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By

Takashi KUMAGAI and Ofer ZEITOUNI

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京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

Fluctuations of recentered maxima of discrete Gaussian Free Fields on a class of recurrent graphs

Takashi Kumagai* **Ofer Zeitouni** †

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Abstract

We provide conditions that ensure that the recentered maximum of the Gaussian free field on a sequence of graphs fluctuates at the same order as the field at the point of maximal variance. In particular, on a sequence of such graphs the recentered maximum is not tight, similarly to the situation in $\mathbb{Z}$ but in contrast with the situation in $\mathbb{Z}^2$. We show that our conditions cover a large class of "fractal" graphs.

1 Introduction

The study of the maxima of Gaussian fields has a rich history, which we will not attempt to survey here. The general theory was developed in the 70s and 80s, and an excellent account can be found in [18]. However, general results concerning the order of fluctuations of the maximum are lacking.

In recent years, a special effort has been directed toward the study of the so called Gaussian free field (GFF) on various graphs. While we postpone the general definition to the next section, we discuss in this introduction the special case of the GFF on subsets $V_N = ([-N, N] \cap \mathbb{Z})^d$, with Dirichlet boundary conditions. These are random fields $\{X_x\}_{x \in V_N}$ indexed by points in V_N, with joint density (with respect to Lebesgue measure) proportional to

$$\exp\left(-c \sum_{x \sim y} (X_x - X_y)^2\right),$$

with the sum over neighbors in V_N, and $X_x = 0$ for $x \in \partial V_N$. (An alternative description involving the Green function of random walk on V_N is given below in Section 2; see also [20]

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for a very readable introduction to GFFs in a continuous setting.) With $X^*_{N,d}$ denoting the maximum of the GFF on V_N in dimension d, it is not hard to see that $X^*_{N,d}$ is of order $\sqrt{N}$ for $d = 1$, order $\log N$ for $d = 2$, and order $(\log N)^{1/2}$ for $d \geq 3$. Moreover, a consequence of the Borell-Tsirelson inequality (see [18]) is that for $d \geq 3$, since simple random walk is transient on $\mathbb{Z}^d$, the fluctuations of $X^*_{N,d}$ are at most of order 1, while for $d = 1$ the fluctuations of $X^*_{N,1}$ are of the same order as $X^*_{N,1}$, i.e. of order $\sqrt{N}$. The critical case $d = 2$ was settled only recently [8], where it was shown that the fluctuations of $X^*_{N,2}$ are also of order 1. This raises naturally the question of determining for which sequences of graphs is the sequence of recentered maxima of the GFF tight.

Our goal in this paper is to exhibit a class of sequences of graphs, which are fractal-like and for which the maximum of the GFF fluctuates at the same order as the maximum itself, and both are of the order of the maximal standard deviation of the GFF in the graph. In that respect, the behavior of the maximum is similar to that of $X^*_{N,1}$. For this class of graphs, we also show that the *cover time* of the graph, measured in terms of the (square root of the) local time at a fixed vertex, also does not concentrate. (We note in passing that for the cover time of V_N in two dimensions, it is, to the best of our knowledge, an open problem to decide whether this quantity concentrates or not.)

The structure of the paper is as follows. In the next section, we introduce the GFF on general graphs and state Assumption 2.1 that characterizes the graphs which we investigate; the main feature is a relation between the graph distance and the resistance, and control of the covering number of the graph in terms of resistance distance. We then state our main result, Theorem 2.2, concerning fluctuations of the maximum of the GFF. We also state Proposition 2.4 concerning the cover time of the graphs. Proofs of the theorem and proposition are given in Section 3. The heart of the paper is then Section 4.1, where we show that certain naturally constructed fractal-like graphs satisfy our assumptions. In particular, this is the case for the standard Sierpinski carpets in two dimensions and gaskets in all dimensions.

Notation Throughout the paper, we use $c_1, c_2, \cdots$ to denote generic constants, independent of N, whose exact values are not important and may change from line to line. We write $a_n \asymp b_n$ if there exist constants $c_1, c_2 > 0$ such that $c_1 b_n \leq a_n \leq c_2 b_n$ for all $n \in \mathbb{N}$.

2 Framework

We first introduce general notation for finite graphs with a 'wired' boundary and their associated resistance. Let $G = (V(G), E(G))$ be a connected (undirected) finite graph with at least two vertices, where $V(G)$ denotes the vertex set and $E(G)$ the edge set of G. Let d_G be the graph distance, that is, $d_G(x, y)$ is the number of edges in the shortest path from x to

y in G. Define a symmetric weight function $\mu^G : V(G) \times V(G) \to \mathbb{R}_+$ that satisfies $\mu^G_{xy} > 0$ if and only if $\{x, y\} \in E(G)$. For $B \subset G$ with $B \neq G$ and for distinct $x, y \in V(G)$ not both in B, we define the *resistance* between x and y by

$$R_B(x,y)^{-1} := \inf\{ \frac{1}{2} \sum_{w,z \in V(G)} (f(w) - f(z))^2 \mu^G_{wz} : f(x) = 1, f(y) = 0, f|_B = \text{constant}\}.$$

We set $R_B(x, x) = 0$, $R_B(x, y) = 0$ if $x, y \in B$ and, for $x \in V(G) \setminus B$, we define $R_B(x, B) = R_B(x, y)$ for any $y \in B$. We write $R(x, y) := R_\emptyset(x, y)$.

The resistance $R_B(\cdot, \cdot)$ is the resistance of the following electrical network with a 'wired' boundary: Consider the graph $\overline{G}$ obtained by combining all vertices in B to a single vertex b, that is $V(\overline{G}) = (V(G) \setminus B) \cup \{b\}$ and

$$\begin{aligned} E(\overline{G}) &= \{\{x, y\} : \{x, y\} \in E(G), x, y \in G \setminus B\} \\ &\quad \bigcup \{\{x, b\} : x \in G \setminus B, \exists y \in B \text{ with } \{x, y\} \in E(G)\}. \end{aligned}$$

Define the modified symmetric weight function

$$\mu^{\overline{G}}_{xy} = \begin{cases} \mu^G_{xy}, & x \in V(\overline{G}) \setminus \{b\},\ y \in V(\overline{G}) \setminus \{b\}, \\ \sum_{z \in B} \mu^G_{xz}, & x \in V(\overline{G}) \setminus \{b\},\ y = b, \end{cases}$$

and set as before $\mu^{\overline{G}}_x = \sum_{y \in V(\overline{G})} \mu^{\overline{G}}_{xy}$. Let $\{\overline{w}_t\}_{t \geq 0}$ be the continuous time random walk on $\overline{G}$ such that the holding time at a vertex is exp(1), and the jump probability is given by $\mu_{x,y}/\mu_x$. Let

$$L^{x,N}_t = \frac{1}{\mu^{\overline{G}}_x} \int_0^t \mathbf{1}_{\{\overline{w}_s = x\}} ds$$

denote the (weight normalized) local time at x.

Now, let $\{G^N\}_{N \geq 1}$ be a sequence of finite connected graphs such that $|G^N| \geq 2$ for all $N \geq 1$ and $\lim_{N \to \infty} |G^N| = \infty$. For each $G^N = (V(G^N), E(G^N))$, we take a symmetric weight function μ^{G^N}, a boundary $B^N \subset G^N$ with $B^N \neq G^N$, and the corresponding continuous time Markov chain $\{\overline{w}^N_t\}_{t \geq 0}$ with the wired boundary condition on B^N as above. We assume that $G^N \setminus B^N$ is connected. Let $T^N := \min\{t \geq 0 : \overline{w}^N_t = b\}$, and define, for each $x, y \in V(G^N) \setminus B^N$, $\mathcal{G}_N(x, y) = E^x_{G^N}[L^{y,N}_{T^N}]$ where $E^x_{G^N}$ denotes the expectation with respect to $\overline{w}^N_t$ started at x. For $z \in B^N$, we set $X^N_z \equiv 0$. The *Gaussian free field* (GFF for short) on G^N (with boundary B^N) is the zero-mean Gaussian field $\{X^N_z\}_{z \in V(G^N)}$ with covariance $\mathcal{G}_N(\cdot, \cdot)$. It can be easily checked (using for instance [9, Lemma 2.1], [15, Proposition 3.6]) that

$$E[(X^N_x - X^N_y)^2] = R_{B^N}(x, y).$$

Let $h : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function with $h(0) = 0$, that satisfies the following doubling property: there exist $0 < \beta_1 \leq \beta_2 < \infty$ and $C > 0$ such that, for all $0 < r \leq R < \infty$,

$$C^{-1} \left(\frac{R}{r}\right)^{\beta_1} \leq \frac{h(R)}{h(r)} \leq C \left(\frac{R}{r}\right)^{\beta_2}. \tag{2.1}$$

We assume the following.

Assumption 2.1. *There exist $\alpha > 0$ and $c_1, c_2, c_3 > 0$ such that the following hold for all large N.*
(i) $R_{B^N}(x,y) \le c_1 h(d_{G^N}(x,y))$ *for all* $x, y \in G^N$.
(ii) $\max_{x\in G^N} R_{B^N}(x, B^N) \ge c_2 \max_{x\in G^N} h(d_{G^N}(x, B^N))$ *for all* $x \in G^N$.
(iii) $\mathcal{N}_{G^N}(\delta d^N_{max}) \le c_3 \delta^{-\alpha}$ *for all* $\delta \in (0,1]$ *where* $d^N_{max} := \max_{x\in G^N} d_{G^N}(x, B^N)$ *and* $\mathcal{N}_{G^N}(\varepsilon)$ *is the minimal number of* d_{G^N}*-balls of radius* ε *needed to cover* G^N. *Furthermore,* $d^N_{max} \to \infty$ *as* $N \to \infty$.

Let $X^*_N = \max_{z\in V(G^N)} X^N_z$ and define $\tilde{X}_N = X^*_N/\overline{\sigma}_N$, where $\overline{\sigma}_N = (\max_{z\in G^N} E[(X^N_z)^2])^{1/2}$. Note that $\overline{\sigma}^2_N = \max_{x\in G^N} R_{B^N}(x, B^N)$, and $\lim_{N\to\infty} \overline{\sigma}_N = \infty$ under Assumption 2.1(iii).

Theorem 2.2. *Under Assumption 2.1, there exist constants $A, B, A' > 0$ and a function $g : (0,\infty) \to (0,1)$ such that the following holds for all N large.*

$$P(\tilde{X}_N < A) > B, \quad P(\tilde{X}_N > c) \ge g(c) \ \ \forall c > 0, \quad E(\tilde{X}_N) \le A'. \tag{2.2}$$

In particular, under Assumption 2.1, $\{X^*_N - EX^*_N\}_N$ fluctuates with order $\overline{\sigma}_N$ and therefore it is not tight.

Remark 2.3. *We stated Assumption 2.1 with respect to the graph distance in G^N, because this will be easiest to check in the applications. However, one should note that the proof of Theorem 2.1 does not depend on the particular metric chosen, as long as the metric satisfies the assumption. In particular, if we choose $R_{B^N}(\cdot,\cdot)$ as the metric, Assumption 2.1 (i), (ii) turns out to be trivial with $h(s) = s$, and the assumption boils down to $\mathcal{N}_{R_{B^N}}(\delta\overline{\sigma}^2_N) \le c_3\delta^{-\alpha}$ for all $\delta \in (0,1]$ and $\lim_{N\to\infty} \overline{\sigma}_N = \infty$, where $\mathcal{N}_{R_{B^N}}(\varepsilon)$ is the minimal number of R_{B^N}-balls of radius ε needed to cover G^N.*

In a recent seminal work, [9] have established a close relation between the expectation of the maximum of the GFF on general graphs and the expected cover time of these graphs by random walk. Under the assumptions of Theorem 2.2, one can also derive information on the fluctuations of the cover time, as follows. Define the *cover time* of $\overline{G}^N$ as

$$\tau^N_{\mathrm{COV}} = \inf\{t > 0 : L^{x,N}_t > 0, \forall x \in \overline{G}^N\}.$$

It is easy to see

$$\tau^N_{\mathrm{COV}} = \inf\{t > 0 : \forall x \in \overline{G}^N, \exists s \le t \text{ such that } \overline{w}^N_s = x\}.$$

We will consider the square-root of the normalized local time at B^N at cover time, i.e. the random variable $L^N := \sqrt{L^{b,N}_{\tau^N_{\mathrm{COV}}}}$. One expects (see [9]) that L^N should behave similarly to $|X^*_N|$. In the special case of G^N being the rooted at b binary tree of depth N, this was confirmed in [7]. In our setup here, this is confirmed in the following proposition.

Proposition 2.4. *With notation as above and under Assumption 2.1, the conclusion of Theorem 2.2 hold with $L^N/\overline{\sigma}_N$ replacing $\tilde{X}_N$.*

3 Proofs of Theorem 2.2 and Proposition 2.4

We begin with the proof of Theorem 2.2.

Proof of Theorem 2.2: Let $\tilde{d}(x,y) = (E[(X_x^N - X_y^N)^2])^{1/2}/\overline{\sigma}_N = R_{B^N}(x,y)^{1/2}/\overline{\sigma}_N$. Then, using Assumption 2.1 (i),(ii), there exists $c > 0$ such that for all $x, y \in G^N$ with $d_{G^N}(x,y) \leq d_{max}^N$ and all $N \in \mathbb{N}$,

$$\tilde{d}(x,y) = \Big(\frac{R_{B^N}(x,y)}{\overline{\sigma}_N^2}\Big)^{1/2} \leq c\Big(\frac{h(d_{G^N}(x,y))}{h(d_{max}^N)}\Big)^{1/2} \leq cC\Big(\frac{d_{G^N}(x,y)}{d_{max}^N}\Big)^{\beta_1/2}.$$

Thus, denoting $\mathcal{N}_{\tilde{d}}(\varepsilon)$ the minimal number of $\tilde{d}$-balls of radius ε needed to cover G^N, we have

$$\mathcal{N}_{\tilde{d}}(cC\delta^{\beta_1/2}) \leq \mathcal{N}_{G^N}(\delta d_{max}^N) \leq c_3\delta^{-\alpha},$$

where we used Assumption 2.1 (iii) in the second inequality. Rewriting this, we have $\mathcal{N}_{\tilde{d}}(\varepsilon) \leq c'\varepsilon^{-2\alpha/\beta_1}$, where $c' > 0$ is independent of N. Set $\gamma = 2\alpha/\beta_1$. We can apply [1, Theorem 5.2] to deduce that there exist $\lambda_0 > 0$ and N_0 such that for all $\lambda > \lambda_0$, $\varepsilon > 0$ and $N > N_0$,

$$P(\tilde{X}_N > \lambda) \leq C_\gamma \lambda^{\gamma+1+\varepsilon}\Psi(\lambda),$$

where $C_\gamma \geq 1$ does not depend on N and $\Psi(\lambda) = (2\pi)^{-1/2}\int_\lambda^\infty e^{-x^2/2}dx$. On the other hand, let x_N^* be such that $E(X_{x_N^*}^2) = \overline{\sigma}_N^2$. Then, for any $\lambda > 0$,

$$P(\tilde{X}_N > \lambda) \geq P(X_{x_N^*}^N > \lambda\overline{\sigma}_N) = \Psi(\lambda)\,.$$

The estimates in (2.2) are easy consequences of the last two displayed inequalities. □

We turn to the analysis of cover times.

Proof of Proposition 2.4: The upper bound in the proposition is a consequence of the Eisenbaum-Kaspi-Marcus-Rosen-Shi isomorphism theorem [10], as was observed in [9]: indeed, by [9, Eq. (20),(21)] and using the last estimate in (2.2), there exist constants $c_1, c_2 > 0$ so that with $t = \theta\overline{\sigma}_N^2$, and all θ large enough,

$$P(\min_x L_{\tau^N(t)}^{x,N} \leq t/2) \leq c_1 e^{-c_2\theta} \tag{3.1}$$

while

$$P(\max_x L_{\tau^N(t)}^{x,N} \geq 2t) \leq c_1 e^{-c_2\theta}\,, \tag{3.2}$$

where $\tau^N(t) := \inf\{s > 0 : L_s^{b,N} > t\}$.

On the event $\{\min_x L^{x,N}_{\tau^N(t)} \geq t/2\}$ we have that $\tau^N(t) \geq \tau^N_{\rm COV}$. Thus, on the event

$$\{\min_x L^{x,N}_{\tau^N(t)} \geq t/2\} \cap \{\max_x L^{x,N}_{\tau^N(t)} \leq 2t\}\,,$$

one has that

$$L^{b,N}_{\tau^N_{\rm COV}} \leq L^{b,N}_{\tau^N(t)} \leq \max_x L^{x,N}_{\tau^N(t)} \leq 2t\,. \tag{3.3}$$

In particular, (3.1), (3.2) and (3.3) imply that $EL^N/\overline{\sigma}_N$ is bounded uniformly.

To estimate L^N from below, we use the Markov property. Let $x^* \in V(\overline{G}^N)$ be such that $R_{B^N}(x^*, B^N) = \overline{\sigma}^2_N$ and let $T_{x^*} = \inf\{t : \overline{w}^N_t = x^*\}$. Since $\tau^N_{\rm COV} \geq T_{x^*}$, we have that $L^N \geq \sqrt{L^{b,N}_{T_{x^*}}}$. We decompose the walk $\overline{w}^N_t$ according to excursions from b: the probability to hit x^* during one excursion (see e.g. [19, Ch. 2]) is

$$p_N = \frac{1}{\overline{\sigma}^2_N \mu_N}\,,$$

where $\mu_N = \mu_b^{\overline{G}^N}$. Therefore,

$$L^{b,N}_{T_{x^*}} \stackrel{d}{=} \frac{1}{\mu_N} \sum_{i=1}^{Z_N} \mathcal{E}_i\,,$$

where Z_N is geometric of parameter p_N and $\mathcal{E}_i$ are standard independent exponential random variables. Note that $EL^{b,N}_{T_{x^*}} = \overline{\sigma}^2_N$.

Consider now a parameter $\xi > 0$. We have that

$$\begin{aligned} P(L^{b,N}_{T_{x^*}} \geq \xi\overline{\sigma}^2_N) &\geq P(Z_N \geq \xi/p_N) P\left(\frac{1}{\mu_N}\sum_{i=1}^{\xi/p_N} \mathcal{E}_i > \xi\overline{\sigma}^2_N\right) \\ &\geq P(Z_N \geq \xi/p_N) P\left(\frac{p_N}{\xi}\sum_{i=1}^{\xi/p_N} \mathcal{E}_i \geq 1\right) =: P_1 P_2\,. \end{aligned}$$

Note that from the properties of the geometric distribution, regardless of p_N we have that $P_1 \geq c_1(\xi) > 0$. On the other hand, if $p_N \to 0$ then $p_N \sum_{i=1}^{1/p_N} \mathcal{E}_i \to 1$ a.s., and in any case we also have that $P_2 \geq c_2(\xi) > 0$. We conclude that

$$P(L^N \geq \sqrt{\xi}\overline{\sigma}_N) \geq c_1(\xi)c_2(\xi)\,.$$

□

4 Examples

4.1 Nested fractal graphs and strongly recurrent Sierpinski carpet graphs

Let $\{\psi_i\}_{i=1}^K$ be a family of L-similitudes on $\mathbb{R}^d$ for some $L > 1$, that is, for each i, ψ_i is a map from $\mathbb{R}^d$ to $\mathbb{R}^d$ such that $\psi_i(x) = L^{-1}U_i x + \gamma_i,\ x \in \mathbb{R}^d$, where U_i is a unitary map and $\gamma_i \in \mathbb{R}^d$. We assume that $\{\psi_i\}_{i=1}^K$ satisfies the open set condition, namely there exists a non-empty bounded set $O \subset \mathbb{R}^d$ such that $\{\psi_i(O)\}_{i=1}^K$ are disjoint and $\cup_{i=1}^K \psi_i(O) \subset O$. Since $\{\psi_i\}_{i=1}^K$ is a family of contraction maps, there exists a unique non-empty compact set F such that $F = \cup_{i=1}^K \psi_i(F)$. We assume that F is connected.

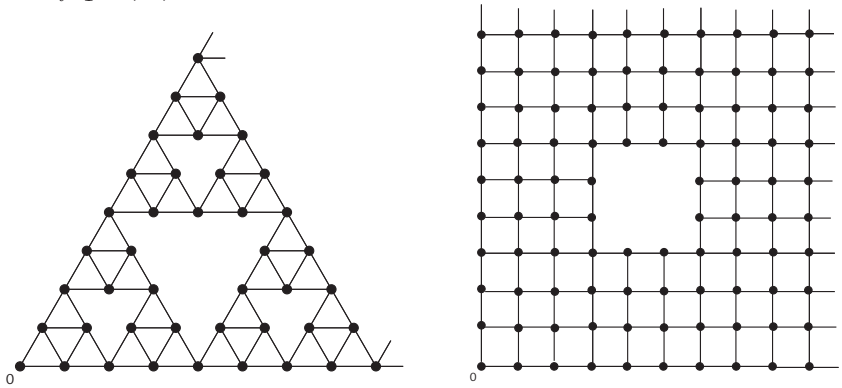

Fig 1: 2-dimensional Sierpinski gasket graph and carpet graph

4.1.1 Nested fractal graphs

Let Ξ be the set of fixed points of $\{\psi_i\}_{i=1}^K$, and define

$$V_0 := \{x \in \Xi :\ \exists i, j \in \{1, \dots, K\},\ i \neq j \text{ and } y \in \Xi \text{ such that } \psi_i(x) = \psi_j(y)\}.$$

Assume that $\#V_0 \geq 2$ and set $\psi_{i_1 \dots i_n} := \psi_{i_1} \circ \cdots \circ \psi_{i_n}$. F is then called a nested fractal if the following holds.

- (Nesting) If $i_1 \dots i_n$ and $j_1 \dots j_n$ are distinct sequences in $\{1, \dots, K\}$, then
$$\psi_{i_1 \dots i_n}(F) \cap \psi_{j_1 \dots j_n}(F) = \psi_{i_1 \dots i_n}(V_0) \cap \psi_{j_1 \dots j_n}(V_0).$$
- (Symmetry) If $x, y \in V_0$, then the reflection in the hyperplane $H_{xy} := \{z \in \mathbb{R}^d : |z - x| = |z - y|\}$ maps $\bigcup_{i_1, \dots, i_n = 1}^K \psi_{i_1 \dots i_n}(V_0)$ to itself.

We assume without loss of generality that $\psi_1(x) = L^{-1}x$ and that the origin belongs to V_0. Let

$$V(G^N) := \bigcup_{i_1, \dots, i_N = 1}^{K} L^N \psi_{i_1 \dots i_N}(V_0), \quad G := \bigcup_{N=1}^{\infty} V(G^N). \tag{4.1}$$

Next, define $B_0 := \{\{x, y\} : x \neq y \in V_0\}$. Then inside each $L^N\psi_{i_1\dots i_N}(V_0)$, $N \geq 0, 1 \leq i_1, \cdots, i_N \leq K$, we place a copy of B_0 and denote by B the set of all the edges determined in this way. Next, we assign $\mu_{xy} = \mu_{yx} > 0$ for each $\{x, y\} \in B$ in such a way that there exist $c_1, c_2 > 0$ such that

$$c_1 \leq \mu_{xy} = \mu_{yx} \leq c_2, \qquad \forall \{x, y\} \in B.$$

We call the graph (G, μ) a nested fractal graph. A typical example is the 2-dimensional Sierpinski gasket graph in Fig 1 (where $L = 2$). Let $d(\cdot, \cdot)$ be the graph distance on G, $\{w_k\}_k$ the Markov chain for (X, μ), and define the heat kernel as $p_k(x, y) = P^x(w_k = y)/\mu_y$. (Note that we consider the discrete time Markov chain here in order to apply the results in [5] to derive the resistance estimates (4.5). Indeed, (4.5) can be obtained through both discrete and continuous time Markov chains.) It is known (see [12] (also [16] for the continuous setting)) that there exist constants $c_3, \dots, c_6$ such that for all $x, y \in G, k > 0$

$$p_k(x, y) \leq c_3 k^{-d_f/d_w} \exp\left(-c_4 \left(\frac{d(x, y)^{d_w}}{k}\right)^{1/(d_w-1)}\right), \tag{4.2}$$

and for $k > d(x, y)$,

$$p_k(x, y) + p_{k+1}(x, y) \geq c_5 k^{-d_f/d_w} \exp\left(-c_6 \left(\frac{d(x, y)^{d_w}}{k}\right)^{1/(d_w-1)}\right), \tag{4.3}$$

where $d_w = \log(\rho K)/\log(L\eta)$, $d_f = \log K/\log(L\eta)$ with some constants $\rho > 1$, $\eta \geq 1$. d_f is called the *Hausdorff dimension* and d_w is called the *walk dimension*. For the 2-dimensional Sierpinski gasket graph, $L = 2, \eta = 1$, $K = 3$ and $\rho = 5/3$. Noting that $d_w > d_f$ and that

$$c_7 R^{d_f} \leq \mu(B(x, R)) \leq c_8 R^{d_f}, \quad \forall x \in G, R \geq 1, \tag{4.4}$$

(4.2), (4.3) implies (see [5, Theorem 1.3, Lemma 2.4])

$$R(x, y) \leq c_9 d(x, y)^{d_w - d_f}, \quad R(x, B^c(x, R)) \geq c_{10} R^{d_w - d_f}, \ \ \forall x, y \in G, \ \forall R \geq 1. \tag{4.5}$$

We now define a sequence of graphs $\{G^N\}_{N\geq 0}$ by setting $V(G^N)$ as above and $E(G^N) := \{\{x, y\} \in B : x, y \in V(G^N)\}$. Let $d_{G^N}(\cdot, \cdot)$ be the graph distance on G^N; one can easily see that $d(x, y) \leq d_{G^N}(x, y)$ for $x, y \in G^N$. (Note that $|x - y| \asymp d_{G^N}(x, y)^{\log L/\log(L\eta)}$ for $x, y \in G^N$ (cf. [16, Section 3]) and $\log L/\log(L\eta)$ is called the chemical-distance exponent.)

Let $B^N := L^N V_0$. Clearly $R_{B^N}(x, y) \leq R(x, y)$ for $x, y \in G^N$ and $d^N_{max} \asymp d_{G^N}(0, B^N) \asymp (L\eta)^N$. So (4.5) implies Assumption 2.1 (i),(ii) with $h(s) = s^{d_w - d_f}$, and (4.4) with the self-similarity of the graph imply Assumption 2.1 (iii) with $\alpha = d_f$. We note that we can actually take B^N arbitrary as long as $d^N_{max} \asymp (L\eta)^N$.

4.1.2 Strongly recurrent Sierpinski carpet graphs

Let $H_0 = [0,1]^d$, and let $L \in \mathbb{N}$, $L \geq 2$ be fixed. Set $\mathcal{Q} = \{\Pi_{i=1}^d[(k_i-1)/L, k_i/L] : 1 \leq k_i \leq L\ (1 \leq i \leq d)\}$, let $L \leq K \leq L^d$ and let $\{\psi_i\}_{i=1}^K$ be a family of L-similitudes of H_0 onto some element of $\mathcal{Q}$. We assume that the sets $\psi_i(H_0)$ are distinct, and as before assume $\psi_1(x) = L^{-1}x$. Set $H_1 = \cup_{i=1}^K \psi_i(H_0)$. Then, there exists a unique non-void compact set $F \subset H_0$ such that $F = \cup_{i=1}^K \psi_i(F)$. We assume F is connected. F is called a (generalized) Sierpinski carpet if the following hold (cf. [4]):
(SC1) (Symmetry) H_1 is preserved by all the isometries of the unit cube H_0.
(SC2) (Non-diagonality) Let B be a cube in H_0 which is the union of 2^d distinct elements of $\mathcal{Q}$. (So B has side length $2L^{-1}$.) Then if $\mathrm{Int}(H_1 \cap B)$ is non-empty, it is connected.
(SC3) (Borders included) H_1 contains the line segment $\{x : 0 \leq x_1 \leq 1, x_2 = \cdots = x_d = 0\}$.

The main difference from nested fractals is that Sierpinski carpets are infinitely ramified, i.e. F cannot be disconnected by removing a finite number of points.

Let V_0 be a set of vertices in H_0 and define $V(G^N)$ and G as in (4.1). Set $B_0 := \{\{x,y\} : x \neq y \in V_0, |x-y| = 1\}$, and define B and μ_{xy} as in the case of nested fractal graphs. We call the graph (G,μ) a Sierpinski carpet graph. A typical example is the 2-dimensional Sierpinski carpet graph in Fig 1.

It is known, see [3] and also [4] for the continuous setting, that (4.2), (4.3) hold, where $d_w = \log(\rho K)/\log L$, $d_f = \log K/\log L$ with some constant $\rho > 0$. For the 2-dimensional Sierpinski gasket graph, $L = 3$, $K = 8$ and $\rho > 1$. Let us restrict ourselves to the case $\rho > 1$, namely $d_w > d_f$. In this case, since (4.4) holds, we can show that (4.2) and (4.3) imply (4.5) as before. Arguing further as before, we have Assumption 2.1 (i)–(iii) with $h(s) = s^{d_w - d_f}$ and $\alpha = d_f$.

4.2 Homogeneous random Sierpinski carpet graphs

Let $\ell \geq 2$ and $I := \{1, \cdots, \ell\}$. For each $k \in I$, let $\{\psi_i^k\}_{i=1}^{K_k}$ be a family of L_k-similitudes as in the definition of the Sierpinski carpet graphs. As before, we assume $\psi_1^k(x) = L_k^{-1}x$. For $\xi = (k_1, \cdots, k_n, \cdots) \in I^\infty$ and $n \in \mathbb{N}$, write $\xi|_N = (k_1, \cdots, k_N) \in I^N$, and let

$$V(G^N_{\xi|_N}) := \bigcup_{\substack{i_j \in \{1,\cdots,K_{k_j}\},\\ 1 \leq j \leq N}} L_{k_1}\cdots L_{k_N}\psi_{i_N}^{k_N} \circ \cdots \circ \psi_{i_1}^{k_1}(V_0), \quad G_\xi := \bigcup_{N=1}^{\infty} V(G^N_{\xi|_N}). \tag{4.6}$$

Let $B_0 := \{\{x,y\} : x \neq y \in V_0, |x-y| = 1\}$, and define $B = B_\xi$ as in the cases of nested fractal graphs and carpet graphs. For simplicity, put weight $\mu_{xy} \equiv 1$ for each $\{x,y\} \in B$. We call the graph (G_ξ, μ_ξ) a homogeneous (random) Sierpinski carpet graph.

Fix $n \in \mathbb{N}$, $\xi|_n = (k_1, \cdots, k_n) \in I^n$, and let $B_n = L_{k_1}\cdots L_{k_n}$, $M_n = K_{k_1}\cdots K_{k_n}$. We write R_n for the effective resistance between $\{0\} \times [0, B_n]^{d-1} \cap G^n_{\xi|_n}$ and $\{B_n\} \times [0, B_n]^{d-1} \cap G^n_{\xi|_n}$

in $G^n_{\xi|_n}$, and define $T_n = R_n M_n$. Now set

$$d_f(n) = \frac{\log M_n}{\log B_n}, \qquad d_w(n) = \frac{\log T_n}{\log B_n}.$$

For $x \in G_\xi$ and $r \geq 1$, let $V_d(x,r)$ be the number of vertices in the ball of radius r centered at x w.r.t. the graph distance. It can be easily seen that

$$c_1 r^{d_f(n)} \leq V_d(x,r) \leq c_2 r^{d_f(n)} \quad \text{if} \quad B_n \leq r < B_{n+1},\ x \in G_\xi. \tag{4.7}$$

Define a time scale function $\tau : [1,\infty) \to [1,\infty)$ and resistance scale factor $h : [1,\infty) \to [1,\infty)$ as

$$\tau(s) = s^{d_w(n)},\ h(s) = s^{d_w(n)-d_f(n)} \quad \text{if} \quad T_n \leq s < T_{n+1}.$$

We set $\tau(0) = h(0) = 0$. Note that τ and h satisfy the property in (2.1) since $\ell < \infty$.

Given these, it is possible to obtain heat kernel estimates similar to those in Theorem 6.3 and Lemma 6.7 of [13] by tracking the proof in [13] faithfully (see the Appendix for a sketch). By making additional computations (similar to those in [11, Lemma 3.19]) in the proof of [13, Lemma 3.10], we can obtain the following heat kernel estimates (cf. Remark after Theorem 24.6 in [14]): There exist $c_3, \cdots, c_6 > 0$ such that if $k \in \mathbb{N}$, $x, y \in G_\xi$, then

$$p_k(x,y) \leq \frac{c_3}{V_d(x,\tau^{-1}(k))} \exp\Big(-c_4\big(\frac{\tau(d(x,y))}{k}\big)^{1/(\beta_1-1)}\Big), \tag{4.8}$$

$$p_k(x,y) + p_{k+1}(x,y) \geq \frac{c_5}{V_d(x,\tau^{-1}(k))} \quad \text{for} \quad k \geq c_6\tau(d(x,y)). \tag{4.9}$$

Now assume the following limits exist and the inequality holds.

$$d_f := \lim_{n\to\infty} d_f(n), \quad d_w := \lim_{n\to\infty} d_w(n), \quad d_w > d_f. \tag{4.10}$$

Under this assumption, we have

$$c_7 \frac{\tau(d(x,y))}{V_d(x,d(x,y))} \leq R(x,y) \leq c_8 \frac{\tau(d(x,y))}{V_d(x,d(x,y))}, \quad \forall x, y \in G_\xi. \tag{4.11}$$

The equivalence of (4.8)+(4.9) and (4.11) is proved in [5] when $\tau(s) = s^\beta$ for some $\beta \geq 2$ under some volume growth condition referred as $(VG(\beta_-))$. Here we need a generalized version of this under the doubling property of τ. In fact, we only need (4.8)+(4.9) $\Rightarrow$ (4.11), and the generalization of this direction is easy. Indeed, using (4.8) and (4.9), we can obtain the scaled Poincaré inequality and the lower bound of (4.11) similarly to the proof of [5, Proposition 4.2] (with $\tau(s)$ replacing s^β there). Under (4.10), a condition corresponding to $(VG((d_w)_-))$ in [5] holds, so together with the scaled Poincaré inequality, we can obtain the upper bound of (4.11) similarly to the proof of [5, Lemma 2.3 (b)].

Now let $B^{\xi|_N} := B_N V_0$. Clearly $R_{B^{\xi|_N}}(x,y) \le R(x,y)$ for $x, y \in G^N_{\xi|_N}$ and $d^N_{max} \asymp B_N$. So (4.11) implies Assumption 2.1 (i),(ii), and (4.7), (4.10) with the homogeneity of the graph imply Assumption 2.1 (iii) with $\alpha = \max_n d_f(n)$. As before we can take $B^{\xi|_N}$ arbitrary as long as $d^N_{max} \asymp B_N$.

Finally we will introduce randomness on this graph. Let $(I^{\mathbb{N}}, \mathcal{F}, \mathbb{P})$ be a Borel probability space where the measure $\mathbb{P}$ is stationary and ergodic for the shift operator $\theta : I^{\mathbb{N}} \to I^{\mathbb{N}}$ defined by $\theta((k_1, \cdots, k_n, \cdots)) = (k_2, \cdots, k_n, \cdots)$. Then, by [13, Proposition 7.1] and the sub-additive ergodic theorem, one can prove the existence of the first two limits in (4.10). Let d^i_f, d^i_w be the Hausdorff dimension and the walk dimension for $G_{\mathbf{i}}$ where $\mathbf{i} = (i, i, i, \cdots)$ for $i \in I$. Let us consider a special case when $d = 3, \ell = 2$, and $\mathbb{P}$ is the Bernoulli probability measure with $\mathbb{P}(\xi_1 = 1) = p$, $\mathbb{P}(\xi_1 = 2) = 1 - p$ for some $p \in [0,1]$. One can see that d_f/d_w is a continuous function of p. Indeed, it can be easily seen that it is enough to prove $\lim_{n\to\infty} R_n/n$ is continuous for p. By the proof of [13, Proposition 7.1], there exist $c_1, c_2 > 0$ such that we have

$$\frac{1}{k}\mathbb{E}\log(c_1 R_k) \le \lim_{n\to\infty}\frac{1}{n}R_n \le \frac{1}{k}\mathbb{E}\log(c_2 R_k), \quad \mathbb{P}-\text{a.s.},$$

for any $k \ge 1$ where $\mathbb{E}$ is the average over $\mathbb{P}$. Since $\mathbb{E}\log(c_i R_k)$, $i = 1, 2$ are continuous for p (because the graph is finite), we obtain the desired continuity of $\lim_{n\to\infty} R_n/n$. So, when we choose the two carpets in such a way that $d^1_w > d^1_f$ and $d^2_w < d^2_f$ (which is possible, see [4, Section 9]), we are able to construct a one parameter family of homogeneous random Sierpinski carpet graphs where d_f/d_w is $\mathbb{P}$-a.e. an arbitrary fixed number between d^1_f/d^1_w and d^2_f/d^2_w. In particular, there exists $p_* \in (0,1)$ such that (4.10) holds $\mathbb{P}$-a.e. for all $p < p_*$.

A Appendix: Heat kernel estimates for Markov chains on homogeneous random Sierpinski carpet graphs

In this appendix, we will briefly sketch the proof of (4.8) and (4.9). The Markov chain we consider here is the discrete time Markov chain.

Set $V_n := V(G^n_{\xi|_n})$. We first define the Dirichlet form as follows.

$$\mathcal{E}_n(f,g) := \sum_{\substack{x,y\in V_n\\ \{x,y\}\in B}} (f(x)-f(y))(g(x)-g(y)), \qquad \forall f, g : V_n \to \mathbb{R}.$$

Given two processes Y^1, Y^2, defined on the same state space, we define a coupling time of Y^1 and Y^2 as

$$T_C(Y^1, Y^2) = \inf\{t \ge 0 : Y^1_t = Y^2_t\}.$$

Let $m \le n$. We call sets of the form $L_{k_1} \cdots L_{k_n} \psi_{i_{n-m}}^{k_{n-m}} \circ \cdots \circ \psi_{i_1}^{k_1}([0,1]^d) \cap V_n$ *m-complexes*. For $A \subset G_\xi$, define

$$\begin{aligned} D_m^0(A) &= \{m\text{-complex which contains } A\}, \\ D_m^1(A) &= D_m^0(A) \cup \{B : B \text{ is a } m\text{-complex}, D_m^0(A) \cap B \neq \emptyset\}. \end{aligned}$$

Let S_B^z denote the exit time from the set B, when the process is started from the point z.

Theorem A.1. *(Coupling) There exist $0 < p_0 < 1$ and $K_0 \in \mathbb{N}$ such that for each $x, y \in G_\xi$, there exist Markov chains $\overline{w}_t^x, \overline{w}_t^y$ with $\overline{w}_0^x = x$, $\overline{w}_0^y = y$ on G_ξ whose laws are equal to the simple random walk that satisfy the following: For $n > K_0$ and $y \in D_{n-K_0}^0(x)$,*

$$P(T_C(\overline{w}_t^x, \overline{w}_t^y) < \min\{S_{D_n^1(x)}^x, S_{D_n^1(x)}^y\}) > p_0.$$

The proof of the theorem follows in the same way as [4, Section 3], as G_ξ and $\overline{w}_t^x$ have enough symmetries for the argument there to work.

Once we have the coupling estimate, we can deduce the uniform (elliptic) Harnack inequality as in [4, Section 4]. Let $\mathcal{L}$ be the infinitesimal generator associated with the simple random walk.

Theorem A.2. *There exists $c_1 > 0$ such that for each $x_0 \in G_\xi$, and each $f : B(x_0, 2R) \to [0, \infty)$ with $\mathcal{L}f(x) = 0$ for all $x \in B(x_0, 2R) = 0$, $R \ge 1$, it holds that*

$$\max_{x \in B(x_0,R)} f(x) \le c_1 \min_{x \in B(x_0,R)} f(x). \tag{A.1}$$

We next introduce the following Poincaré constant:

$$\lambda_n = \sup\{\sum_{x \in V_n} (u(x) - \langle u \rangle_{V_n})^2 \mid u : V_n \to \mathbb{R},\ \mathcal{E}_n(u,u) = 1\},$$

where $\langle u \rangle_A = (\sharp A)^{-1} \sum_{x \in A} u(x)$ for any finite set A and $u : V_n \to \mathbb{R}$.

The following proposition can be proved similarly to Proposition 3.1, Corollary 3.7 of [13] and (2.3), (4.4) of [17]. (Note that Theorem A.2 is needed in the proof of (A.3).)

Proposition A.3. *There exist constants $c_1, \cdots, c_4 > 0$ such that for each $n, m \in \mathbf{N}$,*

$$c_1 R_n R_{\theta^n \xi | m} \le R_{n+m} \le c_2 R_n R_{\theta^n \xi | m}, \tag{A.2}$$

$$c_3 \lambda_n \le T_n \le c_4 \lambda_n. \tag{A.3}$$

Lemma A.4. *There is a constant c such that if $T_{n-1} \le t \le T_n$, then*

$$p_t(x,y) \le c M_n^{-1}. \tag{A.4}$$

Proof. From the definition of the Dirichlet form and the Poincaré constant, the proof is similar to [17, Theorem 3.3] by using Proposition A.3. □

The next lemma can be proved similarly to [13, Lemma 3.8].

Lemma A.5. *There exist $c_1, c_2 > 0$ such that*

$$c_1 T_r \le E S^z_{D^1_r(x)} \text{ for all } z \in D^0_r(x), \qquad E S^z_{D^1_r(x)} \le c_2 T_r \text{ for all } z \in D^1_r(x).$$

Since $S_{D^1_l(x)} \le t + 1_{(S_{D^1_l} > t)}(S_{D^1_l} - t)$ we have, from Lemma A.5,

$$c_1 T_l \le E S^z_{D^1_l} \le t + E\big[1_{(S^z_{D^1_l} > t)} E[S^{X_t}_{D^1_l}]\big] \le t + P(S^z_{D^1_l} > t) c_2 T_l \quad \text{for } t \ge 0, z \in D^0_l(x).$$

Thus, we deduce the following: there exist $c_3 > 0$, $c_4 \in (0,1)$ such that

$$P(S^z_{D^1_l(x)} \le t) \le c_3 T_l^{-1} t + c_4 \quad \text{for } t \ge 0, z \in D^0_l(x). \tag{A.5}$$

We can improve this to an exponential estimate on $P(S^z_{D^1_l(x)} \le t)$. In order to do this we define the following function of time and space,

$$k = k(n,l) = \inf\big\{l' \le n : \frac{T_{l'}}{B_{l'}} \ge \frac{T_l}{B_n}\big\}. \tag{A.6}$$

The next lemma corresponds to [13, Lemma 3.10]. Since the labeling here differs from that in [13], we give the proof.

Lemma A.6. *There exist constants c_1, c_2 such that if $k = k(n,l)$ as in* (A.6) *then for all $x \in E$, and $n, l \ge 0$,*

$$P(S^x_{D^1_n(x)} \le T_l) \le c_1 \exp\left(-c_2 B_n / B_k\right). \tag{A.7}$$

Proof. If $l' \le n$, then for the simple random walk to cross one n-complex it must cross at least $N = B_n/B_{l'}$, l'-complexes. So, there exists $0 < c < 1$ such that

$$S^x_{D^1_n(x)} \ge \sum_{i=1}^{cB_n/B_{l'}} V_i^{x_i},$$

where x_i depend only on $V_1^{x_1}, \dots, V_{i-1}^{x_{i-1}}$, and V_i^y have the same distribution as $S^y_{D^1_{l'}(y)}$. The deviation estimate [2, Lemma 1.1] states that if $P(V_i^y < s) \le p_0 + \alpha s$, where $p_0 \in (0,1)$ and $\alpha > 0$, then

$$\log P\big(\sum_1^{cN} V_i^{x_i} \le t\big) \le 2(\alpha c_1 N t/p_0)^{1/2} - c_2 N \log(1/p_0). \tag{A.8}$$

Thus, using (A.5) and (A.8), we have

$$\log P(S^x_{D^1_n(x)} \le T_l) \le c_3 (B_n/B_{l'})^{1/2} [(T_l/T_{l'})^{1/2} - c_4 (B_n/B_{l'})^{1/2}]. \tag{A.9}$$

Given $k = k(n,l)$ as above, there exists c_5 and k_0 such that $k \le k_0 \le k + c_5$, and

$$(T_l/T_{k_0})^{1/2} < \frac{1}{2} c_4 (B_n/B_{k_0})^{1/2}.$$

Provided $k_0 \le n$ we deduce

$$\log P(S^x_{D^1_n(x)} \le T_l) \le -\frac{1}{2} c_3 c_4 B_n/B_{k_0}.$$

Choosing c_6 large enough we have $1 < c_6 \exp(-c_2 B_n/B_k)$ whenever $k > n - c_5$, so that (A.7) holds in all cases. □

Theorem A.7. *There exist constants c_1, c_2 such that if $k \in \mathbb{N}$, $x, y \in G_\xi$, and n, m satisfy*

$$T_{n-1} \le t < T_n, \quad B_{m-1} \le d(x,y) < B_m, \tag{A.10}$$

and $k = k(m,n)$, then

$$p_t(x,y) \le c_1 t^{-d_f(n)/d_w(n)} \exp\Big(-c_2 \big(\frac{d(x,y)^{d_w(k)}}{t}\big)^{1/(d_w(k)-1)}\Big). \tag{A.11}$$

Proof. Noting that $M_n^{-1} \le c t^{-d_f(n)/d_w(n)}$, this is proved from Lemma A.4 and Lemma A.6 by the same argument as in Theorem 6.9 of [4]. □

Note that the bound (A.11) may also be written in the form

$$p_t(x,y) \le c M_n^{-1} \exp(-c' B_n/B_k),$$

where m, n satisfy (A.10), and $k = k(m,n)$ as in (A.6). The upper bound (4.8) can be obtained from this using (4.7).

The lower bound is obtained in the following procedure.

Lemma A.8. *There exists a constant c_1 such that if $T_n \le t$ then*

$$p_t(x,x) \ge c_1 M_n^{-1} \quad \text{for all } x \in G_\xi. \tag{A.12}$$

Proof. Using Lemma A.4 and (4.7), a standard argument gives the desired estimate. See for instance [6, Lemma 5.1]. □

Lemma A.9. *There exist c_1, c_2 such that if $T_{n-1} < t \le T_n$, then*

$$p_t(x,y) \ge c_1 M_n^{-1} \quad \text{whenever } d(x,y) \le c_2 B_n.$$

Proof. Using Theorem A.2 and Lemma A.8, this can be proved similarly to the proof of [3, Proposition 6.4]. □

We can deduce (4.9) from this and (4.7). □

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References

[1] R.J. Adler. *An introduction to continuity, extrema, and related topics for general Gaussian processes.* IMS Lecture Notes-Monograph Series, 1990.

[2] M.T. Barlow and R.F. Bass. Construction of Brownian motion on the Sierpinski carpet. *Ann. Inst. H. Poincaré*, **25**, 225–257, 1989.

[3] M.T. Barlow and R.F. Bass. *Random walks on graphical Sierpinski carpets.* In: Random walks and discrete potential theory, pp. 26-55, Cambridge, Cambridge Univ. Press, 1999.

[4] M.T. Barlow and R.F. Bass. Brownian motion and harmonic analysis on Sierpinski carpets. *Canadian J. of Math.* **51** (1999), 673–744.

[5] M.T. Barlow, T. Coulhon and T. Kumagai. Characterization of sub-Gaussian heat kernel estimates on strongly recurrent graphs. *Comm. Pure Appl. Math.,* **58** (2005), 1642–1677.

[6] M.T. Barlow and B.M. Hambly. Transition density estimates for Brownian motion on scale irregular Sierpinski gasket *Ann. Inst. H. Poincaré*, **33**, 531–557, 1997.

[7] M. Bramson and O. Zeitouni. Tightness for a family of recursion equations. *Ann. Probab.* **37** (2009), 615–653.

[8] M. Bramson and O. Zeitouni. Tightness of the recentered maximum of the two-dimensional discrete Gaussian Free Field. *Comm. Pure Appl. Math.* **65** (2012), 1–20.

[9] J. Ding, J. R. Lee and Y. Peres, Cover times, blanket times and majorizing measures. *Annals Math.* **175** (2012), 1409–1471.

[10] N. Eisenbaum, H. Kaspi, M. B. Marcus, J. Rosen and Z. Shi. A Ray-Knight theorem for symmetric Markov processes. *Annals. Probab.* **28** (2000), 1781–1796.

[11] A. Grigor'yan and A. Telcs. Two-sided estimates of heat kernels on metric measure spaces. *Ann. Probab.* **40** (2012), 1212–1284.

[12] B.M. Hambly and T. Kumagai. *Heat kernel estimates for symmetric random walks on a class of fractal graphs and stability under rough isometries.* In: Fractal geometry and applications, Proc. of Symposia in Pure Math. **72**, Part 2, pp. 233–260, Amer. Math. Soc. 2004.

[13] B.M. Hambly, T. Kumagai, S. Kusuoka and X.Y. Zhou. Transition density estimates for diffusion processes on homogeneous random Sierpinski carpets. *J. Math. Soc. Japan* **52** (2000), no. 2, 373–408.

[14] J. Kigami. Resistance forms, quasisymmetric maps and heat kernel estimates. *Memoirs Amer. Math. Soc.* **216**, no. 1015 (2012).

[15] T. Kumagai. Random walks on disordered media and their scaling limits.St. Flour Lecture Notes (2010). `http://www.kurims.kyoto-u.ac.jp/~kumagai/StFlour-TK.pdf`.

[16] T. Kumagai. Estimates of transition densities for Brownian motion on nested fractals. *Probab. Theory Relat. Fields* **96** (1993), 205–224.

[17] S. Kusuoka and X.Y. Zhou. Dirichlet forms on fractals: Poincaré constant and resistance. *Probab. Theory Relat. Fields*, **93**, 169–196, 1992.

[18] M. Ledoux and M. Talagrand. *Probability in Banach spaces. Isoperimetry and processes.* Springer-Verlag, Berlin, (1991).

[19] Lyons, R. with Peres, Y. (2012). *Probability on Trees and Networks.* Cambridge University Press. In preparation. Current version available at `http://mypage.iu.edu/~rdlyons/`.

[20] S. Sheffield. Gaussian free fields for mathematicians. *Probab. Theory Rel. Fields.* **139** (2007), 521–541.

Takashi Kumagai
Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan.
E-mail: `kumagai@kurims.kyoto-u.ac.jp`

Ofer Zeitouni University of Minnesota, School of Mathematics, 206 Church Street SE, Minneapolis, MN 55455, USA.
and
Weizmann Institute of Science, Faculty of Mathematics, POB 26, Rehovot 76100, Israel.
E-mail: `zeitouni@math.umn.edu`