zeta_0 \quad \text{and} \quad \zeta_{2-0} := \zeta_2 - \zeta_0 \tag{4.10}$$

and (4.9) has only one turning point at $x = -1$. (The other turning point $x = 1$ is a kind of singular points in discussing (4.9).) We also set

$$f_{1-0}(x) = -\int_{x_0}^{x} \zeta_{2-0}\, dx \quad \text{and} \quad f_{2-0}(x) = -\int_{x_0}^{x} \zeta_{1-0}\, dx. \tag{4.11}$$

From now on, applying Theorem 1.2, we investigate the Borel summability of the formal solution (1.2) of (4.8) (or equivalently (4.9)) when x_0 lies in the region Ω specified in Figure 5 or its boundary.

Case (I) (*when x_0 lies in the interior of Ω.*)

When x_0 lies in the interior of Ω, the configuration of the steepest descent paths $\Gamma^{(0)}_{1-0}$ of Re f_{1-0} and $\Gamma^{(0)}_{2-0}$ of Re f_{2-0} passing through x_0 become as is indicated in Figure 6. While $\Gamma^{(0)}_{2-0}$ is prolonged to $x = \infty$ without crossing a Stokes curve of (4.9) of

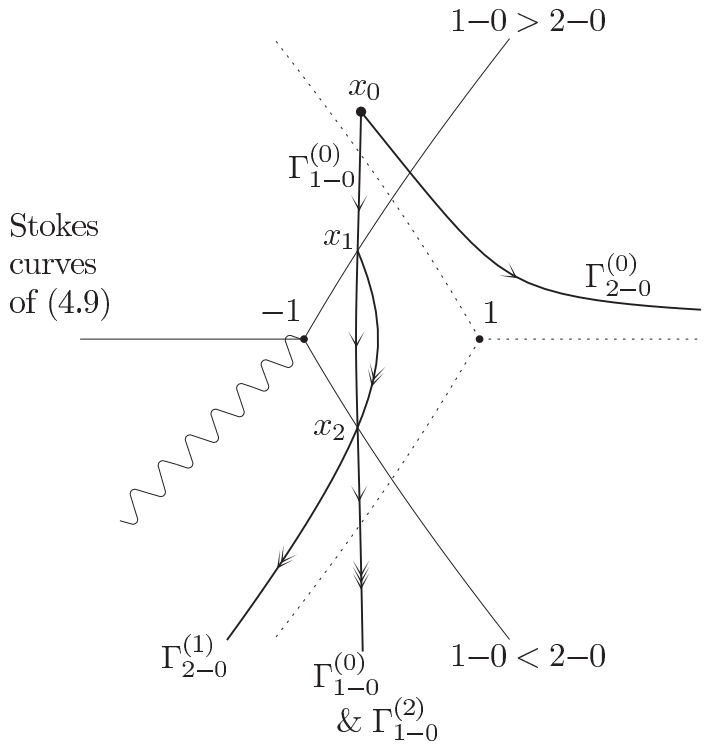

Figure 6. Exact steepest descent path passing through x_0 in Case (I).

type $2-0 > 1-0$, $\Gamma^{(0)}_{1-0}$ crosses a Stokes curve of type $1-0 > 2-0$ at x_1. Thus we need to take into account also a bifurcated steepest descent path $\Gamma^{(1)}_{2-0}$ passing through x_1. Since $\Gamma^{(1)}_{2-0}$ again crosses a Stokes curve of type $2-0 > 1-0$ at x_2, we should consider another bifurcated steepest descent path $\Gamma^{(2)}_{1-0}$, which coincides with the original $\Gamma^{(0)}_{1-0}$ (thanks to the symmetry of the equation with respect to the real axis) in this case.

As is clearly visualized in Figure 6, all the steepest descent paths $\Gamma^{(0)}_{1-0}$, $\Gamma^{(0)}_{2-0}$, $\Gamma^{(1)}_{2-0}$ and $\Gamma^{(2)}_{1-0}$ are prolonged to $x=\infty$. Thus Theorem 1.2 guarantees the Borel summability of the formal solution (1.2) in this case.

Case (II) (*when x_0 lies on the left boundary of Ω, i.e., on a Stokes curve of* (4.1).)

If we pick up a point x_0 from the left boundary of Ω, that is, from a Stokes curve of (4.1), then one of the steepest descent paths $\Gamma^{(0)}_{2-0}$ passing through x_0 flows into $x=1$ and cannot be prolonged to $x=\infty$ (cf. Figure 7). Thus the Borel summability of the

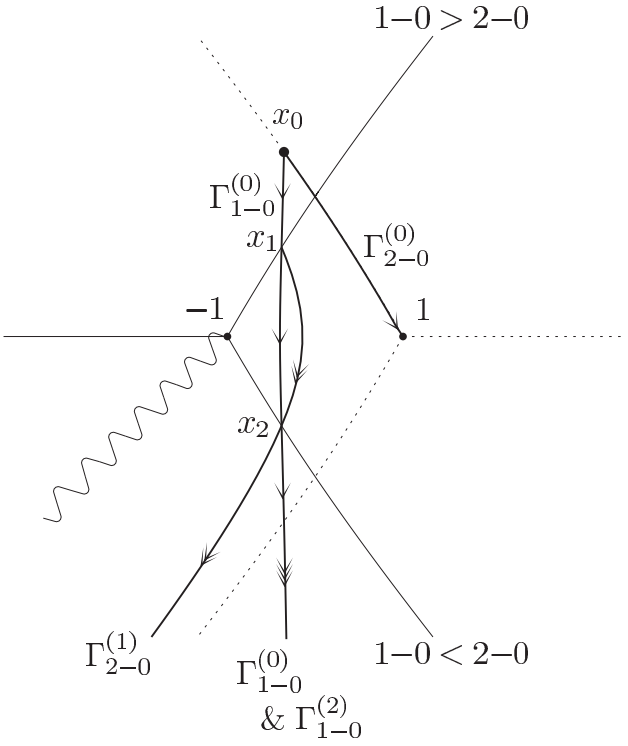

Figure 7. Exact steepest descent path passing through x_0 in Case (II).

formal solution (1.2) is not expected to hold in this case.

Case (III) (*when x_0 lies on the right boundary of Ω, i.e., on a new Stokes curve of* (4.1).)

Finally we consider the case where x_0 lies on the right boundary of Ω, that is, on a new Stokes curve of the BNR equation. In this case there is no problem with $\Gamma^{(0)}_{2-0}$, but the steepest descent path $\Gamma^{(1)}_{2-0}$ bifurcated from $\Gamma^{(0)}_{1-0}$ at x_1 flows into $x=1$ and cannot be prolonged to $x=\infty$, as is shown in Figure 8. Hence, similarly to Case (II) (but due to different geometric obstruction), the Borel summability of the formal solution (1.2) is not expected to hold in Case (III) as well. In this way the new Stokes curve of the BNR equation (4.1) is also captured through the analysis of the non-homogeneous

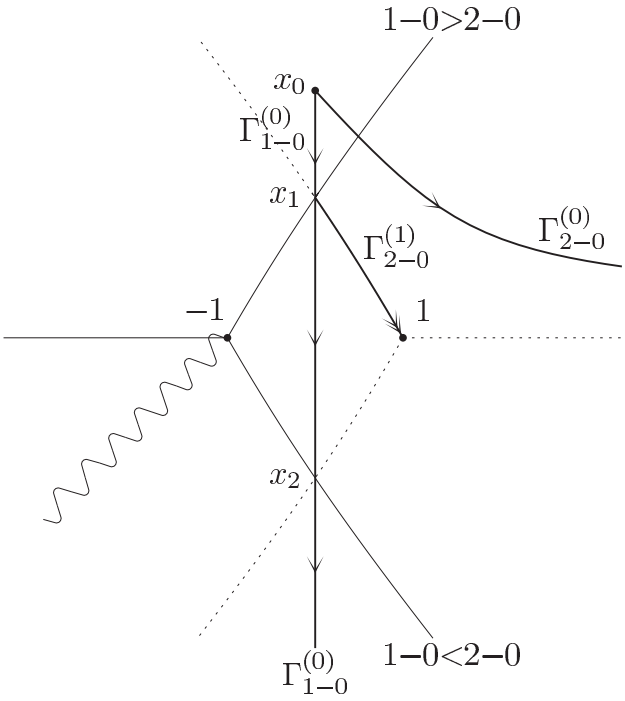

Figure 8. Exact steepest descent path passing through x_0 in Case (III).

second-order equation (4.9).

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