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1 Introduction

In this paper, we study the exact WKB analysis for a completely integrable system (a holonomic system) of two variables with a large parameter $\eta > 0$:

(1.1) $$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial x_1}\Psi = P(x_1, x_2, \eta)\Psi, \quad P(x_1, x_2, \eta) = \displaystyle\sum_{n\geq 0} \eta^{-n} P_n(x_1, x_2), \\ \eta^{-1}\dfrac{\partial}{\partial x_2}\Psi = Q(x_1, x_2, \eta)\Psi, \quad Q(x_1, x_2, \eta) = \displaystyle\sum_{n\geq 0} \eta^{-n} Q_n(x_1, x_2), \end{cases}$$

where $P_n(x_1, x_2)$ and $Q_n(x_1, x_2)$ $(n = 0, 1, 2, \cdots)$ are 3×3 matrices with holomorphic entries.

As in the case of ordinary differential equations, the Stokes geometry (i.e., turning points and Stokes surfaces) is an important ingredient of the exact WKB analysis for completely integrable systems. In [4], we discussed the Stokes geometry for two concrete completely integrable systems, that is, the Pearcey system and the (1,4) hypergeometric system. For example, the Pearcey system is a completely integrable system of the following form:

(1.2) $$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial x_1}\Psi = P_0(x_1, x_2)\Psi, \\ \eta^{-1}\dfrac{\partial}{\partial x_2}\Psi = \left\{Q_0(x_1, x_2) + \eta^{-1}Q_1(x_1, x_2)\right\}\Psi, \end{cases}$$

where

(1.3) $$P_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -x_1/4 & -x_2/2 & 0 \end{pmatrix}, \quad Q_0 = P_0^2 + \frac{x_2}{3}, \quad Q_1 = \frac{\partial P_0}{\partial x_1}.$$

This system is obtained from a system of partial differential equations

(1.4) $$\begin{cases} \left(\dfrac{\partial^3}{\partial x_1^3} + \dfrac{x_2}{2}\eta^2\dfrac{\partial}{\partial x_1} + \dfrac{x_1}{4}\eta^3\right)\psi = 0, \\ \left(\eta\dfrac{\partial}{\partial x_2} - \dfrac{\partial^2}{\partial x_1^2}\right)\psi = 0, \end{cases}$$

which the Pearcey integral ([7])

(1.5) $$\psi = \int \exp\left\{\eta\left(t^4 + x_2 t^2 + x_1 t\right)\right\} dt$$

satisfies, through the transformation

$$\Psi = \exp\left(\eta \frac{x_2^2}{6}\right) \begin{pmatrix} \psi \\ \eta^{-1} \dfrac{\partial}{\partial x_1} \psi \\ \eta^{-2} \dfrac{\partial^2}{\partial x_1^2} \psi \end{pmatrix}. \tag{1.6}$$

As discussed in [4], the point $(x_1, x_2) = (0, 0)$ plays a crucially important role in the global study of the Stokes geometry for the Pearcey system (1.2)–(1.3). In fact, the point $(x_1, x_2) = (0, 0)$ is a degenerate point where the characteristic equation

$$\xi_1^3 + \frac{x_2}{2}\xi_1 + \frac{x_1}{4} = 0 \tag{1.7}$$

of the first equation

$$\left(\frac{\partial^3}{\partial x_1^3} + \frac{x_2}{2}\eta^2 \frac{\partial}{\partial x_1} + \frac{x_1}{4}\eta^3\right)\psi = 0 \tag{1.8}$$

of (1.4) has a triple root. Furthermore, at $(x_1, x_2) = (0, 0)$ we observe a phenomenon that two turning points $x_1 = \tau^{(1)}(x_2)$ and $x_1 = \tau^{(2)}(x_2)$ of different types of (1.8) for $x_2 \neq 0$ (i.e., $x_1 = \tau^{(i)}(x_2)$ $(i = 1, 2)$ are points where (1.7) has a double root and, further, their types are different in the sense that pairs of roots merging at $x_1 = \tau^{(i)}(x_2)$ are different with sharing only one common root) coalesce at $x_1 = 0$ when x_2 tends to 0. This degeneracy is related to the fact that a new Stokes curve of (1.8) discussed by Berk et al. ([3]) is included in the Stokes surface for the Pearcey system and plays a central role in the global study of the Stokes geometry for the Pearcey system (1.2)–(1.3).

As shown in [4], similar results hold also for the (1,4) hypergeometric system. Thus, in this paper, we study the behavior of a general completely integrable system (1.1) of two variables with 3×3 matrix coefficients near such a degenerate point where two turning points of different types coalesce.

The main theorem of this paper is as follows: We consider a completely integrable system (1.1) near $(x_1, x_2) = (0, 0)$ where two turning points of different types coalesce. Here we assume that $P_n(x_1, x_2)$ and $Q_n(x_1, x_2)$ $(n = 0, 1, 2, \cdots)$ are 3×3 matrices with holomorphic entries in $D_{\rho_0} = \{(x_1, x_2) \in \mathbb{C}^2 \; ; \; |x_j| \leq \rho_0\}$ satisfying

$$\|P_n\|_{\rho_0,\rho_0}, \|Q_n\|_{\rho_0,\rho_0} \leq C\alpha^n n! \tag{1.9}$$

with some norm $\|\cdot\|_{\rho_0,\rho_0}$ (see (2.3), (2.4) for the precise definition) and some positive constants C and α. As we will see later (cf. (2.8)), we may assume $\mathrm{tr} P_0 = \mathrm{tr} Q_0 = 0$ without loss of generality. Then, letting $a_2(x)$ and $a_3(x)$ be holomorphic functions defined by $\det(\xi_1 - P_0(x)) = \xi_1^3 + a_2(x)\xi_1 + a_3(x)$, we may assume that $a_2(x)$ and $a_3(x)$ satisfy

(i) $a_2(0) = a_3(0) = 0$,

since the characteristic equation $\xi_1^3 + a_2(x)\xi_1 + a_3(x) = 0$ is expected to have a triple root at a degenerate point where two turning points of different types coalesce. In addition, we suppose

(ii) $\dfrac{\partial a_3}{\partial x_1}(0) \neq 0$.

(iii) $\det \begin{pmatrix} \dfrac{\partial a_2}{\partial x_1}(0) & \dfrac{\partial a_2}{\partial x_2}(0) \\ \dfrac{\partial a_3}{\partial x_1}(0) & \dfrac{\partial a_3}{\partial x_2}(0) \end{pmatrix} \neq 0$.

(In fact, under these assumptions, we can show that two turning points of different types coalesce at $(x_1, x_2) = (0, 0)$.) Then there exists a WKB theoretic transformation

(1.10)
$$\Psi(x_1, x_2, \eta) = T(x_1, x_2, \eta)\widetilde{\Psi}(\widetilde{x}_1(x_1, x_2), \widetilde{x}_2(x_1, x_2), \eta), \quad T(x_1, x_2, \eta) = \sum_{n\geq 0} \eta^{-n} T_n(x_1, x_2)$$

near $(x_1, x_2) = (0, 0)$ such that the completely integrable system (1.1) is transformed into the Pearcey system (1.2)–(1.3) by the transformation (1.10). That is, if $\Psi(x_1, x_2, \eta)$ is a solution of (1.1), then $\widetilde{\Psi}(\widetilde{x}, \widetilde{x}_2, \eta)$ defined by (1.10) is a solution of the Pearcey system (1.2)–(1.3) with the independent variable $(\widetilde{x}_1, \widetilde{x}_2)$. Here $(\widetilde{x}_1(x_1, x_2), \widetilde{x}_2(x_1, x_2))$ is a biholomorphic coordinate transformation near $(x_1, x_2) = (0, 0)$ and $T_n(x_1, x_2)$ $(n = 0, 1, 2, \cdots)$ are 3×3 matrices with holomorphic entries in D_ρ for some $0 < \rho < \rho_0$ satisfying

$$\|T_n\|_{\rho,\rho} \leq \widetilde{C}\widetilde{\alpha}^n n! \tag{1.11}$$

with some positive constants $\widetilde{C}$ and $\widetilde{\alpha}$.

We explain the meaning of the assumptions (ii) and (iii). First, the assumption (ii) (together with the assumption (i)) means that P_0 and Q_0 satisfy a relation similar to the second relation of (1.3) near $(x_1, x_2) = (0, 0)$. To be more specific, there exist holomorphic functions $b_k(x_1, x_2)$ $(k = 0, 1, 2)$ near $(x_1, x_2) = (0, 0)$ such that

$$Q_0 = b_2(x_1, x_2)P_0^2 + b_1(x_1, x_2)P_0 + b_0(x_1, x_2) \tag{1.12}$$

holds (cf. Lemma 2.2). Furthermore, by using this relation we can verify that the set of turning points for the completely integrable system (1.1) is given by the zeros of the discriminant (with respect to ξ_1) of the characteristic equation $\det(\xi_1 - P_0(x_1, x_2))$ of $P_0(x_1, x_2)$ (cf. Proposition 2.1). We can also confirm that the assumption (ii) automatically follows from the assumptions (i) and (iii) by using the compatibility condition. (see Proposition 2.2.) Next, the assumption (iii) means that the set of turning points for (1.1) is analytically equivalent to the set of turning points for the Pearcey system $\{27x_1^2 + 8x_2^3 = 0\}$ near $(x_1, x_2) = (0, 0)$. The claim of the main theorem is that, under these geometric assumptions, the completely integrable system (1.1) is transformed into the Pearcey system (1.2)–(1.3) by a WKB theoretic transformation near a degenerate point where two turning points of different types coalesce. In particular, near such a degenerate point, the Stokes geometry for (1.1) is analytically equivalent to the Stokes geometry for the Pearcey system (1.2)–(1.3) by the coordinate transformation $(\widetilde{x}_1(x_1, x_2), \widetilde{x}_2(x_1, x_2))$ in the main theorem.

The WKB theoretic transformation (1.10) is, as a formal transformation, the same as the one that Wasow [9], [10] used in transforming a system of ordinary differential equations

$$\eta^{-1}\frac{d}{dx}\Psi = P(x, \eta)\Psi, \quad P(x, \eta) = \sum_{n\geq 0} \eta^{-n} P_n(x) \tag{1.13}$$

into a canonical form. Aoki-Kawai-Takei [2] first used this type of transformations in the framework of the exact WKB analysis, that is, in connection with the Borel resummation method with respect to the large parameter η. In [2], Aoki-Kawai-Takei considered a WKB theoretic transformation for a second-order linear ordinary differential equation

$$\left(\frac{d^2}{dx^2} - \eta^2 Q(x)\right)\psi = 0 \tag{1.14}$$

near a turning point and showed that (1.14) is transformed into the Airy equation near a simple turning point by a WKB theoretic transformation, such a WKB theoretic transformation acts analytically on Borel transformed WKB solutions, and that several properties (for example, the connection formula on Stokes curves) of the Borel sum of WKB solutions can be obtained by these facts. In this paper, suggested by [8], we employ the WKB theoretic transformation (1.10) for the analysis of a completely integrable system (1.1) of two variables. In particular, the estimate (1.11) guarantees that $T(x_1, x_2, \eta)$ acts analytically on the Borel transform of a WKB solution $\widetilde{\Psi}$.

The paper is constructed as follows: In Section 2, we explain the precise statement of the main theorem and discuss a generalization of the main theorem to completely integrable systems of two variables with $m \times m$ matrix coefficients. Furthermore, we give some remarks on the main theorem in Subsection 2.1. In particular, we show an important property of the set of turning points and the Stokes surface for (1.1). Lemmas shown in Subsection 2.1 play an important role throughout the paper. In Subsection 2.2, we give a proof of the generalization of the main theorem to completely integrable systems with $m \times m$ matrix coefficients by using the main theorem. Then in Section 3 we consider some examples to which the main theorem is applicable. We discuss the (1,4) hypergeometric system in Subsection 3.1 and the (2,3) hypergeometric system in Subsection 3.2. In Sections 4 and 5, we give a proof of the main theorem. We construct a WKB theoretic transformation in Section 4 and prove the estimate (1.11) for it in Section 5.

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2 Main theorem

In this section, we first consider the completely integrable system

$$(2.1)\qquad \begin{cases} \eta^{-1}\dfrac{\partial}{\partial x_1}\Psi = P(x,\eta)\Psi, \quad P(x,\eta) = \displaystyle\sum_{n\geq 0}\eta^{-n}P_n(x), \\ \eta^{-1}\dfrac{\partial}{\partial x_2}\Psi = Q(x,\eta)\Psi, \quad Q(x,\eta) = \displaystyle\sum_{n\geq 0}\eta^{-n}Q_n(x), \end{cases}$$

where $P_n(x)$ and $Q_n(x)$ are 3×3 matrices with holomorphic entries in $D_{\rho_0} = \{x = (x_1, x_2) \in \mathbb{C}^2\ ;\ |x_j| \leq \rho_0\}$ for some $\rho_0 > 0$ and satisfy an estimate

$$(2.2)\qquad \|P_n\|_{\rho_0,\rho_0}, \|Q_n\|_{\rho_0,\rho_0} \leq C\alpha^n n! \quad (n \geq 0)$$

with some positive constants C and α. Here we define $\|A\|_{\rho_1,\rho_2}$ for a matrix $A(x) = (a_{ij}(x))$ with holomorphic entries in D_{ρ_0} $(0 < \rho_1, \rho_2 < \rho_0)$ as follows:

$$(2.3)\qquad \|A\|_{\rho_1,\rho_2} = \sup_{|x_1|\leq\rho_1, |x_2|\leq\rho_2} \|A(x)\|,$$

$$(2.4)\qquad \|A(x)\| = \sum_{i,j}|a_{ij}(x)|.$$

Note that the system satisfies the compatibility condition

$$(2.5)\qquad [P, Q] + \eta^{-1}\frac{\partial P}{\partial x_2} - \eta^{-1}\frac{\partial Q}{\partial x_1} = 0.$$

As its consequence we have

$$(2.6)\qquad [P_0, Q_0] = 0,$$

$$[P_1, Q_0] + [P_0, Q_1] + \frac{\partial P_0}{\partial x_2} - \frac{\partial Q_0}{\partial x_1} = 0. \tag{2.7}$$

Without loss of generality we may assume that P_0 and Q_0 satisfy

$$\mathrm{tr} P_0 = \mathrm{tr} Q_0 = 0. \tag{2.8}$$

In fact, taking the trace of (2.5), we obtain

$$\mathrm{tr}\left(\frac{\partial P}{\partial x_2} - \frac{\partial Q}{\partial x_1}\right) = 0. \tag{2.9}$$

In particular, $\mathrm{tr}\,(P_0(x)dx_1 + Q_0(x)dx_2)$ is a closed one form. Then, by a gauge transformation $\Psi = \exp\left\{\eta \int^x \mathrm{tr}\,(P_0(x)dx_1 + Q_0(x)dx_2)\,/3\right\}\Phi$, (2.1) is transformed into

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial x_1}\Phi = \left(P(x,\eta) - \dfrac{\mathrm{tr}P_0(x)}{3}\right)\Phi, \\ \eta^{-1}\dfrac{\partial}{\partial x_2}\Phi = \left(Q(x,\eta) - \dfrac{\mathrm{tr}Q_0(x)}{3}\right)\Phi. \end{cases} \tag{2.10}$$

Our main theorem is the following

Theorem 2.1. *Let $a_2(x)$ and $a_3(x)$ be holomorphic functions defined by $\det(\xi_1 - P_0(x)) = \xi_1^3 + a_2(x)\xi_1 + a_3(x)$. Suppose that $a_2(x)$ and $a_3(x)$ satisfy the following conditions:*

(i) $a_2(0) = a_3(0) = 0.$

(ii) $\dfrac{\partial a_3}{\partial x_1}(0) \neq 0.$

(iii) $\det\begin{pmatrix} \dfrac{\partial a_2}{\partial x_1}(0) & \dfrac{\partial a_2}{\partial x_2}(0) \\ \dfrac{\partial a_3}{\partial x_1}(0) & \dfrac{\partial a_3}{\partial x_2}(0) \end{pmatrix} \neq 0.$

Then there exist a sufficiently small positive constant $0 < \rho < \rho_0$, holomorphic functions $\widetilde{x}_i(x)$ $(i = 1, 2)$ in D_ρ, and an infinite series of 3×3 matrices $\{T_n(x)\}_{n \geq 0}$ which satisfy the following properties:

- $\widetilde{x}_1(0) = \widetilde{x}_2(0) = 0.$
- *$\widetilde{x}(x) = (\widetilde{x}_1(x), \widetilde{x}_2(x))$ is a biholomorphic map from D_ρ to $\widetilde{x}(D_\rho)$.*
- *Every entry of $T_n(x)$ is holomorphic in D_ρ and $\det T_0(x) \neq 0$ $(x \in D_\rho)$.*
- *$T_n(x)$ $(n \geq 0)$ satisfy*

  $$\|T_n\|_{\rho,\rho} \leq \widetilde{C}\widetilde{\alpha}^n n! \tag{2.11}$$

  with some positive constants $\widetilde{C}$ and $\widetilde{\alpha}$.
- *By a formal transformation*

  $$\Psi(x,\eta) = T(x,\eta)\widetilde{\Psi}(\widetilde{x}(x),\eta), \quad T(x,\eta) = \sum_{n\geq 0}\eta^{-n}T_n(x), \tag{2.12}$$

(2.1) *is transformed into the following system of equations:*

$$
\begin{cases}
\eta^{-1}\dfrac{\partial}{\partial \widetilde{x}_1}\widetilde{\Psi} = \widetilde{P}(\widetilde{x},\eta)\widetilde{\Psi}, \quad \widetilde{P}(\widetilde{x},\eta) = \displaystyle\sum_{n\geq 0}\eta^{-n}\widetilde{P}_n(\widetilde{x}), \\
\eta^{-1}\dfrac{\partial}{\partial \widetilde{x}_2}\widetilde{\Psi} = \widetilde{Q}(\widetilde{x},\eta)\widetilde{\Psi}, \quad \widetilde{Q}(\widetilde{x},\eta) = \displaystyle\sum_{n\geq 0}\eta^{-n}\widetilde{Q}_n(\widetilde{x}),
\end{cases} \tag{2.13}
$$

where $\widetilde{P}_n(\widetilde{x})$ *and* $\widetilde{Q}_n(\widetilde{x})$ *are given as follows:*

$$
\widetilde{P}_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\widetilde{x}_1/4 & -\widetilde{x}_2/2 & 0 \end{pmatrix}, \quad \widetilde{P}_n = 0 \quad (n \geq 1), \tag{2.14}
$$

$$
\widetilde{Q}_0 = \widetilde{P}_0^2 + \frac{\widetilde{x}_2}{3}, \quad \widetilde{Q}_1 = \frac{\partial \widetilde{P}_0}{\partial \widetilde{x}_1}, \quad \widetilde{Q}_n = 0 \quad (n \geq 2). \tag{2.15}
$$

Note that the system (2.13) is equivalent to the Pearcey system

$$
\begin{cases}
\left(\dfrac{\partial^3}{\partial \widetilde{x}_1^3} + \dfrac{\widetilde{x}_2}{2}\eta^2\dfrac{\partial}{\partial \widetilde{x}_1} + \dfrac{\widetilde{x}_1}{4}\eta^3\right)\widetilde{\psi} = 0, \\
\left(\eta\dfrac{\partial}{\partial \widetilde{x}_2} - \dfrac{\partial^2}{\partial \widetilde{x}_1^2}\right)\widetilde{\psi} = 0
\end{cases} \tag{2.16}
$$

through the transformation

$$
\widetilde{\Psi} = \exp\left(\eta\frac{\widetilde{x}_2^2}{6}\right)\begin{pmatrix} \widetilde{\psi} \\ \eta^{-1}\dfrac{\partial}{\partial \widetilde{x}_1}\widetilde{\psi} \\ \eta^{-2}\dfrac{\partial^2}{\partial \widetilde{x}_1^2}\widetilde{\psi} \end{pmatrix}. \tag{2.17}
$$

We call (2.13) also the Pearcey system in this paper.

We next consider a generalization of Theorem 2.1 to a completely integrable system of two variables with $m \times m$ matrix coefficients:

$$
\begin{cases}
\eta^{-1}\dfrac{\partial}{\partial x_1}\Psi = P(x,\eta)\Psi, \quad P(x,\eta) = \displaystyle\sum_{n\geq 0}\eta^{-n}P_n(x), \\
\eta^{-1}\dfrac{\partial}{\partial x_2}\Psi = Q(x,\eta)\Psi, \quad Q(x,\eta) = \displaystyle\sum_{n\geq 0}\eta^{-n}Q_n(x),
\end{cases} \tag{2.18}
$$

where $P_n(x)$ and $Q_n(x)$ are $m \times m$ matrices with holomorphic entries in $D_{\rho_0} = \{x = (x_1, x_2) \in \mathbb{C}^2 \; ; \; |x_j| \leq \rho_0\}$ for some $\rho_0 > 0$ and satisfy an estimate

$$
\|P_n\|_{\rho_0,\rho_0}, \|Q_n\|_{\rho_0,\rho_0} \leq C\alpha^n n! \quad (n \geq 0) \tag{2.19}
$$

with some positive constants C and α.

Theorem 2.2. *Suppose that* $D(x,\xi_1) = \det(\xi_1 - P_0(x))$ *satisfies the following conditions:*

(i)* *The equation* $D(0,\xi_1) = 0$ *has a triple root* $\xi_1^\star$*, that is,* $\xi_1^\star$ *satisfies*

$$
D(0,\xi_1^\star) = \frac{\partial D}{\partial \xi_1}(0,\xi_1^\star) = \frac{\partial^2 D}{\partial \xi_1^2}(0,\xi_1^\star) = 0, \quad \frac{\partial^3 D}{\partial \xi_1^3}(0,\xi_1^\star) \neq 0. \tag{2.20}
$$

(ii)* $\dfrac{\partial D}{\partial x_1}(0,\xi_1^\star) \neq 0.$

(iii)* $\det \begin{pmatrix} \dfrac{\partial D}{\partial x_1} & \dfrac{\partial D}{\partial x_2} \\ \dfrac{\partial^2 D}{\partial x_1 \partial \xi_1} & \dfrac{\partial^2 D}{\partial x_2 \partial \xi_1} \end{pmatrix} |_{(x,\xi_1)=(0,\xi_1^\star)} \neq 0.$

Then there exist a sufficiently small positive constant $0 < \rho < \rho_0$ and an infinite series of $m \times m$ matrices $\{T_n^\star(x)\}_{n \geq 0}$ which satisfy the following properties:

- *Every entry $T_n^\star(x)$ is holomorphic in D_ρ and $\det T_0^\star(x) \neq 0$ $(x \in D_\rho)$.*
- *$T_n^\star(x)$ $(n \geq 0)$ satisfy*

  $$\|T_n^\star\|_{\rho,\rho} \leq C^\star \, (\alpha^\star)^n \, n! \tag{2.21}$$

  with some positive constants $C^\star$ and $\alpha^\star$.
- *By a formal transformation*

  $$\Psi = T^\star(x,\eta)\Psi^\star, \quad T^\star(x,\eta) = \sum_{n\geq 0} \eta^{-n} T_n^\star(x), \tag{2.22}$$

  (2.18) is transformed into the following system of equations:

  $$\begin{cases} \eta^{-1} \dfrac{\partial}{\partial x_1} \Psi^\star = P^\star(x,\eta)\Psi^\star, & P^\star(x,\eta) = \displaystyle\sum_{n\geq 0} \eta^{-n} P_n^\star(x), \\ \eta^{-1} \dfrac{\partial}{\partial x_2} \Psi^\star = Q^\star(x,\eta)\Psi^\star, & Q^\star(x,\eta) = \displaystyle\sum_{n\geq 0} \eta^{-n} Q_n^\star(x), \end{cases} \tag{2.23}$$

  where $P_n^\star(x)$ and $Q_n^\star(x)$ are block-diagonal matrices

  $$P_n^\star(x) = \begin{pmatrix} P_n^{\star(1)}(x) & \\ & P_n^{\star(2)}(x) \end{pmatrix}, \quad Q_n^\star(x) = \begin{pmatrix} Q_n^{\star(1)}(x) & \\ & Q_n^{\star(2)}(x) \end{pmatrix}, \tag{2.24}$$

  $P_n^{\star(1)}(x), Q_n^{\star(1)}(x)$ (resp., $P_n^{\star(2)}(x), Q_n^{\star(2)}(x)$) are 3×3 matrices (resp., $(m-3)\times(m-3)$ matrices), and all coefficients $P_n^{\star(j)}(x)$ and $Q_n^{\star(j)}(x)$ $(j = 1, 2, n = 0, 1, 2, \cdots)$ are holomorphic in D_ρ and satisfy an estimate

  $$\left\| P_n^{\star(j)} \right\|_{\rho,\rho}, \left\| Q_n^{\star(j)} \right\|_{\rho,\rho} \leq \widetilde{C}^\star \, (\widetilde{\alpha}^\star)^n \, n! \quad (j = 1, 2) \tag{2.25}$$

  with some positive constants $\widetilde{C}^\star$ and $\widetilde{\alpha}^\star$. Furthermore we have

  $$\det\left(\xi_1 - P_0^{\star(1)}(x)\right) |_{(x,\xi_1)=(0,\xi_1^\star)} = 0, \quad \det\left(\xi_1 - P_0^{\star(2)}(x)\right) |_{(x,\xi_1)=(0,\xi_1^\star)} \neq 0. \tag{2.26}$$

- *The subsystem*

  $$\begin{cases} \eta^{-1} \dfrac{\partial}{\partial x_1} \Psi^{\star(1)} = P^{\star(1)}(x,\eta)\Psi^{\star(1)}, & P^{\star(1)}(x,\eta) = \displaystyle\sum_{n\geq 0} \eta^{-n} P_n^{\star(1)}(x), \\ \eta^{-1} \dfrac{\partial}{\partial x_2} \Psi^{\star(1)} = Q^{\star(1)}(x,\eta)\Psi^{\star(1)}, & Q^{\star(1)}(x,\eta) = \displaystyle\sum_{n\geq 0} \eta^{-n} Q_n^{\star(1)}(x), \end{cases} \tag{2.27}$$

  of (2.23) can be transformed into the Pearcey system (2.13) near $x = 0$ in the sense of Theorem 2.1.

2.1 Some remarks on Theorem 2.1

In this subsection, we give some remarks on Theorem 2.1. Let us begin with the following lemmas, which play an important role throughout this paper.

Lemma 2.1. *Let $a_2(x)$ and $a_3(x)$ be holomorphic functions defined by* $\det(\xi_1 - P_0(x)) = \xi_1^3 + a_2(x)\xi_1 + a_3(x)$. *Suppose that $a_2(x)$ and $a_3(x)$ satisfy the following conditions:*

(i) $a_2(0) = a_3(0) = 0$,

(ii) $\dfrac{\partial a_3}{\partial x_1}(0) \neq 0$.

Then there exists a 3×3 matrix $T^\dagger(x)$ which satisfies the following properties:

- *Every entry of $T^\dagger(x)$ is holomorphic near $x = 0$ and* $\det T^\dagger(0) \neq 0$.
- $\left(T^\dagger(x)\right)^{-1} P_0(x) T^\dagger(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3(x) & -a_2(x) & 0 \end{pmatrix}$.

Proof. Since $P_0(0)^3 = 0$ by the assumption, there exists a constant matrix $T_1^\dagger \in GL(3;\mathbb{C})$ such that $(T_1^\dagger)^{-1} P_0(0) T_1^\dagger$ can be expressed as one of the following:

$$\text{Case 1}: \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{Case 2}: \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{Case 3}: \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Case 1 : Let $p_i(x)$ be a $\mathbb{C}^3$-valued function defined by $(T_1^\dagger)^{-1} P_0(x) T_1^\dagger = (p_1(x), p_2(x), p_3(x))$. Then we have $p_i(0) = 0$ $(i = 1, 2, 3)$ and

$$\begin{aligned} \frac{\partial a_3}{\partial x_1}(0) &= -\frac{\partial}{\partial x_1} \det P_0(x)\big|_{x=0} \\ &= -\det\left(\frac{\partial p_1}{\partial x_1}(0), 0, 0\right) - \det\left(0, \frac{\partial p_2}{\partial x_1}(0), 0\right) - \det\left(0, 0, \frac{\partial p_3}{\partial x_3}(0)\right) \\ &= 0. \end{aligned} \tag{2.28}$$

This contradicts the assumption.

Case 2 : Let $p_i(x)$ be a $\mathbb{C}^3$-valued function defined by $(T_1^\dagger)^{-1} P_0(x) T_1^\dagger = (p_1(x), p_2(x), p_3(x))$. Then we have $p_i(0) = 0$ $(i = 1, 3)$ and

$$\begin{aligned} \frac{\partial a_3}{\partial x_1}(0) &= -\frac{\partial}{\partial x_1} \det P_0(x)\big|_{x=0} \\ &= -\det\left(\frac{\partial p_1}{\partial x_1}(0), p_2(0), 0\right) - \det\left(0, \frac{\partial p_2}{\partial x_1}(0), 0\right) - \det\left(0, p_2(0), \frac{\partial p_3}{\partial x_3}(0)\right) \\ &= 0. \end{aligned} \tag{2.29}$$

This contradicts the assumption.

Hence $(T_1^\dagger)^{-1} P_0(0) T_1^\dagger$ must be of the form of Case 3. We next define a 3×3 matrix $T_2^\dagger(x)$ with holomorphic entries near $x = 0$ as follows:

$$T_2^\dagger(x) = \begin{pmatrix} (1,0,0) \\ (1,0,0)\,(T_1^\dagger)^{-1} P_0(x) T_1^\dagger \\ (1,0,0)\,(T_1^\dagger)^{-1} P_0(x)^2 T_1^\dagger \end{pmatrix}. \tag{2.30}$$

Since $(T_1^\dagger)^{-1}P_0(0)T_1^\dagger$ is of the form of Case 3, $T_2^\dagger(0)$ is the identity matrix. Hence $T_2^\dagger(x)$ is invertible near $x = 0$. Furthermore, we have

$$T_2^\dagger(x)(T_1^\dagger)^{-1}P_0(x)T_1^\dagger = \begin{pmatrix} (1,0,0)\,(T_1^\dagger)^{-1}P_0(x)T_1^\dagger \\ (1,0,0)\,(T_1^\dagger)^{-1}P_0(x)^2T_1^\dagger \\ (1,0,0)\,(T_1^\dagger)^{-1}P_0(x)^3T_1^\dagger \end{pmatrix}$$
$$= \begin{pmatrix} (1,0,0)\,(T_1^\dagger)^{-1}P_0(x)T_1^\dagger \\ (1,0,0)\,(T_1^\dagger)^{-1}P_0(x)^2T_1^\dagger \\ -a_3(x)\,(1,0,0) - a_2(x)\,(1,0,0)\,(T_1^\dagger)^{-1}P_0(x)T_1^\dagger \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3(x) & -a_2(x) & 0 \end{pmatrix} T_2^\dagger(x). \tag{2.31}$$

Therefore $T^\dagger(x) = T_1^\dagger T_2^\dagger(x)^{-1}$ satisfies the properties in Lemma 2.1. □

Using this lemma, we prove

Lemma 2.2. *Under the same assumptions in Lemma 2.1, there exist unique holomorphic functions $b_k(x)$ $(k = 0, 1, 2)$ near $x = 0$ such that*

$$Q_0 = b_2(x)P_0^2 + b_1(x)P_0 + b_0(x). \tag{2.32}$$

Proof. Since $[P_0, Q_0] = 0$ follows from the compatibility condition, we have the following relation:

$$\left[(T^\dagger)^{-1}P_0T^\dagger, (T^\dagger)^{-1}Q_0T^\dagger\right] = (T^\dagger)^{-1}\,[P_0, Q_0]\,T^\dagger = 0, \tag{2.33}$$

where $T^\dagger = T^\dagger(x)$ is the one given in Lemma 2.1. We denote by $b_0(x)$ (resp., $b_1(x)$, $b_2(x)$) the $(1,1)$ (resp., $(1,2)$, $(1,3)$) entry of $T^\dagger(x)^{-1}Q_0(x)T^\dagger(x)$ and consider the following matrix

$$(T^\dagger)^{-1}\left(Q_0 - b_2(x)P_0^2 - b_1(x)P_0 - b_0(x)\right)T^\dagger. \tag{2.34}$$

For this matrix, we have

$$\left[(T^\dagger)^{-1}P_0T^\dagger, (T^\dagger)^{-1}\left(Q_0 - b_2(x)P_0^2 - b_1(x)P_0 - b_0(x)\right)T^\dagger\right] = 0, \tag{2.35}$$

and the $(1,1)$, $(1,2)$ and $(1,3)$ entries of (2.34) are all zero. Thanks to this fact and Lemma 2.1, the $(1,1)$ (resp., $(1,2)$, $(1,3)$) entry of the left-hand side of (2.35) is the $(2,1)$ (resp., $(2,2)$, $(2,3)$) entry of (2.34), and hence the $(2,1)$, $(2,2)$ and $(2,3)$ entries of (2.34) are all zero. By the same argument, we can verify that the $(3,1)$, $(3,2)$ and $(3,3)$ entries of (2.34) are also zero. Thus we have proved that (2.34) is the zero matrix.

We next prove the uniqueness of $b_k(x)$ $(k = 0, 1, 2)$. If holomorphic functions $\widetilde{b}_k(x)$ $(k = 0, 1, 2)$ near $x = 0$ satisfy

$$\widetilde{b}_2(x)P_0^2 + \widetilde{b}_1(x)P_0 + \widetilde{b}_0(x) = 0, \tag{2.36}$$

then we have

$$\widetilde{b}_2(x)\left((T^\dagger)^{-1}P_0T^\dagger\right)^2 + \widetilde{b}_1(x)(T^\dagger)^{-1}P_0T^\dagger + \widetilde{b}_0(x) = 0. \tag{2.37}$$

By the choice of $T^\dagger(x)$, the $(1,1)$, $(1,2)$ and $(1,3)$ entries of the left-hand side of (2.37) is given by $\widetilde{b}_0(x)$, $\widetilde{b}_1(x)$ and $\widetilde{b}_2(x)$, respectively. Therefore we have $\widetilde{b}_k(x) = 0$ $(k = 0, 1, 2)$. This means the uniqueness of $b_k(x)$ $(k = 0, 1, 2)$. □

Note that, by the definition of $a_2(x)$ and $a_3(x)$, P_0 satisfies the following relation

$$P_0^3 + a_2(x)P_0 + a_3(x) = 0. \tag{2.38}$$

Similarly, for $\widetilde{P}_0$ we have

$$\widetilde{P}_0^3 + \frac{\widetilde{x}_2}{2}\widetilde{P}_0 + \frac{\widetilde{x}_1}{4} = 0. \tag{2.39}$$

We now discuss the Stokes geometry for the system (2.1) near $x = 0$, using Theorem 2.1. First, let us recall the definition of a turning point for the system (2.1). Let $\xi_{1,i}$ $(i = 1, 2, 3)$ be three roots of the algebraic equation $\det(\xi_1 - P_0(x)) = \xi_1^3 + a_2(x)\xi_1 + a_3(x) = 0$. In view of Lemma 2.2, we find that $b_2(x)\xi_{1,i}^2 + b_1(x)\xi_{1,i} + b_0(x)$ is a root of the algebraic equation of $\det(\xi_2 - Q_0(x)) = 0$. By this fact, we label three roots of the algebraic equation $\det(\xi_2 - Q_0(x)) = 0$ as follows:

$$\xi_{2,i} = b_2(x)\xi_{1,i}^2 + b_1(x)\xi_{1,i} + b_0(x) \quad (i = 1, 2, 3). \tag{2.40}$$

Definition 2.1. *A point $c \in \mathbb{C}^2$ is called a turning point for the system (2.1) if there exist $i, i' \in \{1, 2, 3\}$ $(i \neq i')$ such that*

$$\xi_{1,i}(c) = \xi_{1,i'}(c), \quad \xi_{2,i}(c) = \xi_{2,i'}(c). \tag{2.41}$$

Proposition 2.1. *The set of the turning points of the system (2.1) is explicitly given by*

$$\left\{x\ ;\ 27a_3(x)^2 + 4a_2(x)^3 = 0\right\}. \tag{2.42}$$

Proof. By (2.40), the condition (2.41) is equivalent to

$$\xi_{1,i}(c) = \xi_{1,i'}(c). \tag{2.43}$$

Hence a turning point for the system (2.1) is a point where the algebraic equation $\xi_1^3 + a_2(x)\xi_1 + a_3(x) = 0$ has a multiple root, that is, a zero of the discriminant of $\xi_1^3 + a_2(x)\xi_1 + a_3(x) = 0$. This completes the proof . □

Remark 2.1. *Under the assumptions of Theorem 2.1 we find that the set of the turning points (2.42) is transformed into the set of the turning point of the Pearcey system $\left\{z\ ;\ 27z_1^2 + 8z_2^3 = 0\right\}$ by a coordinate transformation $z(x) = (4a_3(x), 2a_2(x))$ near $x = 0$. In particular, the set of the turning points (2.42) has a cusp at $x = 0$.*

We have the following relations between $a_k(x)$ and $b_k(x)$ as a consequence of the compatibility condition.

Lemma 2.3. *Suppose that the compatibility condition (2.5) is satisfied. Then $a_k(x)$ and $b_k(x)$ satisfy the following relations.*

$$\begin{cases} \dfrac{\partial}{\partial x_1}(2b_2a_2 - 3b_0) = 0, \\ \dfrac{\partial a_2}{\partial x_2} = 2b_2\dfrac{\partial a_3}{\partial x_1} + b_1\dfrac{\partial a_2}{\partial x_1} + 3\dfrac{\partial b_2}{\partial x_1}a_3 + 2\dfrac{\partial b_1}{\partial x_1}a_2, \\ \dfrac{\partial a_3}{\partial x_2} = b_1\dfrac{\partial a_3}{\partial x_1} + 3\dfrac{\partial b_1}{\partial x_1}a_3 - \dfrac{\partial b_0}{\partial x_1}a_2. \end{cases} \tag{2.44}$$

Proof. By a transformation $\Psi = T^\dagger(x)\Psi^\dagger$ with $T^\dagger(x)$ being given by Lemma 2.1, (2.1) is transformed into the following form:

$$
\begin{cases}
\eta^{-1}\dfrac{\partial}{\partial x_1}\Psi^\dagger = P^\dagger(x,\eta)\Psi^\dagger, \quad P^\dagger(x,\eta) = \displaystyle\sum_{n\geq 0}\eta^{-n}P_n^\dagger(x), \\
\eta^{-1}\dfrac{\partial}{\partial x_2}\Psi^\dagger = Q^\dagger(x,\eta)\Psi^\dagger, \quad Q^\dagger(x,\eta) = \displaystyle\sum_{n\geq 0}\eta^{-n}Q_n^\dagger(x),
\end{cases} \tag{2.45}
$$

where

$$
P^\dagger(x,\eta) =(T^\dagger)^{-1}P(x,\eta)T^\dagger - \eta^{-1}(T^\dagger)^{-1}\frac{\partial T^\dagger}{\partial x_1}, \tag{2.46}
$$

$$
Q^\dagger(x,\eta) =(T^\dagger)^{-1}Q(x,\eta)T^\dagger - \eta^{-1}(T^\dagger)^{-1}\frac{\partial T^\dagger}{\partial x_2}. \tag{2.47}
$$

In particular,

$$
P_0^\dagger = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & 0 \end{pmatrix}, \quad Q_0^\dagger = \begin{pmatrix} b_0 & b_1 & b_2 \\ -b_2a_3 & -b_2a_2 + b_0 & b_1 \\ -b_1a_3 & -b_2a_3 - b_1a_2 & -b_2a_2 + b_0 \end{pmatrix} \tag{2.48}
$$

hold in view of Lemmas 2.1 and 2.2. Since the system (2.1) satisfies the compatibility condition, the system (2.45) also satisfies the compatibility condition

$$
\left[P^\dagger, Q^\dagger\right] + \eta^{-1}\frac{\partial P^\dagger}{\partial x_2} - \eta^{-1}\frac{\partial Q^\dagger}{\partial x_1} = 0. \tag{2.49}
$$

In particular, we have

$$
\left[P_0^\dagger, Q_0^\dagger\right] = 0, \tag{2.50}
$$

$$
\left[P_1^\dagger, Q_0^\dagger\right] + \left[P_0^\dagger, Q_1^\dagger\right] + \frac{\partial P_0^\dagger}{\partial x_2} - \frac{\partial Q_0^\dagger}{\partial x_1} = 0. \tag{2.51}
$$

For $k = 0, 1, 2$, we have

$$
\begin{aligned}
&\operatorname{tr}\left\{(P_0^\dagger)^k\left(\left[P_1^\dagger, Q_0^\dagger\right] + \left[P_0^\dagger, Q_1^\dagger\right]\right)\right\} \\
=&\operatorname{tr}\left\{(P_0^\dagger)^kP_1^\dagger Q_0^\dagger - (P_0^\dagger)^kQ_0^\dagger P_1^\dagger\right\} - \operatorname{tr}\left\{(P_0^\dagger)^kP_0^\dagger Q_1^\dagger - (P_0^\dagger)^kQ_1^\dagger P_0^\dagger\right\} \\
=&\operatorname{tr}\left[(P_0^\dagger)^kP_1^\dagger, Q_0^\dagger\right] - \operatorname{tr}\left[P_0^\dagger, (P_0^\dagger)^kQ_1^\dagger\right] \\
=&0
\end{aligned} \tag{2.52}
$$

in view of (2.50). Hence

$$
\operatorname{tr}\left\{(P_0^\dagger)^k\left(\frac{\partial P_0^\dagger}{\partial x_2} - \frac{\partial Q_0^\dagger}{\partial x_1}\right)\right\} = 0 \tag{2.53}
$$

holds for $k = 0, 1, 2$. On the other hand, using (2.48), we obtain by straightforward computations

$$
\operatorname{tr}\left\{\left(\frac{\partial P_0^\dagger}{\partial x_2} - \frac{\partial Q_0^\dagger}{\partial x_1}\right)\right\} =\frac{\partial}{\partial x_1}\left(2b_2a_2 - 3b_0\right), \tag{2.54}
$$

$$
\operatorname{tr}\left\{P_0^\dagger\left(\frac{\partial P_0^\dagger}{\partial x_2} - \frac{\partial Q_0^\dagger}{\partial x_1}\right)\right\} = -\frac{\partial a_2}{\partial x_2} + 2b_2\frac{\partial a_3}{\partial x_1} + b_1\frac{\partial a_2}{\partial x_1} + 3\frac{\partial b_2}{\partial x_1}a_3 + 2\frac{\partial b_1}{\partial x_1}a_2, \tag{2.55}
$$

$$
\operatorname{tr}\left\{(P_0^\dagger)^2\left(\frac{\partial P_0^\dagger}{\partial x_2} - \frac{\partial Q_0^\dagger}{\partial x_1}\right)\right\} = -\frac{\partial a_3}{\partial x_2} + b_1\frac{\partial a_3}{\partial x_1} + 3\frac{\partial b_1}{\partial x_1}a_3 + 2a_2\frac{\partial}{\partial x_1}\left(b_0 - b_2a_2\right). \tag{2.56}
$$

Using (2.53), (2.54), (2.55) and (2.56), we obtain (2.44). □

Proposition 2.2. *The assumption* (ii) *of Theorem 2.1 follows from the assumptions* (i) *and* (iii) *of Theorem 2.1.*

Proof. This proposition is verified by using the compatibility condition for the system (2.1). As a matter of fact, by the assumption (i) of Theorem 2.1 and (2.44), we find

$$\det\begin{pmatrix} \frac{\partial a_2}{\partial x_1}(0) & \frac{\partial a_2}{\partial x_2}(0) \\ \frac{\partial a_3}{\partial x_1}(0) & \frac{\partial a_3}{\partial x_2}(0) \end{pmatrix} = \det\begin{pmatrix} \frac{\partial a_2}{\partial x_1}(0) & 2b_2(0)\frac{\partial a_3}{\partial x_1}(0) + b_1(0)\frac{\partial a_2}{\partial x_1}(0) \\ \frac{\partial a_3}{\partial x_1}(0) & b_1(0)\frac{\partial a_3}{\partial x_1}(0) \end{pmatrix} = -2b_2(0)\left(\frac{\partial a_3}{\partial x_1}(0)\right)^2. \tag{2.57}$$

Hence, using the assumption (iii) of Theorem 2.1, we get the assumption (ii) of Theorem 2.1. □

Remark 2.2. *By Proposition 2.2, the assumptions of Theorem 2.1 is equivalent to*

(i)′ $a_2(0) = a_3(0) = 0.$

(ii)′ $\det\begin{pmatrix} \frac{\partial a_2}{\partial x_1}(0) & \frac{\partial a_2}{\partial x_2}(0) \\ \frac{\partial a_3}{\partial x_1}(0) & \frac{\partial a_3}{\partial x_2}(0) \end{pmatrix} \neq 0.$

Remark 2.3. *By Remark 2.1, we find that*

(i)″ $a_2(0) = a_3(0) = 0,$

(ii)″ $\{x\ ;\ 27a_3(x)^2 + 4a_2(x)^3 = 0\}$ *is transformed into* $\{z\ ;\ 27z_1^2 + 8z_2^3 = 0\}$ *by some coordinate transformation* $z : \mathbb{C}_x^2 \to \mathbb{C}_z^2$ *near* $x = 0$,

are necessary conditions for (i)′ *and* (ii)′ *in Remark 2.2. Furthermore, we can prove that* (i)″ *and* (ii)″ *are necessary and sufficient conditions for* (i)′ *and* (ii)′.

We now see that the Stokes geometry for (2.1) is transformed into the Stokes geometry for the Pearcey system near $x = 0$ by the coordinate transformation $\widetilde{x}(x)$ given by Theorem 2.1.

Lemma 2.4. *Let* $\widetilde{x}(x)$ *be the coordinate transformation given by Theorem 2.1. Let* $\widetilde{\xi}_{1,i}(\widetilde{x})$ $(i = 1, 2, 3)$ *be roots of*

$$\det\left(\widetilde{\xi}_1 - \widetilde{P}_0(\widetilde{x})\right) = \left(\widetilde{\xi}_1\right)^3 + \frac{\widetilde{x}_2}{2}\widetilde{\xi}_1 + \frac{\widetilde{x}_1}{4} = 0, \tag{2.58}$$

and let $\widetilde{\xi}_{2,i}(\widetilde{x})$ *be roots of* $\det\left(\widetilde{\xi}_2 - \widetilde{Q}_0(\widetilde{x})\right)$, *that is,*

$$\widetilde{\xi}_{2,i}(\widetilde{x}) = \left(\widetilde{\xi}_{1,i}(\widetilde{x})\right)^2 + \frac{\widetilde{x}_2}{3}. \tag{2.59}$$

Then

$$\xi_{1,i}(x) = \widetilde{\xi}_{1,i}(\widetilde{x}(x))\frac{\partial \widetilde{x}_1}{\partial x_1} + \widetilde{\xi}_{2,i}(\widetilde{x}(x))\frac{\partial \widetilde{x}_2}{\partial x_1}, \quad \xi_{2,i}(x) = \widetilde{\xi}_{1,i}(\widetilde{x}(x))\frac{\partial \widetilde{x}_1}{\partial x_2} + \widetilde{\xi}_{2,i}(\widetilde{x}(x))\frac{\partial \widetilde{x}_2}{\partial x_2} \tag{2.60}$$

satisfy

$$\det(\xi_1 - P_0(x)) = \xi_1^3 + a_2(x)\xi_1 + a_3(x) = 0, \quad \xi_2 = b_2(x)\xi_1^2 + b_1(x)\xi_1 + b_0(x). \tag{2.61}$$

Conversely, any solution of (2.61) *is given by* (2.60).

Using this lemma, we have

Proposition 2.3. *A point $c \in D_\rho$ is a turning point of the system* (2.1) *if and only if $\widetilde{x}(c)$ is a turning point of the Pearcey system* (2.13)*.*

The proof of Lemma 2.4 and Proposition 2.3 is given in Subsection 4.1.

Finally, we consider the Stokes surface for the system (2.1).

Lemma 2.5. *For $i = 1, 2, 3$, we have*

$$\frac{\partial \xi_{2,i}}{\partial x_1} = \frac{\partial \xi_{1,i}}{\partial x_2}. \tag{2.62}$$

Proof. For simplicity, $\xi_{1,i}$ and $\xi_{2,i}$ are denoted in the proof by ξ_1 and ξ_2, respectively. We show the lemma near a point $x \in \mathbb{C}^2$ which satisfies $3\xi_1(x)^2 + a_2(x) \neq 0$, that is, outside the set of the turning points for the system.

Taking the partial derivative of $\xi_1^3 + a_2(x)\xi_1 + a_3(x) = 0$ with respect to the variable x_1 and x_2, we obtain

$$\left(3\xi_1^2 + a_2\right)\frac{\partial \xi_1}{\partial x_1} + \frac{\partial a_2}{\partial x_1}\xi_1 + \frac{\partial a_3}{\partial x_1} = 0, \tag{2.63}$$

$$\left(3\xi_1^2 + a_2\right)\frac{\partial \xi_1}{\partial x_2} + \frac{\partial a_2}{\partial x_2}\xi_1 + \frac{\partial a_3}{\partial x_2} = 0. \tag{2.64}$$

Since $3\xi_1(x)^2 + a_2(x) \neq 0$, it suffices to prove that $\left(\partial\xi_2/\partial x_1\right)(x)$ also satisfies the following equation

$$\left(3\xi_1^2 + a_2\right)\frac{\partial \xi_2}{\partial x_1} + \frac{\partial a_2}{\partial x_2}\xi_1 + \frac{\partial a_3}{\partial x_2} = 0. \tag{2.65}$$

Using (2.40), (2.63) and Lemma 2.3, we obtain

(2.66)

$$\begin{aligned}
&\left(3\xi_1^2 + a_2\right)\frac{\partial \xi_2}{\partial x_1} + \frac{\partial a_2}{\partial x_2}\xi_1 + \frac{\partial a_3}{\partial x_2} \\
=& \left(3\xi_1^2 + a_2\right)\left\{(2b_2\xi_1 + b_1)\frac{\partial \xi_1}{\partial x_1} + \frac{\partial b_2}{\partial x_1}\xi_1^2 + \frac{\partial b_1}{\partial x_1}\xi_{1,i} + \frac{\partial b_0}{\partial x_1}\right\} + \frac{\partial a_2}{\partial x_2}\xi_1 + \frac{\partial a_3}{\partial x_2} \\
=& \left(2b_2\xi_1 + b_1\right)\left(-\frac{\partial a_2}{\partial x_1}\xi_1 - \frac{\partial a_3}{\partial x_1}\right) + \left(3\xi_1^2 + a_2\right)\left(\frac{\partial b_2}{\partial x_1}\xi_1^2 + \frac{\partial b_1}{\partial x_1}\xi_1 + \frac{\partial b_0}{\partial x_1}\right) + \frac{\partial a_2}{\partial x_2}\xi_1 + \frac{\partial a_3}{\partial x_2} \\
=& \left(-2b_2\frac{\partial a_2}{\partial x_1} - 2\frac{\partial b_2}{\partial x_1}a_2 + 3\frac{\partial b_0}{\partial x_1}\right)\xi_1^2 + \left(-2b_2\frac{\partial a_3}{\partial x_1} - b_1\frac{\partial a_2}{\partial x_1} - 3\frac{\partial b_2}{\partial x_1}a_3 - 2\frac{\partial b_1}{\partial x_1}a_2 + \frac{\partial a_2}{\partial x_2}\right)\xi_1 \\
&+ \left(-b_1\frac{\partial a_3}{\partial x_1} - 3\frac{\partial b_1}{\partial x_1}a_3 + \frac{\partial b_0}{\partial x_1}a_2 + \frac{\partial a_3}{\partial x_2}\right) \\
=&0.
\end{aligned}$$

□

Note that Lemma 2.5 guarantees that $\xi_{1,i}dx_1 + \xi_{2,i}dx_2$ is a closed 1-form. Based on this fact, the Stokes surface for (2.1) is defined as follows:

Definition 2.2. *Let $c \in \mathbb{C}^2$ be a turning point which satisfies* (2.41)*. A Stokes surface for the system* (2.1) *emanating from $x = c$ is a real 3-dimensional surface defined by*

$$\Im \int_c^x \left\{(\xi_{1,i}dx_1 + \xi_{2,i}dx_2) - \left(\xi_{1,i'}dx_1 + \xi_{2,i'}dx_2\right)\right\} = 0. \tag{2.67}$$

By using the coordinate transformation $\widetilde{x}(x)$ of Theorem 2.1, we have

Proposition 2.4. *Let x_0 be a point in D_ρ. Then we have*

$$\int_{x_0}^{x} (\xi_{1,i}dx_1 + \xi_{2,i}dx_2) = \int_{\widetilde{x}(x_0)}^{\widetilde{x}(x)} \left(\widetilde{\xi}_{1,i}d\widetilde{x}_1 + \widetilde{\xi}_{2,i}d\widetilde{x}_2\right). \tag{2.68}$$

In particular, a point $x \in D_\rho$ is in the Stokes surface for the system (2.1) *if and only if $\widetilde{x}(x)$ is in the Stokes surface for the Pearcey system* (2.13).

Proposition 2.4 is an immediate consequence of Lemma 2.4 and Proposition 2.3.

2.2 Derivation of Theorem 2.2 from Theorem 2.1

In this subsection, we give the proof of Theorem 2.2.

By the assumption (i)* and using the arguments of [9], [10] and [8], we can verify that there exist a sufficiently small positive constant $0 < \rho < \rho_0$ and an infinite series of $m \times m$ matrices $\{T_n^\star(x)\}_{n\geq 0}$ satisfying

- Every entry of $T_n^\star(x)$ is holomorphic in D_ρ and $\det T_0^\star(x) \neq 0$ ($x \in D_\rho$).
- $T_n^\star(x)$ ($n \geq 0$) satisfy

  $$\|T_n^\star(x)\|_{\rho,\rho} \leq C^\star (\alpha^\star)^n\, n! \tag{2.69}$$

  with some positive constants $C^\star$ and $\alpha^\star$.
- By a formal transformation

  $$\Psi = T^\star(x,\eta)\Psi^\star, \quad T^\star(x,\eta) = \sum_{n\geq 0} \eta^{-n} T_n^\star(x), \tag{2.70}$$

  the first equation of (2.18) is transformed into the following equation:

  $$\eta^{-1}\frac{\partial}{\partial x_1}\Psi^\star = P^\star(x,\eta)\Psi^\star, \quad P^\star(x,\eta) = \sum_{n\geq 0}\eta^{-n}P_n^\star(x) \tag{2.71}$$

  where $P_n^\star(x)$ is a block-diagonal matrix

  $$P_n^\star(x) = \begin{pmatrix} P_n^{\star(1)}(x) & \\ & P_n^{\star(2)}(x) \end{pmatrix}, \tag{2.72}$$

  $P_n^{\star(1)}(x)$ is a 3×3 matrix, $P_n^{\star(2)}(x)$ is an $(m-3)\times(m-3)$ matrix, and all coefficients $P_n^{\star(1)}(x)$ and $P_n^{\star(2)}(x)$ are holomorphic in D_ρ and satisfy an estimate

  $$\left\|P_n^{\star(j)}\right\|_{\rho,\rho} \leq \widetilde{C}^\star(\widetilde{\alpha}^\star)^n n! \quad (j = 1,2) \tag{2.73}$$

  with some positive constants $\widetilde{C}^\star$ and $\widetilde{\alpha}^\star$.
- $D^{(1)}(x,\xi_1) = \det\left(\xi_1 - P_0^{\star(1)}(x)\right)$ and $D^{(2)}(x,\xi_1) = \det\left(\xi_1 - P_0^{\star(2)}(x)\right)$ satisfy

  $$D(x,\xi_1) = D^{(1)}(x,\xi_1)D^{(2)}(x,\xi_1), \tag{2.74}$$

  $$D^{(1)}(0,\xi_1^\star) = \frac{\partial D^{(1)}}{\partial \xi_1}(0,\xi_1^\star) = \frac{\partial^2 D^{(1)}}{\partial \xi_1^2}(0,\xi_1^\star) = 0, \quad \frac{\partial^3 D^{(1)}}{\partial \xi_1^3}(0,\xi_1^\star) \neq 0, \tag{2.75}$$

  $$D^{(2)}(0,\xi_1^\star) \neq 0. \tag{2.76}$$

(The proof of the estimate (2.73) is provided in [8].)

Furthermore, by using the argument of [8], we also find that the second equation of (2.18) is transformed by the transformation (2.70) into the following equation:

$$\eta^{-1}\frac{\partial}{\partial x_2}\Psi^\star = Q^\star(x,\eta)\Psi^\star, \quad Q^\star(x,\eta) = \sum_{n\geq 0}\eta^{-n}Q_n^\star(x) \tag{2.77}$$

where $Q_n^\star(x)$ is a block-diagonal matrix

$$Q_n^\star(x) = \begin{pmatrix} Q_n^{\star(1)}(x) & \\ & Q_n^{\star(2)}(x)\end{pmatrix}, \tag{2.78}$$

$Q_n^{\star(1)}(x)$ is a 3×3 matrix, $Q_n^{\star(2)}(x)$ is an $(m-3)\times(m-3)$ matrix, and (by replacing ρ, $\widetilde{C}^\star$ and $\widetilde{\alpha}^\star$ if necessary) all coefficients $Q_n^{\star(1)}(x)$ and $Q_n^{\star(2)}(x)$ are holomorphic in D_ρ and satisfy an estimate

$$\left\| Q_n^{\star(j)} \right\|_{\rho,\rho} \leq \widetilde{C}^\star(\widetilde{\alpha}^\star)^n n! \quad (j=1,2). \tag{2.79}$$

Thus what remains to be proved is the last claim of Theorem 2.2 for the completely integrable system

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial x_1}\Psi^{\star(1)} = P^{\star(1)}(x,\eta)\Psi^{\star(1)}, & P^{\star(1)}(x,\eta) = \displaystyle\sum_{n\geq 0}\eta^{-n}P_n^{\star(1)}(x), \\ \eta^{-1}\dfrac{\partial}{\partial x_2}\Psi^{\star(1)} = Q^{\star(1)}(x,\eta)\Psi^{\star(1)}, & Q^{\star(1)}(x,\eta) = \displaystyle\sum_{n\geq 0}\eta^{-n}Q_n^{\star(1)}(x). \end{cases} \tag{2.80}$$

Let $a_k^\star(x)$ $(k=1,2,3)$ be holomorphic functions near $x=0$ defined by

$$\begin{aligned} D^{(1)}(x,\xi_1) =& \det\left(\xi_1 - P_0^{\star(1)}(x)\right) \\ =& (\xi_1-\xi_1^\star)^3 + a_1^\star(x)(\xi_1-\xi_1^\star)^2 + a_2^\star(x)(\xi_1-\xi_1^\star) + a_3^\star(x). \end{aligned} \tag{2.81}$$

Then, by (2.75), $a_k^\star(x)$ $(k=1,2,3)$ satisfy

$$a_k^\star(0) = 0. \tag{2.82}$$

Using (2.74), (2.75) and (2.81), we have

$$\begin{aligned} \frac{\partial D}{\partial x_1}(0,\xi_1^\star) =& \frac{\partial D^{(1)}}{\partial x_1}(0,\xi_1^\star)D^{(2)}(0,\xi_1^\star) + D^{(1)}(0,\xi_1^\star)\frac{\partial D^{(2)}}{\partial x_1}(0,\xi_1^\star) \\ =& \frac{\partial a_3^\star}{\partial x_1}(0)D^{(2)}(0,\xi_1^\star). \end{aligned} \tag{2.83}$$

Hence, by the assumption (ii)* and (2.76), we find

$$\frac{\partial a_3^\star}{\partial x_1}(0) \neq 0. \tag{2.84}$$

On the other hand, using (2.74) (2.75), (2.81) and

$$\frac{\partial^2 D^{(1)}}{\partial x_j \partial \xi_1}(0,\xi_1^\star) = \frac{\partial a_2^\star}{\partial x_j}(0), \tag{2.85}$$

we have

$$\det \begin{pmatrix} \dfrac{\partial D}{\partial x_1} & \dfrac{\partial D}{\partial x_2} \\ \dfrac{\partial^2 D}{\partial x_1 \partial \xi_1} & \dfrac{\partial^2 D}{\partial x_2 \partial \xi_1} \end{pmatrix} \Big|_{(x,\xi_1)=(0,\xi_1^\star)} = D^{(2)}(0,\xi_1^\star)^2 \det \begin{pmatrix} \dfrac{\partial a_3^\star}{\partial x_1} & \dfrac{\partial a_3^\star}{\partial x_2} \\ \dfrac{\partial a_2^\star}{\partial x_1} & \dfrac{\partial a_2^\star}{\partial x_2} \end{pmatrix} \Big|_{x=0}. \tag{2.86}$$

Hence, by the assumption (iii)* and (2.76), we find

$$\det \begin{pmatrix} \dfrac{\partial a_3^\star}{\partial x_1} & \dfrac{\partial a_3^\star}{\partial x_2} \\ \dfrac{\partial a_2^\star}{\partial x_1} & \dfrac{\partial a_2^\star}{\partial x_2} \end{pmatrix} \Big|_{x=0} \neq 0. \tag{2.87}$$

By using a gauge transformation $\Psi^{\star(1)} = \exp\left\{\eta \int^x \operatorname{tr}\left(P_0^{\star(1)}(x)dx_1 + Q_0^{\star(1)}(x)dx_2\right)/3\right\}\Phi^{\star(1)}$, (2.80) is transformed into

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial x_1}\Phi^{\star(1)} = \left(P^{\star(1)}(x,\eta) - \dfrac{\operatorname{tr}P_0^{\star(1)}(x)}{3}\right)\Phi^{\star(1)}, \\ \eta^{-1}\dfrac{\partial}{\partial x_2}\Phi^{\star(1)} = \left(Q^{\star(1)}(x,\eta) - \dfrac{\operatorname{tr}Q_0^{\star(1)}(x)}{3}\right)\Phi^{\star(1)}. \end{cases} \tag{2.88}$$

Let $\widetilde{a}_2^\star(x)$ and $\widetilde{a}_3^\star(x)$ be holomorphic functions defined by

$$\det\left\{\xi_1 - \left(P_0^{\star(1)}(x) - \frac{\operatorname{tr}P_0^{\star(1)}(x)}{3}\right)\right\} = \xi_1^3 + \widetilde{a}_2^\star(x)\xi_1 + \widetilde{a}_3^\star(x). \tag{2.89}$$

Then we have

$$\widetilde{a}_2^\star(x) = a_2^\star(x) - \frac{a_1^\star(x)^2}{3}, \quad \widetilde{a}_3^\star(x) = a_3^\star(x) - \frac{a_1^\star(x)a_2^\star(x)}{3} + \frac{2a_1^\star(x)^3}{27}. \tag{2.90}$$

Hence, by (2.82), (2.84) and (2.87), $\widetilde{a}_2^\star(x)$ and $\widetilde{a}_3^\star(x)$ satisfy

$$\widetilde{a}_2^\star(0) = \widetilde{a}_3^\star(0) = 0, \tag{2.91}$$

$$\frac{\partial \widetilde{a}_3^\star}{\partial x_1}(0) \neq 0, \tag{2.92}$$

$$\det \begin{pmatrix} \dfrac{\partial \widetilde{a}_2^\star}{\partial x_1} & \dfrac{\partial \widetilde{a}_2^\star}{\partial x_2} \\ \dfrac{\partial \widetilde{a}_3^\star}{\partial x_1} & \dfrac{\partial \widetilde{a}_3^\star}{\partial x_2} \end{pmatrix} \Big|_{x=0} \neq 0. \tag{2.93}$$

That is, (2.88) satisfies the assumptions (i), (ii) and (iii) of Theorem 2.1. Thus the system (2.80) is transformed into the Pearcey system near $x = 0$ thanks to Theorem 2.1. This completes the proof of Theorem 2.2.

3 Examples

3.1 The (1,4) hypergeometric system

Let us consider the following system

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial x_1}\Psi = P_0(x)\Psi, \\ \eta^{-1}\dfrac{\partial}{\partial x_2}\Psi = \left(Q_0(x) + \eta^{-1}Q_1(x)\right)\Psi, \end{cases} \tag{3.1}$$

$$(3.2)\qquad P_0 = \begin{pmatrix} 2x_2/9 & 1 & 0 \\ 0 & 2x_2/9 & 1 \\ \alpha/3 & -x_1/3 & -4x_2/9 \end{pmatrix}, \quad Q_0 = P_0^2 - \frac{4}{9}x_2 P_0 + \frac{2}{9}x_1 - \frac{8}{81}x_2^2, \quad Q_1 = \frac{\partial P_0}{\partial x_1},$$

where α is a complex constant. This system is equivalent to the following holonomic system

$$(3.3)\qquad \begin{cases} \left(\dfrac{\partial^3}{\partial x_1^3} + \dfrac{2}{3}x_2\eta\dfrac{\partial^2}{\partial x_1^2} + \dfrac{1}{3}x_1\eta^2\dfrac{\partial}{\partial x_1} - \dfrac{\alpha}{3}\eta^3\right)\psi = 0, \\ \left(\eta\dfrac{\partial}{\partial x_2} - \dfrac{\partial^2}{\partial x_1^2}\right)\psi = 0, \end{cases}$$

which belongs to the class of the hypergeometric systems of two variables studied in [6]. Note that (3.3) has the following solution

$$(3.4)\qquad \psi = \int \exp\left\{\eta\left(t^3 + x_2 t^2 + x_1 t\right)\right\} t^{-\eta\alpha-1} dt.$$

Since the system (3.3) and its solution (3.4) are determined by the partition "(1,4)" of the natural number 5, we call (3.1) the (1,4) hypergeometric system. In what follows, we assume $\alpha \neq 0$.

We first study turning points for the (1,4) hypergeometric system. Let ξ_1 (resp., ξ_2) be an eigenvalue of P_0 (resp., Q_0). Then ξ_1 and ξ_2 satisfy

$$(3.5)\qquad \det(\xi_1 - P_0(x)) = \xi_1^3 + a_2(x)\xi_1 + a_3(x) = 0,$$

$$(3.6)\qquad a_2(x) = \frac{1}{3}x_1 - \frac{4}{27}x_2^2, \quad a_3(x) = \frac{16}{729}x_2^3 - \frac{2}{27}x_1x_2 - \frac{\alpha}{3},$$

$$(3.7)\qquad \xi_2 = \xi_1^2 - \frac{4}{9}x_2\xi_1 + \frac{2}{9}x_1 - \frac{8}{81}x_2^2.$$

In view of (3.7) we find that a turning point for the (1,4) hypergeometric system is given by a zero of the discriminant $27a_3(x)^2 + 4a_2(x)^3$ of (3.5). A singular point of the set of turning points for the (1,4) hypergeometric system is given by

$$(3.8)\qquad a_2(x) = a_3(x) = 0,$$

that is,

$$(3.9)\qquad x = \left(3^{4/3}\alpha^{2/3}e^{4\pi\sqrt{-1}(j-1)/3}, -\frac{3^{5/3}}{2}\alpha^{1/3}e^{2\pi\sqrt{-1}(j-1)/3}\right) =: c_j \quad (j = 1, 2, 3).$$

Furthermore, we have

$$(3.10)\qquad \frac{\partial a_3}{\partial x_1}(c_j) = 3^{-4/3}\alpha^{2/3}e^{4\pi\sqrt{-1}(j-1)/3} \neq 0,$$

$$(3.11)\qquad \det\begin{pmatrix} \dfrac{\partial a_2}{\partial x_1}(c_j) & \dfrac{\partial a_2}{\partial x_2}(c_j) \\ \dfrac{\partial a_3}{\partial x_1}(c_j) & \dfrac{\partial a_3}{\partial x_2}(c_j) \end{pmatrix} = -\frac{2}{81}\left(3^{4/3}\alpha^{2/3}e^{4\pi\sqrt{-1}(j-1)/3}\right) \neq 0.$$

Hence in a sufficiently small neighborhood of $x = c_j$ $(j = 1, 2, 3)$ the (1,4) hypergeometric system can be transformed into the Pearcey system.

3.2 The (2,3) hypergeometric system

Next we consider the following system

$$\text{(3.12)} \qquad \begin{cases} \eta^{-1}\dfrac{\partial}{\partial x_1}\Psi = \left(P_0(x) + \eta^{-1}P_1(x)\right)\Psi, \\ \eta^{-1}\dfrac{\partial}{\partial x_2}\Psi = Q_0(x)\Psi, \end{cases}$$

(3.13)

$$P_0 = \begin{pmatrix} x_1/6 & 1 & 0 \\ 0 & x_1/6 & 1 \\ x_2/2 & \alpha/2 & -x_1/3 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1/2 & 0 \end{pmatrix}, \quad Q_0 = \frac{2}{x_2}P_0^2 + \frac{x_1}{3x_2}P_0 - \frac{x_1^2}{9x_2} - \frac{2\alpha}{3x_2},$$

where α is a complex constant. This system is equivalent to the following holonomic system

$$\text{(3.14)} \qquad \begin{cases} \left(\dfrac{\partial^3}{\partial x_1^3} + \dfrac{x_1}{2}\eta\dfrac{\partial^2}{\partial x_1^2} + \dfrac{-\alpha+\eta^{-1}}{2}\eta^2\dfrac{\partial}{\partial x_1} - \dfrac{x_2}{2}\eta^3\right)\psi = 0, \\ \left(\dfrac{\partial^2}{\partial x_1 \partial x_2} - \eta^2\right)\psi = 0, \end{cases}$$

which belongs also to the class of the hypergeometric systems of two variables studied in [6]. Since (3.14) has the following solution

$$\text{(3.15)} \qquad \psi = \int \exp\left\{\eta\left(t^2 + x_1 t + \frac{x_2}{t}\right)\right\} t^{-\eta\alpha-1}dt$$

and it is determined by the partition "(2,3)" of the natural number 5, we call (3.12) the (2,3) hypergeometric system. In what follows, we assume $x_2 \neq 0$ and $\alpha \neq 0$.

Similarly to the preceding subsection we study turning points for the (2,3) hypergeometric system. Let ξ_1 (resp., ξ_2) be an eigenvalue of P_0 (resp., Q_0). Then ξ_1 and ξ_2 satisfy

$$\text{(3.16)} \qquad \det\left(\xi_1 - P_0(x)\right) = \xi_1^3 + a_2(x)\xi_1 + a_3(x),$$

$$\text{(3.17)} \qquad a_2(x) = -\frac{x_1^2}{12} - \frac{\alpha}{2}, \quad a_3(x) = \frac{x_1^3}{108} + \frac{\alpha x_1}{12} - \frac{x_2}{2},$$

$$\text{(3.18)} \qquad \xi_2 = \frac{2}{x_2}\xi_1^2 + \frac{x_1}{3x_2}\xi_1 - \frac{x_1^2}{9x_2} - \frac{2\alpha}{3x_2}.$$

Thanks to (3.18) a turning point for the (2,3) hypergeometric system is given by a zero of the discriminant $27a_3(x)^2 + 4a_2(x)^3$ of (3.16). A singular point of the set of turning points for the (2,3) hypergeometric system is given by

$$\text{(3.19)} \qquad a_2(x) = a_3(x) = 0,$$

that is,

$$\text{(3.20)} \qquad x = \pm\left(\sqrt{-6\alpha}, \frac{\alpha}{18}\sqrt{-6\alpha}\right) =: c_\pm.$$

Since

$$\text{(3.21)} \qquad \frac{\partial a_3}{\partial x_1}(c_\pm) = -\frac{\alpha}{12} \neq 0,$$

$$\text{(3.22)} \qquad \det\begin{pmatrix} \dfrac{\partial a_2}{\partial x_1}(c_\pm) & \dfrac{\partial a_2}{\partial x_2}(c_\pm) \\ \dfrac{\partial a_3}{\partial x_1}(c_\pm) & \dfrac{\partial a_3}{\partial x_2}(c_\pm) \end{pmatrix} = \pm\frac{\sqrt{-6\alpha}}{12} \neq 0$$

hold, in a sufficiently small neighborhood of $x = c_\pm$ the (2,3) hypergeometric system can be transformed into the Pearcey system.

4 Construction of the transformation

In this section, we discuss the construction of $\widetilde{x}(x)$ and $\{T_n(x)\}_{n\geq 0}$. The estimate (2.11) for $T_n(x)$ will be verified in Section 5.

Suppose that the system (2.13) is obtained from (2.1) through the transformation

$$\Psi(x,\eta) = T(x,\eta)\widetilde{\Psi}(\widetilde{x}(x),\eta), \quad \widetilde{x}(x) = (\widetilde{x}_1(x),\widetilde{x}_2(x)), \quad T(x,\eta) = \sum_{n\geq 0}\eta^{-n}T_n(x). \tag{4.1}$$

Then $\widetilde{x}_i(x)$ $(i=1,2)$ and $T(x,\eta)$ should satisfy the following relation:

$$\begin{cases} T\left(\dfrac{\partial \widetilde{x}_1}{\partial x_1}\widetilde{P} + \dfrac{\partial \widetilde{x_2}}{\partial x_1}\widetilde{Q}\right) + \eta^{-1}\dfrac{\partial T}{\partial x_1} = PT, \\ T\left(\dfrac{\partial \widetilde{x}_1}{\partial x_2}\widetilde{P} + \dfrac{\partial \widetilde{x_2}}{\partial x_2}\widetilde{Q}\right) + \eta^{-1}\dfrac{\partial T}{\partial x_2} = QT, \end{cases} \tag{4.2}$$

that is,

$$\begin{cases} T_0\left(\dfrac{\partial \widetilde{x}_1}{\partial x_1}\widetilde{P}_0 + \dfrac{\partial \widetilde{x}_2}{\partial x_1}\widetilde{Q}_0\right) = P_0T_0, \\ T_0\left(\dfrac{\partial \widetilde{x}_1}{\partial x_2}\widetilde{P}_0 + \dfrac{\partial \widetilde{x_2}}{\partial x_2}\widetilde{Q}_0\right) = Q_0T_0, \end{cases} \tag{4.3}$$

$$\begin{cases} \displaystyle\sum_{j=0}^{n} T_j\left(\dfrac{\partial \widetilde{x}_1}{\partial x_1}\widetilde{P}_{n-j} + \dfrac{\partial \widetilde{x}_2}{\partial x_1}\widetilde{Q}_{n-j}\right) + \dfrac{\partial T_{n-1}}{\partial x_1} = \sum_{j=0}^{n} P_{n-j}T_j, \\ \displaystyle\sum_{j=0}^{n} T_j\left(\dfrac{\partial \widetilde{x}_1}{\partial x_2}\widetilde{P}_{n-j} + \dfrac{\partial \widetilde{x}_2}{\partial x_2}\widetilde{Q}_{n-j}\right) + \dfrac{\partial T_{n-1}}{\partial x_2} = \sum_{j=0}^{n} Q_{n-j}T_j, \end{cases} \quad (n\geq 1). \tag{4.4}$$

4.1 Construction of the transformation, I

In this subsection, we prove

Proposition 4.1. *Let $a_2(x)$ and $a_3(x)$ be holomorphic functions defined by* $\det(\xi_1 - P_0(x)) = \xi_1^3 + a_2(x)\xi_1 + a_3(x)$. *Suppose that $a_2(x)$ and $a_3(x)$ satisfy the following conditions:*

(i) $a_2(0) = a_3(0) = 0$,

(ii) $\dfrac{\partial a_3}{\partial x_1}(0) \neq 0$,

(iii) $\det\begin{pmatrix} \dfrac{\partial a_2}{\partial x_1}(0) & \dfrac{\partial a_2}{\partial x_2}(0) \\ \dfrac{\partial a_3}{\partial x_1}(0) & \dfrac{\partial a_3}{\partial x_2}(0)\end{pmatrix} \neq 0.$

Then there exist a sufficiently small positive constant $0 < \rho < \rho_0$, holomorphic functions $\widetilde{x}_i(x)$ $(i=1,2)$ in D_ρ, and a 3×3 matrix $T_0(x)$ which satisfy the following properties:

- $\widetilde{x}_1(0) = \widetilde{x}_2(0) = 0$.
- *$\widetilde{x}(x) = (\widetilde{x}_1(x),\widetilde{x}_2(x))$ is a biholomorphic map from D_ρ to $\widetilde{x}(D_\rho)$.*
- *Every entry of $T_0(x)$ is holomorphic in D_ρ and* $\det T_0(x) \neq 0$ $(x\in D_\rho)$.

- *By a transformation*

$$\Psi(x,\eta) = T_0(x)\widetilde{\Psi}(\widetilde{x}(x),\eta), \tag{4.5}$$

(2.1) *is transformed into the following form:*

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial \widetilde{x}_1}\widetilde{\Psi} = \widetilde{P}(\widetilde{x},\eta)\widetilde{\Psi}, & \widetilde{P}(\widetilde{x},\eta) = \displaystyle\sum_{n\geq 0}\eta^{-n}\widetilde{P}_n(\widetilde{x}), \\ \eta^{-1}\dfrac{\partial}{\partial \widetilde{x}_2}\widetilde{\Psi} = \widetilde{Q}(\widetilde{x},\eta)\widetilde{\Psi}, & \widetilde{Q}(\widetilde{x},\eta) = \displaystyle\sum_{n\geq 0}\eta^{-n}\widetilde{Q}_n(\widetilde{x}), \end{cases} \tag{4.6}$$

where

$$\widetilde{P}_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\widetilde{x}_1/4 & -\widetilde{x}_2/2 & 0 \end{pmatrix}, \quad \widetilde{Q}_0 = \widetilde{P}_0^2 + \frac{\widetilde{x}_2}{3}. \tag{4.7}$$

Here $\widetilde{P}_n(x)$ *and* $\widetilde{Q}_n(x)$ *are* 3×3 *matrices with holomorphic entries in* D_ρ *and satisfy an estimate*

$$\left\|\widetilde{P}_n\right\|_{\rho,\rho}, \left\|\widetilde{Q}_n\right\|_{\rho,\rho} \leq \widetilde{C}\widetilde{\alpha}^n n! \quad (n \geq 0) \tag{4.8}$$

with some positive constants $\widetilde{C}$ *and* $\widetilde{\alpha}$.

Since the system (4.6) is obtained from (2.1) through the transformation (4.5), $\widetilde{x}_i(x)$ $(i=1,2)$ and $T_0(x)$ should satisfy the following relations:

$$\begin{cases} T_0\left(\dfrac{\partial \widetilde{x}_1}{\partial x_1}\widetilde{P} + \dfrac{\partial \widetilde{x_2}}{\partial x_1}\widetilde{Q}\right) + \eta^{-1}\dfrac{\partial T_0}{\partial x_1} = PT_0, \\ T_0\left(\dfrac{\partial \widetilde{x}_1}{\partial x_2}\widetilde{P} + \dfrac{\partial \widetilde{x_2}}{\partial x_2}\widetilde{Q}\right) + \eta^{-1}\dfrac{\partial T_0}{\partial x_2} = QT_0, \end{cases} \tag{4.9}$$

that is,

$$\begin{cases} T_0\left(\dfrac{\partial \widetilde{x}_1}{\partial x_1}\widetilde{P}_0 + \dfrac{\partial \widetilde{x}_2}{\partial x_1}\widetilde{Q}_0\right) = P_0T_0, \\ T_0\left(\dfrac{\partial \widetilde{x}_1}{\partial x_2}\widetilde{P}_0 + \dfrac{\partial \widetilde{x}_2}{\partial x_2}\widetilde{Q}_0\right) = Q_0T_0, \end{cases} \tag{4.10}$$

$$\begin{cases} T_0\left(\dfrac{\partial \widetilde{x}_1}{\partial x_1}\widetilde{P}_1 + \dfrac{\partial \widetilde{x}_2}{\partial x_1}\widetilde{Q}_1\right) + \dfrac{\partial T_0}{\partial x_1} = P_1T_0, \\ T_0\left(\dfrac{\partial \widetilde{x}_1}{\partial x_2}\widetilde{P}_1 + \dfrac{\partial \widetilde{x}_2}{\partial x_2}\widetilde{Q}_1\right) + \dfrac{\partial T_0}{\partial x_2} = Q_1T_0, \end{cases} \tag{4.11}$$

$$\begin{cases} T_0\left(\dfrac{\partial \widetilde{x}_1}{\partial x_1}\widetilde{P}_n + \dfrac{\partial \widetilde{x}_2}{\partial x_1}\widetilde{Q}_n\right) = P_nT_0, \\ T_0\left(\dfrac{\partial \widetilde{x}_1}{\partial x_2}\widetilde{P}_n + \dfrac{\partial \widetilde{x}_2}{\partial x_2}\widetilde{Q}_n\right) = Q_nT_0, \end{cases} \quad (n \geq 2). \tag{4.12}$$

We first show

Lemma 4.1. $\widetilde{x}_i(x)$ $(i=1,2)$ *satisfy the following system of nonlinear partial differential equations:*

(4.13)

$$\begin{cases} \dfrac{\widetilde{x}_2^2}{12}\left(\dfrac{\partial \widetilde{x}_2}{\partial x_1}\right)^2 - \dfrac{3}{4}\widetilde{x}_1 \dfrac{\partial \widetilde{x}_1}{\partial x_1}\dfrac{\partial \widetilde{x}_2}{\partial x_1} - \dfrac{\widetilde{x}_2}{2}\left(\dfrac{\partial \widetilde{x}_1}{\partial x_1}\right)^2 + a_2(x) = 0, \\ \dfrac{\widetilde{x}_1^2}{16}\left(\dfrac{\partial \widetilde{x}_2}{\partial x_1}\right)^3 + \dfrac{\widetilde{x}_2^3}{108}\left(\dfrac{\partial \widetilde{x}_2}{\partial x_1}\right)^3 - \dfrac{\widetilde{x}_1}{4}\left(\dfrac{\partial \widetilde{x}_1}{\partial x_1}\right)^3 + \dfrac{\widetilde{x}_1\widetilde{x}_2}{8}\dfrac{\partial \widetilde{x}_1}{\partial x_1}\left(\dfrac{\partial \widetilde{x}_2}{\partial x_1}\right)^2 + \dfrac{\widetilde{x}_2^2}{6}\left(\dfrac{\partial \widetilde{x}_1}{\partial x_1}\right)^2\dfrac{\partial \widetilde{x}_2}{\partial x_1} + a_3(x) = 0, \\ \dfrac{\partial \widetilde{x}_1}{\partial x_2} = b_2(x)\left\{-\dfrac{\widetilde{x}_2}{3}\dfrac{\partial \widetilde{x}_1}{\partial x_1}\dfrac{\partial \widetilde{x}_2}{\partial x_1} - \dfrac{\widetilde{x}_1}{4}\left(\dfrac{\partial \widetilde{x}_2}{\partial x_1}\right)^2\right\} + b_1(x)\dfrac{\partial \widetilde{x}_1}{\partial x_1}, \\ \dfrac{\partial \widetilde{x}_2}{\partial x_2} = b_2(x)\left\{\left(\dfrac{\partial \widetilde{x}_1}{\partial x_1}\right)^2 + \dfrac{\widetilde{x}_2}{6}\left(\dfrac{\partial \widetilde{x}_2}{\partial x_1}\right)^2\right\} + b_1(x)\dfrac{\partial \widetilde{x}_2}{\partial x_1}. \end{cases}$$

Proof. It follows from (4.7) and the first equation of (4.10) that

$$P_0T_0 = T_0\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\widetilde{P}_0^2 + \frac{\partial \widetilde{x}_1}{\partial x_1}\widetilde{P}_0 + \frac{\widetilde{x}_2}{3}\frac{\partial \widetilde{x}_2}{\partial x_1}\right). \tag{4.14}$$

It follows from this relation, (2.38) and (2.39) that

(4.15)

$$\begin{aligned} 0 =& \left(P_0^3 + a_2(x)P_0 + a_3(x)\right)T_0 \\ =& T_0\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\widetilde{P}_0^2 + \frac{\partial \widetilde{x}_1}{\partial x_1}\widetilde{P}_0 + \frac{\widetilde{x}_2}{3}\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^3 + a_2(x)T_0\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\widetilde{P}_0^2 + \frac{\partial \widetilde{x}_1}{\partial x_1}\widetilde{P}_0 + \frac{\widetilde{x}_2}{3}\frac{\partial \widetilde{x}_2}{\partial x_1}\right) + a_3(x)T_0 \\ =& T_0\left[\left\{\frac{\widetilde{x}_2^2}{12}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^3 - \frac{3}{4}\widetilde{x}_1\frac{\partial \widetilde{x}_1}{\partial x_1}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^2 - \frac{\widetilde{x}_2}{2}\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\right)^2\frac{\partial \widetilde{x}_2}{\partial x_1} + a_2(x)\frac{\partial \widetilde{x}_2}{\partial x_1}\right\}\widetilde{P}_0^2\right. \\ &+ \left\{\frac{\widetilde{x}_2^2}{12}\frac{\partial \widetilde{x}_1}{\partial x_1}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^2 - \frac{3}{4}\widetilde{x}_1\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\right)^2\frac{\partial \widetilde{x}_2}{\partial x_1} - \frac{\widetilde{x}_2}{2}\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\right)^3 + a_2(x)\frac{\partial \widetilde{x}_1}{\partial x_1}\right\}\widetilde{P}_0 \\ &\left. + \left\{\left(\frac{\widetilde{x}_1^2}{16} + \frac{\widetilde{x}_2^3}{27}\right)\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^3 - \frac{\widetilde{x}_1}{4}\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\right)^3 - \frac{\widetilde{x}_1\widetilde{x}_2}{8}\frac{\partial \widetilde{x}_1}{\partial x_1}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^2 + a_2(x)\frac{\widetilde{x}_2}{3}\frac{\partial \widetilde{x}_2}{\partial x_1} + a_3(x)\right\}\right]. \end{aligned}$$

Hence, by the same reasoning as in the proof of Lemma 2.2, we get

$$\frac{\partial \widetilde{x}_2}{\partial x_1}\left\{\frac{\widetilde{x}_2^2}{12}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^2 - \frac{3}{4}\widetilde{x}_1\frac{\partial \widetilde{x}_1}{\partial x_1}\frac{\partial \widetilde{x}_2}{\partial x_1} - \frac{\widetilde{x}_2}{2}\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\right)^2 + a_2(x)\right\} = 0, \tag{4.16}$$

$$\frac{\partial \widetilde{x}_1}{\partial x_1}\left\{\frac{\widetilde{x}_2^2}{12}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^2 - \frac{3}{4}\widetilde{x}_1\frac{\partial \widetilde{x}_1}{\partial x_1}\frac{\partial \widetilde{x}_2}{\partial x_1} - \frac{\widetilde{x}_2}{2}\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\right)^2 + a_2(x)\right\} = 0, \tag{4.17}$$

$$\left(\frac{\widetilde{x}_1^2}{16} + \frac{\widetilde{x}_2^3}{27}\right)\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^3 - \frac{\widetilde{x}_1}{4}\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\right)^3 - \frac{\widetilde{x}_1\widetilde{x}_2}{8}\frac{\partial \widetilde{x}_1}{\partial x_1}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^2 + a_2(x)\frac{\widetilde{x}_2}{3}\frac{\partial \widetilde{x}_2}{\partial x_1} + a_3(x) = 0. \tag{4.18}$$

Here, if $(\partial \widetilde{x}_1/\partial x_1)(0)$ and $(\partial \widetilde{x}_2/\partial x_1)(0)$ simultaneously vanish, then we have $(\partial a_3/\partial x_1)(0) = 0$ by the assumption $a_2(0) = 0$ and (4.18). This contradicts the assumption $(\partial a_3/\partial x_1)(0) \neq 0$, that is, $(\partial \widetilde{x}_1/\partial x_1) \neq 0$ or $(\partial \widetilde{x}_2/\partial x_1) \neq 0$ holds near $x = 0$. Hence we get the first equation of (4.13) by (4.16) and (4.17). Furthermore, using the first equation of (4.13) and (4.18), we obtain the second equation of (4.13).

Next, using (2.32), (2.39) and (4.14), we obtain

(4.19)

$$Q_0T_0 = \left(b_2(x)P_0^2 + b_1(x)P_0 + b_0(x)\right)T_0$$

$$
\begin{aligned}
=&b_2(x)T_0\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\widetilde{P}_0^2+\frac{\partial\widetilde{x}_1}{\partial x_1}\widetilde{P}_0+\frac{\widetilde{x}_2}{3}\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^2+b_1(x)T_0\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\widetilde{P}_0^2+\frac{\partial\widetilde{x}_1}{\partial x_1}\widetilde{P}_0+\frac{\widetilde{x}_2}{3}\frac{\partial\widetilde{x}_2}{\partial x_1}\right)+b_0(x)T_0\\
=&T_0\left(\left[b_2(x)\left\{\left(\frac{\partial\widetilde{x}_1}{\partial x_1}\right)^2+\frac{\widetilde{x}_2}{6}\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^2\right\}+b_1(x)\frac{\partial\widetilde{x}_2}{\partial x_1}\right]\widetilde{P}_0^2\right.\\
&+\left[b_2(x)\left\{-\frac{\widetilde{x}_2}{3}\frac{\partial\widetilde{x}_1}{\partial x_1}\frac{\partial\widetilde{x}_2}{\partial x_1}-\frac{\widetilde{x}_1}{4}\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^2\right\}+b_1(x)\frac{\partial\widetilde{x}_1}{\partial x_1}\right]\widetilde{P}_0\\
&\left.+\left[b_2(x)\left\{\frac{\widetilde{x}_2^2}{9}\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^2-\frac{\widetilde{x}_1}{2}\frac{\partial\widetilde{x}_1}{\partial x_1}\frac{\partial\widetilde{x}_2}{\partial x_1}\right\}+b_1(x)\frac{\widetilde{x}_2}{3}\frac{\partial\widetilde{x}_2}{\partial x_1}+b_0(x)\right]\right).
\end{aligned}
$$

On the other hand, the left-hand side of this equation can be expressed as

$$
Q_0T_0=T_0\left(\frac{\partial\widetilde{x}_2}{\partial x_2}\widetilde{P}_0^2+\frac{\partial\widetilde{x}_1}{\partial x_2}\widetilde{P}_0+\frac{\widetilde{x}_2}{3}\frac{\partial\widetilde{x}_2}{\partial x_2}\right) \tag{4.20}
$$

in view of (4.7) and the second equation of (4.10). Hence, we get

$$
\frac{\partial\widetilde{x}_2}{\partial x_2}=b_2(x)\left\{\left(\frac{\partial\widetilde{x}_1}{\partial x_1}\right)^2+\frac{\widetilde{x}_2}{6}\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^2\right\}+b_1(x)\frac{\partial\widetilde{x}_2}{\partial x_1}, \tag{4.21}
$$

$$
\frac{\partial\widetilde{x}_1}{\partial x_2}=b_2(x)\left\{-\frac{\widetilde{x}_2}{3}\frac{\partial\widetilde{x}_1}{\partial x_1}\frac{\partial\widetilde{x}_2}{\partial x_1}-\frac{\widetilde{x}_1}{4}\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^2\right\}+b_1(x)\frac{\partial\widetilde{x}_1}{\partial x_1}, \tag{4.22}
$$

$$
\frac{\widetilde{x}_2}{3}\frac{\partial\widetilde{x}_2}{\partial x_2}=b_2(x)\left\{\frac{\widetilde{x}_2^2}{9}\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^2-\frac{\widetilde{x}_1}{2}\frac{\partial\widetilde{x}_1}{\partial x_1}\frac{\partial\widetilde{x}_2}{\partial x_1}\right\}+b_1(x)\frac{\widetilde{x}_2}{3}\frac{\partial\widetilde{x}_2}{\partial x_1}+b_0(x). \tag{4.23}
$$

Thus we get the third and the fourth equations of (4.13).

Finally, we show that (4.23) follows from (4.13). Using the fourth equation of (4.13), we find that (4.23) is equivalent to

$$
b_2(x)\left\{\frac{\widetilde{x}_2^2}{18}\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^2-\frac{\widetilde{x}_1}{2}\frac{\partial\widetilde{x}_1}{\partial x_1}\frac{\partial\widetilde{x}_2}{\partial x_1}-\frac{\widetilde{x}_2}{3}\left(\frac{\partial\widetilde{x}_1}{\partial x_1}\right)^2\right\}+b_0(x)=0. \tag{4.24}
$$

On the other hand, Lemmas 2.1 and 2.2 imply

$$
\begin{aligned}
0=&\operatorname{tr}Q_0\\
=&\operatorname{tr}\left(b_2(x)P_0^2+b_1(x)P_0+b_0(0)\right)\\
=&\operatorname{tr}\left[b_2(x)\left\{(T^\dagger)^{-1}P_0T^\dagger\right\}^2+b_1(x)(T^\dagger)^{-1}P_0T^\dagger+b_0(0)\right]\\
=&-2b_2(x)a_2(x)+3b_0(x).
\end{aligned} \tag{4.25}
$$

Hence (4.24) is equivalent also to

$$
b_2(x)\left\{\frac{\widetilde{x}_2^2}{18}\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^2-\frac{\widetilde{x}_1}{2}\frac{\partial\widetilde{x}_1}{\partial x_1}\frac{\partial\widetilde{x}_2}{\partial x_1}-\frac{\widetilde{x}_2}{3}\left(\frac{\partial\widetilde{x}_1}{\partial x_1}\right)^2+\frac{2}{3}a_2(x)\right\}=0. \tag{4.26}
$$

This relation exactly coincides with the first equation of (4.13). □

In what follows we discuss the existence of a holomorphic solution $(\widetilde{x}_1(x),\widetilde{x}_2(x))$ of (4.13) near $x=0$.

Let F (resp., G) denote the left-hand side of the first (resp., second) equation of (4.13). Then we can first prove the following

Lemma 4.2. *Let us consider a system of ordinary differential equations*

$$\begin{cases} F\big|_{x_2=0} = 0, \\ G\big|_{x_2=0} = 0, \end{cases} \tag{4.27}$$

obtained by restricting $F = 0$ and $G = 0$ to $x_2 = 0$. Then this system has a holomorphic solution $(\widetilde{x}_1(x_1, 0), \widetilde{x}_2(x_1, 0))$ near $x_1 = 0$ satisfying

$$\widetilde{x}_1(0) = 0, \quad \widetilde{x}_2(0) = 0, \quad \frac{\partial \widetilde{x}_1}{\partial x_1}(0) \neq 0. \tag{4.28}$$

In the proof of this lemma we use the following Briot-Bouquet type theorem for a system of ordinary differential equations verified by Kaneko and Ohyama ([5]).

Theorem 4.1. *Let $h(t, u, v)$ and $k(t, u, v)$ be holomorphic functions near $(t, u, v) = 0$ satisfying*

- *$h(0) = k(0) = 0$.*
- *For any non-negative integer n,*

$$\det \begin{pmatrix} n - \dfrac{\partial h}{\partial u}(0) & -\dfrac{\partial h}{\partial v}(0) \\ -\dfrac{\partial k}{\partial u}(0) & n - \dfrac{\partial k}{\partial v}(0) \end{pmatrix} \neq 0. \tag{4.29}$$

Then the system of ordinary differential equations

$$\begin{cases} t\dfrac{du}{dt} = h(t, u, v), \\ t\dfrac{dv}{dt} = k(t, u, v) \end{cases} \tag{4.30}$$

has a holomorphic solution $(u(t), v(t))$ near $t = 0$ satisfying $u(0) = v(0) = 0$.

Proof of Lemma 4.2. Let $Y(x_1)$, $\widetilde{Y}(x_1)$, $Z(x_1)$ and $\widetilde{Z}(x_1)$ be holomorphic functions defined by

$$\widetilde{x}_1\big|_{x_2=0} = Y = x_1\widetilde{Y}, \quad \widetilde{x}_2\big|_{x_2=0} = Z = x_1\widetilde{Z}. \tag{4.31}$$

By the assumption $a_2(0) = a_3(0) = 0$, $a_2\big|_{x_2=0}$ and $a_3\big|_{x_2=0}$ can be expressed as follows:

$$a_2\big|_{x_2=0} = \sum_{n\geq 1} a_{2,n}x_1^n, \quad a_3\big|_{x_2=0} = \sum_{n\geq 1} a_{3,n}x_1^n. \tag{4.32}$$

Using these symbols, we can write (4.27) as

$$\begin{cases} f(x_1, \widetilde{Y}, \widetilde{Z}, Y_1, Z_1) := -\dfrac{1}{2}\widetilde{Z}Y_1^2 - \dfrac{3}{4}\widetilde{Y}Y_1Z_1 + \dfrac{x_1}{12}\widetilde{Z}^2Z_1^2 + \displaystyle\sum_{n\geq 1} a_{2,n}x_1^{n-1} = 0, \\ g(x_1, \widetilde{Y}, \widetilde{Z}, Y_1, Z_1) := -\dfrac{1}{4}\widetilde{Y}Y_1^3 + \dfrac{x_1}{6}\widetilde{Z}^2Y_1^2Z_1 + \dfrac{x_1}{8}\widetilde{Y}\widetilde{Z}Y_1Z_1^2 + \dfrac{x_1}{16}\widetilde{Y}^2Z_1^3 + \dfrac{x_1^2}{108}\widetilde{Z}^3Z_1^3 + \displaystyle\sum_{n\geq 1} a_{3,n}x_1^{n-1} \\ \qquad =0, \end{cases} \tag{4.33}$$

where $Y_1 = dY/dx_1$ and $Z_1 = dZ/dx_1$.

First, by the assumption $(\partial a_3/\partial x_1)(0) = a_{3,1} \neq 0$, a solution of

$$\begin{cases} f(0, p, q, p, q) = 0, \\ g(0, p, q, p, q) = 0 \end{cases} \tag{4.34}$$

is given by

$$p = (4a_{3,1})^{1/4} \neq 0, \quad q = \frac{2a_{2,1}}{5\,(a_{3,1})^{1/2}}. \tag{4.35}$$

Using this fact, we have

$$\det \begin{pmatrix} \dfrac{\partial f}{\partial Y_1} & \dfrac{\partial f}{\partial Z_1} \\ \dfrac{\partial g}{\partial Y_1} & \dfrac{\partial g}{\partial Z_1} \end{pmatrix} \Big|_{(x_1,\widetilde{Y},\widetilde{Z},Y_1,Z_1)=(0,p,q,p,q)} = -\frac{9}{16}p^5 \neq 0. \tag{4.36}$$

Then, thanks to the implicit function theorem, we can rewrite (4.33) as

$$\begin{cases} Y_1 = H(x_1,\widetilde{Y},\widetilde{Z}), \\ Z_1 = K(x_1,\widetilde{Y},\widetilde{Z}) \end{cases} \tag{4.37}$$

with some holomorphic functions $H(x_1,\widetilde{Y},\widetilde{Z})$, $K(x_1,\widetilde{Y},\widetilde{Z})$ near $(x_1,\widetilde{Y},\widetilde{Z}) = (0,p,q)$. Introducing new unknown functions $\widehat{Y} = \widetilde{Y} - p$ and $\widehat{Z} = \widetilde{Z} - q$, we further rewrite (4.37) as

$$\begin{cases} x_1 \dfrac{d\widehat{Y}}{dx_1} = H(x_1, p+\widehat{Y}, q+\widehat{Z}) - p - \widehat{Y} =: \widehat{H}(x_1,\widehat{Y},\widehat{Z}), \\ x_1 \dfrac{d\widehat{Z}}{dx_1} = K(x_1, p+\widehat{Y}, q+\widehat{Z}) - q - \widehat{Z} =: \widehat{K}(x_1,\widehat{Y},\widehat{Z}), \end{cases} \tag{4.38}$$

where $\widehat{H}(x_1,\widehat{Y},\widehat{Z})$ and $\widehat{K}(x_1,\widehat{Y},\widehat{Z})$ are holomorphic functions near $(x_1,\widehat{Y},\widehat{Z}) = 0$. Substituting these relations into (4.33), we have

$$\widehat{H}(0) = 0, \quad \frac{\partial \widehat{H}}{\partial \widehat{Y}}(0) = -\frac{4}{3}, \quad \frac{\partial \widehat{H}}{\partial \widehat{Z}}(0) = 0, \quad \widehat{K}(0) = 0, \quad \frac{\partial \widehat{K}}{\partial \widehat{Y}}(0) = -\frac{2q}{9p}, \quad \frac{\partial \widehat{K}}{\partial \widehat{Z}}(0) = -\frac{5}{3}. \tag{4.39}$$

This condition means that (4.38) satisfies the assumptions of Theorem 4.1. Hence, applying Theorem 4.1, we find that (4.38) has a holomorphic solution $\left(\widehat{Y}(x_1), \widehat{Z}(x_1)\right)$ with $\widehat{Y}(0) = \widehat{Z}(0) = 0$, that is, (4.27) has a holomorphic solution $(\widetilde{x}_1(x_1,0), \widetilde{x}_2(x_1,0))$ near $x_1 = 0$. The construction of the solution also verifies

$$\widetilde{x}_1(x_1,0) = x_1\left(p + \mathcal{O}(x_1)\right), \quad \widetilde{x}_2(x_1,0) = x_1\left(q + \mathcal{O}(x_1)\right), \tag{4.40}$$

that is, (4.28) holds. □

For this solution $(\widetilde{x}_1(x_1,0), \widetilde{x}_2(x_1,0))$, we have

Lemma 4.3. *Let* $(\widetilde{x}_1(x_1,0), \widetilde{x}_2(x_1,0))$ *be the holomorphic solution of* (4.27) *near* $x_1 = 0$ *obtained in Lemma 4.2. Then the following initial value problem*

$$\begin{cases} \dfrac{\partial \widetilde{x}_1}{\partial x_2} = b_2(x)\left\{ -\dfrac{\widetilde{x}_2}{3}\dfrac{\partial \widetilde{x}_1}{\partial x_1}\dfrac{\partial \widetilde{x}_2}{\partial x_1} - \dfrac{\widetilde{x}_1}{4}\left(\dfrac{\partial \widetilde{x}_2}{\partial x_1}\right)^2 \right\} + b_1(x)\dfrac{\partial \widetilde{x}_1}{\partial x_1}, \\ \dfrac{\partial \widetilde{x}_2}{\partial x_2} = b_2(x)\left\{ \left(\dfrac{\partial \widetilde{x}_1}{\partial x_1}\right)^2 + \dfrac{\widetilde{x}_2}{6}\left(\dfrac{\partial \widetilde{x}_2}{\partial x_1}\right)^2 \right\} + b_1(x)\dfrac{\partial \widetilde{x}_2}{\partial x_1}, \\ \widetilde{x}_i\big|_{x_2=0} = \widetilde{x}_i(x_1,0), \quad (i=1,2) \end{cases} \tag{4.41}$$

has a holomorphic solution near $x = 0$.

This lemma is an immediate consequence of the Cauchy-Kowalevski theorem.

Lemma 4.4. *The holomorphic solution* $(\widetilde{x}_1(x), \widetilde{x}_2(x))$ *given by Lemma 4.3 satisfies*

$$F = G = 0. \tag{4.42}$$

Proof. For simplicity, in this proof $F\big|_{(\widetilde{x}_1,\widetilde{x}_2)=(\widetilde{x}_1(x),\widetilde{x}_2(x))}$ and $G\big|_{(\widetilde{x}_1,\widetilde{x}_2)=(\widetilde{x}_1(x),\widetilde{x}_2(x))}$ are denoted by F and G, respectively. We prove that F and G satisfy the following system of partial differential equations

$$\begin{cases} \dfrac{\partial F}{\partial x_2} = 2b_2\dfrac{\partial G}{\partial x_1} + b_1\dfrac{\partial F}{\partial x_1} + 3\dfrac{\partial b_2}{\partial x_1}G + 2\dfrac{\partial b_1}{\partial x_1}F, \\ \dfrac{\partial G}{\partial x_2} = b_1\dfrac{\partial G}{\partial x_1} + \dfrac{1}{3}b_2\dfrac{\partial}{\partial x_1}F\left(F-2a_2\right) + 3\dfrac{\partial b_1}{\partial x_1}G + \dfrac{2}{3}\dfrac{\partial b_2}{\partial x_1}F\left(F-2a_2\right). \end{cases} \tag{4.43}$$

Using the definition of F and G, we have

(4.44)

$$\begin{aligned} \frac{\partial}{\partial x_1}\left(F-a_2\right) = &\left\{\frac{\widetilde{x}_2}{6}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^3 - \frac{5}{4}\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\right)^2\frac{\partial \widetilde{x}_2}{\partial x_1}\right\} + \left(-\frac{3}{4}\widetilde{x}_1\frac{\partial \widetilde{x}_2}{\partial x_1} - \widetilde{x}_2\frac{\partial \widetilde{x}_1}{\partial x_1}\right)\frac{\partial^2 \widetilde{x}_1}{\partial x_1^2} \\ &+ \left(\frac{\widetilde{x}_2^2}{6}\frac{\partial \widetilde{x}_2}{\partial x_1} - \frac{3}{4}\widetilde{x}_1\frac{\partial \widetilde{x}_1}{\partial x_1}\right)\frac{\partial^2 \widetilde{x}_2}{\partial x_1^2}, \end{aligned}$$

(4.45)

$$\begin{aligned} \frac{\partial}{\partial x_1}\left(G-a_3\right) = &\left\{\frac{\widetilde{x}_1}{4}\frac{\partial \widetilde{x}_1}{\partial x_1}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^3 + \frac{\widetilde{x}_2^2}{36}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^4 - \frac{1}{4}\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\right)^4 + \frac{11}{24}\widetilde{x}_2\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\right)^2\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^2\right\} \\ &+ \left\{-\frac{3}{4}\widetilde{x}_1\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\right)^2 + \frac{\widetilde{x}_1\widetilde{x}_2}{8}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^2 + \frac{\widetilde{x}_2^2}{3}\frac{\partial \widetilde{x}_1}{\partial x_1}\frac{\partial \widetilde{x}_2}{\partial x_1}\right\}\frac{\partial^2 \widetilde{x}_1}{\partial x_1^2} \\ &+ \left\{\frac{3}{16}\widetilde{x}_1^2\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^2 + \frac{\widetilde{x}_2^3}{36}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^2 + \frac{\widetilde{x}_1\widetilde{x}_2}{4}\frac{\partial \widetilde{x}_1}{\partial x_1}\frac{\partial \widetilde{x}_2}{\partial x_1} + \frac{\widetilde{x}_2^2}{6}\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\right)^2\right\}\frac{\partial^2 \widetilde{x}_2}{\partial x_1^2}, \end{aligned}$$

(4.46)

$$\begin{aligned} \frac{\partial}{\partial x_2}\left(F-a_2\right) = &\left(-\frac{3}{4}\frac{\partial \widetilde{x}_1}{\partial x_1}\frac{\partial \widetilde{x}_2}{\partial x_1}\right)\frac{\partial \widetilde{x}_1}{\partial x_2} + \left\{\frac{\widetilde{x}_2}{6}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^2 - \frac{1}{2}\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\right)^2\right\}\frac{\partial \widetilde{x}_2}{\partial x_2} \\ &+ \left(-\frac{3}{4}\widetilde{x}_1\frac{\partial \widetilde{x}_2}{\partial x_1} - \widetilde{x}_2\frac{\partial \widetilde{x}_1}{\partial x_1}\right)\frac{\partial^2 \widetilde{x}_1}{\partial x_1\partial x_2} + \left(\frac{\widetilde{x}_2^2}{6}\frac{\partial \widetilde{x}_2}{\partial x_1} - \frac{3}{4}\widetilde{x}_1\frac{\partial \widetilde{x}_1}{\partial x_1}\right)\frac{\partial^2 \widetilde{x}_2}{\partial x_1\partial x_2}, \end{aligned}$$

(4.47)

$$\begin{aligned} \frac{\partial}{\partial x_2}\left(G-a_3\right) = &\left\{\frac{\widetilde{x}_1}{8}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^3 - \frac{1}{4}\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\right)^3 + \frac{\widetilde{x}_2}{8}\frac{\partial \widetilde{x}_1}{\partial x_1}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^2\right\}\frac{\partial \widetilde{x}_1}{\partial x_2} \\ &+ \left\{\frac{\widetilde{x}_2^2}{36}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^3 + \frac{\widetilde{x}_1}{8}\frac{\partial \widetilde{x}_1}{\partial x_1}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^2 + \frac{\widetilde{x}_2}{3}\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\right)^2\frac{\partial \widetilde{x}_2}{\partial x_1}\right\}\frac{\partial \widetilde{x}_2}{\partial x_2} \\ &+ \left\{-\frac{3}{4}\widetilde{x}_1\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\right)^2 + \frac{\widetilde{x}_1\widetilde{x}_2}{8}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^2 + \frac{\widetilde{x}_2^2}{3}\frac{\partial \widetilde{x}_1}{\partial x_1}\frac{\partial \widetilde{x}_2}{\partial x_1}\right\}\frac{\partial^2 \widetilde{x}_1}{\partial x_1\partial x_2} \\ &+ \left\{\frac{3}{16}\widetilde{x}_1^2\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^2 + \frac{\widetilde{x}_2^3}{36}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^2 + \frac{\widetilde{x}_1\widetilde{x}_2}{4}\frac{\partial \widetilde{x}_1}{\partial x_1}\frac{\partial \widetilde{x}_2}{\partial x_1} + \frac{\widetilde{x}_2^2}{6}\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\right)^2\right\}\frac{\partial^2 \widetilde{x}_2}{\partial x_1\partial x_2}. \end{aligned}$$

Here, by (4.41), we find

(4.48)

$$\frac{\partial^2 \widetilde{x}_1}{\partial x_1\partial x_2} = \frac{\partial b_2}{\partial x_1}(x)\left\{-\frac{\widetilde{x}_2}{3}\frac{\partial \widetilde{x}_1}{\partial x_1}\frac{\partial \widetilde{x}_2}{\partial x_1} - \frac{\widetilde{x}_1}{4}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^2\right\}$$

$$+ b_2(x)\left\{-\frac{\widetilde{x}_2}{3}\frac{\partial\widetilde{x}_2}{\partial x_1}\frac{\partial^2\widetilde{x}_1}{\partial x_1^2} - \frac{7}{12}\frac{\partial\widetilde{x}_1}{\partial x_1}\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^2 - \frac{\widetilde{x}_2}{3}\frac{\partial\widetilde{x}_1}{\partial x_1}\frac{\partial^2\widetilde{x}_2}{\partial x_1^2} - \frac{\widetilde{x}_1}{2}\frac{\partial\widetilde{x}_2}{\partial x_1}\frac{\partial^2\widetilde{x}_2}{\partial x_1^2}\right\}$$

$$+ \frac{\partial b_1}{\partial x_1}(x)\frac{\partial\widetilde{x}_1}{\partial x_1} + b_1(x)\frac{\partial^2\widetilde{x}_1}{\partial x_1^2},$$

(4.49)

$$\frac{\partial^2\widetilde{x}_2}{\partial x_1\partial x_2} = \frac{\partial b_2}{\partial x_1}(x)\left\{\left(\frac{\partial\widetilde{x}_1}{\partial x_1}\right)^2 + \frac{\widetilde{x}_2}{6}\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^2\right\} + b_2(x)\left\{2\frac{\partial\widetilde{x}_1}{\partial x_1}\frac{\partial^2\widetilde{x}_1}{\partial x_1^2} + \frac{1}{6}\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^3 + \frac{\widetilde{x}_2}{3}\frac{\partial\widetilde{x}_2}{\partial x_1}\frac{\partial^2\widetilde{x}_2}{\partial x_1^2}\right\}$$

$$+ \frac{\partial b_1}{\partial x_1}(x)\frac{\partial\widetilde{x}_2}{\partial x_1} + b_1(x)\frac{\partial^2\widetilde{x}_2}{\partial x_1^2}.$$

Then, using these relations, we can confirm by a straightforward computation that

(4.50)

$$\begin{cases} \dfrac{\partial}{\partial x_2}(F - a_2) - 2b_2\dfrac{\partial}{\partial x_1}(G - a_3) - b_1\dfrac{\partial}{\partial x_1}(F - a_2) - 3\dfrac{\partial b_2}{\partial x_1}(G - a_3) - 2\dfrac{\partial b_1}{\partial x_1}(F - a_2) = 0, \\ \dfrac{\partial}{\partial x_2}(G - a_3) - b_1\dfrac{\partial}{\partial x_1}(G - a_3) - \dfrac{1}{3}b_2\dfrac{\partial}{\partial x_1}(F - a_2)^2 - 3\dfrac{\partial b_1}{\partial x_1}(G - a_3) - \dfrac{2}{3}\dfrac{\partial b_2}{\partial x_1}(F - a_2)^2 = 0. \end{cases}$$

In view of (2.44) and (4.25), (4.43) immediately follows from (4.50).

On the other hand, by the construction of $(\widetilde{x}_1(x), \widetilde{x}_2(x))$, F and G satisfy

$$F\big|_{x_2=0} = G\big|_{x_2=0} = 0. \tag{4.51}$$

Hence, by the uniqueness of solutions for the initial value problem (4.43) and (4.51), we obtain (4.42). □

We have thus confirmed that the system (4.13) has a holomorphic solution $(\widetilde{x}_1(x), \widetilde{x}_2(x))$ near $x = 0$. This solution has the following properties.

Lemma 4.5. *Let $(\widetilde{x}_1(x), \widetilde{x}_2(x))$ be the solution given by the above argument. Then $(\widetilde{x}_1(x), \widetilde{x}_2(x))$ satisfies*

- $\widetilde{x}_1(0) = \widetilde{x}_2(0) = 0.$
- *The Jacobian*

$$|J_{\widetilde{x}}|(x) = \det\begin{pmatrix} \dfrac{\partial\widetilde{x}_1}{\partial x_1}(x) & \dfrac{\partial\widetilde{x}_1}{\partial x_2}(x) \\ \dfrac{\partial\widetilde{x}_2}{\partial x_1}(x) & \dfrac{\partial\widetilde{x}_2}{\partial x_2}(x) \end{pmatrix} \tag{4.52}$$

  of $\widetilde{x}(x) = (\widetilde{x}_1(x), \widetilde{x}_2(x))$ is non-zero near $x = 0$.

Proof. By Lemma 2.3 and the assumption $a_2(0) = a_3(0) = 0$, we find

$$\det\begin{pmatrix} \dfrac{\partial a_2}{\partial x_1}(0) & \dfrac{\partial a_2}{\partial x_2}(0) \\ \dfrac{\partial a_3}{\partial x_1}(0) & \dfrac{\partial a_3}{\partial x_2}(0) \end{pmatrix} = \det\begin{pmatrix} \dfrac{\partial a_2}{\partial x_1}(0) & 2b_2(0)\dfrac{\partial a_3}{\partial x_1}(0) + b_1(0)\dfrac{\partial a_2}{\partial x_1}(0) \\ \dfrac{\partial a_3}{\partial x_1}(0) & b_1(0)\dfrac{\partial a_3}{\partial x_1}(0) \end{pmatrix} \tag{4.53}$$

$$= -2b_2(0)\left(\frac{\partial a_3}{\partial x_1}(0)\right)^2.$$

Hence, using the assumptions of Proposition 4.1, we get

$$b_2(0) \neq 0. \tag{4.54}$$

It follows from the third and the fourth equations of (4.13) and (4.28) that

$$\begin{aligned}
\frac{\partial \widetilde{x}_1}{\partial x_2}(0) =& b_2(0)\left\{-\frac{\widetilde{x}_2(0)}{3}\frac{\partial \widetilde{x}_1}{\partial x_1}(0)\frac{\partial \widetilde{x}_2}{\partial x_1}(0)-\frac{\widetilde{x}_1(0)}{4}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}(0)\right)^2\right\}+b_1(0)\frac{\partial \widetilde{x}_1}{\partial x_1}(0) \\
=& b_1(0)\frac{\partial \widetilde{x}_1}{\partial x_1}(0),
\end{aligned} \tag{4.55}$$

$$\begin{aligned}
\frac{\partial \widetilde{x}_2}{\partial x_2}(0) =& b_2(0)\left\{\left(\frac{\partial \widetilde{x}_1}{\partial x_1}(0)\right)^2+\frac{\widetilde{x}_2(0)}{6}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}(0)\right)^2\right\}+b_1(0)\frac{\partial \widetilde{x}_2}{\partial x_1}(0) \\
=& b_2(0)\left(\frac{\partial \widetilde{x}_1}{\partial x_1}(0)\right)^2+b_1(0)\frac{\partial \widetilde{x}_2}{\partial x_1}(0).
\end{aligned} \tag{4.56}$$

We thus obtain

$$|J_{\widetilde{x}}|(0) = b_2(0)\left(\frac{\partial \widetilde{x}_1}{\partial x_1}(0)\right)^3 \neq 0, \tag{4.57}$$

that is, $|J_{\widetilde{x}}|(x)$ is non-zero near $x = 0$. □

Next, we construct $T_0(x)$ which satisfies (4.10).

Lemma 4.6. *There exists a 3×3 matrix $T_0(x)$ with holomorphic entries which is invertible near $x = 0$ and satisfies*

$$T_0\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\widetilde{P}_0+\frac{\partial \widetilde{x}_2}{\partial x_1}\widetilde{Q}_0\right) = P_0 T_0. \tag{4.58}$$

Proof. Using (2.39), (4.7) and the first and the second equations of (4.13), we have

$$\begin{aligned}
&\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\widetilde{P}_0+\frac{\partial \widetilde{x}_2}{\partial x_1}\widetilde{Q}_0\right)^3+a_2(x)\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\widetilde{P}_0+\frac{\partial \widetilde{x}_2}{\partial x_1}\widetilde{Q}_0\right)+a_3(x) \\
=&\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\widetilde{P}_0^2+\frac{\partial \widetilde{x}_1}{\partial x_1}\widetilde{P}_0+\frac{\widetilde{x}_2}{3}\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^3+a_2(x)\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\widetilde{P}_0^2+\frac{\partial \widetilde{x}_1}{\partial x_1}\widetilde{P}_0+\frac{\widetilde{x}_2}{3}\frac{\partial \widetilde{x}_2}{\partial x_1}\right)+a_3(x) \\
=&\left\{\frac{\widetilde{x}_2^2}{12}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^3-\frac{3}{4}\widetilde{x}_1\frac{\partial \widetilde{x}_1}{\partial x_1}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^2-\frac{\widetilde{x}_2}{2}\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\right)^2\frac{\partial \widetilde{x}_2}{\partial x_1}+a_2(x)\frac{\partial \widetilde{x}_2}{\partial x_1}\right\}\widetilde{P}_0^2 \\
&+\left\{\frac{\widetilde{x}_2^2}{12}\frac{\partial \widetilde{x}_1}{\partial x_1}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^2-\frac{3}{4}\widetilde{x}_1\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\right)^2\frac{\partial \widetilde{x}_2}{\partial x_1}-\frac{\widetilde{x}_2}{2}\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\right)^3+a_2(x)\frac{\partial \widetilde{x}_1}{\partial x_1}\right\}\widetilde{P}_0 \\
&+\left\{\left(\frac{\widetilde{x}_1^2}{16}+\frac{\widetilde{x}_2^3}{27}\right)\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^3-\frac{\widetilde{x}_1}{4}\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\right)^3-\frac{\widetilde{x}_1\widetilde{x}_2}{8}\frac{\partial \widetilde{x}_1}{\partial x_1}\left(\frac{\partial \widetilde{x}_2}{\partial x_1}\right)^2+\frac{a_2(x)}{3}\widetilde{x}_2\frac{\partial \widetilde{x}_2}{\partial x_1}+a_3(x)\right\} \\
=&0.
\end{aligned} \tag{4.59}$$

Then, by the same argument as in the proof of Lemma 2.1, we can find a 3×3 invertible matrix $\widetilde{T}^{\dagger}(x)$ with holomorphic entries near $x = 0$ which satisfies

$$(\widetilde{T}^{\dagger})^{-1}\left(\frac{\partial \widetilde{x}_1}{\partial x_1}\widetilde{P}_0+\frac{\partial \widetilde{x}_2}{\partial x_1}\widetilde{Q}_0\right)\widetilde{T}^{\dagger} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & 0 \end{pmatrix}. \tag{4.60}$$

If we set $T_0(x) = T^{\dagger}(x)\widetilde{T}^{\dagger}(x)^{-1}$, where $T^{\dagger}(x)$ is a 3×3 matrix given in Lemma 2.1, $T_0(x)$ satisfies the properties of this lemma. □

Furthermore, we have

Lemma 4.7.

(4.61)
$$T_0\left(\frac{\partial\widetilde{x}_1}{\partial x_2}\widetilde{P}_0+\frac{\partial\widetilde{x}_2}{\partial x_2}\widetilde{Q}_0\right)=Q_0T_0.$$

Proof. Using (2.32), (2.39), (4.7), the third and the fourth equations of (4.13), (4.14) and (4.23), we obtain

(4.62)
$$\begin{aligned}
&Q_0T_0\\
&=\left(b_2(x)P_0^2+b_1(x)P_0+b_0(x)\right)T_0\\
&=b_2(x)T_0\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\widetilde{P}_0^2+\frac{\partial\widetilde{x}_1}{\partial x_1}\widetilde{P}_0+\frac{\widetilde{x}_2}{3}\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^2+b_1(x)T_0\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\widetilde{P}_0^2+\frac{\partial\widetilde{x}_1}{\partial x_1}\widetilde{P}_0+\frac{\widetilde{x}_2}{3}\frac{\partial\widetilde{x}_2}{\partial x_1}\right)+b_0(x)T_0\\
&=T_0\left(\left[b_2(x)\left\{\left(\frac{\partial\widetilde{x}_1}{\partial x_1}\right)^2+\frac{\widetilde{x}_2}{6}\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^2\right\}+b_1(x)\frac{\partial\widetilde{x}_2}{\partial x_1}\right]\widetilde{P}_0^2\right.\\
&\quad+\left[b_2(x)\left\{-\frac{\widetilde{x}_2}{3}\frac{\partial\widetilde{x}_1}{\partial x_1}\frac{\partial\widetilde{x}_2}{\partial x_1}-\frac{\widetilde{x}_1}{4}\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^2\right\}+b_1(x)\frac{\partial\widetilde{x}_1}{\partial x_1}\right]\widetilde{P}_0\\
&\quad\left.+\left[b_2(x)\left\{\frac{\widetilde{x}_2^2}{9}\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^2-\frac{\widetilde{x}_1}{2}\frac{\partial\widetilde{x}_1}{\partial x_1}\frac{\partial\widetilde{x}_2}{\partial x_1}\right\}+\frac{b_1(x)}{3}\widetilde{x}_2\frac{\partial\widetilde{x}_2}{\partial x_1}+b_0(x)\right]\right)\\
&=T_0\left(\frac{\partial\widetilde{x}_2}{\partial x_2}\widetilde{P}_0^2+\frac{\partial\widetilde{x}_1}{\partial x_2}\widetilde{P}_0+\frac{\widetilde{x}_2}{3}\frac{\partial\widetilde{x}_2}{\partial x_2}\right)\\
&=T_0\left(\frac{\partial\widetilde{x}_1}{\partial x_2}\widetilde{P}_0+\frac{\partial\widetilde{x}_2}{\partial x_2}\widetilde{Q}_0\right).
\end{aligned}$$

□

Using these lemmas, we now prove Proposition 4.1.

Proof of Proposition 4.1. Let $(\widetilde{x}_1(x),\widetilde{x}_2(x))$ be a holomorphic solution of (4.13) constructed in Lemmas 4.2, 4.3, 4.4 and 4.5, and let $T_0(x)$ be a 3×3 invertible matrix with holomorphic entries given in Lemmas 4.6 and 4.7. Let us define 3×3 matrices $\widetilde{P}_n(x)$ and $\widetilde{Q}_n(x)$ by

(4.63)
$$\begin{cases}\widetilde{P}_1=(|J_{\widetilde{x}}|\,T_0)^{-1}\left\{\dfrac{\partial\widetilde{x}_2}{\partial x_2}\left(P_1T_0-\dfrac{\partial T_0}{\partial x_1}\right)-\dfrac{\partial\widetilde{x}_2}{\partial x_1}\left(Q_1T_0-\dfrac{\partial T_0}{\partial x_2}\right)\right\},\\[2ex]
\widetilde{Q}_1=(|J_{\widetilde{x}}|\,T_0)^{-1}\left\{\dfrac{\partial\widetilde{x}_1}{\partial x_1}\left(Q_1T_0-\dfrac{\partial T_0}{\partial x_2}\right)-\dfrac{\partial\widetilde{x}_1}{\partial x_2}\left(P_1T_0-\dfrac{\partial T_0}{\partial x_1}\right)\right\},\end{cases}$$

(4.64)
$$\begin{cases}\widetilde{P}_n=(|J_{\widetilde{x}}|\,T_0)^{-1}\left(\dfrac{\partial\widetilde{x}_2}{\partial x_2}P_n-\dfrac{\partial\widetilde{x}_2}{\partial x_1}Q_n\right)T_0,\\[2ex]
\widetilde{Q}_n=(|J_{\widetilde{x}}|\,T_0)^{-1}\left(\dfrac{\partial\widetilde{x}_1}{\partial x_1}Q_n-\dfrac{\partial\widetilde{x}_1}{\partial x_2}P_n\right)T_0,\end{cases}\qquad(n\geq 2).$$

Then there exists a positive constant $\rho>0$ such that all the required properties in Proposition 4.1 except for the estimate (4.8) are satisfied.

We finally prove the estimate (4.8). Since $(\partial\widetilde{x}_i/\partial x_j)(x)$ $(i,j=1,2)$, $T_0(x)$, $(|J_{\widetilde{x}}|\,(x)T_0(x))^{-1}$, $\widetilde{P}_i(x)$ $(i=0,1)$ and $\widetilde{Q}_i(x)$ $(i=0,1)$ are holomorphic in D_ρ, we have

(4.65)
$$\left|\frac{\partial\widetilde{x}_i}{\partial x_j}\right|_{\rho,\rho},\ \|T_0\|_{\rho,\rho},\ \left\|(|J_{\widetilde{x}}|\,T_0)^{-1}\right\|_{\rho,\rho},\ \left\|\widetilde{P}_i\right\|_{\rho,\rho},\ \left\|\widetilde{Q}_i\right\|_{\rho,\rho}\leq C^{\dagger}$$

with some positive constant $C^\dagger$. Here we define $|f|_{\rho_1,\rho_2}$ for a holomorphic function $f(x)$ in D_{ρ_0} $(0<\rho_1,\rho_2<\rho_0)$ as follows:

$$|f|_{\rho_1,\rho_2} = \sup_{|x_1|\le\rho_1, |x_2|\le\rho_2} |f(x)|\,. \tag{4.66}$$

Then, if we define positive constants $\widetilde{C}$ and $\widetilde{\alpha}$ by

$$\widetilde{C} = \max\left\{C^\dagger,\ 2\left(C^\dagger\right)^3 C\right\}, \quad \widetilde{\alpha} = \max\{1,\ \alpha\}, \tag{4.67}$$

the estimate (4.8) does really hold. In fact,

$$\left\|\widetilde{P}_0\right\|_{\rho,\rho}, \left\|\widetilde{Q}_0\right\|_{\rho,\rho} \le C^\dagger \le \widetilde{C}, \tag{4.68}$$

$$\left\|\widetilde{P}_1\right\|_{\rho,\rho}, \left\|\widetilde{Q}_1\right\|_{\rho,\rho} \le C^\dagger \le \widetilde{C}\widetilde{\alpha} \tag{4.69}$$

trivially hold and it follows from (4.64) that

$$\left\|\widetilde{P}_n\right\|_{\rho,\rho}, \left\|\widetilde{Q}_n\right\|_{\rho,\rho} \le 2\left(C^\dagger\right)^3 C\alpha^n n! \le \widetilde{C}\widetilde{\alpha}^n n! \tag{4.70}$$

for $n \ge 2$. □

Finally, we prove Lemma 2.4 and Proposition 2.3 by using the construction of $\widetilde{x}(x)$.

Proof of Lemma 2.4. For simplicity, $\xi_{1,i}(x)$, $\xi_{2,i}(x)$, $\widetilde{\xi}_{1,i}(\widetilde{x})$ and $\widetilde{\xi}_{2,i}(\widetilde{x})$ are denoted by $\xi_1(x)$, $\xi_2(x)$, $\widetilde{\xi}_1(\widetilde{x})$ and $\widetilde{\xi}_2(\widetilde{x})$, respectively. Combining (2.58), (2.59) and (2.60) with (4.13), (4.18) and (4.23), we have

$$\begin{aligned}
&\xi_1(x)^3 + a_2(x)\xi_1(x) + a_3(x) \\
&=\left\{\widetilde{\xi}_1(\widetilde{x}(x))\frac{\partial\widetilde{x}_1}{\partial x_1} + \left(\widetilde{\xi}_1(\widetilde{x}(x))\right)^2\frac{\partial\widetilde{x}_2}{\partial x_1} + \frac{\widetilde{x}_2}{3}\frac{\partial\widetilde{x}_2}{\partial x_1}\right\}^3 \\
&\quad + a_2(x)\left\{\widetilde{\xi}_1(\widetilde{x}(x))\frac{\partial\widetilde{x}_1}{\partial x_1} + \left(\widetilde{\xi}_1(\widetilde{x}(x))\right)^2\frac{\partial\widetilde{x}_2}{\partial x_1} + \frac{\widetilde{x}_2}{3}\frac{\partial\widetilde{x}_2}{\partial x_1}\right\} + a_3(x) \\
&=\left\{\frac{\widetilde{x}_2^2}{12}\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^3 - \frac{3}{4}\widetilde{x}_1\frac{\partial\widetilde{x}_1}{\partial x_1}\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^2 - \frac{\widetilde{x}_2}{2}\left(\frac{\partial\widetilde{x}_1}{\partial x_1}\right)^2\frac{\partial\widetilde{x}_2}{\partial x_1} + a_2(x)\frac{\partial\widetilde{x}_2}{\partial x_1}\right\}\left(\widetilde{\xi}_1(\widetilde{x}(x))\right)^2 \\
&\quad + \left\{\frac{\widetilde{x}_2^2}{12}\frac{\partial\widetilde{x}_1}{\partial x_1}\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^2 - \frac{3}{4}\widetilde{x}_1\left(\frac{\partial\widetilde{x}_1}{\partial x_1}\right)^2\frac{\partial\widetilde{x}_2}{\partial x_1} - \frac{\widetilde{x}_2}{2}\left(\frac{\partial\widetilde{x}_1}{\partial x_1}\right)^3 + a_2(x)\frac{\partial\widetilde{x}_1}{\partial x_1}\right\}\widetilde{\xi}_1(\widetilde{x}(x)) \\
&\quad + \left\{\left(\frac{\widetilde{x}_1^2}{16} + \frac{\widetilde{x}_2^3}{27}\right)\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^3 - \frac{\widetilde{x}_1}{4}\left(\frac{\partial\widetilde{x}_1}{\partial x_1}\right)^3 - \frac{\widetilde{x}_1\widetilde{x}_2}{8}\frac{\partial\widetilde{x}_1}{\partial x_1}\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^2 + \frac{a_2(x)}{3}\widetilde{x}_2\frac{\partial\widetilde{x}_2}{\partial x_1} + a_3(x)\right\} \\
&=0,
\end{aligned} \tag{4.71}$$

$$\begin{aligned}
&b_2(x)\xi_1(x)^2 + b_1(x)\xi_1(x) + b_0(x) \\
&=b_2(x)\left\{\widetilde{\xi}_1(\widetilde{x}(x))\frac{\partial\widetilde{x}_1}{\partial x_1} + \left(\widetilde{\xi}_1(\widetilde{x}(x))\right)^2\frac{\partial\widetilde{x}_2}{\partial x_1} + \frac{\widetilde{x}_2}{3}\frac{\partial\widetilde{x}_2}{\partial x_1}\right\}^2 \\
&\quad + b_1(x)\left\{\widetilde{\xi}_1(\widetilde{x}(x))\frac{\partial\widetilde{x}_1}{\partial x_1} + \left(\widetilde{\xi}_1(\widetilde{x}(x))\right)^2\frac{\partial\widetilde{x}_2}{\partial x_1} + \frac{\widetilde{x}_2}{3}\frac{\partial\widetilde{x}_2}{\partial x_1}\right\} + b_0(x) \\
&=\left[b_2(x)\left\{\left(\frac{\partial\widetilde{x}_1}{\partial x_1}\right)^2 + \frac{\widetilde{x}_2}{6}\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^2\right\} + b_1(x)\frac{\partial\widetilde{x}_2}{\partial x_1}\right]\left(\widetilde{\xi}_1(\widetilde{x}(x))\right)^2
\end{aligned} \tag{4.72}$$

$$
\begin{aligned}
&+\left[b_2(x)\left\{-\frac{\widetilde{x}_2}{3}\frac{\partial\widetilde{x}_1}{\partial x_1}\frac{\partial\widetilde{x}_2}{\partial x_1}-\frac{\widetilde{x}_1}{4}\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^2\right\}+b_1(x)\frac{\partial\widetilde{x}_1}{\partial x_1}\right]\widetilde{\xi}_1(\widetilde{x}(x))\\
&+\left[b_2(x)\left\{\frac{\widetilde{x}_2^2}{9}\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^2-\frac{\widetilde{x}_1}{2}\frac{\partial\widetilde{x}_1}{\partial x_1}\frac{\partial\widetilde{x}_2}{\partial x_1}\right\}+\frac{b_1(x)}{3}\widetilde{x}_2\frac{\partial\widetilde{x}_2}{\partial x_1}+b_0(x)\right]\\
=&\frac{\partial\widetilde{x}_2}{\partial x_2}\left(\widetilde{\xi}_1(\widetilde{x})\right)^2+\frac{\partial\widetilde{x}_1}{\partial x_2}\widetilde{\xi}_1(\widetilde{x}(x))+\frac{\widetilde{x}_2}{3}\frac{\partial\widetilde{x}_2}{\partial x_2}\\
=&\frac{\partial\widetilde{x}_1}{\partial x_2}\widetilde{\xi}_1(\widetilde{x}(x))+\frac{\partial\widetilde{x}_2}{\partial x_2}\widetilde{\xi}_2(\widetilde{x}(x))\\
=&\xi_2(x).
\end{aligned}
$$

□

Proof of Proposition 2.3. Using (2.60), we have

$$
\begin{aligned}
&\left(\xi_{1,i}(x)-\xi_{1,i'}(x),\xi_{2,i}(x)-\xi_{2,i'}(x)\right)\\
&=\left(\widetilde{\xi}_{1,i}(\widetilde{x}(x))-\widetilde{\xi}_{1,i'}(\widetilde{x}(x)),\widetilde{\xi}_{2,i}(\widetilde{x}(x))-\widetilde{\xi}_{2,i'}(\widetilde{x}(x))\right)\begin{pmatrix}\dfrac{\partial\widetilde{x}_1}{\partial x_1}(x) & \dfrac{\partial\widetilde{x}_1}{\partial x_2}(x)\\ \dfrac{\partial\widetilde{x}_2}{\partial x_1}(x) & \dfrac{\partial\widetilde{x}_2}{\partial x_2}(x)\end{pmatrix}.
\end{aligned}
\tag{4.73}
$$

Hence, by the definition of turning points and Lemma 4.5, we immediately obtain Proposition 2.3. □

4.2 Construction of the transformation, II

As a consequence of the argument of the previous subsection, we may assume that the completely integrable system in question has the following form

$$
\begin{cases}
\eta^{-1}\dfrac{\partial}{\partial x_1}\Psi=P(x,\eta)\Psi, & P(x,\eta)=\displaystyle\sum_{n\geq 0}\eta^{-n}P_n(x),\\
\eta^{-1}\dfrac{\partial}{\partial x_2}\Psi=Q(x,\eta)\Psi, & Q(x,\eta)=\displaystyle\sum_{n\geq 0}\eta^{-n}Q_n(x).
\end{cases}
\tag{4.74}
$$

Here

$$
P_0=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -x_1/4 & -x_2/2 & 0\end{pmatrix},\quad Q_0=P_0^2+\frac{x_2}{3}
\tag{4.75}
$$

and $P_n(x)$ and $Q_n(x)$ are 3×3 matrices with holomorphic entries in D_{ρ_0} satisfying an estimate

$$
\|P_n\|_{\rho_0,\rho_0},\|Q_n\|_{\rho_0,\rho_0}\leq C\alpha^n n!\quad (n\geq 0)
\tag{4.76}
$$

with some positive constants C and α.

In this subsection, we prove

Proposition 4.2. *There exist a sufficiently small positive constant $0<\rho<\rho_0$ and 3×3 matrices $T_0(x)$, $T_1(x)$ which satisfy the following properties:*

- *Every entry of $T_0(x)$ and $T_1(x)$ is holomorphic in D_ρ and $\det T_0(x)\neq 0$ $(x\in D_\rho)$.*

- *By a transformation*

$$\Psi(x,\eta) = T(x,\eta)\widetilde{\Psi}(x,\eta), \quad T(x,\eta) = T_0(x) + \eta^{-1}T_1(x), \tag{4.77}$$

(4.74) is transformed into the following form:

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial x_1}\widetilde{\Psi} = \widetilde{P}(x,\eta)\widetilde{\Psi}, & \widetilde{P}(x,\eta) = \displaystyle\sum_{n\geq 0}\eta^{-n}\widetilde{P}_n(x), \\ \eta^{-1}\dfrac{\partial}{\partial x_2}\widetilde{\Psi} = \widetilde{Q}(x,\eta)\widetilde{\Psi}, & \widetilde{Q}(x,\eta) = \displaystyle\sum_{n\geq 0}\eta^{-n}\widetilde{Q}_n(x), \end{cases} \tag{4.78}$$

where

$$\widetilde{P}_0 = P_0, \quad \widetilde{P}_1 = 0, \quad \widetilde{Q}_0 = P_0^2 + \frac{x_2}{3}, \quad \widetilde{Q}_1 = \frac{\partial P_0}{\partial x_1}, \tag{4.79}$$

and $\widetilde{P}_n(x)$ and $\widetilde{Q}_n(x)$ are 3×3 matrices with holomorphic entries in D_ρ satisfying an estimate

$$\left\|\widetilde{P}_n\right\|_{\rho,\rho}, \left\|\widetilde{Q}_n\right\|_{\rho,\rho} \leq \widetilde{C}\widetilde{\alpha}^n n! \quad (n \geq 0) \tag{4.80}$$

with some positive constants $\widetilde{C}$ and $\widetilde{\alpha}$.

Since the system (4.78) is obtained from (4.74) through the transformation (4.77), $T(x,\eta)$ should satisfy the following relation:

$$\begin{cases} T\widetilde{P} + \eta^{-1}\dfrac{\partial T}{\partial x_1} = PT, \\ T\widetilde{Q} + \eta^{-1}\dfrac{\partial T}{\partial x_2} = QT, \end{cases} \tag{4.81}$$

that is,

$$\begin{cases} [P_0, T_0] = 0, \\ [Q_0, T_0] = 0, \end{cases} \tag{4.82}$$

$$\begin{cases} [P_0, T_1] = \dfrac{\partial T_0}{\partial x_1} - P_1 T_0, \\ [Q_0, T_1] = \dfrac{\partial T_0}{\partial x_2} + T_0\dfrac{\partial P_0}{\partial x_1} - Q_1 T_0, \end{cases} \tag{4.83}$$

$$\begin{cases} \dfrac{\partial T_1}{\partial x_1} + T_0\widetilde{P}_2 - P_2 T_0 - P_1 T_1 = 0, \\ \dfrac{\partial T_1}{\partial x_2} + T_0\widetilde{Q}_2 - Q_2 T_0 + T_1\dfrac{\partial P_0}{\partial x_1} - Q_1 T_1 = 0, \end{cases} \tag{4.84}$$

$$\begin{cases} T_0\widetilde{P}_n - P_n T_0 + T_1\widetilde{P}_{n-1} - P_{n-1}T_1 = 0, \\ T_0\widetilde{Q}_n - Q_n T_0 + T_1\widetilde{Q}_{n-1} - Q_{n-1}T_1 = 0, \end{cases} \quad (n \geq 3). \tag{4.85}$$

First, by $[T_0, P_0] = 0$ and the same argument as in the proof of Lemma 2.2, we find that $T_0(x)$ is of the form

$$T_0 = c_{0,2}(x)P_0^2 + c_{0,1}(x)P_0 + c_{0,0}(x) \tag{4.86}$$

with some holomorphic functions $c_{0,k}(x)$ $(k = 0, 1, 2)$ near $x = 0$. Note that $T_0(x)$ then automatically satisfies $[T_0, Q_0] = 0$ since $[P_0, Q_0] = 0$.

We next consider

$$
(4.87)\qquad \begin{cases} [P_0, T_1] = \dfrac{\partial T_0}{\partial x_1} - P_1 T_0 =: F_1, \\ [Q_0, T_1] = \dfrac{\partial T_0}{\partial x_2} + T_0 \dfrac{\partial P_0}{\partial x_1} - Q_1 T_0 =: F_2. \end{cases}
$$

For this system, we have

Lemma 4.8. *If F_1 and F_2 satisfy*

$$
(4.88)\qquad \begin{cases} \operatorname{tr}\left(F_1 P_0^k\right) = 0 \quad (k = 0, 1, 2), \\ F_2 = P_0 F_1 + F_1 P_0, \end{cases}
$$

then the system (4.87) *has a solution T_1.*

Proof. Let $T_1(x)$ be a 3×3 matrix with holomorphic entries near $x = 0$ defined by

$$
(4.89)\qquad T_1 = \Lambda F_1 + \Lambda^2 F_1 P_0, \quad \Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},
$$

then, by the explicit calculation of matrix entries, we find

$$
(4.90)\qquad [P_0, T_1] = F_1 - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \operatorname{tr}\left(F_1 P_0^2\right) + \dfrac{x_2}{2} \operatorname{tr}(F_1) & \operatorname{tr}(F_1 P_0) & \operatorname{tr}(F_1) \end{pmatrix}.
$$

Hence, using $\operatorname{tr}\left(F_1 P_0^k\right) = 0$ $(k = 0, 1, 2)$, we find that $T_1(x)$ satisfies the first equation of (4.87). Furthermore, using $F_2 = P_0 F_1 + F_1 P_0$, we obtain

$$
\begin{aligned}
(4.91)\qquad [Q_0, T_1] &= \left[P_0^2, T_1\right] \\
&= P_0^2 T_1 - T_1 P_0^2 \\
&= P_0 [P_0, T_1] + [P_0, T_1] P_0 \\
&= P_0 F_1 + F_1 P_0 \\
&= F_2.
\end{aligned}
$$

□

In what follows, we try to determine the matrix T_0 of the form (4.86) so that the relations (4.88) may be satisfied.

Sublemma 4.1. *There exist holomorphic functions $d_k(x)$ $(k = 0, 1, 2)$ near $x = 0$ which satisfy*

$$
(4.92)\qquad Q_1 - \frac{\partial P_0}{\partial x_1} - P_0 P_1 - P_1 P_0 = d_2(x) P_0^2 + d_1(x) P_0 + d_0(x).
$$

Proof. Using the compatibility condition of the system (4.74) and the explicit form of P_0 and Q_0, we have

$$
\begin{aligned}
(4.93)\qquad \left[P_0, Q_1 - \frac{\partial P_0}{\partial x_1} - (P_0 P_1 + P_1 P_0)\right] &= \left[P_0, Q_1 - \frac{\partial P_0}{\partial x_1}\right] - [P_0, P_0 P_1 + P_1 P_0] \\
&= [P_0, Q_1] + \frac{\partial P_0}{\partial x_2} - \frac{\partial Q_0}{\partial x_1} - [Q_0, P_1] \\
&= 0.
\end{aligned}
$$

Then, by the same argument as in the proof of Lemma 2.2, we get (4.92). □

Using this sublemma, we find

Lemma 4.9. *The relation* (4.88) *implies that* $(c_{0,0}(x), c_{0,1}(x), c_{0,2}(x))$ *satisfies the following system of partial differential equations:*

(4.94)

$$\begin{cases} \left(3x_1 + 4x_2 P_0^T\right) \dfrac{\partial}{\partial x_1} \begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{0,2} \end{pmatrix} = \left\{\operatorname{diag}(0,-1,-2) + x_1 e_0(x) - 4e_2(x)P_0 - 4e_1(x)P_0^2\right\}^T \begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{0,2} \end{pmatrix}, \\ \dfrac{\partial}{\partial x_2} \begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{0,2} \end{pmatrix} = 2P_0^T \dfrac{\partial}{\partial x_1} \begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{0,2} \end{pmatrix} + \left(d_2(x)P_0^2 + d_1(x)P_0 + d_0(x)\right)^T \begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{0,2} \end{pmatrix}, \end{cases}$$

where A^T *designates the transposed matrix of* A*,* $\operatorname{diag}(a_1, a_2, a_3)$ *is a diagonal matrix whose* (i,i) *entry is* a_i*, and* $e_k(x) = \operatorname{tr}\left(P_1 P_0^k\right)$ $(k = 0,1,2)$*. Conversely, if* $(c_{0,0}(x), c_{0,1}(x), c_{0,2}(x))$ *is a holomorphic solution of* (4.94)*, then the relation* (4.88) *holds.*

Proof. First, we consider

$$\operatorname{tr}\left(F_1 P_0^k\right) = 0 \quad (k = 0,1,2). \tag{4.95}$$

By the explicit form of P_0, we have

(4.96)

$$\operatorname{tr}(P_0) = 0, \quad \operatorname{tr}\left(P_0^2\right) = -x_2, \quad \operatorname{tr}\left(\frac{\partial P_0}{\partial x_1}\right) = 0, \quad \operatorname{tr}\left(P_0 \frac{\partial P_0}{\partial x_1}\right) = 0, \quad \operatorname{tr}\left(P_0^2 \frac{\partial P_0}{\partial x_1}\right) = -\frac{1}{4}.$$

It follows from these relations that

(4.97)

$$\operatorname{tr}(F_1) = -x_2 \frac{\partial c_{0,2}}{\partial x_1} + 3\frac{\partial c_{0,0}}{\partial x_1} - e_2(x)c_{0,2} - e_1(x)c_{0,1} - e_0(x)c_{0,0},$$

(4.98)

$$\operatorname{tr}(F_1 P_0) = -\frac{3}{4}x_1 \frac{\partial c_{0,2}}{\partial x_1} - x_2 \frac{\partial c_{0,1}}{\partial x_1} - \left(\frac{1}{2} - \frac{x_2}{2}e_1(x) - \frac{x_1}{4}e_0(x)\right) c_{0,2} - e_2(x)c_{0,1} - e_1(x)c_{0,0},$$

(4.99)

$$\begin{aligned} \operatorname{tr}\left(F_1 P_0^2\right) =& \frac{x_2^2}{2}\frac{\partial c_{0,2}}{\partial x_1} - \frac{3}{4}x_1 \frac{\partial c_{0,1}}{\partial x_1} - x_2 \frac{\partial c_{0,0}}{\partial x_1} \\ & - \left(-\frac{x_2}{2}e_2(x) - \frac{x_1}{4}e_1(x)\right) c_{0,2} - \left(\frac{1}{4} - \frac{x_2}{2}e_1(x) - \frac{x_1}{4}e_0(x)\right) c_{0,1} - e_2(x)c_{0,0}. \end{aligned}$$

If we multiply (4.97) (resp., (4.98), (4.99)) by x_1 (resp., (-4), (-4)), we get the first equation of (4.94).

Next, we consider

$$F_2 = P_0 F_1 + F_1 P_0. \tag{4.100}$$

Using (4.92), we find that this relation is equivalent to

(4.101)

$$\begin{aligned} \frac{\partial T_0}{\partial x_2} + \left[T_0, \frac{\partial P_0}{\partial x_1}\right] - P_0 \frac{\partial T_0}{\partial x_1} - \frac{\partial T_0}{\partial x_1} P_0 &= \left(F_2 + Q_1 T_0 - \frac{\partial P_0}{\partial x_1} T_0\right) - P_0\left(F_1 + P_1 T_0\right) - \left(F_1 + P_1 T_0\right) P_0 \\ &= \left(Q_1 - \frac{\partial P_0}{\partial x_1} - P_0 P_1 - P_1 P_0\right) T_0 \end{aligned}$$

$$= \left(d_2(x)P_0^2 + d_1(x)P_0 + d_0(x)\right) T_0.$$

Note that

$$\frac{\partial P_0}{\partial x_2} - 2\frac{\partial P_0}{\partial x_1} P_0 = 0 \tag{4.102}$$

holds by the explicit form of P_0. Hence, as

$$P_0^3 + \frac{x_2}{2} P_0 + \frac{x_1}{4} = 0 \tag{4.103}$$

holds, the left-hand side of (4.101) becomes

$$\begin{aligned}
&\frac{\partial T_0}{\partial x_2} + \left[T_0, \frac{\partial P_0}{\partial x_1}\right] - P_0 \frac{\partial T_0}{\partial x_1} - \frac{\partial T_0}{\partial x_1} P_0 \\
&= \left(\frac{\partial c_{0,2}}{\partial x_2} - 2\frac{\partial c_{0,1}}{\partial x_1}\right) P_0^2 + \left(\frac{\partial c_{0,1}}{\partial x_2} + x_2 \frac{\partial c_{0,2}}{\partial x_1} - 2\frac{\partial c_{0,0}}{\partial x_1}\right) P_0 + \left(\frac{\partial c_{0,0}}{\partial x_2} + \frac{x_1}{2}\frac{\partial c_{0,2}}{\partial x_1}\right) \\
&\quad + c_{0,2}(x) \left\{ \left(\frac{\partial P_0}{\partial x_2} - 2\frac{\partial P_0}{\partial x_1} P_0\right) P_0 + P_0 \left(\frac{\partial P_0}{\partial x_2} - 2\frac{\partial P_0}{\partial x_1} P_0\right)\right\} + c_{0,1}(x) \left(\frac{\partial P_0}{\partial x_2} - 2\frac{\partial P_0}{\partial x_1} P_0\right) \\
&= \left(\frac{\partial c_{0,2}}{\partial x_2} - 2\frac{\partial c_{0,1}}{\partial x_1}\right) P_0^2 + \left(\frac{\partial c_{0,1}}{\partial x_2} + x_2 \frac{\partial c_{0,2}}{\partial x_1} - 2\frac{\partial c_{0,0}}{\partial x_1}\right) P_0 + \left(\frac{\partial c_{0,0}}{\partial x_2} + \frac{x_1}{2}\frac{\partial c_{0,2}}{\partial x_1}\right).
\end{aligned} \tag{4.104}$$

On the other hand, in view of (4.103), we can rewrite the right-hand side of (4.101) as

$$\begin{aligned}
&\left(d_2(x)P_0^2 + d_1(x)P_0 + d_0(x)\right) T_0 \\
&= \left\{ \left(d_0(x) - \frac{x_2}{2} d_2(x)\right) c_{0,2} + d_1(x) c_{0,1} + d_2(x) c_{0,0} \right\} P_0^2 \\
&\quad + \left\{ \left(-\frac{x_2}{2} d_1(x) - \frac{x_1}{4} d_2(x)\right) c_{0,2} + \left(d_0(x) - \frac{x_2}{2} d_2(x)\right) c_{0,1} + d_1(x) c_{0,0} \right\} P_0 \\
&\quad + \left\{ \left(-\frac{x_1}{4} d_1(x)\right) c_{0,2} + \left(-\frac{x_1}{4} d_2(x)\right) c_{0,1} + d_0(x) c_{0,0} \right\}.
\end{aligned} \tag{4.105}$$

Comparing (4.104) and (4.105), we obtain

$$\frac{\partial c_{0,2}}{\partial x_2} - 2\frac{\partial c_{0,1}}{\partial x_1} = \left(d_0(x) - \frac{x_2}{2} d_2(x)\right) c_{0,2} + d_1(x) c_{0,1} + d_2(x) c_{0,0}, \tag{4.106}$$

$$\frac{\partial c_{0,1}}{\partial x_2} + x_2 \frac{\partial c_{0,2}}{\partial x_1} - 2\frac{\partial c_{0,0}}{\partial x_1} = \left(-\frac{x_2}{2} d_1(x) - \frac{x_1}{4} d_2(x)\right) c_{0,2} + \left(d_0(x) - \frac{x_2}{2} d_2(x)\right) c_{0,1} + d_1(x) c_{0,0}, \tag{4.107}$$

$$\frac{\partial c_{0,0}}{\partial x_2} + \frac{x_1}{2}\frac{\partial c_{0,2}}{\partial x_1} = \left(-\frac{x_1}{4} d_1(x)\right) c_{0,2} + \left(-\frac{x_1}{4} d_2(x)\right) c_{0,1} + d_0(x) c_{0,0}. \tag{4.108}$$

Thus we also get the second equation of (4.94).

The "converse" part is clear from the above argument. □

We now prove the existence of a holomorphic solution of the system (4.94).

Lemma 4.10. *Let*

$$3x_1 \frac{d}{dx_1} \begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{0,2} \end{pmatrix} = \left\{ \mathrm{diag}\,(0, -1, -2) + x_1 e_0(x) - 4e_2(x) P_0 - 4e_1(x) P_0^2 \right\}^T \Big|_{x_2=0} \begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{0,2} \end{pmatrix} \tag{4.109}$$

be a system of ordinary differential equations obtained by restricting the first equation of (4.94) *to* $x_2 = 0$*. Then* (4.109) *has a holomorphic solution* $(c_{0,0}(x_1,0), c_{0,1}(x_1,0), c_{0,2}(x_1,0))$ *near* $x_1 = 0$ *satisfying*

$$\begin{pmatrix} c_{0,0}(0) \\ c_{0,1}(0) \\ c_{0,2}(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -4e_2(0) \\ -2e_1(0) + 8e_2(0)^2 \end{pmatrix}. \tag{4.110}$$

In the proof of this lemma we use the following

Theorem 4.2. ([9]) *Let* $A(t)$ *be an* $n \times n$ *matrix with holomorphic entries near* $t = 0$*. Suppose that* $A(0)$ *has no eigenvalues that differ from each other by positive integers. Then there exists an* $n \times n$ *matrix* $B(t)$ *with holomorphic entries near* $t = 0$ *with* $B(0) = I$*, such that a transformation*

$$W = B(t)\widetilde{W} \tag{4.111}$$

reduces a system of ordinary differential equations

$$t\frac{dW}{dt} = A(t)W \tag{4.112}$$

to the following system of ordinary differential equations

$$t\frac{d\widetilde{W}}{dt} = A(0)\widetilde{W}. \tag{4.113}$$

Proof of Lemma 4.10. By the explicit form of P_0, the eigenvalues of $(1/3)\left\{\mathrm{diag}\,(0,-1,-2) + x_1 e_0(x) - 4e_2(x)P_0 - 4e_1(x)P_0^2\right\}^T\big|_{x=0}$ are given by 0, $-1/3$ and $-2/3$. Hence, applying Theorem 4.2, we find a 3×3 matrix $B(x_1)$ which satisfy the properties in Theorem 4.2. By the transformation

$$\begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{0,2} \end{pmatrix} = B(x_1)\widetilde{\begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{0,2} \end{pmatrix}}, \tag{4.114}$$

the system (4.109) is transformed into the following system:

$$x_1\frac{d}{dx_1}\widetilde{\begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{0,2} \end{pmatrix}} = \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 \\ -4e_2(0) & -1 & 0 \\ -4e_1(0) & -4e_2(0) & -2 \end{pmatrix}\widetilde{\begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{0,2} \end{pmatrix}}. \tag{4.115}$$

Since

$$\widetilde{\begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{0,2} \end{pmatrix}} = \begin{pmatrix} 1 \\ -4e_2(0) \\ -2e_1(0) + 8e_2(0)^2 \end{pmatrix} \tag{4.116}$$

is a solution of the system (4.115),

$$\begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{0,2} \end{pmatrix} = B(x_1)\begin{pmatrix} 1 \\ -4e_2(0) \\ -2e_1(0) + 8e_2(0)^2 \end{pmatrix} \tag{4.117}$$

is a solution of the system (4.109). Since $B(0) = I$, (4.117) also satisfies (4.110). □

Lemma 4.11. *Let $(c_{0,0}(x_1,0), c_{0,1}(x_1,0), c_{0,2}(x_1,0))$ be a holomorphic solution near $x_1 = 0$ given in Lemma 4.10. Then the following initial value problem*

$$\left\{\begin{aligned} &\frac{\partial}{\partial x_2}\begin{pmatrix} c_{0,0}\\ c_{0,1}\\ c_{0,2}\end{pmatrix} = 2P_0^T\frac{\partial}{\partial x_1}\begin{pmatrix} c_{0,0}\\ c_{0,1}\\ c_{0,2}\end{pmatrix} + \left(d_2(x)P_0^2 + d_1(x)P_0 + d_0(x)\right)^T\begin{pmatrix} c_{0,0}\\ c_{0,1}\\ c_{0,2}\end{pmatrix},\\ &\begin{pmatrix} c_{0,0}\\ c_{0,1}\\ c_{0,2}\end{pmatrix}\Big|_{x_2=0} = \begin{pmatrix} c_{0,0}(x_1,0)\\ c_{0,1}(x_1,0)\\ c_{0,2}(x_1,0)\end{pmatrix}\end{aligned}\right. \tag{4.118}$$

has a holomorphic solution near $x = 0$.

This lemma is an immediate consequence of the Cauchy-Kowalevski theorem.

Furthermore, we can verify

Lemma 4.12. *The holomorphic function $(c_{0,0}(x), c_{0,1}(x), c_{0,2}(x))$ given by Lemma 4.11 satisfies the first equation of* (4.94).

Proof. By the argument of Lemma 4.9, it suffices to prove that

$$T_0 = c_{0,2}(x)P_0^2 + c_{0,1}(x)P_0 + c_{0,0}(x) \tag{4.119}$$

satisfies

$$\mathrm{tr}\left\{\left(\frac{\partial T_0}{\partial x_1} - P_1T_0\right)P_0^k\right\} = 0 \quad (k = 0,1,2). \tag{4.120}$$

Let $f_{0,k}(x)$ $(k = 0,1,2)$ be holomorphic functions defined by

$$f_{0,k} = \mathrm{tr}\left\{\left(\frac{\partial T_0}{\partial x_1} - P_1T_0\right)P_0^k\right\} \quad (k = 0,1,2). \tag{4.121}$$

We prove that $(f_{0,0}(x), f_{0,1}(x), f_{0,2}(x))$ satisfies the following system of partial differential equations

$$\frac{\partial}{\partial x_2}\begin{pmatrix} f_{0,0}\\ f_{0,1}\\ f_{0,2}\end{pmatrix} = 2\frac{\partial}{\partial x_1}\left\{P_0\begin{pmatrix} f_{0,0}\\ f_{0,1}\\ f_{0,2}\end{pmatrix}\right\} + \left(d_2(x)P_0^2 + d_1(x)P_0 + d_0(x)\right)\begin{pmatrix} f_{0,0}\\ f_{0,1}\\ f_{0,2}\end{pmatrix}. \tag{4.122}$$

For $k = 0,1,2$, we have

$$\begin{aligned} \mathrm{tr}\left([P_2,Q_0]\,T_0P_0^k\right) =& \mathrm{tr}\left(P_2Q_0T_0P_0^k - Q_0P_2T_0P_0^k\right)\\ =& \mathrm{tr}\left[P_2T_0P_0^k, Q_0\right]\\ =& 0, \end{aligned} \tag{4.123}$$

$$\begin{aligned} \mathrm{tr}\left([P_0,Q_2]\,T_0P_0^k\right) =& \mathrm{tr}\left(P_0Q_2T_0P_0^k - Q_2P_0T_0P_0^k\right)\\ =& \mathrm{tr}\left[P_0, Q_2T_0P_0^k\right]\\ =& 0 \end{aligned} \tag{4.124}$$

in view of $[P_0,Q_0] = 0$ and (4.86). Furthermore, it follows from (4.92) that

$$\mathrm{tr}\left([P_1,Q_1]\,T_0P_0^k\right) = \mathrm{tr}\left(\left[P_1, \frac{\partial P_0}{\partial x_1} + P_0P_1 + P_1P_0 + d_2(x)P_0^2 + d_1(x)P_0 + d_0(x)\right]T_0P_0^k\right) \tag{4.125}$$

$$
\begin{aligned}
=&\mathrm{tr}\left(\left[P_1, \frac{\partial P_0}{\partial x_1} + d_2(x)P_0^2 + d_1(x)P_0 + d_0(x)\right] T_0 P_0^k\right)\\
&+ \mathrm{tr}\left\{\left(P_1P_0P_1 + P_1^2P_0 - P_0P_1^2 - P_1P_0P_1\right) T_0P_0^k\right\}\\
=&\mathrm{tr}\left(\left[P_1, \frac{\partial P_0}{\partial x_1} + d_2(x)P_0^2 + d_1(x)P_0 + d_0(x)\right] T_0 P_0^k\right)\\
&+ \mathrm{tr}\left[P_1^2T_0P_0^k, P_0\right]\\
=&\mathrm{tr}\left(\left[P_1, \frac{\partial P_0}{\partial x_1} + d_2(x)P_0^2 + d_1(x)P_0 + d_0(x)\right] T_0 P_0^k\right).
\end{aligned}
$$

By using these relations, the compatibility conditions, (4.92), (4.101) and (4.102), we obtain

(4.126)

$$
\begin{aligned}
\frac{\partial f_{0,0}}{\partial x_2} =&\mathrm{tr}\left(\frac{\partial^2 T_0}{\partial x_1 \partial x_2} - \frac{\partial P_1}{\partial x_2}T_0 - P_1\frac{\partial T_0}{\partial x_2}\right)\\
=&\mathrm{tr}\left[\frac{\partial}{\partial x_1}\left\{-\left[T_0, \frac{\partial P_0}{\partial x_1}\right] + P_0\frac{\partial T_0}{\partial x_1} + \frac{\partial T_0}{\partial x_1}P_0 + \left(Q_1 - \frac{\partial P_0}{\partial x_1} - P_0P_1 - P_1P_0\right)T_0\right\}\right.\\
&+\left\{[P_2, Q_0] + [P_1, Q_1] + [P_0, Q_2] - \frac{\partial Q_1}{\partial x_1}\right\}T_0\\
&\left.-P_1\left\{-\left[T_0, \frac{\partial P_0}{\partial x_1}\right] + P_0\frac{\partial T_0}{\partial x_1} + \frac{\partial T_0}{\partial x_1}P_0 + \left(Q_1 - \frac{\partial P_0}{\partial x_1} - P_0P_1 - P_1P_0\right)T_0\right\}\right]\\
=&2\frac{\partial}{\partial x_1}\mathrm{tr}\left\{\left(\frac{\partial T_0}{\partial x_1} - P_1T_0\right)P_0\right\} + \mathrm{tr}\left\{\left(\frac{\partial T_0}{\partial x_1} - P_1T_0\right)\left(d_2(x)P_0^2 + d_1(x)P_0 + d_0(x)\right)\right\}\\
=&2\frac{\partial f_{0,1}}{\partial x_1} + d_2(x)f_{0,2} + d_1(x)f_{0,1} + d_0(x)f_{0,0},
\end{aligned}
$$

(4.127)

$$
\begin{aligned}
\frac{\partial f_{0,1}}{\partial x_2} =&\mathrm{tr}\left\{\left(\frac{\partial^2 T_0}{\partial x_1 \partial x_2} - \frac{\partial P_1}{\partial x_2}T_0 - P_1\frac{\partial T_0}{\partial x_2}\right)P_0 + \left(\frac{\partial T_0}{\partial x_1} - P_1T_0\right)\frac{\partial P_0}{\partial x_2}\right\}\\
=&\mathrm{tr}\left[\frac{\partial}{\partial x_1}\left\{-\left[T_0, \frac{\partial P_0}{\partial x_1}\right] + P_0\frac{\partial T_0}{\partial x_1} + \frac{\partial T_0}{\partial x_1}P_0 + \left(Q_1 - \frac{\partial P_0}{\partial x_1} - P_0P_1 - P_1P_0\right)T_0\right\}P_0\right.\\
&+\left\{[P_2, Q_0] + [P_1, Q_1] + [P_0, Q_2] - \frac{\partial Q_1}{\partial x_1}\right\}T_0P_0\\
&-P_1\left\{-\left[T_0, \frac{\partial P_0}{\partial x_1}\right] + P_0\frac{\partial T_0}{\partial x_1} + \frac{\partial T_0}{\partial x_1}P_0 + \left(Q_1 - \frac{\partial P_0}{\partial x_1} - P_0P_1 - P_1P_0\right)T_0\right\}P_0\\
&\left.+\left(\frac{\partial T_0}{\partial x_1} - P_1T_0\right)\left(2\frac{\partial P_0}{\partial x_1}P_0\right)\right]\\
=&2\frac{\partial}{\partial x_1}\mathrm{tr}\left\{\left(\frac{\partial T_0}{\partial x_1} - P_1T_0\right)P_0^2\right\} + \mathrm{tr}\left\{\left(\frac{\partial T_0}{\partial x_1} - P_1T_0\right)\left(d_2(x)P_0^3 + d_1(x)P_0^2 + d_0(x)P_0\right)\right\}\\
=&2\frac{\partial f_{0,2}}{\partial x_1} + d_1(x)f_{0,2} + \left(-\frac{x_2}{2}d_2(x) + d_0(x)\right)f_{0,1} + \left(-\frac{x_1}{4}d_2(x)\right)f_{0,0},
\end{aligned}
$$

(4.128)

$$
\begin{aligned}
\frac{\partial f_{0,2}}{\partial x_2} =&\mathrm{tr}\left\{\left(\frac{\partial^2 T_0}{\partial x_1 \partial x_2} - \frac{\partial P_1}{\partial x_2}T_0 - P_1\frac{\partial T_0}{\partial x_2}\right)P_0^2 + \left(\frac{\partial T_0}{\partial x_1} - P_1T_0\right)\left(\frac{\partial P_0}{\partial x_2}P_0 + P_0\frac{\partial P_0}{\partial x_2}\right)\right\}\\
=&\mathrm{tr}\left[\frac{\partial}{\partial x_1}\left\{-\left[T_0, \frac{\partial P_0}{\partial x_1}\right] + P_0\frac{\partial T_0}{\partial x_1} + \frac{\partial T_0}{\partial x_1}P_0 + \left(Q_1 - \frac{\partial P_0}{\partial x_1} - P_0P_1 - P_1P_0\right)T_0\right\}P_0^2\right.\\
&+\left\{[P_2, Q_0] + [P_1, Q_1] + [P_0, Q_2] - \frac{\partial Q_1}{\partial x_1}\right\}T_0P_0^2\\
&-P_1\left\{-\left[T_0, \frac{\partial P_0}{\partial x_1}\right] + P_0\frac{\partial T_0}{\partial x_1} + \frac{\partial T_0}{\partial x_1}P_0 + \left(Q_1 - \frac{\partial P_0}{\partial x_1} - P_0P_1 - P_1P_0\right)T_0\right\}P_0^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{\partial T_0}{\partial x_1}-P_1T_0\right)\left(2\frac{\partial P_0}{\partial x_1}P_0^2+2P_0\frac{\partial P_0}{\partial x_1}P_0\right)\Bigg]\\
=&2\frac{\partial}{\partial x_1}\mathrm{tr}\left\{\left(\frac{\partial T_0}{\partial x_1}-P_1T_0\right)P_0^3\right\}+\mathrm{tr}\left\{\left(\frac{\partial T_0}{\partial x_1}-P_1T_0\right)\left(d_2(x)P_0^4+d_1(x)P_0^3+d_0(x)P_0^2\right)\right\}\\
=&-x_2\frac{\partial f_{0,1}}{\partial x_1}-\frac{x_1}{2}\frac{\partial f_{0,0}}{\partial x_1}\\
&+\left(-\frac{x_2}{2}d_2(x)+d_0(x)\right)f_{0,2}+\left(-\frac{x_1}{4}d_2(x)-\frac{x_2}{2}d_1(x)\right)f_{0,1}+\left(-\frac{x_1}{4}d_1(x)-\frac{1}{2}\right)f_{0,0}.
\end{aligned}
$$

Thus we get (4.122).

Note that the system (4.109) means $f_{0,k}(x)$ $(k=0,1,2)$ vanishes on $\{(x_1,0)\ ;\ x_1\neq 0\}$. Thus, by Lemmas 4.10 and 4.11 and the holomorphy of $f_{0,k}(x)$, we find that

$$
\begin{pmatrix} f_{0,0}\\ f_{0,1}\\ f_{0,2}\end{pmatrix}\Big|_{x_2=0}=0. \tag{4.129}
$$

Hence, by the uniqueness of solutions for the initial value problem (4.122) and (4.129), we obtain (4.120). □

We have thus obtained a holomorphic solution of (4.94) near $x=0$, that is, we have constructed a 3×3 matrix $T_0(x)$ with holomorphic entries satisfying the relation (4.88). Using this $T_0(x)$, we prove Proposition 4.2.

Proof of Proposition 4.2. Let $T_0(x)$ be a 3×3 matrix thus obtained. By (4.110), we have

$$
\det T_0(0)=\det\left(c_{0,2}(0)P_0(0)^2+c_{0,1}(0)P_0(0)+c_{0,0}(0)\right)=1. \tag{4.130}
$$

Hence, there exists some positive constant $0<\rho<\rho_0$ such that $T_0(x)$ is holomorphic in D_ρ and $\det T_0(x)\neq 0$ $(x\in D_\rho)$.

Next, define a 3×3 matrix $T_1(x)$ by

$$
T_1=\Lambda\left(\frac{\partial T_0}{\partial x_1}-P_1T_0\right)+\Lambda^2\left(\frac{\partial T_0}{\partial x_1}-P_1T_0\right)P_0,\quad \Lambda=\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}. \tag{4.131}
$$

Then $T_1(x)$ is holomorphic in D_ρ and it follows from the argument in the proof of Lemma 4.8 that $T_1(x)$ satisfies the relation (4.83).

Furthermore, define 3×3 matrices $\widetilde{P}_n(x)$ and $\widetilde{Q}_n(x)$ as follows:

$$
\begin{cases}
\widetilde{P}_2=T_0^{-1}\left(P_2T_0+P_1T_1-\dfrac{\partial T_1}{\partial x_1}\right),\\
\widetilde{Q}_2=T_0^{-1}\left(Q_2T_0+Q_1T_1-T_1\dfrac{\partial P_0}{\partial x_1}-\dfrac{\partial T_1}{\partial x_2}\right),
\end{cases} \tag{4.132}
$$

$$
\begin{cases}
\widetilde{P}_n=T_0^{-1}\left(P_nT_0-T_1\widetilde{P}_{n-1}+P_{n-1}T_1\right),\\
\widetilde{Q}_n=T_0^{-1}\left(Q_nT_0-T_1\widetilde{Q}_{n-1}+Q_{n-1}T_1\right),
\end{cases}\qquad (n\geq 3). \tag{4.133}
$$

Then these matrices are holomorphic in D_ρ and satisfy the relations (4.84) and (4.85).

Finally, we prove the estimate (4.80) for $\widetilde{P}_n(x)$ and $\widetilde{Q}_n(x)$. Since $T_i(x)$ $(i=0,1)$, $T_0(x)^{-1}$, $\widetilde{P}_i(x)$ $(i=0,1,2)$ and $\widetilde{Q}_i(x)$ $(i=0,1,2)$ are holomorphic in D_ρ, we have

$$
\|T_i\|_{\rho,\rho},\ \left\|T_0^{-1}\right\|_{\rho,\rho},\ \left\|\widetilde{P}_i\right\|_{\rho,\rho},\ \left\|\widetilde{Q}_i\right\|_{\rho,\rho}\leq C^{\dagger} \tag{4.134}
$$

with some positive constant $C^\dagger > 0$. Then, if we define positive constants $\widetilde{C}$ and $\widetilde{\alpha}$ by

$$\widetilde{C} = \max\left\{C^\dagger,\ C(\alpha+1)\right\}, \quad \widetilde{\alpha} = \max\left\{1,\ \alpha,\ 2\left(C^\dagger\right)^2\right\}, \tag{4.135}$$

the estimate (4.80) holds. In fact, we have

$$\left\|\widetilde{P}_0\right\|_{\rho,\rho}, \left\|\widetilde{Q}_0\right\|_{\rho,\rho} \le C^\dagger \le \widetilde{C}, \tag{4.136}$$

$$\left\|\widetilde{P}_1\right\|_{\rho,\rho}, \left\|\widetilde{Q}_1\right\|_{\rho,\rho} \le C^\dagger \le \widetilde{C}\widetilde{\alpha}, \tag{4.137}$$

$$\left\|\widetilde{P}_2\right\|_{\rho,\rho}, \left\|\widetilde{Q}_2\right\|_{\rho,\rho} \le C^\dagger \le \widetilde{C}\widetilde{\alpha}^2 2!, \tag{4.138}$$

and, using the definition (4.133) of $\widetilde{P}_n(x)$ and $\widetilde{Q}_n(x)$ and an induction on $n \ge 3$, we obtain

$$\begin{aligned} \left\|\widetilde{P}_n\right\|_{\rho,\rho}, \left\|\widetilde{Q}_n\right\|_{\rho,\rho} &\le \left(C^\dagger\right)^2 \left(C\alpha^n n! + \widetilde{C}\widetilde{\alpha}^{n-1}(n-1)! + C\alpha^{n-1}(n-1)!\right) \\ &\le \frac{\widetilde{\alpha}}{2}\left(C\alpha + \widetilde{C} + C\right)\widetilde{\alpha}^{n-1} n! \\ &\le \widetilde{C}\widetilde{\alpha}^n n!. \end{aligned} \tag{4.139}$$

□

4.3 Construction of the transformation, III

Thanks to the argument so far, we may assume that the completely integrable system in question has the following form

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial x_1}\Psi = P(x,\eta)\Psi, \quad P(x,\eta) = \displaystyle\sum_{n\ge 0}\eta^{-n}P_n(x), \\ \eta^{-1}\dfrac{\partial}{\partial x_2}\Psi = Q(x,\eta)\Psi, \quad Q(x,\eta) = \displaystyle\sum_{n\ge 0}\eta^{-n}Q_n(x). \end{cases} \tag{4.140}$$

Here

$$P_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -x_1/4 & -x_2/2 & 0 \end{pmatrix}, \quad P_1 = 0, \quad Q_0 = P_0^2 + \frac{x_2}{3}, \quad Q_1 = \frac{\partial P_0}{\partial x_1} \tag{4.141}$$

and $P_n(x)$ and $Q_n(x)$ are 3×3 matrices with holomorphic entries in D_ρ satisfying an estimate

$$\|P_n\|_{\rho,\rho}, \|Q_n\|_{\rho,\rho} \le C\alpha^n n! \quad (n \ge 0) \tag{4.142}$$

with some positive constants C and α.

In this subsection, we prove

Proposition 4.3. *There exists an infinite series of 3×3 matrices $\{T_n(x)\}_{n\ge 0}$ which satisfies the following properties:*

- *Every entry of $T_n(x)$ is holomorphic in D_ρ and $\det T_0(x) \ne 0$ $(x \in D_\rho)$.*
- *$T_n(x)$ satisfies*

  $$\|T_n\|_{\rho,\rho} \le \widetilde{C}\widetilde{\alpha}^n n! \tag{4.143}$$

  with some positive constants $\widetilde{C}$ and $\widetilde{\alpha}$.

- *By a formal transformation*

$$\Psi(x,\eta) = T(x,\eta)\widetilde{\Psi}(x,\eta), \quad T(x,\eta) = \sum_{n\geq 0} \eta^{-n} T_n(x), \tag{4.144}$$

(4.140) *is transformed into the following form:*

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial x_1}\widetilde{\Psi} = \widetilde{P}(x,\eta)\widetilde{\Psi}, & \widetilde{P}(x,\eta) = \displaystyle\sum_{n\geq 0}\eta^{-n}\widetilde{P}_n(x), \\ \eta^{-1}\dfrac{\partial}{\partial x_2}\widetilde{\Psi} = \widetilde{Q}(x,\eta)\widetilde{\Psi}, & \widetilde{Q}(x,\eta) = \displaystyle\sum_{n\geq 0}\eta^{-n}\widetilde{Q}_n(x), \end{cases} \tag{4.145}$$

where $\widetilde{P}_n(x)$ *and* $\widetilde{Q}_n(x)$ *are given as follows:*

$$\widetilde{P}_0 = P_0, \quad \widetilde{P}_n = 0 \quad (n \geq 1), \tag{4.146}$$

$$\widetilde{Q}_0 = Q_0, \quad \widetilde{Q}_1 = Q_1, \quad \widetilde{Q}_n = 0 \quad (n \geq 2). \tag{4.147}$$

In this subsection we only discuss the construction of $\{T_n(x)\}_{n\geq 0}$. The estimate (4.143) will be proved in the next section.

Since the system (4.145) is obtained from (4.140) through the transformation (4.144), $T(x,\eta)$ should satisfy the following relation:

$$\begin{cases} T\widetilde{P} + \eta^{-1}\dfrac{\partial T}{\partial x_1} = PT, \\ T\widetilde{Q} + \eta^{-1}\dfrac{\partial T}{\partial x_2} = QT, \end{cases} \tag{4.148}$$

that is,

$$\begin{cases} [P_0, T_n] = F_{n,1}, \\ [Q_0, T_n] = F_{n,2}, \end{cases} \qquad (n \geq 0), \tag{E_n}$$

where

$$F_{0,1} = F_{0,2} = 0, \tag{4.149}$$

$$\begin{cases} F_{1,1} = \dfrac{\partial T_0}{\partial x_1}, \\ F_{1,2} = \dfrac{\partial T_0}{\partial x_2} + \left[T_0, \dfrac{\partial P_0}{\partial x_1}\right], \end{cases} \tag{4.150}$$

$$\begin{cases} F_{n,1} = \dfrac{\partial T_{n-1}}{\partial x_1} - \displaystyle\sum_{i=0}^{n-2} P_{n-i}T_i, \\ F_{n,2} = \dfrac{\partial T_{n-1}}{\partial x_2} - \displaystyle\sum_{i=0}^{n-2} Q_{n-i}T_i + \left[T_{n-1}, \dfrac{\partial P_0}{\partial x_1}\right], \end{cases} \qquad (n \geq 2). \tag{4.151}$$

We now discuss the construction of $\{T_n(x)\}_{n\geq 0}$ satisfying the relations (E_n). Having Lemma 4.8 in mind, we prove the following assertions $(\mathcal{S}_n)$ $(n = 0, 1, 2, \cdots)$ by an induction on n.

$(\mathcal{S}_n)$ There exist 3×3 matrices $T_i(x)$ for $i = 0, 1, \cdots, n$ that satisfy the following:

- $T_i(x)$ $(i = 0, 1, \cdots, n)$ are holomorphic in D_ρ.
- The relations (E_j) hold for $j = 0, 1, \cdots, n$.

- $\operatorname{tr}\left(F_{n+1,1}P_0^k\right) = 0 \quad (k = 0, 1, 2)$.
- $F_{n+1,2} = P_0 F_{n+1,1} + F_{n+1,1} P_0$.

We trivially have $(\mathcal{S}_0)$ by setting $T_0(x) = I$. Let us prove $(\mathcal{S}_n)$ provided that $(\mathcal{S}_{n-1})$ should hold. For this purpose it suffices to construct $T_n(x)$ that satisfies the following conditions:

- $T_n(x)$ is holomorphic in D_ρ.
- The relation E_n holds.
- $\operatorname{tr}\left(F_{n+1,1}P_0^k\right) = 0 \quad (k = 0, 1, 2)$.
- $F_{n+1,2} = P_0 F_{n+1,1} + F_{n+1,1} P_0$.

It follows from the induction hypothesis that $F_{n,1}$ is holomorphic in D_ρ and satisfies:

$$\begin{cases} \operatorname{tr}\left(F_{n,1}P_0^k\right) = 0 \quad (k = 0, 1, 2), \\ F_{n,2} = P_0 F_{n,1} + F_{n,1} P_0. \end{cases} \tag{4.152}$$

We now define a 3×3 matrix $T_n(x)$ as follows:

$$T_n = T_{n,1} + T_{n,2}, \tag{4.153}$$
$$T_{n,1} = c_{n,2}(x) P_0^2 + c_{n,1}(x) P_0 + c_{n,0}(x), \tag{4.154}$$
$$T_{n,2} = \Lambda F_{n,1} + \Lambda^2 F_{n,1} P_0, \tag{4.155}$$

where $c_{n,k}(x)$ $(k = 0, 1, 2)$ are holomorphic functions in D_ρ which will be determined later. Then $T_n(x)$ is holomorphic in D_ρ and satisfies

$$[P_0, T_n] = [P_0, T_{n,2}] = F_{n,1}, \tag{4.156}$$
$$[Q_0, T_n] = [Q_0, T_{n,2}] = F_{n,2} \tag{4.157}$$

by the same argument as in the proof of Lemma 4.8. Hence (E_n) is satisfied. Thus, in what follows, we determine holomorphic functions $c_{n,k}(x)$ $(k = 0, 1, 2)$ in D_ρ so that

$$\begin{cases} \operatorname{tr}\left(F_{n+1,1}P_0^k\right) = 0 \quad (k = 0, 1, 2), \\ F_{n+1,2} = P_0 F_{n+1,1} + F_{n+1,1} P_0 \end{cases} \tag{4.158}$$

may be satisfied.

Let $F^\dagger_{n+1,1}$ and $F^\dagger_{n+1,2}$ be 3×3 matrices defined by

$$F^\dagger_{n+1,1} = F_{n+1,1} - \frac{\partial T_{n,1}}{\partial x_1}, \tag{4.159}$$
$$F^\dagger_{n+1,2} = F_{n+1,2} - \frac{\partial T_{n,1}}{\partial x_2} - \left[T_{n,1}, \frac{\partial P_0}{\partial x_1}\right], \tag{4.160}$$

that is, $F^\dagger_{n+1,1}$ (resp., $F^\dagger_{n+1,2}$) is the part of $F_{n+1,1}$ (resp., $F_{n+1,2}$) which is independent of $T_{n,1}$. For $F^\dagger_{n+1,1}$ and $F^\dagger_{n+1,1}$, we have

Sublemma 4.2. *There exist holomorphic functions $g_k(x)$ $(k = 0, 1, 2)$ in D_ρ which satisfy*

$$-F^\dagger_{n+1,2} + F^\dagger_{n+1,1} P_0 + P_0 F^\dagger_{n+1,1} = g_2(x) P_0^2 + g_1(x) P_0 + g_0(x). \tag{4.161}$$

Proof. By the definition, $F^{\dagger}_{n+1,1}$ and $F^{\dagger}_{n+1,2}$ have the following forms

$$F^{\dagger}_{n+1,1} = \frac{\partial T_{n,2}}{\partial x_1} - \sum_{i=0}^{n-1} P_{n+1-i} T_i, \tag{4.162}$$

$$F^{\dagger}_{n+1,2} = \frac{\partial T_{n,2}}{\partial x_2} - \sum_{i=0}^{n-1} Q_{n+1-i} T_i + \left[T_{n,2}, \frac{\partial P_0}{\partial x_1}\right]. \tag{4.163}$$

Using the compatibility condition, (E_j) $(j = 0, 1, \cdots, n-1)$, (4.156) and (4.157), we have

(4.164)

$$\begin{aligned}
&\left[-F^{\dagger}_{n+1,2} + F^{\dagger}_{n+1,1} P_0 + P_0 F^{\dagger}_{n+1,1}, P_0\right] \\
=&\left[P_0, \frac{\partial T_{n,2}}{\partial x_2}\right] + \left[\frac{\partial T_{n,2}}{\partial x_1}, Q_0\right] + P_0 \left[T_{n,2}, \frac{\partial P_0}{\partial x_1}\right] - \left[T_{n,2}, \frac{\partial P_0}{\partial x_1}\right] P_0 \\
&- \sum_{i=0}^{n-1} P_0 Q_{n+1-i} T_i + \sum_{i=0}^{n-1} Q_0 P_{n+1-i} T_i - \sum_{i=0}^{n-1} P_{n+1-i} T_i Q_0 + \sum_{i=0}^{n-1} Q_{n+1-i} T_i P_0 \\
=&\left(- \left[\frac{\partial P_0}{\partial x_2}, T_{n,2}\right] + \frac{\partial^2 T_{n-1}}{\partial x_1 \partial x_2} - \sum_{i=0}^{n-2} \frac{\partial P_{n-i}}{\partial x_2} T_i - \sum_{i=0}^{n-2} P_{n-i} \frac{\partial T_i}{\partial x_2} \right) \\
&+ \left(\left[\frac{\partial Q_0}{\partial x_1}, T_{n,2}\right] - \frac{\partial^2 T_{n-1}}{\partial x_1 \partial x_2} + \sum_{i=0}^{n-2} \frac{\partial Q_{n-i}}{\partial x_1} T_i + \sum_{i=0}^{n-2} Q_{n-i} \frac{\partial T_i}{\partial x_1} - \left[\frac{\partial T_{n-1}}{\partial x_1}, \frac{\partial P_0}{\partial x_1}\right] \right) \\
&+ P_0 \left[T_{n,2}, \frac{\partial P_0}{\partial x_1}\right] - \left[T_{n,2}, \frac{\partial P_0}{\partial x_1}\right] P_0 - \sum_{i=0}^{n-1} P_0 Q_{n+1-i} T_i + \sum_{i=0}^{n-1} Q_0 P_{n+1-i} T_i \\
&+ \left(- \sum_{i=0}^{n-1} P_{n+1-i} Q_0 T_i + \sum_{i=0}^{n-2} P_{n-i} \frac{\partial T_i}{\partial x_2} + \sum_{i=0}^{n-2} P_{n-i} \left[T_i, \frac{\partial P_0}{\partial x_1}\right] - \sum_{i=2}^{n-1} \sum_{k=0}^{i-2} P_{n+1-i} Q_{i-k} T_k \right) \\
&+ \left(\sum_{i=0}^{n-1} Q_{n+1-i} P_0 T_i - \sum_{i=0}^{n-2} Q_{n-i} \frac{\partial T_i}{\partial x_1} + \sum_{i=2}^{n-1} \sum_{k=0}^{i-2} Q_{n+1-i} P_{i-k} T_k \right) \\
=&\left[T_{n,2}, \frac{\partial P_0}{\partial x_2} - \frac{\partial Q_0}{\partial x_1} + \left[P_0, \frac{\partial P_0}{\partial x_1}\right]\right] + \left[[P_0, T_{n,2}] - \frac{\partial T_{n-1}}{\partial x_1}, \frac{\partial P_0}{\partial x_1}\right] \\
&- \sum_{i=0}^{n-2} \left(\frac{\partial P_{n-i}}{\partial x_2} - \frac{\partial Q_{n-i}}{\partial x_1} \right) T_i - \sum_{i=0}^{n-1} \left([P_0, Q_{n+1-i}] + [P_{n+1-i}, Q_0] \right) T_i + \sum_{i=0}^{n-2} P_{n-i} \left[T_i, \frac{\partial P_0}{\partial x_1}\right] \\
&- \sum_{i=2}^{n-1} \sum_{k=2}^{i-2} \left(P_{n+1-i} Q_{i-k} - Q_{n+1-i} P_{i-k} \right) T_k \\
=&\left[[P_0, T_{n,2}] - \frac{\partial T_{n-1}}{\partial x_1}, \frac{\partial P_0}{\partial x_1}\right] + \sum_{i=0}^{n-2} \sum_{k=0}^{n+1-i} [P_k, Q_{n+1-i-k}] T_i \\
&- \sum_{i=0}^{n-1} \left([P_0, Q_{n+1-i}] + [P_{n+1-i}, Q_0] \right) T_i + \sum_{i=0}^{n-2} P_{n-i} \left[T_i, \frac{\partial P_0}{\partial x_1}\right] - \sum_{i=0}^{n-3} \sum_{k=2}^{n-i-1} [P_k, Q_{n+1-i-k}] T_i \\
=&\left[- \sum_{i=0}^{n-2} P_{n-i} T_i, \frac{\partial P_0}{\partial x_1}\right] + \sum_{i=0}^{n-2} P_{n-i} \left[T_i, \frac{\partial P_0}{\partial x_1}\right] + \sum_{i=0}^{n-2} [P_{n-i}, Q_1] T_i \\
=&- \sum_{i=0}^{n-2} \left[P_{n-i}, \frac{\partial P_0}{\partial x_1}\right] T_i + \sum_{i=0}^{n-2} [P_{n-i}, Q_1] T_i \\
=&0.
\end{aligned}$$

Hence, by the same argument as in the proof of Lemma 2.2, we get (4.161). □

Using this lemma, we find

Lemma 4.13. *The relation* (4.158) *implies that* $(c_{n,0}(x), c_{n,1}(x), c_{n,2}(x))$ *satisfies the following system of partial differential equations:*

$$\tag{4.165} \begin{cases} \left(3x_1 + 4x_2 P_0^T\right) \dfrac{\partial}{\partial x_1} \begin{pmatrix} c_{n,0} \\ c_{n,1} \\ c_{n,2} \end{pmatrix} = \mathrm{diag}\,(0, -1, -2) \begin{pmatrix} c_{n,0} \\ c_{n,1} \\ c_{n,2} \end{pmatrix} + \begin{pmatrix} -x_1 h_0(x) \\ 4h_1(x) \\ 4h_2(x) \end{pmatrix}, \\ \dfrac{\partial}{\partial x_2} \begin{pmatrix} c_{n,0} \\ c_{n,1} \\ c_{n,2} \end{pmatrix} = 2P_0^T \dfrac{\partial}{\partial x_1} \begin{pmatrix} c_{n,0} \\ c_{n,1} \\ c_{n,2} \end{pmatrix} + \begin{pmatrix} g_0(x) \\ g_1(x) \\ g_2(x) \end{pmatrix}, \end{cases}$$

where

$$\tag{4.166} h_0(x) = \mathrm{tr}\left(F_{n+1,1}^\dagger\right), \quad h_1(x) = \mathrm{tr}\left(F_{n+1,1}^\dagger P_0^2\right), \quad h_2(x) = \mathrm{tr}\left(F_{n+1,1}^\dagger P_0\right).$$

Conversely, if $(c_{n,0}(x), c_{n,1}(x), c_{n,2}(x))$ *is a holomorphic solution of* (4.165)*, then the relation* (4.158) *holds.*

Proof. First, we consider

$$\tag{4.167} \mathrm{tr}\left(F_{n+1,1} P_0^k\right) = 0 \quad (k = 0, 1, 2).$$

By the relations (4.96), we obtain

$$\tag{4.168} \mathrm{tr}\left(F_{n+1,1}\right) = -x_2 \frac{\partial c_{n,2}}{\partial x_1} + 3\frac{\partial c_{n,0}}{\partial x_1} + h_0(x),$$

$$\tag{4.169} \mathrm{tr}\left(F_{n+1,1} P_0\right) = -\frac{3}{4} x_1 \frac{\partial c_{n,2}}{\partial x_1} - x_2 \frac{\partial c_{n,1}}{\partial x_1} - \frac{1}{2} c_{n,2} + h_2(x),$$

$$\tag{4.170} \mathrm{tr}\left(F_{n+1,1} P_0^2\right) = \frac{x_2^2}{2} \frac{\partial c_{n,2}}{\partial x_1} - \frac{3}{4} x_1 \frac{\partial c_{n,1}}{\partial x_1} - x_2 \frac{\partial c_{n,0}}{\partial x_1} - \frac{1}{4} c_{n,1} + h_1(x).$$

If we multiply (4.168) (resp., (4.169), (4.170)) by x_1 (resp., (-4), (-4)), we get the first equation of (4.165).

Next, we consider

$$\tag{4.171} F_{n+1,2} = P_0 F_{n+1,1} + F_{n+1,1} P_0.$$

Using (4.161), this relation is equivalent to

$$\tag{4.172} \frac{\partial T_{n,1}}{\partial x_2} - \left[\frac{\partial P_0}{\partial x_1}, T_{n,1}\right] - P_0 \frac{\partial T_{n,1}}{\partial x_1} - \frac{\partial T_{n,1}}{\partial x_1} P_0 = g_2(x) P_0^2 + g_1(x) P_0 + g_0(x).$$

By (4.102) and (4.103), we have

$$\tag{4.173} \begin{aligned} &\frac{\partial T_{n,1}}{\partial x_2} + \left[T_{n,1}, \frac{\partial P_0}{\partial x_1}\right] - P_0 \frac{\partial T_{n,1}}{\partial x_1} - \frac{\partial T_{n,1}}{\partial x_1} P_0 \\ &= \left(\frac{\partial c_{n,2}}{\partial x_2} - 2\frac{\partial c_{n,1}}{\partial x_1}\right) P_0^2 + \left(\frac{\partial c_{n,1}}{\partial x_2} + x_2 \frac{\partial c_{n,2}}{\partial x_1} - 2\frac{\partial c_{n,0}}{\partial x_1}\right) P_0 + \left(\frac{\partial c_{n,0}}{\partial x_2} + \frac{x_1}{2} \frac{\partial c_{n,2}}{\partial x_1}\right). \end{aligned}$$

Comparing the left-hand and right-hand sides of (4.172), we obtain

$$\frac{\partial c_{n,2}}{\partial x_2} - 2\frac{\partial c_{n,1}}{\partial x_1} = g_2(x), \tag{4.174}$$

$$\frac{\partial c_{n,1}}{\partial x_2} + x_2\frac{\partial c_{n,2}}{\partial x_1} - 2\frac{\partial c_{n,0}}{\partial x_1} = g_1(x), \tag{4.175}$$

$$\frac{\partial c_{n,0}}{\partial x_2} + \frac{x_1}{2}\frac{\partial c_{n,2}}{\partial x_1} = g_0(x). \tag{4.176}$$

Thus we also get the second equation of (4.165).

The "converse" part is clear from the above argument. □

We now prove the existence of a holomorphic solution of the system (4.165).

Lemma 4.14. *Let*

$$3x_1\frac{d}{dx_1}\begin{pmatrix} c_{n,0} \\ c_{n,1} \\ c_{n,2} \end{pmatrix} = \operatorname{diag}(0,-1,-2)\begin{pmatrix} c_{n,0} \\ c_{n,1} \\ c_{n,2} \end{pmatrix} + \begin{pmatrix} -x_1h_0(x) \\ 4h_1(x) \\ 4h_2(x) \end{pmatrix}\Big|_{x_2=0} \tag{4.177}$$

be the system of ordinary differential equation obtained by restricting the first equation of (4.165) *to* $x_2 = 0$. *Then* (4.177) *has a holomorphic solution* $(c_{n,0}(x_1,0), c_{n,1}(x_1,0), c_{n,2}(x_1,0))$ *near* $x_1 = 0$ *satisfying*

$$c_{n,0}(0) = 0. \tag{4.178}$$

Proof. By the method of variation of constants, we find

$$\begin{cases} c_{n,0}(x_1,0) = \dfrac{1}{3}\displaystyle\int_0^1 \{-x_1h_0(x_1s,0)\}\,ds, \\ c_{n,1}(x_1,0) = \dfrac{1}{3}\displaystyle\int_0^1 \left\{4s^{-2/3}h_1(x_1s,0)\right\}ds, \\ c_{n,2}(x_1,0) = \dfrac{1}{3}\displaystyle\int_0^1 \left\{4s^{-1/3}h_2(x_1s,0)\right\}ds \end{cases} \tag{4.179}$$

is a holomorphic solution of the system (4.177) satisfying (4.178). □

Lemma 4.15. *Let* $(c_{n,0}(x_1,0), c_{n,1}(x_1,0), c_{n,2}(x_1,0))$ *be a holomorphic solution near* $x_1 = 0$ *given in Lemma 4.14. Then the following initial value problem*

$$\begin{cases} \dfrac{\partial}{\partial x_2}\begin{pmatrix} c_{n,0} \\ c_{n,1} \\ c_{n,2} \end{pmatrix} = 2P_0^T\dfrac{\partial}{\partial x_1}\begin{pmatrix} c_{n,0} \\ c_{n,1} \\ c_{n,2} \end{pmatrix} + \begin{pmatrix} g_0(x) \\ g_1(x) \\ g_2(x) \end{pmatrix}, \\ \begin{pmatrix} c_{n,0} \\ c_{n,1} \\ c_{n,2} \end{pmatrix}\Big|_{x_2=0} = \begin{pmatrix} c_{n,0}(x_1,0) \\ c_{n,1}(x_1,0) \\ c_{n,2}(x_1,0) \end{pmatrix} \end{cases} \tag{4.180}$$

has a holomorphic solution near $x = 0$.

This lemma immediately follows from the Cauchy-Kowalevski theorem.

Furthermore, we can prove

Lemma 4.16. *The holomorphic function* $(c_{n,0}(x_1,0), c_{n,1}(x_1,0), c_{n,2}(x_1,0))$ *given by Lemma 4.15 satisfies the first equation of* (4.165).

In the proof of this lemma we use the following

Sublemma 4.3. *For $k=0,1,2$, we have*

(4.181)
$$\mathrm{tr}\left[\left\{-\frac{\partial}{\partial x_2}\left(\sum_{i=0}^{n-1}P_{n+1-i}T_i\right)+\frac{\partial}{\partial x_1}\left(\sum_{i=0}^{n-1}Q_{n+1-i}T_i\right)\right\}P_0^k\right]=\mathrm{tr}\left\{\frac{\partial P_0}{\partial x_1}\left[P_0^k,\sum_{i=0}^{n-1}P_{n+1-i}T_i\right]\right\}.$$

Proof. By the compatibility condition, (4.141) and (E_j) $(j=0,1,\cdots,n)$, we have

(4.182)
$$\begin{aligned}
&\mathrm{tr}\left\{\sum_{i=0}^{n}\left(-\frac{\partial P_{n+1-i}}{\partial x_2}+\frac{\partial Q_{n+1-i}}{\partial x_1}\right)T_iP_0^k\right\}\\
=&\mathrm{tr}\left(\sum_{i=0}^{n}\sum_{j=0}^{n+2-i}\left[P_j,Q_{n+2-i-j}\right]T_iP_0^k\right)\\
=&\mathrm{tr}\left\{\sum_{i=0}^{n}\left(\sum_{j=1}^{n+1-i}\left[P_j,Q_{n+2-i-j}\right]T_i+P_{n+2-i}\left[Q_0,T_i\right]-Q_{n+2-i}\left[P_0,T_i\right]\right)P_0^k\right\}\\
=&\mathrm{tr}\left(\sum_{i=0}^{n}\sum_{j=1}^{n+1-i}\left[P_j,Q_{n+2-i-j}\right]T_iP_0^k\right)\\
&+\mathrm{tr}\left[\sum_{i=1}^{n}\left\{P_{n+2-i}\left(\frac{\partial T_{i-1}}{\partial x_2}-\sum_{j=0}^{i-1}Q_{i-j}T_j+T_{i-1}\frac{\partial P_0}{\partial x_1}\right)-Q_{n+2-i}\left(\frac{\partial T_{i-1}}{\partial x_1}-\sum_{j=0}^{i-1}P_{i-j}T_j\right)\right\}P_0^k\right]\\
=&\mathrm{tr}\left(\sum_{i=0}^{n-1}\sum_{j=2}^{n+1-i}\left[P_j,Q_{n+2-i-j}\right]T_iP_0^k\right)\\
&+\mathrm{tr}\left\{\sum_{i=1}^{n}\left(P_{n+2-i}\frac{\partial T_{i-1}}{\partial x_2}+P_{n+2-i}T_{i-1}\frac{\partial P_0}{\partial x_1}-Q_{n+2-i}\frac{\partial T_{i-1}}{\partial x_1}\right)P_0^k\right\}\\
&+\mathrm{tr}\left\{\sum_{i=0}^{n-1}\left(\sum_{j=2}^{n-i}\left[Q_{n+2-i-j},P_j\right]-P_{n+1-i}Q_1\right)T_iP_0^k\right\}\\
=&\mathrm{tr}\left(\sum_{i=0}^{n-1}\left[P_{n+1-i},Q_1\right]T_iP_0^k\right)+\mathrm{tr}\left\{\sum_{i=0}^{n-1}\left(P_{n+1-i}\frac{\partial T_i}{\partial x_2}+P_{n+1-i}T_i\frac{\partial P_0}{\partial x_1}-Q_{n+1-i}\frac{\partial T_i}{\partial x_1}\right)P_0^k\right\}\\
&-\mathrm{tr}\left(\sum_{i=0}^{n-1}P_{n+1-i}Q_1T_iP_0^k\right)\\
=&\mathrm{tr}\left\{\sum_{i=0}^{n-1}\left(P_{n+1-i}\frac{\partial T_i}{\partial x_2}-Q_{n+1-i}\frac{\partial T_i}{\partial x_1}\right)P_0^k\right\}+\mathrm{tr}\left\{\sum_{i=0}^{n-1}\left(P_{n+1-i}T_i\frac{\partial P_0}{\partial x_1}-Q_1P_{n+1-i}T_i\right)P_0^k\right\}\\
=&\mathrm{tr}\left\{\sum_{i=0}^{n-1}\left(P_{n+1-i}\frac{\partial T_i}{\partial x_2}-Q_{n+1-i}\frac{\partial T_i}{\partial x_1}\right)P_0^k\right\}+\mathrm{tr}\left\{\frac{\partial P_0}{\partial x_1}\left[P_0^k,\sum_{i=0}^{n-1}P_{n+1-i}T_i\right]\right\}.
\end{aligned}$$

□

Proof of Lemma 4.16. By the argument of Lemma 4.13, it suffices to prove that

(4.183)
$$T_{n,1}=c_{n,2}(x)P_0^2+c_{n,1}(x)P_0+c_{n,0}(x)$$

satisfies

$$\operatorname{tr}\left(F_{n+1,1}P_0^k\right)=0 \quad (k=0,1,2). \tag{4.184}$$

Let $f_{n,k}(x)$ $(k=0,1,2)$ be holomorphic functions defined by

$$f_{n,k}=\operatorname{tr}\left(F_{n+1,1}P_0^k\right) \quad (k=0,1,2). \tag{4.185}$$

We prove that $(f_{n,0}(x),f_{n,1}(x),f_{n,2}(x))$ satisfies the following system of partial differential equations

$$\frac{\partial}{\partial x_2}\begin{pmatrix} f_{n,0}\\ f_{n,1}\\ f_{n,2}\end{pmatrix}=2\frac{\partial}{\partial x_1}\left\{P_0\begin{pmatrix} f_{n,0}\\ f_{n,1}\\ f_{n,2}\end{pmatrix}\right\}. \tag{4.186}$$

By the second equation of (4.158), we have

$$\operatorname{tr}\left(F_{n+1,2}\right)=2\operatorname{tr}\left(F_{n+1,1}P_0\right)=2f_{n,1}, \tag{4.187}$$

$$\operatorname{tr}\left(F_{n+1,2}P_0\right)=2\operatorname{tr}\left(F_{n+1,1}P_0^2\right)=2f_{n,2}, \tag{4.188}$$

$$\operatorname{tr}\left(F_{n+1,2}P_0^2\right)=2\operatorname{tr}\left(F_{n+1,1}P_0^3\right)=-x_2f_{n,1}-\frac{x_1}{2}f_{n,0}, \tag{4.189}$$

that is,

$$\begin{pmatrix}\operatorname{tr}\left(F_{n+1,2}\right)\\ \operatorname{tr}\left(F_{n+1,2}P_0\right)\\ \operatorname{tr}\left(F_{n+1,2}P_0^2\right)\end{pmatrix}=2P_0\begin{pmatrix} f_{n,0}\\ f_{n,1}\\ f_{n,2}\end{pmatrix}. \tag{4.190}$$

Hence (4.186) is equivalent to

$$\frac{\partial}{\partial x_2}\left\{\operatorname{tr}\left(F_{n+1,1}P_0^k\right)\right\}=\frac{\partial}{\partial x_1}\left\{\operatorname{tr}\left(F_{n+1,2}P_0^k\right)\right\} \quad (k=0,1,2). \tag{4.191}$$

By using Sublemma 4.3, we can confirm these relations as follows:

$$\begin{aligned}
&\frac{\partial}{\partial x_2}\left\{\operatorname{tr}\left(F_{n+1,1}\right)\right\}-\frac{\partial}{\partial x_1}\left\{\operatorname{tr}\left(F_{n+1,2}\right)\right\}\\
=&\operatorname{tr}\left(\frac{\partial F_{n+1,1}}{\partial x_2}-\frac{\partial F_{n+1,2}}{\partial x_1}\right)\\
=&\operatorname{tr}\left[\left\{\frac{\partial^2 T_n}{\partial x_2\partial x_1}-\frac{\partial}{\partial x_2}\left(\sum_{i=0}^{n-1}P_{n+1-i}T_i\right)\right\}\right.\\
&\left.-\left\{\frac{\partial^2 T_n}{\partial x_1\partial x_2}-\frac{\partial}{\partial x_1}\left(\sum_{i=0}^{n-1}Q_{n+1-i}T_i\right)+\left[\frac{\partial T_n}{\partial x_1},\frac{\partial P_0}{\partial x_1}\right]\right\}\right]\\
=&\operatorname{tr}\left\{\frac{\partial P_0}{\partial x_1}\left[I,\sum_{i=0}^{n-1}P_{n+1-i}T_i\right]-\left[\frac{\partial T_n}{\partial x_1},\frac{\partial P_0}{\partial x_1}\right]\right\}\\
=&0,
\end{aligned} \tag{4.192}$$

$$\begin{aligned}
&\frac{\partial}{\partial x_2}\left\{\operatorname{tr}\left(F_{n+1,1}P_0\right)\right\}-\frac{\partial}{\partial x_1}\left\{\operatorname{tr}\left(F_{n+1,2}P_0\right)\right\}\\
=&\operatorname{tr}\left(\frac{\partial F_{n+1,1}}{\partial x_2}P_0-\frac{\partial F_{n+1,2}}{\partial x_1}P_0+F_{n+1,1}\frac{\partial P_0}{\partial x_2}-F_{n+1,2}\frac{\partial P_0}{\partial x_1}\right)\\
=&\operatorname{tr}\left[\left\{\frac{\partial^2 T_n}{\partial x_2\partial x_1}-\frac{\partial}{\partial x_2}\left(\sum_{i=0}^{n-1}P_{n+1-i}T_i\right)\right\}P_0\right.
\end{aligned} \tag{4.193}$$

$$
\begin{aligned}
&-\left\{\frac{\partial^2 T_n}{\partial x_1 \partial x_2} - \frac{\partial}{\partial x_1}\left(\sum_{i=0}^{n-1} Q_{n+1-i} T_i\right) + \left[\frac{\partial T_n}{\partial x_1}, \frac{\partial P_0}{\partial x_1}\right]\right\} P_0 \\
&+2F_{n+1,1}\frac{\partial P_0}{\partial x_1} P_0 - (P_0 F_{n+1,1} + F_{n+1,1} P_0)\frac{\partial P_0}{\partial x_1}\Bigg] \\
=&\mathrm{tr}\left\{\frac{\partial P_0}{\partial x_1}\left[P_0, \sum_{i=0}^{n-1} P_{n+1-i} T_i\right] - \left[\frac{\partial T_n}{\partial x_1}, \frac{\partial P_0}{\partial x_1}\right] P_0 + [P_0, F_{n+1,1}]\frac{\partial P_0}{\partial x_1}\right\} \\
=&\mathrm{tr}\left\{\frac{\partial P_0}{\partial x_1}\left[P_0, \frac{\partial T_n}{\partial x_1}\right] - \left[\frac{\partial T_n}{\partial x_1}, \frac{\partial P_0}{\partial x_1}\right] P_0\right\} \\
=&\mathrm{tr}\left[\frac{\partial P_0}{\partial x_1} P_0, \frac{\partial T_n}{\partial x_1}\right] \\
=&0,
\end{aligned}
$$

$$
\begin{aligned}
(4.194)\quad &\frac{\partial}{\partial x_2}\left\{\mathrm{tr}\left(F_{n+1,1} P_0^2\right)\right\} - \frac{\partial}{\partial x_1}\left\{\mathrm{tr}\left(F_{n+1,2} P_0^2\right)\right\} \\
=&\mathrm{tr}\left(\frac{\partial F_{n+1,1}}{\partial x_2} P_0^2 - \frac{\partial F_{n+1,2}}{\partial x_1} P_0^2 + F_{n+1,1}\frac{\partial}{\partial x_2}\left(P_0^2\right) - F_{n+1,2}\frac{\partial}{\partial x_1}\left(P_0^2\right)\right) \\
=&\mathrm{tr}\Bigg[\left\{\frac{\partial^2 T_n}{\partial x_2 \partial x_1} - \frac{\partial}{\partial x_2}\left(\sum_{i=0}^{n-1} P_{n+1-i} T_i\right)\right\} P_0^2 \\
&-\left\{\frac{\partial^2 T_n}{\partial x_1 \partial x_2} - \frac{\partial}{\partial x_1}\left(\sum_{i=0}^{n-1} Q_{n+1-i} T_i\right) + \left[\frac{\partial T_n}{\partial x_1}, \frac{\partial P_0}{\partial x_1}\right]\right\} P_0^2 \\
&+F_{n+1,1}\left(2P_0\frac{\partial P_0}{\partial x_1} P_0 + 2\frac{\partial P_0}{\partial x_1} P_0^2\right) - (P_0 F_{n+1,1} + F_{n+1,1} P_0)\left(\frac{\partial P_0}{\partial x_1} P_0 + P_0\frac{\partial P_0}{\partial x_1}\right)\Bigg] \\
=&\mathrm{tr}\left\{\frac{\partial P_0}{\partial x_1}\left[P_0^2, \sum_{i=0}^{n-1} P_{n+1-i} T_i\right] - \left[\frac{\partial T_n}{\partial x_1}, \frac{\partial P_0}{\partial x_1}\right] P_0^2 + \left[P_0^2, F_{n+1,1}\right]\frac{\partial P_0}{\partial x_1}\right\} \\
=&\mathrm{tr}\left\{\frac{\partial P_0}{\partial x_1}\left[P_0^2, \frac{\partial T_n}{\partial x_1}\right] - \left[\frac{\partial T_n}{\partial x_1}, \frac{\partial P_0}{\partial x_1}\right] P_0^2\right\} \\
=&\mathrm{tr}\left[\frac{\partial P_0}{\partial x_1} P_0^2, \frac{\partial T_n}{\partial x_1}\right] \\
=&0.
\end{aligned}
$$

Thus we obtain (4.186).

The system (4.177) means that $f_{n,k}(x)$ $(k = 0, 1, 2)$ vanishes on $\{(x_1, 0)\ ;\ x_1 \neq 0\}$. Thus, by Lemmas 4.14 and 4.15 and the holomorphy of $f_{n,k}(x)$, we find that

$$
(4.195)\qquad \begin{pmatrix} f_{n,0} \\ f_{n,1} \\ f_{n,2} \end{pmatrix}\Big|_{x_2=0} = 0.
$$

Hence, by the uniqueness of solutions for the initial value problem (4.186) and (4.195), we obtain (4.184). □

We have thus obtained a holomorphic solution of (4.165) near $x = 0$, that is, we have constructed $T_n(x)$ satisfying (4.158).

The solution $c_{n,k}(x)$ $(k = 0, 1, 2)$ of (4.165) thus obtained enjoys the following property:

Lemma 4.17. *The solution* $(c_{n,0}(x), c_{n,1}(x), c_{n,2}(x))$ *of the system* (4.165) *is holomorphic in* D_ρ

and has the following integral representation

$$\text{(4.196)} \qquad \begin{cases} c_{n,0}(x) = \dfrac{1}{3}\displaystyle\int_0^1 \left\{ -x_1 h_0(x_1 s, x_2 s^{2/3}) + 2x_2 s^{-1/3} g_0(x_1 s, x_2 s^{2/3}) \right\} ds, \\ c_{n,1}(x) = \dfrac{1}{3}\displaystyle\int_0^1 \left\{ 4s^{-2/3} h_1(x_1 s, x_2 s^{2/3}) + 2x_2 g_1(x_1 s, x_2 s^{2/3}) \right\} ds, \\ c_{n,2}(x) = \dfrac{1}{3}\displaystyle\int_0^1 \left\{ 4s^{-1/3} h_2(x_1 s, x_2 s^{2/3}) + 2x_2 s^{1/3} g_2(x_1 s, x_2 s^{2/3}) \right\} ds. \end{cases}$$

Proof. Using (4.165), we find that $(c_{n,0}(x), c_{n,1}(x), c_{n,2}(x))$ satisfies

$$\text{(4.197)} \qquad \left\{ 3x_1 \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_2} + \operatorname{diag}(0,1,2) \right\} \begin{pmatrix} c_{n,0} \\ c_{n,1} \\ c_{n,2} \end{pmatrix} = \begin{pmatrix} -x_1 h_0(x) + 2x_2 g_0(x) \\ 4h_1(x) + 2x_2 g_1(x) \\ 4h_2(x) + 2x_2 g_2(x) \end{pmatrix}.$$

Since a unique holomorphic solution of this system near $x = 0$ with $c_{n,0}(0) = 0$ is represented by the right-hand side of (4.196), we obtain the integral representation.

Furthermore, by this integral representation (4.196) and the holomorphy of $g_k(x)$, $h_k(x)$ in D_ρ, we also find that $(c_{n,0}(x), c_{n,1}(x), c_{n,2}(x))$ is holomorphic in D_ρ. □

This completes the proof of the assertion $(\mathcal{S}_n)$. In other words, our induction on n is now completed.

5 Estimate of the transformation

In this section we prove the estimate (4.143) of the transformation constructed in Subsection 4.3.

Before proving the estimate, we prepare the following

Lemma 5.1. *Let $A(x)$ be a 3×3 matrix with holomorphic entries in D_ρ. Suppose that there exist a positive constant C and an integer $n \geq 1$ such that for any $\rho/2 < \sigma < \rho$*

$$\text{(5.1)} \qquad \|A\|_{\sigma,\sigma} \leq C n! \, (\rho - \sigma)^{-n}$$

holds. Then, for any $\rho/2 < \sigma < \rho$, we have

$$\text{(5.2)} \qquad \left\| \frac{\partial A}{\partial x_j} \right\|_{\sigma,\sigma} \leq eC \, (n+1)! \, (\rho - \sigma)^{-n-1} \quad (j = 1, 2).$$

Proof. Since the proof is similar for $j = 1, 2$, we show (5.2) only for $j = 1$. For $|x_1|, |x_2| \leq \sigma$ we have

$$\text{(5.3)} \qquad \begin{aligned} \frac{\partial A}{\partial x_1}(x) =& \frac{1}{2\pi\sqrt{-1}} \int_{|t - x_1| = \frac{\rho-\sigma}{n+1}} \frac{A(t, x_2)}{(t - x_1)^2} dt \\ =& \frac{n+1}{2\pi(\rho - \sigma)} \int_0^{2\pi} A\left(x_1 + \frac{\rho - \sigma}{n+1} e^{\sqrt{-1}\theta}, x_2 \right) e^{-\sqrt{-1}\theta} d\theta. \end{aligned}$$

Hence we get

$$\text{(5.4)} \qquad \begin{aligned} \left\| \frac{\partial A}{\partial x_1}(x) \right\|_{\sigma,\sigma} \leq& \frac{n+1}{2\pi(\rho - \sigma)} \int_0^{2\pi} \|A\|_{\frac{n\sigma+\rho}{n+1}, \sigma} \, d\theta \\ \leq& \frac{n+1}{\rho - \sigma} \|A\|_{\frac{n\sigma+\rho}{n+1}, \frac{n\sigma+\rho}{n+1}} \end{aligned}$$

$$
\begin{aligned}
&\leq \left(1-\frac{1}{n+1}\right)^{-n} C\,(n+1)!\,(\rho-\sigma)^{-n-1} \\
&\leq eC\,(n+1)!\,(\rho-\sigma)^{-n-1}.
\end{aligned}
$$

□

Combining (4.142) and Lemma 5.1, we can prove

Lemma 5.2. *Replacing C and α in* (4.142) *by new ones, we have the following estimates for any $\rho/2 < \sigma < \rho$:*

$$
\|P_0\|_{\sigma,\sigma}\,,\ \left\|\frac{\partial P_0}{\partial x_j}\right\|_{\sigma,\sigma},\ \|Q_0\|_{\sigma,\sigma}\,,\ \left\|\frac{\partial Q_0}{\partial x_j}\right\|_{\sigma,\sigma} \leq C, \tag{5.5}
$$

$$
\|P_1\|_{\sigma,\sigma}\,,\ \left\|\frac{\partial P_1}{\partial x_j}\right\|_{\sigma,\sigma},\ \|Q_1\|_{\sigma,\sigma}\,,\ \left\|\frac{\partial Q_1}{\partial x_j}\right\|_{\sigma,\sigma} \leq C\,(\rho-\sigma)^{-1}, \tag{5.6}
$$

$$
\|P_n\|_{\sigma,\sigma}\,,\ \left\|\frac{\partial P_n}{\partial x_j}\right\|_{\sigma,\sigma},\ \|Q_n\|_{\sigma,\sigma}\,,\ \left\|\frac{\partial Q_n}{\partial x_j}\right\|_{\sigma,\sigma} \leq C\alpha^{n-2}\,(n-1)!\,(\rho-\sigma)^{-n+1} \quad (n\geq 2). \tag{5.7}
$$

Proof. Since $n \leq 2^n$ and $(\rho-\sigma)^{n-1} \leq \rho^{n-1}$ hold for any $n \geq 2$, we have

$$
\begin{aligned}
\|P_n\|_{\sigma,\sigma} &\leq C\alpha^n n! \\
&\leq C\alpha^2\alpha^{n-2} n\,(n-1)!\,(\rho-\sigma)^{n-1}(\rho-\sigma)^{-n+1} \\
&\leq 4C\alpha^2\rho\,(2\alpha\rho)^{n-2}\,(n-1)!\,(\rho-\sigma)^{-n+1}.
\end{aligned} \tag{5.8}
$$

Similarly, since $n\,(n-1) \leq 4^n$ and $(\rho-\sigma)^{n-2} \leq \rho^{n-2}$ hold for any $n \geq 2$, we also have

$$
\begin{aligned}
\|P_n\|_{\sigma,\sigma} &\leq C\alpha^n n! \\
&\leq C\alpha^2\alpha^{n-2} n\,(n-1)\,(n-2)!\,(\rho-\sigma)^{n-2}(\rho-\sigma)^{-n+2} \\
&\leq 16C\alpha^2\,(4\alpha\rho)^{n-2}\,(n-2)!\,(\rho-\sigma)^{-n+2}.
\end{aligned} \tag{5.9}
$$

Hence, by Lemma 5.1, we obtain

$$
\left\|\frac{\partial P_n}{\partial x_j}\right\|_{\sigma,\sigma} \leq e16C\alpha^2\,(4\alpha\rho)^{n-2}\,(n-1)!\,(\rho-\sigma)^{-n+1}. \tag{5.10}
$$

The estimates for $Q_n(x)$ and $(\partial Q_n/\partial x_j)\,(x)$ can be obtained in a similar way. □

Furthermore, we may assume $T_1(x)$ and $(\partial T_1/\partial x_j)\,(x)$ satisfy the following estimate

$$
\|T_1\|_{\sigma,\sigma}\,,\ \left\|\frac{\partial T_1}{\partial x_j}\right\|_{\sigma,\sigma} \leq C\,(\rho-\sigma)^{-1}. \tag{5.11}
$$

Note that, since $T_0(x) = I$, we also have

$$
\|T_0\|_{\sigma,\sigma} = 3, \qquad \left\|\frac{\partial T_0}{\partial x_j}\right\|_{\sigma,\sigma} = 0. \tag{5.12}
$$

Making use of these lemmas, we prove the estimate (4.143) for $\{T_n(x)\}_{n\geq 0}$. In what follows, we may assume (by replacing ρ if necessary) $0 < \rho < 27/8$.

Proposition 5.1. *There exist some positive constants $\widetilde{C} \geq C$ and $\widetilde{\alpha} \geq \alpha$ such that for any $n \geq 2$ and $\rho/2 < \sigma < \rho$*

$$
\|T_n\|_{\sigma,\sigma}\,,\ \left\|\frac{\partial T_n}{\partial x_j}\right\|_{\sigma,\sigma} \leq \widetilde{C}\widetilde{\alpha}^{n-2} n!\,(\rho-\sigma)^{-n}. \tag{5.13}
$$

We prove Proposition 5.1 by an induction on n. We now assume that (5.13) holds up to $n-1$, that is, for any $i = 2, 3, \cdots, n-1$ and $\rho/2 < \sigma < \rho$ we have

$$\|T_i\|_{\sigma,\sigma}, \left\|\frac{\partial T_i}{\partial x_j}\right\|_{\sigma,\sigma} \le \widetilde{C}\widetilde{\alpha}^{j-2} i! \, (\rho-\sigma)^{-i}. \tag{5.14}$$

Under these assumptions we prove the following estimates for any $\rho/2 < \sigma < \rho$

$$\|T_n\|_{\sigma,\sigma}, \left\|\frac{\partial T_n}{\partial x_j}\right\|_{\sigma,\sigma} \le \widetilde{C}\widetilde{\alpha}^{n-2} n! \, (\rho-\sigma)^{-n} \tag{5.15}$$

(by choosing $\widetilde{C}$ and $\widetilde{\alpha}$ suitably).

We first show

Lemma 5.3. *For any $\rho/2 < \sigma < \rho$, we have*

$$\|T_{n,2}\|_{\sigma,\sigma} \le (2+C)\left(\widetilde{C}\widetilde{\alpha}^{-1} + C\widetilde{C}\widetilde{\alpha}^{-2} + C^2\widetilde{\alpha}^{-1} + C\right)\widetilde{\alpha}^{n-2}(n-1)!\,(\rho-\sigma)^{-n+1}, \tag{5.16}$$

$$\left\|\frac{\partial T_{n,2}}{\partial x_j}\right\|_{\sigma,\sigma} \le e\,(2+C)\left(\widetilde{C}\widetilde{\alpha}^{-1} + C\widetilde{C}\widetilde{\alpha}^{-2} + C^2\widetilde{\alpha}^{-1} + C\right)\widetilde{\alpha}^{n-2} n!\,(\rho-\sigma)^{-n}. \tag{5.17}$$

Proof. First, we consider the estimate for $F_{n,1}$:

(5.18)

$$\begin{aligned}
\|F_{n,1}\|_{\sigma,\sigma} =& \left\|\frac{\partial T_{n-1}}{\partial x_1} - \sum_{i=0}^{n-2} P_{n-i}T_i\right\|_{\sigma,\sigma} \\
\le& \left\|\frac{\partial T_{n-1}}{\partial x_1}\right\|_{\sigma,\sigma} + \sum_{i=2}^{n-2}\|P_{n-i}\|_{\sigma,\sigma}\|T_i\|_{\sigma,\sigma} + \|P_{n-1}\|_{\sigma,\sigma}\|T_1\|_{\sigma,\sigma} + \|P_n\|_{\sigma,\sigma} \\
\le& \widetilde{C}\widetilde{\alpha}^{n-3}(n-1)!\,(\rho-\sigma)^{-n+1} \\
&+ \sum_{i=2}^{n-2}\left\{C\alpha^{n-i-2}(n-i-1)!\,(\rho-\sigma)^{-n+i+1}\right\} \times \left\{\widetilde{C}\widetilde{\alpha}^{i-2} i!\,(\rho-\sigma)^{-i}\right\} \\
&+ \left\{C\alpha^{n-3}(n-2)!\,(\rho-\sigma)^{-n+2}\right\} \times \left\{C(\rho-\sigma)^{-1}\right\} + C\alpha^{n-2}(n-1)!\,(\rho-\sigma)^{-n+1} \\
\le& \widetilde{C}\widetilde{\alpha}^{n-3}(n-1)!\,(\rho-\sigma)^{-n+1} + C\widetilde{C}\widetilde{\alpha}^{n-4}(\rho-\sigma)^{-n+1}\sum_{i=2}^{n-2}(n-i-1)!i! \\
&+ C^2\widetilde{\alpha}^{n-3}(n-2)!\,(\rho-\sigma)^{-n+1} + C\widetilde{\alpha}^{n-2}(n-1)!\,(\rho-\sigma)^{-n+1} \\
\le& \left(\widetilde{C}\widetilde{\alpha}^{-1} + C\widetilde{C}\widetilde{\alpha}^{-2} + C^2\widetilde{\alpha}^{-1} + C\right)\widetilde{\alpha}^{n-2}(n-1)!\,(\rho-\sigma)^{-n+1}.
\end{aligned}$$

Hence we get

$$\begin{aligned}
\|T_{n,2}\|_{\sigma,\sigma} &= \left\|\Lambda F_{n,1} + \Lambda^2 F_{n,1}P_0\right\|_{\sigma,\sigma} \\
&\le \|\Lambda\|_{\sigma,\sigma}\|F_{n,1}\|_{\sigma,\sigma} + \left\|\Lambda^2\right\|_{\sigma,\sigma}\|F_{n,1}\|_{\sigma,\sigma}\|P_0\|_{\sigma,\sigma} \\
&\le (2+C)\|F_{n,1}\|_{\sigma,\sigma} \\
&\le (2+C)\left(\widetilde{C}\widetilde{\alpha}^{-1} + C\widetilde{C}\widetilde{\alpha}^{-2} + C^2\widetilde{\alpha}^{-1} + C\right)\widetilde{\alpha}^{n-2}(n-1)!\,(\rho-\sigma)^{-n+1}.
\end{aligned} \tag{5.19}$$

The estimate (5.17) can be obtained by (5.19) and Lemma 5.1. □

Next, we consider the estimates for $T_{n,1}(x)$ and $(\partial T_{n,1}/\partial x_j)(x)$. For this purpose, we first prove

Lemma 5.4. *Let C_1 be a positive constant defined by*

$$C_1 = \max\left\{1+2C,\ C^2\right\} \times \left\{(e+2C\rho)(2+C)\left(\widetilde{C}\widetilde{\alpha}^{-1} + C\widetilde{C}\widetilde{\alpha}^{-2} + C^2\widetilde{\alpha}^{-1} + C\right) + C\widetilde{C}\widetilde{\alpha}^{-1} + C^2 + C\alpha\right\}. \tag{5.20}$$

Then we have

$$|g_k|_{\sigma,\sigma},\ |h_k|_{\sigma,\sigma} \le C_1\widetilde{\alpha}^{n-2} n!\,(\rho-\sigma)^{-n} \tag{5.21}$$

for any $k = 0, 1, 2$ and $\rho/2 < \sigma < \rho$.

Proof. We first give the estimates for $F^{\dagger}_{n+1,1}$ and $F^{\dagger}_{n+1,2}$:

(5.22)

$$\begin{aligned}
\left\|F^{\dagger}_{n+1,1}\right\|_{\sigma,\sigma} =& \left\|\frac{\partial T_{n,2}}{\partial x_1} - \sum_{i=0}^{n-1} P_{n+1-i}T_i\right\|_{\sigma,\sigma} \\
\le& \left\|\frac{\partial T_{n,2}}{\partial x_1}\right\|_{\sigma,\sigma} + \sum_{i=2}^{n-1}\|P_{n+1-i}\|_{\sigma,\sigma}\|T_i\|_{\sigma,\sigma} + \|P_n\|_{\sigma,\sigma}\|T_1\|_{\sigma,\sigma} + \|P_{n+1}\|_{\sigma,\sigma} \\
\le& e(2+C)\left(\widetilde{C}\widetilde{\alpha}^{-1} + C\widetilde{C}\widetilde{\alpha}^{-2} + C^2\widetilde{\alpha}^{-1} + C\right)\widetilde{\alpha}^{n-2} n!\,(\rho-\sigma)^{-n} \\
&+ \sum_{i=2}^{n-2}\left\{C\alpha^{n-i-1}(n-i)!\,(\rho-\sigma)^{-n+i}\right\} \times \left\{\widetilde{C}\widetilde{\alpha}^{i-2} i!\,(\rho-\sigma)^{-i}\right\} \\
&+ \left\{C\alpha^{n-2}(n-1)!\,(\rho-\sigma)^{-n+1}\right\} \times \left\{C(\rho-\sigma)^{-1}\right\} + C\alpha^{n-1} n!\,(\rho-\sigma)^{-n} \\
\le& e(2+C)\left(\widetilde{C}\widetilde{\alpha}^{-1} + C\widetilde{C}\widetilde{\alpha}^{-2} + C^2\widetilde{\alpha}^{-1} + C\right)\widetilde{\alpha}^{n-2} n!\,(\rho-\sigma)^{-n} \\
&+ C\widetilde{C}\widetilde{\alpha}^{n-3}(\rho-\sigma)^{-n}\sum_{i=2}^{n-1}(n-i)!i! + C^2\widetilde{\alpha}^{n-2}(n-1)!\,(\rho-\sigma)^{-n} \\
&+ C\widetilde{\alpha}^{n-2}\alpha n!\,(\rho-\sigma)^{-n} \\
\le& \left\{e(2+C)\left(\widetilde{C}\widetilde{\alpha}^{-1} + C\widetilde{C}\widetilde{\alpha}^{-2} + C^2\widetilde{\alpha}^{-1} + C\right) + C\widetilde{C}\widetilde{\alpha}^{-1} + C^2 + C\alpha\right\} \\
&\times \widetilde{\alpha}^{n-2} n!\,(\rho-\sigma)^{-n}.
\end{aligned}$$

(5.23)

$$\begin{aligned}
\left\|F^{\dagger}_{n+1,2}\right\|_{\sigma,\sigma} =& \left\|\frac{\partial T_{n,2}}{\partial x_2} + \left[T_{n,2}, \frac{\partial P_0}{\partial x_1}\right] - \sum_{i=0}^{n-1} Q_{n+1-i}T_i\right\|_{\sigma,\sigma} \\
\le& \left\|\frac{\partial T_{n,2}}{\partial x_2}\right\|_{\sigma,\sigma} + 2\|T_{n,2}\|_{\sigma,\sigma}\left\|\frac{\partial P_0}{\partial x_1}\right\|_{\sigma,\sigma} \\
&+ \sum_{i=2}^{n-1}\|Q_{n+1-i}\|_{\sigma,\sigma}\|T_i\|_{\sigma,\sigma} + \|Q_n\|_{\sigma,\sigma}\|T_1\|_{\sigma,\sigma} + \|Q_{n+1}\|_{\sigma,\sigma} \\
\le& e(2+C)\left(\widetilde{C}\widetilde{\alpha}^{-1} + C\widetilde{C}\widetilde{\alpha}^{-2} + C^2\widetilde{\alpha}^{-1} + C\right)\widetilde{\alpha}^{n-2} n!\,(\rho-\sigma)^{-n} \\
&+ 2C(2+C)\left(\widetilde{C}\widetilde{\alpha}^{-1} + C\widetilde{C}\widetilde{\alpha}^{-2} + C^2\widetilde{\alpha}^{-1} + C\right)\widetilde{\alpha}^{n-2}(n-1)!\,(\rho-\sigma)^{-n+1} \\
&+ \sum_{i=2}^{n-2}\left\{C\alpha^{n-i-1}(n-i)!\,(\rho-\sigma)^{-n+i}\right\} \times \left\{\widetilde{C}\widetilde{\alpha}^{i-2} i!\,(\rho-\sigma)^{-i}\right\} \\
&+ \left\{C\alpha^{n-2}(n-1)!\,(\rho-\sigma)^{-n+1}\right\} \times \left\{C(\rho-\sigma)^{-1}\right\} + C\alpha^{n-1} n!\,(\rho-\sigma)^{-n}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(e+2C\frac{\rho-\sigma}{n}\right)(2+C)\left(\widetilde{C}\widetilde{\alpha}^{-1}+C\widetilde{C}\widetilde{\alpha}^{-2}+C^2\widetilde{\alpha}^{-1}+C\right)\widetilde{\alpha}^{n-2}n!\,(\rho-\sigma)^{-n}\\
&\quad +C\widetilde{C}\widetilde{\alpha}^{n-3}(\rho-\sigma)^{-n}\sum_{i=2}^{n-1}(n-i)!i!+C^2\widetilde{\alpha}^{n-2}(n-1)!\,(\rho-\sigma)^{-n}\\
&\quad +C\widetilde{\alpha}^{n-2}\alpha n!\,(\rho-\sigma)^{-n}\\
&\leq \left\{(e+2C\rho)(2+C)\left(\widetilde{C}\widetilde{\alpha}^{-1}+C\widetilde{C}\widetilde{\alpha}^{-2}+C^2\widetilde{\alpha}^{-1}+C\right)+C\widetilde{C}\widetilde{\alpha}^{-1}+C^2+C\alpha\right\}\\
&\quad \times \widetilde{\alpha}^{n-2}n!\,(\rho-\sigma)^{-n}.
\end{aligned}
$$

Hence, if we define $C_1^{\dagger}$ by

$$
C_1^{\dagger}=\left\{(e+2C\rho)(2+C)\left(\widetilde{C}\widetilde{\alpha}^{-1}+C\widetilde{C}\widetilde{\alpha}^{-2}+C^2\widetilde{\alpha}^{-1}+C\right)+C\widetilde{C}\widetilde{\alpha}^{-1}+C^2+C\alpha\right\}, \tag{5.24}
$$

then we obtain

$$
\left\|F_{n+1,1}^{\dagger}\right\|_{\sigma,\sigma},\ \left\|F_{n+1,2}^{\dagger}\right\|_{\sigma,\sigma}\leq C_1^{\dagger}\widetilde{\alpha}^{n-2}n!\,(\rho-\sigma)^{-n}. \tag{5.25}
$$

Using these estimates, we prove the estimates (5.21). Since the $(1,1)$, $(1,2)$ and $(1,3)$ entries of $g_2(x)P_0^2+g_1(x)P_0+g_0(x)$ are $g_0(x)$, $g_1(x)$, $g_2(x)$, respectively, we get

$$
\begin{aligned}
|g_k|_{\sigma,\sigma} &\leq \left\|g_2P_0^2+g_1P_0+g_0\right\|_{\sigma,\sigma}\\
&\leq \left\|-F_{n+1,2}^{\dagger}+F_{n+1,1}^{\dagger}P_0+P_0F_{n+1,1}^{\dagger}\right\|_{\sigma,\sigma}\\
&\leq \left\|F_{n+1,2}^{\dagger}\right\|_{\sigma,\sigma}+2\left\|F_{n+1,1}^{\dagger}\right\|_{\sigma,\sigma}\|P_0\|_{\sigma,\sigma}\\
&\leq (1+2C)\,C_1^{\dagger}\widetilde{\alpha}^{n-2}n!\,(\rho-\sigma)^{-n}.
\end{aligned} \tag{5.26}
$$

Furthermore,

$$
\begin{aligned}
|h_0|_{\sigma,\sigma} &\leq \left|\mathrm{tr}\left(F_{n+1,1}^{\dagger}\right)\right|_{\sigma,\sigma}\\
&\leq \left\|F_{n+1,1}^{\dagger}\right\|_{\sigma,\sigma}\\
&\leq C_1^{\dagger}\widetilde{\alpha}^{n-2}n!\,(\rho-\sigma)^{-n},
\end{aligned} \tag{5.27}
$$

$$
\begin{aligned}
|h_1|_{\sigma,\sigma} &\leq \left|\mathrm{tr}\left(F_{n+1,1}^{\dagger}P_0^2\right)\right|_{\sigma,\sigma}\\
&\leq \left\|F_{n+1,1}^{\dagger}P_0^2\right\|_{\sigma,\sigma}\\
&\leq \left\|F_{n+1,1}^{\dagger}\right\|_{\sigma,\sigma}\|P_0\|_{\sigma,\sigma}^2\\
&\leq C^2C_1^{\dagger}\widetilde{\alpha}^{n-2}n!\,(\rho-\sigma)^{-n},
\end{aligned} \tag{5.28}
$$

$$
\begin{aligned}
|h_2|_{\sigma,\sigma} &\leq \left|\mathrm{tr}\left(F_{n+1,1}^{\dagger}P_0\right)\right|_{\sigma,\sigma}\\
&\leq \left\|F_{n+1,1}^{\dagger}P_0\right\|_{\sigma,\sigma}\\
&\leq \left\|F_{n+1,1}^{\dagger}\right\|_{\sigma,\sigma}\|P_0\|_{\sigma,\sigma}\\
&\leq CC_1^{\dagger}\widetilde{\alpha}^{n-2}n!\,(\rho-\sigma)^{-n}.
\end{aligned} \tag{5.29}
$$

Thus the estimates (5.21) immediately follow if we define C_1 by (5.20). □

We next consider the estimates for $c_{n,k}(x)$ $(k=0,1,2)$.

Lemma 5.5. *Let C_2 be a positive constant defined by*

(5.30)
$$C_2 = \max\left\{4,\ \frac{1000}{27\rho - 8\rho^2},\ \frac{1000}{27(\rho/2) - 8(\rho/2)^2},\ \frac{6000C}{27\rho - 8\rho^2} + 1,\ \frac{6000C}{27(\rho/2) - 8(\rho/2)^2} + 1\right\}.$$

Then we have

$$|c_{n,k}|_{\sigma,\sigma},\ \left|\frac{\partial c_{n,k}}{\partial x_j}\right|_{\sigma,\sigma} \le C_2 \sum_{i=0}^{2}\left(|g_i|_{\sigma,\sigma} + |h_i|_{\sigma,\sigma}\right) \tag{5.31}$$

for any $k = 0, 1, 2$ and $\rho/2 < \sigma < \rho$.

In the proof of Lemma 5.5, we need the following two sublemmas.

Sublemma 5.1. *For any $\rho/2 < \sigma < \rho$, $c_{n,k}(x)$ $(k = 0, 1, 2)$ satisfy the following estimates.*

$$|c_{n,0}|_{\sigma,\sigma} \le \frac{\sigma}{3}|h_0|_{\sigma,\sigma} + \sigma|g_0|_{\sigma,\sigma}, \tag{5.32}$$

$$|c_{n,1}|_{\sigma,\sigma} \le 4|h_1|_{\sigma,\sigma} + \frac{2\sigma}{3}|g_1|_{\sigma,\sigma}, \tag{5.33}$$

$$|c_{n,2}|_{\sigma,\sigma} \le 2|h_2|_{\sigma,\sigma} + \frac{\sigma}{2}|g_2|_{\sigma,\sigma}. \tag{5.34}$$

Proof. Thanks to the integral representations (4.196) for $c_{n,k}(x)$, we obtain the following estimates for $|x_j| \le \sigma$ $(j = 1, 2)$:

$$\begin{aligned}
|c_{n,0}(x)| &= \left|\frac{1}{3}\int_0^1 \left\{-x_1 h_0(x_1 s, x_2 s^{2/3}) + 2x_2 s^{-1/3} g_0(x_1 s, x_2 s^{2/3})\right\} ds\right| \\
&\le \frac{1}{3}\int_0^1 \left\{\sigma\left|h_0(x_1 s, x_2 s^{2/3})\right| + 2\sigma s^{-1/3}\left|g_0(x_1 s, x_2 s^{2/3})\right|\right\} ds \\
&\le \frac{1}{3}\int_0^1 \left\{\sigma|h_0|_{\sigma,\sigma} + 2\sigma s^{-1/3}|g_0|_{\sigma,\sigma}\right\} ds \\
&= \frac{\sigma}{3}|h_0|_{\sigma,\sigma} + \sigma|g_0|_{\sigma,\sigma},
\end{aligned} \tag{5.35}$$

$$\begin{aligned}
|c_{n,1}(x)| &= \left|\frac{1}{3}\int_0^1 \left\{4s^{-2/3} h_1(x_1 s, x_2 s^{2/3}) + 2x_2 g_1(x_1 s, x_2 s^{2/3})\right\} ds\right| \\
&\le \frac{1}{3}\int_0^1 \left\{4s^{-2/3}\left|h_1(x_1 s, x_2 s^{2/3})\right| + 2\sigma\left|g_1(x_1 s, x_2 s^{2/3})\right|\right\} ds \\
&\le \frac{1}{3}\int_0^1 \left\{4s^{-2/3}|h_1|_{\sigma,\sigma} + 2\sigma|g_1|_{\sigma,\sigma}\right\} ds \\
&= 4|h_1|_{\sigma,\sigma} + \frac{2\sigma}{3}|g_1|_{\sigma,\sigma},
\end{aligned} \tag{5.36}$$

$$\begin{aligned}
|c_{n,2}(x)| &= \left|\frac{1}{3}\int_0^1 \left\{4s^{-1/3} h_2(x_1 s, x_2 s^{2/3}) + 2x_2 s^{1/3} g_2(x_1 s, x_2 s^{2/3})\right\} ds\right| \\
&\le \frac{1}{3}\int_0^1 \left\{4s^{-1/3}\left|h_2(x_1 s, x_2 s^{2/3})\right| + 2\sigma s^{1/3}\left|g_2(x_1 s, x_2 s^{2/3})\right|\right\} ds \\
&\le \frac{1}{3}\int_0^1 \left\{4s^{-1/3}|h_2|_{\sigma,\sigma} + 2\sigma s^{1/3}|g_2|_{\sigma,\sigma}\right\} ds \\
&= 2|h_2|_{\sigma,\sigma} + \frac{\sigma}{2}|g_2|_{\sigma,\sigma}.
\end{aligned} \tag{5.37}$$

□

Sublemma 5.2. *Let $f(x)$ be a holomorphic function in D_ρ. For any $\rho/2 < \sigma < \rho$, we have*

$$\sup_{|x_1|,|x_2|\le\sigma} |f(x)| = \sup_{|x_1|=|x_2|=\sigma} |f(x)| . \tag{5.38}$$

Sublemma 5.2 is an immediate consequence of the maximum modulus principle for holomorphic functions.

Using these sublemmas, we prove Lemma 5.5.

Proof of Lemma 5.5. Let $R(x)$ be an adjugate matrix of $3x_1 + 4x_2 P_0^T$, that is,

$$\begin{aligned} R(x) =& 16x_2^2 \left(P_0^T\right)^2 - 12x_1x_2P_0^T + 9x_1^2 + 8x_2^3 \\ =& \begin{pmatrix} 9x_1^2+8x_2^3 & -4x_1x_2^2 & 3x_1^2x_2 \\ -12x_1x_2 & 9x_1^2 & 2x_1x_2^2 \\ 16x_2^2 & -12x_1x_2 & 9x_1^2 \end{pmatrix}. \end{aligned} \tag{5.39}$$

Then $R(x)$ satisfies the following relation:

$$R\left(3x_1 + 4x_2P_0^T\right) = \left(3x_1 + 4x_2P_0^T\right) R = x_1\left(27x_1^2 + 8x_2^3\right). \tag{5.40}$$

It follows from this relation and the first equation of (4.165) that

$$\begin{aligned} x_1\left(27x_1^2+8x_2^3\right)\frac{\partial}{\partial x_1}\begin{pmatrix} c_{n,0} \\ c_{n,1} \\ c_{n,2}\end{pmatrix} =& R\left(3x_1+4x_2P_0^T\right)\frac{\partial}{\partial x_1}\begin{pmatrix} c_{n,0} \\ c_{n,1} \\ c_{n,2}\end{pmatrix} \\ =& R\begin{pmatrix} -x_1h_0 \\ -c_{n,1}+4h_1 \\ -2c_{n,2}+4h_2 \end{pmatrix}. \end{aligned} \tag{5.41}$$

Furthermore, by the explicit form of $R(x)$, we have the following estimate:

$$\|R\|_{\sigma,\sigma} \le 67\sigma^2 + 17\sigma^3. \tag{5.42}$$

Combining these properties of $R(x)$ with Sublemmas 5.1 and 5.2, we find

$$\begin{aligned} \left|\frac{\partial c_{n,k}}{\partial x_1}\right|_{\sigma,\sigma} =& \sup_{|x_1|=|x_2|=\sigma}\left|\frac{\partial c_{n,k}}{\partial x_1}(x)\right| \\ =& \sup_{|x_1|=|x_2|=\sigma}\left|\frac{1}{x_1\left(27x_1^2+8x_2^3\right)}\right| \cdot \left|x_1\left(27x_1^2+8x_2^3\right)\frac{\partial c_{n,k}}{\partial x_1}\right|_{\sigma,\sigma} \\ \le& \frac{\|R\|_{\sigma,\sigma}}{27\sigma^3-8\sigma^4}\left(|x_1h_0|_{\sigma,\sigma} + |-c_{n,1}+4h_1|_{\sigma,\sigma} + |-2c_{n,2}+4h_2|_{\sigma,\sigma}\right) \\ \le& \frac{67+17\sigma}{27\sigma-8\sigma^2}\left(\sigma|h_0|_{\sigma,\sigma} + 8|h_1|_{\sigma,\sigma} + 8|h_2|_{\sigma,\sigma} + \frac{2\sigma}{3}|g_1|_{\sigma,\sigma} + \sigma|g_2|_{\sigma,\sigma}\right) \\ \le& \frac{1000}{27\sigma-8\sigma^2}\sum_{i=0}^{2}\left(|g_i|_{\sigma,\sigma} + |h_i|_{\sigma,\sigma}\right) \end{aligned} \tag{5.43}$$

in view of $\sigma < \rho < 27/8$.

On the other hand, by the second equation of (4.165), we find

$$\left|\frac{\partial c_{n,k}}{\partial x_2}\right|_{\sigma,\sigma} \le 2\|P_0\|_{\sigma,\sigma}\sum_{i=0}^{2}\left|\frac{\partial c_{n,i}}{\partial x_1}\right|_{\sigma,\sigma} + \sum_{i=0}^{2}|g_i|_{\sigma,\sigma} \tag{5.44}$$

$$\leq \left(\frac{6000C}{27\sigma - 8\sigma^2} + 1 \right) \sum_{i=0}^{2} \left(|g_i|_{\sigma,\sigma} + |h_i|_{\sigma,\sigma} \right).$$

Hence we obtain the estimates (5.31) for C_2 defined by (5.30). □

Using Lemmas 5.4 and 5.5, we can give the estimates for $T_{n,1}(x)$ and $(\partial T_{n,1}/\partial x_j)(x)$.

Lemma 5.6. *For any $\rho/2 < \sigma < \rho$, we have*

$$\|T_{n,1}\|_{\sigma,\sigma} \leq 6\left(C^2 + C + 3\right) C_1 C_2 \widetilde{\alpha}^{n-2} n! \,(\rho - \sigma)^{-n}, \tag{5.45}$$

$$\left\| \frac{\partial T_{n,1}}{\partial x_j} \right\|_{\sigma,\sigma} \leq 6\left(3C^2 + 2C + 3\right) C_1 C_2 \widetilde{\alpha}^{n-2} n! \,(\rho - \sigma)^{-n}. \tag{5.46}$$

Proof. Using Lemmas 5.4 and 5.5, we find

$$\begin{aligned}
\|T_{n,1}\|_{\sigma,\sigma} &\leq \left\| c_{n,2} P_0^2 + c_{n,1} P_0 + c_{n,0} \right\|_{\sigma,\sigma} \\
&\leq |c_{n,2}|_{\sigma,\sigma} \|P_0\|_{\sigma,\sigma}^2 + |c_{n,1}|_{\sigma,\sigma} \|P_0\|_{\sigma,\sigma} + 3\,|c_{n,0}|_{\sigma,\sigma} \\
&\leq \left(C^2 + C + 3\right) C_2 \sum_{i=0}^{2} \left(|g_i|_{\sigma,\sigma} + |h_i|_{\sigma,\sigma} \right) \\
&\leq 6\left(C^2 + C + 3\right) C_1 C_2 \widetilde{\alpha}^{n-2} n! \,(\rho - \sigma)^{-n},
\end{aligned} \tag{5.47}$$

$$\begin{aligned}
\left\| \frac{\partial T_{n,1}}{\partial x_j} \right\|_{\sigma,\sigma} &\leq \left\| \frac{\partial c_{n,2}}{\partial x_j} P_0^2 + \frac{\partial c_{n,1}}{\partial x_j} P_0 + \frac{\partial c_{n,0}}{\partial x_j} + c_{n,2} \frac{\partial}{\partial x_j} P_0^2 + c_{n,1} \frac{\partial}{\partial x_j} P_0 \right\|_{\sigma,\sigma} \\
&\leq \left| \frac{\partial c_{n,2}}{\partial x_j} \right|_{\sigma,\sigma} \|P_0\|_{\sigma,\sigma}^2 + \left| \frac{\partial c_{n,1}}{\partial x_j} \right|_{\sigma,\sigma} \|P_0\|_{\sigma,\sigma} + 3 \left| \frac{\partial c_{n,0}}{\partial x_j} \right|_{\sigma,\sigma} \\
&\quad + 2\,|c_{n,2}|_{\sigma,\sigma} \|P_0\|_{\sigma,\sigma} \left\| \frac{\partial P_0}{\partial x_j} \right\|_{\sigma,\sigma} + |c_{n,1}|_{\sigma,\sigma} \left\| \frac{\partial P_0}{\partial x_j} \right\|_{\sigma,\sigma} \\
&\leq \left(3C^2 + 2C + 3\right) C_2 \sum_{i=0}^{2} \left(|g_i|_{\sigma,\sigma} + |h_i|_{\sigma,\sigma} \right) \\
&\leq 6\left(3C^2 + 2C + 3\right) C_1 C_2 \widetilde{\alpha}^{n-2} n! \,(\rho - \sigma)^{-n}.
\end{aligned} \tag{5.48}$$

□

Using these lemmas, we now prove Proposition 5.1.

Proof of Proposition 5.1. By Lemmas 5.3 and 5.6, we obtain

$$\begin{aligned}
\|T_n\|_{\sigma,\sigma} &\leq \|T_{n,1}\|_{\sigma,\sigma} + \|T_{n,2}\|_{\sigma,\sigma} \\
&\leq \widehat{C}_1 \widetilde{\alpha}^{n-2} n! \,(\rho - \sigma)^{-n},
\end{aligned} \tag{5.49}$$

$$\begin{aligned}
\left\| \frac{\partial T_n}{\partial x_j} \right\|_{\sigma,\sigma} &\leq \left\| \frac{\partial T_{n,1}}{\partial x_j} \right\|_{\sigma,\sigma} + \left\| \frac{\partial T_{n,2}}{\partial x_j} \right\|_{\sigma,\sigma} \\
&\leq \widehat{C}_2 \widetilde{\alpha}^{n-2} n! \,(\rho - \sigma)^{-n},
\end{aligned} \tag{5.50}$$

where

$$\widehat{C}_1 = \left(\frac{\widehat{C}_{1,1}}{\widetilde{\alpha}} + \frac{\widehat{C}_{1,2}}{\widetilde{\alpha}^2} \right) \widetilde{C} + \left(\widehat{C}_{1,3} + \frac{\widehat{C}_{1,4}}{\widetilde{\alpha}} \right), \quad \widehat{C}_2 = \left(\frac{\widehat{C}_{2,1}}{\widetilde{\alpha}} + \frac{\widehat{C}_{2,2}}{\widetilde{\alpha}^2} \right) \widetilde{C} + \left(\widehat{C}_{2,3} + \frac{\widehat{C}_{2,4}}{\widetilde{\alpha}} \right) \tag{5.51}$$

with some positive constants $\widehat{C}_{i,j}$ $(i = 1, 2, j = 1, 2, 3, 4)$ being independent of $\widetilde{C}$, $\widetilde{\alpha}$, σ, n. Take a constant $\widetilde{\alpha}$ largely enough so that

$$\frac{\widehat{C}_{1,1}}{\widetilde{\alpha}} + \frac{\widehat{C}_{1,2}}{\widetilde{\alpha}^2}, \frac{\widehat{C}_{2,1}}{\widetilde{\alpha}} + \frac{\widehat{C}_{2,2}}{\widetilde{\alpha}^2} < 1 \tag{5.52}$$

hold. Then there exists a positive constant $\widetilde{C}$ satisfying

$$\widehat{C}_1, \widehat{C}_2 \leq \widetilde{C}. \tag{5.53}$$

This completes the proof of Proposition 5.1. □

Using Proposition 5.1, we can verify the estimate (4.143). In fact, taking $\sigma = (3/4)\rho$, we obtain from (5.13)

$$\begin{aligned} \|T_n\|_{(3/4)\rho,(3/4)\rho} \leq & \widetilde{C}\widetilde{\alpha}^{n-2} n! \left(\frac{\rho}{4}\right)^{-n} \\ = & \widetilde{C}\widetilde{\alpha}^{-2} \left(\frac{4\widetilde{\alpha}}{\rho}\right)^n n!. \end{aligned} \tag{5.54}$$

Hence, if we define positive constants $\widetilde{C}^\dagger$, $\widetilde{\alpha}^\dagger$ and $\rho^\dagger$ by

$$\widetilde{C}^\dagger = \widetilde{C}\widetilde{\alpha}^{-2}, \quad \widetilde{\alpha}^\dagger = \frac{4\widetilde{\alpha}}{\rho}, \quad \rho^\dagger = \frac{3\rho}{4}, \tag{5.55}$$

the estimate (4.143) holds for $\widetilde{C} = \widetilde{C}^\dagger$, $\widetilde{\alpha} = \widetilde{\alpha}^\dagger$ and $\rho = \rho^\dagger$.

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