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AN EXPLICIT FORMULA FOR THE GENERIC NUMBER OF DORMANT INDIGENOUS BUNDLES

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Abstract. A dormant indigenous bundle is an integrable $\mathbb{P}^1$-bundle on a proper hyperbolic curve of positive characteristic satisfying certain conditions. Dormant indigenous bundles were introduced and studied in the p-adic Teichmüller theory developed by S. Mochizuki. K. Joshi proposed a conjecture concerning an explicit formula for the degree over the moduli stack of curves of the moduli stack classifying dormant indigenous bundles. In this paper, we give a proof for this conjecture of Joshi.

Contents

Introduction

Let

$$\mathcal{M}^{\text{Zzz...}}_{g,\mathbb{F}_p}$$

be the moduli stack classifying proper smooth curves of genus $g > 1$ over $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ together with a *dormant* indigenous bundle (cf. the notation "Zzz..."!). It is known (cf. Theorem 3.3) that $\mathcal{M}^{\text{Zzz...}}_{g,\mathbb{F}_p}$ is represented by a smooth, geometrically connected Deligne-Mumford stack over $\mathbb{F}_p$ of dimension $3g-3$. Moreover, if we denote by $\mathcal{M}_{g,\mathbb{F}_p}$ the moduli stack classifying proper smooth curves of genus g over $\mathbb{F}_p$, then the natural projection $\mathcal{M}^{\text{Zzz...}}_{g,\mathbb{F}_p} \to \mathcal{M}_{g,\mathbb{F}_p}$ is finite, faithfully flat, and generically étale. The main theorem of the present paper, which was conjectured by K. Joshi, asserts that if $p > 2(g-1)$, then the degree $\deg_{\mathcal{M}_{g,\mathbb{F}_p}}(\mathcal{M}^{\text{Zzz...}}_{g,\mathbb{F}_p})$ of $\mathcal{M}^{\text{Zzz...}}_{g,\mathbb{F}_p}$ over $\mathcal{M}_{g,\mathbb{F}_p}$ may be calculated as follows:

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Theorem A (= Corollary 5.4)**.**

$$\deg_{\mathcal{M}_{g,\mathbb{F}_p}}(\mathcal{M}^{\text{Zzz...}}_{g,\mathbb{F}_p}) = \frac{p^{g-1}}{2^{2g-1}} \cdot \sum_{\theta=1}^{p-1} \frac{1}{\sin^{2g-2}(\frac{\pi\cdot\theta}{p})}.$$

Here, recall that an indigenous bundle on a proper smooth curve X is a $\mathbb{P}^1$-bundle on X, together with a connection, that satisfy certain properties (cf. Definition 2.1). The notion of an indigenous bundle was originally introduced and studied by Gunning in the context of compact hyperbolic Riemann surfaces (cf. [10]). One may think of an indigenous bundle as an algebraic object encoding uniformization data for X. It may be interpreted as a projective structure, i.e., a maximal atlas covered by coordinate charts on X such that the transition functions are expressed as Möbius transformations. Also, various equivalent mathematical objects, including certain kinds of differential operators or kernel functions, have been studied by many mathematicians.

In the present paper, we focus on indigenous bundles in *positive characteristic*. Just as in the case of the theory over $\mathbb{C}$, one may define the notion of an indigenous bundle and the moduli space classifying indigenous bundles. Various properties of such objects were firstly discussed in the context of the p-adic Teichmüller theory developed by S. Mochizuki (cf. [26], [27]). One of the key ingredients in the development of this theory is the study of the p-curvature of indigenous bundles in characteristic p. Recall that the p-curvature of a connection may be thought of as the obstruction to the compatibility of p-power structures that appear in certain associated spaces of infinitesimal (i.e., "Lie") symmetries. We say that an indigenous bundle is *dormant* (cf. Definition 3.1) if its p-curvature vanishes identically. This condition on an indigenous bundle implies, in particular, the existence of "sufficiently many" horizontal sections locally in the Zariski topology. Moreover, a dormant indigenous bundle corresponds, in a certain sense, to a certain type of rank 2 semistable bundle. Such semistable bundles have been studied in a different context (cf. § 6.1). This sort of phenomenon is peculiar to the theory of indigenous bundles in *positive characteristic*.

In this context, one natural question is the following:

> *Can one calculate explicitly the number of dormant indigenous bundles on a general curve?*

Since (as discussed above) $\mathcal{M}^{\text{Zzz...}}_{g,\mathbb{F}_p}$ is finite, faithfully flat, and generically étale over $\mathcal{M}_{g,\mathbb{F}_p}$, the task of resolving this question may be reduced to the explicit computation of the degree $\deg_{\mathcal{M}_{g,\mathbb{F}_p}}(\mathcal{M}^{\text{Zzz...}}_{g,\mathbb{F}_p})$ of $\mathcal{M}^{\text{Zzz...}}_{g,\mathbb{F}_p}$ over $\mathcal{M}_{g,\mathbb{F}_p}$.

In the case of $g = 2$, S. Mochizuki (cf. [27], Chap. V, Corollary 3.7), H. Lange-C. Pauly (cf. [22], Theorem 2), and B. Osserman (cf. [30], Theorem 1.2) verified (by applying different methods) the equality

$$\deg_{\mathcal{M}_{2,\mathbb{F}_p}}(\mathcal{M}^{\text{Zzz...}}_{2,\mathbb{F}_p}) = \frac{1}{24} \cdot (p^3 - p).$$

For general g, K. Joshi conjectured, with his amazing insight, an explicit description, as asserted in Theorem A, of the value $\deg_{\mathcal{M}_{g,\mathbb{F}_p}}(\mathcal{M}^{\mathrm{Zzz...}}_{g,\mathbb{F}_p})$. (In fact, Joshi has proposed a somewhat more general conjecture. In the present paper, however, we shall restrict our attention to a certain special case of this more general conjecture.) The goal of the present paper is to verify this conjecture of Joshi.

Our discussion in the present paper follows, to a substantial extent, the ideas discussed in [16], as well as in personal communication to the author by K. Joshi. Indeed, certain of the results obtained in the present paper are mild generalizations of the results obtained in [16] concerning rank 2 opers to the case of *families of curves* over quite general base schemes. (Such *relative* formulations are necessary in the theory of the present paper, in order to consider *deformations* of various types of data.) For example, Lemma 4.1 in the present paper corresponds to [16], Theorem 3.1.6 (or [17], § 5.3; [32], Lemma 2.1); Lemma 4.2 corresponds to [16], Theorem 5.4.1; and Proposition 4.3 corresponds to [16], Proposition 5.4.2. Moreover, the insight concerning the connection with the formula of Holla (cf. Theorem 5.1), which is a special case of the Vafa-Intriligator formula, is due to Joshi.

On the other hand, the new ideas introduced in the present paper may be summarized as follows. First, we verify the *vanishing of obstructions* to deformation to characteristic zero of a certain Quot-scheme that is related to $\mathcal{M}^{\mathrm{Zzz...}}_{g,\mathbb{F}_p}$ (cf. Proposition 4.3, Lemma 4.4, and the discussion in the proof of Lemma 5.2). Then we relate the value $\deg_{\mathcal{M}_{g,\mathbb{F}_p}}(\mathcal{M}^{\mathrm{Zzz...}}_{g,\mathbb{F}_p})$ to the degree of the result of base-changing this Quot-scheme to $\mathbb{C}$ by applying the formula of Holla (cf. Theorem 5.1, the proof of Lemma 5.2) *directly*.

Finally, F. Liu and B. Osserman have shown (cf. [19], Theorem 2.1) that the value $\deg_{\mathcal{M}_{g,\mathbb{F}_p}}(\mathcal{M}^{\mathrm{Zzz...}}_{g,\mathbb{F}_p})$ may expressed as a polynomial with respect to the characteristic of the base field. This was done by applying Ehrhart's theory concerning the cardinality of the set of lattice points inside a polytope. In § 6, we shall discuss the relation between this result and the main theorem of the present paper.

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1. Preliminaries

1.1. Throughout this paper, we fix an *odd* prime number p.

1.2. We shall denote by (Set) the category of (small) sets. If S is a Deligne-Mumford stack, then we shall denote by $(Sch)_S$ the category of schemes over S.

1.3. If S is a scheme and $\mathcal{F}$ an $\mathcal{O}_S$-module, then we shall denote by $\mathcal{F}^\vee$ its dual sheaf, i.e., $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{O}_S)$. If $f : T \to S$ is a finite flat scheme over a connected scheme S, then we shall denote by $\deg_S(T)$ the degree of T over S, i.e., the rank of locally free $\mathcal{O}_S$-module $f_* \mathcal{O}_T$.

1.4. If S is a scheme (or more generally, a Deligne-Mumford stack), then we define a *curve* over S to be a geometrically connected and flat (relative) scheme $f : X \to S$ over S of relative dimension 1. Denote by $\Omega_{X/S}$ the sheaf of 1-differentials of X over S and $\mathcal{T}_{X/S}$ the dual sheaf of $\Omega_{X/S}$ (i.e., the sheaf of derivations of X over S). We shall say that a proper smooth curve $f : X \to S$ over S is *of genus* g if the direct image $f_* \Omega_{X/S}$ is locally free of constant rank g.

1.5. Let S be a scheme over a field k, X a smooth scheme over S, G an algebraic group over k, and $\mathfrak{g}$ the Lie algebra of G. Suppose that $\pi : \mathcal{E} \to X$ is a G-torsor on X. Then we may associate to π a short exact sequence

$$0 \to \mathcal{E} \wedge^G \mathfrak{g} \to \widetilde{\mathcal{T}}_{\mathcal{E}/S} \stackrel{\alpha_{\mathcal{E}}}{\to} \mathcal{T}_{X/S} \to 0$$

— where $\mathcal{E} \wedge^G \mathfrak{g}$ denotes the adjoint bundle associated to $\mathcal{E}$, and $\widetilde{\mathcal{T}}_{\mathcal{E}/S}$ denotes the subsheaf of G-invariant sections $(\pi_* \mathcal{T}_{\mathcal{E}/S})^G$ of $\pi_* \mathcal{T}_{\mathcal{E}/S}$. An *$S$-connection* on $\mathcal{E}$ is a split injection $\nabla : \mathcal{T}_{X/S} \to \widetilde{\mathcal{T}}_{\mathcal{E}/S}$ of the above short exact sequence (i.e., $\alpha_{\mathcal{E}} \circ \nabla = \mathrm{id}$). If X is of relative dimension 1 over S, then any such S-connection is necessarily *integrable*, i.e., compatible with the Lie bracket structures on $\mathcal{T}_{X/S}$ and $\widetilde{\mathcal{T}}_{\mathcal{E}/S} = (\pi_* \mathcal{T}_{\mathcal{E}/S})^G$.

1.6. Let S be a scheme of characteristic p (cf. §1, 1.1) and $f : X \to S$ a scheme over S. The *Frobenius twist of X over S* is the base-change of $f : X \to S$ via the absolute Frobenius morphism $F_S : S \to S$ of S. Denote by $f_F : X_F \to S$ the structure morphism of X_F over S. The *relative Frobenius morphism of X over S* is the unique morphism $F_{X/S} : X \to X_F$ over S that fits into a commutative diagram of the form

$$\begin{array}{ccccc} X & \xrightarrow{F_{X/S}} & X_F & \longrightarrow & X \\ {\scriptstyle f}\downarrow & & {\scriptstyle f_F}\downarrow & & {\scriptstyle f}\downarrow \\ S & \xrightarrow{\mathrm{id}} & S & \longrightarrow & S, \end{array}$$

where the upper (respectively, the lower) composite is the absolute Frobenius morphism of X (respectively, S). If $f : X \to S$ is smooth, geometrically connected and of relative dimension n, then the relative Frobenius morphism $F_{X/S} : X \to X_F$ is finite and faithfully flat of degree p^n. In particular, the $\mathcal{O}_{X_F}$-module $F_{X/S,*}\mathcal{O}_X$ is locally free of rank p^n.

2. Indigenous bundles

In this section, we recall the notion of an indigenous bundle on a curve. Much of the content of this section is implicit in [26].

First, we discuss the definition of an indigenous bundle on a curve. Fix a scheme S of characteristic p (cf. §1.1) and a proper smooth curve $X \to S$ of genus $g > 1$ (cf. §1.2).

Definition 2.1. (cf. [7], §4; [26], Chap. I, Def. 2.2)

(i) Let $\mathcal{P}^{\circledast} = (\mathcal{P}, \nabla)$ be a pair consisting of a PGL_2-torsor $\mathcal{P}$ on X and an (integrable) S-connection ∇ on $\mathcal{P}$. We shall say that $\mathcal{P}^{\circledast}$ is an *indigenous bundle* on X/S if there exists a globally defined section σ of the associated $\mathbb{P}^1$-bundle $\mathbb{P}^1_{\mathcal{P}} := \mathcal{P} \wedge^{\mathrm{PGL}_2} \mathbb{P}^1$ which has a nowhere vanishing derivative with respect to the connection ∇. We shall refer to the section σ as the *Hodge section* of $\mathcal{P}^{\circledast}$ (cf. Remark 2.1.1 (i)).

(ii) Let $\mathcal{P}^{\circledast}_1 = (\mathcal{P}_1, \nabla_1)$, $\mathcal{P}^{\circledast}_2 = (\mathcal{P}_2, \nabla_2)$ be indigenous bundles on X/S. An *isomorphism* from $\mathcal{P}^{\circledast}_1$ to $\mathcal{P}^{\circledast}_2$ is an isomorphism $\mathcal{P}_1 \xrightarrow{\sim} \mathcal{P}_2$ of PGL_2-torsors on X that is compatible with the respective connections (cf. Remark 2.1.1 (iii)).

Remark 2.1.1.
Let $\mathcal{P}^{\circledast} = (\mathcal{P}, \nabla)$ be an indigenous bundle on X/S.

(i) The Hodge section σ of $\mathcal{P}^{\circledast}$ is uniquely determined by the condition that σ have a nowhere vanishing derivative with respect to ∇ (cf. [26], Chap. I, Proposition 2.4).

(ii) The underlying PGL_2-torsors of any two indigenous bundles on X/S are isomorphic (cf. [26], Chap. I, Proposition 2.5). If there is a spin structure $\mathbb{L} = (\mathcal{L}, \eta_{\mathcal{L}})$ on X/S (cf. Definition 2.2), then $\mathcal{P}$ is isomorphic to the projectivization of an $\mathbb{L}$-bundle $\mathcal{F}$ as in Definition 2.3 (i), and the subbundle $\mathcal{L} \subseteq \mathcal{F}$ (cf. Definition 2.3 (i)) induces the Hodge section σ (cf. Proposition 2.4).

(iii) If two indigenous bundles on X/S are isomorphic, then any isomorphism between them is unique. In particular, an indigenous bundle has no nontrivial automorphisms (cf. [26], Chap. I, Theorem 2.8).

Next, we consider a certain class of rank 2 vector bundles with an integrable connection (cf. Definition 2.3 (ii)) associated to a specific choice of a spin structure (cf. Definition 2.2). In particular, we show (cf. Proposition 2.4) that such objects correspond to indigenous bundles bijectively.

Definition 2.2. (cf., e.g., [15])
A *spin structure* on X/S is a pair

$$\mathbb{L} := (\mathcal{L}, \eta_{\mathcal{L}})$$

consisting of an invertible sheaf $\mathcal{L}$ on X and an isomorphism $\eta_{\mathcal{L}} : \Omega_{X/S} \xrightarrow{\sim} \mathcal{L}^{\otimes 2}$.
A *spin curve* is a pair

$$(Y/S, \mathbb{L})$$

consisting of a proper smooth curve Y/S of genus $g > 1$ and a spin structure $\mathbb{L}$ on Y/S.

Remark 2.2.1.

(i) X/S necessarily admits, at least étale locally on S, a spin structure. Indeed, let us denote by $Pic^d_{X/S}$ the relative Picard scheme of X/S classifying the set of (equivalence classes, relative to the equivalence relation determined by tensoring with a line bundle pulled back from the base S, of) degree d invertible sheaves on X. Then the morphism

$$\mathrm{Pic}^{g-1}_{X/S} \to \mathrm{Pic}^{2g-2}_{X/S} : [\mathcal{L}] \mapsto [\mathcal{L}^{\otimes 2}]$$

given by multiplication by 2 is finite and étale. Thus, the S-rational point of $\mathrm{Pic}^{2g-2}_{X/S}$ classifying the equivalence class $[\Omega_{X/S}]$ determined by $\Omega_{X/S}$ lifts, étale locally, to a point of $\mathrm{Pic}^{g-1}_{X/S}$.

(ii) Let $(\mathcal{L}, \eta_{\mathcal{L}})$ be a spin structure on X/S and T an S-scheme. Then by pulling back the structures $\mathcal{L}$, $\eta_{\mathcal{L}}$ via the natural projection $X \times_S T \to X$, we obtain a spin structure on the curve $X \times_S T$ over T, which, by abuse of notation, we shall also denote by $\mathbb{L}$.

In the following, let us fix a spin structure $\mathbb{L} = (\mathcal{L}, \eta_{\mathcal{L}})$ on X/S.

Definition 2.3.

(i) An $\mathbb{L}$-*bundle* on X/S is an extension, in the category of $\mathcal{O}_X$-modules,

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{F} \longrightarrow \mathcal{L}^\vee \longrightarrow 0$$

of $\mathcal{L}^\vee$ by $\mathcal{L}$ whose restriction to each fiber over S is nontrivial (cf. Remark 2.3.1 (i)). We shall regard the underlying rank 2 vector bundle associated to an $\mathbb{L}$-bundle as being equipped with a 2-step decreasing filtration $\{\mathcal{F}^i\}_{i=0}^2$, namely, the filtration defined as follows:

$$\mathcal{F}^2 := 0 \quad \subseteq \quad \mathcal{F}^1 := \mathrm{Im}(\mathcal{L}) \quad \subseteq \quad \mathcal{F}^0 := \mathcal{F}.$$

(ii) An $\mathbb{L}$-*indigenous vector bundle* on X/S is a triple

$$\mathcal{F}^{\circledast} := (\mathcal{F}, \nabla, \{\mathcal{F}^1\}_{i=0}^2)$$

consisting of an $\mathbb{L}$-bundle $(\mathcal{F}, \{\mathcal{F}^i\}_{i=0}^2)$ on X/S and an S-connection $\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega_{X/S}$ on $\mathcal{F}$ satisfying the following two conditions.

(1) If we equip $\mathcal{O}_X$ with the trivial connection and the determinant bundle $\det(\mathcal{F})$ with the natural connection induced by ∇, then the natural composite isomorphism

$$\det(\mathcal{F}) \overset{\sim}{\to} \mathcal{L} \otimes \mathcal{L}^\vee \overset{\sim}{\to} \mathcal{O}_X$$

is horizontal.

(2) The composite

$$\mathcal{L} \overset{\nabla|_{\mathcal{L}}}{\to} \mathcal{F} \otimes \Omega_{X/S} \twoheadrightarrow \mathcal{L}^\vee \otimes \Omega_{X/S}$$

of the restriction $\nabla|_{\mathcal{L}}$ of ∇ to $\mathcal{L}$ $(\subseteq \mathcal{F})$ and the morphism $\mathcal{F} \otimes \Omega_{X/S} \twoheadrightarrow \mathcal{L}^\vee \otimes \Omega_{X/S}$ induced by the quotient $\mathcal{F} \twoheadrightarrow \mathcal{L}^\vee$ is an isomorphism. This composite is often referred to as the *Kodaira-Spencer map*.

(iii) Let $\mathcal{F}_1^{\circledast} = (\mathcal{F}_1, \nabla_1, \{\mathcal{F}_1^1\}_{i=0}^2)$, $\mathcal{F}_2^{\circledast} = (\mathcal{F}_2, \nabla_2, \{\mathcal{F}_2^1\}_{i=0}^2)$ on X/S be $\mathbb{L}$-indigenous bundles on X/S. Then an *isomorphism* from $\mathcal{F}_1^{\circledast}$ to $\mathcal{F}_2^{\circledast}$ is an isomorphism $\mathcal{F}_1 \overset{\sim}{\to} \mathcal{F}_2$ of $\mathcal{O}_X$-modules that is compatible with the respective connections and filtrations and induces the identity morphism of $\mathcal{O}_X$ (relative to the respective natural composite isomorphisms discussed in (i)) upon taking determinants.

Remark 2.3.1.

(i) X/S always admits an $\mathbb{L}$-bundle. Moreover, any two $\mathbb{L}$-bundles on X/S are isomorphic Zariski locally on S. Indeed, since $f : X \to S$ is of relative dimension 1, the Leray-Serre spectral sequence associated to the morphism $f : X \to S$ yields an exact sequence

$$H^1(S, f_*\Omega_{X/S}) \to \mathrm{Ext}^1(\mathcal{L}^\vee, \mathcal{L}) \to H^0(S, \mathbb{R}^1 f_*\Omega_{X/S}(\cong \mathcal{O}_S)) \to 0,$$

where the set $\mathrm{Ext}^1(\mathcal{L}^\vee, \mathcal{L})$ corresponds to the set of extension classes of $\mathcal{L}^\vee$ by $\mathcal{L}$. In particular, if S is an affine scheme, then the set of nontrivial extension classes corresponds bijectively to the subset $H^0(S, \mathcal{O}_S) \setminus \{0\} \subseteq H^0(S, \mathcal{O}_S) \cong H^0(S, \mathbb{R}^1 f_*\Omega_{X/S})$.

Also, we note that it follows immediately from the fact that the degree of the line bundle $\mathcal{L}$ on each fiber over S is *positive* that the structure of $\mathbb{L}$-bundle on the underlying rank 2 vector bundle of an $\mathbb{L}$-bundle is *unique*.

(ii) If two $\mathbb{L}$-indigenous vector bundles on X/S are isomorphic, then any isomorphism between them is unique up to multiplication by an element of $\Gamma(S, \mathcal{O}_S)$ whose square is equal to 1 (i.e, ± 1 if S is connected). In particular, the group of automorphisms of an $\mathbb{L}$-indigenous vector bundle may be identified with the group of elements of $\Gamma(S, \mathcal{O}_S)$ whose square is equal to 1. (Indeed, these facts follow from an argument similar to the argument given in the proof in [26], Theorem 2.8.)

(iii) One may define, in an evident fashion, the pull-back of an $\mathbb{L}$-indigenous vector bundles on X/S with respect to a morphism of schemes $S' \rightarrow S$; this notion of pull-back is compatible, in the evident sense, with composites $S'' \rightarrow S' \rightarrow S$.

Let $\mathcal{F}^{\circledast} = (\mathcal{F}, \nabla, \{\mathcal{F}^i\}_{i=0}^2)$ be an $\mathbb{L}$-indigenous vector bundle on X/S. By executing a change of structure group via the natural map $\mathrm{SL}_2 \rightarrow \mathrm{PGL}_2$, one may construct, from the pair $(\mathcal{F}, \nabla)$, a PGL-torsor $\mathcal{P}_{\mathcal{F}}$ together with an S-connection $\nabla_{\mathcal{P}_{\mathcal{F}}}$ on $\mathcal{P}_{\mathcal{F}}$. Moreover, the subbundle $\mathcal{L}$ $(\subseteq \mathcal{F})$ determines a globally defined section σ of the associated $\mathbb{P}^1$-bundle $\mathbb{P}^1_{\mathcal{F}} := \mathcal{P}_{\mathcal{F}} \wedge^{\mathrm{PGL}_2} \mathbb{P}^1$ on X. One may verify easily from the condition given in Definition 2.3 (ii) (2) that the pair $\mathcal{P}^{\circledast} := (\mathcal{P}_{\mathcal{F}}, \nabla_{\mathcal{P}_{\mathcal{F}}})$ forms an indigenous bundle on X/S, whose Hodge section is given by σ (cf. Definition 2.1 (i)). Then, we have the following:

Proposition 2.4. *(cf.* [26]*, Chap. I, §2, Proposition 2.6)*
If $(X/S, \mathbb{L})$ is a spin curve, then the assignment $\mathcal{F}^{\circledast} \mapsto \mathcal{P}^{\circledast}$ discussed above determines a functor from the groupoid of $\mathbb{L}$-indigenous vector bundles on X/S to the groupoid of indigenous bundles on X/S. Moreover, this functor induces a bijective correspondence between the set of isomorphism classes of $\mathbb{L}$-indigenous vector bundles on X/S (cf. Remark 2.3.1 (ii)) and the set of isomorphism classes of indigenous bundles on X/S (cf. Remark 2.1.1 (iii)). Finally, this correspondence is functorial with respect to S (cf. Remark 2.3.1 (iii)).

Proof. The construction of a functor as asserted in the statement of Proposition 2.4 is routine. The asserted (bijective) correspondence follows from [26], Chap. I, §2, Proposition 2.6. (Here, we note that Proposition 2.6 in loc. cit. states only that an indigenous bundle determines an *indigenous vector bundle* (cf. [26], Chap. I, §2, Definition 2.2) up to tensor product with a line bundle together with a connection whose square is trivial. But one may eliminate such an indeterminacy by the condition that the underlying vector bundle be an $\mathbb{L}$-bundle.) The asserted functoriality with respect to S follows immediately from the construction of the assignment $\mathcal{F}^{\circledast} \mapsto \mathcal{P}^{\circledast}$ (cf. Remark 2.3.1 (iii)). □

3. Dormant indigenous bundles

In this section, we recall the notion of a dormant indigenous bundle and discuss various moduli functors related to this notion.

Let S be a scheme over a field k of characteristic p (cf. § 1.1) and $f : X \to S$ a proper smooth curve of genus $g > 1$. Denote by X_F the Frobenius twist of X over S and $F_{X/S} : X \to X_F$ the relative Frobenius morphism of X over S (cf. § 1.6).

First, we recall the definition of the p-curvature map. Let us fix an algebraic group G over k and denote by $\mathfrak{g}$ the Lie algebra of G. Let $(\pi : \mathcal{E} \to X, \nabla)$ be a pair consisting of a G-torsor $\mathcal{E}$ on X and an S-connection $\nabla : \mathcal{T}_{X/S} \to \widetilde{\mathcal{T}}_{\mathcal{E}/S}$ on $\mathcal{E}$, i.e., a section of the natural quotient $\alpha_{\mathcal{E}} : (\pi_* \mathcal{T}_{\mathcal{E}/S})^G =: \widetilde{\mathcal{T}}_{\mathcal{E}/S} \to \mathcal{T}_{X/S}$ (cf. § 1.5). If ∂ is a derivation corresponding to a local section of $\mathcal{T}_{X/S}$ (respectively, $\widetilde{\mathcal{T}}_{\mathcal{E}} := (\pi_* \mathcal{T}_{\mathcal{E}/S})^G$), then we shall denote by $\partial^{[p]}$ the p-th iterate of ∂, which is also a derivation corresponding to a local section of $\mathcal{T}_{X/S}$ (respectively, $\widetilde{\mathcal{T}}_{\mathcal{E}}$). Since $\alpha_{\mathcal{E}}(\partial^{[p]}) = (\alpha_{\mathcal{E}}(\partial))^{[p]}$ for any local section of $\mathcal{T}_{X/S}$, the image of the p-linear map from $\mathcal{T}_{X/S}$ to $\widetilde{\mathcal{T}}_{\mathcal{E}/S}$ defined by assigning $\partial \mapsto \nabla(\partial^{[p]}) - (\nabla(\partial))^{[p]}$ is contained in $\mathcal{E} \wedge^G \mathfrak{g} (= \ker(\alpha_{\mathcal{E}}))$. Thus, we obtain an $\mathcal{O}_X$-linear morphism

$$\psi_{(\mathcal{E},\nabla)} : \mathcal{T}_{X/S}^{\otimes p} \to \mathcal{E} \wedge^G \mathfrak{g}$$

determined by assigning

$$\partial^{\otimes p} \mapsto \nabla(\partial^{[p]}) - (\nabla(\partial))^{[p]}.$$

We shall refer to the morphism $\psi_{(\mathcal{E},\nabla)}$ as the *p-curvature map* of $(\mathcal{E}, \nabla)$.

If $\mathcal{V}$ is a vector bundle on X_F (i.e., a GL_n-torsor on X_F for some $n \geq 1$), then we may define an S-connection

$$\nabla_{\mathcal{V}}^{\mathrm{can}}$$

on the pull-back $F_{X/S}^* \mathcal{V}$ of $\mathcal{V}$, which is uniquely determined by the condition that the sections of the subsheaf $F_{X/S}^{-1}(\mathcal{V})$ be horizontal. It is easily verified that the p-curvature map of the connection $\nabla_{\mathcal{V}}^{\mathrm{can}}$ vanishes identically on X.

Remark 3.0.1.
Let $(\mathcal{E}, \nabla)$ be a pair consisting of a G-torsor $\mathcal{E}$ on X and an S-connection ∇ on $\mathcal{E}$.

(i) Assume that G is a subgroup of GL_n for $n \geq 1$. Then the p-curvature map $\psi_{(\mathcal{E},\nabla)}$ of $(\mathcal{E}, \nabla)$ is compatible, in the evident sense, with the classical p-curvature map (cf., e.g., [18], §5) associated to the vector bundle equipped with an S-connection obtained by executing a change of structure group via the natural injective morphism $G \hookrightarrow \mathrm{GL}_n$. In this situation, we shall not distinguish between these definitions of the p-curvature map.

(ii) The sheaf $\mathcal{E}^{\nabla}$ of horizontal sections in $\mathcal{E}$ may be considered as an $\mathcal{O}_{X_F}$-module via the underlying homeomorphism of the relative Frobenius morphism $F_{X/S} : X \to X_F$. Thus, we have a natural horizontal morphism

$$\nu_{(\mathcal{E},\nabla)} : (F_{X/S}^* \mathcal{E}^{\nabla}, \nabla_{\mathcal{E}^{\nabla}}^{\mathrm{can}}) \longrightarrow (\mathcal{E}, \nabla)$$

of $\mathcal{O}_X$-modules. It is known (cf. [18], Theorem 5.1) that the p-curvature map of $(\mathcal{E}, \nabla)$ vanishes identically on X if and only if $\nu_{(\mathcal{E},\nabla)}$ is an isomorphism. In particular, the assignment $\mathcal{V} \mapsto (F^*_{X/S}\mathcal{V}, \nabla^{\mathrm{can}}_{\mathcal{V}})$ determines an equivalence, which is compatible with the formation of tensor products (hence also symmetric and exterior products), between the category of vector bundles on X_F and the category of vector bundles on X equipped with an S-connection whose p-curvature vanishes identically.

Definition 3.1.
We shall say that an indigenous bundle $(\mathcal{P}, \nabla)$ (respectively, an $\mathbb{L}$-indigenous vector bundle $(\mathcal{F}, \nabla, \{\mathcal{F}^i\}_{i=0}^2))$ on X/S is *dormant* if the p-curvature map of $(\mathcal{P}, \nabla)$ (respectively, $(\mathcal{F}, \nabla)$) vanishes identically on X.

Next, we shall define a certain class of dormant indigenous bundles, which we shall refer to as *dormant ordinary.* Let $\mathcal{P}^\circledast = (\mathcal{P}, \nabla)$ be a dormant indigenous bundle on X/S. Denote by

$$\mathrm{Ad}(\mathcal{P}^\circledast) := (\mathcal{P} \wedge^{\mathrm{PGL}_2} \mathfrak{sl}_2, \nabla_{\mathrm{Ad}})$$

the pair consisting of the adjoint bundle $\mathcal{P} \wedge^{\mathrm{PGL}_2} \mathfrak{sl}_2$ (cf. the discussion preceding [26], Chap. I, Definition 1.8) associated to the PGL_2-torsor $\mathcal{P}$ and the connection ∇_{Ad} on $\mathcal{P} \wedge^{\mathrm{PGL}_2} \mathfrak{sl}_2$ naturally induced by ∇. Let us consider the 1-st relative de Rham cohomology sheaf $\mathcal{H}^1_{\mathrm{dR}}(\mathrm{Ad}(\mathcal{P}^\circledast))$ of $\mathrm{Ad}(\mathcal{P}^\circledast)$, that is,

$$\mathcal{H}^1_{\mathrm{dR}}(\mathrm{Ad}(\mathcal{P}^\circledast)) := \mathbb{R}^1 f_*((\mathcal{P} \wedge^{\mathrm{PGL}_2} \mathfrak{sl}_2) \otimes \Omega^\bullet_{X/S}),$$

where $(\mathcal{P} \wedge^{\mathrm{PGL}_2} \mathfrak{sl}_2) \otimes \Omega^\bullet_{X/S}$ denotes the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{P} \wedge^{\mathrm{PGL}_2} \mathfrak{sl}_2 \xrightarrow{\nabla_{\mathrm{Ad}}} (\mathcal{P} \wedge^{\mathrm{PGL}_2} \mathfrak{sl}_2) \otimes \Omega_{X/S} \longrightarrow 0 \longrightarrow \cdots$$

concentrated in degrees 0 and 1. Recall (cf. [26], §2, Theorem 2.8) that there is a natural exact sequence

$$0 \to f_*(\Omega^{\otimes 2}_{X/S}) \to \mathcal{H}^1_{\mathrm{dR}}(\mathrm{Ad}(\mathcal{P}^\circledast)) \to \mathbb{R}^1 f_*(\mathcal{T}_{X/S}) \to 0$$

On the other hand, the natural inclusion $(\mathcal{P} \wedge^{\mathrm{PGL}_2} \mathfrak{sl}_2)^\nabla \hookrightarrow \mathcal{P} \wedge^{\mathrm{PGL}_2} \mathfrak{sl}_2$ of the subsheaf of horizontal sections induces a morphism of $\mathcal{O}_S$-modules

$$\mathbb{R}^1 f_{F*}((\mathcal{P} \wedge^{\mathrm{PGL}_2} \mathfrak{sl}_2)^\nabla) \to \mathcal{H}^1_{\mathrm{dR}}(\mathrm{Ad}(\mathcal{P}^\circledast)),$$

where $f_F : X_F \to S$ denotes the structure morphism of X_F/S (§ 1.6). Thus, by composing this morphism with the right-hand epimorphism in the above short exact sequence, we obtain a morphism

$$\gamma_{\mathcal{P}^\circledast} : \mathbb{R}^1 f_{F*}((\mathcal{P} \wedge^{\mathrm{PGL}_2} \mathfrak{sl}_2)^\nabla) \to \mathbb{R}^1 f_*(\mathcal{T}_{X/S})$$

of $\mathcal{O}_S$-modules.

Definition 3.2.
We shall say that an indigenous bundle $\mathcal{P}^\circledast$ is *dormant ordinary* if $\gamma_{\mathcal{P}^\circledast}$ is an isomorphism.

Next, let us introduce notations for various moduli functors classifying the objects discussed above. Let $\mathcal{M}_{g,\mathbb{F}_p}$ be the moduli stack of proper smooth curves of genus $g > 1$ *over* $\mathbb{F}_p$. Denote by

$$\mathcal{S}_{g,\mathbb{F}_p} : (Sch)_{\mathcal{M}_{g,\mathbb{F}_p}} \longrightarrow (Set)$$

(cf. the discussion preceding [26], Chap. I, Lemma 3.2) the set-valued functor on $(Sch)_{\mathcal{M}_{g,\mathbb{F}_p}}$ (cf. § 1.2) which, to any $\mathcal{M}_{g,\mathbb{F}_p}$-scheme T, classifying a curve Y/T, assigns the set of isomorphism classes of indigenous bundles on Y/T. Also, denote by

$$\mathcal{M}^{\mathrm{Zzz...}}_{g,\mathbb{F}_p} \quad (\text{resp., } {}^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,\mathbb{F}_p})$$

the subfunctor of $\mathcal{S}_{g,\mathbb{F}_p}$ classifying the set of isomorphism classes of dormant indigenous bundles (resp., dormant ordinary indigenous bundles). By forgetting the datum of an indigenous bundle, we obtain natural transformations

$$\mathcal{S}_{g,\mathbb{F}_p} \longrightarrow \mathcal{M}_{g,\mathbb{F}_p}, \qquad \mathcal{M}^{\mathrm{Zzz...}}_{g,\mathbb{F}_p} \longrightarrow \mathcal{M}_{g,\mathbb{F}_p}.$$

Next, if $(X/S, \mathbb{L})$ is a spin curve, then we shall denote by

$$\mathcal{M}^{\mathrm{Zzz...}}_{X/S,\mathbb{L}} : (Sch)_S \longrightarrow (Set)$$

the set-valued functor on $(Sch)_S$ which, to any S-scheme T, assigns the set of isomorphism classes of dormant $\mathbb{L}$-indigenous bundles on the curve $X \times_S T$ over T. It follows from Proposition 2.4 that there is a natural isomorphism of functors on $(Sch)_S$

$$\mathcal{M}^{\mathrm{Zzz...}}_{X/S,\mathbb{L}} \xrightarrow{\sim} \mathcal{M}^{\mathrm{Zzz...}}_{g,\mathbb{F}_p} \times_{\mathcal{M}_{g,\mathbb{F}_p}} S,$$

where $\mathcal{M}^{\mathrm{Zzz...}}_{g,\mathbb{F}_p} \times_{\mathcal{M}_{g,\mathbb{F}_p}} S$ denotes the fiber product of the natural projection $\mathcal{M}^{\mathrm{Zzz...}}_{g,\mathbb{F}_p} \rightarrow \mathcal{M}_{g,\mathbb{F}_p}$ and the classifying morphism $S \rightarrow \mathcal{M}_{g,\mathbb{F}_p}$ of X/S.

Next, we quote a result from p-adic Teichmüller theory due to S. Mochizuki concerning the moduli stacks (i.e., which are in fact *schemes*, relatively speaking, over $\mathcal{M}_{g,\mathbb{F}_p}$) that represent the functors discussed above. Here, we wish to emphasize the importance of the *open density* of the dormant ordinary locus. As we shall see in Proposition 4.2 and its proof, the properties stated in the following Theorem 3.3 enable us to relate a numerical calculation in *characteristic zero* to the degree of certain moduli spaces of interest in *positive characteristic.*

Theorem 3.3 (cf. [26], Chap. I, Corollary 2.9; [27], Introduction, § 1.2, Theorem 1.3 (ii); [27], Chap. II, Lemma 2.7; [27], Chap. II, §2.3, Theorem 2.8 (and its proof))**.**

The functor $\mathcal{S}_{g,\mathbb{F}_p}$ *is represented by a relative affine space over* $\mathcal{M}_{g,\mathbb{F}_p}$ *of relative dimension* $3g-3$. *The functor* $\mathcal{M}^{\mathrm{Zzz...}}_{g,\mathbb{F}_p}$ *is represented by a closed substack of* $\mathcal{S}_{g,\mathbb{F}_p}$ *which is finite and faithfully flat over* $\mathcal{M}_{g,\mathbb{F}_p}$, *and which is smooth and geometrically irreducible over* $\mathbb{F}_p$. *The functor* ${}^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,\mathbb{F}_p}$ *is an open dense substack of* $\mathcal{M}^{\mathrm{Zzz...}}_{g,\mathbb{F}_p}$ *and coincides with the étale locus of* $\mathcal{M}^{\mathrm{Zzz...}}_{g,\mathbb{F}_p}$ *over* $\mathcal{M}_{g,\mathbb{F}_p}$.

In particular, it follows that it makes sense to speak of the *degree*

$$\deg_{\mathcal{M}_{g,\mathbb{F}_p}}(\mathcal{M}^{\text{Zzz...}}_{g,\mathbb{F}_p})$$

of $\mathcal{M}^{\text{Zzz...}}_{g,\mathbb{F}_p}$ over $\mathcal{M}_{g,\mathbb{F}_p}$. The generic étaleness of $\mathcal{M}^{\text{Zzz...}}_{g,\mathbb{F}_p}$ over $\mathcal{M}_{g,\mathbb{F}_p}$ implies that if X is a sufficiently generic proper smooth curve of genus g over an algebraically closed field of characteristic p, then the number of dormant indigenous bundles on X is exactly $\deg_{\mathcal{M}_{g,\mathbb{F}_p}}(\mathcal{M}^{\text{Zzz...}}_{g,\mathbb{F}_p})$. As we explained in the Introduction, our main interest in the present paper is the *explicit computation* of the value $\deg_{\mathcal{M}_{g,\mathbb{F}_p}}(\mathcal{M}^{\text{Zzz...}}_{g,\mathbb{F}_p})$.

4. Quot-schemes

To calculate the value of $\deg_{\mathcal{M}_{g,\mathbb{F}_p}}(\mathcal{M}^{\text{Zzz...}}_{g,\mathbb{F}_p})$, it will be necessary to relate $\mathcal{M}^{\text{Zzz...}}_{g,\mathbb{F}_p}$ to certain Quot-schemes. Here, to prepare for the discussion in §5 below, we introduce notions for Quot-schemes in arbitrary characteristic.

Let T be a noetherian scheme, Y a proper smooth curve over T of genus $g > 1$ and $\mathcal{E}$ a vector bundle on Y. Denote by

$$\mathcal{Q}^{2,0}_{\mathcal{E}/Y/T} : (Sch)_T \longrightarrow (Set)$$

the set-valued functor on $(Sch)_T$ which to any $f : T' \to T$ associates the set of isomorphism classes of injective morphisms of coherent $\mathcal{O}_{Y\times_T T'}$-modules

$$i : \mathcal{F} \to \mathcal{E}_{T'},$$

where $\mathcal{E}_{T'}$ denotes the pull-back of $\mathcal{E}$ via the projection $Y \times_T T' \to Y$, such that the quotient $\mathcal{E}_{T'}/i(\mathcal{F})$ is flat over T' (which, since Y/T is smooth of relative dimension 1, implies that $\mathcal{F}$ is *locally free*), and $\mathcal{F}$ is of rank 2 and degree 0. It is known (cf. [8], Theorem 5.14) that $\mathcal{Q}^{2,0}_{\mathcal{E}/Y/T}$ is represented by a proper scheme over T.

Now let $(X/S, \mathbb{L} = (\mathcal{L}, \eta_{\mathcal{L}}))$ be a spin curve of characteristic p and denote, for simplicity, the relative Frobenius morphism $F_{X/S} : X \to X_F$ by F. Then in the following discussion, we consider the Quot-scheme discussed above

$$\mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X_F/S}$$

in the case where the data "$(Y/T, \mathcal{E})$" is taken to be $(X_F/S, F_*(\mathcal{L}^\vee))$. If we denote by $\widetilde{i} : \widetilde{\mathcal{F}} \to (F_*(\mathcal{L}^\vee))_{\mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X_F/S}}$ the tautological injective morphism of sheaves on $X_F \times_S \mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X_F/S}$, then the determinant bundle $\det(\widetilde{\mathcal{F}}) := \wedge^2(\widetilde{\mathcal{F}})$ determines a classifying morphism

$$\det : \mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X_F/S} \to Pic^0_{X_F/S}$$

to the relative Picard scheme $Pic^0_{X_F/S}$ (cf. Remark 2.2.1 (i)) classifying the set of equivalence classes of degree 0 line bundles on X_F/S. We shall denote by

$$\mathcal{Q}^{2,\mathcal{O}}_{F_*(\mathcal{L}^\vee)/X_F/S}$$

the scheme-theoretic inverse image, via det, of the identity section of $Pic^0_{X_F/S}$.

Next, we discuss a certain relationship between $\mathcal{M}^{\mathrm{Zzz...}}_{X/S,\mathbb{L}}$ and $\mathcal{Q}^{2,\mathcal{O}}_{F_*(\mathcal{L}^\vee)/X_F/S}$. To this end, we introduce a certain filtered vector bundle with connection as follows. Let us consider the rank p vector bundle

$$\mathcal{A}_{\mathcal{L}} := F^* F_*(\mathcal{L}^\vee)$$

on X (cf. § 1.6), which has the canonical S-connection

$$\nabla^{\mathrm{can}}_{F_*(\mathcal{L}^\vee)}$$

(cf. the discussion preceding Remark 3.0.1). By using this connection, we may define a *p-step decreasing filtration*

$$\{\mathcal{A}^i_{\mathcal{L}}\}^p_{i=0}$$

on $\mathcal{A}_{\mathcal{L}}$ as follows.

$$\mathcal{A}^0_{\mathcal{L}} := \mathcal{A}_{\mathcal{L}},$$

$$\mathcal{A}^1_{\mathcal{L}} := \ker(\mathcal{A}_{\mathcal{L}} \overset{q}{\twoheadrightarrow} \mathcal{L}^\vee),$$

$$\mathcal{A}^j_{\mathcal{L}} := \ker(\mathcal{A}^{j-1}_{\mathcal{L}} \xrightarrow{\nabla^{\mathrm{can}}_{F_*(\mathcal{L}^\vee)}|_{\mathcal{A}^{j-1}_{\mathcal{L}}}} \mathcal{A}_{\mathcal{L}} \otimes \Omega_{X/S} \twoheadrightarrow \mathcal{A}_{\mathcal{L}}/\mathcal{A}^{j-1}_{\mathcal{L}} \otimes \Omega_{X/S})$$

($j = 2, \cdots, p$), where $\mathcal{A}_{\mathcal{L}}(= F^*F_*(\mathcal{L}^\vee)) \overset{q}{\twoheadrightarrow} \mathcal{L}^\vee$ denotes the natural quotient determined by the adjunction relation "$F^*(-) \dashv F_*(-)$".

Lemma 4.1. *(cf.* [16]*, Theorem 3.1.6)*

(i) *For each $j = 1, \cdots p-1$, the map*

$$\mathcal{A}^{j-1}_{\mathcal{L}}/\mathcal{A}^j_{\mathcal{L}} \to \mathcal{A}^j_{\mathcal{L}}/\mathcal{A}^{j+1}_{\mathcal{L}} \otimes \Omega_{X/S}$$

defined by assigning $\overline{a} \mapsto \overline{\nabla^{\mathrm{can}}_{F_*(\mathcal{L}^\vee)}(a)}$ ($a \in \mathcal{A}^{j-1}_{\mathcal{L}}$), where the "bars" denote the images in the respective quotients, is well-defined and determines an isomorphism of $\mathcal{O}_X$-modules.

(ii) *Let us identify $\mathcal{A}^1_{\mathcal{L}}/\mathcal{A}^2_{\mathcal{L}}$ with $\mathcal{L}$ via the isomorphism*

$$\mathcal{A}^1_{\mathcal{L}}/\mathcal{A}^2_{\mathcal{L}} \xrightarrow{\sim} \mathcal{A}^0_{\mathcal{L}}/\mathcal{A}^1_{\mathcal{L}} \otimes \Omega_{X/S} \xrightarrow{\sim} \mathcal{L}^\vee \otimes \Omega_{X/S} \xrightarrow{\sim} \mathcal{L},$$

obtained by composing the isomorphism of (i) (i.e., the first isomorphism of the display) with the tautological isomorphism arising from the definition of $\mathcal{A}^1_{\mathcal{L}}$ (i.e., the second isomorphism of the display), followed by the isomorphism determined by the given spin structure (i.e., the third isomorphism of the display). Then the natural extension structure

$$0 \to \mathcal{A}^1_{\mathcal{L}}/\mathcal{A}^2_{\mathcal{L}} \to \mathcal{A}_{\mathcal{L}}/\mathcal{A}^2_{\mathcal{L}} \to \mathcal{A}_{\mathcal{L}}/\mathcal{A}^1_{\mathcal{L}} \to 0$$

determines a structure of $\mathbb{L}$-bundle on $\mathcal{A}_{\mathcal{L}}/\mathcal{A}^2_{\mathcal{L}}$.

Proof. The various assertions of Lemma 4.1 follow from an argument (in the case where S is an arbitrary scheme) similar to the argument (in the case where $S = \mathrm{Spec}(k)$ for an algebraically closed field k) given in the proofs of [17], § 5.3 and [32], Lemma 2.1. □

Lemma 4.2 (cf. [16], Theorem 5.4.1)**.**

Let $g : \mathcal{V} \to F_*(\mathcal{L}^\vee)$ *be an injective morphism classified by an* S*-rational point of* $\mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X_F/S}$ *and denote by* $\{(F^*\mathcal{V})^i\}_{i=0}^p$ *the filtration on the pull-back* $F^*\mathcal{V}$ *defined by setting*

$$(F^*\mathcal{V})^i := (F^*\mathcal{V}) \cap (F^*g)^{-1}(\mathcal{A}^i_{\mathcal{L}}),$$

where we denote by F^*g *the pull-back of* g *via* F*.*

(i) *The composite*

$$F^*\mathcal{V} \to \mathcal{A}_{\mathcal{L}} \twoheadrightarrow \mathcal{A}_{\mathcal{L}}/\mathcal{A}^2_{\mathcal{L}}$$

of F^*g *with the natural quotient* $\mathcal{A}_{\mathcal{L}} \twoheadrightarrow \mathcal{A}_{\mathcal{L}}/\mathcal{A}^2_{\mathcal{L}}$ *is an isomorphism of* $\mathcal{O}_X$*-modules.*

(ii) *If, moreover,* g *corresponds to an* S*-rational point of* $\mathcal{Q}^{2,\mathcal{O}}_{F_*(\mathcal{L}^\vee)/X_F/S}$*, then the triple*

$$(F^*\mathcal{V}, \nabla^{\mathrm{can}}_{\mathcal{V}}, \{(F^*\mathcal{V})^i\}_{i=0}^2),$$

where $\nabla^{\mathrm{can}}_{\mathcal{V}}$ *denotes the canonical connection on* $F^*\mathcal{V}$ *(cf. the discussion preceding Remark 3.0.1), forms a dormant* $\mathbb{L}$*-indigenous bundle on* X/S*.*

Proof. First, we consider assertion (i). Since $F^*\mathcal{V}$ and $\mathcal{A}_{\mathcal{L}}/\mathcal{A}^2_{\mathcal{L}}$ are flat over S, it suffices, by considering the various fibers over S, to verify the case where $S = \mathrm{Spec}(k)$ for a field k. If we write $\mathrm{gr}^i := (F^*\mathcal{V})^i/(F^*\mathcal{V})^{i+1}$ ($i = 0, \cdots, p-1$), then it follows immediately from the definitions that the coherent $\mathcal{O}_X$-module gr^i admits a natural embedding

$$\mathrm{gr}^i \hookrightarrow \mathcal{A}^i_{\mathcal{L}}/\mathcal{A}^{i+1}_{\mathcal{L}}$$

into the subquotient $\mathcal{A}^i_{\mathcal{L}}/\mathcal{A}^{i+1}_{\mathcal{L}}$. Since this subquotient is a line bundle (cf. Lemma 4.1 (i), (ii)), one verifies easily that gr^i is either trivial or a line bundle. In particular, since $F^*\mathcal{V}$ is of rank 2, the cardinality of the set $I := \{i \mid \mathrm{gr}^i \neq 0\}$ is exactly 2. Next, let us observe that the pull-back F^*g of g via F is compatible with the respective connections $\nabla^{\mathrm{can}}_{\mathcal{V}}$ (cf. the statement of assertion (ii)), $\nabla^{\mathrm{can}}_{F_*(\mathcal{L}^\vee)}$. Thus, it follows from Lemma 4.1 (i) that $\mathrm{gr}^{i+1} \neq 0$ implies $\mathrm{gr}^i \neq 0$. But this implies that $I = \{0, 1\}$, and hence that the composite

$$F^*\mathcal{V} \to \mathcal{A}_{\mathcal{L}} \twoheadrightarrow \mathcal{A}_{\mathcal{L}}/\mathcal{A}^2_{\mathcal{L}}$$

is an isomorphism at the generic point of X. On the other hand, observe that

$$\deg(F^*\mathcal{V}) = p \cdot \deg(\mathcal{V}) = p \cdot 0 = 0$$

and

$$\deg(\mathcal{A}_{\mathcal{L}}/\mathcal{A}^2_{\mathcal{L}}) = \deg(\mathcal{A}_{\mathcal{L}}/\mathcal{A}^1_{\mathcal{L}}) + \deg(\mathcal{A}^1_{\mathcal{L}}/\mathcal{A}^2_{\mathcal{L}}) = \deg(\mathcal{L}^\vee) + \deg(\mathcal{L}) = 0$$

(cf. Lemma 4.1 (i)). Thus, by comparing the respective degrees of $F^*\mathcal{V}$ and $\mathcal{A}_{\mathcal{L}}/\mathcal{A}^2_{\mathcal{L}}$, we conclude that the above composite is an isomorphism of $\mathcal{O}_X$-modules. This completes the proof of assertion (i). Assertion (ii) follows immediately from the definition of an $\mathbb{L}$-indigenous bundle, assertion (i), and Lemma 4.1 (i), (ii). □

By applying the above lemma, we may conclude that the moduli space $\mathcal{M}^{\mathrm{Zzz...}}_{X/S,\mathbb{L}}$ is isomorphic to the Quot-scheme $\mathcal{Q}^{2,\mathcal{O}}_{F_*(\mathcal{L}^\vee)/X/S}$ as follows.

Proposition 4.3 (cf. [16], Proposition 5.4.2)**.**
Let $(X/S, \mathbb{L})$ be a spin curve. Then there is an isomorphism of S-schemes

$$\mathcal{Q}^{2,\mathcal{O}}_{F_*(\mathcal{L}^\vee)/X/S} \xrightarrow{\sim} \mathcal{M}^{\mathrm{Zzz...}}_{X/S,\mathbb{L}}.$$

Proof. The assignment

$$[g : \mathcal{V} \to F_*(\mathcal{L}^\vee)] \mapsto (F^*\mathcal{V}, \nabla^{\mathrm{can}}_{F^*\mathcal{V}}, \{(F^*\mathcal{V})^i\}_{i=0}^2),$$

discussed in Lemma 4.2, determines (by Lemma 4.2 (ii)) a map

$$\alpha_S : \mathcal{Q}^{2,\mathcal{O}}_{F_*(\mathcal{L}^\vee)/X/S}(S) \to \mathcal{M}^{\mathrm{Zzz...}}_{X/S,\mathbb{L}}(S)$$

between the respective sets of S-rational points. By the functoriality of the construction of α_S with respect to S, it suffices to prove the bijectivity of α_S.

The *injectivity* of α_S follows from the observation that any element $[g : \mathcal{V} \to F_*(\mathcal{L}^\vee)] \in \mathcal{Q}^{2,\mathcal{O}}_{F_*(\mathcal{L}^\vee)/X/S}(S)$ is, by adjunction, determined by the morphism $F^*\mathcal{V} \to \mathcal{L}^\vee$, i.e., the natural surjection, as in Definition 2.3 (i), arising from the fact that $F^*\mathcal{V}$ is an $\mathbb{L}$-bundle (cf. Lemma 4.2 (ii)).

Next, we consider the *surjectivity* of α_S. Let $(\mathcal{F}, \nabla, \{\mathcal{F}^i\}_i)$ be a dormant $\mathbb{L}$-indigenous bundle on X/S. Consider the composite $F^*\mathcal{F}^\nabla \xrightarrow{\sim} \mathcal{F} \twoheadrightarrow \mathcal{L}^\vee$ of the natural horizontal isomorphism $F^*\mathcal{F}^\nabla \xrightarrow{\sim} \mathcal{F}$ (cf. Remark 3.0.1 (ii)) with the natural surjection $\mathcal{F} \twoheadrightarrow \mathcal{F}/\mathcal{F}^1 = \mathcal{L}^\vee$. This composite determines a morphism

$$g_{\mathcal{F}} : (\mathcal{F} \cong) F^*\mathcal{F}^\nabla \to F^*F_*(\mathcal{L}^\vee) (=: \mathcal{A}_{\mathcal{L}})$$

via the adjunction relation "$F^*(-) \dashv F_*(-)$" and pull-back by F.

Next, we claim that $g_{\mathcal{F}}$ is *injective*. Indeed, since $g_{\mathcal{F}}$ is (tautologically, by construction!) compatible with the respective surjections $\mathcal{F} \twoheadrightarrow \mathcal{L}^\vee$, $\mathcal{A}_{\mathcal{L}} \twoheadrightarrow \mathcal{L}^\vee$ to $\mathcal{L}^\vee$, we conclude that $g_{\mathcal{F}}(\mathcal{F}^1) \subseteq \mathcal{A}^1_{\mathcal{L}}$, and $\ker(g_{\mathcal{F}}) \subseteq \mathcal{F}^1$. Since $g_{\mathcal{F}}$ is *manifestly horizontal* (by construction), $\ker(g_{\mathcal{F}})$ is stabilized by ∇, hence contained in the kernel of the Kodaira-Spencer map $\mathcal{F}^1 \to \mathcal{F}/\mathcal{F}^1 \otimes \Omega_{X/S}$(cf. Definition 2.3 (ii) (2)), which is an isomorphism by the definition of an $\mathbb{L}$-indigenous bundle (cf. Definition 2.3 (ii)). This implies that $g_{\mathcal{F}}$ is injective and completes the proof of the claim. Moreover, by applying a similar argument to the pull-back of $g_{\mathcal{F}}$ via any base-change over S, one concludes that $g_{\mathcal{F}}$ is *universally injective* with respect to base-change over S. This implies that $\mathcal{A}_{\mathcal{L}}/g_{\mathcal{F}}(\mathcal{F})$ is *flat* over S (cf. [23], p.17, Theorem 1).

Now denote by $g^\nabla_{\mathcal{F}} : \mathcal{F}^\nabla \to F_*(\mathcal{L}^\vee)$ the morphism obtained by restricting $g_{\mathcal{F}}$ to the respective subsheaves of horizontal sections in $\mathcal{F}$, $\mathcal{A}_{\mathcal{L}}$. Observe that the pull-back of $g^\nabla_{\mathcal{F}}$ via F may be identified with $g_{\mathcal{F}}$, and that $F^*(F_*(\mathcal{L}^\vee)/g^\nabla_{\mathcal{F}}(\mathcal{F}^\nabla))$ is naturally isomorphic to $\mathcal{A}_{\mathcal{L}}/g_{\mathcal{F}}(\mathcal{F})$. Thus, it follows from the faithful flatness of F that $g^\nabla_{\mathcal{F}}$ is injective, and $F_*(\mathcal{L}^\vee)/g^\nabla_{\mathcal{F}}(\mathcal{F}^\nabla)$ is flat over S. On the other hand, since the determinant of $(\mathcal{F}, \nabla)$ is trivial, $\det(\mathcal{F}^\nabla)$ is isomorphic to the trivial $\mathcal{O}_{X_F}$-module (cf. Remark 3.0.1 (ii)). Thus, $g^\nabla_{\mathcal{F}}$ determines an S-rational point of $\mathcal{Q}^{2,\mathcal{O}}_{F_*(\mathcal{L}^\vee)/X/S}$ that is mapped by α_S to the S-rational point of $\mathcal{M}^{\mathrm{Zzz...}}_{X/S,\mathbb{L}}$

corresponding to $(\mathcal{F}, \nabla, \{\mathcal{F}^i\}_i)$. This implies that α_S is surjective and hence completes the proof of Proposition 4.3. $\square$

Next, we relate $\mathcal{Q}^{2,\mathcal{O}}_{F_*(\mathcal{L}^\vee)/X_F/S}$ to $\mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X_F/S}$. By pulling back line bundles on X_F via the relative Frobenius $F : X \to X_F$, we obtain a morphism

$$\begin{aligned} \mathrm{Pic}^0_{X_F/S} &\to \mathrm{Pic}^0_{X/S} \\ [\mathcal{N}] &\mapsto [F^*\mathcal{N}]. \end{aligned}$$

We shall denote by

$$\mathrm{Ver}_{X/S}$$

the scheme-theoretic inverse image, via this morphism, of the identity section of $\mathrm{Pic}^0_{X/S}$. It is well-known (cf. [4], EXPOSE VII, § 4.3; [25], Proposition 8.1, Theorem 8.2; [24], APPENDIX, Lemma (1.0)) that $\mathrm{Ver}_{X/S}$ is finite and faithfully flat over S of degree p^g and, moreover, étale over the points s of S such that the fiber of X/S at s is ordinary. (Recall that the locus of $\mathcal{M}_{g,\mathbb{F}_p}$ classifying ordinary curves is open and dense.) Then we have the following

Lemma 4.4.
There is an isomorphism of S-schemes

$$\mathcal{Q}^{2,\mathcal{O}}_{F_*(\mathcal{L}^\vee)/X_F/S} \times_S \mathrm{Ver}_{X/S} \xrightarrow{\sim} \mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X_F/S}.$$

Proof. It suffices to prove that there is a bijection between the respective sets of S-rational points that is functorial with respect to S.

Let $(g : \mathcal{V} \to F_*(\mathcal{L}^\vee), \mathcal{N})$ be an element of $(\mathcal{Q}^{2,\mathcal{O}}_{F_*(\mathcal{L}^\vee)/X_F/S} \times_S \mathrm{Ver}_{X/S})(S)$. It follows from the projection formula that the composite

$$g_\mathcal{N} : \mathcal{V} \otimes \mathcal{N} \to F_*(\mathcal{L}^\vee) \otimes \mathcal{N} \to F_*(\mathcal{L}^\vee \otimes F^*\mathcal{N}) \xrightarrow{\sim} F_*(\mathcal{L}^\vee \otimes \mathcal{O}_X) = F_*(\mathcal{L}^\vee)$$

determines an element of $\mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X_F/S}(S)$. Thus, we obtain a functorial (with respect to S) map

$$\gamma_S : (\mathcal{Q}^{2,\mathcal{O}}_{F_*(\mathcal{L}^\vee)/X_F/S} \times_S \mathrm{Ver}_{X/S})(S) \to \mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X_F/S}(S).$$

Conversely, let $g : \mathcal{V} \to F_*(\mathcal{L}^\vee)$ be an injective morphism classified by an element of $\mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X_F/S}(S)$. Consider the injective morphism $g_{\det(\mathcal{V})^{\otimes \frac{p-1}{2}}}$, i.e., the morphism $g_\mathcal{N}$ constructed above in the case where "$\mathcal{N}$" is taken to be $\mathcal{N} = \det(\mathcal{V})^{\otimes \frac{p-1}{2}}$. Here, we observe that

$$\det(\mathcal{V} \otimes \det(\mathcal{V})^{\otimes \frac{p-1}{2}}) \cong \det(\mathcal{V}) \otimes \det(\mathcal{V})^{\otimes 2 \cdot \frac{p-1}{2}} \cong \det(\mathcal{V})^{\otimes p} \cong F_S^*(F^*(\det(\mathcal{V}))),$$

where $F_S^*(-)$ denotes the pull-back by the morphism $X_F \to X$ obtained by base-change of X/S via the absolute Frobenius morphism $F_S : S \to S$ of S (cf. § 1.6). On the other hand, since $F^*(\det(\mathcal{V})) \cong (\mathcal{A}_\mathcal{L}/\mathcal{A}^1_\mathcal{L}) \otimes (\mathcal{A}^1_\mathcal{L}/\mathcal{A}^2_\mathcal{L}) \cong \mathcal{L}^\vee \otimes \mathcal{L} \cong \mathcal{O}_X$ (cf. Lemmas 4.1 (ii), 4.2 (i)), it follows that the determinant of $\mathcal{V} \otimes \det(\mathcal{V})^{\otimes \frac{p-1}{2}}$ is trivial. Thus the pair $(g_{\det(\mathcal{V})^{\otimes \frac{p-1}{2}}}, \det(\mathcal{V}))$ determines an element of $(\mathcal{Q}^{2,\mathcal{O}}_{F_*(\mathcal{L}^\vee)/X_F/S} \times_S \mathrm{Ver}_{X/S})(S)$. One verifies easily that this assignment determines an inverse to γ_S. This completes the proof of Lemma 4.4. $\square$

5. Computation via the Vafa-Intriligator formula

By combining Proposition 4.3, Lemma 4.4, and the discussions preceding Theorem 3.3 and Lemma 4.4, we obtain the following equalities:

$$\mathrm{deg}_{\mathcal{M}_{g,\mathbb{F}_p}}(\mathcal{M}^{\mathrm{Zzz...}}_{g,\mathbb{F}_p}) = \mathrm{deg}_S(\mathcal{M}^{\mathrm{Zzz...}}_{X/S,\mathbb{L}}) = \mathrm{deg}_S(\mathcal{Q}^{2,\mathcal{O}}_{F_*(\mathcal{L}^\vee)/X/S}) = \frac{1}{p^g}\cdot \mathrm{deg}_S(\mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X/S}).$$

Therefore, to determine the value of $\mathrm{deg}_{\mathcal{M}_{g,\mathbb{F}_p}}(\mathcal{M}^{\mathrm{Zzz...}}_{g,\mathbb{F}_p})$, it suffices to calculate the value $\mathrm{deg}_S(\mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X/S})$ (for an *arbitrary* spin curve $(X/S, \mathbb{L})$).

In this section, we review a numerical formula concerning the degree of a certain Quot-scheme over the field of complex number $\mathbb{C}$ and relate it to the degree of the Quot-scheme in positive characteristic.

Let C be a smooth proper curve over $\mathbb{C}$ of genus $g > 1$. If r is an integer, and $\mathcal{E}$ is a vector bundle on C of rank n and degree d with $1 \leq r \leq n$, then we define invariants

$$e_{\mathrm{max}}(\mathcal{E}, r) := \max\{\mathrm{deg}(\mathcal{F}) \in \mathbb{Z} \mid \mathcal{F} \text{ is a subbundle of } \mathcal{E} \text{ of rank } r \},$$
$$s_r(\mathcal{E}) := d\cdot r - n\cdot e_{\mathrm{max}}(\mathcal{E}, r).$$

(Here, we recall that one verifies immediately, for instance, by considering an embedding of $\mathcal{E}$ into a direct sum of n line bundles, that $e_{\mathrm{max}}(\mathcal{E}, r)$ is well-defined.)

In the following, we review some facts concerning these invariants (cf. [11]; [20]; [12]). Denote by ${}^{\mathrm{s}}\mathcal{N}^{n,d}_C$ the moduli space of stable bundles on C of rank n and degree d (cf. [20], § 1). It is known that ${}^{\mathrm{s}}\mathcal{N}^{n,d}_C$ is irreducible (cf. [20], the discussion at the beginning of § 2). Thus, it makes sense to speak of a "sufficiently general" stable bundle in ${}^{\mathrm{s}}\mathcal{N}^{n,d}_C$, i.e., a stable bundle that corresponds to a point of the scheme ${}^{\mathrm{s}}\mathcal{N}^{n,d}_C$ that lies outside some fixed closed subscheme. If $\mathcal{E}$ is a sufficiently general stable bundle in ${}^{\mathrm{s}}\mathcal{N}^{n,d}_C$, then it holds (cf. [20], §1) that $s_r(\mathcal{E}) = r(n-r)(g-1)+\epsilon$, where ϵ is the unique integer such that $0 \leq \epsilon < n$ and $s_r(\mathcal{E}) = r\cdot d \mod n$. Also, the number ϵ coincides (cf. [12], § 1) with the dimension of every irreducible component of the Quot-scheme $\mathcal{Q}^{r,e_{\mathrm{max}}(\mathcal{E},r)}_{\mathcal{E}/C/\mathbb{C}}$ (cf. § 4). If, moreover, the equality $s_r(\mathcal{E}) = r(n-r)(g-1)$ holds (i.e., $\dim(\mathcal{Q}^{r,e_{\mathrm{max}}(\mathcal{E},r)}_{\mathcal{E}/C/\mathbb{C}}) = 0$), then $\mathcal{Q}^{r,e_{\mathrm{max}}(\mathcal{E},r)}_{\mathcal{E}/C/\mathbb{C}}$ is étale over $\mathrm{Spec}(\mathbb{C})$ (cf. [12], § 1). Finally, under this particular assumption, a formula for the degree of this Quot-scheme was given by Holla as follows.

Theorem 5.1 (cf. [12], Theorem 4.2, where "k" (respectively, "r") corresponds to our r (respectively, n))**.**

Let C be a proper smooth curve over $\mathbb{C}$ of genus $g > 1$, $\mathcal{E}$ a sufficiently general stable bundle in ${}^{s}\mathcal{N}^{n,d}_C$. Write (a, b) for the unique pair of integers such that $d = an - b$ with $0 \leq b < n$. Also, we suppose that the equality $s_r(\mathcal{E}) = r(n-r)(g-1)$ (equivalently, $e_{\mathrm{max}}(\mathcal{E}, r) = (dr - r(n-r)(g-1))/n$)

holds. Then we have

$$\deg_{\mathbb{C}}(\mathcal{Q}^{r,e_{\max}(\mathcal{E},r)}_{\mathcal{E}/C/\mathbb{C}}) = \frac{(-1)^{(r-1)(br-(g-1)r^2)/n}n^{r(g-1)}}{r!}\sum_{\rho_1,\cdots,\rho_r}\frac{(\prod_{i=1}^r\rho_i)^{b-g+1}}{\prod_{i\neq j}(\rho_i-\rho_j)^{g-1}},$$

where $\rho_i^n = 1$, *for* $1 \leq i \leq r$ *and the sum is over tuples* $(\rho_1, \cdots, \rho_r)$ *with* $\rho_i \neq \rho_j$.

By applying this formula, we conclude the same kind of formula for certain vector bundles in positive characteristic, as follows.

Lemma 5.2.
Let k *an algebraically closed field of characteristic* p *and* $(X/k, \mathbb{L} = (\mathcal{L}, \eta_{\mathcal{L}}))$ *a spin curve of genus* $g > 1$. *Suppose that* X/k *is sufficiently general in* $\mathcal{M}_{g,\mathbb{F}_p}$. *(Here, we recall that* $\mathcal{M}_{g,\mathbb{F}_p}$ *is irreducible (cf.* [3], § 5*); thus, it makes sense to speak of a "sufficiently general"* X/k, *i.e., an* X/k *that determines a point of* $\mathcal{M}_{g,\mathbb{F}_p}$ *that lies outside some fixed closed substack.) Then* $\mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X_F/k}$ *is finite and étale over* k. *If, moreover, we suppose that* $p > 2(g-1)$, *then the degree* $\deg_k(\mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X_F/k})$ *of* $\mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X_F/k}$ *over* $\mathrm{Spec}(k)$ *is given by the following formula:*

$$\begin{aligned}\deg_k(\mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X_F/k}) &= \frac{p^{2g-1}}{2^{2g-1}}\cdot\sum_{\theta=1}^{p-1}\frac{1}{\sin^{2g-2}(\frac{\pi\cdot\theta}{p})}\\ &\Big(= \frac{(-1)^{g-1}\cdot p^{2g-1}}{2}\cdot\sum_{\zeta^p=1,\zeta\neq 1}\frac{\zeta^{g-1}}{(\zeta-1)^{2g-2}}\Big).\end{aligned}$$

Proof. Suppose that X is an ordinary (cf. the discussion preceding Lemma 4.4) proper smooth curve over k classified by a k-rational point of $\mathcal{M}_{g,\mathbb{F}_p}$ which lies in the complement of the image of $\mathcal{M}^{\mathrm{Zzz...}}_{g,\mathbb{F}_p} \setminus {}^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,\mathbb{F}_p}$ via the natural projection $\mathcal{M}^{\mathrm{Zzz...}}_{g,\mathbb{F}_p} \to \mathcal{M}_{g,\mathbb{F}_p}$ (cf. Theorem 3.3; the discussion preceding Theorem 3.3). Then it follows from Theorem 3.3, Proposition 4.3, and Lemma 4.4 that $\mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X_F/k}$ is finite and étale over k.

Next, we determine the value of $\deg_k(\mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X_F/k})$. Denote by W the ring of Witt vectors with coefficients in k and K the fraction field of W. Since $\dim(X_F) = 1$, which implies that $H^2(X_F, \Omega^\vee_{X_F}) = 0$, it follows from well-known generalities concerning deformation theory that X_F may be lifted to a smooth proper curve $(X_F)_W$ over W of genus g. In a similar vein, the fact that $H^2(X_F, \mathcal{E}nd_{\mathcal{O}_{X_F}}(F_*(\mathcal{L}^\vee))) = 0$ implies that $F_*(\mathcal{L}^\vee)$ may be lifted to a vector bundle $\mathcal{E}$ on $(X_F)_W$.

Now let η be a k-rational point of $\mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X_F/k}$ classifying an injective morphism $i : \mathcal{F} \to F_*(\mathcal{L}^\vee)$. The tangent space to $\mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X_F/k}$ at η may be naturally identified with the k-vector space $\mathrm{Hom}_{\mathcal{O}_{X_F}}(\mathcal{F}, F_*(\mathcal{L}^\vee)/i(\mathcal{F}))$, and the obstruction to lifting η to any first order thickening of $\mathrm{Spec}(k)$ is given by an element of $\mathrm{Ext}^1_{\mathcal{O}_{X_F}}(\mathcal{F}, F_*(\mathcal{L}^\vee)/i(\mathcal{F}))$. On the other hand, since, as was observed above, $\mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X_F/k}$ is étale over $\mathrm{Spec}(k)$, it holds that $\mathrm{Hom}_{\mathcal{O}_{X_F}}(\mathcal{F}, F_*(\mathcal{L}^\vee)/i(\mathcal{F})) = 0$, and hence $\mathrm{Ext}^1_{\mathcal{O}_{X_F}}(\mathcal{F}, F_*(\mathcal{L}^\vee)/i(\mathcal{F})) = 0$ by Lemma 5.3 below. This implies that η may be lifted to a W-rational point of $\mathcal{Q}^{2,0}_{\mathcal{E}/(X_F)_W/W}$, and hence that $\mathcal{Q}^{2,0}_{\mathcal{E}/(X_F)_W/W}$ is finite and étale over W. Now it follows from a routine argument that K may be supposed to be a subfield of $\mathbb{C}$. Denote by $(X_F)_\mathbb{C}$ the base-change of $(X_F)_W$ via the morphism $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(W)$ induced by the composite embedding $W \hookrightarrow K \hookrightarrow \mathbb{C}$, and $\mathcal{E}_\mathbb{C}$ the pull-back of $\mathcal{E}$ via the natural morphism $(X_F)_\mathbb{C} \to (X_F)_W$. Thus, we obtain equalities

$$\deg_k(\mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X_k/k}) = \deg_W(\mathcal{Q}^{2,0}_{\mathcal{E}/(X_F)_W/W}) = \deg_\mathbb{C}(\mathcal{Q}^{2,0}_{\mathcal{E}_\mathbb{C}/(X_F)_\mathbb{C}/C}).$$

To prove the required formula, we calculate the degree $\deg_\mathbb{C}(\mathcal{Q}^{2,0}_{\mathcal{E}_\mathbb{C}/(X_F)_\mathbb{C}/\mathbb{C}})$ by applying Theorem 5.1.

By [32], Theorem 2.2, $F_*(\mathcal{L}^\vee)$ is stable. Since the degree of $\mathcal{E}_\mathbb{C}$ coincides with the degree of $F_*(\mathcal{L}^\vee)$, $\mathcal{E}_\mathbb{C}$ is a vector bundle of degree $\deg(\mathcal{E}_\mathbb{C}) = (p-2)(g-1)$ (cf. the proof of Lemma 5.3). On the other hand, one verifies easily from the definition of stability and the properness of Quot schemes (cf. [8], Theorem 5.14) that $\mathcal{E}_\mathbb{C}$ is a stable vector bundle. Next, let us observe that $\mathcal{Q}^{2,0}_{\mathcal{E}_\mathbb{C}/(X_F)_\mathbb{C}/C}$ is zero-dimensional (cf. the discussion above), which, by the discussion preceding Theorem 5.1, implies that $s_2(\mathcal{E}_\mathbb{C}) = 2(p-2)(g-1)$. Thus, by choosing the deformation $\mathcal{E}$ of $F_*(\mathcal{L}^\vee)$ appropriately, we may assume, without loss of generality, that $\mathcal{E}_\mathbb{C}$ is sufficiently general in ${}^{\mathrm{s}}\mathcal{N}^{p,(p-2)(g-1)}_{(X_F)_\mathbb{C}}$ that Theorem 5.1 holds. Now we compute (cf. the discussion preceding Theorem 5.1):

$$\begin{aligned} e_{\max}(\mathcal{E}_\mathbb{C}, 2) &= \frac{1}{p} \cdot (\deg_\mathbb{C}(\mathcal{E}_\mathbb{C}) \cdot 2 - s_2(\mathcal{E}_\mathbb{C})) \\ &= \frac{1}{p} \cdot ((p-2)(g-1) \cdot 2 - 2 \cdot (p-2)(g-1)) \\ &= 0. \end{aligned}$$

If, moreover, we write (a, b) for the unique pair of integers such that $\deg_\mathbb{C}(\mathcal{E}_\mathbb{C}) = p \cdot a - b$ with $0 \leq b < p$, then it follows from the hypothesis $p > 2(g-1)$ that $a = g-1$ and $b = 2(g-1)$. Thus, by applying Theorem 5.1 in the case where the data

$$\text{“}(C, \mathcal{V}, n, d, r, a, b, e_{\max}(\mathcal{V}, r))\text{”}$$

is taken to be

$$((X_F)_\mathbb{C}, \mathcal{E}_\mathbb{C}, p, (g-1)(p-2), 2, g-1, 2(g-1), 0),$$

we obtain that

$$\deg_\mathbb{C}(\mathcal{Q}^{2,0}_{\mathcal{E}_\mathbb{C}/(X_F)_\mathbb{C}/\mathbb{C}}) = \frac{(-1)^{(2-1)(2(g-1)2-(g-1)2^2)/p} p^{2(g-1)}}{2!} \cdot \sum_{\rho_1,\rho_2} \frac{(\prod_{i=1}^2 \rho_i)^{2(g-1)-g+1}}{\prod_{i \neq j} (\rho_i - \rho_j)^{g-1}}$$

$$
\begin{aligned}
&= \frac{(-1)^{g-1}\cdot p^{2g-1}}{2}\cdot \sum_{\zeta^p=1,\zeta\neq 1}\frac{\zeta^{g-1}}{(\zeta-1)^{2g-2}}\\
&= \frac{p^{2g-1}}{2^g}\cdot \sum_{\zeta^p=1,\zeta\neq 1}\frac{1}{(1-\frac{\zeta+\zeta^{-1}}{2})^{g-1}}\\
&= \frac{p^{2g-1}}{2^{2g-1}}\cdot \sum_{\theta=1}^{p-1}\frac{1}{\sin^{2g-2}(\frac{\pi\cdot\theta}{p})}.
\end{aligned}
$$

This completes the proof of the required equality. □

The following lemma was used in the proof of Lemma 5.2.

Lemma 5.3.

Let k be a field of characteristic p, $(X/k, \mathbb{L} := (\mathcal{L}, \eta_{\mathcal{L}}))$ a spin curve, and $i : \mathcal{F} \to F_*(\mathcal{L}^\vee)$ an injective morphism classified by a k-rational point of $\mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X_F/k}$. Write $\mathcal{G} := F_*(\mathcal{L}^\vee)/i(\mathcal{F})$. Then $\mathcal{G}$ is a vector bundle on X_F, and it holds that

$$\dim_k(\mathrm{Hom}_{\mathcal{O}_{X_F}}(\mathcal{F},\mathcal{G})) = \dim_k(\mathrm{Ext}^1_{\mathcal{O}_{X_F}}(\mathcal{F},\mathcal{G})).$$

Proof. First, we verify that $\mathcal{G}$ is a vector bundle. Since $F : X \to X_F$ is faithfully flat, it suffices to verify that the pull-back $F^*\mathcal{G}$ is a vector bundle on X. Recall (cf. Lemma 4.2 (i)) that the composite $F^*\mathcal{F} \to \mathcal{A}_{\mathcal{L}}(= F^*F_*(\mathcal{L}^\vee)) \to \mathcal{A}_{\mathcal{L}}/\mathcal{A}^2_{\mathcal{L}}$ of the pull-back of i with the natural surjection $\mathcal{A}_{\mathcal{L}} \to \mathcal{A}_{\mathcal{L}}/\mathcal{A}^2_{\mathcal{L}}$ is an isomorphism. One verifies easily that this implies that the natural composite $\mathcal{A}^2_{\mathcal{L}} \to \mathcal{A}_{\mathcal{L}} \to F^*\mathcal{G}$ is an isomorphism, and hence that $F^*\mathcal{G}$ is a vector bundle, as desired.

Next we consider the asserted equality. Since the morphism $F : X \to X_F$ is finite, it follows from well-known generalities concerning cohomology that we have an equality of Euler characteristics $\chi(F_*(\mathcal{L}^\vee)) = \chi(\mathcal{L}^\vee)$. Thus, it follows from the Riemann-Roch theorem that

$$
\begin{aligned}
\deg(F_*(\mathcal{L}^\vee)) &= \chi(F_*(\mathcal{L}^\vee)) - \mathrm{rk}(F_*(\mathcal{L}^\vee))(1-g)\\
&= \chi(\mathcal{L}^\vee) - p(1-g)\\
&= (p-2)(g-1),
\end{aligned}
$$

and, since $\mathrm{rk}(\mathcal{H}om_{\mathcal{O}_{X_F}}(\mathcal{F},\mathcal{G})) = 2(p-2)$, that

$$
\begin{aligned}
\deg(\mathcal{H}om_{\mathcal{O}_{X_F}}(\mathcal{F},\mathcal{G})) &= 2\cdot\deg(\mathcal{G}) - (p-2)\cdot\deg(\mathcal{F})\\
&= 2\cdot\deg(F_*(\mathcal{L}^\vee)) - 0\\
&= 2(p-2)(g-1).
\end{aligned}
$$

Finally, by applying the Riemann-Roch theorem again, we obtain equalities

$$
\begin{aligned}
&\dim_k(\mathrm{Hom}_{\mathcal{O}_{X_F}}(\mathcal{F},\mathcal{G})) - \dim_k(\mathrm{Ext}^1_{\mathcal{O}_{X_F}}(\mathcal{F},\mathcal{G}))\\
&= \deg(\mathcal{H}om_{\mathcal{O}_{X_F}}(\mathcal{F},\mathcal{G})) + \mathrm{rk}(\mathcal{H}om_{\mathcal{O}_{X_F}}(\mathcal{F},\mathcal{G}))(1-g)\\
&= 2(p-2)(g-1) + 2(p-2)(1-g)\\
&= 0.
\end{aligned}
$$

□

Thus, we conclude the main result of the present paper.

Corollary 5.4.
Suppose that $p > 2(g-1)$ $(\Longleftrightarrow \frac{1}{2}\cdot p+1 > g)$. *Then the degree* $\deg_{\mathcal{M}_{g,\mathbb{F}_p}}(\mathcal{M}^{Zzz...}_{g,\mathbb{F}_p})$ *of* $\mathcal{M}^{Zzz...}_{g,\mathbb{F}_p}$ *over* $\mathcal{M}_{g,\mathbb{F}_p}$ *is given by the following formula:*

$$\deg_{\mathcal{M}_{g,\mathbb{F}_p}}(\mathcal{M}^{Zzz...}_{g,\mathbb{F}_p}) = \frac{p^{g-1}}{2^{2g-1}} \cdot \sum_{\theta=1}^{p-1} \frac{1}{\sin^{2g-2}(\frac{\pi\cdot\theta}{p})}$$
$$\Big(= \frac{(-1)^{g-1}\cdot p^{g-1}}{2} \cdot \sum_{\zeta^p=1, \zeta\neq 1} \frac{\zeta^{g-1}}{(\zeta-1)^{2g-2}} \Big).$$

Proof. Let us fix a spin curve $(X/k, \mathbb{L})$ for which Lemma 5.2 holds. Then it follows from Lemma 5.2 and the discussion at the beginning of § 5 that

$$\deg_{\mathcal{M}_{g,\mathbb{F}_p}}(\mathcal{M}^{Zzz...}_{g,\mathbb{F}_p}) = \frac{1}{p^g} \cdot \mathcal{Q}^{2,0}_{F_*(\mathcal{L}^\vee)/X/k}$$
$$= \frac{p^{g-1}}{2^{2g-1}} \cdot \sum_{\theta=1}^{p-1} \frac{1}{\sin^{2g-2}(\frac{\pi\cdot\theta}{p})}$$
$$\Big(= \frac{(-1)^{g-1}\cdot p^{g-1}}{2} \cdot \sum_{\zeta^p=1, \zeta\neq 1} \frac{\zeta^{g-1}}{(\zeta-1)^{2g-2}} \Big).$$

□

6. Relation with other results

In the present section, which forms the final section of the present paper, we discuss some topics related to the main result of the present paper.

6.1. Let k be an algebraically closed field of characteristic p and X a proper smooth curve over k of genus g with $\frac{1}{2}\cdot p + 1 > g > 1$. Denote by $F : X \to X_F$ the relative Frobenius morphism. Let $\mathcal{E}$ be an indecomposable vector bundle on X of rank 2 and degree 0. If $\mathcal{E}$ admits a rank one subbundle of positive degree, then it follows from the definition of semistability that $\mathcal{E}$ is not semistable. On the other hand, since $\mathcal{E}$ is indecomposable, a computation of suitable Ext^1 groups via Serre duality shows that the degree of any rank one subbundle of $\mathcal{E}$ is at most $g-1$. We shall say that $\mathcal{E}$ is *maximally unstable* if $\mathcal{E}$ admits a rank one subbundle of degree $g-1 (> 0)$. Let us denote by B the set of isomorphism classes of rank 2 semistable bundles $\mathcal{V}$ on X_F such that $\det(\mathcal{V}) \cong \mathcal{O}_X$, and $F^*\mathcal{V}$ is indecomposable and maximally unstable. Then it is well-known (cf., e.g., [29], Proposition 4.2) that there is a natural 2^{2g}-to-1 correspondence between B and the set of isomorphism classes of dormant indigenous bundles on X/k. Thus, Corollary 5.4 of the present paper enables us to calculate the cardinality of B, i.e., to conclude that $\sharp B = 2 \cdot p^{g-1} \cdot \sum_{\theta=1}^{p-1} \frac{1}{\sin^{2g-2}(\frac{\pi \cdot \theta}{p})}$. In the case where $g = 2$, this result is consistent with the result obtained in [22], Theorem 2.

6.2. F. Liu and B. Osserman have shown (cf. [19], Theorem 2.1) that the value $\deg_{\mathcal{M}_{g,\mathbb{F}_p}}(\mathcal{M}^{\text{Zzz...}}_{g,\mathbb{F}_p})$ may be expressed as a polynomial with respect to the characteristic p of degree $3g-3$ (e.g., $\deg_{\mathcal{M}_{2,\mathbb{F}_p}}(\mathcal{M}^{\text{Zzz...}}_{2,\mathbb{F}_p}) = \frac{1}{24} \cdot (p^3 - p)$, as referred to in Introduction). In fact, this result may also be obtained as a consequence of Corollary 5.4. This may not be apparent at first glance, but nevertheless may be verified by applying either of the following two different (but, closely related) arguments.

(1) Let C be a connected compact Riemann surface of genus $g > 1$. Then it is known that the moduli space of S-equivalence classes (cf. [13], Definition 1.5.3) of rank 2 semistable bundles on C with trivial determinant

$$^{\text{ss}}\mathcal{N}_C^{2,\mathcal{O}}$$

may be represented by a projective algebraic variety of dimension $3g-3$ (cf. [31], Theorem 8.1; [2], § 1; [28], Introduction), and that $\mathrm{Pic}(^{\text{ss}}\mathcal{N}_C^{2,\mathcal{O}}) \cong \mathbb{Z} \cdot [\mathcal{L}]$ for a certain ample line bundle $\mathcal{L}$ (cf. [5], Theorem B; [2], Theorem 1; [2], the discussion at the beginning of § 4). The Verlinde formula, introduced in [34] and proved, e.g., in [6], implies that, for $k = 0, 1, \cdots$, we have an equality

$$\dim_{\mathbb{C}}(H^0(^{\text{ss}}\mathcal{N}_C^{2,\mathcal{O}}, \mathcal{L}^{\otimes k})) = \frac{(k+2)^{g-1}}{2^{g-1}} \cdot \sum_{\theta=1}^{k+1} \frac{1}{\sin^{2g-2}(\frac{\pi \cdot \theta}{k+2})}$$

(cf. [2], § 5, Corollary). Thus, for sufficiently large k, the value at k of the Hilbert polynomial $\mathrm{Hilb}_{\mathcal{L}}(t) \in \mathbb{Q}[t]$ of $\mathcal{L}$ coincides with the RHS of the above equality. On the other hand, it follows from Corollary 5.4 that for an odd prime p, the value at $k = p-2$ of this RHS divided by 2^g coincides with the value $\deg_{\mathcal{M}_{g,\mathbb{F}_p}}(\mathcal{M}^{\text{Zzz...}}_{g,\mathbb{F}_p})$. Thus, the value $\deg_{\mathcal{M}_{g,\mathbb{F}_p}}(\mathcal{M}^{\text{Zzz...}}_{g,\mathbb{F}_p})$ (for sufficiently large p) may be expressed as $\mathrm{Hilb}_{\mathcal{L}}(p-2)$ for a suitable polynomial $\mathrm{Hilb}_{\mathcal{L}}(t) \in \mathbb{Q}[t]$ of degree $3g-3$ $\ (= \dim(^{\text{ss}}\mathcal{N}_C^{2,\mathcal{O}}))$.

(2) By comparison to the discussion of (1), the approach of the following discussion yields a more concrete expression for $\deg_{\mathcal{M}_{g,\mathbb{F}_p}}(\mathcal{M}^{\text{Zzz...}}_{g,\mathbb{F}_p})$. For a pair of positive integers (n, k), we set

$$V(n,k) := \sum_{\theta=1}^{k-1} \frac{1}{\sin^{2n}(\frac{\pi \cdot \theta}{k})}.$$

Then it follows from [35], Theorem 1 (i), (ii); [35], the proof of Theorem 1 (iii), that

$$V(n,k) = -\mathrm{Res}_{x=0}\Big[\frac{k \cdot \cot(kx)}{\sin^{2n}(x)} dx\Big],$$

where $\mathrm{Res}_{x=0}(f)$ denotes the residue of f at $x=0$. Thus, $V(n,k)$ may be computed by considering the relation $\frac{1}{\sin^2(x)} = 1+\cot^2(x)$ and the coefficient of the Laurent expansion (cf. [35], the proof of Theorem 1 (iii))

$$\cot(x) = \frac{1}{x} + \sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j} B_{2j}}{(2j)!} x^{2j-1}$$

where B_{2j} denotes the $(2j)$-th Bernoulli number, i.e.,

$$\frac{w}{e^w - 1} = 1 - \frac{w}{2} + \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} w^{2j}.$$

In particular, it follows from an explicit computation that $V(n,k)$ may be expressed as a polynomial of degree $2n$ with respect to k. Thus, the value $\deg_{\mathcal{M}_{g,\mathbb{F}_p}}(\mathcal{M}^{\text{Zzz...}}_{g,\mathbb{F}_p})$ $(= \frac{p^{g-1}}{2^{2g-1}} V(g-1,p)$ by Corollary 5.4) may be expressed as a polynomial with respect to p of degree $2(g-1)+(g-1) = 3g-3$. Moreover, by applying the above discussion to our calculations, we obtain the following explicit expressions for the polynomials under consideration:

$$\begin{aligned}
\deg_{\mathcal{M}_{2,\mathbb{F}_p}}(\mathcal{M}^{\text{Zzz...}}_{2,\mathbb{F}_p}) =& \frac{1}{24} \cdot (p^3 - p), \\
\deg_{\mathcal{M}_{3,\mathbb{F}_p}}(\mathcal{M}^{\text{Zzz...}}_{3,\mathbb{F}_p}) =& \frac{1}{1440} \cdot (p^6 + 10p^4 - 11p^2), \\
\deg_{\mathcal{M}_{4,\mathbb{F}_p}}(\mathcal{M}^{\text{Zzz...}}_{4,\mathbb{F}_p}) =& \frac{1}{120960} \cdot (2p^9 + 21p^7 + 168p^5 - 191p^3), \\
\deg_{\mathcal{M}_{5,\mathbb{F}_p}}(\mathcal{M}^{\text{Zzz...}}_{5,\mathbb{F}_p}) =& \frac{1}{7257600} \cdot (3p^{12} + 40p^{10} + 294p^8 + 2160p^6 - 2497p^4), \\
\deg_{\mathcal{M}_{6,\mathbb{F}_p}}(\mathcal{M}^{\text{Zzz...}}_{6,\mathbb{F}_p}) =& \frac{1}{2048} \cdot \Big(\frac{2}{93555}p^{15} + \frac{1}{2835}p^{13} + \frac{26}{8505}p^{11} + \frac{164}{8505}p^9 \\
& + \frac{128}{945}p^7 - \frac{14797}{93555}p^5\Big), \\
\deg_{\mathcal{M}_{7,\mathbb{F}_p}}(\mathcal{M}^{\text{Zzz...}}_{7,\mathbb{F}_p}) =& \frac{1}{8192} \cdot \Big(\frac{1382}{638512875}p^{18} + \frac{4}{93555}p^{16} + \frac{31}{70875}p^{14} \\
& + \frac{556}{178605}p^{12} + \frac{3832}{212625}p^{10} + \frac{256}{2079}p^8 - \frac{92427157}{638512875}p^6\Big) \\
\deg_{\mathcal{M}_{8,\mathbb{F}_p}}(\mathcal{M}^{\text{Zzz...}}_{8,\mathbb{F}_p}) =& \frac{1}{32768}p^7 \cdot \Big(\frac{4}{18243225}p^{14} + \frac{1382}{273648375}p^{12} + \frac{4}{66825}p^{10} \\
& + \frac{311}{637875}p^8 + \frac{1184}{382725}p^6 + \frac{1888}{111375}p^4 + \frac{1024}{9009}p^2 \\
& - \frac{36740617}{273648375}\Big)
\end{aligned}$$

$$\begin{aligned}\deg_{\mathcal{M}_{9,\mathbb{F}_p}}(\mathcal{M}^{\mathrm{Zzz...}}_{9,\mathbb{F}_p}) =& \frac{1}{131072}p^8 \cdot \Big(\frac{3617}{162820783125}p^{16} + \frac{32}{54729675}p^{14} \\ &+ \frac{226648}{28733079375}p^{12} + \frac{2144}{29469825}p^{10} + \frac{4946}{9568125}p^8 \\ &+ \frac{268864}{88409475}p^6 + \frac{17067584}{1064188125}p^4 + \frac{2048}{19305}p^2 \\ &- \frac{61430943169}{488462349375}\Big) \\ \deg_{\mathcal{M}_{10,\mathbb{F}_p}}(\mathcal{M}^{\mathrm{Zzz...}}_{10,\mathbb{F}_p}) =& \frac{1}{524288}p^9 \cdot \Big(\frac{87734}{38979295480125}p^{18} + \frac{3617}{54273594375}p^{16} \\ &+ \frac{92}{91216125}p^{14} + \frac{2092348}{201131555625}p^{12} + \frac{4042}{49116375}p^{10} \\ &+ \frac{18716}{35083125}p^8 + \frac{119654944}{40226311125}p^6 + \frac{16229632}{1064188125}p^4 \\ &+ \frac{32768}{328185}p^2 - \frac{23133945892303}{194896477400625}\Big).\end{aligned}$$

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