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Bethe subalgebras in Hecke algebra and Gaudin models [1]

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Abstract. The generating function for elements of the Bethe subalgebra of Hecke algebra is constructed as Sklyanin's transfer-matrix operator for Hecke chain. We show that in a special classical limit $q \to 1$ the Hamiltonians of the Gaudin model can be derived from the transfer-matrix operator of Hecke chain. We consruct a non-local analogue of the Gaudin Hamiltonians for the case of Hecke algebras.

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1 Introduction

The Gaudin models were firstly introduced by M.Gaudin in [1]. These models were also investigated as limiting cases of integrable quantum inhomogeneous $su(2)$-chains in [2]. Here we use an algebraic approach and obtain Gaudin's Hamiltonians from the transfer-matrix operator for open inhomogeneous chain models which formulated in terms of generators of affine Hecke algebra $\hat{H}_{M+1}(q)$. In our chain model an inhomogeneity appears (as well as in [2]) as different shifts in spectral parameters related to different sites of the Hecke chain. The Gaudin Hamiltonians are obtained from the generating function which defines a Bethe subalgebra in the Hecke algebra $\hat{H}_{M+1}(q)$, by taking a special "classical limit" $q \to 1$.

The Bethe subalgebras in the group ring of symmetric groups have been studied, for example, in [13], [11]. In the present paper we construct a **lift** of the Bethe subalgebras studied in the papers mentioned above, to the cases of the Hecke and affine Hecke algebras. Our construction of the Bethe subalgebras is based on some special properties of the **trace** maps [7],[8],[9] in the *tower* of the (affine) Hecke algebras, see Section 3, formulae (15). Non formally speaking, the main idea behind our construction, is to define a set of "baxterized" Jucys–Murphy elements in the (affine) Hecke algebra. To realize this idea we treat the Jucys–Murphy elements in the (affine) Hecke algebra as a "classical limit" of the canonical free abelian subgroup in the (affine) braid group, see Section 2.

The plan of the paper is as follows. In Section 2 we review some basic facts about braid and affine braid groups we need, namely, definitions and the construction of the maximal free abelian subgroups in these groups.
Sections 3,4 contain our main results, namely, the construction of Bethe's subalgebras in the Hecke and affine Hecke algebras. In particular, Theorem 1 in Section 4 describes the Hecke version of the Gaudin Hamiltonians. We treat the Bethe subalgebras obtained as a "baxterization" of the canonical free abelian subgroup in the corresponding braid group. In other words, we introduce a spectral parameter dependences in definition of the Jucys–Murphy elements keeping the commutativity property of the deformed elements. A similar construction can be done for the Birman–Murakami-Wenzl algebras, cyclotomic Hecke algebras and some other quotients of braid groups.

In Section 5 we study the classical and Yangian limits of the Bethe subalgebras in the Hecke and affine Hecke algebras correspondingly.

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2 Braid group

Denote by $\mathbb{S}_n$ **the symmetric group** on n letters, and by s_i the simple transposition $(i, i+1)$ for $1 \le i \le n-1$.

The well-known Moore–Coxeter presentation of the symmetric group has the form

$$\langle s_1, \dots, s_{n-1} \mid s_i^2 = 1, \;\; s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \; s_i s_j = s_j s_i, \quad if \quad |i-j| \ge 2 \rangle.$$

Transpositions $s_{ij} := s_i \; s_{i+1} \cdots s_{j-2} \; s_{j-1} \; s_{j-2} \cdots s_{i+1} \; s_i$, $1 \le i < j < j \le n$, satisfy the following set of (defining) relations:
$s_{ij}^2 = 1$, $s_{ij} \; s_{kl} = s_{kl} \; s_{ij}$, if $\{i,j\} \bigcap \{k,l\} = \emptyset$,
$s_{ij} \; s_{ik} = s_{jk} \; s_{ij} = s_{ik} \; s_{jk}$, $s_{ik} \; s_{ij} = s_{ij} \; s_{jk} = s_{jk} \; s_{ik}$, $i < j < k$.

The Artin braid group on n strands B_n is defined by generators $\sigma_1, \dots, \sigma_{n-1}$ and relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad 1 \le i \le n-2, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \;\; if \;\; |i-j| \ge 2. \tag{1}$$

Proposition 2.1 *Let us introduce elements*

$$D_{i,j} := \sigma_{j-1} \sigma_{j-2} \cdots \sigma_{i+1} \; \sigma_i^2 \; \sigma_{i+1} \cdots \sigma_{j-2} \sigma_{j-1},$$

$$F_{i,j} := \sigma_{n-j} \sigma_{n-j+1} \dots \sigma_{n-i-1} \; \sigma_{n-i}^2 \; \sigma_{n-i-1} \dots \sigma_{n-j+1} \sigma_{n-j},$$

where $1 \le i < j \le n$.

For example,
$D_{i,i+1}=\sigma_i^2$, $D_{i,i+2}=\sigma_{i+1}\sigma_i^2\sigma_{i+1}$, $F_{i,i+1}=\sigma_{n-i}^2$, $F_{i,i+2}=\sigma_{n-i-1}\,\sigma_{n-i}^2\,\sigma_{n-i-1}$, *and so on.*
Then

- *For each* $j=3,\ldots,n$, *the element* $D_{1,j}$ *commutes with* $\sigma_1,\ldots,\sigma_{j-2}$.
- *The elements* $D_{i,i+1},D_{i,i+2},\ldots,D_{i,n}$ *(resp.* $F_{i,i+1},F_{i,i+2},\ldots,F_{i,n}$*)* $1\le i\le n-1$, *generate a free abelian subgroup in* B_n.
- *The elements* $D_{1,2},D_{2,3},\ldots,D_{1,n}$ *(resp.* $F_{1,2},F_{2,3},\ldots,F_{1,n}$*) generate a maximal free abelian subgroup in* B_n.
- *If* $n\ge 3$, *the element*

$$\prod_{2\le j\le n} D_{1,j}=\prod_{2\le j\le n} F_{1,j}=(\sigma_1\cdots\sigma_{n-1})^n$$

generates the center of the braid group B_n.
- $D_{i,j}D_{i,j+1}D_{j,j+1}=D_{j,j+1}D_{i,j+1}D_{i,j}$, *if* $i<j$.

- Consider the elements $s:=\sigma_1\,\sigma_2\,\sigma_1$, $t:=\sigma_1\,\sigma_2$ in the braid group B_3. Then $s^2=t^3$ and the element $c:=s^2$ generates the center of the group B_3. Moreover,

$$B_3/\langle c\rangle\cong PSL_2(\mathbb{Z}),\quad B_3/\langle c^2\rangle\cong SL_2(\mathbb{Z}).$$

The affine Artin braid group B_n^{aff}, is an extension of the Artin Braid group on n strands B_n by the element τ subject to the set of crossing relations

$$\sigma_1\;\tau\;\sigma_1\;\tau=\tau\;\sigma_1\;\tau\;\sigma_1,\quad \sigma_i\;\tau=\tau\;\sigma_i\quad for\quad 2\le i\le n-1.$$

Proposition 2.2 *The elements*

$$\hat{D}_1:=\tau,\;\hat{D}_j=\sigma_{j-1}\;\hat{D}_{j-1}\;\sigma_{j-1},\quad 2\le j\le n,$$

generate a free abelian subgroup in B_n^{aff}.

Therefore, for a unital commutative algebra F, any quotient $F[B_n]/J$ of the group algebra $F[B_n]$ of the braid group B_n (resp. a quotient $F[B_n^{aff}]/I$ of the affine braid group B_n^{aff} group algebra $F[B_n^{aff}]$) by a two-sided ideal $J\subset F[B_n]$ (resp $I\subset F[B_n^{aff}]$) contains distinguish commutative subalgebra generated by the images of elements $D_{1,2},\ldots,D_{1,n-1}$ (resp. $\hat{D}_1,\ldots,\hat{D}_n$). It is well-known that the Hecke and affine Hecke algebras are certain quotients of the braid and affine braid groups correspondingly, see the next Section for details. In these cases the images of elements $D_{1,2},\ldots,D_{1,n-1}$ and those $\hat{D}_1,\ldots,\hat{D}_n$ coincide with the Jucys– Murphy elements in the Hecke and affine Hecke algebras correspondingly. Our main objective of the next Section is to construct an analogue of the Bethe subalgebras in the affine Hecke algebras.

3 Bethe subalgebras for affine Hecke algebra

The **Hecke algebra** $H_{M+1}(q)$ (see, e.g., [5] and [7]) is generated by invertible elements T_i $(i=1,\ldots,M)$ subject to the set of relations:

$$T_i\,T_{i+1}\,T_i=T_{i+1}\,T_i\,T_{i+1}\;,\quad T_i\,T_j=T_j\,T_i\;\text{ for }\;i\neq j\pm 1\;. \tag{2}$$

$$T_i^2-1=(q-q^{-1})\,T_i\;, \tag{3}$$

Let x be a spectral parameter. We define Baxterized elements

$$T_i(x):=T_i-xT_i^{-1}=(1-x)T_i+\lambda x\in H_{M+1}(q)\;, \tag{4}$$

which in view of (2) and (3) solve the Yang-Baxter equation

$$T_i(x)\,T_{i-1}(xz)\,T_i(z)=T_{i-1}(z)\,T_i(xz)\,T_{i-1}(x)\;, \tag{5}$$

and satisfy relations

$$T_i(x)T_i(z) = \lambda T_i(xz) + (1-x)(1-z) \; , \tag{6}$$

$$T_i(1) = \lambda \; , \quad T_i(x) = \tfrac{(1-x)}{(1-z)} T_i(z) + \lambda \tfrac{(x-z)}{(1-z)} \; , 0 \tag{7}$$

where $\lambda := (q - q^{-1})$. Note that from (6) for Baxterized elements we have the condition

$$T_i(x)\, T_i(x^{-1}) = (q^{-1}x - q)(q^{-1}x^{-1} - q) \; ,$$

which can be written as unitarity condition $\widetilde{T}_i(x)\widetilde{T}_i(x^{-1}) = 1$ for modified baxterized elements

$$\widetilde{T}_i(x) = \frac{1}{(q - q^{-1}x)}\, T_i(x) \; . \tag{8}$$

The <u>**affine Hecke algebra** $\hat{H}_{M+1}(q)$</u> (see, e.g., Chapter 12.3 in [5] and [10]) is an extension of the Hecke algebra$H_{M+1}(q)$ by additional affine elements $\hat{y}_k$ $(k = 1, \ldots, M+1)$ subject to relations:

$$\hat{y}_{k+1} = T_k\, \hat{y}_k\, T_k \; , \quad \hat{y}_k\, \hat{y}_j = \hat{y}_j\, \hat{y}_k \; , \quad \hat{y}_j\, T_i = T_i\, \hat{y}_j \;\; (j \neq i, i+1) \; . \tag{9}$$

The elements $\{\hat{y}_k\}$ form a commutative subalgebra in $\hat{H}_{M+1}$, while the symmetric functions in $\hat{y}_k$ form the center in $\hat{H}_{M+1}$. The Jucys–Murphy elements $\{\hat{y}_k\}$ coincide with the images of elements $\hat{D}_1, \ldots, \hat{D}_n$ considered in previous Section. Here and below we omit the dependence on q in the notations $H_{M+1}(q)$ and $\hat{H}_{M+1}(q)$ of the Hecke algebras.

The <u>**Ariki-Koike algebra [3],[4]** $\mathcal{H}_{M+1}(q, Q_1, ..., Q_m)$</u> is the quotient of the affine Hecke algebra $\hat{H}_{M+1}$ by the characteristic identity

$$(\hat{y}_1 - Q_1) \cdots (\hat{y}_1 - Q_m) = 0 \; , \tag{10}$$

where $Q_1, \ldots, Q_m$ are parameters.

Definition 3.1 Let $\vec{\xi}_{(n)} = (\xi_1, \ldots, \xi_n)$ be n parameters and $y_1(x) \in \hat{H}_{M+1}$, define the elements

$$\begin{aligned} y_n(x; \vec{\xi}_{(n-1)}) &= T_{n-1}(\tfrac{x}{\xi_{n-1}}) \cdots T_2(\tfrac{x}{\xi_2}) T_1(\tfrac{x}{\xi_1})\, y_1(x)\, T_1(x\xi_1) T_2(x\xi_2) \cdots T_{n-1}(x\xi_{n-1}) = \\ &= T_{n-1}(\tfrac{x}{\xi_{n-1}}) y_{n-1}(x; \vec{\xi}_{(n-2)}) T_{n-1}(x\xi_{n-1}) \; , \end{aligned} \tag{11}$$

which we call as "baxterized" Jucys–Murphy elements.

Proposition 3.1 *Assume that the element $y_1(x) \in \hat{H}_{M+1}$ in (11) is any local (i.e., $[y_1(x), T_k] = 0$, $\forall k > 1$) solution of the reflection equation*

$$T_1\,(x/z)\; y_1(x)\, T_1(x\,z)\, y_1(z)) = y_1(z)\, T_1(x\,z)\, y_1(x)\, T_1\,(x/z) \;\; , \tag{12}$$

Then the elements (11) satisfy the <u>reflection equation</u>

$$T_n\,(x/z)\; y_n(x; \vec{\xi}_{(n-1)})\, T_n(x\,z)\, y_n(z; \vec{\xi}_{(n-1)}) = y_n(z; \vec{\xi}_{(n-1)})\, T_n(x\,z)\, y_n(x; \vec{\xi}_{(n-1)})\, T_n\,(x/z) \;\; , \tag{13}$$

Proof The case $n = 1$ of the equation (13) corresponds to our assumption that $y_1(x)$ satisfies the equation (12). The general case follows by induction using the definition (11) of elements $y_n(x; \vec{\xi}_{(n-1)})$.
■

For example, in the case of the affine Hecke algebra, one can use the local solution (see [8]):

$$y_1(x) = \frac{\hat{y}_1 - \xi x}{\hat{y}_1 - \xi x^{-1}}, \tag{14}$$

where ξ is a parameter. In the case of the Ariki-Koike algebra this rational solution is represented in the polynomial form by writing the characteristic identity (10) as

$$\frac{1}{\hat{y}_1 - \xi x^{-1}} = v_1\, \hat{y}_1^{m-1} + v_2\, \hat{y}_1^{m-2} + \cdots + v_{m-1}\, \hat{y}_1 + v_m \; ,$$

where $v_1, \ldots, v_m$ are functions of $\xi, x, Q_1, \ldots, Q_m$.

Consider the following inclusions of the subalgebras $\hat{H}_1 \subset \hat{H}_2 \subset \cdots \subset \hat{H}_{M+1}$:

$$\{\hat{y}_1; T_1, \ldots, T_{n-1}\} \in \hat{H}_n \subset \hat{H}_{n+1} \ni \{\hat{y}_1; T_1, \ldots, T_{n-1}, T_n\} \; .$$

Define for the algebra $\hat{H}_{M+1}$ linear mappings

$$\mathrm{Tr}_{(n+1)} : \;\; \hat{H}_{n+1} \to \hat{H}_n \; , \quad (n = 1, 2, \ldots, M) \; ,$$

such that for all $X, X' \in \hat{H}_n$ and $Y \in \hat{H}_{n+1}$ we have

$$\begin{gathered} \mathrm{Tr}_{(n+1)}(T_n^{\pm 1} \cdot X \cdot T_n^{\mp 1}) = \mathrm{Tr}_{(n)}(X) \, , \;\; \mathrm{Tr}_{(n+1)}(X \cdot Y \cdot X') = X \cdot \mathrm{Tr}_{(n+1)}(Y) \cdot X' \;\; , \\ \mathrm{Tr}_{(n)}\mathrm{Tr}_{(n+1)}(T_n \cdot Y) = \mathrm{Tr}_{(n)}\mathrm{Tr}_{(n+1)}(Y \cdot T_n) \; , \\ \mathrm{Tr}_{(n+1)}(T_n) = 1 \; , \mathrm{Tr}_{(1)}(y_1^k) = D^{(k)} \; , \;\; \mathrm{Tr}_{(n+1)}(X) = D^{(0)} \, X \; , \end{gathered} \tag{15}$$

where $k \in \mathbb{Z}$ and $D^{(k)} \in \mathbb{C}\backslash\{0\}$ are constants. Note that $D^{(0)}$ is independent of n and all $D^{(k)}$ can be considered as central elements for certain central extension $\mathrm{Ext}(\hat{H}_{M+1})$ of $\hat{H}_{M+1}$. The elements $D^{(k)}$ generate an abelian subalgebra (we denote this subalgebra $\hat{H}_0$) in $\mathrm{Ext}(\hat{H}_{M+1})$.

Using the properties (15) of the map $\mathrm{Tr}_{_{(n+1)}}$ and relations (6), one can show

Lemma 3.1 *For all $X \in \hat{H}_n$ and $\forall x, z$, the following identity is true:*

$$\mathrm{Tr}_{_{(n+1)}}\Big(T_n(x) \cdot X \cdot T_n(z)\Big) = (1-x)\,(1-z)\,\mathrm{Tr}_{_{(n)}}(X) + \lambda\,(1 - p\,x\,z)\;X \; , \tag{16}$$

where $T_n(x)$ are Baxterized elements (4) and

$$p = 1 - \lambda D^{(0)} = 1 - (q - q^{-1})\mathrm{Tr}_{_{(n+1)}}(1) \; .$$

■

From eq. (16), for $p\,x\,z = 1$, we obtain the "crossing-symmetry relation"

$$\mathrm{Tr}_{_{(n+1)}}\Big(T_n(x) \cdot X \cdot T_n\left(1/(px)\right)\Big) = \frac{1}{F_p(x)}\,\mathrm{Tr}_{_{(n)}}(X) \; , \tag{17}$$

where $F_p(x) = \frac{p\,x}{(1-x)(p\,x-1)}$.

Proposition 3.2 (see also [8], [9]). *Let $y_n(x) \in \hat{H}_n$ be any solution of the RE (13). The operators*

$$\tau_{n-1}(x) = \mathrm{Tr}_{_{(n)}}\left(y_n(x)\right) \in \hat{H}_{n-1} \; , \tag{18}$$

form a commutative family of operators

$$\Big[\tau_{n-1}(x) \, , \; \tau_{n-1}(z)\Big] = 0 \quad (\forall x, z) \; , \tag{19}$$

in the subalgebra $\hat{H}_{n-1} \subset \hat{H}_{M+1}$.

Proof. Using (15), (17) and (13) we find

$$\begin{gathered} \tau_{n-1}(x)\,\tau_{n-1}(z) = Tr_{_{(n)}}\left(y_n(x)\,\tau_{n-1}(z)\right) = \\ = F_p(x\,z)\,\mathrm{Tr}_{_{(n)}}\left(y_n(x)\,\mathrm{Tr}_{_{(n+1)}}\left(T_n(xz)\,y_n(z)T_n((pxz)^{-1})\right)\right) = \\ = F_p(x\,z)\,\mathrm{Tr}_{_{(n)}}\mathrm{Tr}_{_{(n+1)}}\left(T_n^{-1}(x/z)\,y_n(x)\,T_n(xz)\,y_n(z)\,T_n(x/z)\,T_n((pxz)^{-1})\right) = \\ = F_p(x\,z)\,\mathrm{Tr}_{_{(n)}}\mathrm{Tr}_{_{(n+1)}}\left(y_n(z)\,T_n(xz)\,y_n(x)\,T_n((pxz)^{-1})\right) = \\ = \mathrm{Tr}_{_{(n)}}\left(y_n(z)\,\tau_{n-1}(x)\right) = \tau_{n-1}(z)\,\tau_{n-1}(x) \; , \end{gathered}$$

where $F_p(x)$ is defined in (17).

■

Now we consider the operators (18), where solution $y_n(x)$ of the reflection equation is taken in the form (11):

$$\tau_n(x;\vec{\xi}_{(n)}) = \mathrm{Tr}_{(n+1)}\Big(y_{n+1}(x;\vec{\xi}_{(n)})\Big) \in \hat{H}_n \tag{20}$$

We stress that the elements (20) are nothing but the analogs of Sklyanin's transfer-matrices [12] and the coefficients in the expansion of $\tau_n(x;\vec{\xi}_{(n)})$ over the variable x (for the homogeneous case $\xi_k = 1$) are the Hamiltonians for the open Hecke chain models with nontrivial boundary conditions which was considered e.g. in [9]. Consider this expansion of $\tau_n(x;\vec{\xi}_{(n)})$ for inhomogeneous case:

$$\tau_n(x;\vec{\xi}_{(n)}) = \sum_{k=-\infty}^{\infty} \Phi_k(\vec{\xi}_{(n)})\, x^k \ \in \hat{H}_n\ . \tag{21}$$

According to the Proposition 3.2, for fixed parameters $\vec{\xi}_{(n)} = (\xi_1,\dots,\xi_n)$, the elements $\Phi_k(\vec{\xi}_{(n)})$ generate a commutative subalgebra $\hat{\mathcal{B}}_n(\vec{\xi}_{(n)}) \subset \hat{H}_n$. These elements are interpreted as Hamiltonians for the inhomogeneous open Hecke chain models. Following [13] we call the subalgebras $\hat{\mathcal{B}}_n(\vec{\xi}_{(n)})$ as Bethe subalgebras of the affine Hecke algebra $\hat{H}_n$.

First we obtain more explicit form for the generating function of the elements $\Phi_k(\vec{\xi}_{(n)}) \in \hat{\mathcal{B}}_n(\vec{\xi}_{(n)})$. For this we substitute the solution $y_{n+1}(x;\vec{\xi}_{(n)})$ of the reflection equation in the form (11) to the transfer-matrix operators (20). Using relation (16) we obtain

$$\begin{gathered}
\tau_n(x;\vec{\xi}_{(n)}) = \mathrm{Tr}_{(n+1)}\Big(T_n(\tfrac{x}{\xi_n})\cdots T_2(\tfrac{x}{\xi_2})T_1(\tfrac{x}{\xi_1})\, y_1(x)\, T_1(x\xi_1)T_2(x\xi_2)\cdots T_n(x\xi_n)\Big) = \\
= (\xi_n - x)(\xi_n^{-1} - x)\, \tau_{n-1}(x;\vec{\xi}_{(n-1)}) + \lambda(1 - p\,x^2)\, y_n(x;\vec{\xi}_{(n-1)}) = \\
= \prod_{k=n-1}^{n} (\xi_k - x)(\xi_k^{-1} - x)\, \tau_{n-2}(x;\vec{\xi}_{(n-2)}) + \\
+\lambda(1 - p\,x^2)\Big((\xi_n - x)(\xi_n^{-1} - x) y_{n-1}(x;\vec{\xi}_{(n-2)}) + y_n(x;\vec{\xi}_{(n-1)})\Big) = \cdots = \\
= \Big(\prod_{k=1}^{n} (\xi_k - x)(\xi_k^{-1} - x)\Big)\, \tau_0(x) + \lambda(1 - p\,x^2)\, J_n(x;\vec{\xi}_{(n)})\ ,
\end{gathered} \tag{22}$$

where the element $\tau_0(x) = \mathrm{Tr}_{(1)}\,(y_1(x)) \in \hat{H}_0$ by definition is the central element in $\hat{H}_n$. In equation (22) we have introduced the notation $J_n(x;\vec{\xi}_{(n)})$ for new explicit generating function of the commutative elements $\Phi_k(\vec{\xi}_{(n)}) \in \hat{H}_n$:

$$\begin{gathered}
J_n(x;\vec{\xi}_{(n)}) = \sum_{r=1}^{n} d_r^n(x;\vec{\xi}\,)\ y_r(x;\vec{\xi}_{(r-1)}) = \\
= \sum_{r=1}^{n} d_r^n(x;\vec{\xi}\,)\ T_{r-1}(\tfrac{x}{\xi_{r-1}})\cdots T_1(\tfrac{x}{\xi_1})\, y_1(x)\, T_1(x\xi_1)\cdots T_{r-1}(x\xi_{r-1})\ .
\end{gathered} \tag{23}$$

where we have used the concise notation $d_r^n(x;\vec{\xi}\,)$ for coefficient functions

$$d_r^n(x;\vec{\xi}_{(n)}) = \prod_{k=r+1}^{n} (x - \xi_k)(x - \xi_k^{-1}) = \prod_{k=r+1}^{n} \left(1 - \rho_k\, x + x^2\right)\ , \tag{24}$$

$$\rho_k = (\xi_k + \xi_k^{-1})\ . \tag{25}$$

Remark. From Yang-Baxter equation (5) and reflection equation (13) we deduce

$$\begin{gathered}
\tau_n(x;\vec{\xi}_{(n)})\, T_k(\xi_{k+1}/\xi_k) = T_k(\xi_{k+1}/\xi_k)\, \tau_n(x;\mathbf{s}_k\cdot\vec{\xi}_{(n)})\ , \quad k = 1,\dots,n-1\ , \\
\tau_n(x;\vec{\xi}_{(n)})\, \bar{y}_k(\xi_k;\vec{\xi}_{(k-1)}) = \bar{y}_k(\xi_k;\vec{\xi}_{(k-1)})\, \tau_n(x;\mathbf{I}_k\cdot\vec{\xi}_{(n)})\ , \quad k = 1,\dots,n\ ,
\end{gathered} \tag{26}$$

where $\mathbf{s}_k\cdot\vec{\xi}_{(n)} \equiv (\xi_1,\dots,\xi_{k-1},\xi_{k+1},\xi_k,\xi_{k+2},\dots\xi_n)$, i.e. $\mathbf{s}_k$ is the transposition of two parameters ξ_k and ξ_{k+1}, and $\mathbf{I}_k\cdot\vec{\xi}_{(n)} \equiv (\xi_1,\dots,\xi_{k-1},\xi_k^{-1},\xi_{k+1},\dots\xi_n)$. It means that the Bethe subalgebras generated by transfer-matrix type elements $\tau_n(x;\vec{\xi}_{(n)})$, $\tau_n(x;\mathbf{s}_k\cdot\vec{\xi}_{(n)})$ and $\tau_n(x;\mathbf{I}_k\cdot\vec{\xi}_{(n)})$ are equivalent. It is clear that the symmetry (26) is also valid for the generating functions (23).

4 Bethe subalgebra for the Hecke algebra

The Hecke algebra H_n is the quotient of the affine Hecke algebra $\hat{H}_n$ by the relation $\hat{y}_1 = 1$. Thus, one can obtain the generating function for the elements of Bethe subalgebra of usual Hecke algebra H_n if we substitute into (23) the trivial solution $y_1(x) = 1$ of the reflection equation. Then to simplify the function (23) for the case of H_n we first transform the elements $y_r(x; \vec{\xi}_{(r-1)})$ given in (11). For this we use relations (4) and identities

$$T_k(\tfrac{x}{\xi_k})\, T_k(x\xi_k) = \lambda T_k(x^2) + (x-\xi_k)(x-\xi_k^{-1}) = \\ = \lambda(1-x^2)T_k + [\lambda^2 x^2 + (x-\xi_k)(x-\xi_k^{-1})] \; , \tag{27}$$

which can be deduced from (6). For $y_1(x) = 1$, applying (27) many times, we obtain new representation for the elements (11):

$$y_{n+1}(x; \vec{\xi}_{(n)}) = T_n(\tfrac{x}{\xi_n}) \cdots T_2(\tfrac{x}{\xi_2}) T_1(\tfrac{x}{\xi_1})\, T_1(x\xi_1) T_2(x\xi_2) \cdots T_n(x\xi_n) = \\ = T_n(\tfrac{x}{\xi_n}) \cdots T_2(\tfrac{x}{\xi_2}) \left(\lambda(1-x^2)T_1 + [\lambda^2 x^2 + (x-\xi_1)(x-\xi_1^{-1})]\right) T_2(x\xi_2) \cdots T_n(x\xi_n) = \\ = \lambda(1-x^2)\, \widetilde{y}_{n+1}(x; \vec{\xi}_{(n)}) + c_{n+1}(x; \vec{\rho}_{(n)}) \; , \tag{28}$$

where

$$\widetilde{y}_n(x; \vec{\xi}_{(n-1)}) = \sum_{k=1}^{n-1} c_k(x; \vec{\rho}_{(k-1)}) T_{n-1}(\tfrac{x}{\xi_{n-1}}) \cdots T_{k+1}(\tfrac{x}{\xi_{k+1}}) T_k T_{k+1}(x\xi_{k+1}) \cdots T_{n-1}(x\xi_{n-1}) \; , \tag{29}$$

parameters ρ_k were defined in (25) and for the coefficient functions c_k we have $c_1 = 1$,

$$c_k(x; \vec{\rho}_{(k-1)}) \equiv \prod_{j=1}^{k-1} \left(1 - x\rho_j + (1+\lambda^2)x^2\right) \qquad (\forall k \geq 2) \; .$$

Using representation (28) it is convenient to redefine the generating function (23) once again

$$J_n(x; \vec{\xi}_{(n)}) = \sum_{r=1}^{n} d_r^n(x; \vec{\xi}\,) \; \Big(\lambda(1-x^2)\widetilde{y}_r(x; \vec{\xi}_{(r-1)}) + c_r(x; \vec{\rho}_{(r-1)})\Big) = \\ = \lambda(1-x^2)\widetilde{J}_n(x; \vec{\xi}_{(n)}) + \sum_{r=1}^{n} d_r^n(x; \vec{\xi}\,)\; c_r(x; \vec{\rho}_{(r-1)}) \; . \tag{30}$$

For new function $\widetilde{J}_n(x; \vec{\xi}_{(n)})$ which generate elements of the Bethe subalgebra $\mathcal{B}_n(\vec{\xi}_{(n)}) \subset H_n$ we obtain the recurrent relations

$$\widetilde{J}_2 = \widetilde{y}_2 = T_1 \; , \quad \widetilde{J}_n = (1 - x\rho_n + x^2)\widetilde{J}_{n-1} + \widetilde{y}_n \; = \; \sum_{k=2}^{n} d_k^n(x; \vec{\rho})\, \widetilde{y}_k \; , \tag{31}$$

where coefficients $d_k^n(x; \vec{\rho})$ were defined in (24). Using the recurrence relation (30) we can compute the Hamiltonian of our (integrable) system, namely,

$$\frac{\partial}{\partial x} \widetilde{J}_n(x; \vec{\xi}_{(n)})|_{x=0} = \sum_{1 \leq i < j \leq n} (\rho_i + \rho_j) T_{(ij)} + \lambda \sum_{1 \leq i < j < k < n} \left(\xi_j^{-1} T_{(ij)} T_{(jk)} + \xi_j T_{(jk)} T_{(ij)}\right).$$

At the end of this Section we present the explicit expressions for first few elements $\widetilde{y}_n$ and $\widetilde{J}_n$ for $n \geq 2$:

$$\widetilde{y}_2 = T_1 \; , \quad \widetilde{y}_3 = T_2(\tfrac{x}{\xi_2}) T_1 T_2(x\xi_2) + \left(1 - x\rho_1 + (1+\lambda^2)x^2\right) T_2 \; , \\ \widetilde{y}_4 = T_3(\tfrac{x}{\xi_3}) T_2(\tfrac{x}{\xi_2}) T_1 T_2(x\xi_2) T_3(x\xi_3) + \left(1 - x\rho_1 + (1+\lambda^2)x^2\right) T_3(\tfrac{x}{\xi_3}) T_2 T_3(x\xi_3) \\ + [1 - x\rho_1 + (1+\lambda^2)x^2][1 - x\rho_2 + (1+\lambda^2)x^2] T_3 \; , \\ \widetilde{y}_5 = T_4(\tfrac{x}{\xi_4}) ... T_2(\tfrac{x}{\xi_2}) T_1 T_2(x\xi_2) ... T_4(x\xi_4) + c_2(x; \rho_1)\, T_4(\tfrac{x}{\xi_4}) T_3(\tfrac{x}{\xi_3}) T_2 T_3(x\xi_3) T_4(x\xi_4) + \\ + c_3(x; \rho_1, \rho_2)\, T_4(\tfrac{x}{\xi_4}) T_3 T_4(x\xi_4) + c_4(x; \rho_1, \rho_2, \rho_3)\, T_4 \; . \tag{32}$$

$$\widetilde{J}_2 = T_1 \ , \quad \widetilde{J}_3 = \Big(1 - x\rho_3 + (1+\lambda^2)x^2\Big)T_1 + \Big(1 - x\rho_1 + (1+\lambda^2)x^2\Big)T_2 +$$
$$+ \left(1 - x\rho_2 + x^2\right) T_2T_1T_2 + \lambda x(\xi_2 - x)\, T_2T_1 + \lambda x\, (\xi_2^{-1} - x)\, T_1T_2 =$$
$$T_1 + T_2 + T_1T_2T_1 - (\rho_3 T_1 + \rho_2 T_1T_2T_1 + \rho_1 T_2 - \lambda\xi_2\ T_2T_1 - \lambda\xi_2^{-1}T_1T_2)x +$$
$$\Big((1+\lambda^2)(T_1 + T_2 + T_1T_2T_1) - \lambda(T_1T_2 + T_2T_1 + \lambda\ T_1T_2T_1)\Big)x^2\ . \tag{33}$$

Therefore the Bethe subalgebra $\mathcal{B}_3(\vec{\xi}_{(3)})$ is generated by the central elements $C_1 = T_1 + T_2 + T_{(13)}$ and $C_2 = T_1T_2 + T_2T_1 + \lambda T_{(13)}$, and the element

$$D = \rho_3 T_1 + \rho_2 T_{(13)} + \rho_1 T_2 - \lambda\xi_2^{-1} T_2T_1 - \lambda\xi_2\ T_1T_2.$$

Here we used notation $T_{(13)} := T_1T_2T_1$. Now let us introduce the following elements

$$\theta_1 := \theta_1^{\vec{\xi}_{(3)}} = \frac{D - \rho_1 C_1}{(\rho_2-\rho_1)(\rho_3-\rho_1)} = \frac{T_1}{\rho_2-\rho_1} + \frac{T_{(13)}}{\rho_3-\rho_1} - \frac{\lambda\xi_2\ T_2T_1 + \lambda\xi_2^{-1}T_1T_2}{(\rho_2-\rho_1)(\rho_3-\rho_1)},$$

$$\theta_2 := \theta_1^{\vec{\xi}_{(3)}} = \frac{D - \rho_2 C_1}{(\rho_1-\rho_2)(\rho_3-\rho_2)} = \frac{T_1}{\rho_1-\rho_2} + \frac{T_2}{\rho_3-\rho_2} - \frac{\lambda\xi_2\ T_2T_1 + \lambda\xi_2^{-1}T_1T_2}{(\rho_1-\rho_2)(\rho_3-\rho_2)},$$

$$\theta_3 := \theta_1^{\vec{\xi}_{(3)}} = \frac{D - \rho_3 C_1}{(\rho_1-\rho_3)(\rho_2-\rho_3)} = \frac{T_2}{\rho_2-\rho_3} + \frac{T_{(13)}}{\rho_1-\rho_3} - \frac{\lambda\xi_2\ T_2T_1 + \lambda\xi_2^{-1}T_1T_2}{(\rho_2-\rho_3)(\rho_1-\rho_3)}.$$

One can check that

$$\theta_1 + \theta_2 + \theta_3 = 0, \quad \rho_1\theta_1 + \rho_2\theta_2 + \rho_3\theta_3 = C_1, \quad (\lambda^2+3)C_2 = C_1^2 - 2\lambda C_1 - 3,$$

and the elements $\theta_1^{\vec{\xi}_{(3)}}, \theta_2^{\vec{\xi}_{(3)}}, \theta_3^{\vec{\xi}_{(3)}}$ pairwise commute and generate the Bethe subalgebra $\mathcal{B}_3(\vec{\xi}_{(3)})$. Our goal is to show that a similar set of generators exist for the Bethe algebra $\mathcal{B}_n(\vec{\xi}_{(n)})$ for arbitrary n.

To state our main result of this Section we need to introduce a bit of notation. First of all, for a pair of integers $1 \le i < j \le n$ let us introduce the elements $T_{(ij)} := T_{j-1} \cdots T_{i+1} T_i T_{i+1} \cdots T_{j-1}$, $1 \le i < j \le n$. Now let $B \subset [1, 2, \ldots, n]$ be a subset, define inductively the elements $T(B) := T^{\vec{\xi}_{(n)}}(B)$ as follows

- $T(\{b\}) = 0$, $T(\{a < b\}) = T_{(ab)}$,
- $T(\{a < b < c < \ldots < d\}) = \xi_a\ T(\{b < c < \ldots < d\})\ T_{(ab)} + \xi_a^{-1} T_{(ab)}\ T(\{b < c < \ldots < d\})$.

For example,
$T(\{a<b<c<d\}) = \xi_a\xi_b\ T_{(cd)}T_{(bc)}T_{(ab)} + \xi_a\xi_b^{-1}T_{(bc)}T_{(cd)}T_{(ab)} + \xi_a^{-1}\xi_b\ T_{(ab)}T_{(cd)}T_{(bc)} + \xi_a^{-1}\xi_b^{-1}T_{(ab)}T_{(bc)}T_{(cd)}$.

Using the notation introduced above, let us define the following elements

$$\theta_a^{\vec{\xi}_{(n)}} = \sum_{\substack{B \subset [1,\ldots,n] \\ a \in B}} \lambda^{|B|-2}\ \frac{T(B)}{\prod_{\substack{b\in B \\ b \ne a}} (\rho_a - \rho_b)}, \quad a = 1, \ldots, n.$$

Theorem 1

The elements $\theta_a^{\vec{\xi}_{(n)}}$, $a = 1, \ldots, n$ *mutually commute, generate the Bethe subalgebra* $\mathcal{B}_n(\vec{\xi}_{(n)})$ *and satisfy the following properties*

- $\theta_1^{\vec{\xi}_{(n)}} + \ldots + \theta_n^{\vec{\xi}_{(n)}} = 0$, $\sum_{j=1}^n \xi_j \theta_j^{\vec{\xi}_{(n)}} = \sum_{1 \le i < j \le n} T_{(ij)}$,
- *the elementary symmetric polynomials* $e_j(\theta_1^{\vec{\xi}_{(n)}}, \ldots, \theta_n^{\vec{\xi}_{(n)}})$, $j = 2, \ldots, n$, *generate the center of the Hecke algebra* H_n,

Clearly,

$$\theta_a^{\vec{\xi}_{(n)}} = \sum_{b \ne a} \frac{T_{(ab)}}{\rho_a - \rho_b} + \lambda(\ldots).$$

In other words the elements $\{\theta_a^{\vec{\xi}_{(n)}},\ a = 1, \ldots, n\}$ are a *lift* of the Gaudin elements $\{g_a(\vec{\xi}_{(n)}) := \sum_{j \ne a} (\rho_a - \rho_j)^{-1}\ s_{aj},\ a = 1, \ldots, n\}$ from the group algebra $\mathbb{C}[\mathbb{S}_n]$ of the symmetric group $\mathbb{S}_n$ to the Hecke algebra $H_n \otimes \mathbb{C}$.

Using the recurrence relation (30) we can compute the Hamiltonian $\mathcal{H}^{\vec{\xi}_{(n)}}$ of our (integrable) model, namely,

$$\mathcal{H}^{\vec{\xi}_{(n)}} = \frac{\partial}{\partial x} \widetilde{J}_n(x;\vec{\xi}_{(n)})|_{x=0} = \sum_{1\le i<j\le n} (\rho_i+\rho_j)T_{(ij)} + \sum_{\substack{B\subset[1,\dots,n]\\|B|\ge 3}} (-\lambda)^{|B|-2}\, T(B).$$

Finally we remark that the example above shows that the set of all Bethe's subalgebras in the Hecke algebra H_n does not coincide with the set of all maximal commutative subalgebras in H_n, if $n\ge 3$.

5 Symmetric group limit and Gaudin model.

Let us consider the special classical limit when $q\to 1$ while parameters x and ξ_k are fixed. For $q=1$ or $\lambda=0$ in view of (3), (27) the Hecke algebra H_{M+1} is degenerated to the symmetric group algebra $\mathcal{S}_{M+1}$, i.e. $T_k = T_k^{-1} = s_{k,k+1} = s_k$ are elementary transpositions of k and $k+1$. In this limit we have $T_k(x) = (1-x)s_k$ and formulas (29) and (32) are simplified

$$\begin{gathered} \widetilde{y}_2 = s_1\ ,\ \ \widetilde{y}_3 = [1-x\rho_2+x^2]s_2s_1s_2 + [1-x\rho_1+x^2]s_2\ , \\ \widetilde{y}_4 = [1-x\rho_2+x^2][1-x\rho_3+x^2]s_3s_2s_1s_2s_3 + [1-x\rho_1+x^2][1-x\rho_3+x^2]s_3s_2s_3 \\ +[1-x\rho_1+x^2][1-x\rho_2+x^2]s_3\ ,\ \dots, \end{gathered} \tag{34}$$

$$\widetilde{y}_k = \left(\prod_{m=1}^{k-1}[1-x\rho_k+x^2]\right)\sum_{j=1}^{k-1}\frac{s_{j,k}}{[1-x\rho_j+x^2]}\ , \tag{35}$$

where $s_{j,k} = s_{k-1}...s_{j+1}s_js_{j+1}...s_{k-1}$ are transpositions in $\mathcal{S}_{M+1}$, i.e. $s_{j,k}=s_{k,j}$. Substitution of (35) into (31) gives

$$\begin{gathered} \widetilde{J}_n(x;\vec{\rho}_{(n)}) = \sum_{k=2}^{n}\prod_{m=k+1}^{n}\left(1-x\rho_m+x^2\right)\widetilde{y}_k = \\ = \sum_{k=2}^{n}\sum_{j=1}^{k-1}\prod_{\substack{m=1\\m\neq j,k}}^{n}\left(1-x\rho_m+x^2\right)s_{j,k} = \prod_{m=1}^{n}\left(1-x\rho_m+x^2\right)\sum_{k>j}^{n}\frac{s_{j,k}}{[1-x\rho_j+x^2][1-x\rho_k+x^2]}\ , \end{gathered}$$

After the renormalization

$$\widetilde{J}_n(x;\vec{\rho}_{(n)}) \to \widetilde{J}'_n(x;\vec{\rho}_{(n)}) = \frac{x^2\ \widetilde{J}_n(x;\vec{\rho}_{(n)})}{\prod\limits_{m=1}^{n}(1-x\rho_m+x^2)}$$

and change of variables $u = x+1/x$ we obtain the generating function for Bethe subalgebra of symmetric group algebra $\mathcal{S}_n$ in the form

$$\begin{gathered} \widetilde{J}'_n(x;\vec{\rho}_{(n)}) = \frac{x^2}{2}\sum_{\substack{k,j=1\\k\neq j}}^{n}\frac{s_{j,k}}{[1-x\rho_j+x^2][1-x\rho_k+x^2]} = \frac{1}{2}\sum_{\substack{k,j=1\\k\neq j}}^{n}\frac{s_{j,k}}{[u-\rho_j][u-\rho_k]} = \frac{1}{2}\sum_{\substack{k,j=1\\k\neq j}}^{n}\frac{s_{j,k}}{\rho_j-\rho_k}\frac{\rho_j-\rho_k}{[u-\rho_j][u-\rho_k]} = \\ = \frac{1}{2}\sum_{\substack{k,j=1\\k\neq j}}^{n}\frac{s_{j,k}}{\rho_j-\rho_k}\left(\frac{1}{[u-\rho_j]}-\frac{1}{[u-\rho_k]}\right) = \sum_{\substack{k,j=1\\k\neq j}}^{n}\frac{s_{j,k}}{\rho_j-\rho_k}\frac{1}{[u-\rho_j]}\ . \end{gathered} \tag{36}$$

The commuting Hamiltonians $H_j^{[n]}$ for Gaudin model are obtained from (36) as residues for $u\to\rho_j$:

$$H_j^{[n]} = \mathrm{res}(\widetilde{J}'_n(x;\vec{\rho}_{(n)}))\Big|_{u=\rho_j} = \sum_{\substack{k=1\\k\neq j}}^{n}\frac{s_{j,k}}{\rho_j-\rho_k}\ .$$

The right hand side of (36), after the change of variables $\rho_k\to z_k$, can be represented in the form

$$\frac{1}{\prod\limits_{m}[u-z_m]}\left(\sum_{\substack{k,j=1\\k\neq j}}^{n}\frac{s_{j,k}}{z_j-z_k}\prod_{\substack{m\\m\neq j}}[u-z_m]\right)\ ,$$

and the expression in the brackets is related (up to the shift by a scalar function) to the generating function $\Phi_2^{[n]}(u)$ of the Bethe subalgebra which was presented in [13] (see Remark after the Theorem 4.3 in [13]).

Remark 1. There is another semi-classical limit $q \to 1$ for the constructions considered above which is called Yangian limit. In this case we have to consider substitution

$$x = q^{-2u} \; , \quad \xi_j = q^{-2z_j} \; , \quad q = e^h \; ,$$

and take the limit $h \to 0$. Then we have

$$\begin{array}{c} x = 1 - 2hu + \dots \, , \quad \lambda = 2h + \dots \, , \quad T_k(x) = 2h(us_k + 1) + \dots \, , \\ T_k = s_k + h\, s'_k + \dots \; , \quad T_k^2 = 1 + 2\, h\, s_k + \dots \; , \quad s_k s'_k + s'_k s_k = 2 s_k \; , \\ (1 - \rho_j x + x^2) = (1 - \xi_j x)(1 - \frac{x}{\xi_j}) = 4h^2(u^2 - z_j^2) + \dots \, , \\ (1 - \rho_j x + (1+\lambda^2) x^2) = 4h^2(u^2 - z_j^2 + 1) + \dots \, , \\ \hat{y}_1 = 1 + 2h\, \ell_1 + \dots \; , \quad \hat{y}_k = 1 + 2h\, \ell'_k + \dots \; , \\ T_k(\frac{x}{\xi_k}) = 2h((u - z_k)s_k + 1) \; , \quad T_k(x\xi_k) = 2h((u + z_k)s_k + 1) \; , \end{array} \tag{37}$$

where $s_k \in \mathbb{S}_{M+1}$ are elementary transpositions, dots $\dots$ means the higher orders in h and we have used notation

$$\ell'_1 = \ell_1 \; , \quad \ell'_{k+1} = s_k + s_k \ell'_k s_k \; , \tag{38}$$

for commutative set of elements $\{\ell'_k\}$. Moreover it is easy to see that $s_k \ell'_k = \ell'_{k+1} s_k - 1$, $k = 1, \dots, M$, and the elements $\{s_k, \ell'_1, \ell'_{k+1}, \; k = 1, \dots, M\}$ generate the degenerate affine Hecke algebra [6].

We call the limit (37) as Yangian limit since for the modified Baxterized element $\widetilde{T}_k(x)$ given in (8) we obtain (see the first line in (37)):

$$\widetilde{T}_k(x) = \frac{us_k + 1}{u + 1} + h \, \dots \; . \tag{39}$$

In the matrix representation ρ of the symmetric group $\mathbb{S}_{M+1}$ we have $\rho(s_k) = P_{kk+1}$, where P_{kk+1} are permutation matrices which act in the products of n-dimensional vector spaces $V^{\otimes(M+1)}$ and permute factors with numbers k and $k+1$. In this representation the Baxterized element (39) is nothing but Yangian R-matrix which is used for the definition of the $g\ell(n)$-type Yangian.

Remark 2. Now we show that the elements ℓ_1 commute as following:

$$[\ell_1, \ell_2] = s_1 \ell_1 - \ell_1 s_1 \; , \tag{40}$$

where we denoted $\ell_2 := s_1\, \ell_1\, s_1$. It follows from the relation $[\ell'_1, \ell'_2] = 0$, but we check this identity directly. First, we start with the relation (see (9)) in the affine Hecke algebra

$$T_1\, \hat{y}_1\, T_1\, \hat{y}_1 = \hat{y}_1\, T_1\, \hat{y}_1\, T_1 \; , \tag{41}$$

and change here the affine generators $\hat{y}_1 \to K_1$:

$$\hat{y}_1 = 1 + (q - q^{-1}) K_1 \; . \tag{42}$$

The substitution of (42) into (41) and using of (3) gives for $(q - q^{-1}) \neq 0$:

$$T_1 K_1 T_1 K_1 - K_1 T_1 K_1 T_1 = K_1 T_1 - T_1 K_1 \; . \tag{43}$$

Now we take here the Yangian limit (37), i.e. we put $K_1 = \ell_1 + h \dots$ and $T_1 = s_1 + h \dots$. Then, in the zero order approximation in h, eq. (43) is converted to (40). In the matrix representation (see the end of the Remark 1.) when $\rho(s_k) = P_{kk+1}$,

$$\rho(\ell_1) = L \otimes \underbrace{1 \otimes \cdots \otimes 1}_{M} \in End(V^{\otimes(M+1)})$$

and $L = ||L_{ij}||_{i,j=1,\dots,n}$ is a $n \times n$ matrix with noncommutative entries, relation (40) gives the defining relations for generators L_{ij} of $g\ell(n)$ Lie algebra.

Remark 3. The using of the Yangian limit (37) in (23) will give the generating function for generators of Bethe subalgebras for degenerate affine Hecke algebra. Here we present first few cases for $n = 1, 2, 3$:

$$J_1(x, \vec{\xi}_{(1)}) = y_1(x) = \frac{\hat{y}_1 - \xi\, x}{\hat{y}_1 - \xi\, x^{-1}} \stackrel{h\to 0}{\longrightarrow} \frac{\ell_1 + (z+u)}{\ell_1 + (z-u)} + h \dots\ ,$$

$$J_2(x, \vec{\xi}_{(2)}) = (x^2 - \rho_2 x + 1)\, y_1(x) + T_1(x/\xi_1)\, y_1(x)\, T_1(x\xi_1) \stackrel{h\to 0}{\longrightarrow}$$
$$4h^2 \left((u^2 - z_2^2) \frac{\ell_1 + (z+u)}{\ell_1 + (z-u)} + ((u - z_1)s_1 + 1) \frac{\ell_1 + (z+u)}{\ell_1 + (z-u)} ((u + z_1)s_1 + 1) \right) + h^3 \dots\ ,$$

$$J_3(x, \vec{\xi}_{(3)}) = (x^2 - \rho_3 x + 1)\Big((x^2 - \rho_2 x + 1)\, y_1(x) + T_1(x/\xi_1)\, y_1(x)\, T_1(x\xi_1) \Big) +$$
$$+ T_2(x/\xi_2)\, T_1(x/\xi_1)\, y_1(x)\, T_1(x\xi_1)\, T_2(x\xi_2) \stackrel{h\to 0}{\longrightarrow}$$
$$(2h)^4 \left((u^2 - z_3^2)(u^2 - z_2^2) \frac{\ell_1 + (z+u)}{\ell_1 + (z-u)} + (u^2 - z_3^2)((u - z_1)s_1 + 1) \frac{\ell_1 + (z+u)}{\ell_1 + (z-u)} ((u + z_1)s_1 + 1) + \right.$$
$$\left. + ((u - z_2)s_2 + 1)((u - z_1)s_1 + 1) \frac{\ell_1 + (z+u)}{\ell_1 + (z-u)} ((u + z_1)s_1 + 1)((u + z_2)s_2 + 1) \right) + h^5 \dots ,$$

where we have used (14) and taken $\xi = q^{-2z}$. The expansion of elements $J_n(x, \vec{\xi}_{(n)})\Big|_{h\to 0}$ over the spectral parameter u gives the set of commuting elements $\Phi_k(z_1, \dots, z_n)$ (cf. eq. (21)):

$$J_n(x, \vec{\xi}_{(n)})\Big|_{h\to 0} = (2h)^{2(n-1)} \sum_{k=0}^{\infty} \Phi_k(z_1, \dots, z_n)\, u^k + h^{2n-1} \dots\ .$$

The elements $\Phi_k(z_1, \dots, z_n)$ are generators of the Bethe subalgebras in the degenerate affine Hecke algebra $\{s_m, \ell_1,\ m = 1, \dots, n-1\}$ and $\Phi_k(z_1, \dots, z_n)$ can be interpreted as Hamiltonians for inhomogeneous open spin chains. In particular for homogeneous case $z_m = 0$, $z \neq 0$ of such open spin chain we obtain the local Hamiltonian $\mathcal{H}(z)$:

$$\partial_u\ J_n\Big(q^{-2u}, (1, ..., 1) \Big)\Big|_{h\to 0, u=0} \sim h^{2(n-1)}\, \mathcal{H}(z) + h^{2n-1} \dots\ , \qquad \mathcal{H}(z) = \left(\sum_{m=1}^{n-1} s_m + \frac{1}{\ell_1 + z} \right) .$$

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