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Abstract

Inter-universal Teichmüller theory may be described as a sort of arithmetic version of Teichmüller theory that concerns a certain type of **canonical deformation** associated to an *elliptic curve* over a *number field* and a *prime number* $l \geq 5$. We begin our survey of inter-universal Teichmüller theory with a review of the technical difficulties that arise in applying **scheme-theoretic Hodge-Arakelov theory** to diophantine geometry. It is precisely the goal of overcoming these technical difficulties that motivated the author to construct the **non-scheme-theoretic deformations** that form the content of inter-universal Teichmüller theory. Next, we discuss generalities concerning **"Teichmüller-theoretic deformations"** of various familiar geometric and arithmetic objects which at first glance appear *one-dimensional*, but in fact have *two underlying dimensions*. We then proceed to discuss in some detail the various components of the **log-theta-lattice**, which forms the central stage for the various constructions of inter-universal Teichmüller theory. Many of these constructions may be understood to a certain extent by considering the analogy of these constructions with such *classical* results as **Jacobi's identity** *for the* **theta function** and the **integral** *of the* **Gaussian distribution** *over the real line*. We then discuss the **"inter-universal"** aspects of the theory, which lead naturally to the introduction of **anabelian** techniques. Finally, we summarize the main abstract theoretic and diophantine consequences of inter-universal Teichmüller theory, which include a verification of the **ABC/Szpiro Conjecture**.

Contents

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§ 1. Hodge-Arakelov-theoretic Motivation

The *inter-universal Teichmüller theory* developed in [IUTchI], [IUTchII], [IUTchIII], [IUTchIV] arose from attempts by the author, which began around the summer of 2000, to overcome certain **technical obstructions** in the **scheme-theoretic Hodge-Arakelov theory** of [HASurI], [HASurII] to applying this theory to diophantine geometry [cf. the discussion of [HASurI], §1.5.1; [HASurII], Remark 3.7; [IUTchI], Remark 4.3.1]. Thus, we begin our overview of inter-universal Teichmüller theory with a brief review of the essential content of those aspects of scheme-theoretic Hodge-Arakelov theory that are relevant to the development of inter-universal Teichmüller theory.

Let l be a prime number. Then we recall that the module $E[l]$ of l-torsion points of a **Tate curve** $E \stackrel{\mathrm{def}}{=} \mathbb{G}_m/q^{\mathbb{Z}}$ [over, say, a p-adic field or the complex field $\mathbb{C}$], whose *q-parameter* we denote by q, fits into a natural exact sequence

$$0 \longrightarrow \boldsymbol{\mu}_l \longrightarrow E[l] \longrightarrow \mathbb{Z}/l\mathbb{Z} \longrightarrow 0.$$

That is to say, one has **canonical** objects as follows:

a "**multiplicative subspace**" $\boldsymbol{\mu}_l \subseteq E[l]$ and "**generators**" $\pm 1 \in \mathbb{Z}/l\mathbb{Z}$.

In the following discussion, we fix an **elliptic curve** E over a **number field** F and a *prime number* $l \geq 5$. For convenience, we shall use the notation

$$l^{*} \stackrel{\mathrm{def}}{=} \tfrac{l-1}{2}.$$

Also, we suppose that E has *stable reduction* at all nonarchimedean primes of F. Then, in general, the module $E[l]$ [i.e., more precisely, the finite étale group scheme over F] of l-torsion points of E does **not** admit

a **"global multiplicative subspace"** or **"global canonical generators"**

— i.e., a rank one submodule $M \subseteq E[l]$ or a pair of generators of the quotient $E[l]/M$ that coincide with the above *canonical multiplicative subspace* and *generators* at all nonarchimedean primes of F where E has *bad multiplicative reduction.* Nevertheless, let us

suppose *that such global objects do in fact exist!*

Also, let us write

$$K \stackrel{\mathrm{def}}{=} F(E[l])$$

for the extension field of F generated by the fields of definition of the l-torsion points of E, $\mathbb{V}(K)$ for the set of [archimedean and nonarchimedean] valuations of K, and

$\mathbb{V}(K)^{\mathrm{bad}} \subseteq \mathbb{V}(K)$ for the set of nonarchimedean valuations where E has *bad multiplicative reduction.* For $v \in \mathbb{V}(K)$, we shall write K_v for the *completion* of K at v, $\mathcal{O}_v \subseteq K_v$ for the subset of elements $f \in K_v$ such that $|f|_v \leq 1$, and $\mathfrak{m}_v \subseteq \mathcal{O}_v$ for the subset of elements $f \in K_v$ such that $|f|_v < 1$. Let $N \subseteq E[l]$ be a rank one submodule such that the natural morphism of finite groups schemes over F

$$M \times N \quad \stackrel{\sim}{\to} \quad E[l]$$

is an *isomorphism.* Then the *Fundamental Theorem of Hodge-Arakelov Theory* may be formulated as follows [cf. [HASurI], §1; [HASurII], §1, §3; the explicit series representation of the *theta function* on a *Tate curve* given in [EtTh], Proposition 1.4]:

Theorem 1.1. ("Idealized" Version of Fundamental Theorem of Hodge-Arakelov Theory) *We maintain the notation of the above discussion. In particular, we assume the existence of a* **global multiplcative subspace** *and* **global canonical generators** *[as described above] for $E[l]$. Write ${}^*E \stackrel{\mathrm{def}}{=} E_K/N$ for the elliptic curve over K obtained by forming the quotient of $E_K \stackrel{\mathrm{def}}{=} E \times_F K$ by N; ${}^*E^\dagger \to {}^*E$ for the universal vectorial extension of ${}^*E$; $\mathcal{L}$ for the line bundle of degree one on ${}^*E$ determined by some nontrivial 2-torsion point of ${}^*E$; $\mathcal{L}|_{{}^*E^\dagger}$ for the restriction of $\mathcal{L}$ to ${}^*E^\dagger$. Then restriction of sections of $\mathcal{L}|_{{}^*E^\dagger}$ of relative degree [i.e., relative to ${}^*E^\dagger \to {}^*E$] $< l$ — a condition which we shall denote by means of a superscript "$< l$" — to $M_K \stackrel{\mathrm{def}}{=} M \times_F K$ via the composite inclusion $M_K \hookrightarrow E_K[l] \to {}^*E[l]$, followed by application of a suitable* **theta trivialization** *of the restriction of $\mathcal{L}$ to M yields* **functions** *on M; consideration of the Fourier coefficients of such functions determines an* **isomorphism** *of K-vector spaces of dimension l*

$$\Gamma({}^*E^\dagger, \mathcal{L}|_{{}^*E^\dagger})^{<l} \quad \stackrel{\sim}{\to} \quad \bigoplus_{j=-l^*}^{l^*} (\underline{\underline{q}}^{j^2} \cdot \mathcal{O}_K) \otimes_{\mathcal{O}_K} K \qquad (*_{\mathrm{HA}})$$

— where we write

$$q \stackrel{\mathrm{def}}{=} \{q_v\}_{v \in \mathbb{V}(K)^{\mathrm{bad}}}$$

for the collection of **q-parameters** *of E_K at $v \in \mathbb{V}(K)^{\mathrm{bad}}$ [so $q_v \in \mathfrak{m}_v$] and*

$$\underline{\underline{q}} \stackrel{\mathrm{def}}{=} \{\underline{\underline{q}}_v\}_{v \in \mathbb{V}(K)^{\mathrm{bad}}}$$

for some collection of $2l$-th roots "$\underline{\underline{q}}_v = q_v^{1/2l}$" of the q_v; the direct summand labeled j of the codomain of the isomorphism $(*_{\mathrm{HA}})$ is to be understood as a copy of K which we regard as being equipped with the **integral structure** *obtained from the integral structure given by the ring of integers $\mathcal{O}_K \subseteq K$ by replacing, for each $v \in \mathbb{V}(K)^{\mathrm{bad}}$, the integral structure given by $\mathcal{O}_v$ by the integral structure given by $\underline{\underline{q}}_v^{j^2} \cdot \mathcal{O}_v$. The domain* *of this isomorphism admits a natural* **Hodge filtration** F^{-i}*, for* $i \in \mathbb{N}$*, given by the sections of relative degree* $\leq i$*; the subquotients of this Hodge filtration admit* **natural isomorphisms**

$$F^{-i}/F^{-i+1} \quad \overset{\sim}{\to} \quad \omega_{^*E}^{\otimes i} \otimes_K F^0$$

— where we write $\omega_{^*E}$ *for the cotangent space at the origin of* $^*E$*. The codomain of the isomorphism* $(*_{\mathrm{HA}})$ *admits a natural* **Galois action***, i.e., an action by* $\mathrm{Gal}(K/F)$*. Finally, this isomorphism* $(*_{\mathrm{HA}})$ *of* K*-vector spaces is in fact* **compatible***, up to relatively mild discrepancies, with the natural* **integral structures** *(respectively,* **metrics***) at nonarchimedean (respectively, archimedean) elements of* $\mathbb{V}(K)$*; here, the integral structure of the codomain at nonarchimedean* $v \in \mathbb{V}(K)$ *is as described above.*

Over the complement in $\mathrm{Spec}(\mathcal{O}_K)$ of $\mathbb{V}(K)^{\mathrm{bad}}$, as well as at the archimedean primes of K, the content of Theorem 1.1 is essentially equivalent to the content of the ["non-idealized"] *Fundamental Theorem of Hodge-Arakelov Theory* given in [HASurI], Theorem A; [HASurII], Theorem 1.1 [cf. the discussion preceding [HASurII], Definition 3.1] — i.e., "non-idealized" in the sense that it holds even in the *absence* of the assumption of the existence of the *global multiplicative subspace* and *global canonical generators*. On the other hand, the portion of Theorem 1.1 that concerns the *integral structures* at the valuations $\in \mathbb{V}(K)^{\mathrm{bad}}$ follows immediately [i.e., in light of the theory reviewed in [HASurI], §1] from the *explicit series representation* of the *theta function* on a *Tate curve* given in [EtTh], Proposition 1.4.

One way to understand [both the "idealized" and "non-idealized" versions of] the Fundamental Theorem of Hodge-Arakelov Theory is as a sort of **scheme-theoretic** version of the classical computation of the **Gaussian integral**

$$\int_{-\infty}^{\infty} e^{-x^2} dx \ = \ \sqrt{\pi}$$

by applying a coordinate transformation from **cartesian** to **polar** coordinates — cf. the discussion of [IUTchII], Remark 1.12.5, (i). Indeed, the function "$j \mapsto \underline{\underline{q}}^{j^2}$" — which may be thought of as a sort of *discrete version of the Gaussian distribution* "e^{-x^2}" — appears quite *explicitly* in the *codomain* of the isomorphism $(*_{\mathrm{HA}})$ of Theorem 1.1. On the other hand, the value "$\sqrt{\pi}$" may be thought of as corresponding to the [negative] *tensor powers of the sheaf "ω"* that arise in the subquotients of the Hodge filtration that appear in the *domain* of this isomorphism. Indeed, if, in the domain of this isomorphism, one omits the restriction "$< l$", then one obtains a natural *crystal* — cf. the theory of the **"crystalline theta object"** discussed in [HASurII], §2. Unlike the crystals that typically arise in the case of the de Rham cohomology associated to a family of varieties, which satisfy the property of *Griffiths transversality*, this crystal exhibits a property that we refer to as **Griffiths semi-transversality** [cf. [HASurII], Theorem

2.8], i.e., the crystal corresponds to a connection whose application has the effect of *shifting* the Hodge filtration [not by one, but rather] by *two steps*! This property of Griffiths semi-transversality gives rise to a *Kodaira-Spencer isomorphism* [cf. [HASurII], Theorem 2.10] between $\omega^{\otimes 2}$ and the restriction via the classifying morphism associated to the elliptic curve under consideration of the *sheaf of logarithmic differentials on the moduli stack of elliptic curves* — i.e., in effect, between ω and the *"square root"* of this sheaf of logarithmic differentials. Here, it is useful to recall that this sheaf of logarithmic differentials admits a *canonical generator* in a neighborhood of the cusp at infinity of the moduli stack of elliptic curves, namely, the logarithmic differential $q^{-1} \cdot dq$ of the q-parameter, which, if one thinks in terms of the classical complex theory and integrates this logarithmic differential once over a loop surrounding the cusp at infinity, has an associated *period* equal to 2π. This provides the justification for thinking of "ω" as corresponding to "$\sqrt{\pi}$". Finally, we recall that this relationship between the Fundamental Theorem of Hodge-Arakelov Theory and the classical Gaussian integral may be seen more *explicitly*, via the classical theory of *Hermite polynomials*, when this theory is restricted to the archimedean primes of a number field via the *"Hermite model"* [cf. the discussion of [HASurI], §1.1].

In this context, it is also of interest to note that unlike many aspects of the *classical theory* of *theta functions*, the Hodge-Arakelov theory discussed in [HASurI], [HASurII] does **not** admit a natural generalization to the case of **higher-dimensional abelian varieties** [cf. the discussion of [HASurI], §1.5.2]. This is perhaps not so surprising in light of the analogy discussed above with the classical computation of the Gaussian integral reviewed above, which does not admit, at least in any immediate, naive sense, a generalization to higher dimension. This phenomenon of a lack of any immediate generalization to higher dimension may also be seen in the theory of the **étale theta function** developed in [EtTh], which plays a central role in inter-universal Teichmüller theory. In the case of the étale theta function, this phenomenon is essentially a reflection of the fact that, unlike the case with the complement of an ample divisor in an abelian variety of arbitrary dimension, the complement of the origin in an elliptic curve may be regarded as a **hyperbolic curve**, i.e., as an object for which there exists an extensive and well-developed theory of **anabelian geometry**, which may be [and indeed is, in [EtTh]] applied to obtain various important **rigidity** properties involving the étale theta function [cf. the discussion of the Introduction to [EtTh]].

As discussed in [HASurI], Theorem A, the isomorphism of Theorem 1.1 may be considered over the **moduli stack of elliptic curves** $\overline{\mathcal{M}}_{\mathrm{ell}}$ over $\mathbb{Z}$. Indeed, as discussed in [HASurI], §1, a proof of the *characteristic zero* portion of the isomorphism of Theorem 1.1 [or [HASurI], Theorem A] may be given by considering the corresponding map between *vector bundles of rank l* over $(\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{Q}} \stackrel{\mathrm{def}}{=} \overline{\mathcal{M}}_{\mathrm{ell}} \times_{\mathbb{Z}} \mathbb{Q}$: That is to say, an easy explicit computation involving the explicit series representation of the theta function on a Tate curve [cf., e.g., [EtTh], Proposition 1.4] shows that this map between vector bundles on $(\overline{\mathcal{M}}_{\text{ell}})_{\mathbb{Q}}$ is an isomorphism at the generic point of $(\overline{\mathcal{M}}_{\text{ell}})_{\mathbb{Q}}$. Then one concludes that this map between vector bundles is in fact an *isomorphism* on $(\overline{\mathcal{M}}_{\text{ell}})_{\mathbb{Q}}$ by computing the **degrees** of the domain and codomain of the map under consideration and observing that these two degrees **coincide**. Although the precise computation of these degrees is rather involved, from the point of view of the present discussion it suffices to observe that the **highest order** portions of the *average degrees* [i.e., where by "average", we mean the result of dividing by the rank l] of the domain [i.e., "LHS"] and codomain [i.e., "RHS"] vector bundles on $(\overline{\mathcal{M}}_{\text{ell}})_{\mathbb{Q}}$ of the map under consideration **coincide**:

$$\tfrac{1}{l} \cdot \text{LHS} \;\approx\; -\tfrac{1}{l} \cdot \sum_{i=0}^{l-1} i \cdot [\omega_E] \;\approx\; -\tfrac{l}{2} \cdot [\omega_E]$$

$$\tfrac{1}{l} \cdot \text{RHS} \;\approx\; -\tfrac{1}{l^2} \cdot \sum_{j=1}^{l^*} j^2 \cdot [\log(q)] \;\approx\; -\tfrac{l}{24} \cdot [\log(q)] \;=\; -\tfrac{l}{2} \cdot [\omega_E]$$

— where we write $[\omega_E]$ for the degree on $(\overline{\mathcal{M}}_{\text{ell}})_{\mathbb{Q}}$ of the line bundle ω_E [i.e., the line bundle determined by the relative cotangent bundle at the origin of the tautological semi-abelian scheme over $(\overline{\mathcal{M}}_{\text{ell}})_{\mathbb{Q}}$]; we write $[\log(q)]$ for the degree on $(\overline{\mathcal{M}}_{\text{ell}})_{\mathbb{Q}}$ of the divisor at infinity of $(\overline{\mathcal{M}}_{\text{ell}})_{\mathbb{Q}}$; we recall the elementary fact [a consequence of the existence of the modular form typically referred to as the "discrimant"] that $[\log(q)] = 12 \cdot [\omega_E]$. In this context, we remark that the line bundle F^0 [i.e., the line bundle on $(\overline{\mathcal{M}}_{\text{ell}})_{\mathbb{Q}}$ that corresponds to the line bundle denoted "F^0" in Theorem 1.1] is *"sufficiently small"* that it may be ignored, i.e., from the point of view of computing portions of highest order.

Now let us return to considering *arithmetic vector bundles* on $\text{Spec}(\mathcal{O}_K)$ in the context of Theorem 1.1. The *Hodge filtration* F^{-i} is **not compatible** with the **direct sum decomposition** of the codomain of the isomorphism $(*_{\text{HA}})$ of Theorem 1.1. This fact may be derived, for instance, from the explicit series representation of the theta function on a Tate curve [cf., e.g., [EtTh], Proposition 1.4] and is closely related to the theory of the **arithmetic Kodaira-Spencer morphism** arising from this isomorphism [cf. the discussion of [HASurII], §3, especially, [HASurII], Corollary 3.6]. In particular, it follows that, for most j, by *projecting* to the factor labeled j of this direct sum decomposition, one may construct a [nonzero!] *morphism of arithmetic line bundles* [i.e., of rank one K-vector spaces that is compatible "from below" with the integral structures at nonarchimedean primes and the metrics at archimedean primes — cf., e.g., the discussion of [GenEll], §1]

$$(\mathcal{O}_K \otimes_{\mathcal{O}_K} K \quad \approx) \quad F^0 \quad \hookrightarrow \quad (\underline{\underline{q}}^{j^2} \cdot \mathcal{O}_K) \otimes_{\mathcal{O}_K} K$$

— where we remark that, from the point of view of computations to **highest order**, one may think of the arithmetic line bundle corresponding to F^0 as being, essentially the **trivial** *arithmetic line bundle*, and one may **ignore** *the "relatively mild discrepancies"* referred to in Theorem 1.1. Such a nonzero morphism of arithmetic line bundles implies an **inequality** between **arithmetic degrees** "$\deg_{\mathrm{arith}}(-)$" of arithmetic line bundles, i.e., if we write $\Omega^{\log}_{\mathcal{M}}|_E$ for the arithmetic line bundle on $\mathrm{Spec}(\mathcal{O}_K)$ determined by restricting the line bundle of logarithmic differentials on $\overline{\mathcal{M}}_{\mathrm{ell}}$ [relative to the log structure determined by the divisor at infinity] via the classifying morphism associated to E_K, then

$$\tfrac{1}{6} \cdot \deg_{\mathrm{arith}}(\log(q)) \;=\; \mathrm{ht}_E \;<\; \text{constant}$$

— where we write $\mathrm{ht}_E \overset{\mathrm{def}}{=} 2 \cdot \deg_{\mathrm{arith}}(\omega_E) = \deg_{\mathrm{arith}}(\Omega^{\log}_{\mathcal{M}}|_E)$ for the **canonical height** of the elliptic curve E_K and $\deg_{\mathrm{arith}}(\log(q))$ for the arithmetic degree of the arithmetic divisor on $\mathrm{Spec}(\mathcal{O}_K)$ determined by restricting the divisor at infinity of $\overline{\mathcal{M}}_{\mathrm{ell}}$ via the classifying morphism associated to E_K [cf., e.g., the discussion of [GenEll], §1, §3]. Here, the arithmetic degrees are to be understood as being *normalized* [i.e., by dividing by the degree over $\mathbb{Q}$ of the number field under consideration — cf. the discussion of [GenEll], §1] so as to be invariant with respect to the operation of passing to a finite extension of the number field under consideration.

Before proceeding, we pause to discuss the meaning of the *"constant"* that appears in the inequality of the preceding display. First of all, to simplify the discussion, we assume that the *complex moduli* of the elliptic curve E_K at the archimedean primes of K are subject to the restriction that they are only allowed to vary within some **fixed compact subset** of $(\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{C}} \overset{\mathrm{def}}{=} (\overline{\mathcal{M}}_{\mathrm{ell}}) \times_{\mathbb{Z}} \mathbb{C}$ that does not contain the *divisor at infinity*. It follows from [GenEll], Theorem 2.1, that, from the point of view of verifying the *ABC Conjecture*, this does not result in any essential loss of generality. Then it follows from the discussion of the *"étale integral structure"* in [HASurI], §1 [cf. also [HASurI], Theorem A], together with elementary estimates via *Stirling's formula*, that this "constant" is *roughly of the order of* $\log(l)$. In particular, if [as is done, for instance, in [GenEll], §4; [IUTchIV], §2] one assumes that l *is roughly of the order of* ht_E, then, by possibly enlarging the "constant" of the inequality under consideration,

> one may assume without loss of generality — i.e., so long as one respects the **restrictions** just imposed on the **complex moduli** and the **size** of l relative to the height! — that this constant is, in fact, **independent** of the *elliptic curve* E, the *number field* F, and the *prime number* l.

For a more precise statement, we refer to [GenEll], Lemma 3.5, where an inequality is derived by assuming, in effect, only the *existence of a global multiplicative subspace* [i.e., without assuming the existence of global canonical generators!] for $E[l]$; moreover, the proof of [GenEll], Lemma 3.5, is entirely elementary and does not require the use of the scheme-theoretic Hodge-Arakelov theory surveyed in [HASurI], [HASurII]. Thus, one may conclude — i.e., either from the above discussion of the inequality arising from Theorem 1.1 or from [GenEll], Lemma 3.5 — that

> if one **bounds** the **degree** $[F : \mathbb{Q}]$ of the number field F over $\mathbb{Q}$ and the **complex moduli** of the elliptic curve E at the archimedean primes of F, then the assumption that $E[l]$ admits a **global multiplicative subspace** and **global canonical generators** [i.e., where, in fact, the latter is unnecessary, if one applies [GenEll], Lemma 3.5, as is done, for instance, in the proof of [IUTchIV], Corollary 2.2, (ii)] implies that there are **only finitely many possibilities** for the j-invariant of E [cf. [GenEll], Propositions 1.4, (iv); 3.4].

In this context, it is of interest to consider the following **complex analytic analogue** of the above discussion. Let $E_S \to S$ be a family of one-dimensional semi-abelian varieties over a connected smooth proper algebraic curve S over $\mathbb{C}$ that restricts to a family of elliptic curves $E_U \to U$ over some nonempty open subscheme $U \subseteq S$. Write $\mathcal{E}_{\mathcal{U}} \to \mathcal{U}$ for the **complex analytic family of complex tori** determined by $E_U \to U$. Let $\widetilde{\mathcal{U}} \to \mathcal{U}$ be a universal covering space of the Riemann surface $\mathcal{U}$. Then the classifying morphism of the family $\mathcal{E}_{\mathcal{U}} \to \mathcal{U}$ determines a holomorphic map $\phi : \widetilde{\mathcal{U}} \to \mathfrak{H}$ to the *upper half-plane* $\mathfrak{H}$ which is well-defined up to composition with an automorphism of $\mathfrak{H}$ induced by the well-known action of $SL_2(\mathbb{Z})$ on $\mathfrak{H}$. Now one verifies immediately that this map $\phi : \widetilde{\mathcal{U}} \to \mathfrak{H}$ is either *constant* or has *open dense image* in $\mathfrak{H}$. In particular, if ϕ is nonconstant, then it follows that *every point in the boundary* $\partial\mathfrak{H}$ of $\mathfrak{H}$ [i.e., where we regard $\mathfrak{H}$ as being embedded, in the usual way, in the complex projective line $\mathbb{P}^1_{\mathbb{C}}$] lies in the closure of the image $\mathrm{Im}(\phi)$ of ϕ. On the other hand, the natural *complex analytic analogue* of the condition that there exist a **global multiplicative subspace** in the sense of the above discussion concerning elliptic curves over number fields may be formulated as follows:

> The local system on $\mathcal{U}$ in rank two free $\mathbb{Z}$-modules determined by the [abelian!] fundamental groups of the fibers of $\mathcal{E}_{\mathcal{U}} \to \mathcal{U}$ admits a *rank one subspace* that *coincides* with the subspace of the *"complex Tate curve $\mathbb{G}_m/q^{\mathbb{Z}}$"* determined by the *fundamental group of* $\mathbb{G}_m$ in every sufficiently small neighborhood of a point of the Riemann suface $\mathcal{S}$ associated to S where the family $\mathcal{E}_{\mathcal{U}} \to \mathcal{U}$ *degenerates*.

Moreover, one verifies immediately that this condition is equivalent to the condition that the intersection with $\partial\mathfrak{H}$ of the closure of $\mathrm{Im}(\phi)$ be equal to a *single cusp* [i.e., a single point of the $SL_2(\mathbb{Z})$-orbit of the point at infinity of $\mathbb{P}^1_{\mathbb{C}}$]. In particular, one concludes that this condition can only hold if ϕ is *constant*, i.e., if the original family of elliptic curves $E_U \to U$ is **isotrivial**.

Now let us return to our discussion of *elliptic curves over number fields*. Then, roughly speaking, the above discussion may summarized as follows:

> The assumption of the *existence of a global multiplicative subspace and global canonical generators* may be thought of as a sort of arithmetic analogue of the geometric notion of an *isotrivial family of elliptic curves* and, moreover, implies *bounds on the height* of the elliptic curve under consideration.

This state of affairs motivates the following *question*:

> Is it possible to obtain **bounds** — perhaps **weaker** in some suitable sense! — on the **height** of **arbitrary** elliptic curves over number fields, i.e., without assuming the existence of a global multiplicative subspace or global canonical generators?

Put another way, one would like to somehow carry out the above derivation of bounds on the height from the isomorphism ($*_{\mathrm{HA}}$) of Theorem 1.1 without assuming the existence of a global multiplicative subspace or global canonical generators. As discussed in [HASurI], §1.5.1, if one tries to carry out the above derivation by applying the *"non-idealized"* version of Theorem 1.1 [i.e., which holds without assuming the existence of a global multiplicative subspace or global canonical generators!], then one must contend with *"Gaussian poles"* — i.e., with a situation in which the critical *morphism of arithmetic line bundles* that appears in the above discussion has *poles* of relatively large order! — which have the effect of rendering the resulting inequality essentially *vacuous*. That is to say, from the point of view of generalizing the derivation of bounds on the height given above, the above question may be *reinterpreted* as follows:

> Is it possible to somehow **reformulate** *the scheme-theoretic Hodge-Arakelov theory* of [HASurI], [HASurII] in such a way that one may *circumvent the* **technical obstruction** constituted by the **Gaussian poles**?

It was this state of affairs that motivated the author in the summer of 2000 to initiate the development of the **inter-universal Teichmüller theory** established in [IUTchI], [IUTchII], [IUTchIII], [IUTchIV].

From an *extremely naive* point of view, the approach that underlies the development of inter-universal Teichmüller theory may be described as follows:

> Suppose that the **assignment**
>
> $$\left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\dots,l^*} \quad \mapsto \quad \underline{\underline{q}} \qquad (*_{\mathrm{KEY}})$$
>
> — that is to say, for each valuation $v \in \mathbb{V}(K)^{\mathrm{bad}}$, one considers the assignment $\{\underline{\underline{q}}_v^{j^2}\}_{j=1,\dots,l^*} \quad \mapsto \quad \underline{\underline{q}}_v$ — somehow determines an **automorphism** of the **number field** K!

Before proceeding, we remark that the assignment $(*_{\mathrm{KEY}})$ may be thought of as a sort of **tautological solution** to the problem discussed above of resolving the *technical obstruction* constituted by the *Gaussian poles* — i.e., in short, the problem of contending with the *absence* of the *domain* of the assignment $(*_{\mathrm{KEY}})$ in a situation in which the *presence* of the *codomain* of the assignment $(*_{\mathrm{KEY}})$ may be taken for granted.

This approach may seem somewhat *far-fetched* at first glance but is in fact motivated by the classical analogy between *number fields* and *function fields*: that is to say, if one thinks of a typical $2l$-th root of a q-parameter $\underline{\underline{q}}_v$ as being, roughly speaking, like some *positive power of a prime number* p, and one thinks of, say, the rational number field $\mathbb{Q}$ as corresponding to the one-dimensional function field $\mathbb{F}_p(t)$, i.e., so p corresponds to the indeterminate t, then the assignment $(*_{\mathrm{KEY}})$ considered above corresponds, roughly speaking [i.e., if one reverses the direction of "$\mapsto$"], to an assignment of the form

$$t \quad \mapsto \quad t^a$$

for some positive integer a, i.e., an assignment which does indeed determine a homomorphism of fields $\mathbb{F}_p(t) \hookrightarrow \mathbb{F}_p(t)$. Also, in this context, it is of interest to observe that the proof of the result quoted earlier [i.e., [GenEll], Lemma 3.5] to the effect that *the existence of a global multiplicative subspace implies bounds on the height* proceeds precisely by *observing* that the existence of a global multiplicative subspace implies that the elliptic curve under consideration is *isogenous* to — i.e., in effect, has roughly [that is to say, up to terms of relatively negligible order] the *same height* as — an elliptic curve whose q-parameters are *precisely the l-th powers* of the q-parameters of the original elliptic curve [cf. the application of [GenEll], Lemma 3.2, (ii), in the proof of [GenEll], Lemma 3.5]: that is to say, in short, the existence of a global multiplicative subspace implies that one is in a situation that is **invariant**, so to speak, with respect to the transformation "$q \quad \mapsto \quad q^l$"! As discussed in the introduction to [GenEll], §3, this technique is, in essence, a sort of miniature, or simplified, version of a technique which dates back to Tate for proving "Tate conjecture-type results" and may be seen, for instance, in Faltings' work on the Tate conjecture.

At any rate, if one assumes that one has such an *automorphism* of the *number field* K, then since such an automorphism necessarily *preserves* **degrees** *of* **arithmetic line bundles**, it follows — since the absolute value of the degree of the **RHS** of the assignment $(*_{\mathrm{KEY}})$ under consideration is **"small"** by comparison to the absolute value of the [*average*!] degree of the **LHS** of this assignment — in a similar fashion to the discussion of the derivation of a bound on the height from Theorem 1.1 that a similar **inequality**, i.e., a bound on the height of E_K, holds:

$$\frac{1}{6} \cdot \deg_{\mathrm{arith}}(\log(q)) \;=\; \mathrm{ht}_E \;<\; \text{constant}.$$

Of course, needless to say, such *automorphisms of number fields* do **not** in fact

exist!! On the other hand, the *starting point* of inter-universal Teichmüller theory lies in adopting the following point of view:

> We regard the "$\{\underline{\underline{q}}^{j^2}\}$" on the LHS and the "$\underline{\underline{q}}$" on the RHS of the **assignment** $(*_{\mathrm{KEY}})$
>
> $$\left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\ldots,l^*} \quad \mapsto \quad \underline{\underline{q}}$$
>
> as belonging to **distinct** copies of **"conventional ring/scheme theory"**, i.e., **"distinct arithmetic holomorphic structures"**, and we think of the assignment $(*_{\mathrm{KEY}})$ as a sort of arithmetic version of the notion of a **quasiconformal map** between Riemann surfaces equipped with **distinct holomorphic structures**.

That is to say, this approach allows us to realize the assignment $(*_{\mathrm{KEY}})$, albeit at the cost of **partially dismantling** *conventional ring/scheme theory.* On the other hand, this approach requires us

to compute just how much of a **distortion** *occurs*

as a result of **deforming** *conventional ring/scheme theory.* This *vast computation* is the essential content of inter-universal Teichmüller theory. In the remainder of the present paper, we intend to survey the ideas surrounding this dismantling/deformation of conventional ring/scheme theory.

We begin by observing that one way to approach the issue of understanding such dismantling/deformation operations is to focus on that portion of the objects under consideration which is **invariant** with respect to — i.e., *"immune"* to — the dismantling/deformation operations under consideration. For instance, in the case of the classical theory of quasiconformal maps between Riemann surfaces equipped with distinct holomorphic structures, the **underlying real analytic surface** of the Riemann surfaces under consideration constitutes just such an *invariant.* In inter-universal Teichmüller theory, such invariant mathematical structures are referred to as **cores**, and the corresponding property of invariance is referred to as **coricity**. In the case of the deformations of the **arithmetic holomorphic structure** — i.e., the conventional ring/scheme-theoretic structure — of a number field equipped with an elliptic curve that are studied in inter-universal Teichmüller theory, the structures that correspond, relative to the analogy with quasiconformal maps between Riemann surface, to the "underlying real analytic surface" are referred to as **mono-analytic**. That is to say, the term "mono-analytic" may be understood as a shorthand for the expression "the arithmetic analogue of the term real analytic". Thus, to summarize, in the terminology of [IUTchI], [IUTchII], [IUTchIII], [IUTchIV], the **coricity** of **mono-analytic** structures plays a central role in the theory.

The approach taken in inter-universal Teichmüller theory to estimating the *distortion* discussed above consists of constructing **"mono-analytic containers"** in which the **alien arithmetic holomorphic structures** that appear — e.g., the arithmetic holomorphic structure surrounding the *domain* of the assignment ($*_{\mathrm{KEY}}$) discussed above, when considered from the point of view of the arithmetic holomorphic structure surrounding the *codomain* of this assignment — may be *embedded*, up to certain **relatively mild indeterminacies** — cf. Theorem 4.1 below; [IUTchIII], Theorem A. Once such an embedding is obtained, one may then proceed to **estimate** *the* **log-volume** *of a region which is sufficiently large as to cover* **all the possibilities** that arise from such indeterminacies — cf. Fig. 1.1 below; [IUTchIII], Theorem B.

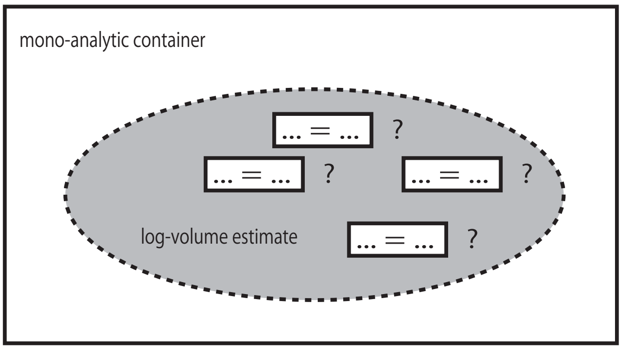

Fig. 1.1: Log-volume estimate region inside a mono-analytic container

In this context, it is of interest to note that the idea of constructing appropriate *"mono-analytic containers"* has numerous classical antecedents. Perhaps the most fundamental example is the idea of studying the variation of the **complex holomorphic moduli** of an **elliptic curve** by studying the way in which the [complex] Hodge filtration is *embedded* within the topological invariant constituted by the first singular cohomology module with complex coefficients of the underlying torus. A slightly less classical example may be seen in conventional Arakelov theory, in which one studies the *metric aspects* of — i.e., the **analytic torsion** associated to — the holomorphic variation of a complex variety by *embedding* the space of holomorphic sections of an ample line bundle into the space of real analytic [or, more generally, L^2-] sections of the line bundle. That is to say, the theory of analytic torsion may be thought of as a sort of approach to measuring the metric aspects of this embedding [cf. the discussion of [IUTchIV], Remark 1.10.4, (a)]. Indeed, **scheme-theoretic Hodge-Arakelov theory** was *originally conceived* precisely as a sort of *arithmetic analogue* of these two relatively classical examples [cf. the discussion of [HASurI], §1.3, §1.4; [IUTchIV], Remark 1.10.4, (b)].

At a concrete level, the *log-volume estimates* discussed above may be summarized as asserting that the "distortion" that occurs at the portion labeled by the index j of the LHS of the assignment $(*_{\mathrm{KEY}})$ is [roughly]

$$\leq \; j \cdot (\text{log-diff}_F + \text{log-cond}_E)$$

— i.e., where we write log-diff_F for the *log-different* of the number field F and log-cond_E for the *log-conductor* of the elliptic curve E [cf. [GenEll], Definition 1.5, (iii), (iv), for more details]. In particular, by the **exact same** computation [cf. the discussion of [IUTchIV], Remark 1.10.1] — i.e., of the term of *highest order* of the *average* over j — as the computation discussed above in the case of degrees of vector bundles on the moduli stack of elliptic curves over $\mathbb{Q}$, we obtain the following **inequality**:

$$\frac{1}{6} \cdot \deg_{\mathrm{arith}}(\log(q)) \; = \; \mathrm{ht}_E \; \leq \; (1+\epsilon) \cdot (\text{log-diff}_F + \text{log-cond}_E) \; + \; \text{constant}$$

This inequality is the content of the so-called **Szpiro Conjecture**, or, equivalently [cf., e.g., [GenEll], Theorem 2.1], the **ABC Conjecture** — cf. Corollary 4.2 below; [IUTchIV], Theorem A.

§ 2. Teichmüller-theoretic Deformations

In §1, we discussed in some detail the *Hodge-Arakelov-theoretic motivation* that underlies the **deformations** *of conventional ring/scheme theory* — i.e., of **arithmetic holomorphic structure** — that form the principal content of inter-universal Teichmüller theory. In the present §2, we begin to take a closer look at certain qualitative aspects of these deformations of arithmetic holomorphic structure.

The ultimate motivating example that lies behind these deformations considered in inter-universal Teichmüller theory is the theory of **deformations** *of holomorphic structure of a* **Riemann surface** that are studied in classical complex Teichmüller theory. Here, we recall that such classical Teichmüller deformations are associated to a **nonzero square differential** on a Riemann surface. Relative to the *canonical holomorphic coordinates* obtained locally on the Riemann surface by integrating various square roots of the given square differential, these classical Teichmüller deformations may be written in the form [cf. Fig. 2.1 below]

$$z \; \mapsto \; \zeta \; = \; \xi + i\eta = \; \lambda x + iy$$

— where $1 < \lambda < \infty$ is the **dilation** factor [cf., e.g., [Lehto], Chapter V, §8]. The *key qualitative feature* of such deformations that is shared by the deformations of arithmetic holomorphic structure that occur in inter-universal Teichmüller theory may be described as follows:

> The **two** *underlying real dimensions* of the single holomorphic dimension under consideration are **"decoupled"** from one another; then **one** of these two

underlying real dimension is **dilated/deformed**, while the **other** underlying real dimension is left **fixed/undeformed**.

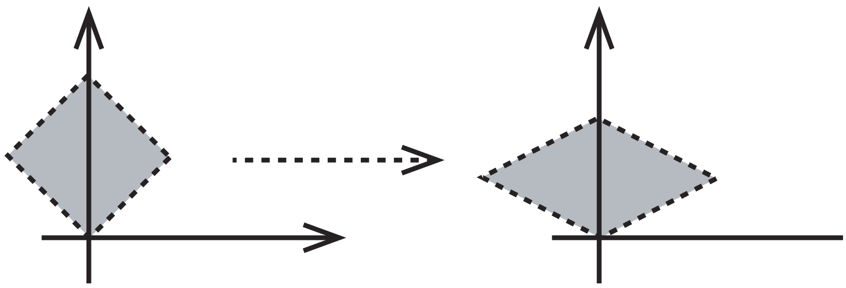

Fig. 2.1: *One* dimension is *dilated*, while the *other* is left *fixed*

Before proceeding, we introduce some terminology which will be useful in our discussion of *deformations* of various types of "holomorphic" structure. To get a sense of the *qualitative* aspects of the terminology introduced, it is useful to keep in mind the *fundamental example* of the situation in which one considers various **complex linear structures** [i.e., "copies of $\mathbb{C}$"] on a **two-dimensional real vector space** [i.e., "$\mathbb{R}^2$"] — cf. the discussion surrounding [IUTchII], Introduction, Fig. I.3; [Quasicon], Appendix. This fundamental example is illustrated in Fig. 2.2 below. As discussed in §1, structures which are *common* to the various distinct "holomorphic" structures under consideration — i.e., such as the underlying *real analytic structure* in the context of deformations of the holomorphic structure of a Riemann surface or the underlying *real vector space* structure in the context of various one-dimensional complex linear structures on a two-dimensional real vector space — will be referred to as **coric**. On the other hand, structures which depend on a *specific* choice of "holomorphic" structure — i.e., a specific choice of a **"spoke"** in the diagram of Fig. 2.2 — will be referred to as **uniradial**.

In this context, perhaps the most subtle notion is the notion of a **multiradial** structure. This term refers to structures that depend on a choice of "holomorphic" structure, but which are described in terms of underlying coric structures in such a way as to be unaffected by alterations in the "holomorphic" structure. In the case of the example illustrated in Fig. 2.2, a typical [albeit somewhat *tautological!*] example of a multiradial structure is given by the $GL_2(\mathbb{R})$*-orbit* of an $\mathbb{R}$-linear isomorphism $\mathbb{C} \xrightarrow{\sim} \mathbb{R}^2$. This terminology is summarized in Fig. 2.3 below. If one thinks of "holomorphic" structures as *"fibers"* and of the underlying coric structure as a *"base space"*, then multiradial structures may be thought of as fiber spaces equipped with a *"connection"* that may be applied to execute **"parallel transport"** operations of fibers corresponding to *arbitrary motions* in the base space [cf. the discussion of [IUTchII], Remark 1.7.1].

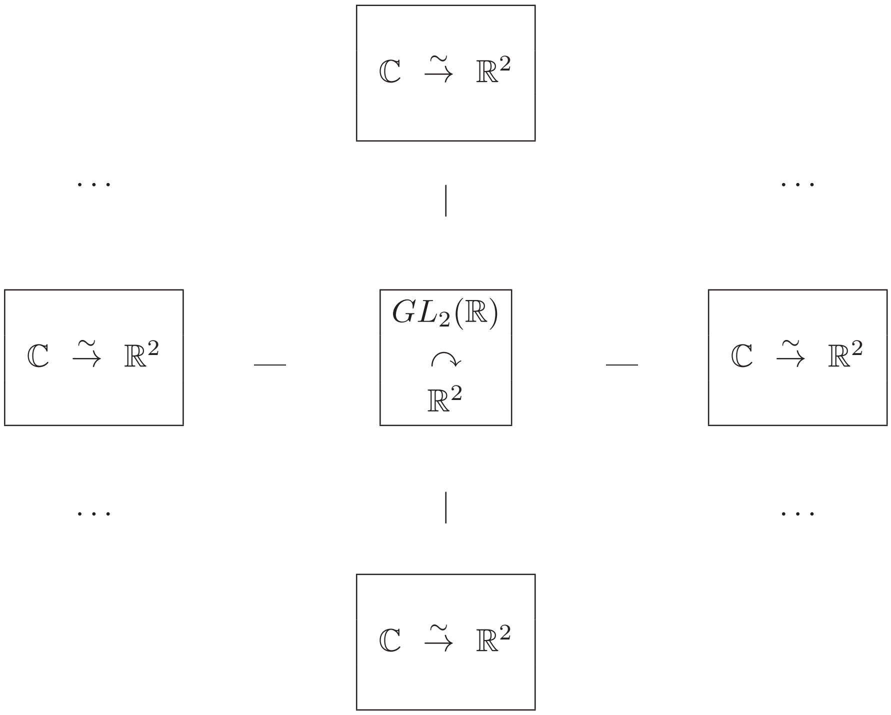

Fig. 2.2: Numerous *one-dimensional complex linear* structures "$\mathbb{C}$" on a single *two-dimensional real vector space* "$\mathbb{R}^2$"

coric structure	underlying common structure	$\mathbb{R}^2$
multiradial structure	"holomorphic" structure described in terms of underlying coric structure	$\curvearrowleft GL_2(\mathbb{R})$ $\mathbb{C} \stackrel{\sim}{\to} \mathbb{R}^2$
uniradial structure	"holomorphic" structure	$\mathbb{C}$

Fig. 2.3: *Coric*, *multiradial*, and *uniradial* structures

Another important motivating example may be seen in the ***p*-adic Teichmüller theory** of [pOrd], [pTeich], [CanLift]. This theory concerns ***p*-adic canonical liftings** of a *hyperbolic curve* over a perfect field of positive characteristic equipped with a *nilpotent ordinary indigenous bundle*. Such canonical liftings of hyperbolic curves are equipped with **canonical Frobenius liftings** — i.e., canonical liftings of the Frobenius morphism in positive characteristic — which are compatible with certain canonical Frobenius liftings on certain p-adic stacks that may be thought of as p-adic étale localizations of the **moduli stack of hyperbolic curves** of a given type. These canonical Frobenius liftings on hyperbolic curves and [certain p-adic étale localizations of] moduli stacks of hyperbolic curves may be regarded as *p-adic analogues* of the well-known metric on the **Poincaré upper half-plane** and the **Weil-Petersson metric** on complex Teichmüller space — cf. Fig. 2.4 below. We refer to the Introductions of [pOrd], [pTeich] for more detailed descriptions of p-adic Teichmüller theory.

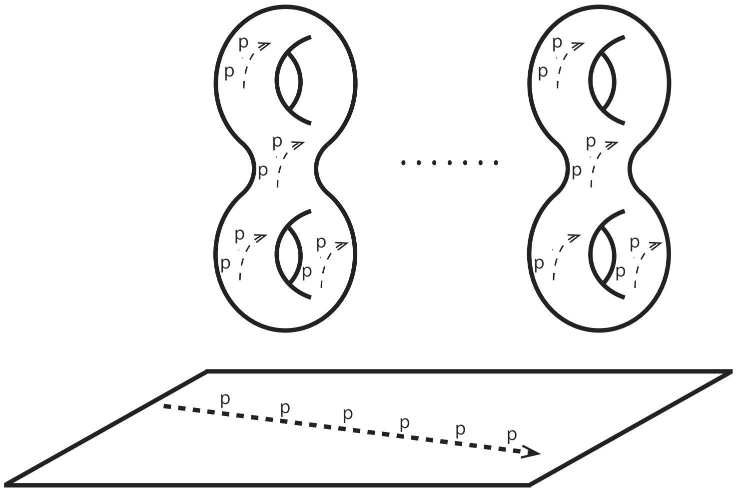

Fig. 2.4: Frobenius liftings on hyperbolic curves and their moduli thought of as "p-adic metric flows"

The analogy between *inter-universal Teichmüller theory* and *p-adic Teichmüller theory* is as summarized in Fig. 2.5 below. In this analogy, the *number field* equipped with a(n) *[once-punctured] elliptic curve* that appears in inter-universal Teichmüller theory [cf. the discussion of §1!] corresponds to the positive characteristic hyperbolic curve equipped with a nilpotent ordinary indigenous bundle that appears in p-adic Teichmüller theory. That is to say, the *number field*, equipped with the finite set of valuations which are either archimedean valuations or nonarchimedean valuations at which the given elliptic curve has bad multiplicative reduction, corresponds to the *positive characteristic hyperbolic curve* [which may be thought of as a one-dimensional function field, equipped with finitely many valuations, i.e., the "cusps at infinity"]. The *[once-punctured] elliptic curve* over the number field then corresponds to the *nilpotent ordinary indigenous bundle* over the positive characteristic hyperbolic curve. Then just as inter-universal Teichmüller theory concerns the issue of **deforming** *conventional ring/scheme theory* [i.e., "over $\mathbb{Z}$"], p-adic Teichmüller theory concerns the issue of ***p*-adically deforming** *certain given objects in positive characteristic scheme theory*. Finally, the **canonical Frobenius liftings** that play a central role in p-adic Teichmüller theory may be thought of as corresponding to the **log-theta-lattice** in inter-universal Teichmüller theory, which we shall discuss in more detail in §3 below. From this point of view, it is of interest to recall the transformations

$$\text{“}t \quad \mapsto \quad t^{a}\text{”} \qquad \text{and} \qquad \text{“}q \quad \mapsto \quad q^{l}\text{”}$$

— which appeared in the discussion of §1, and which are *somewhat reminiscent* in form of the usual *Frobenius morphism in positive characteristic*. Indeed, the *key assignment*

$(*_{\text{KEY}})$

$$\left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\ldots,l^*} \quad \mapsto \quad \underline{\underline{q}}$$

— i.e., which played a central role in the discussion of §1 and will play a central role in the discussion of the *log-theta-lattice* in §3 — may be thought of as a sort of *"deformation"* from the *identity* assignment

$$\underline{\underline{q}} \quad \mapsto \quad \underline{\underline{q}}$$

to the assignment

$$\underline{\underline{q}}^{(l^*)^2} \quad \mapsto \quad \underline{\underline{q}}$$

— i.e., which, if one reverses the direction of the arrow, is reminiscent of the *Frobenius morphism in positive characteristic.* That is to say, the assignment $(*_{\text{KEY}})$ may be thought of as a sort of *"abstract formal analogue"* of the notion of a *Frobenius lifting* in the p-adic theory [cf. the discussion of [IUTchII], Remark 3.6.2, (iii)].

Inter-universal Teichmüller theory	***p*-adic Teichmüller theory**
number field, equipped with a finite set of valuations	*hyperbolic curve* over a perfect field of positive characteristic
[once-punctured] elliptic curve	*nilpotent ordinary indigenous bundle*
conventional ring/scheme theory over $\mathbb{Z}$	*positive characteristic scheme theory*
the *log-theta-lattice*	*canonical Frobenius liftings*

Fig. 2.5: The analogy between inter-universal Teichmüller theory and p-adic Teichmüller theory

At a very rough, qualitative level, p-adic Teichmüller theory may be thought of as a canonical analogue for hyperbolic curves over a perfect field of positive characteristic equipped with a nilpotent ordinary indigenous bundle of the well-known classical theory of the ring of **Witt vectors** $W(\mathbb{F})$ associated to a **perfect field** $\mathbb{F}$ of positive characteristic. That is to say, the ring of Witt vectors may be thought of as a *"p-adic canonical lifting"* of the given perfect field of positive characteristic. Moreover, this "p-adic canonical lifting" is equipped with a **canonical Frobenius lifting** $\Phi_{W(\mathbb{F})} : W(\mathbb{F}) \to W(\mathbb{F})$, i.e., a lifting of the usual Frobenius morphism on the given perfect field of positive characteristic. One central object in the theory of Witt vectors is the multiplicative group of **Teichmüller representatives**

$$[\mathbb{F}^{\times}] \quad \subseteq \quad W(\mathbb{F})$$

of the nonzero elements of the field $W(\mathbb{F})/(p) \xrightarrow{\sim} \mathbb{F}$, which may be characterized by the property $\Phi_{W(\mathbb{F})}(\lambda) = \lambda^p$, for $\lambda \in [\mathbb{F}^{\times}]$. From this point of view, one central aspect of the theory of canonical liftings in p-adic Teichmüller theory that sets this theory apart from the classical theory of Witt vectors is the existence of **canonical coordinates**, i.e., ***q*-parameters** "q_x", in the completion of each closed point x of the **ordinary locus** of a **canonically lifted curve** X over $W(\mathbb{F})$ which may be characterized by the property

$$\Phi_X(q_x) \quad = \quad q_x^p$$

— where we write $\Phi_X : X^{\mathrm{ord}} \to X^{\mathrm{ord}}$ for the canonical Frobenius lifting on the ordinary locus X^{ord} of [the p-adic formal scheme associated to] X. These q-parameters may be regarded as generalizations of the q-parameters on the moduli stack of elliptic curves over $\mathbb{Z}_p$ that appear in **Serre-Tate theory**. From the point of view of the analogy between *p-adic Teichmüller theory* and *classical complex Teichmüller theory* [i.e., as reviewed above!],

> these q-parameters "q_x" may be thought of as corresponding to the *underlying real dimension* that is subject to **dilation**.

From the point of view of the theory of the theory of ***p*-adic Galois representations** of the arithmetic fundamental group of $X \otimes \mathbb{Q}_p$, the q-parameters "q_x" — i.e., which have the effect of "diagonalizing" the Frobenius lifting Φ_X and exhibiting it as a **"mixed characteristic *p*-adic flow"** [cf. Fig. 2.4] — may be thought of as corresponding to **positive slope** Galois representations, i.e., Galois representations that *"straddle the gap"* between "mod p^n" and "mod p^m" for $n \neq m$ [cf. the discussion of the *"positive slope version of Hensel's lemma"* in [AbsTopII], Lemma 2.1; [AbsTopII], Remarks 2.1.1, 2.1.2]. By contrast, the group of Teichmüller representatives "$[\mathbb{F}^\times]$" may be thought of as corresponding to [a certain portion of] the function theory on the given positive characteristic hyperbolic curve that is **held fixed** — i.e., is **coric** — with respect to the deformation to mixed characteristic. From the point of view of the theory of the theory of ***p*-adic Galois representations**, the group of Teichmüller representatives "$[\mathbb{F}^\times]$" may be thought of as corresponding to [a certain portion of the] **slope zero** Galois representations, i.e., at a more concrete level, to *invariants of the* **Frobenius morphism** *in* **positive characteristic**. Relative to the analogy with *classical complex Teichmüller theory* [i.e., as reviewed above!],

> such slope zero Galois representations may be thought of as corresponding to the *underlying real dimension* that is **held fixed**.

The above discussion may be related to the theory of **absolute Galois groups** of p-adic local fields, as follows. Let G_k be the absolute Galois group of a p-adic local field k [i.e., a finite extension of $\mathbb{Q}_p$]. Write $\mathcal{O}_k \subseteq k$ for the ring of integers of k, $\mathcal{O}_k^\times \subseteq \mathcal{O}_k$ for the subgroup of units, and $I_k \subseteq G_k$ for the *inertia subgroup* of G_k. Then at the level of absolute Galois groups, *slope zero* and *positive slope* Galois representations correspond, respectively, to the *maximal unramified quotient* G_k/I_k of G_k and the *inertia subgroup* $I_k \subseteq G_k$. From the point of view of **cohomological dimension**, the **two** cohomological dimensions of G_k may be thought of as consisting precisely of the **one** cohomological dimension of G_k/I_k ($\cong \widehat{\mathbb{Z}}$) and the **one** cohomological dimension of I_k. From the point of **local class field theory**, the one cohomological dimension of G_k/I_k corresponds to the **value group** $k^\times/\mathcal{O}_k^\times$, while the one cohomological dimension of I_k corresponds to the **group of units** $\mathcal{O}_k^\times$.

The above discussion of p-adic local fields may be related naturally to the discussion of **complex Teichmüller theory** at the beginning of the present §2. That is to say, by applying the *exponential function* on $\mathbb{C}$, the **one** real dimension consituted by the *real axis* may be thought of as corresponding to the first factor "$\mathbb{R}_{>0}$" of the natural product decomposition

$$\mathbb{C}^\times \quad = \quad \mathbb{R}_{>0} \times \mathbb{S}^1,$$

while the **one** real dimension constituted by the *complex axis* may be thought of as corresponding to the second factor "$\mathbb{S}^1$" of this product decomposition. If one thinks in terms of the *singular cohomology with compact supports* of $\mathbb{C}$ or $\mathbb{C}^\times$, then the **two** underlying real dimensions of $\mathbb{C}$ or $\mathbb{C}^\times$ may be thought of consisting precisely of the *"cycles with compact supports"* determined by these two real dimensions "$\mathbb{R}_{>0}$" and "$\mathbb{S}^1$", i.e., put another way, of the **value group** and **group of units** of the complex archimedean local field $\mathbb{C}$. That is to say, one obtains an entirely analogous description to the description discussed above in the case of p-adic local fields.

One important point of view in the context of the above discussion of the *"two underlying arithmetic dimensions"* of p-adic and complex archimedean local fields is the following:

> p-adic and complex **Tate curves** "$\mathbb{G}_m/q^{\mathbb{Z}}$" allow one to relate, in a natural fashion, the **"two underlying arithmetic dimensions"** of the local field under consideration to the **"two underlying geometric dimensions"** of the [elliptic curve constituted by such a] Tate curve.

This point of view plays an important role throughout *inter-universal Teichmüller theory*. At a more concrete level, the Tate curve "$\mathbb{G}_m/q^{\mathbb{Z}}$" admits a *natural covering*, in a suitable sense, by "$\mathbb{G}_m$". Then by considering the points of this copy of "$\mathbb{G}_m$" valued in the local field under consideration, this natural covering serves to map the *two underlying arithmetic dimensions* of the above discussion onto the *two underlying geometric dimensions* of the elliptic curve given by such a Tate curve. These two underlying dimensions may be seen concretely in the *usual topological* [i.e., in the complex case] or *étale* [i.e., in the p-adic case] *fundamental group* of this elliptic curve. Moreover, if one thinks of this elliptic curve as the compactification of the once-punctured elliptic curve obtained by removing the origin, then the highly nonabelian structure of the resulting *nonabelian fundamental group* may be thought of as representing the **"intertwining"** of these two underlying dimensions — cf. Fig. 2.6 below. In this context, it is of interest to recall that

> a suitable quotient of this **nonabelian fundamental group** of a **once-punctured elliptic curve** is naturally isomorphic to the **theta group** associated to the ample line bundle on the elliptic curve determined by the origin

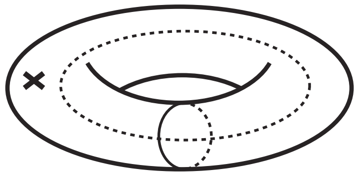

Fig. 2.6: Intertwining cycles on a once-punctured elliptic curve

— cf. the discussion at the beginning of [EtTh], §1, §2. Indeed, the above chain of observations may be thought of as the starting point for the introduction of the theory of **theta functions** as developed in [EtTh] in inter-universal Teichmüller theory. In fact, the review of *complex Teichmüller theory* given at the beginning of the present §2 was included precisely to motivate the following fundamental aspect of inter-universal Teichmüller theory:

> The local portions of the **deformations** of a *number field* equipped with an *elliptic curve* that are constructed in inter-universal Teichmüller theory are obtained precisely by **dilating** the *"one underlying arithmetic dimension"* constituted by the **value groups** by means of a **theta function**, while the *"other underlying arithmetic dimension"* constituted by the **groups of units** is **left fixed**.

We refer to §3 below for a more detailed discussion.

In the context of the above discussion, it is of interest to recall that unlike *theta groups*, the **arithmetic fundamental group** of a once-punctured elliptic curve over an "arithmetic" field such as a number field or a p-adic local field, satisfies highly nontrivial **rigidity properties**, which are the topic of [EtTh], and which play a central role in inter-universal Teichmüller theory [cf. the discussion of §3, §4 below]. For instance, one verifies immediately that a **pro-l theta group** — i.e., the pro-l group generated by three generators α, β, γ satisfying the relations

$$\alpha \cdot \gamma \ = \ \gamma \cdot \alpha, \quad \beta \cdot \gamma \ = \ \gamma \cdot \beta, \quad \alpha \cdot \beta \ = \ \beta \cdot \alpha \cdot \gamma$$

— admits **automorphisms** of the form

$$\alpha \mapsto \alpha^{\lambda}, \quad \beta \mapsto \beta^{\lambda}, \quad \gamma \mapsto \gamma^{\lambda^2},$$

for $\lambda \in \mathbb{Z}_l^{\times}$. Such automorphisms *cease to exist* if one regards this theta group as a *subquotient* of the arithmetic fundamental group of a once-punctured elliptic curve over a number field or a p-adic local field.

So far in our discussion of "two underlying arithmetic dimensions", we have concentrated on local fields, i.e., on the various archimedean and nonarchimedean localizations "F_v" of a [say, for simplicity, *totally complex*] number field F. Note, however, that by considering the *second Galois cohomology module* of the absolute Galois group G_F of F — i.e., in essence, the *Brauer group* of F — one may relate, via the various restriction maps in Galois cohomology, the **two cohomological dimensions** of G_F to the two cohomological dimensions of the various nonarchimedean F_v.

On the other hand, in the various constructions of inter-universal Teichmüller theory, the phenomenon of "two underlying arithmetic dimensions" in the context of *global* number fields will also appear in a somewhat different incarnation, which we describe as follows. The two underlying cohomological dimensions of the absolute Galois group G_k of a p-adic local field k are easiest to understand explicitly if one restricts oneself to the *maximal tamely ramified quotient* of G_k, i.e., which may be described explicitly as a ["representatively large"] closed subgroup of the product over prime numbers $l \neq p$ of profinite groups of the form

$$\mathbb{Z}_l \rtimes \mathbb{Z}_l^{\times}$$

— where the semi-direct product arises from the natural action of the **multiplicative** group $\mathbb{Z}_l^{\times}$ on the **additive** group $\mathbb{Z}_l$. This motivates the point of view that the *semi-direct product of monoids*

$$(\mathbb{Z}, +) \rtimes (\mathbb{Z}, \times)$$

— where the semi-direct product arises from the natural action of the **multiplicative** monoid $(\mathbb{Z}, \times)$ [i.e., obtained by considering the multiplicative portion of the structure of the ring of integers $\mathbb{Z}$] on the **additive** group $(\mathbb{Z}, +)$ [i.e., obtained by considering the additive portion of the structure of the ring of integers $\mathbb{Z}$] — may be thought of as a sort of **"fundamental underlying combinatorial prototype"** of the *two underlying arithmetic dimensions* of a local field, as discussed above.

Moreover, from the point of view of **Hodge-Arakelov theory** [cf. the discussion of §1], it is natural to attempt to **"approximate $\mathbb{Z}$"** by means of the **finite field $\mathbb{F}_l$**, for some prime number l, which one thinks of being **"large"**, i.e., roughly of the order of the **height** of the elliptic curve over a number field under consideration — cf. the discussion of [HASurI], §1.3.4; [IUTchI], Remark 6.12.3, (i). In particular, we shall think of the semi-direct product of monoids $(\mathbb{Z},+) \rtimes (\mathbb{Z},\times)$ as being approximated by the semi-direct product of groups

$$\mathbb{F}_l \rtimes \mathbb{F}_l^{\times}$$

— where the semi-direct product arises from the natural action of the **multiplicative** group $\mathbb{F}_l^{\times}$ on the **additive** group $\mathbb{F}_l$. In fact, in the context of inter-universal Teichmüller theory, this copy of $\mathbb{F}_l = \mathbb{Z}/l \cdot \mathbb{Z}$ will appear as a *quotient* of the module of ***l*-torsion points** of the elliptic curve over a number field under consideration. We refer to the portion of §3 below concerning the structure of Hodge theaters for a more detailed discussion.

§ 3. The Log-theta-lattice

The starting point of the various constructions of inter-universal Teichmüller theory is a collection of **initial Θ-data** [cf. [IUTchI], §I1; [IUTchI], Definition 3.1]. Roughly speaking, this data consists, essentially, of

- an **elliptic curve** E_F over a **number field** F,
- an **algebraic closure** $\overline{F}$ of F,
- a **prime number** $l \geq 5$,
- a *collection of* **valuations** $\underline{\mathbb{V}}$ of a certain subfield $K \subseteq \overline{F}$, and
- a *collection of* **valuations** $\mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}}$ of a certain subfield $F_{\mathrm{mod}} \subseteq F$

that satisfy certain technical conditions — we refer to [IUTchI], Definition 3.1 for more details. Here, we write $F_{\mathrm{mod}} \subseteq F$ for the *field of moduli* of E_F, $K \subseteq \overline{F}$ for the extension field of F determined by the *l-torsion points* of E_F, $X_F \subseteq E_F$ for the *once-punctured elliptic curve* obtained by removing the origin from E_F, and $X_F \to C_F$ for the *hyperbolic orbicurve* obtained by forming the stack-theoretic quotient of X_F by the natural action of $\{\pm 1\}$. Then F is assumed to be *Galois* over F_{mod}, $\mathrm{Gal}(K/F)$ is assumed to be isomorphic to a subgroup of $GL_2(\mathbb{F}_l)$ that *contains* $SL_2(\mathbb{F}_l)$, E_F is assumed to have *stable reduction* at all of the nonarchimedean valuations of F, $C_K \stackrel{\mathrm{def}}{=} C_F \times_F K$ is assumed to be a *K-core* [cf. [CanLift], Remark 2.1.1], $\underline{\mathbb{V}}$ is assumed to be a collection of valuations of K such that the natural inclusion $F_{\mathrm{mod}} \subseteq F \subseteq K$ induces a *bijection* $\underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$ between $\underline{\mathbb{V}}$ and the set $\mathbb{V}_{\mathrm{mod}}$ of all valuations of the number field F_{mod}, and $\mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}} \subseteq \mathbb{V}_{\mathrm{mod}}$ is assumed to be some *nonempty set* of nonarchimedean valuations of *odd* residue characteristic over which E_F has *bad [i.e., multiplicative] reduction* — i.e., roughly speaking, the *subset* of the set of valuations where E_F has bad multiplicative reduction that will be *"of interest"* to us in the context of the constructions of inter-universal Teichmüller theory. Then we shall write $\underline{\mathbb{V}}^{\mathrm{bad}} \stackrel{\mathrm{def}}{=} \mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}} \times_{\mathbb{V}_{\mathrm{mod}}} \underline{\mathbb{V}} \subseteq \underline{\mathbb{V}}$, $\mathbb{V}_{\mathrm{mod}}^{\mathrm{good}} \stackrel{\mathrm{def}}{=} \mathbb{V}_{\mathrm{mod}} \setminus \mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}}$, $\underline{\mathbb{V}}^{\mathrm{good}} \stackrel{\mathrm{def}}{=} \underline{\mathbb{V}} \setminus \underline{\mathbb{V}}^{\mathrm{bad}}$. Also, we shall apply the superscripts "non" and "arc" to $\underline{\mathbb{V}}$, $\mathbb{V}_{\mathrm{mod}}$ to denote the subsets of *nonarchimedean* and *archimedean* valuations, respectively. This *initial* Θ*-data* determines a *finite étale covering* $\underline{C}_K \to C_K$ of degree l such that the base-changed covering

$$\underline{X}_K \stackrel{\mathrm{def}}{=} \underline{C}_K \times_{C_F} X_F \quad \to \quad X_K \stackrel{\mathrm{def}}{=} X_F \times_F K$$

arises from a *rank one quotient* $E_K[l] \twoheadrightarrow Q$ $(\cong \mathbb{Z}/l\mathbb{Z})$ of the module $E_K[l]$ of l-torsion points of $E_K(K)$ [where we write $E_K \stackrel{\mathrm{def}}{=} E_F \times_F K$] which, at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, restricts to the quotient arising from *coverings of the dual graph of the special fiber*.

The constructions of inter-universal Teichmüller theory may, in some sense, be summarized as a *step-by-step* **dismantling** of the conventional scheme-theoretic arithmetic geometry surrounding the given *initial* Θ*-data* which is

> **sufficiently drastic** as to allow us to realize the *key assignment* $(*_{\mathrm{KEY}})$ discussed in §1, but nevertheless **sufficiently controlled** so as to allow us to **compute** precisely the resulting **distortion** of various objects and constructions from conventional arithmetic geometry.

The *central stage* on which these constructions are performed is the **log-theta-lattice** [cf. Fig. 3.1 below], various versions of which are defined in [IUTchIII]. Each "$\bullet$" in Fig. 3.1 represents a $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theater**, which may be thought of as a sort of *miniature model* of the conventional arithmetic geometry surrounding the given initial Θ-data. Each vertical arrow "$\uparrow$" in Fig. 3.1 represents a **$\mathfrak{log}$-link**, i.e., a certain type of **gluing** between various portions of the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters that constitute the domain and codomain of the arrow. Each horizontal arrow "$\to$" in Fig. 3.1 represents a **Θ-link** [various versions of which are defined in [IUTchI], [IUTchII], [IUTchIII]], i.e., another type of **gluing** between various portions of the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters that constitute the domain and codomain of the arrow.

Relative to the analogy between inter-universal Teichmüller theory and p-adic Teichmüller theory [cf. the discussion of §2], each $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater corresponds to a model of **positive characteristic scheme theory** surrounding the given hyperbolic curve over a perfect field equipped with a nilpotent ordinary indigenous bundle; each **$\mathfrak{log}$-link** corresponds to the **Frobenius morphism** in **positive characteristic**; each **Θ-link** corresponds to a **"mixed characteristic transition"** from $p^n\mathbb{Z}/p^{n+1}\mathbb{Z}$ to $p^{n-1}\mathbb{Z}/p^n\mathbb{Z}$, for some positive integer n. In particular, the **two dimensions** of the diagram constituted by the log-theta-lattice may be thought of as corresponding to the two underlying arithmetic dimensions — i.e., *"slope zero"* and *"positive slope"* — of a p-adic local field discussed in §2.

Next, we proceed to take a closer look at the various components "$\bullet$", "$\uparrow$", "$\to$" of the log-theta-lattice. Each $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater* — i.e., each "$\bullet$" — consists of a complicated system of **Frobenioids** equipped with various auxiliary data. A Frobenioid is a category satisfying certain abstract properties, which *imply*, via the theory of [FrdI], that it looks, roughly speaking, like the sort of category that arises from considering the geometry of *divisors* and *line bundles* on the various connected coverings of an irreducible normal noetherian scheme. Put another way, a Frobenioid may be thought of, roughly speaking, as the result of *"deleting the underlying scheme portion of a log scheme"* — i.e., so to speak

$$\text{(Frobenioid)} \;\approx\; \text{(log scheme)} \;-\; \text{(underlying scheme)}.$$

For instance, If $\overline{k}$ is an algebraic closure of a *local p-adic field k*, and we write $G_k \stackrel{\text{def}}{=} \text{Gal}(\overline{k}/k)$ for the *absolute Galois group* of k and $\mathcal{O}^{\triangleright}_{\overline{k}}$ for the *multiplicative monoid* determined by the nonzero integers of $\overline{k}$, then the **local data**

$$G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$$

— i.e., which we think of as consisting of some *abstract topological group* acting on an *abstract topological monoid* — determines [and, indeed, is *equivalent* to the datum of] a certain type of Frobenioid. On the other hand, if we write $G_F \stackrel{\text{def}}{=} \text{Gal}(\overline{F}/F)$ for the *absolute Galois group* of F and $\overline{F}^{\times} \cup \{0\}$ for the multiplicative monoid determined by the elements of $\overline{F}$, then the **global data**

$$G_F \curvearrowright \overline{F}^{\times} \cup \{0\}$$

— i.e., which we think of as consisting of some *abstract topological group* acting on an *abstract monoid* equipped with the various *topologies* determined by the valuations of $\overline{F}$ — determines [and, indeed, is *equivalent* to the datum of] a certain type of Frobenioid. *Variants* of the above examples of Frobenioids may be obtained if G_F (respectively, G_k) is replaced by one of the *arithmetic fundamental groups* $\Pi_{\underline{X}_K}$, $\Pi_{\underline{C}_K}$ of the orbicurves $\underline{X}_K$, $\underline{C}_K$ discussed above (respectively, by the arithmetic fundamental group "$\Pi_{\underline{v}}$" of a certain finite étale covering "$\underline{\underline{X}}_{\underline{v}}$" [when $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$], "$\underline{\overrightarrow{X}}_{\underline{v}}$" [when $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$] of the localization of $\underline{X}_K$ at $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$). Moreover, at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, it will turn out to be more natural to take this "$\Pi_{\underline{v}}$" to be the corresponding *tempered arithmetic fundamental group*.

> The **local** and **global Frobenioids** that appear in a $\Theta^{\pm\text{ell}}$NF-Hodge theater correspond essentially to this sort of *variant* of the examples of Frobenioids just discussed.

That is to say, at $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$, the field k is taken to be the *completion* $K_{\underline{v}}$ of K at $\underline{v}$; by abuse of notation, we shall write "$\overline{F}_{\underline{v}}$" for any of the algebraic closures of $K_{\underline{v}}$ determined by $\overline{F}$ and $G_{\underline{v}} \stackrel{\text{def}}{=} \text{Gal}(\overline{F}_{\underline{v}}/K_{\underline{v}})$; we observe that $G_{\underline{v}}$ may be thought of as a *quotient* $\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$ of $\Pi_{\underline{v}}$. An analogous theory exists at *archimedean* $\underline{v} \in \underline{\mathbb{V}}$.

Throughout the theory of [IUTchI], [IUTchII], and [IUTchIII], we shall often work with *collections of data* of a certain type *indexed* by the elements of $\underline{\mathbb{V}}$. Such collections of data will be referred to as **prime-strips**. For instance, by considering collections of data constituted by the various *"local data"* — i.e., $G_{\underline{v}} \curvearrowright \mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}$, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ — or *"variant local data"* — i.e., $\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}$, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ — discussed above, we obtain $\mathcal{F}^{\vdash}$*-prime-strips* [in the case of the "local data"] or $\mathcal{F}$*-prime-strips* [in the case of the "variant local data"]. Alternatively, the topological group portion of such collections of data — i.e., $G_{\underline{v}}$ or $\Pi_{\underline{v}}$, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ – gives rise to $\mathcal{D}^{\vdash}$*-prime-strips* [in the case of "$G_{\underline{v}}$"] or $\mathcal{D}$*-prime-strips* [in the case of "$\Pi_{\underline{v}}$"]. Here, we remark that the notation

$$\vdash$$

is used as an abbreviation for the term **mono-analytic**, i.e., "the arithmetic analogue of the term real analytic" [cf. the discussion of §1]. We refer to [IUTchI], Fig. I1.2, and the surrounding discussion for a more detailed discussion of the various types of prime-strips that appear in the theory.

$$\begin{array}{ccccc}
 & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & \\
\ldots \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow \ldots \\
 & \uparrow & & \uparrow & \\
\ldots \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow \ldots \\
 & \uparrow & & \uparrow & \\
 & \vdots & & \vdots &
\end{array}$$

Fig. 3.1: The *log-theta-lattice* consists of "$\bullet$" [i.e., $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters*], "$\uparrow$" [i.e., $\mathfrak{log}$-*links*], and "$\rightarrow$" [i.e., Θ-*links*]

The significance of working with various types of prime-strips lies in the fact that **prime-strips** allow one to

> **simulate** a situation in which the natural map $\mathrm{Spec}(K) \to \mathrm{Spec}(F)$ admits **"sections"** which make it possible to consider **global multiplicative subspaces** and **global canonical generators** of the l-torsion points of E_K

— cf. the discussion of §1; Fig. 3.2 below. Indeed, the essential content of the various technical conditions imposed on the collection of valuations $\underline{\mathbb{V}}$ of K may be summarized as the requirement that $\underline{\mathbb{V}}$ *constitutes just such a* **section** [cf. the bijection $\underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$ discussed above]! Put another way, the geometry of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater is arranged precisely so as enable one to construct **tautological** *global multiplicative subspaces* and *global canonical generators* — cf. the discussion of [IUTchI], §I1, surrounding [IUTchI], Fig. I1.4. On the other hand, since such "sections" do not in fact respect the ring structure of K, F — i.e., put another way, they amount to a *partial dismantling* of this ring structure — it is necessary to keep track *explicitly* of the various constructions and objects arising from this ring structure that are of interest in the development of the theory. One important example of such a construction is the theory of *global arithmetic line bundles* on the various number fields under consideration, which may be kept track of by considering suitable *global Frobenioids*, as discussed above. In the case of *mono-analytic prime-strips* [i.e., prime-strips whose type includes the symbol "$\vdash$"], we shall work with the **global realified Frobenioids** associated to the global Frobenioids discussed above — i.e., roughly speaking, Frobenioids that correspond to considering arithmetic divisors with coefficients $\in \mathbb{R}$. The resulting collections of data [i.e., consisting of mono-analytic prime-strips and global realified Frobenioids] are denoted by means of the symbol "$\Vdash$" [cf. [IUTchI], Fig. I1.2, and the surrounding discussion].

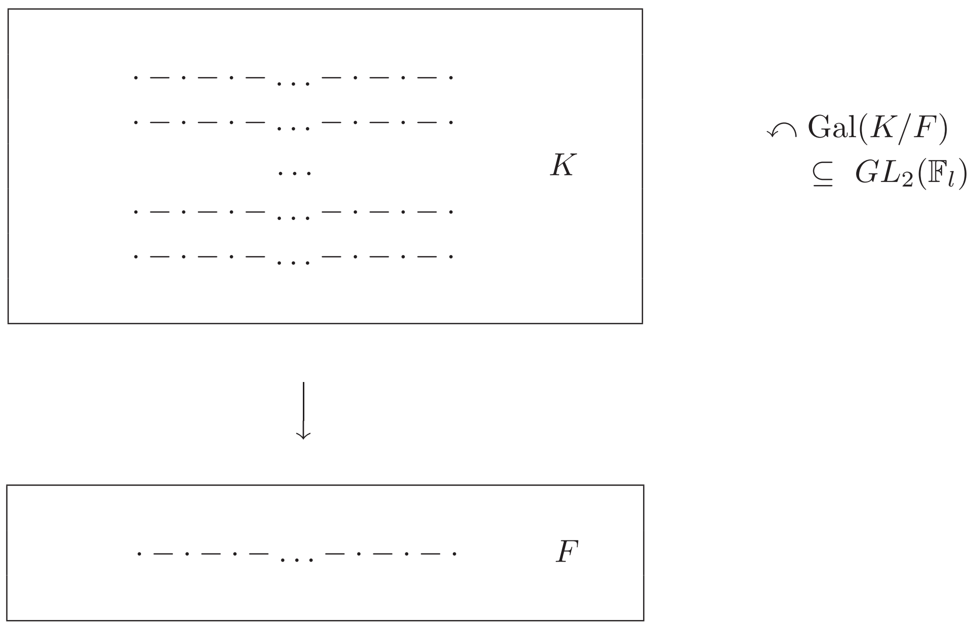

Fig. 3.2: Prime-strips as "sections" of $\mathrm{Spec}(K) \to \mathrm{Spec}(F)$

In the discussion to follow, we shall use the following notation:

$$l^{*} \stackrel{\mathrm{def}}{=} (l-1)/2; \quad l^{\pm} \stackrel{\mathrm{def}}{=} (l+1)/2; \quad \mathbb{F}_l^{*} \stackrel{\mathrm{def}}{=} \mathbb{F}_l^{\times}/\{\pm 1\}; \quad \mathbb{F}_l^{\rtimes\pm} \stackrel{\mathrm{def}}{=} \mathbb{F}_l \rtimes \{\pm 1\}$$

— cf. the discussion of [IUTchI], §I1. The natural action of the stabilizer in $\mathrm{Gal}(K/F)$ of the quotient $E_K[l] \twoheadrightarrow Q$ [cf. the discussion at the beginning of the present §3] on Q determines a *natural "action up to certain indeterminacies"* of $\mathbb{F}_l^{*}$ on $\underline{C}_K$, i.e., a natural isomorphism of $\mathbb{F}_l^{*}$ with some *subquotient* [that is to say, where the "quotient" corresponds to the "indeterminacies" just mentioned] of the automorphism group $\mathrm{Aut}(\underline{C}_K)$

of the algebraic stack $\underline{C}_K$. The $\mathbb{F}_l^{\circledast}$**-symmetry** constituted by this action up to indeterminacies of $\mathbb{F}_l^{\circledast}$ may be thought of as being essentially **arithmetic** in nature, in the sense that the subquotient of $\mathrm{Aut}(\underline{C}_K)$ that gives rise to this poly-action of $\mathbb{F}_l^{\circledast}$ is induced, via the natural map $\mathrm{Aut}(\underline{C}_K) \to \mathrm{Aut}(K)$, by a subquotient of $\mathrm{Gal}(K/F) \subseteq \mathrm{Aut}(K)$. In a similar vein, the natural action of the automorphisms of the scheme $\underline{X}_K$ on the *cusps* of $\underline{X}_K$ determines a *natural "action up to indeterminacies"* of $\mathbb{F}_l^{\rtimes\pm}$ on $\underline{X}_K$, i.e., a natural isomorphism of $\mathbb{F}_l^{\rtimes\pm}$ with some *subquotient* of $\mathrm{Aut}(\underline{X}_K)$. The $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** constituted by this action up to indeterminacies of $\mathbb{F}_l^{\rtimes\pm}$ may be thought of as being essentially **geometric** in nature, in the sense that the subgroup $\mathrm{Aut}_K(\underline{X}_K) \subseteq \mathrm{Aut}(\underline{X}_K)$ [i.e., of K-linear automorphisms] maps *isomorphically* onto the subquotient of $\mathrm{Aut}(\underline{X}_K)$ that gives rise to this $\mathbb{F}_l^{\rtimes\pm}$-symmetry. Finally, the **global** $\mathbb{F}_l^{\circledast}$*-symmetry* of $\underline{C}_K$ only extends to a "$\{1\}$-symmetry" [i.e., in essence, fails to extend!] of the *local* coverings $\underline{\underline{X}}_{\underline{v}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$] and $\underline{X}_{\to_{\underline{v}}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$], while the **global** $\mathbb{F}_l^{\rtimes\pm}$*-symmetry* of $\underline{X}_K$ only extends to a "$\{\pm 1\}$-symmetry" [i.e., in essence, fails to extend!] of the *local* coverings $\underline{\underline{X}}_{\underline{v}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$] and $\underline{X}_{\to_{\underline{v}}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$] — cf. Fig. 3.3 below.

Ultimately, the LHS

$$\left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\ldots,l^{\circledast}}$$

of the **key assignment** $(*_{\mathrm{KEY}})$ discussed in §1 is obtained, in inter-universal Teichmüller theory, by *evaluating the [reciprocal of the l-th root of the] theta function* on the *l-torsion points* corresponding to the quotient $E_K[l] \twoheadrightarrow Q$. One important aspect of this crucial operation of **"Hodge-Arakelov-theoretic evaluation"** is the issue of keeping track of the **labels** $j = 1, \ldots, l^{\circledast}$ of these theta values. At a purely technical level, one of the main functions of the apparatus constituted by a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater is precisely the task of serving as a **bookkeeping device** for these labels. In fact, a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater consists of two portions, namely, a $\Theta^{\pm\mathrm{ell}}$**-Hodge theater** and a ΘNF**-Hodge theater**. The *scheme-theoretic* $\mathbb{F}_l^{\circledast}$-symmetry discussed above induces an $\mathbb{F}_l^{\circledast}$**-symmetry** on certain of the structures [essentially systems of Frobenioids] of a ΘNF-Hodge theater and, in particular, on the labels $j = 1, \ldots, l^{\circledast}$, which we think of [i.e., by reducing modulo l] as elements of $\mathbb{F}_l^{\circledast}$ via the natural action of the group $\mathbb{F}_l^{\circledast}$ on itself. In a similar vein, the *scheme-theoretic* $\mathbb{F}_l^{\rtimes\pm}$-symmetry discussed above induces an $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** on certain of the structures [essentially systems of Frobenioids] of a $\Theta^{\pm\mathrm{ell}}$-Hodge theater and, in particular, on the labels $t = -l^{\circledast}, \ldots, -1, 0, 1, \ldots, l^{\circledast}$, which we think of [i.e., by reducing modulo l] as elements of $\mathbb{F}_l$ via the natural action of the group $\mathbb{F}_l^{\rtimes\pm}$ on $\mathbb{F}_l$. Finally, a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater is obtained by **gluing** a ΘNF-Hodge theater to a $\Theta^{\pm\mathrm{ell}}$-Hodge theater along **prime-strips** equipped with *labels* as discussed above by means of the correspondence of labels given by

$$(\mathbb{F}_l^{\rtimes\pm} \curvearrowright) \qquad \mathbb{F}_l \supseteq \mathbb{F}_l^{\times} \twoheadrightarrow \mathbb{F}_l^{\times}/\{\pm 1\} = \mathbb{F}_l^{\circledast} \qquad (\curvearrowleft \mathbb{F}_l^{\circledast})$$

— cf. [IUTchI], §I1; [IUTchI], §4, §5, §6, for more details.

$$\begin{tikzcd} & \{\pm1\} \curvearrowright \{\underline{\underline{X}}_{\underline{v}} \text{ or } \underline{\overrightarrow{X}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} \arrow[dl, dash] \arrow[dr, dash] & \\ \mathbb{F}_l^{\rtimes\pm} \curvearrowright \underline{X}_K & & \underline{C}_K \curvearrowleft \mathbb{F}_l^{\circledast} \end{tikzcd}$$

Fig. 3.3: Symmetries of coverings of X_F

The $\mathbb{F}_l^{\rtimes\pm}$-*symmetry* has the advantange that, being *geometric* in nature [cf. the above discussion], it allows one to *permute* various copies of "$G_{\underline{v}}$" [where $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$] associated to distinct *labels* $\in \mathbb{F}_l$ without inducing *conjugacy indeterminacies*. This phenomenon, which we shall refer to as **conjugate synchronization**, plays a key role in the **Kummer theory** surrounding the *Hodge-Arakelov-theoretic evaluation of the* **theta function** *at* $\boldsymbol{l}$**-torsion points** that is developed in [IUTchII]— cf. the discussion of [IUTchI], Remark 6.12.6; [IUTchII], Remark 3.5.2, (ii), (iii); [IUTchII], Remark 4.5.3, (i). By contrast, the $\mathbb{F}_l^{\circledast}$-*symmetry* is more suited to situations in which one must *descend* from K to F_{mod}. In inter-universal Teichmüller theory, the most important such situation involves the **Kummer theory** surrounding the **reconstruction** of the **number field** F_{mod} from $\Pi_{\underline{C}_K}$ — cf. [IUTchI], Example 5.1; the discussion of [IUTchI], Remark 6.12.6; [IUTchII], Remark 4.7.6. Here, we note that such situations necessarily induce *global Galois permutations* of the various copies of "$G_{\underline{v}}$" [where $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$] associated to distinct *labels* $\in \mathbb{F}_l^{\circledast}$ that are *only well-defined up to conjugacy indeterminacies*. In particular, the $\mathbb{F}_l^{\circledast}$-symmetry is ill-suited to situations, such as those that appear in the theory of *Hodge-Arakelov-theoretic evaluation* that is developed in [IUTchII], that require one to establish *conjugate synchronization*.

From the point of view of the discussion of §2 concerning the **two underlying arithmetic dimensions** of the ring structure of a ring, the structure of a $\Theta^{\pm\text{ell}}$NF-Hodge theater may be regarded as a decomposition of the *ring* $\mathbb{F}_l$ — which, as discussed in §2, is often regarded, in the context of inter-universal Teichmüller theory, as a *finite approximation* of the *ring of integers* $\mathbb{Z}$ — into its two underlying arithmetic dimensions, i.e., *addition* [cf. the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**] and *multiplication* [cf. the $\mathbb{F}_l^{\circledast}$**-symmetry**]. Morever, this decomposition into additive and multiplicative symmetries in the theory of $\Theta^{\pm\text{ell}}$NF-Hodge theaters may be compared to groups of **additive** and **multiplicative symmetries** of the **upper half-plane** [cf. Fig. 3.4 below]. Here, the **"cuspidal"** geometry expressed by the additive symmetries of the upper half-plane admits a natural "associated coordinate", namely, the classical $\boldsymbol{q}$**-parameter**, which is reminiscent of the way in which the $\mathbb{F}_l^{\rtimes\pm}$-symmetry is well-adapted to the **Kummer theory** surrounding the *Hodge-Arakelov-theoretic evaluation of the* **theta function** *at* $\boldsymbol{l}$**-torsion points** [cf. the above discussion]. By contrast, the **"toral"**, or **"nodal"** [cf. the classical theory of the structure of *Hecke correspondences* modulo p], geometry expressed by the multiplicative symmetries of the upper half-plane admits a natural "associated coordinate", namely, the classical biholomorphic isomorphism of the upper half-plane with the **unit disc**, which is reminiscent of the way in which the $\mathbb{F}_l^{\circledast}$-symmetry is well-adapted to the **Kummer theory** surrounding the **number field** F_{mod} [cf. the above discussion]. For more details, we refer to the discussion of [IUTchI], Remark 6.12.3, (iii).

	Upper half-plane	$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theaters*
("Cuspidal") Additive symmetry	$z \mapsto z + a$, $z \mapsto -\overline{z} + a \quad (a \in \mathbb{R})$	$\mathbb{F}_l^{\rtimes\pm}$-**symmetry**
"Functions" assoc'd to *add. symm.*	$q \stackrel{\mathrm{def}}{=} e^{2\pi i z}$	**theta fn.** evaluated at ***l*-tors.**
("Toral/nodal") Multiplicative symmetry	$z \mapsto \frac{z\cdot\cos(t)-\sin(t)}{z\cdot\sin(t)+\cos(t)}$, $z \mapsto \frac{\overline{z}\cdot\cos(t)+\sin(t)}{\overline{z}\cdot\sin(t)-\cos(t)}$ $(t \in \mathbb{R})$	$\mathbb{F}_l^{\circledast}$**-symmetry**
"Functions" assoc'd to *mult. symm.*	$w \stackrel{\mathrm{def}}{=} \frac{z-i}{z+i}$	elements of the **number field** F_{mod}

Fig. 3.4: Comparison of $\mathbb{F}_l^{\rtimes\pm}$-, $\mathbb{F}_l^{\circledast}$-symmetries with the geometry of the upper half-plane

Each $\mathfrak{log}$**-link** — i.e., each "$\uparrow$" — of the log-theta-lattice may be described as follows. At each $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, if we write $p_{\underline{v}}$ for the prime number determined by $\underline{v}$, then the *ring structures* in the domain and codomain of the $\mathfrak{log}$-link are related by means of the **non-ring-homomorphism** given by the [classical] $\boldsymbol{p_{\underline{v}}}$**-adic logarithm**

$$\log_{\underline{v}} : \overline{F}_{\underline{v}}^{\times} \rightarrow \overline{F}_{\underline{v}}$$

— where the *field* $\overline{F}_{\underline{v}}$ is reconstructed from the *topological monoid* equipped with the action of a *topological group* $\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}_{\overline{F}_{\underline{v}}}$ [cf. the above discussion], in a fashion discussed in detail in [AbsTopIII], §3; the "$\overline{F}_{\underline{v}}$" in the domain (respectively, codomain) of the map $\log_{\underline{v}}$ arises from the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater in the domain (respectively, codomain) of the $\mathfrak{log}$-link [i.e., the arrow "$\uparrow$"]. The *absolute anabelian theory* surrounding such $p_{\underline{v}}$-adic logarithms is one of the *central themes* of [AbsTopIII]. The starting point of this theory of [AbsTopIII] consists of the *key observation* that the $\mathfrak{log}$-link is **compatible** with *"some"* isomorphism

$$\Pi_{\underline{v}} \stackrel{\sim}{\rightarrow} \Pi_{\underline{v}}$$

between the **arithmetic fundamental groups** $\Pi_{\underline{v}}$ on either side of the $\mathfrak{log}$-link, relative to the *natural actions* via $\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$ of $\Pi_{\underline{v}}$ on the domain and codomain of the map

$\log_{\underline{v}}$. At first glance, it may appear as though this *unspecified isomorphism* may be regarded as the *identity* map. In fact, however, the *gap* between the ring/scheme-theoretic *basepoints* in the domain and codomain necessitates the point of view that corresponding arithmetic fundamental groups in the domain and codomain may only be related by means of *some* **indeterminate isomorphism** *of abstract topological groups*. This topic will be discussed in more detail in §4 below.

At $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$, the $\mathfrak{log}$-link admits an *analogous description*. Moreover, if one allows $\underline{v}$ to vary, then the $\mathfrak{log}$-link is also compatible with corresponding **global arithmetic fundamental groups** — i.e., $\Pi_{\underline{X}_K}$, $\Pi_{\underline{C}_K}$ [cf. the above discussion] — in the domain and codomain of the $\mathfrak{log}$-link. Indeed, this sort of *global compatibility* with the $\mathfrak{log}$-link is one of the *central themes* of [AbsTopIII]. Thus, in summary,

> corresponding arithmetic fundamental groups of the $\Theta^{\pm\text{ell}}$NF-Hodge theaters in the domain and codomain of the $\mathfrak{log}$-link are related by means of **indeterminate isomorphisms** *of abstract topological groups*
>
> $$\Pi_{\underline{v}} \xrightarrow{\sim} \Pi_{\underline{v}}, \text{ for } \underline{v} \in \underline{\mathbb{V}}^{\text{non}}; \qquad \Pi_{\underline{X}_K} \xrightarrow{\sim} \Pi_{\underline{X}_K}; \qquad \Pi_{\underline{C}_K} \xrightarrow{\sim} \Pi_{\underline{C}_K}.$$

An analogous statement holds at $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$. This **invariance** of the various arithmetic fundamental groups that appear in a $\Theta^{\pm\text{ell}}$NF-Hodge theater — i.e., when regarded as **abstract topological groups** — is referred to, in inter-universal Teichmüller theory, as the **vertical coricity** [i.e., coricity with respect to the vertical arrows "$\uparrow$"] of these arithmetic fundamental groups, up to indeterminate isomorphism. In this context, we recall that it follows from the **absolute anabelian geometry** developed in [AbsTopIII] that the *ring structure* — i.e., "arithmetic holomorphic structure" — of the various local and global base fields involved may be *reconstructed group-theoretically* from these arithmetic fundamental groups, considered up to isomorphism. That is to say, this **arithmetic holomorphic structure** is an invariant of **each vertical line** of the log-theta-lattice.

Each **Θ-link** — i.e., each "$\rightarrow$" — of the log-theta-lattice may be described as follows. At each $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, the *ring structures* in the domain and codomain of the Θ-link are related by means of the **non-ring-homomorphism** given as follows:

$$(G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}) \xrightarrow{\sim} (G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}); \qquad \left\{ \underline{\underline{q}}^{j^2} \right\}_{j=1,\dots,l^*} \mapsto \underline{\underline{q}}$$

— we write "$\mathcal{O}^{\times\boldsymbol{\mu}} \stackrel{\text{def}}{=} \mathcal{O}^{\times}/\mathcal{O}^{\boldsymbol{\mu}}$" for the quotient of $\mathcal{O}^{\times}$ by its torsion subgroup "$\mathcal{O}^{\boldsymbol{\mu}} \subseteq \mathcal{O}^{\times}$"; we regard $\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$ as an *abstract topological module* equipped with the action of an *abstract topological group* "$G_{\underline{v}}$", as well as with a certain **"integral structure"**, which we shall not describe here in detail [cf. [IUTchII], Definition 4.9, for a more detailed technical discussion]; the collection $\left\{ \underline{\underline{q}}^{j^2} \right\}_{j=1,\dots,l^*}$ may be regarded as a collection of certain values of the [reciprocal of the l-th root of the] **theta function** [cf. the discussion surrounding Fig. 3.3]; the latter portion of the assignment of the above display is to be regarded as an **isomorphism of abstract monoids** [i.e., both of which are isomorphic to the monoid $\mathbb{N}$] generated, respectively, by the LHS and RHS of this assignment. Thus, the latter portion of the Θ-link consists precisely of the **key assignment** $(*_{\text{KEY}})$ that played a central role in the discussion of §1. One *key observation* concerning the Θ-link is that since "$\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$" is regarded as an *abstract topological module* [equipped with a certain "integral structure"], it is subject to the $\widehat{\mathbb{Z}}^{\times}$**-indeterminacies**

$$\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}} \quad \curvearrowleft \quad \lambda \in \widehat{\mathbb{Z}}^{\times}$$

— i.e., given by "raising to the λ-th power". Another *key observation* concerning the Θ-link at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ is that it is **compatible** with *"some"* isomorphism

$$G_{\underline{v}} \ \stackrel{\sim}{\to}\ G_{\underline{v}}$$

between the **arithmetic fundamental groups** $\Pi_{\underline{v}}$ on either side of the $\mathfrak{log}$-link, relative to the *natural actions* of $G_{\underline{v}}$ on the domain and codomain of the Θ-link. At first glance, it may appear as though this *unspecified isomorphism* may be regarded as the *identity* map. In fact, however, the *gap* between the ring/scheme-theoretic *basepoints* in the domain and codomain necessitates the point of view that corresponding arithmetic fundamental groups in the domain and codomain may only be related by means of *some* **indeterminate isomorphism** *of abstract topological groups*. This topic will be discussed in more detail in §4 below.

One central aspect of the theory of Frobenioids developed in [FrdI] is the **fundamental dichotomy** [cf. the discussion of [FrdI], §I4; [IUTchI], §I2, §I3; the beginning of [IUTchII], Introduction] between **"étale-like"** and **"Frobenius-like"** structures. In the discussion above of various examples of Frobenioids, the *étale-like* portion of the corresponding Frobenioids consists of the various *arithmetic fundamental groups* G_k, G_F, $G_{\underline{v}}$, $\Pi_{\underline{v}}$, $\Pi_{\underline{X}_K}$, $\Pi_{\underline{C}_K}$. Thus, the étale-like portion of a $\Theta^{\pm\text{ell}}$NF-Hodge theater, when regarded up to indeterminate isomorphism, is **vertically coric**. In a similar vein, at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, the pair [i.e., regarded as an abstract *topological module* equipped with the action of an abstract *topological group*] $(G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}})$, when regarded up to indeterminate isomorphism, is **horizontally coric**, i.e., invariant with respect to the [horizontal arrow "$\to$" constituted by the] Θ-link. Unlike the case with the vertically coric arithmetic fundamental groups $\Pi_{\underline{v}}$, $\Pi_{\underline{X}_K}$, $\Pi_{\underline{C}_K}$, it is *not* possible to reconstruct the *ring structure* of the local base fields involved from the absolute Galois group $G_{\underline{v}}$. That is to say, the vertically coric arithmetic holomorphic structures in each vertical line of the log-theta-lattice [cf. the above discussion] are **not compatible** with the **horizontal arrows** of the log-theta-lattice. On the other hand, the *Frobenius-like* portion of the various examples of Frobenioids given above consists of the abstract monoids, i.e., $\mathcal{O}^{\rhd}_{\overline{k}}$, $\mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}$, $\overline{F}^{\times} \cup \{0\}$. Here, we recall that such monoids are often *reconstructed* by means of *anabelian algorithms* from various *arithmetic fundamental groups*, i.e., from various *étale-like* structures. In such a situation, if one considers the reconstructed monoid *together* with the arithmetic fundamental group and reconstruction algorithm from which it was reconstructed, then the resulting collection of data constitutes an *étale-like* structure. On the other hand, if one *forgets* the portion of such a collection of data that concerns the way in which the monoid was reconstructed from an arithmetic fundamental group, i.e., if one regards the monoid just as *some abstract monoid*, then the resulting collection of data constitutes a *Frobenius-like* structure.

One important observation in the context of the $\mathfrak{log}$- and Θ-links is the following:

> The **"wall"** — i.e., relative to the *conventional ring/scheme structures* in the domain and codomain — constituted by the assignment that gives rise to the Θ-link at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ can only be defined by working with **abstract topological monoids**, i.e., with **Frobenius-like** structures. A similar statement holds for the $\mathfrak{log}$-link.

Put another way, it is only by *stripping* the abstract topological monoids that occur in the LHS and RHS of the assignment that gives rise to the Θ-link of any information concerning how those abstract topological monoids were, for instance, reconstructed from various arithmetic fundamental groups that appear [cf. the above discussion!] that one may obtain **isomorphic** mathematical objects that allow one to define the **gluing** of $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters constituted by the Θ-link [cf. the discussion of [IUTchII], Remark 1.11.3, (i)]. A similar statement holds for the $\mathfrak{log}$-link. By contrast,

> the importance of **étale-like** structures arises from the fact that they have the power to **penetrate** these "walls"

— cf. the key **vertical** and **horizontal coricity** properties discussed above.

The Θ-link is defined for [both nonarchimedean and archimedean] $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$ in such a way as to be **compatible** with the definition at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, relative to the **"product formula"** [i.e., of elementary number theory, that relates the values that arise as a result of applying various valuations of a number field]. This compatibility with the product formula allows one to define the Θ-link also for certain **global realified Frobenioids**.

One fundamental consequence of the definition of the vertical and horizontal arrows of the log-theta-lattice by means of various *Frobenius-like* structures — i.e., such as *abstract monoids* — is the following observation: the log-theta-lattice is **not symmetric** with respect to permutations that *switch* the domain and codomain of the various arrows that appear in the log-theta-lattice. By contrast, if, for some $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, one considers the "vertically coric copy of $\Pi_{\underline{v}}$ considered up to isomorphism" associated to each vertical line of the log-theta-lattice, and one regards such copies ${}^\dagger\Pi_{\underline{v}}$, ${}^\ddagger\Pi_{\underline{v}}$ arising from *distinct vertical lines* — i.e., **distinct arithmetic holomorphic structures** — as being related by means of *some indeterminate isomorphism* [i.e., of topological groups] of the associated quotients ${}^\dagger\Pi_{\underline{v}} \twoheadrightarrow {}^\dagger G_{\underline{v}}$, ${}^\ddagger\Pi_{\underline{v}} \twoheadrightarrow {}^\ddagger G_{\underline{v}}$, then one obtains a *diagram* [cf. Fig. 3.5 below] in which the **spokes** — i.e., each of which arises from a *vertical line* of the log-theta-lattice — satisfy the property that the diagram is *invariant* with respect to **arbitrary permutations** of the spokes. This diagram of *étale-like structures* [i.e., arithmetic fundamental groups] is referred to as the associated **étale-picture**. In this context, we remark that the diagram of Fig. 2.2 may be thought of as a sort of *"elementary prototype"* of the diagram of Fig. 3.5.

$$\begin{array}{ccccc}
 & & \boxed{\begin{array}{c}\text{arith. hol.}\\ \text{str. } \Pi_{\underline{v}}\end{array}} & & \\
\dots & & | & & \dots \\
\boxed{\begin{array}{c}\text{arith.}\\ \text{hol. str.}\\ \Pi_{\underline{v}}\end{array}} & — & \boxed{\begin{array}{c}\text{mono-}\\ \text{analytic}\\ \text{core } G_{\underline{v}}\end{array}} & — & \boxed{\begin{array}{c}\text{arith.}\\ \text{hol. str.}\\ \Pi_{\underline{v}}\end{array}} \\
\dots & & | & & \dots \\
 & & \boxed{\begin{array}{c}\text{arith. hol.}\\ \text{str. } \Pi_{\underline{v}}\end{array}} & &
\end{array}$$

Fig. 3.5: The portion of the *étale-picture* at $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$

The *transition* from the *Frobenius-like structures* that play a central role in the definition of the **log-theta-lattice** — which is reminiscent of the **cartesian coordinates** in the elementary geometry of the plane — to the *étale-like structures* that give rise to the **étale-picture** — whose symmetries are reminiscent of the **polar coordinates** in the elementary geometry of the plane — is referred to, in inter-universal Teichmüller theory, as the operation of **Kummer-detachment** and is effected by applying various versions of **Kummer theory**. That is to say, this process of *Kummer-detachment* is reminiscent of the transition from *cartesian* to *polar coordinates* in the classical computation of the **Gaussian integral**

$$\int_{-\infty}^{\infty} e^{-x^2} dx \;=\; \sqrt{\pi}.$$

In particular, the log-volume computations that constitute the "final output" of inter-universal Teichmüller theory [cf. Corollary 4.2 below] may be thought of as a sort of **global** *analogue over* **number fields** of this classical computation of the Gaussian integral.

The theory of Kummer-detachment developed in inter-universal Teichmüller theory centers around the **Kummer theory** that concerns each of the *three Frobenius-like components* of the collection of data that appears in the *domain of the* Θ**-link**, namely:

(a) the **units** modulo torsion $\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$ at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [and analogous data at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$];

(b) the **theta values** $\left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\dots,l^*}$;

(c) the corresponding **global realified Frobenioids**.

In general, Kummer theory amounts to essentially a *formality* [cf. the discussion at the beginning of [FrdII], §2] — i.e., by considering the *Galois action* on the various *roots* of some element of a *multiplicative monoid* — once one establishes an appropriate **cyclotomic rigidity isomorphism** between the *cyclotome* [i.e., copy of some quotient of "$\widehat{\mathbb{Z}}(1)$"] that arises from the multiplicative monoid under consideration and some *cyclotome* that arises from the Galois group or arithmetic fundamental group under consideration. For instance, in the case of the natural Galois action of $G_{\underline{v}}$, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, on the topological monoid $\mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}$, the corresponding Kummer theory is established by applying the **conjugate synchronization** arising from the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** [cf. the above discussion], together with a certain *natural cyclotomic rigidity isomorphism* that arises from a suitable interpretation of the well-known content of **local class field theory** [cf. the discussion of [IUTchI], §I2]. On the other hand, the $\widehat{\mathbb{Z}}^\times$**-indeterminacy** [cf. the above discussion] acting on the topological module $\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$ of (a) has the effect of *invalidating this cyclotomic rigidity isomorphism* in the sense that it renders the isomorphism "valid" *only up to a* $\widehat{\mathbb{Z}}^\times$*-indeterminacy.* A substantial portion of inter-universal Teichmüller theory is devoted to overcoming this obstruction to cyclotomic rigidity by establishing **"non-classical cyclotomic rigidity isomorphisms"** for the data of (b), (c). A closely related issue is the issue of **decoupling** the data of (b), (c) from the data of (a), i.e., so as to render the data of (b), (c) **immune** *to the* $\widehat{\mathbb{Z}}^\times$*-indeterminacies* that act on the data of (a).

In the case of the **global number fields** that give rise to the data of (c), this cyclotomic ridigity/decoupling is achieved, in essence, by suitably interpreting various well-known facts from **elementary algebraic number theory** — cf. [IUTchI], Example 5.1, (v); [IUTchIII], Proposition 3.10. The related Kummer theory is performed in the ΘNF-*Hodge theater* portion of a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater, i.e., the portion where the $\mathbb{F}_l^{*}$**-symmetry** appears [cf. the above discussion]. In the case of the **theta functions** that give rise to the theta values that appear in (b), this cyclotomic ridigity/decoupling is achieved by applying the various **ridigity** properties of the **étale theta function**

— cf. [EtTh], Introduction; [IUTchII], §1, §2, §3; [IUTchIII], §2. The related Kummer theory is performed in the portion of a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater constituted by a $\Theta^{\pm\mathrm{ell}}$-*Hodge theater*, i.e., the portion where the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** appears [cf. the above discussion]. In this situation, we remark that when one performs the *Hodge-Arakelov theoretic evaluation* of the theta functions involved at l-torsion points whose *labels* are subject to the bookkeeping apparatus provided by some $\Theta^{\pm\mathrm{ell}}$-Hodge theater [cf. the above discussion], it is necessary to verify that **profinite conjugates** [arising from the *profinite geometric fundamental groups* that occur in the *global portion* of such a $\Theta^{\pm\mathrm{ell}}$-Hodge theater] of the various **tempered** arithmetic fundamental groups at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ do not *intersect* one another in an "unexpected fashion". This non-existence of "unexpected intersections" is obtained by applying a certain technical result [cf. [IUTchI], Theorem B; [IUTchI], §2; [IUTchII], §2] which follows from the theory of [Semi].

One important aspect of this Kummer theory involving the data of (b), (c) is the interpretation of the various rigidity properties in terms of the notion of **multiradiality**, i.e., roughly speaking, as a sort of **rigidity** with respect to **"parallel transport"** between *spokes* of the étale-picture — cf. the discussion of multiradiality in §2; [IUTchIII], Fig. 3.5, and the surrounding discussion. Relative to the analogy with *p-adic Teichmüller theory*, this multiradial, or "parallel transport", interpretation is reminiscent of the **Frobenius crystal structure** on the *canonical indigenous bundles* of p-adic Teichmüller theory. Alternatively, from the point of view of the *scheme-theoretic Hodge-Arakelov theory* reviewed in §1, this multiradial, or "parallel transport", interpretation is reminiscent of theory of the **crystalline theta object** discussed in [HASurII], §2.

In this context, we remark that one important consequence of the theory surrounding the *multiradial interpretation* of the rigidity properties of the étale theta function in the case of the data of (b), together with the closely related $\mathbb{F}_l^{\rtimes\pm}$-*symmetry*, is the **highly distinguished** nature of the theta values $\left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\dots,l^*}$ that occur in (b). That is to say, the intricate technical apparatus that constitutes the *technical backbone* of inter-universal Teichmüller theory would *collapse* if one attempted instead to work with, say, *arbitrary powers* "$\underline{\underline{q}}^N$" of $\underline{\underline{q}}$.

Finally, we observe that there is an interesting analogy between this *multiradial intepretation* and the classical theory of the **functional equation** of the theta function, i.e., **"Jacobi's identity"** — cf., e.g., [DmMn], §1.7.5, as well as [HASurI], §2.1; [EtTh], Remarks 1.6.2, 1.6.3. First of all, we recall that this functional equation is in essence a formal consequence of the **invariance** of the **Gaussian distribution** with respect to the **Fourier transform** on the real line [cf., e.g., [DmMn], §1.7.5]. Moreover, this invariance property is closely related to and, indeed, may be thought of as a sort of *"function-theoretic version"* of the computation of the Gaussian integral discussed above [cf., e.g., [DmMn], §1.7.5]. From this point of view, we observe that just as the *compu-*

tation of the Gaussian integral may be thought of as being analogous to the *log-volume* computation that yields Corollary 4.2 below [cf. the above discussion], the *invariance of the Gaussian distribution with respect to the Fourier transform* may be thought of as being analogous to the *multiradial representation* given in Theorem 4.1 below.

At a more technical level, we observe that the *multiradiality* of the étale theta function may be thought of as a sort of *"parallel transport"* — i.e., a sort of **"integration"** — that exhibits the *rigidity* of the theta functions involved, relative to a situation in which the **units** modulo torsion "$\mathcal{O}^{\times\boldsymbol{\mu}}$" are subject to a **mono-analytic** $\widehat{\mathbb{Z}}^{\times}$*-indeterminacy* — i.e., an indeterminacy with respect to an **exponentiation** operation on the units with respect to some $\lambda \in \widehat{\mathbb{Z}}^{\times}$. This description is reminiscent of the **Fourier transform**, i.e., an **integral** of the Gaussian distribution subject to the effect of a [non-holomorphic!] **real analytic** factor constituted by a **unit** "$e^{it} \in \mathbb{S}^1 \subseteq \mathbb{C}^{\times}$" which is subject to an **exponentiation** operation with respect to some $\lambda \in \mathbb{R}$. In the case of the theory of [EtTh], the desired rigidity properties are proven, in effect, by considering the **quadratic** structure of the commutator relation in the theta group [cf. the discussion of [IUTchIII], Remark 2.1.1]. In the case of the computation of the Fourier transform of the Gaussian distribution, the essence of the computation consists of an analysis, by *"completing the square"*, of the **quadratic** expression that appears in the exponent.

From the point of view of the discussion of §1, the *theta values* $\left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\dots,l^*}$ that occur in (b) may be thought of as a sort of **function-theoretic** expression of the **canonical multiplicative subspace** and **generators** that occur in a neighborhood of a fixed cusp of the moduli stack of elliptic curves. Then, relative to the scheme-theoretic Hodge-Arakelov theory of [HASurI], [HASurII], inter-universal Teichmüller theory provides a means of *globally extending* [cf. the ΘNF-Hodge theater portion of a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater] this function-theoretic expression of the canonical multiplicative subspace and generators [i.e., which is most naturally defined in the $\Theta^{\pm\mathrm{ell}}$-Hodge theater portion of a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater]. This state of affairs is *strongly reminiscent* of the essential content of Jacobi's identity, i.e., which asserts that the global extension to the upper half-plane of the **Gaussian expansion** of the theta function in a neighborhood of the **cusp at infinity** of the upper half-plane admits a *similar Gaussian expansion* in a neighborhood of the **cusp at zero**. Here, the theory of **analytic continuation** of the function *a priori* only defined to a neighborhood of the cusp at infinity may be thought of as corresponding to the **absolute anabelian** representation of local and global arithmetic holomorphic structures afforded by the theory of [AbsTopIII] [cf. the discussion of [IUTchI], Remarks 4.3.1, 4.3.2; [HASurII], Remark 3.7; [IUTchIV], Remark 3.3.2; the theory of Aut-*holomorphic structures* given in [AbsTopIII], §2; the theory of *Belyi cuspidalizations* given in [AbsTopII], §3, which underlies the theory of [AbsTopIII], §1].

§ 4. Inter-universality and Anabelian Geometry

One important aspect of the $\mathfrak{log}$- and Θ-links — i.e., the vertical and horizontal arrows of the log-theta-lattice — is the property that the *assignments* that constitute the $\mathfrak{log}$- and Θ-links [cf. the discussion of §3] are **not compatible** with the **ring structures** in the *domain* and *codomain* of these assignments. On the other hand, from the point of view of the classical theory of **Galois groups** — or, more generally, **étale fundamental groups** of schemes — a Galois group is typically regarded not as some "abstract topological group", but rather as a group of *field automorphisms*

$$\mathrm{Aut}_{\mathrm{field}}(\overline{k})$$

of some *algebraic closure* $\overline{k}$ of a given [say, perfect] field k. In the classical theory of étale fundamental groups of schemes, a morphism between the schemes under consideration — i.e., in essence, up to Zariski localization, a *ring homomorphism* — gives rise, in a natural fashion, to outer homomorphisms between the corresponding étale fundamental groups by considering the *homomorphism* between *algebraically closed fields* associated [up to suitable indeterminacies], relative to the geometric basepoints under consideration, to such a ring homomorphism. That is to say, at least from an *a priori* point of view, there is *no natural functorial way* to define an "induced homomorphism of Galois groups" arising from an assignment which is *not a ring homomorphism*. In particular, when one considers the relationship between the Galois groups in the domain and codomain of such a non-ring/scheme-theoretic assignment, one must sacrifice the datum consisting of the **embedding** of such a Galois group into some "$\mathrm{Aut}_{\mathrm{field}}(\overline{k})$". That is to say, in this sort of a situation, one must work with the Galois groups involved as **abstract topological groups**, which are **not equipped** with the **"labeling apparatus"** [i.e., on elements of such a Galois group] arising from the action via field automorphisms [i.e., the embedding into "$\mathrm{Aut}_{\mathrm{field}}(\overline{k})$"]; it is precisely this state of affairs that requires one to apply various results in **absolute anabelian geometry** in order to *describe ring/scheme-theoretic structures* on the opposite side of the *"wall"* constituted by the $\mathfrak{log}$-, Θ-links [cf. the discussion of §3; Theorem 4.1 below]. Such a labeling apparatus may also be thought of as the labeling apparatus determined by a *specific choice of ring/scheme-theoretic basepoint*, or, alternatively, by the **universe** that gives rise to the *model of set theory* that underlies the codomain of the *fiber functor* determined by such a basepoint. It is for this reason that we refer to this aspect of the theory by the term **"inter-universal"** [cf. the discussion of [IUTchI], §I3].

Thus, if one considers the portion of the log-theta lattice, say, at some *fixed* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, then one must contend with, roughly speaking, a **distinct universe** — i.e., a **distinct labeling apparatus** for the sets that occur in the model of set theory with respect to which one considers some local Galois category at $\underline{v}$ — at **each "•"** of the log-theta-lattice. In a similar vein — cf. the discussion of *prime-strips* as *"sections"* in §3 — one must contend with *"changes of universe"* when one passes back and forth between *local* [i.e., at some $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$] and *global* Galois categories [i.e., Galois categories associated to global arithmetic fundamental groups such as $\Pi_{\underline{X}_K}$, $\Pi_{\underline{C}_K}$].

In this context, it is perhaps useful to observe that in order to regard the correspondence between the "$\Pi_{\underline{v}}$'s" in the domain and codomain of the portion of the $\mathfrak{log}$-*link* at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ as the **"identity"**, it is necessary to **identify** the copies of $\overline{F}_{\underline{v}}$ in the domain and codomain of the $\mathfrak{log}$-link. On the the other hand, such an identification may only be defined if one makes use of the **ring structure** of $\overline{F}_{\underline{v}}$, which is *incompatible* with the Θ-*link*. Put another way, the domain of the Θ-link can only be considered in a meaningful way if one *disables* such an *identification* between the domain and codomain of the $\mathfrak{log}$-link. Thus, in summary, the **highly non-commutative** nature of the **log-theta-lattice** has the effect of obligating one to work with **distinct labeling apparatuses** in the domain and codomain of the vertical arrows of the log-theta-lattice.

From the point of view of the analogy between inter-universal Teichmüller theory and **Jacobi's identity** *for the theta function* discussed at the end of §3, it is perhaps of interest to note that the *labeling apparatus for sets* arising from a *universe* may be thought as corresponding to the *labeling apparatus for the points of a space* [i.e., such as the upper-half plane] provided by a choice of **coordinates**. In particular, "changes of universe" may be regarded as corresponding to **changes of coordinates**. In the context of Jacobi's identity, the *basepoint* given by the *cusp at infinity* gives rise to a *choice of coordinates*, as well as a *canonical multiplicative subspace* and *generator* of the topological fundamental group [i.e., $\cong \mathbb{Z}^2$] of the tautological family of elliptic curves over the upper half-plane. The **Gaussian expansion** of the theta function may then be thought of as a **function-theoretic** representation of this canonical multiplicative subspace and generator, i.e., of the given choice of basepoint. Thus, the *equality of Gaussian expansions* consituted by Jacobi's identity may be thought of as a sort of **"inter-basepoint/inter-coordinate system"** comparison between the basepoints/coordinate systems arising from the *cusp at infinity* and the *cusp at zero*. From an *elementary computational point of view*, this comparison gives rise to a somewhat **startling improvement** in **computational accuracy** [cf. the discussion at the beginning of [DmMn], §1.7.5]! Perhaps it is natural to regard this startling improvement in computational accuracy as a sort of *classical analogue* of the somewhat *"startling" numerical consequence* constituted by Corollary 4.2 of the following abstract result:

Theorem 4.1. **(Multiradial Description of the Domain of the Θ-link)** *The three Frobenius-like components (a), (b), (c) of the collection of data that appears in the* **domain** *of the Θ-link admit a* **multiradial description** *in terms of the [a priori* **"alien"** *]* **arithmetic holomorphic structure** *in the* **codomain** *of the Θ-link.*

We refer to [IUTchIII], Theorem A, for more details. This result may be thought of as the *culmination* of the abstract theory developed in [IUTchI], [IUTchII], [IUTchIII], which, in turn, may be thought of, in essence, as a consequence of the *absolute anabelian* and *Frobenioid-theoretic* theory developed in [Semi], [AbsTopIII], [FrdI], [FrdII], [EtTh].

At first glance, it may appear as though the **$\mathfrak{log}$-link** is *entirely irrelevant* to the content of Theorem 4.1. In fact, however, the theory surrounding the $\mathfrak{log}$-link plays a central role in the construction of the *multiradial description* of Theorem 4.1. That is to say, from a naive, *a priori* point of view, there is no natural way to obtain an *action* of the "$\underline{\underline{q}}^{j^2}$'s" that appear in component (b) on the **multiplicative** monoids "$\mathcal{O}^{\times\boldsymbol{\mu}}$" that appear in component (a). If, on the other hand, one considers the images via the maps "$\log_{\underline{v}}$" [cf. the discussion of §3] of these multiplicative monoids "$\mathcal{O}^{\times\boldsymbol{\mu}}$", then one obtains **additive** monoids on which the "$\underline{\underline{q}}^{j^2}$'s" that appear in component (b) — as well as the *global number fields* that give rise to the global realified Frobenioids that appear in component (c) — do indeed *act naturally.* Since, however, the log-theta-lattice is **highly non-commutative**, it is a *highly nontrivial task* to formulate a "multiradial description" that makes use of the **mono-analytic containers** [cf. the discussion of §1] for the data of (b), (c) that arise, in effect, by taking the *logarithm* of the data of (a). This highly nontrivial task is achieved by applying the interpretation of the theory of [AbsTopIII] that is developed in [IUTchIII], §1, §3 — cf. the discussion surrounding Figs. I.4, I.5, I.6 in [IUTchIII], Introduction. In this context, it is also important to avail oneself of the **compatibility** with the **profinite topology** of the algorithms that give rise to the various **local** *cyclotomic rigidity isomorphisms* [cf. the discussion of §3] that are applied to the data of (a), (b) [cf. the discussion surrounding [IUTchIII], Fig. 3.5]. This compatibility is a property that is *not* shared by the algorithms that give rise to the **global** *cyclotomic rigidity isomorphisms* that are applied to the global number fields that give rise to the data of (c).

From a *numerical* point of view, the significance of the *multiradial description* of Theorem 4.1 is that it yields

> **two tautologically equivalent** *ways to compute the* **log-volume** *— i.e., the* **global arithmetic degree** *— of the arithmetic line bundle in the codomain of the* Θ*-link determined by the [2l-th roots of the] q-parameter* $\underline{\underline{q}}_{\underline{v}}$ *at the* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$.

We refer to the discussion surrounding Fig. I.8 of [IUTchIII], Introduction, for more details. The statement of this numerical consequence of Theorem 4.1 [i.e., [IUTchIII], Theorem A] is given in [IUTchIII], Theorem B. In [IUTchIV], §1, §2, various elementary computations of the *log-volume estimates* obtained in [IUTchIII], Theorem B, are performed. These elementary computations have the following consequence [cf. the discussion of §1; [IUTchIV], Theorem A]:

Corollary 4.2. **(Diophantine Consequences)** *The ABC/Szpiro Conjecture holds.*

Relative to the analogy between *inter-universal Teichmüller theory* and *p-adic/complex Teichmüller theory* [cf. the discussion of §2], these numerical consequences of the *multiradial description* of Theorem 4.1 may be regarded as analogous to the following *inequalities*, which, in essence, reflect the **hyperbolicity** of the curve/hyperbolic Riemann surface under consideration [cf. the discussion of *"negative curvature"* towards the end of [IUTchIII], Introduction]:

(i) the **negativity** of the degree of the **Hasse invariant** associated to a canonically lifted curve of genus g in p-adic Teichmüller theory

$$0 \;>\; (2g-2)(1-p)$$

— i.e., of $\frac{1}{p}$ times the derivative of the **canonical Frobenius lifting**, a quantity which may be thought of as a sort of *numerical comparison* of the *lifting modulo* p^2 [cf. the *domain* of the Θ-link!] of the curve with the original *positive characteristic curve* [cf. the *codomain* of the Θ-link!];

(ii) the **negativity** of the integral that appears in the **Gauss-Bonnet theorem** for a hyperbolic Riemann surface S of genus g

$$0 \;>\; \int_S (\text{curvature of the Poincaré metric}) \;=\; 4\pi(1-g)$$

— a sort of *numerical estimate* in terms of *real analytic* structures such as the Poincaré metric and its curvature concerning the structure of the *holomorphic* Riemann surface S.

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