RIMS-1761

GALOIS ACTION ON MAPPING CLASS GROUPS

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October 2012

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Abstract. Let l be a prime number. In this present paper, we study the outer Galois action on the profinite and the relative pro-l completions of mapping class groups of pointed orientable topological surfaces. In the profinite case, we prove that the outer Galois action is faithful. In the pro-l case, we prove that the kernel of the outer Galois action has certain stability properties with respect to the genus and the number of punctures.

1. Introduction

Let k be a (commutative) field of characteristic zero, X a smooth geometrically connected curve over k, and (g, n) a pair of nonnegative integers such that $2g - 2 + n > 0$ (hyperbolicity). We call X a (g, n)-curve if there exists a proper smooth genus g curve C over k and a closed subscheme $D \subseteq C$ such that $X = C \setminus D$ and the composite $D \hookrightarrow C \to \operatorname{Spec} k$ is a finite étale covering over $\operatorname{Spec} k$ of degree n. Let $\overline{k}$ be an algebraic closure of k. For a (g, n)-curve X, by SGA1 [1], we have a short exact sequence

$$1 \longrightarrow \pi_1(X \otimes_k \overline{k}) \longrightarrow \pi_1(X) \longrightarrow G_k \longrightarrow 1$$

where π_1 denotes the algebraic fundamental groups and $G_k := \operatorname{Gal}(\overline{k}/k)$ is the absolute Galois group of k. Let $\Pi_{g,n}$ denote the profinite completion of the fundamental group $\pi_1(g, n)$ of a compact Riemann surface of genus g with n points punctured. By the comparison theorem, $\pi_1(X \otimes_k \overline{k})$ is isomorphic to $\Pi_{g,n}$. Since $\pi_1(X)$ acts on $\pi_1(X \otimes_k \overline{k})$ by conjugation in the above short exact sequence, $\pi_1(X)$ also acts on $\Pi_{g,n}$. This gives the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{g,n} & \longrightarrow & \pi_1(X) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \operatorname{Inn}(\Pi_{g,n}) & \longrightarrow & \operatorname{Aut}(\Pi_{g,n}) & \longrightarrow & \operatorname{Out}(\Pi_{g,n}) & \longrightarrow & 1, \end{array}$$

where Aut (respectively Inn) denotes the continuous automorphism group (respectively the inner automorphism group) of $\Pi_{g,n}$, and Out denotes the quotient, so that the horizontal sequences are both exact. The right vertical map gives the outer Galois representation

$$\rho_X : G_k \longrightarrow \operatorname{Out}(\Pi_{g,n}).$$

Belyĭ proved that ρ_X is injective when $X = \mathbb{P}^1_k \setminus \{0, 1, \infty\}$ and k is a number field (Corollary to Theorem 4, [6]). Voevodskiĭ proved the injectivity of ρ_X when the genus of X is one and k is a number field, and suggested a conjecture that the ρ_X is

Date: September 18, 2012.

2010 *Mathematics Subject Classification.* Primary 14H30; Secondary 14H10.

injective when X is an affine hyperbolic curve and k is a number field ([34]). This conjecture was solved by Matsumoto ([20]). Moreover, the proper case was proved by Hoshi and Mochizuki ([14]). Therefore, we have the following theorem:

Theorem 1.1. *The outer Galois representation ρ_X is injective when X is a hyperbolic curve and k is a number field.*

Grothendieck considered that any hyperbolic curve over a number field is anabelian, i.e., the geometry of any hyperbolic curve X over a number field is determined by ρ_X (the Grothendieck conjecture for algebraic curves, [12]). This conjecture was proved by Mochizuki ([22, 23]). The above theorem can be regarded as an evidence that ρ_X has high complexity when k is a number field.

On the other hand, Grothendieck considered that the moduli space of hyperbolic curves is also anabelian ([12]). Therefore, it is a natural problem that we consider Voevodskiĭ's conjecture in the case when X is the moduli space of hyperbolic curves. Let $\mathcal{M}_{g,n}$ be the moduli stack over k of smooth geometrically connected proper curves of genus g with n (ordered) marked points ([9, 18]). It is known that $\pi_1(\mathcal{M}_{g,n} \otimes \overline{k})$ is isomorphic to the profinite completion $\Gamma_{g,n}$ of the mapping class group $\mathrm{MCG}_{g,n}$ of an n-pointed genus g topological surface ([30]). As above, we have the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma_{g,n} & \longrightarrow & \pi_1(\mathcal{M}_{g,n}) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}(\Gamma_{g,n}) & \longrightarrow & \mathrm{Aut}(\Gamma_{g,n}) & \longrightarrow & \mathrm{Out}(\Gamma_{g,n}) & \longrightarrow & 1, \end{array}$$

where the horizontal sequences are both exact. The right vertical map gives the outer Galois representation

$$\rho_{g,n} : G_k \longrightarrow \mathrm{Out}(\Gamma_{g,n}).$$

For the injectivity of $\rho_{g,n}$, our result in the present paper is summarized in the following (cf. Theorem 2.3):

Theorem 1.2. *Let k be a number field and (g, n) a pair of nonnegative integers such that $2g - 2 + n > 0$. Then the homomorphism $\rho_{g,n+1}$ is injective.*

Remark 1.3. As $\mathcal{M}_{0,4} = \mathbb{P}^1_k \setminus \{0, 1, \infty\}$, the injectivity of $\rho_{0,4}$ follows from the above theorem of Belyĭ (Corollary to Theorem 4, [6]).

The proof of Theorem 1.2 yields a variant, where we consider an arbitrary family of hyperbolic curves instead of the universal family $\mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$. As above, for any geometrically connected locally noetherian scheme X over k, we can consider the outer Galois representation $\rho_X : G_k \to \mathrm{Out}(\pi_1(X \otimes_k \overline{k}))$ determined by the exact sequence

$$1 \longrightarrow \pi_1(X \otimes_k \overline{k}) \longrightarrow \pi_1(X) \longrightarrow G_k \longrightarrow 1.$$

Grothendieck considered that hyperbolic polycurves (i.e., successive families of hyperbolic curves) are also anabelian ([12]). The injectivity of ρ_X is implicit in [14] when X is a hyperbolic polycurve and k is a number field. We can prove the injectivity of ρ_X when X is an arbitrary family of hyperbolic curves (cf. Theorem 4.3):

Theorem 1.4. *Let k be a number field and (g,n) a pair of nonnegative integers such that $2g-2+n>0$, S a geometrically connected regular scheme of finite type over k and $X \to S$ a family of (g,n)-curves over S. Then the homomorphism ρ_X is injective.*

Hoshi and Tamagawa informed the author of a different proof of Theorem 1.2. In fact, their proof gave a result stronger than Theorem 1.2, as follows. By Oda's theory ([30]) and using the Birman exact sequence (Chapter 4, [10])

$$1 \longrightarrow \pi_1(g,n) \longrightarrow \mathrm{MCG}_{g,n+1} \longrightarrow \mathrm{MCG}_{g,n} \longrightarrow 1,$$

we have the following exact sequence:

$$1 \longrightarrow \Pi_{g,n} \longrightarrow \pi_1(\mathcal{M}_{g,n+1}) \longrightarrow \pi_1(\mathcal{M}_{g,n}) \longrightarrow 1.$$

This exact sequence gives the universal monodromy representation

$$\rho_{g,n}^{univ} : \pi_1(\mathcal{M}_{g,n}) \longrightarrow \mathrm{Out}(\Pi_{g,n}).$$

It is known that the homomorphism $\rho_{g,n}^{univ}$ is injective if and only if $\rho_{g,n}^{univ}|_{\Gamma_{g,n}}$ is injective (Corollary 6.5, [14]).

Remark 1.5. The problem of the injectivity of $\rho_{g,n}^{univ}|_{\Gamma_{g,n}}$ is called the congruence subgroup problem for $\mathrm{MCG}_{g,n}$. The congruence subgroup problem was proved for $g \leq 1$ by Asada ([5]) and for $g=2$, $n>0$ by Boggi ([7]). Boggi called the image of $\rho_{g,n}^{univ}|_{\Gamma_{g,n}}$ the geometric profinite completion of $\mathrm{MCG}_{g,n}$ in [7].

We denote by

$$\rho_{g,n}^{geom} : G_k \longrightarrow G_k^{g,n} \longrightarrow \mathrm{Out}(\rho_{g,n}^{univ}(\Gamma_{g,n}))$$

the natural homomorphism determined by the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Gamma_{g,n} & \longrightarrow & \pi_1(\mathcal{M}_{g,n}) & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \rho_{g,n}^{univ}(\Gamma_{g,n}) & \longrightarrow & \rho_{g,n}^{univ}(\pi_1(\mathcal{M}_{g,n})) & \longrightarrow & G_k^{g,n} & \longrightarrow & 1,
\end{array}$$

where $G_k^{g,n} := \rho_{g,n}^{univ}(\pi_1(\mathcal{M}_{g,n}))/\rho_{g,n}^{univ}(\Gamma_{g,n})$, and the horizontal sequences are exact.

Theorem 1.6 (Hoshi-Tamagawa)**.** *Let k be a number field and (g,n) a pair of nonnegative integers such that $3g-3+n>0$. Then the homomorphism $\rho_{g,n}^{geom}$ is injective. In particular, $\rho_{g,n}$ is injective.*

We remark that Boggi also announced a similar result (Corollary 7.6, [8]).

Next, we consider a pro-l version of Theorem 1.6, which l is a prime number. Let $\Pi_{g,n}^l$ denote the pro-l completion of the fundamental group of a Riemann surface of genus g with n points punctured. For a (g,n)-curve X over k, by the functoriality of pro-l completion, we obtain

$$\rho_X^l : G_k \longrightarrow \mathrm{Out}(\Pi_{g,n}^l).$$

As above, we have the pro-l universal monodromy representation

$$\rho_{g,n}^{univ,l} : \pi_1(\mathcal{M}_{g,n}) \longrightarrow \mathrm{Out}(\Pi_{g,n}^l).$$

Therefore, we also have the natural homomorphism

$$\rho_{g,n}^{geom,l} : G_k \longrightarrow G_k^{l,g,n} \longrightarrow \mathrm{Out}(\rho_{g,n}^{univ,l}(\Gamma_{g,n}))$$

determined by the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma_{g,n} & \longrightarrow & \pi_1(\mathcal{M}_{g,n}) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \rho_{g,n}^{univ,l}(\Gamma_{g,n}) & \longrightarrow & \rho_{g,n}^{univ,l}(\pi_1(\mathcal{M}_{g,n})) & \longrightarrow & G_k^{l,g,n} & \longrightarrow & 1, \end{array}$$

where $G_k^{l,g,n} := \rho_{g,n}^{univ,l}(\pi_1(\mathcal{M}_{g,n}))/\rho_{g,n}^{univ,l}(\Gamma_{g,n})$, and the horizontal sequences are exact. The field determined by $\mathrm{im}(\ker(\rho_{g,n}^{univ,l}) \to G_k)(= \ker(G_k \to G_k^{l,g,n}))$ can be regarded as the field of definition of the Teichmüller modular function field with l-power level structures. Oda conjectured that this field is independent of (g,n) ([29]). This conjecture was proved by using the weight filtration and the universal deformation of a maximally degenerate stable curve ([28, 27, 20, 17, 33]). We prove the second main result in the present paper by using Oda's conjecture (cf. Theorem 3.4):

Theorem 1.7. *Let (g,n) be a pair of nonnegative integers such that $3g-3+n>0$ and either $(g,n) \neq (1,1)$ or $l=2$. Then the kernel of the homomorphism $\rho_{g,n}^{geom,l}$ coincides with the kernel of the homomorphism*

$$\rho^l_{\mathbb{P}^1_k \setminus \{0,1,\infty\}} : G_k \longrightarrow \mathrm{Out}(\Pi^l_{0,3}).$$

We apply Theorem 1.7 to the relative pro-l representation (Corollary 3.8).

The present paper is organized as follows: In section 2, we study the profinite case. Firstly, we prove a technical lemma (Lemma 2.2) in group theory and we derive Theorem 1.2 from this lemma. Secondly, we explain a proof of Theorem 1.6 due to Hoshi and Tamagawa by using a geometric version of the Grothendieck conjecture. In section 3, we prove Theorem 1.7 by using a geometric version of the Grothendieck conjecture and Oda's conjecture. Finally, we study the kernel of the relative pro-l representation. In section 4, we prove a variant of Theorem 1.2 (including Theorem 1.4) which does not follow from the method of Hoshi and Tamagawa.

Acknowledgments. First of all, the author would like to thank Makoto Matsumoto for having introduced him problems around the kernel of outer Galois representations. Also, he would like to thank Yuichiro Hoshi and Akio Tamagawa for informing him of the proof of Theorem 1.6 and allowing him to reproduce it. Finally, the author would like to thank his advisors, Toshiro Hiranouchi, Hiroki Takahashi and Akio Tamagawa for helpful discussions, warm encouragements, and valuable advices.

Notations and Conventions

Numbers: The notation $\mathbb{Z}$ will be used to denote the set, group, or ring of rational integers and the notation $\mathbb{Q}$ will be used to denote the set, group, or field of rational numbers. We shall refer to a finite extension of $\mathbb{Q}$ as a number field. For a prime number l, the notation $\mathbb{Z}_l$ will be used to denote the set, group, or ring of l-adic integers and the notation $\mathbb{Q}_l$ will be used to denote the set, group, or field of l-adic numbers. We shall refer to a finite extension of $\mathbb{Q}_l$ as an l-adic local field. The notation $\mathbb{C}$ will be used to denote the set, group, or field of complex numbers.

Profinite groups: If G is a profinite group, $H \subseteq G$ is a closed subgroup of G, and g is an element of G, then we shall write $Z_G(H)$ for the centralizer of H in G, i.e.,

$$Z_G(H) := \{g \in G \mid ghg^{-1} = h \,\text{for any}\, h \in H\} \subseteq G,$$

and we shall write $N_G(H)$ for the normalizer of H in G, i.e.,

$$N_G(H) := \{g \in G \mid gHg^{-1} = H\} \subseteq G.$$

If G is a profinite group, then we shall denote by $\mathrm{Aut}(G)$ the group of automorphisms of G, by $\mathrm{Inn}(G)$ the group of inner automorphisms of G, by $\mathrm{Out}(G)$ the quotient of $\mathrm{Aut}(G)$ by the normal subgroup $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$.

Surface groups and mapping class groups: For a pair (g, n) of nonnegative integers and a prime number l, the notation $\Pi_{g,n}$ will be used to denote the profinite completion of the fundamental group $\pi_1(g, n)$ of a compact Riemann surface of genus g with n points punctured, the notation $\Pi^l_{g,n}$ will be used to denote the pro-l completion of the fundamental group $\pi_1(g, n)$ of a Riemann surface of genus g with n points punctured, the notation $\mathrm{MCG}_{g,n}$ will be used to denote the mapping class group of (g, n)-type, namely the discrete group of isotopy classes of orientation preserving self-diffeomorphisms of an orientable surface of genus g with n points punctured which fix the n points pointwise, the notation $\mathrm{MCG}_{g,[n]}$ will be used to denote the discrete group of isotopy classes of orientation preserving self-diffeomorphisms of an orientable surface of genus g with n points punctured which preserve the set of punctures, and the notation $\Gamma_{g,n}$ will be used to denote the profinite completion of $\mathrm{MCG}_{g,n}$. We shall denote by $\mathrm{Out}^{\mathrm{C}}(\Pi_{g,n})$ the subgroup of $\mathrm{Out}(\Pi_{g,n})$ consisting of elements which preserve the set of cuspidal inertia subgroups of $\Pi_{g,n}$, and by $\mathrm{Out}^{\mathrm{C}}(\Pi^l_{g,n})$ the subgroup of $\mathrm{Out}(\Pi^l_{g,n})$ consisting of elements which preserve the set of cuspidal inertia subgroups of $\Pi^l_{g,n}$.

Curves: Let $f : X \to S$ be a morphism of schemes. Then for a pair (g, n) of nonnegative integers such that $2g-2+n > 0$, we shall say that f is a family of (g, n)-curves over S if there exist a proper smooth geometrically connected morphism $f^{\mathrm{cpt}} : X^{\mathrm{cpt}} \to S$ whose geometric fibers are of dimension one and of genus g, and a relative divisor $D \subseteq X^{\mathrm{cpt}}$ which is finite étale over S of degree n such that X and $X^{\mathrm{cpt}} \setminus D$ are isomorphic over S. We shall say that $f^{\mathrm{cpt}} : X^{\mathrm{cpt}} \to S$ is a compactification of $f : X \to S$ and $D \subseteq X^{\mathrm{cpt}}$ is a divisor at infinity of $f : X \to S$. We shall say that a family of (g, n)-curves $X \to S$ is split if a finite étale covering $D \to S$ obtained by a divisor at infinity of $X \to S$ is trivial, i.e., D is isomorphic to the disjoint union of n copies of S over S. Note that the pair (X^{cpt}, D) is unique up to canonical isomorphism if S is normal (e.g., Section 0, [24]). In particular, we shall refer to a family of (g, n)-curves over the spectrum of a field k as a (g, n)-curve over k.

Fundamental groups: Let l be a prime number, k a field, and $\overline{k}$ an algebraic closure of k. For a scheme X which is a geometrically connected and of finite type over k, we shall write $\pi_1(X \otimes_k \overline{k})^l$ for the maximal pro-l quotient of $\pi_1(X \otimes_k \overline{k})$, and $\pi_1(X)^{\underline{l}}$ for the quotient of $\pi_1(X)$ by the kernel of the natural surjection $\pi_1(X \otimes_k \overline{k}) \to \pi_1(X \otimes_k \overline{k})^l$.

2. Profinite mapping class groups

In the present section, we prove the main result of the present paper in the profinite case. Let k be a field of characteristic zero, (g, n) a pair of nonnegative integers such that $2g - 2 + n > 0$, $\mathcal{M}_{g,n}$ the moduli stack over k of the smooth geometrically connected proper curves of genus g with n (ordered) marked points, $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ determined by a fixed algebraic closure $\overline{k}$ of k, and $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The following theorem plays an essential role in our proof.

Theorem 2.1 (Corollary 6.4, [14])**.** *Let X be a (g, n)-curve over k. Then the subgroup*

$$\rho_X^{-1}(\rho_{g,n}^{univ}(\Gamma_{g,n})) \subseteq G_k$$

of G_k is contained in the kernel of the homomorphism

$$G_k \longrightarrow G_{\mathbb{Q}}$$

determined by the natural inclusion $\mathbb{Q} \hookrightarrow k$.

Theorem 2.1 was proved by Matsumoto and Tamagawa (Theorem 1.1, [21]) in the affine case, and more recently by Hoshi and Mochizuki (Corollary 6.4, [14]) in the proper case.

Lemma 2.2. *Consider the commutative diagram of groups where the vertical and horizontal sequences are exact:*

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & K & \longrightarrow & \Gamma' & \longrightarrow & \Gamma & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & G & \longrightarrow & G' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H & = & H & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1. & & & &
\end{array}$$

Let $\rho_G : H \to \mathrm{Out}(K), \rho_{G'} : H \to \mathrm{Out}(\Gamma'), \rho_{\Gamma'} : \Gamma \to \mathrm{Out}(K)$ denote the natural homomorphisms determined by the above commutative diagram. Then the subgroup

$$\rho_G(\ker(\rho_{G'})) \subseteq \mathrm{Out}(K)$$

of $\mathrm{Out}(K)$ is contained in the image of $\rho_{\Gamma'}$.

Proof. Let h be an element of the kernel of $\rho_{G'}$. Since G surjects onto H, we can take $h' \in G$ mapped to $h \in H$. By the injectivity of the homomorphism $G \to G'$, we may regard h as an element of H'. Then there exists an element γ of Γ' such that $\mathrm{Inn}(h')$ acts on Γ' by $\mathrm{Inn}(\gamma)$. In particular, $\mathrm{Inn}(h')$ acts on K by $\mathrm{Inn}(\gamma)$. This means $\rho_G(h) \in \mathrm{im}(\rho_{\Gamma'})$. □

Theorem 2.3. *Let (g,n) be a pair of nonnegative integers such that $2g-2+n>0$. Then the kernel of the homomorphism $\rho_{g,n+1}$ is contained in the kernel of the homomorphism*

$$G_k \longrightarrow G_{\mathbb{Q}}$$

determined by the natural inclusion $\mathbb{Q} \hookrightarrow k$.

In particular, if k is a number field or an l-adic local field, then the homomorphism $\rho_{g,n+1}$ is injective.

Proof. By the commutative diagram

$$\begin{array}{ccccc}
1 & & 1 & & \\
\downarrow & & \downarrow & & \\
\Gamma_{g,n+1} & \longrightarrow & \Gamma_{g,n} & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \\
\pi_1(\mathcal{M}_{g,n+1}) & \longrightarrow & \pi(\mathcal{M}_{g,n}) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \\
G_k & = & G_k & & \\
\downarrow & & \downarrow & & \\
1 & & 1, & &
\end{array}$$

where the vertical and horizontal sequences are exact, we may assume that n is small, so that there exists a (g,n)-curve X over k such that a divisor at infinity of $X \to \operatorname{Spec} k$ is split by considering a hyperelliptic curve. Since $\mathcal{M}_{g,n+1}$ is the universal curve over $\mathcal{M}_{g,n}$ (see [18]), we obtain a cartesian square

$$\begin{array}{ccc}
X & \longrightarrow & \operatorname{Spec} k \\
\downarrow & \square & \downarrow \\
\mathcal{M}_{g,n+1} & \longrightarrow & \mathcal{M}_{g,n}.
\end{array}$$

This induces a commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Pi_{g,n} & \longrightarrow & \Gamma_{g,n+1} & \longrightarrow & \Gamma_{g,n} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(\mathcal{M}_{g,n+1}) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G_k & = & G_k & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1, & & & &
\end{array}$$

where the vertical and horizontal sequences are exact. Then Lemma 2.2 implies that

$$\rho_X(\ker(\rho_{g,n+1})) \subseteq \mathrm{im}(\Gamma_{g,n} \longrightarrow \mathrm{Out}(\Pi_{g,n})).$$

By using Theorem 2.1, the result follows. □

Next, we explain a different proof of Theorem 2.3 due to Hoshi and Tamagawa, using a geometric version of the Grothendieck conjecture. In fact, their proof gives a result stronger than Theorem 2.3. The following theorem plays an essential role in their proof.

Theorem 2.4 (Theorem D, [15])**.** *Let (g,n) be a pair of nonnegative integers such that $3g-3+n>0$ and l a prime number.*

(i) *The group $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,n})}(\rho_{g,n}^{univ}(\Gamma_{g,n}))$ is isomorphic to*

$$\begin{cases} \mathbb{Z}/2\times\mathbb{Z}/2 & \text{if } (g,n)=(0,4); \\ \mathbb{Z}/2 & \text{if } (g,n)\in\{(1,1),(1,2),(2,0)\}; \\ \{1\} & \text{if } (g,n)\notin\{(0,4),(1,1),(1,2),(2,0)\}. \end{cases}$$

(ii) *Suppose that*

$$(g,n)\neq(1,1).$$

Then the group $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi^l_{g,n})}(\rho_{g,n}^{univ,l}(\Gamma_{g,n}))$ is isomorphic to

$$\begin{cases} \mathbb{Z}/2\times\mathbb{Z}/2 & \text{if } (g,n)=(0,4); \\ \mathbb{Z}/2 & \text{if } (g,n)\in\{(1,2),(2,0)\}; \\ \{1\} & \text{if } (g,n)\notin\{(0,4),(1,2),(2,0)\}. \end{cases}$$

(iii) *Suppose that $l=2$. Then the group $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi^l_{1,1})}(\rho_{1,1}^{univ,l}(\Gamma_{1,1}))$ is isomorphic to $\mathbb{Z}/2$.*

The proof of Theorem 2.4 is very sophisticated using the theory of profinite Dehn twists developed in [15].

Theorem 2.5 (Hoshi-Tamagawa)**.** *Let (g,n) be a pair of nonnegative integers such that $3g-3+n>0$. Then the kernel of the homomorphism $\rho_{g,n}^{geom}$ is contained in the kernel of the homomorphism*

$$G_k \longrightarrow G_{\mathbb{Q}}$$

determined by the natural inclusion $\mathbb{Q}\hookrightarrow k$.

In particular, if k is a number field or an l-adic local field, then the homomorphisms $\rho_{g,n}^{geom}$ and $\rho_{g,n}$ are injective.

Proof. We may assume that k is $\mathbb{Q}$. Note that $G_{\mathbb{Q}}^{g,n} := \rho_{g,n}^{univ}(\pi_1(\mathcal{M}_{g,n}))/\rho_{g,n}^{univ}(\Gamma_{g,n})$ is isomorphic to $G_{\mathbb{Q}}$ by Theorem 2.1. Also, by Theorem 2.3 and the injectivity of $\rho_{g,n}^{univ}$ when g is zero (Theorem 3A, [5]), we may assume that $g>0$. Then the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \rho_{g,n}^{univ}(\Gamma_{g,n}) & \longrightarrow & \rho_{g,n}^{univ}(\pi_1(\mathcal{M}_{g,n})) & \longrightarrow & G_{\mathbb{Q}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}(\rho_{g,n}^{univ}(\Gamma_{g.n})) & \longrightarrow & \mathrm{Aut}(\rho_{g,n}^{univ}(\Gamma_{g.n})) & \longrightarrow & \mathrm{Out}(\rho_{g,n}^{univ}(\Gamma_{g,n})) & \longrightarrow & 1 \end{array}$$

induces an isomorphism

$$Z_{\rho_{g,n}^{univ}(\pi_1(\mathcal{M}_{g,n}))}(\rho_{g,n}^{univ}(\Gamma_{g,n}))/Z_{\rho_{g,n}^{univ}(\Gamma_{g,n})}(\rho_{g,n}^{univ}(\Gamma_{g,n}))$$

$$\simeq \ker(G_{\mathbb{Q}} \longrightarrow \mathrm{Out}(\rho_{g,n}^{univ}(\Gamma_{g,n}))).$$

Therefore, it is enough to prove

$$Z_{\rho_{g,n}^{univ}(\pi_1(\mathcal{M}_{g,n}))}(\rho_{g,n}^{univ}(\Gamma_{g,n}))/Z_{\rho_{g,n}^{univ}(\Gamma_{g,n})}(\rho_{g,n}^{univ}(\Gamma_{g,n})) = \{1\}.$$

Note that the image of $\rho_{g,n}^{univ}$ is contained in $\mathrm{Out}^{\mathrm{C}}(\Pi_{g,n})$. By the injectivity of $\mathrm{MCG}_{g,[n]} \to \mathrm{Out}(\pi_1(g,n))$ (e.g., Theorem 8.8, in [10]) and $\mathrm{Out}(\pi_1(g,n)) \to \mathrm{Out}(\Pi_{g,n})$ (Lemma 3.2.1 in [3] for $n > 0$ and [11] for $n = 0$), we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{MCG}_{g,n} & \longrightarrow & \mathrm{Out}(\pi_1(g,n)) \\ \downarrow & \nearrow & \downarrow \\ \mathrm{MCG}_{g,[n]} & \hookrightarrow & \mathrm{Out}(\Pi_{g,n}). \end{array}$$

Since an element of $\mathrm{MCG}_{g,[n]}$ induces an action on the set of conjugacy classes of cuspidal inertia subgroups of $\pi_1(g,n)$, an element of $\mathrm{MCG}_{g,[n]}$ induces an action on the set of conjugacy classes of cuspidal inertia subgroups of $\Pi_{g,n}$. Note that there exits a canonical bijection between the set of conjugacy classes of cuspidal inertia subgroups of $\pi_1(g,n)$ and the set of conjugacy classes of cuspidal inertia subgroups of $\Pi_{g,n}$. Hence, the image of $\mathrm{MCG}_{g,[n]} \hookrightarrow \mathrm{Out}(\Pi_{g,n})$ is contained in $\mathrm{Out}^{\mathrm{C}}(\Pi_{g,n})$. In particular, we have the natural inclusion $Z_{\mathrm{MCG}_{g,[n]}}(\mathrm{MCG}_{g,[n]}) \hookrightarrow Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,n})}(\rho_{g,n}^{univ}(\Gamma_{g,n}))$ and this inclusion is isomorphism by Theorem 2.4 (i) and section 4 of Chapter 3 in [10]. If the image σ' of an element σ of $Z_{\mathrm{MCG}_{g,[n]}}(\mathrm{MCG}_{g,[n]})$ is not contained in $\rho_{g,n}^{univ}(\Gamma_{g,n})$, σ is not contained in $\mathrm{MCG}_{g,n}$. Since the action of $\mathrm{MCG}_{g,[n]} / \mathrm{MCG}_{g,n}$ on the set of conjugacy classes of cuspidal inertia subgroups of $\pi_1(g,n)$ is faithful, σ induces a nontrivial action on the set of conjugacy classes of cuspidal inertia subgroups of $\pi_1(g,n)$. Therefore, σ' induces a nontrivial action on the set of conjugacy classes of cuspidal inertia subgroups of $\Pi_{g,n}$. Since the action of $\rho_{g,n}^{univ}(\pi_1(\mathcal{M}_{g,n}))$ on the set of conjugacy classes of cuspidal inertia subgroups of $\Pi_{g,n}$ is trivial by the definition of $\pi_1(\mathcal{M}_{g,n})$, σ' is not contained in $\rho_{g,n}^{univ}(\pi_1(\mathcal{M}_{g,n}))$. Hence, we have $Z_{\rho_{g,n}^{univ}(\pi_1(\mathcal{M}_{g,n}))}(\rho_{g,n}^{univ}(\Gamma_{g,n}))/Z_{\rho_{g,n}^{univ}(\Gamma_{g,n})}(\rho_{g,n}^{univ}(\Gamma_{g,n})) = \{1\}$. □

3. Pro-l mapping class groups

In the present section, we prove the pro-l version of the main result of the present paper. Let l be a prime number and assume that the base field k is a field of characteristic zero.

Lemma 3.1. *Let (g,n) be a pair of nonnegative integers such that $2g-2+n>0$. Then the natural homomorphism $\pi_1(g,n) \to \Pi_{g.n}^{l}$ is injective.*

Proof. It follows immediately from the fact that $\pi_1(g,n)$ is conjugacy l-separable (Theorem 3.2, Theorem 4.1 in [31]). □

By above lemma, we can consider $\pi_1(g,n)$ as a subgroup of $\Pi_{g,n}^{l}$.

Lemma 3.2. *Let (g,n) be a pair of nonnegative integers such that $2g-2+n>0$. Then the group $N_{\Pi^l_{g,n}}(\pi_1(g,n))$ is equal to $\pi_1(g,n)$. In particular, the natural homomorphism $\mathrm{Out}(\pi_1(g,n))\to\mathrm{Out}(\Pi^l_{g,n})$ induced by $\pi_1(g,n)\hookrightarrow\Pi^l_{g,n}$ is injective.*

Proof. It is clear that $N_{\Pi^l_{g,n}}(\pi_1(g,n))\supseteq\pi_1(g,n)$ by the definition of normalizer. Let a be an element of $N_{\Pi^l_{g,n}}(\pi_1(g,n))$. Then, for any element γ of $\pi_1(g,n)$, γ is conjugate to $a\gamma a^{-1}$ in $\pi_1(g,n)$ by the fact that $\pi_1(g,n)$ is conjugacy l-separable (Theorem 3.2, Theorem 4.1 in [31]). Therefore, since $\pi_1(g,n)$ has Property A (Lemma 1, Theorem 3 in [11]), there exists an element h of $\pi_1(g,n)$ such that $a\gamma a^{-1}=h\gamma h^{-1}$ for any element γ of $\pi_1(g,n)$. Since $\Pi^l_{g,n}$ is center-free (Corollary 1.3.4 in [26]) and $\pi_1(g,n)$ is dense in $\Pi^l_{g,n}$, we have $a=h\in\pi_1(g,n)$. □

Remark 3.3. These lemmas may be well-known. At least, Lemma 3.2 was proved for special cases by several people (e.g., Proposition 1, [19], Corollary 2 to Proposition B2, [4]).

Theorem 3.4. *Let (g,n) be a pair of nonnegative integers such that $3g-3+n>0$ and either $(g,n)\neq(1,1)$ or $l=2$. Then the kernel of the homomorphism $\rho^{geom,l}_{g,n}$ coincides with the kernel of the homomorphism*

$$\rho^l_{\mathbb{P}^1_k\setminus\{0,1,\infty\}}:G_k\longrightarrow\mathrm{Out}(\Pi^l_{0,3}).$$

Proof. By the Galois Kernel Theorem in [16] (or Theorem C in [14]) and $\rho^{univ,l}_{g,n}(\Gamma_{g,n})$ is isomorphic to $\Gamma^l_{g,n}$ when g is zero (Remark to Theorem 1, [5]), we may assume that $g>0$. Here, $\Gamma^l_{g,n}$ is the pro-l completion of $\Gamma_{g.n}$. As the proof of Theorem 2.5, we can show that the natural homomorphism

$$G_k^{l,g,n}\longrightarrow\mathrm{Out}(\rho^{univ,l}_{g,n}(\Gamma_{g,n}))$$

is injective. Here, $G_k^{l,g,n}$ is the group

$$\rho^{univ,l}_{g,n}(\pi_1(\mathcal{M}_{g,n}))/\rho^{univ,l}_{g,n}(\Gamma_{g,n}).$$

Indeed, the arguments of the proof of Theorem 2.5 go well as they are, if we replace Theorem 2.4 (i) with Theorem 2.4 (ii), (iii) and the injectivity of $\mathrm{Out}(\pi_1(g,n))\to\mathrm{Out}(\Pi_{g.n})$ with the injectivity of $\mathrm{Out}(\pi_1(g,n))\to\mathrm{Out}(\Pi^l_{g,n})$ (Lemma 3.2). Therefore, it is sufficient to prove that

$$\ker(G_k\longrightarrow G_k^{l,g,n})=\ker(\rho^l_{\mathbb{P}^1_k\setminus\{0,1,\infty\}}).$$

Let $p_{g,n}:\pi_1(\mathcal{M}_{g,n})\to G_k$ be the natural homomorphism. Then we have

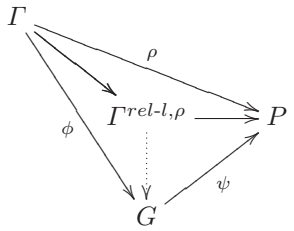
$$\ker(G_k\longrightarrow G_k^{l,g,n})=p_{g,n}(\ker(\rho^{univ,l}_{g,n})).$$

However, it is known that $p_{g,n}(\ker(\rho^{univ,l}_{g,n}))$ coincides with $\ker(\rho^l_{\mathbb{P}^1_k\setminus\{0,1,\infty\}})$ (Oda's conjecture, cf. Theorem 3.3, [33]). This completes the proof. □

Next, we consider the relative pro-l case. Since all mapping class groups in genus g are perfect when $g\geq 3$, their pro-l completions are trivial. However, Hain and Matsumoto developed a theory of relative pro-l completion of groups, and showed that the natural relative pro-l completions of mapping class groups are large and more closely reflect their structure ([13]). We explain below their theory.

Let Γ be a discrete or profinite group, P a profinite group, and $\rho:\Gamma\to P$ a continuous dense homomorphism. (Here, a dense homomorphism means a homomorphism with dense image.) The relative pro-l completion $\Gamma^{rel\text{-}l,\rho}$ of Γ with

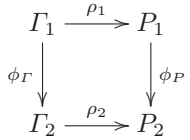
respect to ρ is characterized by a universal mapping property: if G is a profinite group, $\psi : G \to P$ a continuous homomorphism with pro-l kernel, and if $\phi : \Gamma \to G$ is a continuous homomorphism whose composition with ψ is ρ, then there is a unique continuous homomorphism $\Gamma^{rel\text{-}l,\rho} \to G$ that extends ϕ:



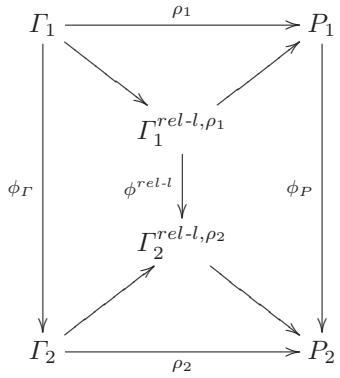
The following properties are direct consequences of the universal mapping property:

Proposition 3.5 (Proposition 2.1, [13]). *A dense homomorphism $\rho : \Gamma \to P$ from a discrete group to a profinite group induces a homomorphism $\overline{\rho} : \hat{\Gamma} \to P$ from the profinite completion of Γ to P. The natural homomorphism $\Gamma \to \hat{\Gamma}$ induces a natural isomorphism $\Gamma^{rel\text{-}l,\rho} \to \hat{\Gamma}^{rel\text{-}l,\overline{\rho}}$.*

Proposition 3.6 (Proposition 2.3, [13]). *Suppose that Γ_1 and Γ_2 are both discrete groups or both profinite groups and that P_1 and P_2 are profinite groups. Suppose that $\rho_j : \Gamma_j \to P_j$ ($j \in \{1, 2\}$) are continuous dense homomorphisms. If*



is a commutative diagram of topological groups, then there is a unique continuous homomorphism $\phi^{rel\text{-}\,l} : \Gamma_1^{rel\text{-}l,\rho_1} \to \Gamma_1^{rel\text{-}l,\rho_2}$ such that the diagram



commutes.

Proposition 3.7 (Proposition 2.4, [13]). *Suppose that P_1, P_2 and P_3 are profinite groups and that $\rho_j : \Gamma_j \to P_j$ ($j \in \{1, 2, 3\}$) are continuous dense homomorphisms of topological groups. Suppose that the Γ_j are all discrete groups or all profinite*

groups. If the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma_1 & \longrightarrow & \Gamma_2 & \longrightarrow & \Gamma_3 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_3} & & \\ 1 & \longrightarrow & P_1 & \longrightarrow & P_2 & \longrightarrow & P_3 & \longrightarrow & 1 \end{array}$$

of topological groups commutes and has two raws exact, then the sequence

$$\Gamma_1^{rel\text{-}l,\rho_1} \longrightarrow \Gamma_2^{rel\text{-}l,\rho_2} \longrightarrow \Gamma_3^{rel\text{-}l,\rho_3} \longrightarrow 1$$

is exact.

Let $\mathcal{A}_g$ be the moduli stack of principally polarized abelian varieties of dimension g. It is known that the orbifold fundamental groups $\pi_1^{\mathrm{orb}}(\mathcal{M}_{g,n}(\mathbb{C}))$ and $\pi_1^{\mathrm{orb}}(\mathcal{A}_g(\mathbb{C}))$ of $\mathcal{M}_{g,n}(\mathbb{C})$ and $\mathcal{A}_g(\mathbb{C})$ are isomorphic to $\mathrm{MCG}_{g,n}$ and $Sp_g(\mathbb{Z})$ respectively. Here, $Sp_g(A)$ is the group of symplectic $2g \times 2g$ matrices with entries in a commutative ring A. Let

$$\rho^{period} : \mathrm{MCG}_{g,n} \longrightarrow Sp_g(\mathbb{Z})$$

be the surjective homomorphism determined by the period map $\mathcal{M}_{g,n}(\mathbb{C}) \to \mathcal{A}_g(\mathbb{C})$ which takes the moduli point $[C]$ of a compact Riemann surface C (equipped with n marked points) to that of its jacobian $[\mathrm{Jac}(C)]$ (also see Chapter 6, [10]). Then ρ^{period} induces the continuous dense homomorphism

$$\rho^{period,l} : \mathrm{MCG}_{g,n} \longrightarrow Sp_g(\mathbb{Z}_l).$$

Hain and Matsumoto defined the relative pro-l completion of mapping class group by

$$\Gamma_{g,n}^{rel\text{-}l} := \mathrm{MCG}_{g,n}^{rel\text{-}l,\rho^{period,l}}.$$

Let $\overline{\rho}^{period,l} : \Gamma_{g,n} \to Sp_g(\mathbb{Z}/l)$ be the homomorphism determined by ρ^{period}. Then, by using Proposition 3.5 and the universal mapping property, we have the natural isomorphism

$$\Gamma_{g,n}^{rel\text{-}l} \simeq \Gamma_{g,n}^{rel\text{-}l,\overline{\rho}^{period,l}}.$$

This means that $\Gamma_{g,n}^{rel\text{-}l}$ is an almost pro-l group (i.e. there exists a closed subgroup of $\Gamma_{g,n}^{rel\text{-}l}$ with finite index that is a pro-l group). Also, Hain and Matsumoto proved that the natural homomorphism $\mathrm{MCG}_{g,n} \to \Gamma_{g,n}^{rel\text{-}l}$ is injective for $n > 0$ (Proposition 3.1, [13]). (In fact, since the injectivity of $\mathrm{MCG}_{g,n} \to \Gamma_{g,n}^{rel\text{-}l}$ is reduced to the injectivity of $\mathrm{MCG}_{g,n+1} \to \Gamma_{g,n+1}^{rel\text{-}l}$ by using Lemma 3.2, we also have the injectivity of $\mathrm{MCG}_{g,0} \to \Gamma_{g,0}^{rel\text{-}l}$ (for $g > 1$).)

The functoriality of relative pro-l completion implies that there is an outer Galois action

$$\rho_{g,n}^{rel\text{-}l} : G_k \longrightarrow \mathrm{Out}(\Gamma_{g,n}^{rel\text{-}l}).$$

Since the representation $\rho_{g,n}^{rel\text{-}l}$ is unramified outside l when k is a number field (Theorem 3, [13]), $\rho_{g,n}^{rel\text{-}l}$ is not injective. By using Theorem 3.4, we have the following corollary.

Corollary 3.8. *Let (g,n) be a pair of natural numbers such that $3g-3+n>0$ and either $(g,n) \neq (1,1)$ or $l = 2$. Then the kernel of the homomorphism $\rho_{g,n}^{rel\text{-}l}$ is contained in the kernel of the homomorphism*

$$\rho^l_{\mathbb{P}^1_k \setminus \{0,1,\infty\}} : G_k \longrightarrow \mathrm{Out}(\Pi^l_{0,3}).$$

Proof. The commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{g,n} & \longrightarrow & \Gamma_{g,n+1} & \longrightarrow & \Gamma_{g,n} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Pi^l_{g,n} & \longrightarrow & \Gamma^{rel\text{-}l}_{g,n+1} & \longrightarrow & \Gamma^{rel\text{-}l}_{g,n} & \longrightarrow & 1, \end{array}$$

where the horizontal sequences are exact (Proposition 3.1 (2), [13]), induces the following commutative diagram

$$\begin{array}{ccc} \Gamma_{g,n} & \longrightarrow & \Gamma^{rel\text{-}l}_{g,n} \\ & \searrow^{\rho^{univ,l}_{g,n}} & \downarrow \\ & & \mathrm{Out}(\Pi^l_{g,n}). \end{array}$$

Therefore, we have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma^{rel\text{-}l}_{g,n} & \longrightarrow & \pi_1(\mathcal{M}_{g,n})/\ker(\Gamma_{g,n} \to \Gamma^{rel\text{-}l}_{g,n}) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \rho^{univ,l}_{g,n}(\Gamma_{g,n}) & \longrightarrow & \rho^{univ,l}_{g,n}(\pi_1(\mathcal{M}_{g,n})) & \longrightarrow & G^{l,g,n}_k & \longrightarrow & 1, \end{array}$$

where the horizontal sequences are exact and the vertical homomorphisms are surjective. Hence, this induces

$$\ker(\rho^{rel\text{-}l}_{g,n}) \subseteq \ker(\rho^{geom,l}_{g,n}) = \ker(\rho^l_{\mathbb{P}^1_k \setminus \{0,1,\infty\}}).$$

□

4. The case of an arbitrary family of hyperbolic curves

In the present section, we prove a variant of Theorem 2.3. Let l be a prime number, k a field of characteristic zero, and $\overline{k}$ an algebraic closure of k. For any geometrically connected regular scheme S of finite type over k and any family $X \to S$ of (g,n)-curves over S, we denote by $\varphi^l_{X/S} : \pi_1(S \otimes_k \overline{k}) \to \mathrm{Aut}(\Pi^{\mathrm{ab}}_{g,n} \otimes_{\mathbb{Z}} (\mathbb{Z}/l))$ the natural monodromy action arising from the family of (g,n)-curves $X \to S$. Here, the group $\Pi^{\mathrm{ab}}_{g,n}$ is the abelianization of $\Pi_{g,n}$.

Proposition 4.1. *Let (g,n) be a pair of nonnegative integers such that $2g-2+n > 0$, S a geometrically connected regular scheme of finite type over k, and $X \to S$ a family of (g,n)-curves over S. Then the natural sequence*

$$1 \longrightarrow \Pi_{g,n} \longrightarrow \pi_1(X) \longrightarrow \pi_1(S) \longrightarrow 1$$

is exact. Moreover, if the image of $\varphi^l_{X/S}$ is an l-group, then the natural sequence

$$1 \longrightarrow \Pi^l_{g,n} \longrightarrow \pi_1(X)^{\underline{l}} \longrightarrow \pi_1(S)^{\underline{l}} \longrightarrow 1$$

is exact.

Proof. It is enough to prove the case for $k = \overline{k}$. First, we prove the profinite case. Then we have the following exact sequence

$$\Pi_{g,n} \longrightarrow \pi_1(X) \longrightarrow \pi_1(S) \longrightarrow 1$$

by [1]. Let $X^{\mathrm{cpt}} \to S$ be the compactification of $X \to S$ and $D \subseteq X^{\mathrm{cpt}}$ the divisor at infinity of $X \to S$. Then we can take a finite étale (connected) Galois covering $S' \to S$ such that the finite étale covering $D \times_S S' \to S'$ is split. We put $X' := X \times_S S'$, $X'^{\mathrm{cpt}} := X^{\mathrm{cpt}} \times_S S'$, $D' := D \times_S S'$. Then the natural projection $X' \to S'$ is a family of (g,n)-curves and X'^{cpt} (respectively D') is the compactification (respectively the divisor at infinity) of $X' \to S'$. Since $D' \to S'$ is split, by Proposition 2.2 in [25], the natural sequence

$$1 \longrightarrow \Pi_{g,n} \longrightarrow \pi_1(X') \longrightarrow \pi_1(S') \longrightarrow 1$$

is exact. Moreover, by the definition of $X' \to S'$, we have the following commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \Pi_{g,n} & \longrightarrow & \pi_1(X') & \longrightarrow & \pi_1(S') \longrightarrow 1 \\ & & \| & & \downarrow & & \downarrow \\ & & \Pi_{g,n} & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(S) \longrightarrow 1. \end{array}$$

Now, since the natural projection $X' \to X$ is a finite étale covering, $\pi_1(X') \to \pi_1(X)$ is injective. This completes the proof for the profinite case.

Next, we consider the pro-l case. Since the image of $\varphi^l_{X/S}$ is an l-group, by using Lemma 4.5.5 in [32] and Théorème 2.3.1 in [2], the natural homomorphism $\pi_1(S) \to \mathrm{Out}(\Pi_{g,n}) \to \mathrm{Out}(\Pi^l_{g,n})$ factors through the maximal pro-l quotient $\pi_1(S)^l$ of $\pi_1(S)$. Therefore, the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{g,n} & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(S) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}(\Pi^l_{g,n}) & \longrightarrow & \mathrm{Aut}(\Pi^l_{g,n}) & \longrightarrow & \mathrm{Out}(\Pi^l_{g,n}) & \longrightarrow & 1 \end{array}$$

induces the following commutative diagram

$$\begin{array}{ccccccccc} & & \Pi^l_{g,n} & \longrightarrow & \pi_1(X)^l & \longrightarrow & \pi_1(S)^l & \longrightarrow & 1 \\ & & \wr\downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}(\Pi^l_{g,n}) & \longrightarrow & \mathrm{Aut}(\Pi^l_{g,n}) & \longrightarrow & \mathrm{Out}(\Pi^l_{g,n}) & \longrightarrow & 1, \end{array}$$

where the horizontal sequences are exact and the left vertical homomorphism is isomorphism by Corollary 1.3.4 in [26]. This completes the proof for the pro-l case. □

In the notation of the above proposition, we have the natural homomorphisms $\varphi_S : \pi_1(S) \to \mathrm{Out}(\Pi_{g,n})$, $\varphi^l_S : \pi_1(S) \to \mathrm{Out}(\Pi^l_{g,n})$ determined by the following exact sequence

$$1 \longrightarrow \Pi_{g,n} \longrightarrow \pi_1(X) \longrightarrow \pi_1(S) \longrightarrow 1.$$

Note that $\Gamma_{0,4}$ (respectively $\Gamma^{rel\text{-}l}_{0,4}$) is canonically isomorphic to $\Pi_{0,3}$ (respectively $\Pi^l_{0,3}$). By a similar argument used in the proof of Theorem 2.1 (Theorem 1.1, [21] or Corollary 6.4, [14]), we can prove the following proposition.

Proposition 4.2. *Let (g,n) be a pair of nonnegative integers such that $2g-2+n>0$, S a geometrically connected regular scheme of finite type over k with a k-rational point s, $X \to S$ a family of (g,n)-curves over S, X_s the fiber of $X \to S$ at s, and ρ_{X_s} (respectively $\rho^l_{X_s}$) the homomorphism $G_k \to \mathrm{Out}(\Pi_{g,n})$ (respectively $G_k \to \mathrm{Out}(\Pi^l_{g,n})$) associated to the (g,n)-curve X_s over k. Then the subgroup*

$$\rho_{X_s}^{-1}(\varphi_S(\pi_1(S \otimes_k \overline{k}))) \subseteq G_k \text{ (respectively } (\rho^l_{X_s})^{-1}(\varphi^l_S(\pi_1(S \otimes_k \overline{k}))) \subseteq G_k)$$

of G_k is contained in the kernel of the homomorphism

$$\rho_{0,4} : G_k \longrightarrow \mathrm{Out}(\Pi_{0,3}) \text{ (respectively } \rho^{rel\text{-}l}_{0,4} : G_k \longrightarrow \mathrm{Out}(\Pi^l_{0,3})).$$

Proof. Since the pro-l case can be proved by exactly the same argument, we prove only the profinite case. Let i_s be the section $G_k \to \pi_1(S)$ induced by the k-rational point s, $k(S)$ the function field of S, $\overline{k(S)}$ an algebraic closure of $k(S)$, $X_{k(S)} := X \times_S \mathrm{Spec}\, k(S)$, $\rho_{X_{k(S)}}$ the homomorphism $G_{k(S)} := \mathrm{Gal}(\overline{k(S)}/k(S)) \to \mathrm{Out}(\Pi_{g,n})$ associated to the (g,n)-curve $X_{k(S)}$ over $k(S)$. Then we have $\varphi_S \circ i_s = \rho_{X_s}$, and the natural (outer) homomorphisms $G_{k(S)} \to \pi_1(S)$ is surjective by the geometrically-connectedness of S. Assume that there exist $\gamma \in \pi_1(S \otimes_k \overline{k})$ and $\sigma \in G_k$ such that $\varphi_S(\gamma)$ is equal to $\rho_{X_s}(\sigma)$. By the surjectivity of the above (outer) homomorphism, we can take $\tilde{\gamma}, \tilde{\sigma} \in G_{k(S)}$ mapped to $\gamma, i_s(\sigma) \in \pi_1(S)$, respectively. Since the following diagram

$$\begin{array}{ccc} G_{k(S)} & \twoheadrightarrow & \pi_1(S) \\ & \searrow^{\rho_{X_{k(S)}}} & \downarrow{\scriptstyle \varphi_S} \\ & & \mathrm{Out}(\Pi_{g,n}) \end{array}$$

is commutative, $\tilde{\gamma}\tilde{\sigma}^{-1}$ is contained in the kernel of $\rho_{X_{k(S)}}$. Hence, by Corollary 6.2, in [14], $\tilde{\gamma}\tilde{\sigma}^{-1}$ is contained in the kernel of the natural homomorphism $G_{k(S)} \to \mathrm{Out}(\Pi_{0,3})$. Now, since the following diagram

$$\begin{array}{ccc} G_{k(S)} & \longrightarrow & G_k \\ & \searrow & \downarrow{\scriptstyle \rho_{0,4}} \\ & & \mathrm{Out}(\Pi_{0,3}) \end{array}$$

is commutative and γ is contained in the kernel of $\pi_1(S) \to G_k$, σ is contained in the kernel of $\rho_{0,4}$. □

For a scheme X which is a geometrically connected and of finite type over k, we denote by $\rho^l_X : G_k \to \mathrm{Out}(\pi_1(X \otimes_k \overline{k})^l)$ the composite of $\rho_X : G_k \to \mathrm{Out}(\pi_1(X \otimes_k \overline{k}))$ and the natural homomorphism $\mathrm{Out}(\pi_1(X \otimes_k \overline{k})) \to \mathrm{Out}(\pi_1(X \otimes_k \overline{k})^l)$. The following theorem is a variant of Theorem 2.3.

Theorem 4.3. *Let (g,n) be a pair of nonnegative integers such that $2g-2+n>0$, S a geometrically connected regular scheme of finite type over k, $X \to S$ a family of (g,n)-curves over S. Then the kernel of the homomorphism ρ_X is contained in the kernel of the homomorphism*

$$\rho_{0,4} : G_k \longrightarrow \mathrm{Out}(\Pi_{0,3}).$$

Moreover, if the image of $\varphi^l_{X/S}$ is an l-group, then the kernel of the homomorphism ρ^l_X is contained in the kernel of the homomorphism

$$\rho^{rel\text{-}l}_{0,4} : G_k \longrightarrow \mathrm{Out}(\Pi^l_{0,3}).$$

In particular, if k is a number field or an l-adic local field, then the homomorphism ρ_X is injective.

Proof. First, we prove the profinite case. Let $k(S)$ be the function field of S, $\overline{k(S)}$ an algebraic closure of $k(S)$, $X_{k(S)} := X \times_S \mathrm{Spec}\, k(S)$, $X_{\overline{k(S)}} := X \times_S \mathrm{Spec}\, \overline{k(S)}$, $S_{k(S)} := S \otimes_k k(S)$. Then the diagonal map $S \to S \times_{\mathrm{Spec}\, k} S$ induces a section $\mathrm{Spec}\, k(S) \to S_{k(S)}$ of the natural projection $S_{k(S)} \to \mathrm{Spec}\, k(S)$. Note that we have the following diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X_{\overline{k(S)}}) & \longrightarrow & \pi_1(X_{k(S)}) & \longrightarrow & G_{k(S)} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \iota} & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(X \otimes_k \overline{k}) & \longrightarrow & \pi(X) & \longrightarrow & G_k & \longrightarrow & 1.
\end{array}$$

This diagram induces the following commutative diagram

$$\begin{array}{ccc}
G_{k(S)} & \longrightarrow & G_k \\
 & {\scriptstyle \rho_{X_{k(S)}}} \searrow & \downarrow{\scriptstyle \rho_X} \\
 & & \mathrm{Out}(\pi_1(X \otimes_k \overline{k})).
\end{array}$$

Also, since S is geometrically connected over k, the natural (outer) homomorphism $G_{k(S)} = \mathrm{Gal}(\overline{k(S)}/k(S)) \to G_k$ is surjective. In particular, $\ker(\rho_{X_{k(S)}})$ surjects onto $\ker(\rho_X)$. Therefore, if $\ker(\rho_{X_{k(S)}})$ is included in $\ker(G_{k(S)} \to \mathrm{Out}(\Pi_{0,3}))$, $\ker(\rho_X)$ is included in $\ker(G_k \to \mathrm{Out}(\Pi_{0,3}))$ by the following commutative diagram

$$\begin{array}{ccc}
G_{k(S)} & \twoheadrightarrow & G_k \\
 & \searrow & \downarrow \\
 & & \mathrm{Out}(\Pi_{0,3}).
\end{array}$$

Hence, replacing $X \to S \to \mathrm{Spec}\, k$ by $X_{k(S)} \to S_{k(S)} \to \mathrm{Spec}\, k(S)$ if necessary, we may assume that S has a k-rational point. Let s be a k-rational point of S, $\overline{s}$ a $\overline{k}$-rational point over s, X_s the fiber of $X \to S$ at s, $X_{\overline{s}}$ the fiber of $X \to S$ at $\overline{s}$. The above k-rational point s of S induces a cartesian square

$$\begin{array}{ccc}
X_s & \longrightarrow & \mathrm{Spec}\, k \\
\downarrow & \square & \downarrow \\
X & \longrightarrow & S.
\end{array}$$

This induces a commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Pi_{g,n} & \longrightarrow & \pi_1(X \otimes_k \overline{k}) & \longrightarrow & \pi_1(S \otimes_k \overline{k}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \pi_1(X_s) & \longrightarrow & \pi_1(X) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G_k & = & G_k & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1, & & & &
\end{array}$$

where the vertical and horizontal sequences are exact. Then Lemma 2.2 implies that

$$\rho_{X_s}(\ker(\rho_X)) \subseteq \mathrm{im}(\varphi_S : \pi_1(S \otimes_k \overline{k}) \longrightarrow \mathrm{Out}(\Pi_{g,n})).$$

Here, ρ_{X_s} is the homomorphism $G_k \to \mathrm{Out}(\Pi_{g,n})$ associated to the hyperbolic curve X_s over k. Hence, by using Proposition 4.2, the result follows for the profinite case.

For the pro-l case, since we have the following commutative diagram

$$\begin{array}{ccc}
\pi_1(S) & \twoheadrightarrow & \pi_1(S)^{\underline{l}} \\
 & \searrow^{\varphi_S^l} & \downarrow \\
 & & \mathrm{Out}(\Pi_{g,n}^l),
\end{array}$$

we can prove by exactly the same argument. □

Remark 4.4. It is trivial that Theorem 2.5 implies Theorem 2.3. However, it seems that Theorem 2.5 (or its proof) does not imply Theorem 4.3.

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