Multi-rogue waves solutions : from NLS to KP-I equation

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Abstract. The discovery of the multi-rogue waves solutions made in 2010 completely changed the vision of the links of the theory of rogue waves and integrable systems allowing to explain many phenomena which were not understood before. It's enough to mention the famous 3-sister waves observed in ocean, the creation of a regular approach to study higher Peregrine breathers and the new understanding of $2+1$ dimensional rogue waves via the NLS-KP correspondence. This article continues the study of the multi-rogue waves solutions of the NLS equation and their links with the KP-I equation started in the series of articles [1, 2, 3, 4]. In particular, it contains the discussion of the large parametric asymptotic of these solutions which was never studied before.

Contents

1. Introduction

In this article we discuss the multi-rogue waves (MRW) solutions of the focusing NLS equation

$$iu_t + u_{xx} + 2|u|^2 u = 0 \tag{1}$$

and the related solutions of the KP-I equation

$$(4v_t + 6vv_x + v_{xxx})_x = 3v_{yy} \tag{2}$$

which we construct below. Equation (1) is obviously invariant with respect to the scaling transformations, phase transformations and Galilean transformations

$$u(x,t) \to Bu(B^2x, Bt) \quad B > O \tag{3}$$

$$u(x,t) \to e^{i\chi}u(x,t) \quad \chi \in \mathbb{R} \tag{4}$$

$$u(x,t) \to u(x - Vt, t)\exp\left(iVx/2 - iV^2t/4\right) \quad V \in \mathbb{R}. \tag{5}$$

MRW solutions of the NLS equation are quasi rational solutions

$$u = e^{2iB^2t}R(x,t), \quad R(x,t) = \frac{N(x,t)}{D(x,t)}, \quad B > 0. \tag{6}$$

Here $N(x,t), D(x,t)$ are polynomials of x and t, and
$\deg N(x,t) = \deg R(x,t) = n(n+1)$, such that

$$|u^2| \to B^2, \quad x^2 + t^2 \to \infty.$$

The rational function $R(x,t)$ obviously satisfies the 1D Gross-Pitaevskii (GP) equation or more precisely the 1D GP equation with zero trapping potential

$$iR_t + 2R(|R|^2 - B^2) + R_{xx} = 0, \quad |R| = |u|. \tag{7}$$

Therefore, rational MRW solutions of the GP equation are trivially connected with quasi-rational MRW solutions of the focusing NLS equation. MRW solutions of the NLS equation and the related (see the explanation below) solutions of the KP-I equation are labeled by an integer n which defines the degree of polynomials N and D. Below we call this integer the rank of the solution. For given n these solutions of the NLS equation depend on $2n+3$ free real parameters $\chi, B, V, \varphi_j, j = 1, \dots 2n$, where χ, B and V correspond to the phase freedom, scaling freedom and a freedom to perform a Galilean transformation with velocity parameter V. Without loss of generality we can always deal with the case $\chi = 0, B = 1, V = 0$. But we will keep B which has a sense of asymptotic magnitude of $u(x,t)$.

2. The main formulas: multi-rogue solutions of the NLS equation.

2.1. Multi-rogue waves solutions of the NLS equation : φ_j parametrization

Let n be any positive integer. Following [5] we define two polynomials q_{2n} and Φ of degree $2n$ by

$$q_{2n}(k) := \prod_{j=1}^{n} \left(k^2 - \frac{\omega^{2m_j+1}+1}{\omega^{2m_j+1}-1} B^2 \right), \quad \omega := \exp\left(\frac{i\pi}{2n+1}\right) \tag{8}$$

$$\Phi(k) := i \sum_{l=1}^{2n} \varphi_l (ik)^l, \quad B > 0, \varphi \in \mathbb{R}. \tag{9}$$

We assume that the integers m_j satisfy

$$0 \le m_j \le 2n-1, \quad m_l \ne 2n - m_j. \tag{10}$$

In particular this condition is satisfied when $m_j = j-1$ and below we use this choice of m_j. In [5] the last choice was replaced by $m_j = j$, which is not valid.‡ It is clear that for any k the function f defined by

$$f(k,x,t) := \frac{\exp(kx + ik^2 t + \Phi(k))}{q_{2n}(k)} \tag{11}$$

is a solution of the non stationary linear Schrödinger equation with zero potential

$$-if_t = f_{xx}. \tag{12}$$

The same is true for the functions $f_1, \dots, f_{2n}$ defined by the formula

$$f_j(x,t) := D_k^{2j-1} f(k,x,t)\,|_{k=B}\,, \quad D_k := \frac{k^2}{k^2+B^2}\frac{\partial}{\partial k} \quad j = 1\dots n; \tag{13}$$

$$f_{n+j}(x,t) := D_k^{2j-1} f(k,x,t)\,|_{k=-B}\,. \tag{14}$$

Suppose that W_1, W_2 are Wronskian determinants composed from the functions f_j and f

$$W_1 := W(f_1, \dots, f_n) \equiv \det A, \quad A_{lj} = \partial_x^{l-1} f_j, \quad W_2 := W(f_1, \dots, f_n, f).$$

Then the following proposition holds.

‡ See [4] for further comments concerning the condition (10).

Theorem 1 *The function*

$$u_n(x,t) := -q_{2n}(0)B^{1-2n}e^{2iB^2t}\frac{W_2\,|_{k=0}}{W_1} \tag{15}$$

represents a $2n+1$ parametric family of solutions of the NLS equation.

We call this solution multi-rogue waves solution of rank n or simply MRW_n.§ We will give a proof of theorem 1 slightly later.

2.2. Non-stationary Schrödinger equation and the KP-I equation

Consider the Lax system for the KP-I equation

$$-i\psi_y = \psi_{xx} + v(x,y,t)\psi \tag{16}$$

$$-4\psi_t = 4\psi_{xxx} + 6v\psi_x + 3w(x,y,t)\psi. \tag{17}$$

In particular the first equation of this system is the non stationary linear Schrödinger equation with potential $v(x,y)$ depending on the parameter t with the "time" evolution variable y.

Suppose $v(x,y,t)$ is any solution of the KP-I equation (2) and $f_1,\dots,f_n,f$ are linearly independent solutions of (16) and (17). Then the following proposition [6, 7, 8] holds.‖

Theorem 2 *The function*

$$\psi := \frac{W(f_1,\dots,f_n,f)}{W(f_1,\dots,f_n)} \tag{18}$$

is a solution of (16) and (17) with potential

$$v_n(x,y,t) := v(x,y,t) + 2\partial_x^2 \log W(f_1,\dots,f_n) \tag{19}$$

and the function $v_n(x,y,t)$ is a new solution of KP-I equation.

In particular, this is true when $v(x,y,t)=0, w=0$.

It is clear that all functions f, f_j, $j=1,\dots 2n$ defined by (11) and (13) will satisfy the Lax system (16) and (17) with $v=0, w=0$ if we denote t by y and φ_3 by $-t$. Therefore we have the following result

Theorem 3 *The function*

$$v_{2n}(x,y,t) := 2\partial_x^2 \log\, W(\tilde f_1,\dots,\tilde f_{2n}), \quad \tilde f_j(x,y,t) := f_j\,|_{t=y,\varphi_3=-t} \tag{20}$$

where f_j are defined in (13) represents a family of smooth real rational solutions of the KP-I equation. This solution satisfies the relation

$$\int_{-\infty}^{\infty} v_{2n}(x,y,t)\,dx = 0. \tag{21}$$

These solutions are also rational functions of $2n-1$ parameters $\varphi_1,\varphi_2,\varphi_4,\dots,\varphi_{2n}$.

§ In a different and more complicated notations, with no use of Wronskian determinants, this solution was first presented in [5] where it was also mentioned that for $n=1$ it reproduces the so called Peregrine breather or P_1 breather [9].

‖ [6] and [7] contain much more general statements directly applicable to the non-Abelian KP-I and KP-II hierarchies and their different reductions.

To understand why these solutions of the KP-I equation are real valued and non-singular we have to remark that an important statement which we call NLS-KP correspondence holds.

Theorem 4 *The solution (20) can be also written as follows :*

$$v_{2n} = 2(|\tilde{u_n}|^2 - B^2), \quad \tilde{u_n}(x,y,t) := u_n(x,t,\varphi_1,\dots,\varphi_{2n})\,|_{t=y,\varphi_3=-t}\,. \tag{22}$$

It turns out that $v_{2n}(x,y,t)$ *satisfies the important inequality*

$$v_{2n} \geq -2B^2. \tag{23}$$

A reasonable conjecture is that the maximum value of $|u_n(x,t)|$ is described by the formula

$$\max_{x,t,\varphi_1,\dots,\varphi_{2n}\in R} |u_n(x,t,\varphi_1,\dots,\varphi_{2n})| = B(2n+1). \tag{24}$$

The solution where the parameters $\varphi_1,\dots,\varphi_{2n}$ are chosen is such a way that this maximum is attained is denoted $P_n(x,t)$ and called a P_n-breather. In the case of P_n-breathers, this conjecture belongs to Akhmediev and was tested by him first for $n \leq 3$ and next by Pierre Gaillard for $n \leq 10$ but the conjecture in its full extent was first introduced in our works where the whole family of solutions was first constructed. Of course it is enough to prove this conjecture for $B = 1$ since the general result then follows from the scaling invariance of the NLS equation. If this conjecture is true the related maximal value of the solution of the KP equation described by (20) is given by the formula

$$\max_{x,y,t\in R} v(x,y,t) = 8B^2 n(n+1), \tag{25}$$

i.e. it is equal to the number of peaks of the generic solution of rank n of the NLS equation times the magnitude of the KP-I image of the P_1 breather.

2.3. Reduction to the nonlinear Schrödinger equation

By theorem 2 the function

$$\psi(x,t,k) := \frac{W(f_1,\dots,f_{2n},f)}{W(f_1,\dots,f_{2n})} \tag{26}$$

is a solution of

$$i\psi_t + \psi_{xx} + v\psi = 0 \tag{27}$$

for the potential

$$v(x,t) := 2\partial_x^2 \log W(f_1,\dots,f_{2n}) \tag{28}$$

and so is

$$R_C(x,t) := C\psi(x,t,0). \tag{29}$$

This is a general result that holds any time the functions f_j are solutions of (12). We can remark that R_C is a rational solution of (27). The particular form of the f_j and the right choice for the constant C allow us to reduce (27) to (1).

Proposition 5 *If $C = e^{i\chi} q_{2n}(0) B^{1-2n}$ then v and R defined by (28) and (29) satisfy*

$$v = 2\left(|R|^2 - B^2\right)$$

¶

Proof

Because of (9) we can define define the following three meromorphic differentials

$$\mathrm{d}\Omega := \frac{(k^2+B^2)^{2n+1} + (k^2-B^2)^{2n+1}}{2k^2(k^2-B^2)^{2n}}\,\mathrm{d}k, \tag{30}$$

$$\mathrm{d}\Omega_1 := \psi(x,t,k)\overline{\psi(x,t,-\overline{k})}\,\mathrm{d}\Omega, \tag{31}$$

$$\mathrm{d}\Omega_2 := (k + \frac{B^2}{k})\,\mathrm{d}\Omega_1. \tag{32}$$

The polynomial q_{2n} defined by (8) satisfy

$$2k^2 q_{2n}(k)\overline{q_{2n}(-\overline{k})} = (k^2+B^2)^{2n+1} + (k^2-B^2)^{2n+1}$$

hence $\mathrm{d}\Omega$ and $\mathrm{d}\Omega_1$ have poles at $k = \pm B$ and $k = \infty$ and $\mathrm{d}\Omega_2$ have poles at $k = 0$, $k = \pm B$ and $k = \infty$.

In the neighborhood of $k = \pm B$ we use the local parameter $z = k - B/k$. $\mathrm{d}\Omega$ can be expressed as

$$\mathrm{d}\Omega = \left(\frac{(z^2+4B^2)^n}{z^{2n}} + \frac{z}{2B}\left(1+\frac{z^2}{4B^2}\right)^{-\frac{1}{2}}\right)\frac{\mathrm{d}z}{2}$$

and it admits the following expansion

$$\mathrm{d}\Omega = \left(\frac{\alpha_0^{\pm}}{z^{2n}} + \frac{\alpha_1^{\pm}}{z^{2n-2}} + \ldots + \frac{\alpha_{n-1}^{\pm}}{z^2} + \mathrm{O}(1)\right)\mathrm{d}z. \tag{33}$$

Given (26) we can check that the odd order derivatives of ψ with respect to z vanish up to order $2n-1$ so ψ has an expansion of the form

$$\psi = \beta_0^{\pm} + \beta_1^{\pm} z^2 + \ldots + \beta_{n-1}^{\pm} z^{2n-2} + \mathrm{O}(z^{2n}). \tag{34}$$

By (33) and (34) we obtain that the residues of $\mathrm{d}\Omega_1$, and subsequently $\mathrm{d}\Omega_2$, at $k = \pm B$ vanish and the remaining residues must satisfy

$$\mathrm{res}_{\infty}\mathrm{d}\Omega_1 = 0, \tag{35}$$

$$\mathrm{res}_0\mathrm{d}\Omega_2 = -\mathrm{res}_{\infty}\mathrm{d}\Omega_2. \tag{36}$$

¶ The proof of this proposition is the same as in [5]. The proof of the smoothness of the solution (15) in [5] was too complicated. It follows directly from the structure of focusing NLS equation and meromorphic nature of the discussed solution considered as a function of x. See [10] for a detailed explanation.

In the neighborhood of $k = \infty$, ψ admits the following expansion

$$\psi = \left(1 + \frac{\xi_1(x,t)}{k} + \frac{\xi_2(x,t)}{k^2} + \dots\right) e^{kx+ik^2t+\Phi(k)} \tag{37}$$

and (35) yields the reality of ξ_1. We can easily check that

$$\mathrm{res}_0 \mathrm{d}\Omega_2 = |\psi(x,t,0)|^2 |q_{2n}(0)|^2/B^{4n-2} = |R|^2 \tag{38}$$

and

$$-\mathrm{res}_\infty \mathrm{d}\Omega_2 = \xi_2 + \overline{\xi_2} - \xi_1^2 + (2n+1)B^2. \tag{39}$$

Substituting (37) into (27) we obtain the relations

$$v = -2\partial_x \xi_1$$

and

$$i\partial_t \xi_1 + 2\partial_x \xi_2 + \partial_x^2 \xi_1 - 2\partial_x \xi_1 \xi_1 = 0. \tag{40}$$

The real part of (40) combined with the reality of ξ_1 yields

$$\partial_x \left(\xi_2 + \overline{\xi_2} - v/2 - \xi_1^2\right) = 0$$

or, equivalently,

$$\xi_2 + \overline{\xi_2} - \xi_1^2 = v/2 + K(t).$$

This latest relation with (36), (38) and (39) gives us

$$|R|^2 = v/2 + K(t) + (2n+1)B^2.$$

A comparison of the behaviors of both sides when $|x| \to \infty$ shows that $K(t) = -2nB^2$ and gives the announced result.

R is now a rational solution of (7) and $u(x,t) := R(x,t)e^{2iB^2t}$ is a solution of (1).

3. α, β-parametrization and P_n-breathers

3.1. First and second order solutions

In this subsection we set $\chi = 0, B = 1$ and $V = 0$ but these parameters can easily be restored performing the scaling transformation (3), the phase transformation (4) and the Galilean transformation (5). Finally, phases φ_1 and φ_2 are simply space and time translation parameters, so we can select their values in order to produce the most compact expressions. Between the $2n+2$ parameters $\chi, B, \varphi_1, \dots, \varphi_{2n}$, only $2n-2$ of them, namely $\varphi_3, \dots \varphi_{2n}$, have a direct influence on the shape of magnitude of the solution.

In the case $n = 1$ it means we essentially get only one solution. By choosing $\varphi_1 = 0$ and $\varphi_2 = \sqrt{3}/4$ we get the Peregrine solution [9] :

$$P_1(x,t) = \left(1 - 4\frac{1+4it}{1+4x^2+16t^2}\right) e^{2it} \equiv \left(1 - 4\frac{1+iT}{1+X^2+T^2}\right) e^{iT/2}, \quad X := 2x,\ T := 4t.$$

Its plot is presented in figure 1.

The case $n = 2$ is the first one where we get a family of solutions depending on 4 parameters including two nontrivial parameters φ_3 and φ_4. We already explained above that φ_3 has a connection with the KP-I equation. For convenience we choose $\varphi_1 = 3\varphi_3$ and $\varphi_2 = 2\varphi_4 + (3+\sqrt{5})\sin(\pi/5)/4$ and we switch from the pair of parameters $\{\varphi_3, \varphi_4\}$ to the pair $\{\alpha, \beta\}$ defined by

$$\alpha := 48\varphi_3$$
$$\beta := 4(5+\sqrt{5})\sin(\pi/5) - 96\varphi_4.$$

The solutions read

$$u_2(x,t,\alpha,\beta) = \left(1 - 12\frac{G(2x,4t) + iH(2x,4t)}{Q(2x,4t)}\right)e^{2it} \tag{41}$$

where

$$\begin{aligned} G(X,T) :=\ & X^4 + 6(T^2+1)X^2 + 4\alpha X + 5T^4 + 18T^2 - 4\beta T - 3 \\ H(X,T) :=\ & TX^4 + 2(T^3 - 3T + \beta)X^2 + 4\alpha TX + T^5 + 2T^3 \\ & - 2\beta T^2 - 15T + 2\beta \\ Q(X,T) :=\ & (1 + X^2 + T^2)^3 - 4\alpha X^3 - 12(2T^2 - \beta T - 2)X^2 \\ & + 4(3\alpha(T^2+1)X + 6T^4 - \beta T^3 + 24T^2 - 9\beta T \\ & + \alpha^2 + \beta^2 + 2). \end{aligned} \tag{42}$$

When $\alpha = \beta = 0$ this solution coincide with the P_2-breather with maximum of magnitude equal 5 obtained at the point $x = t = 0$. When $\alpha^2 + \beta^2$ is small enough the solution is obviously very close to the P_2-breather. Formula (42) shows that the solution $u_2(x,t,\alpha,\beta)$ can be considered as a two-parametric quadratic deformation of the P_2-breather. When the two parameters are large enough we obtain the "generic" form of rank 2 multi-rogue waves : three rogue waves of similar height. These two phenomenon are shown in figure 2. In between we can observe some transition states. Some are represented in figure 3 or figure 4.

Formula (42) which was first discovered in [2, 3, 4] was very important, showing for the first time that, contrary to the genuine Peregrine breathers, its higher versions are not isolated.[+] There exist a families of solutions with very similar properties (obtained by the sufficiently small variation of parameters α and β in vicinity of their zero values) having almost the same extreme rogue wave behaviour as the P_2-breather. Formula (42) also clearly answered the question posed by Eleonski and Kulagin about 28 years ago : how to embed the P_2 breather discovered in [11] into a larger family of quasi rational solutions of the NLS equation.[*]

[+] In [2, 3, 4] the parameters α, β were defined differently. They were proportional to those we use here in order to shorten the writing of the formulas.

[*] It is instructive to see the movies KP2a, KP2e providing at any fixed moment of time the plot of the square of the absolute value of the related solution of the NLS equation. See the subsection "Links with KP-I equations and the related movies" below.

When $\alpha^2 + \beta^2$ tends to ∞, $u_2(x, t, \alpha, \beta)$ tends to e^{2it}. Thus, a simple wave, i.e. rank 0 solution, can be interpreted as a large parametric limit of the rank 2 multi-rogue wave solution. Below we will show that for higher ranks similar but more diversified phenomena take place.

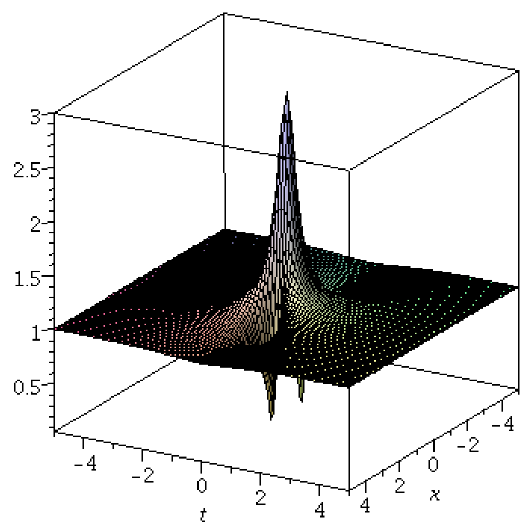

Figure 1. The Peregrine solution

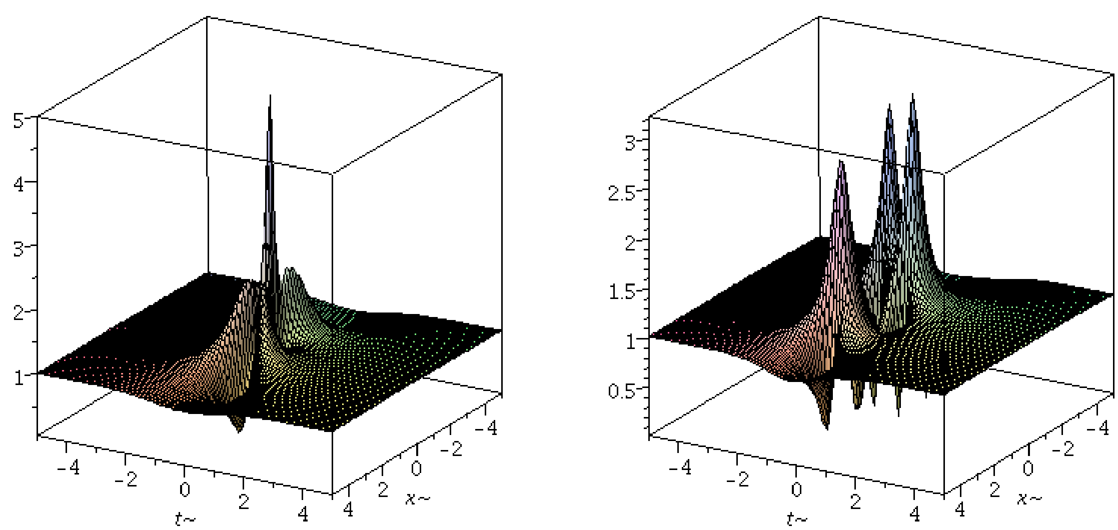

Figure 2. Second order solution (41) for $\alpha = 0$ and $\beta = 0$ on the left and $\alpha = 20$ and $\beta = 20$ on the right

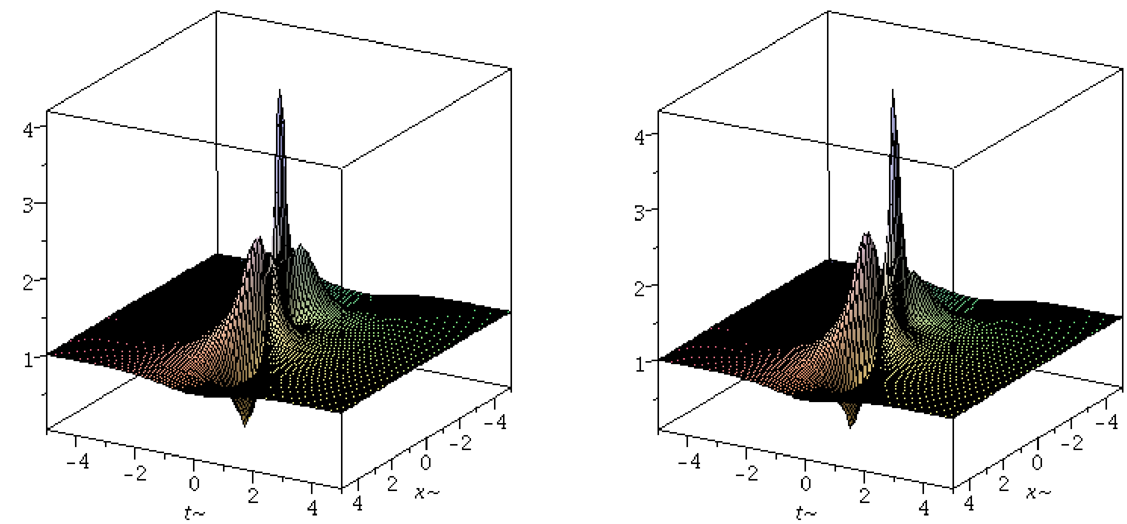

Figure 3. Second order solution (41) for $\alpha = 0$ and $\beta = 1$ on the left and $\alpha = 0$ and $\beta = 3$ on the right

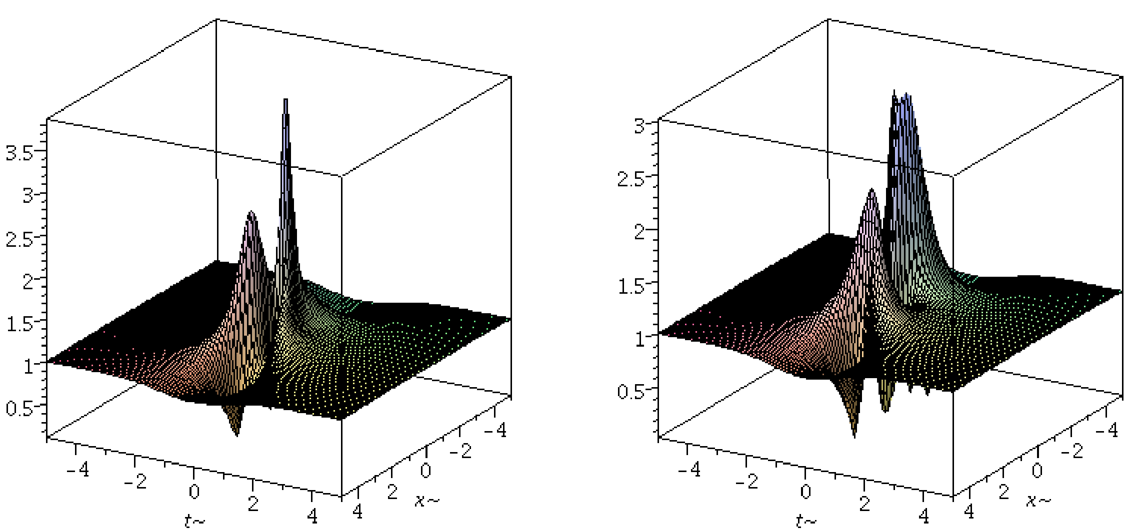

Figure 4. Second order solution (41) for $\alpha = 0$ and $\beta = 6$ on the left and $\alpha = 5$ and $\beta = 0$ on the right

3.2. Rank 3 solutions

The φ-parametrization described above makes it difficult to isolate the values of parameters describing higher Peregrine breathers. Similarly to the previous section here we introduce 4 "essential" parameters $\alpha_1, \beta_1, \alpha_2, \beta_2$. We choose φ_j according to the following linear system

$$\begin{aligned}
\varphi_1 &= 3\varphi_3 - 5\varphi_5 \\
\varphi_2 &= 2\varphi_4 - 3\varphi_6 + \tfrac{\sin(\pi/7)}{4(1-\cos(\pi/7))} \\
768\varphi_3 &= 26\alpha_1 - \alpha_2 \\
1920\varphi_4 &= -40\beta_1 + \beta_2 + 96(3\sin(\pi/7) + 8\sin(2\pi/7) + 2\sin(3\pi/7)) \\
3840\varphi_5 &= 10\alpha_1 - \alpha_2 \\
7680\varphi_6 &= -20\beta_1 + \beta_2 + 32(4\sin(\pi/7) + 14\sin(2\pi/7) + \sin(3\pi/7)).
\end{aligned}$$

Substituting these formulas in expression (15) for $n = 3$, we can after long calculation, using Maple, convert the related solution of NLS equation in the following explicit form :

$$u_3(x,t,\alpha_1,\beta_1,\alpha_2,\beta_2) = \left(1 - 24\frac{G_3(2x,4t) + iH_3(2x,4t)}{Q_3(2x,4t)}\right)e^{2it} \tag{43}$$

with

$$\begin{aligned}
G_3(X,T) &= X^{10} + 15(T^2+1)X^8 + \textstyle\sum_{n=0}^{6} g_n(T)X^n \\
H_3(X,T) &= TX^{10} + 5(T^3 - 3T + \beta_1)X^8 + \textstyle\sum_{n=0}^{6} h_n(T)X^n \\
Q_3(X,T) &= (1 + X^2 + T^2)^6 - 20\alpha_1 X^9 - 60(2T^2 - \beta_1 T - 2)X^8 + 4\textstyle\sum_{n=0}^{7} q_n(T)X^n
\end{aligned}$$

where

$$\begin{aligned}
g_6 &= 50T^4 - 60T^2 + 80\beta_1 T + 210 \\
g_5 &= 120\alpha_1 T^2 - 18\alpha_2 + 300\alpha_1 \\
g_4 &= 70T^6 - 150T^4 + 200\beta_1 T^3 + 450T^2 + 30\beta_2 T - 450 + 150\alpha_1^2 - 50\beta_1^2 \\
g_3 &= 400\alpha_1 T^4 + (3000\alpha_1 - 60\alpha_2)T^2 - 800\alpha_1\beta_1 T - 600\alpha_1 - 60\alpha_2 \\
g_2 &= 45T^8 + 420T^6 + 6750T^4 - (6000\beta_1 - 180\beta_2)T^3 - (300\alpha_1^2 - 900\beta_1^2 + 13500)T^2 \\
&\quad +(3600\beta_1 + 180\beta_2)T - 675 - 300\alpha_1^2 - 300\beta_1^2 \\
g_1 &= 280\alpha_1 T^6 + (150\alpha_2 - 2100\alpha_1)T^4 + 800\alpha_1\beta_1 T^3 - (3600\alpha_1 - 540\alpha_2)T^2 \\
&\quad +(120\beta_2\alpha_1 + 1200\alpha_1\beta_1 - 120\alpha_2\beta_1)T - 200\alpha_1\beta_1^2 - 900\alpha_1 - 90\alpha_2 - 200\alpha_1^3 \\
g_0 &= 11T^{10} + 495T^8 - 120\beta_1 T^7 + 2190T^6 - (42\beta_2 + 1200\beta_1)T^5 \\
&\quad +(350\alpha_1^2 + 150\beta_1^2 - 7650)T^4 + (6600\beta_1 - 420\beta_2)T^3 \\
&\quad -(2100\beta_1^2 + 2025 - 120\beta_2\beta_1 - 120\alpha_2\alpha_1 + 900\alpha_1^2)T^2 + (200\alpha_1^2\beta_1 + 200\beta_1^3 - 90\beta_2)T \\
&\quad +675 + 150\alpha_1^2 + 6\alpha_2^2 + 150\beta_1^2 + 6\beta_2^2
\end{aligned}$$

$$
\begin{aligned}
h_6 &= 10T^5 - 140T^3 + 40\beta_1 T^2 - 150T + 60\beta_1 - 5\beta_2 \\
h_5 &= 40\alpha_1 T^3 + (60\alpha_1 - 18\alpha_2)T + 40\alpha_1\beta_1 \\
h_4 &= 10T^7 - 210T^5 + 50\beta_1 T^4 - 450T^3 + 15\beta_2 T^2 - (50\beta_1^2 + 1350 - 150\alpha_1^2)T \\
&\quad +150\beta_1 - 15\beta_2 \\
h_3 &= 80\alpha_1 T^5 + (1000\alpha_1 - 20\alpha_2)T^3 - 400\alpha_1\beta_1 T^2 - (1800\alpha_1 - 60\alpha_2)T \\
&\quad +200\alpha_1\beta_1 + 20\beta_2\alpha_1 - 20\alpha_2\beta_1 \\
h_2 &= 5T^9 - 60T^7 + 1710T^5 + (45\beta_2 - 2100\beta_1)T^4 + (300\beta_1^2 - 6300 - 100\alpha_1^2)T^3 \\
&\quad +(1800\beta_1 - 90\beta_2)T^2 + (4725 + 300\alpha_1^2 + 300\beta_1^2)T - 135\beta_2 - 100\beta_1^3 \\
&\quad -100\alpha_1^2\beta_1 - 900\beta_1 \\
h_1 &= 40\alpha_1 T^7 + (30\alpha_2 - 1140\alpha_1)T^5 + 200\alpha_1\beta_1 T^4 - (2400\alpha_1 - 60\alpha_2)T^3 \\
&\quad +(60\beta_2\alpha_1 - 60\alpha_2\beta_1 + 600\alpha_1\beta_1)T^2 - (900\alpha_1 + 450\alpha_2 + 200\alpha_1^3 + 200\alpha_1\beta_1^2)T \\
&\quad +60\alpha_2\beta_1 - 60\beta_2\alpha_1 \\
h_0 &= T^{11} + 25T^9 - 15\beta_1 T^8 - 870T^7 + (40\beta_1 - 7\beta_2)T^6 + (70\alpha_1^2 - 9630 + 30\beta_1^2)T^5 \\
&\quad +(5850\beta_1 - 75\beta_2)T^4 + (40\beta_2\beta_1 + 40\alpha_2\alpha_1 - 2475 - 900\alpha_1^2 - 1300\beta_1^2)T^3 \\
&\quad +(100\alpha_1^2\beta_1 + 495\beta_2 + 100\beta_1^3)T^2 + (6\alpha_2^2 + 4725 - 240\alpha_2\alpha_1 - 240\beta_2\beta_1 \\
&\quad +750\beta_1^2 + 6\beta_2^2 + 750\alpha_1^2)T - 20\alpha_1^2\beta_2 - 675\beta_1 - 45\beta_2 - 100\alpha_1^2\beta_1 - 100\beta_1^3 \\
&\quad +40\alpha_2\alpha_1\beta_1 + 20\beta_1^2\beta_2
\end{aligned}
$$

$$
\begin{aligned}
q_7 &= 3\alpha_2 - 30\alpha_1 \\
q_6 &= -60T^4 + 40\beta_1 T^3 + 120T^2 - (15\beta_2 - 60\beta_1)T + 35\beta_1^2 + 15\alpha_1^2 + 580 \\
q_5 &= 30\alpha_1 T^4 - (27\alpha_2 - 90\alpha_1)T^2 + 120\alpha_1\beta_1 T - 27\alpha_2 + 540\alpha_1 \\
q_4 &= 30\beta_1 T^5 - 360T^4 + (15\beta_2 + 600\beta_1)T^3 + (3360 + 225\alpha_1^2 - 75\beta_1^2)T^2 \\
&\quad +(135\beta_2 - 1350\beta_1)T + 225\beta_1^2 - 30\alpha_2\alpha_1 + 525\alpha_1^2 - 30\beta_2\beta_1 + 840 \\
q_3 &= 40\alpha_1 T^6 + (1950\alpha_1 - 15\alpha_2)T^4 - 400\alpha_1\beta_1 T^3 + (90\alpha_2 + 4500\alpha_1)T^2 \\
&\quad +(60\beta_2\alpha_1 - 1800\alpha_1\beta_1 - 60\alpha_2\beta_1)T - 450\alpha_1 + 100\alpha_1^3 + 100\alpha_1\beta_1^2 - 135\alpha_2 \\
q_2 &= 60T^8 + 3360T^6 - (1620\beta_1 - 27\beta_2)T^5 + (225\beta_1^2 - 75\alpha_1^2 + 19560)T^4 \\
&\quad -(16200\beta_1 - 270\beta_2)T^3 + (450\alpha_1^2 - 9120 + 4050\beta_1^2)T^2 \\
&\quad +(675\beta_2 + 2700\beta_1 - 300\beta_1^3 - 300\alpha_1^2\beta_1)T + 3036 + 9\alpha_2^2 - 180\alpha_2\alpha_1 \\
&\quad +225\beta_1^2 + 225\alpha_1^2 + 9\beta_2^2 - 180\beta_2\beta_1 \\
q_1 &= 15\alpha_1 T^8 + (15\alpha_2 - 90\alpha_1)T^6 + 120\alpha_1\beta_1 T^5 + (405\alpha_2 - 5400\alpha_1)T^4 \\
&\quad +(3000\alpha_1\beta_1 - 60\alpha_2\beta_1 + 60\beta_2\alpha_1)T^3 + (1485\alpha_2 - 300\alpha_1\beta_1^2 - 1350\alpha_1 - 300\alpha_1^3)T^2 \\
&\quad +(540\beta_2\alpha_1 - 540\alpha_2\beta_1)T + 300\alpha_1^3 - 120\alpha_1\beta_1\beta_2 - 60\alpha_2\alpha_1^2 + 135\alpha_2 + 60\alpha_2\beta_1^2 \\
&\quad +300\alpha_1\beta_1^2 + 2025\alpha_1 \\
q_0 &= 30T^{10} - 5\beta_1 T^9 + 930T^8 - (240\beta_1 + 3\beta_2)T^7 + (15\beta_1^2 + 3820 + 35\alpha_1^2)T^6 \\
&\quad +(1710\beta_1 - 153\beta_2)T^5 + (30\beta_2\beta_1 + 30\alpha_2\alpha_1 - 975\beta_1^2 + 35940 - 75\alpha_1^2)T^4 \\
&\quad +(100\beta_1^3 + 100\alpha_1^2\beta_1 + 135\beta_2 - 23400\beta_1)T^3 \\
&\quad +(9\beta_2^2 + 23286 + 9\alpha_2^2 - 360\beta_2\beta_1 - 360\alpha_2\alpha_1 + 4725\alpha_1^2 + 8325\beta_1^2)T^2 \\
&\quad +(120\alpha_2\alpha_1\beta_1 - 60\alpha_1^2\beta_2 - 1500\alpha_1^2\beta_1 + 60\beta_1^2\beta_2 - 7425\beta_1 - 675\beta_2 - 1500\beta_1^3)T \\
&\quad +506 + 9\beta_2^2 + 100\beta_1^4 + 675\alpha_1^2 + 100\alpha_1^4 + 9\alpha_2^2 + 90\beta_2\beta_1 + 200\alpha_1^2\beta_1^2 \\
&\quad +675\beta_1^2 + 90\alpha_2\alpha_1.
\end{aligned}
$$

We can easily get the solutions of the first section by choosing the parameters

$$\alpha_1 = 48(\varphi_3 - 5\varphi_5),$$

$$\alpha_2 = 480(\varphi_3 - 13\varphi_5),$$
$$\beta_1 = 96(4\varphi_6 - \varphi_4) + 8(\sin(\pi/7) + 2\sin(2\pi/7) + \sin(3\pi/7)),$$
$$\beta_2 = 1920(8\varphi_6 - \varphi_4) + 32(\sin(\pi/7) - 4\sin(2\pi/7) + 4\sin(3\pi/7))$$

and performing the translation $x, t \rightsquigarrow x - \varphi_1, t - \varphi_2$.

It is easy to see that P_3 can be obtained as $u_3(x, t, 0, 0, 0, 0)$ and in particular $|P_3(0, 0)| = 7$. Let us indicate the particular values of φ_j corresponding to the P_3 breather corresponding to particular solution of the system (3.2) with $\alpha_j = \beta_j = 0,\ j = 1, 2$:

$$\varphi_1 = \varphi_3 = \varphi_5 = 0,$$
$$\varphi_4 = (3\sin(\pi/7) + 8\sin(2\pi/7) + 2\sin(3\pi/7))/20,$$
$$\varphi_6 = (4\sin(\pi/7) + 14\sin(2\pi/7) + \sin(3\pi/7)/240,$$
$$\varphi_2 = 2\varphi_4 - 3\varphi_6 + \frac{\sin(\pi/7)}{4(1 - \cos(\pi/7))}.$$

One of the advantage of this representation of the solution is that we can analyze its limit behavior when one or several parameters tend to infinity and x and t remain bounded.

- If α_2 and β_2 remain finite then $u_3(x, t, \alpha_1, \beta_1, \alpha_2, \beta_2)$ tends to e^{2it} when $\alpha_1^2 + \beta_1^2$ tends to ∞.

- If α_1 and β_1 remain finite then $u_3(x, t, \alpha_1, \beta_1, \alpha_2, \beta_2)$ tends to $P_1(x, t)$ when $\alpha_2^2 + \beta_2^2$ tends to ∞.

- If α_1, β_1, α_2 and β_2 all tend to ∞ in the following way

  $$\beta_1 \sim b\alpha_1,\ \alpha_2 \sim c\alpha_1^r,\ \beta_2 \sim d\alpha_1^r$$

  then the limit of $u_3(x, t, \alpha_1, \beta_1, \alpha_2, \beta_2)$ depends on r according to the following table

r	limit
< 2	e^{2it}
> 2	$u_1(x, t)$
2	$u_1(x - x_1, t - t_1)$

  where x_1 and t_1 are defined by

  $$x_1 = \frac{10(1-b^2)c+20bd}{3(c^2+d^2)}$$
  $$t_1 = \frac{10(1-b^2)d-20bc}{3(c^2+d^2)}$$

Thus we have shown that u_3 contain the solutions of rank 0 and 1 as appropriate large parametric limits. Plots for several choices of parameters are shown in figure 5 and 6.

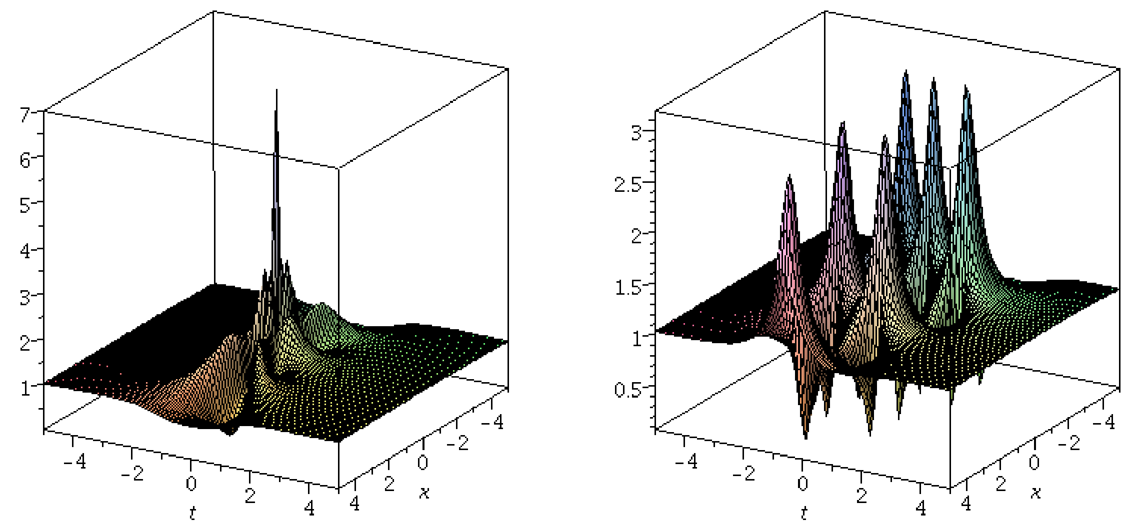

Figure 5. Third order solution (43) for $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$ on the left and $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 50$ on the right

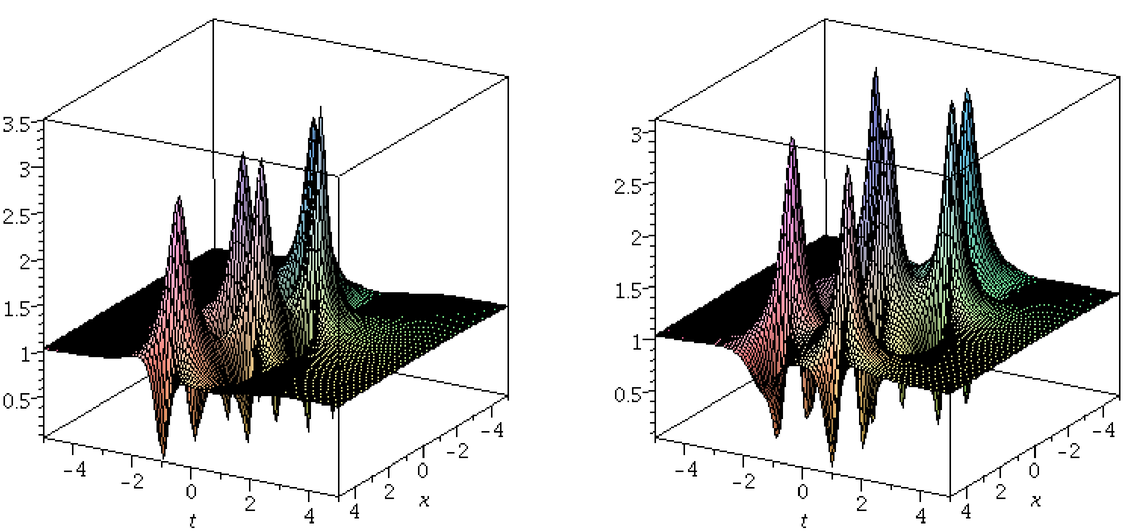

Figure 6. Third order solution (43) for $\alpha_1 = \alpha_2 = 0$ and $\beta_1 = \beta_2 = 50$ on the left and $\alpha_1 = \beta_1 = 0$ and $\alpha_2 = \beta_2 = 5000$ on the right

3.3. Rank 4 solutions

Here we present without details the formulas providing the 6-parametric family of multi-rogue wave solutions similar to the one of the previous section :

$$u_4(x,t,\alpha_1,\beta_1,\alpha_2,\beta_2,\alpha_3,\beta_3) = \left(1 - 40\frac{G_4(2x,4t)+iH_4(2x,4t)}{Q_4(2x,4t)}\right)e^{2it} \quad (44)$$

with

$$\begin{aligned} G_4(X,T) &= X^{18} + 27(T^2+1)X^{16} - 24\alpha_1 X^{15} + \textstyle\sum_{n=0}^{14} g_n(T)X^n \\ H_4(X,T) &= TX^{18} + 9(T^3 - 3T + \beta_1)X^{16} - 24\alpha_1 TX^{15} + \textstyle\sum_{n=0}^{14} h_n(T)X^n \\ Q_4(X,T) &= (1+X^2+T^2)^{10} - 60\alpha_1 X^{17} - 180(2T^2 - \beta_1 T - 2)X^{16} + 4\textstyle\sum_{n=0}^{15} q_n(T)X^n. \end{aligned}$$

The interested reader can find the explicit formulas for the coefficients in appendix. These solutions are polynomials of order 6 with respect to α_1 and β_1, of order 4 with respect to α_2, β_2 and quadratic with respect to α_3, β_3. The P_4-breather is obtained from it by setting $\alpha_j = \beta_j = 0, \forall j$. It is easy to see that $|P_4(0,0)| = 9$. As above we can investigate the limit of this solution when one or several parameter tends to infinity and x and t remain bounded.

- If α_2, β_2, α_3 and β_3 remain finite then $u_4(x,t,\alpha_1,\beta_1,\alpha_2,\beta_2,\alpha_3,\beta_3)$ tends to $P_1(x,t)$ when $\alpha_1^2+\beta_1^2$ tends to ∞.

- If α_1, β_1, α_3 and β_3 remain finite then $u_4(x,t,\alpha_1,\beta_1,\alpha_2,\beta_2,\alpha_3,\beta_3)$ tends to e^{2it} when $\alpha_2^2+\beta_2^2$ tends to ∞.

- If α_1, β_1, α_2 and β_2 remain finite then $u_4(x,t,\alpha_1,\beta_1,\alpha_2,\beta_2,\alpha_3,\beta_3)$ tends to $u_2(x,t,\alpha_1,\beta_1)$ when $\alpha_3^2+\beta_3^2$ tends to ∞.

- If α_3 and β_3 remain finite and α_1, β_1, α_2 and β_2 all tend to ∞ in the following way

  $$\beta_1 \sim b\alpha_1,\ \alpha_2 \sim c\alpha_1^r,\ \beta_2 \sim d\alpha_1^r$$

  then the limit of $u_4(x,t,\alpha_1,\beta_1,\alpha_2,\beta_2,\alpha_3,\beta_3)$ depends on r according to the following table

r	limit
$< 3/2$	$u_1(x,t)$
$> 3/2$	e^{2it}
$3/2$	$u_1(x-x_2, t-t_2)$

  where x_2 and t_2 are defined by

  $$\begin{aligned} x_2 &= \tfrac{3((1-3b^2)(d^2-c^2)+2(b^2-3)bcd)}{50(b^2+1)^3} \\ t_2 &= \tfrac{3((3-b^2)b(d^2-c^2)+2(1-3b^2)cd)}{50(b^2+1)^3}. \end{aligned}$$

- If α_2 and β_2 remain finite and α_1, β_1, α_3 and β_3 all tend to ∞ in the following way

$$\beta_1 \sim b\alpha_1,\ \alpha_3 \sim e\alpha_1^s,\ \beta_3 \sim f\alpha_1^s$$

then the limit of $u_4(x,t,\alpha_1,\beta_1,\alpha_2,\beta_2,\alpha_3,\beta_3)$ depends on s according to the following table

s	limit
< 2	$u_1(x,t)$
> 2	e^{2it}
2	$u_1(x-x_3, t-t_3)$

where x_3 and t_3 are defined by

$$\begin{aligned} x_3 &= \tfrac{(2bf+(1-b^2)e)}{35(b^2+1)^2} \\ t_3 &= \tfrac{(2be-(1-b^2)f)}{35(b^2+1)^2}. \end{aligned}$$

- If α_1 and β_1 remain finite and α_2, β_2, α_3 and β_3 all tend to ∞ in such a way that

$$\beta_2 \sim d\alpha_2,\ \alpha_3 \sim e\alpha_2^p,\ \beta_2 \sim f\alpha_2^p,$$

then, the limit of $u_4(x,t,\alpha_1,\beta_1,\alpha_2,\beta_2,\alpha_3,\beta_3)$ depends on p according to the following table

p	limit
< 2	e^{2it}
> 2	$u_2(x,t,\alpha_1,\beta_1)$
2	$u_2(x,t,\alpha_1-\alpha_0,\beta_1-\beta_0)$

where α_0 and β_0 are defined by

$$\begin{aligned} \alpha_0 &= \tfrac{21(2df+(1-d^2)e)}{10(e^2+f^2)} \\ \beta_0 &= \tfrac{21(2de-(1-d^2)f)}{10(e^2+f^2)}. \end{aligned}$$

- If α_1, β_1, α_2, β_2, α_3 and β_3 all tend to ∞ in the following way

$$\beta_1 \sim b\alpha_1,\ \alpha_2 \sim c\alpha_1^r,\ \beta_2 \sim d\alpha_1^r,\ \alpha_3 \sim e\alpha_1^s,\ \beta_3 \sim f\alpha_1^s.$$

then the limit of $u_4(x,t,\alpha_1,\beta_1,\alpha_2,\beta_2,\alpha_3,\beta_3)$ depends on r and s according to the following table

r	s	limit
$> 3/2$	any	e^{2it}
any	> 2	e^{2it}
$< 3/2$	< 2	$u_1(x,t)$
$3/2$	< 2	$u_1(x-x_2, t-t_2)$
$< 3/2$	2	$u_1(x-x_3, t-t_3)$
$3/2$	2	$u_1(x-x_2-x_3, t-t_2-t_3)$

where x_2, t_2, x_3 and t_3 are defined as above.

It seems that u_n contain all solutions of order 0 to $n-2$ as appropriately chosen large parametric limits. Of course for higher ranks further work has to be produced to get the similar results. In figure 7 to 9 we present some plots of fourth order solutions.

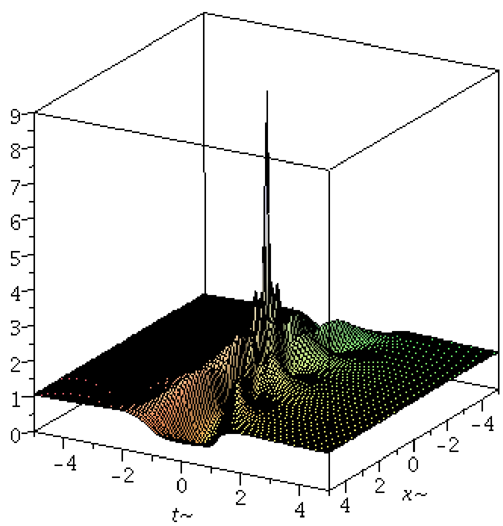

Figure 7. Fourth order solution (44) for $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = \alpha_3 = \beta_3 = 0$

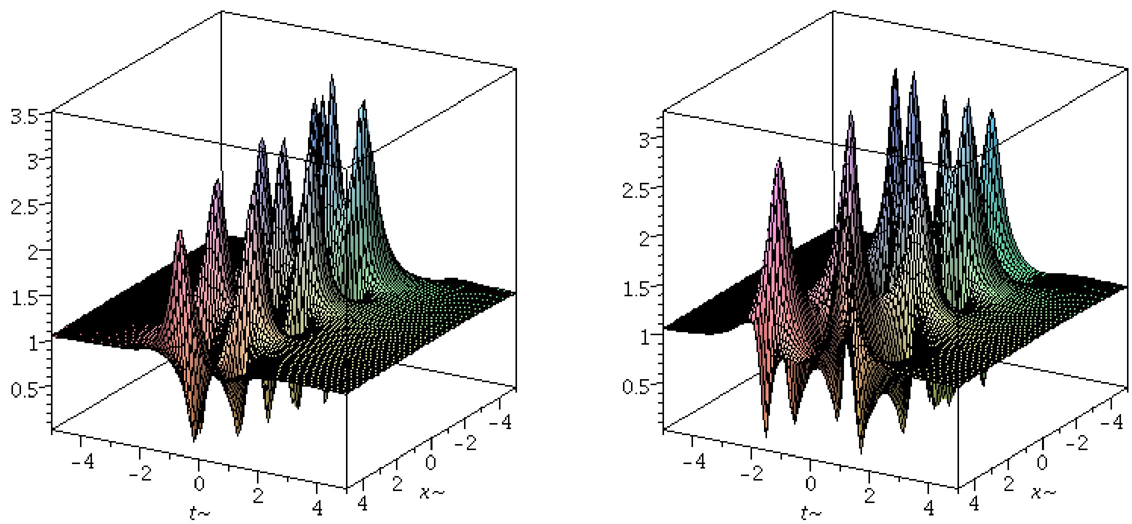

Figure 8. Fourth order solution (44) for $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = \alpha_3 = \beta_3 = 20$ on the left and $\alpha_1 = \beta_1 = 0$ and $\alpha_2 = \beta_2 = \alpha_3 = \beta_3 = 1500$ on the right

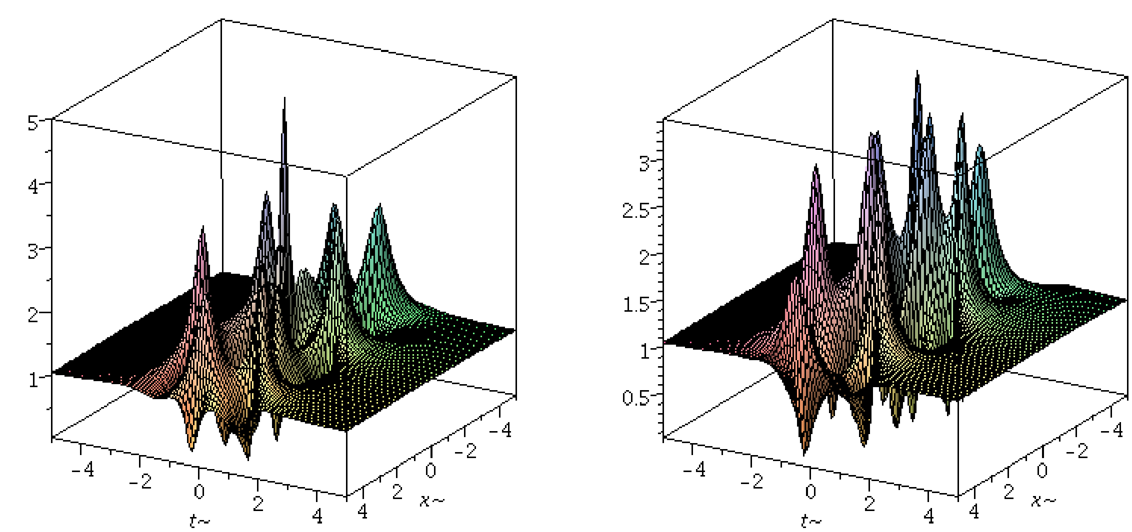

Figure 9. Fourth order solution (44) for $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$ and $\alpha_3 = \beta_3 = 10^5$ on the left and $\alpha_1 = \beta_1 = 20$, $\alpha_2 = \beta_2 = 0$ and $\alpha_3 = \beta_3 = 10^5$ on the right

3.4. Links with the KP-I equation and the related movies

Assume now that $B = 1$ and φ_j, $j \neq 3$ are selected in such a way that when $t = 0$

$$u_n(x, y, \varphi_1, \varphi_2, 0, \varphi_4, \ldots, \varphi_{2n}) = P_n(x, y).$$

Then the related solution of the KP-I equation attains its absolute maximum at the point $x = t = y = 0$ and

$$v(0, 0, 0) = 2[(2n + 1)^2 - 1] = 8n(n + 1).$$

For instance for $n = 2$ this corresponds to

$$\varphi_1 = 0, \quad \varphi_2 = (7 + 5\sqrt{5})\frac{\sin \pi/5}{6}, \quad \varphi_4 = (5 + \sqrt{5})\frac{\sin \pi/5}{24}. \tag{45}$$

Therefore similarly to the P_2-breather the solution of the KP-I equation $v_{p2}(x, y, t)$ generated by the selection of the phases above is rigid. It attains at the point $x = y = t = 0$ the absolute maximum v(0,0,0)=48. This maximum is 3 times greater then the height of the KP-I image of the P_1 breather.

Quite similarly, for the case $n = 3$, from the formulas (44) we see that selecting the phases as

$$\begin{aligned}
&\varphi_1 = \varphi_5 = 0, \varphi_3 = -t,\\
&\varphi_4 = 3\sin(\pi/7) + 8\sin(2\pi/7) + 2\sin(3\pi/7)/20,\\
&\varphi_6 = (4\sin(\pi/7) + 14\sin(2\pi/7) + \sin(3\pi/7)/240,\\
&\varphi_2 = 2\varphi_4 - 3\varphi_6 + \frac{\sin(\pi/7)}{4(1 - \cos(\pi/7))}
\end{aligned} \tag{46}$$

we obtain via the same formula (22) the smooth rational solution of the KP-I equation which we denote $v_{p3}(x, y, t)$ such that $v_{p3}(0, 0, 0) = 96$. The related maximum is 6 times higher then the KP-I image of the P_1-breather.

These solutions of the KP-I equation can also be obtained from the α, β parametrization of the solutions of the NLS equation. Investigating the relation between the parameters φ_j and the parameters α_j, β_j shows that performing the changes of variables $t \rightsquigarrow y$, $x \rightsquigarrow x + 3t$, $\alpha_1 \rightsquigarrow -48t$ and $\alpha_2 \rightsquigarrow -480t + \gamma$ where γ is a free real parameter is equivalent to the transformation presented above. It gives us a family of solutions depending on one parameter β_1 in the case $n = 2$ and three parameters β_1, β_2, γ in the case $n = 3$ that differ from the families constructed above by translations in x, y, t. The time evolution of some of these solutions can be observed on the 5 movies available at http://www.kurims.kyoto-u.ac.jp/∼kirillov/MATVEEV.

The first movie KP2a shows the triangular configuration of 3 peaks corresponding to the choice of parameters $\beta_1 = 20$ which at the moment of their appearance have the average height 16. At the large negative times it is close to an isosceles triangle of growing size when $t \to -\infty$. Close to $t = 0$ the triangle loses its form and rotates while propagating so that the peaks do not produce the confluence but rather the peak, initially coinciding with the left vertex of the back side of the triangle, after some rotation acquires the maximal height 25 at the moment when the distances between the peaks are minimal. After this the peaks diverge forming asymptotically at large times an isosceles triangle of growing size.

For the second movie (KP2e) with $\beta_1 = 0$ the scenario is different. The initial triangle (again almost isosceles for large negative times) does not rotate and contracts progressively so that first happens the confluence of two peaks situated behind a first one going ahead. Next, the so formed higher peak approaches the slower and smaller one, so that at the moment of their confluence they form one peak of the height 48, surrounded by four small peaks.♯ After the full confluence the mirror image of the previous configuration appears forming asymptotically the isosceles triangle of growing size .

The next three movies represent the evolution of a triangular array containing 6 peaks forming for large negative times the almost isosceles triangle. The first of them (KP3a) with $\beta_1 = 20,\ \beta_2 = 20, \gamma = 0$ behaves as follows. First for large negative times we have the almost isosceles triangular array containing 6 peaks. When $t \to 0,\ t < 0$ the triangle rotates and contracts arriving to the maximal height 35 at some moment of time with no confluence between peaks after which it starts to diverge again forming at large positive times the configuration close to an isosceles triangle. For the second movie KP3b with $\beta_1 = 0,\ \beta_2 = 20,\ \gamma = 0$ the maximum of amplitude in the "collision area" is a bit higher but still there is no total confluence of 6 peaks. Finally, in a third movie (KP3e) with $\beta_1 = 0,\ \beta_2 = 0,\ \gamma = 0$ we see the formation of the extremal rogue wave of the height 96 reaching at the moment $t = 0$ absolute maximum of the solution located at the point $(x, y) = (0, 0)$.

It is worthwhile to mention that, taking into account the NLS-KP correspondence (22) the presented movies also show the infinite numbers of space time plots for the square of magnitude of the NLS multi rogue wave solutions of the ranks 2 and 3 in particular illustrating a variety of important non symmetric configurations including 2 peaks configurations.

4. Concluding remarks

1. Our works [1, 2, 3, 4] were the first to explain the concept of the multiple-rogue waves solutions and their links with higher order Peregrine breathers thus, in particular, answering a question posed by Eleonski and Kulagin almost 30 years ago : how to incorporate the second order Peregrine breather discovered in [11] to the larger family of rational solutions. The work [5] was written in 1986 exactly for this purpose but it

♯ this already gives an idea of what is an extremal rogue wave solution of the KP-I equation.

was not properly understood until the appearance of our works written in 2010-2011.†† The explanation of the related formula (15) for the multi-rogue waves solutions takes less than one page and contains all necessary definitions. This formula is of about the same length with respect to other recently appeared works and it was the first which allowed to understand that generic multi-rogue wave solutions (unknown before) can be considered as a simple rational (with respect to free parameters) deformations of the higher Peregrine breathers or P_n breathers as we call them here. In addition this formulation has the advantage of providing the natural construction of the multi-rogue waves smooth rational solutions of the KP-I equation. By a way these solutions are quite different from the so called multi-lumps smooth rational solutions of the KP-I equation found in 1977 in [12]. Already in our first work [1] we mentioned that a number of other approaches namely Darboux transformation formulas for the NLS equation (known since 1882, see for instance [8, 13, 14]) and also a passage to appropriate infinite periods limit in multi-periodic solutions (first obtained in [10, 15] via degeneration of finite gap multi-periodic solutions) should provide another description of the same multi-rogue wave solutions. This prediction was fully realized for DT approach by Guo, Ling and Liu [16] and for the other mentioned approach it was done (to some extent) by P. Gaillard [17, 18, 19, 20]. In these works another formulas for the multi-rogue waves solutions were obtained producing for all tested ranks the higher Peregrine breathers just setting all the parameters to be zeros. In particular, in 2011-2012 Pierre Gaillard computed all the P_n breathers of the rank $n \leq 10$. Before that, only the genuine Peregrine breather, the P_2-breather found in 1985 and the P_3-breather discovered in 2009 [21, 22] were explicitly described.

2. In addition, Hirota direct method was successfully applied to obtain also the description of the multiple rogue waves solutions by Ohta and Yang [23]. The multi-rogue wave solutions in [23] were expressed as a ratio of two tau-functions polynomial with respect to complex parameters $a_j, \overline{a}_j$. In their work the authors considered the complex conjugate of the NLS equation and Gross-Pitaevskii equation used here. For their form of the GP equation they obtained a beautiful explicit formula (3.20) in which dominator and denominator of the solution are polynomials of the complex parameters $a_m, \overline{a}_m$, $a_0 = 1, a_{2j} = 0, \forall j \geq 1$. For particular values of their parameters Ohta and Yang first detected, for the rank 3 solutions, not only the circular arrays found slightly before [18, 19, 24] but also the configurations of the six peaks forming a triangular array close to an equilateral triangle with a slightly curved base (concave or convex) depending on the choice of parameters (see the figure 2 of [23]). This was an indication that the solutions discussed in [17, 18, 24, 25] were not generic contrary to our works [1, 2, 3, 4] and the works [16] and [23]. Our films (see the reference on the related URL above) show that the evolution of these triangular configurations is responsible for formation of extreme rogue waves events described by KP-I equations and provide for ranks 2 and 3 an infinite

†† That's true that every of the formulas describing the multi-rogue wave solutions including our works [1, 2, 3, 4] and those which appeared later [16, 23] , [19] have there own advantages and disadvantages and quite a different analytic and combinatorial structures.

number of such triangular configurations from the point of view of NLS equation. Some triangular static configurations were also found slightly later with respect to [23] by Guo, Ling and Liu [16] using their approach based on Darboux transformation. In [16] similar triangular structure was also first found for the Hirota equation.

3. One further comment concerns two more points in the work by Ohta and Yang. In [23] the values of parameters leading to the higher Peregrines breather were pointed out only for ranks $n = 1, 2, 3$. The connection between the real parameters α_j and β_j used above and the complex valued parameters a_j used in [23] for $j = 2, 3, 4$ is given by the formulas

$$12a_3 = \alpha_1 - 1 - i\beta_1, \quad 240a_5 = \alpha_2 - 1 - i\beta_2, \quad 10080a_7 = \alpha_3 - 1 - i\beta_3.$$

In general the passage from the variables α_j, β_j to the variables a_j of [23] is given by the formula

$$a_{2j+1} = \frac{\alpha_j - 1 - i\beta_j}{2(2j+1)!}. \tag{47}$$

One of us (Ph. Dubard) conjectured that as all P_n-breathers with the maximum of magnitude located at the point $x = -1/2, t = 0$ in the approach of [23] correspond to the choice of a_j given by (47) with $\alpha_j = \beta_j = 0$ i.e.

$$a_{2j+1} = -\frac{1}{2(2j+1)!}, \quad j \leq n, \quad a_j = 0 \quad \forall j \geq 2n+2. \tag{48}$$

This conjecture was confirmed by Ph.Dubard for the ranks $n \leq 7$ although the general proof is still missing .

Ohta and Yang also detailed their formula (3.20) expressing the matrix elements of the related determinants by means of Schur polynomials with the arguments containing some rational numbers r_k, s_k defined by means of simple generating functions (see the formulas (4-9) of [23]), satisfying the condition $r_{2k+1} = s_{2k+1} = 0$. The short calculation proves that r_{2k}, s_{2k} are simply expressed by means of Bernoulli numbers B_{2n}. The laters are defined by the formula

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n\, t^n}{n!}. \tag{49}$$

It follows from the definition above that $B_{2n+1} = 0, \forall n \geq 1$.
Now it is easy to prove the formulas:

$$r_{2n} = \frac{(2^{2n} - 1)}{2n\,(2n)!} B_{2n}, \quad s_{2n} = -\frac{2^{2n} - 2}{2n\,(2n)!} B_{2n}, \quad r_{2n} + s_{2n} = \frac{B_{2n}}{n(2n!)}. \tag{50}$$

Since all Bernoulli numbers are rational this proves that it is also true for s_k and r_k. Also taking into account that Bernoulli numbers are very well studied this gives a simplest way to compute r_{2j}, s_{2j}.

4. Another important comment concerns the works of Pierre Gaillard [17, 18, 19, 20] and several others deposed on Arxiv hal. The key idea of these works suggested by one of us (VM) in 2010 was to obtain the multi-rogue waves solutions considering an

appropriate passage to the limit in the formulas describing the multi-phase trigonometric modulations of the plane wave solutions [10, 15]. In fact in all of these works by P. Gaillard despite the apparent presence of $2n$ or even more free parameters (for the rank n) the related solutions were always 2 parametrical as we will briefly explain below. The reason to think so was the absence in his plots of the triangular arrays similar to those appearing already for $n = 3$ in [16] and [23]. Therefore we asked the author to print out the related analytical expressions for ranks $3, 4$ and 5 including explicitly all his parameters a_j, b_j. At that time we already have got a formulas equivalent to α, β parametrisation of this article for n=3 and 4. The comparison of Gaillard solutions with ours shown that they are:
1. Simply equivalent to solutions of this work with $\alpha_1, \beta_1 = 0$ for $n = 3$ and to the solutions of this work for $n = 4$ with $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$.
2. As we explained to Gaillard his solutions in fact are equivalent to those obtained from his own formulas by setting for instance $a_j = b_j = 0 \quad \forall j > 1$. One of us (VM) showed that this can be proved in intrinsic terms: first of all for any rank n it can be checked that all the terms in numerators and denominators of his formulas are just the quadratic polynomials of the same linear combinations of a_j or b_j, and such a structure can not correspond to generic rational solution for the rank greater then 2. For instance, for one of the works of PG for rank 4 these linear combinations found by VM are:

$$a = (a_1 + 2^6 a_2 + 3^6 a_3 - 2^{12} a_4), \quad b = (b_1 + 2^6 b_2 + 3^6 b_3 - 2^{12} b_4).$$

All coefficients depending on the parameters in this case are proportional to $a^2 + b^2$ or to a and b. This was also checked for higher ranks but of course the coefficients in the aforementioned linear combinations are depending on the rank but qualitatively the fact rest the same. All the formally multi-parametric solutions of the works by Gaillard of the period 2011-2012 were dealing with two parametric quadratic deformations of the higher Peregrine breathers. The work [19] was written already on the base of this understanding which allowed to compactify drastically his previous calculations. Unfortunately he never explained clearly all these points. Therefore the history of coming to modern understanding explained above, which was asking a big amount of work and intelligent study of monstrous formulas obtained as a Maple output having volumes of hundreds of pages already for n=5, was never explained to the reader before. Quite recently in [20] he succeeded to produce, by taking a new kind of long periods limit of [10] a full family of the rank 3 solutions. Initially he supposed that he found a new solution. In fact it is the same solution which he learned from us already in March 2012, corresponding to the formula (43). Exact correspondence between these two solutions computed by Ph. Dubard reads as follows :

$$a_1 = -\frac{\alpha_1}{2}, \quad b_1 = -\frac{\beta_1}{2}, \quad a_2 = 60\alpha_1 - 6\alpha_2, \quad b_2 = \frac{83}{2}\beta_1 - 6\beta_2.$$

Of course this latest result of PG means that in general the method of passage to the limit in [10, 15] produces also the generic solutions under appropriate choice of the limiting procedure but further work in this direction for higher ranks should be done.

5. Recently on arXiv appeared a preprint by He, Fokas et al [26] which can be considered as a natural continuation of [16]. Some of the conclusions of our work here are also relevant to this article, although we considered a more diversified class of large parametric limits of the multi-rogue wave solutions.

6. The writing of MRW solutions to the NLS and KP-I equation by means of explicit polynomials of several variables (independent variables and parameters) allows not only to make a punctual numerical evaluation of these solutions and producing stationary plots but now we can also study the large parametric asymptotic of the solutions and also see some symmetries which were not visible from initial determinant representations. The simplest symmetries concern the P_n-breathers. Namely it can be seen that for all the examples tested in our works and those by P.Gaillard, i.e. for $n \leq 10$, the denominator $D(x,t)$ in (6) is always an even function of both x and t, the real part of N(x,t) is an even function of x and t and the imaginary part of $N(x,t)$ is an even function of x and an odd function of t. Moreover some symmetries can be observed even for the deformations of P_n breathers i.e for generic MRW solutions due to the α, β representations of the solutions. For instance for $n = 2$ it follows from (41) that

$$u_2(x,t,\alpha,\beta) = u_2(-x,t,-\alpha,\beta) = \overline{u}_2(x,-t,\alpha,-\beta)$$

and the same kind of symmetries appear in (43) and (44). These symmetries in the structure of the MRW solutions of the NLS equation give some symmetry to the associated solutions of the KP-I equation. If we denote by v_2 and v_3 the solutions associated to u_2 and u_3 we can easily see that they have the following properties :

$$v_2(x,y,-t,\beta) = v_2(-x,y,t,\beta),$$

$$v_2(x,-y,t,\beta) = v_2(x,y,t,-\beta),$$

$$v_3(x,y,-t,\beta_1,\gamma,\beta_2) = v_3(-x,y,t,\beta_1,-\gamma,\beta_2),$$

$$v_3(x,-y,t,\beta_1,\gamma,\beta_2) = v_3(x,y,t,-\beta_1,\gamma,-\beta_2).$$

7. We can easily construct multi-rogue waves solutions for the Johnson-I aka CKP-I equation or for recently discovered ECKP or Elliptic cylindrical KP-I equation [27, 28] using their links with KP-I equation. For CKP-I equation some plots of the multi-rogue wave solutions with fixed time values can be found in [4] .

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Appendix

$$
\begin{aligned}
g_{14} &= 180T^4 - 360T^2 + 360\beta_1 T + 900\\
g_{13} &= 420\alpha_1 - 42\alpha_2\\
g_{12} &= 588T^6 - 5460T^4 + 3360\beta_1 T^3 - 1260T^2 + (5880\beta_1 - 378\beta_2)T\\
&\quad +18900 + 630\alpha_1^2 + 630\beta_1^2\\
g_{11} &= 2520\alpha_1 T^4 + (22680\alpha_1 - 1764\alpha_2)T^2 + 38808\alpha_1 - 4284\alpha_2 + 72\alpha_3\\
g_{10} &= 1134T^8 - 17640T^6 + 10584\beta_1 T^5 + 18900T^4 - (84\beta_2 + 16800\beta_1)T^3\\
&\quad +(11340\alpha_1^2 - 37800 + 11340\beta_1^2)T^2 + (4620\beta_2 - 168\beta_3 + 37128\beta_1)T\\
&\quad +43260\alpha_1^2 - 3192\alpha_2\alpha_1 + 1176\beta_2\beta_1 - 15540\beta_1^2 - 107730\\
g_9 &= 11760\alpha_1 T^6 + (140700\alpha_1 - 8190\alpha_2)T^4 + (600\alpha_3 + 546000\alpha_1 - 42420\alpha_2)T^2\\
&\quad +(2520\alpha_2\beta_1 - 142800\alpha_1\beta_1 + 17640\beta_2\alpha_1)T + 9800\alpha_1^3 + 600\alpha_3 - 29400\alpha_1\beta_1^2\\
&\quad -342300\alpha_1 - 24150\alpha_2\\
g_8 &= 1386T^{10} - 20790T^8 + 15120\beta_1 T^7 + 235620T^6 + (7938\beta_2 - 219240\beta_1)T^5\\
&\quad +(47250\beta_1^2 + 170100 + 47250\alpha_1^2)T^4 + (149940\beta_2 - 866880\beta_1 - 2520\beta_3)T^3\\
&\quad +(472500\alpha_1^2 - 132300\beta_1^2 - 5698350 - 15120\beta_2\beta_1)T^2 + (1464120\beta_1 - 189000\alpha_1^2\beta_1\\
&\quad +63000\beta_1^3 + 6930\beta_2 - 2520\beta_3)T + 1638\alpha_2^2 + 17640\beta_2\beta_1 - 129150\beta_1^2\\
&\quad +126\beta_2^2 - 481950 - 7560\alpha_2\alpha_1 - 353430\alpha_1^2 - 720\alpha_1\alpha_3\\
g_7 &= 22680\alpha_1 T^8 + (317520\alpha_1 - 10584\alpha_2)T^6 + (47880\alpha_2 + 3119760\alpha_1 - 2160\alpha_3)T^4\\
&\quad +(90720\beta_2\alpha_1 - 2923200\alpha_1\beta_1 - 10080\alpha_2\beta_1)T^3 + (453600\alpha_1\beta_1^2 - 151200\alpha_1^3\\
&\quad -15840\alpha_3 + 425880\alpha_2 - 9621360\alpha_1)T^2 + (5760\alpha_3\beta_1 + 211680\beta_2\alpha_1\\
&\quad -131040\alpha_2\beta_1 + 2600640\alpha_1\beta_1 - 12096\alpha_2\beta_2)T + 12600\alpha_2 + 9360\alpha_3\\
&\quad -352800\alpha_1^3 + 20160\alpha_1^2\alpha_2 - 352800\alpha_1\beta_1^2 + 20160\alpha_2\beta_1^2 - 1317960\alpha_1\\
g_6 &= 1092T^{12} - 1848T^{10} + 9240\beta_1 T^9 + 696780T^8 + (19656\beta_2 - 564480\beta_1)T^7\\
&\quad +(88200\alpha_1^2 + 88200\beta_1^2 + 5609520)T^6 + (518616\beta_2 - 3024\beta_3 - 9603216\beta_1)T^5\\
&\quad +(105840\alpha_2\alpha_1 - 105840\beta_2\beta_1 - 1499400\alpha_1^2 - 34265700 + 2381400\beta_1^2)T^4\\
&\quad +(36960\beta_3 + 18392640\beta_1 + 352800\alpha_1^2\beta_1 - 535080\beta_2 - 117600\beta_1^3)T^3\\
&\quad +(31752\beta_2^2 - 2910600\alpha_1^2 - 20160\beta_1\beta_3 + 423360\alpha_2\alpha_1 + 19845000 - 10584\alpha_2^2\\
&\quad -141120\beta_2\beta_1 - 1428840\beta_1^2)T^2 + (352800\alpha_1^2\beta_1 - 7285320\beta_1 - 588000\beta_1^3\\
&\quad -1464120\beta_2 - 35280\beta_3)T + 112896\alpha_2\alpha_1 + 1176\alpha_2^2 - 452760\alpha_1^2 + 3360\alpha_1\alpha_3\\
&\quad -336\alpha_2\alpha_3 - 117600\alpha_1^4 + 117600\beta_1^4 + 265776\beta_2\beta_1 + 793800\beta_1^2 + 8232\beta_2^2\\
&\quad -336\beta_2\beta_3 - 1190700\\
g_5 &= 22176\alpha_1 T^{10} + (378\alpha_2 + 328860\alpha_1)T^8 + (758520\alpha_2 - 7056\alpha_3 - 2850624\alpha_1)T^6\\
&\quad +(21168\beta_2\alpha_1 - 1058400\alpha_1\beta_1 - 190512\alpha_2\beta_1)T^5 + (2249100\alpha_2 + 529200\alpha_1\beta_1^2\\
&\quad +15120\alpha_3 - 30217320\alpha_1 - 176400\alpha_1^3)T^4 + (84672\alpha_2\beta_2 + 19192320\alpha_1\beta_1 - 70560\beta_2\alpha_1\\
&\quad -1764000\alpha_2\beta_1 - 20160\alpha_1\beta_3 - 20160\alpha_3\beta_1)T^3 + (211680\alpha_1^2\alpha_2 - 5292000\alpha_1\beta_1^2\\
&\quad -45360\alpha_3 - 3880800\alpha_1^3 - 3969000\alpha_2 - 3810240\alpha_1 + 211680\alpha_2\beta_1^2)T^2 + (105840\alpha_2\beta_2\\
&\quad -3024\alpha_3\beta_2 + 1764000\alpha_1\beta_1 - 50400\alpha_1\beta_3 - 818496\beta_2\alpha_1 + 705600\alpha_1\beta_1^3 + 3024\alpha_2\beta_3\\
&\quad +994896\alpha_2\beta_1 + 705600\alpha_1^3\beta_1)T - 7699860\alpha_1 + 39690\alpha_2 + 45360\alpha_3 - 987840\alpha_1^3\\
&\quad +317520\alpha_1^2\alpha_2 + 5040\alpha_1^2\alpha_3 - 14112\alpha_1\alpha_2^2 - 21168\alpha_2\beta_1\beta_2 + 352800\alpha_1\beta_1\beta_2\\
&\quad -987840\alpha_1\beta_1^2 + 7056\alpha_1\beta_2^2 - 35280\alpha_2\beta_1^2 + 5040\alpha_3\beta_1^2
\end{aligned}
$$

$$
\begin{aligned}
g_4 &= 540T^{14} + 16380T^{12} + 903420T^{10} + (16170\beta_2 - 634200\beta_1)T^9 + (85050\beta_1^2 + 10791900 \\
&+85050\alpha_1^2)T^8 + (3600\beta_3 + 68040\beta_2 - 15372000\beta_1)T^7 + (70560\alpha_2\alpha_1 + 5027400\beta_1^2 \\
&-61519500 - 4145400\alpha_1^2)T^6 + (25200\beta_3 + 1587600\alpha_1^2\beta_1 + 52038000\beta_1 - 529200\beta_1^3 \\
&-3501540\beta_2)T^5 + (529200\alpha_2\alpha_1 + 79380\alpha_2^2 + 17860500 - 50400\alpha_1\alpha_3 - 26460\beta_2^2 \\
&-18566100\alpha_1^2 - 16978500\beta_1^2 + 2293200\beta_2\beta_1)T^4 + (1764000\beta_1^3 + 2910600\beta_2 \\
&-22226400\beta_1 + 8820000\alpha_1^2\beta_1 - 352800\beta_1^2\beta_2 - 352800\alpha_1^2\beta_2 - 75600\beta_3)T^3 \\
&+(370440\alpha_2^2 - 52920\beta_2^2 - 100800\alpha_1\alpha_3 + 14200200\beta_1^2 - 2469600\beta_2\beta_1 - 2822400\alpha_2\alpha_1 \\
&+100800\beta_1\beta_3 + 20197800\alpha_1^2 + 65488500)T^2 + (1411200\alpha_1\alpha_2\beta_1 - 25200\alpha_1^2\beta_3 \\
&+105840\alpha_1\alpha_2\beta_2 + 75600\beta_3 + 705600\beta_1^2\beta_2 - 1653750\beta_2 - 32016600\beta_1 - 4762800\alpha_1^2\beta_1 \\
&-105840\alpha_2^2\beta_1 - 4762800\beta_1^3 - 705600\alpha_1^2\beta_2 - 25200\beta_1^2\beta_3)T + 17860500 - 940800\alpha_2\alpha_1 \\
&+14700\alpha_2^2 - 1365630\alpha_1^2 + 11760\alpha_1\alpha_3 - 8400\alpha_2\alpha_3 + 120\alpha_3^2 - 588000\alpha_1^4 + 58800\alpha_1^3\alpha_2 \\
&-176400\alpha_1\alpha_2\beta_1^2 + 176400\alpha_1^2\beta_1\beta_2 - 58800\beta_1^3\beta_2 - 58800\beta_2\beta_1 + 67620\beta_2^2 + 6660570\beta_1^2 \\
&+120\beta_3^2 - 13440\beta_1\beta_3 - 8400\beta_2\beta_3 + 588000\beta_1^4 \\
g_3 &= 10920\alpha_1T^{12} + (157080\alpha_1 + 8316\alpha_2)T^{10} + (795060\alpha_2 - 20661480\alpha_1 - 2520\alpha_3)T^8 \\
&+(11995200\alpha_1\beta_1 - 30240\beta_2\alpha_1 - 211680\alpha_2\beta_1)T^7 + (2640120\alpha_2 - 1764000\alpha_1\beta_1^2 \\
&+588000\alpha_1^3 - 23520\alpha_3 - 97290480\alpha_1)T^6 + (90175680\alpha_1\beta_1 - 84672\alpha_2\beta_2 + 40320\alpha_1\beta_3 \\
&-635040\alpha_2\beta_1 - 776160\beta_2\alpha_1)T^5 + (2910600\alpha_2 - 378000\alpha_3 + 54243000\alpha_1 - 26460000\alpha_1\beta_1^2 \\
&-2940000\alpha_1^3)T^4 + (2352000\alpha_1\beta_1^3 - 10080\alpha_3\beta_2 - 31281600\alpha_1\beta_1 + 94080\beta_2\alpha_1 \\
&+336000\alpha_3\beta_1 + 2352000\alpha_1^3\beta_1 + 23520\alpha_2\beta_1 + 33600\alpha_1\beta_3 + 10080\alpha_2\beta_3 - 776160\alpha_2\beta_2)T^3 \\
&+(211680\alpha_2\beta_1\beta_2 - 11289600\alpha_1^3 - 50400\alpha_3\beta_1^2 + 2116800\alpha_1\beta_1\beta_2 + 756000\alpha_3 \\
&+1764000\alpha_1^2\alpha_2 - 352800\alpha_2\beta_1^2 - 211680\alpha_1\beta_2^2 - 11289600\alpha_1\beta_1^2 - 6218100\alpha_2 \\
&-109809000\alpha_1 - 50400\alpha_1^2\alpha_3)T^2 + (2352000\alpha_1^3\beta_1 + 2352000\alpha_1\beta_1^3 + 211680\alpha_2\beta_2 \\
&+100800\alpha_1\beta_3 + 10080\alpha_2\beta_3 - 201600\alpha_3\beta_1 + 1270080\beta_2\alpha_1 - 10080\alpha_3\beta_2 + 16228800\alpha_1\beta_1 \\
&+1199520\alpha_2\beta_1)T + 17728200\alpha_1 + 1455300\alpha_2 + 37800\alpha_3 + 2587200\alpha_1^3 - 211680\alpha_1^2\alpha_2 \\
&+16800\alpha_1^2\alpha_3 + 23520\alpha_1\alpha_2^2 - 6720\alpha_1\alpha_2\alpha_3 + 7056\alpha_2^3 - 70560\alpha_2\beta_1\beta_2 - 6720\alpha_1\beta_2\beta_3 \\
&-329280\alpha_1\beta_1\beta_2 + 2587200\alpha_1\beta_1^2 + 94080\alpha_1\beta_2^2 + 117600\alpha_2\beta_1^2 + 7056\alpha_2\beta_2^2 + 16800\alpha_3\beta_1^2 \\
g_2 &= 153T^{16} + 13320T^{14} - 2520\beta_1T^{13} + 550620T^{12} + (3276\beta_2 - 332640\beta_1)T^{11} + (41580\beta_1^2 \\
&-11680200 + 41580\alpha_1^2)T^{10} + (11357640\beta_1 + 3960\beta_3 - 477540\beta_2)T^9 + (4063500\alpha_1^2 \\
&-52920\alpha_2\alpha_1 + 143640\beta_2\beta_1 - 3723300\beta_1^2 - 173133450)T^8 + (256757760\beta_1 + 352800\beta_1^3 \\
&-1058400\alpha_1^2\beta_1 + 102240\beta_3 - 8031240\beta_2)T^7 + (14376600\alpha_1^2 + 253222200 + 3386880\beta_2\beta_1 \\
&+3528\alpha_2^2 - 20160\beta_1\beta_3 - 125261640\beta_1^2 + 45864\beta_2^2 - 564480\alpha_2\alpha_1)T^6 + (25048800\beta_1^3 \\
&+559440\beta_3 - 9261000\beta_2 - 197232840\beta_1 - 423360\beta_1^2\beta_2 - 3175200\alpha_1^2\beta_1 - 423360\alpha_1^2\beta_2)T^5 \\
&+(5040\beta_2\beta_3 + 5040\alpha_2\alpha_3 + 55654200\beta_1^2 - 1764000\beta_1^4 + 405720\beta_2^2 - 1693440\alpha_2\alpha_1 \\
&+1764000\alpha_1^4 - 335160\alpha_2^2 - 89302500 - 403200\beta_1\beta_3 - 50400\alpha_1\alpha_3 - 458640\beta_2\beta_1 \\
&-1146600\alpha_1^2)T^4 + (70560\alpha_2^2\beta_1 - 4586400\beta_1^3 + 2116800\beta_1^2\beta_2 + 50400\alpha_1^2\beta_3 + 60328800\beta_1 \\
&+50400\beta_1^2\beta_3 - 1360800\beta_3 - 3528000\alpha_1^2\beta_2 - 211680\alpha_1\alpha_2\beta_2 - 4586400\alpha_1^2\beta_1 + 9128700\beta_2 \\
&-141120\beta_1\beta_2^2 + 5644800\alpha_1\alpha_2\beta_1)T^3 + (826560\beta_1\beta_3 - 1058400\alpha_1\alpha_2\beta_1^2 + 675360\alpha_1\alpha_3 \\
&+1781640\alpha_2^2 - 80640\beta_2\beta_3 + 1058400\alpha_1^2\beta_1\beta_2 + 352800\alpha_1^3\alpha_2 + 720\alpha_3^2 - 14535360\beta_2\beta_1 \\
&+53193420\alpha_1^2 + 1464120\beta_2^2 + 720\beta_3^2 + 52664220\beta_1^2 - 12418560\alpha_2\alpha_1 - 80640\alpha_2\alpha_3 \\
&+464373000 - 352800\beta_1^3\beta_2)T^2 + (50400\alpha_1^2\beta_3 - 21168\beta_2^3 - 21168\alpha_2^2\beta_2 + 3810240\alpha_1\alpha_2\beta_1 \\
&-705600\alpha_1^2\beta_2 - 17992800\alpha_1^2\beta_1 - 493920\alpha_2^2\beta_1 + 113400\beta_3 + 211680\alpha_1\alpha_2\beta_2 \\
&-201600\alpha_1\alpha_3\beta_1 - 17992800\beta_1^3 + 20160\alpha_2\alpha_3\beta_1 - 137327400\beta_1 - 5159700\beta_2 - 282240\beta_1\beta_2^2 \\
&+20160\beta_1\beta_2\beta_3 + 3104640\beta_1^2\beta_2 - 151200\beta_1^2\beta_3)T + 22325625 - 511560\alpha_2\alpha_1 + 88200\alpha_2^2 \\
&+16149420\alpha_1^2 - 80640\alpha_1\alpha_3 - 5040\alpha_2\alpha_3 + 720\alpha_3^2 + 1764000\alpha_1^4 - 352800\alpha_1^3\alpha_2 + 35280\alpha_1^2\alpha_2^2 \\
&-352800\alpha_1\alpha_2\beta_1^2 - 352800\alpha_1^2\beta_1\beta_2 + 35280\beta_1^2\beta_2^2 - 352800\beta_1^3\beta_2 - 511560\beta_2\beta_1 + 88200\beta_2^2 \\
&+16149420\beta_1^2 + 720\beta_3^2 - 80640\beta_1\beta_3 - 5040\beta_2\beta_3 + 1764000\beta_1^4 + 3528000\alpha_1^2\beta_1^2 \\
&+35280\alpha_1^2\beta_2^2 + 35280\alpha_2^2\beta_1^2
\end{aligned}
$$

$$
\begin{aligned}
g_1 \quad = \quad & 2160\alpha_1 T^{14} + (27300\alpha_1 + 3822\alpha_2)T^{12} + (3510192\alpha_1 + 1848\alpha_3 - 6468\alpha_2)T^{10} + (64680\beta_2\alpha_1 \\
& +9240\alpha_2\beta_1 - 2629200\alpha_1\beta_1)T^9 + (90088740\alpha_1 - 5448870\alpha_2 + 83160\alpha_3 + 340200\alpha_1\beta_1^2 \\
& -113400\alpha_1^3)T^8 + (12096\alpha_2\beta_2 + 14400\alpha_1\beta_3 + 151200\beta_2\alpha_1 - 20160\alpha_3\beta_1 - 85760640\alpha_1\beta_1 \\
& +2610720\alpha_2\beta_1)T^7 + (367920\alpha_3 + 23637600\alpha_1\beta_1^2 + 8114400\alpha_1^3 - 352800\alpha_2\beta_1^2 \\
& -24307920\alpha_1 - 352800\alpha_1^2\alpha_2 - 25525080\alpha_2)T^6 + (42406560\alpha_1\beta_1 + 7056\alpha_2\beta_3 \\
& -16962624\beta_2\alpha_1 + 359856\alpha_2\beta_2 + 18409104\alpha_2\beta_1 + 30240\alpha_1\beta_3 - 7056\alpha_3\beta_2 - 201600\alpha_3\beta_1 \\
& -2116800\alpha_1^3\beta_1 - 2116800\alpha_1\beta_1^3)T^5 + (25200\alpha_3\beta_1^2 - 105840\alpha_1\beta_2^2 + 22954050\alpha_2 \\
& -211680\alpha_1\alpha_2^2 - 19051200\alpha_1^3 - 41145300\alpha_1 - 1285200\alpha_3 + 25200\alpha_1^2\alpha_3 + 5115600\alpha_1^2\alpha_2 \\
& +10231200\alpha_1\beta_1\beta_2 - 105840\alpha_2\beta_1\beta_2 - 5115600\alpha_2\beta_1^2 - 19051200\alpha_1\beta_1^2)T^4 + (70560\alpha_2\beta_3 \\
& -1411200\alpha_1^2\alpha_2\beta_1 + 1108800\alpha_3\beta_1 + 470400\alpha_1^3\beta_2 + 2352000\alpha_1^3\beta_1 + 470400\alpha_2\beta_1^3 \\
& -20532960\alpha_2\beta_1 - 70560\alpha_3\beta_2 + 20180160\beta_2\alpha_1 + 2352000\alpha_1\beta_1^3 + 15523200\alpha_1\beta_1 \\
& -1008000\alpha_1\beta_3 - 1411200\alpha_1\beta_1^2\beta_2 - 211680\alpha_2\beta_2)T^3 + (28224000\alpha_1\beta_1^2 + 211680\alpha_2\beta_1\beta_2 \\
& -13829760\alpha_1\beta_1\beta_2 + 604800\alpha_1\beta_1\beta_3 - 20160\alpha_2\beta_1\beta_3 - 7761600\alpha_1^2\alpha_2 - 340200\alpha_3 \\
& +20160\alpha_3\beta_1\beta_2 + 211680\alpha_1\alpha_2^2 + 28224000\alpha_1^3 + 233377200\alpha_1 + 5953500\alpha_2 - 21168\alpha_2\beta_2^2 \\
& +252000\alpha_1^2\alpha_3 + 6068160\alpha_2\beta_1^2 - 352800\alpha_3\beta_1^2 - 21168\alpha_2^3)T^2 + (33600\alpha_1^3\beta_3 \\
& +2822400\alpha_1\beta_1^2\beta_2 - 317520\alpha_2\beta_2 - 2063880\beta_2\alpha_1 - 470400\alpha_1^3\beta_2 - 100800\alpha_1^2\alpha_3\beta_1 \\
& -10348800\alpha_1^3\beta_1 - 10348800\alpha_1\beta_1^3 - 15120\alpha_3\beta_2 - 79909200\alpha_1\beta_1 - 4815720\alpha_2\beta_1 \\
& +151200\alpha_1\beta_3 - 1176000\alpha_2\beta_1^3 + 15120\alpha_2\beta_3 + 2116800\alpha_1^2\alpha_2\beta_1 - 100800\alpha_1\beta_1^2\beta_3 \\
& +33600\alpha_3\beta_1^3)T + 23417100\alpha_1 + 3770550\alpha_2 + 113400\alpha_3 + 12001080\alpha_1^3 + 929040\alpha_1^2\alpha_2 \\
& -28560\alpha_1^2\alpha_3 + 44688\alpha_1\alpha_2^2 - 13440\alpha_1\alpha_2\alpha_3 + 480\alpha_1\alpha_3^2 + 14112\alpha_2^3 - 1008\alpha_2^2\alpha_3 \\
& -2016\alpha_2\beta_2\beta_3 + 7056\alpha_2\beta_1\beta_2 - 13440\alpha_1\beta_2\beta_3 - 53760\alpha_1\beta_1\beta_3 - 305760\alpha_1\beta_1\beta_2 \\
& +12001080\alpha_1\beta_1^2 + 37632\alpha_1\beta_2^2 + 480\alpha_1\beta_3^2 + 1234800\alpha_2\beta_1^2 + 14112\alpha_2\beta_2^2 + 25200\alpha_3\beta_1^2 \\
& +1008\alpha_3\beta_2^2 + 117600\alpha_2\beta_1^4 + 1176000\alpha_1^5 - 117600\alpha_1^4\alpha_2 + 2352000\alpha_1^3\beta_1^2 - 235200\alpha_1^3\beta_1\beta_2 \\
& +1176000\alpha_1\beta_1^4 - 235200\alpha_1\beta_1^3\beta_2
\end{aligned}
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\begin{aligned}
g_0 &= 19T^{18} + 3315T^{16} - 816\beta_1 T^{15} + 129420T^{14} - (66360\beta_1 + 882\beta_2)T^{13} + (8190\beta_1^2 \\
&\quad +8190\alpha_1^2 + 328860)T^{12} - (678048\beta_1 + 312\beta_3 + 43260\beta_2)T^{11} + (11088\beta_2\beta_1 - 170940\alpha_1^2 \\
&\quad +4862970 + 22176\alpha_2\alpha_1 + 198660\beta_1^2)T^{10} + (46200\alpha_1^2\beta_1 - 31883880\beta_1 + 1218210\beta_2 \\
&\quad -15400\beta_1^3 - 21720\beta_3)T^9 + (8694\alpha_2^2 + 5760\beta_1\beta_3 + 7182\beta_2^2 + 5040\alpha_1\alpha_3 - 17640\alpha_2\alpha_1 \\
&\quad +18473490\beta_1^2 - 902790\alpha_1^2 - 201540150 - 677880\beta_2\beta_1)T^8 + (90720\beta_1^2\beta_2 - 3645600\beta_1^3 \\
&\quad +90720\alpha_1^2\beta_2 - 1539720\beta_2 + 152767440\beta_1 - 91440\beta_3 - 352800\alpha_1^2\beta_1)T^7 + (4704\beta_2\beta_3 \\
&\quad -184558500 + 7074816\alpha_2\alpha_1 - 39954600\alpha_1^2 - 51361800\beta_1^2 + 33600\beta_1\beta_3 + 6604416\beta_2\beta_1 \\
&\quad -323400\beta_2^2 + 4704\alpha_2\alpha_3 - 252840\alpha_2^2 + 235200\beta_1^4 - 235200\alpha_1^4)T^6 + (21168\alpha_1\alpha_2\beta_2 \\
&\quad -5040\beta_1^2\beta_3 + 55658610\beta_2 + 63504\alpha_2^2\beta_1 + 8925840\beta_1^3 + 1975680\alpha_1^2\beta_2 - 650160\beta_3 \\
&\quad +8925840\alpha_1^2\beta_1 - 5040\alpha_1^2\beta_3 + 84672\beta_1\beta_2^2 - 4233600\alpha_1\alpha_2\beta_1 - 2257920\beta_1^2\beta_2 \\
&\quad +67367160\beta_1)T^5 + (198413250\beta_1^2 - 57765120\alpha_2\alpha_1 + 206263050\alpha_1^2 + 600\alpha_3^2 + 588000\alpha_1^4 \\
&\quad -52080\alpha_2\alpha_3 + 1142400\beta_1\beta_3 + 600\beta_3^2 - 70642320\beta_2\beta_1 + 529200\alpha_1\alpha_2\beta_1^2 - 588000\beta_1^4 \\
&\quad +1167600\alpha_1\alpha_3 + 1458607500 - 176400\alpha_1^3\alpha_2 + 1484700\alpha_2^2 - 52080\beta_2\beta_3 - 529200\alpha_1^2\beta_1\beta_2 \\
&\quad +1537620\beta_2^2 + 176400\beta_1^3\beta_2)T^4 + (38478720\alpha_1\alpha_2\beta_1 - 806400\alpha_1\alpha_3\beta_1 + 420000\alpha_1^2\beta_3 \\
&\quad -149469600\beta_1^3 - 6656160\alpha_1^2\beta_2 - 1075334400\beta_1 - 36647100\beta_2 - 823200\alpha_2^2\beta_1 + 7056\alpha_2^2\beta_2 \\
&\quad +7056\beta_2^3 - 149469600\alpha_1^2\beta_1 - 893760\beta_1\beta_2^2 - 386400\beta_1^2\beta_3 + 13440\alpha_2\alpha_3\beta_1 + 20160\alpha_1\alpha_3\beta_2 \\
&\quad -20160\alpha_1\alpha_2\beta_3 + 2759400\beta_3 + 31822560\beta_1^2\beta_2 + 13440\beta_1\beta_2\beta_3 - 70560\alpha_1\alpha_2\beta_2)T^3 \\
&\quad +(423360\alpha_2\alpha_1 + 793800\alpha_2^2 + 145291860\alpha_1^2 - 695520\alpha_1\alpha_3 - 120960\alpha_2\alpha_3 + 2160\alpha_3^2 \\
&\quad +15993600\alpha_1^4 - 2704800\alpha_1^3\alpha_2 - 33600\alpha_1^3\alpha_3 + 105840\alpha_1^2\alpha_2^2 + 100800\alpha_1\alpha_3\beta_1^2 - 8820000\alpha_1\alpha_2\beta_1^2 \\
&\quad -100800\alpha_1^2\beta_1\beta_3 + 352800\alpha_1^2\beta_1\beta_2 + 105840\beta_1^2\beta_2^2 + 33600\beta_1^3\beta_3 - 5762400\beta_1^3\beta_2 + 17886960\beta_2\beta_1 \\
&\quad +1746360\beta_2^2 + 321515460\beta_1^2 + 2160\beta_3^2 - 1149120\beta_1\beta_3 - 120960\beta_2\beta_3 + 40454400\beta_1^4 \\
&\quad +56448000\alpha_1^2\beta_1^2 + 105840\alpha_1^2\beta_2^2 + 105840\alpha_2^2\beta_1^2 + 209860875)T^2 + (129360\beta_1^2\beta_3 - 432768\beta_1\beta_2^2 \\
&\quad -480\beta_1\beta_3^2 + 1008\beta_2^2\beta_3 - 480\alpha_3^2\beta_1 - 1008\alpha_2^2\beta_3 - 35280\alpha_2^2\beta_2 - 214032\alpha_2^2\beta_1 - 25200\alpha_1^2\beta_3 \\
&\quad -2822400\alpha_1^2\beta_2 - 41459880\alpha_1^2\beta_1 - 352800\alpha_1^4\beta_2 + 352800\beta_1^4\beta_2 - 218736\alpha_1\alpha_2\beta_2 \\
&\quad -1176000\alpha_1\alpha_2\beta_1 + 2016\alpha_2\alpha_3\beta_2 + 33600\alpha_2\alpha_3\beta_1 + 154560\alpha_1\alpha_3\beta_1 + 33600\beta_1\beta_2\beta_3 \\
&\quad +705600\alpha_1\alpha_2\beta_1^3 + 705600\alpha_1^3\alpha_2\beta_1 - 53184600\beta_1 - 4961250\beta_2 - 113400\beta_3 - 35280\beta_2^3 - 4704000\beta_1^5 \\
&\quad -41459880\beta_1^3 - 9408000\alpha_1^2\beta_1^3 - 4704000\alpha_1^4\beta_1 - 3998400\beta_1^2\beta_2)T - 4465125 + 1076040\alpha_2\alpha_1 \\
&\quad -57330\alpha_2^2 + 1847790\alpha_1^2 + 40320\alpha_1\alpha_3 - 5040\alpha_2\alpha_3 - 360\alpha_3^2 + 1764000\alpha_1^4 + 364560\alpha_1^3\alpha_2 \\
&\quad +11760\alpha_1^2\alpha_2^2 - 3360\alpha_1^2\alpha_2\alpha_3 + 7056\alpha_1\alpha_2^3 - 3360\alpha_2\alpha_3\beta_1^2 + 7056\alpha_2^2\beta_1\beta_2 + 7056\alpha_1\alpha_2\beta_2^2 \\
&\quad +364560\alpha_1\alpha_2\beta_1^2 - 3360\alpha_1^2\beta_2\beta_3 + 364560\alpha_1^2\beta_1\beta_2 + 7056\beta_1\beta_2^3 + 11760\beta_1^2\beta_2^2 + 364560\beta_1^3\beta_2 \\
&\quad +1076040\beta_2\beta_1 - 57330\beta_2^2 + 1847790\beta_1^2 - 360\beta_3^2 + 40320\beta_1\beta_3 - 5040\beta_2\beta_3 + 1764000\beta_1^4 \\
&\quad -3360\beta_1^2\beta_2\beta_3 + 3528000\alpha_1^2\beta_1^2 + 11760\alpha_1^2\beta_2^2 + 11760\alpha_2^2\beta_1^2 + 196000\alpha_1^6 + 588000\alpha_1^4\beta_1^2 \\
&\quad +588000\alpha_1^2\beta_1^4 + 196000\beta_1^6 \\
h_{14} &= 36T^5 - 600T^3 + 180\beta_1 T^2 - 540T + 240\beta_1 - 21\beta_2 \\
h_{13} &= (420\alpha_1 - 42\alpha_2)T \\
h_{12} &= 84T^7 - 2940T^5 + 840\beta_1 T^4 + 1260T^3 - (2940\beta_1 + 189\beta_2)T^2 + (630\beta_1^2 + 630\alpha_1^2 - 13860)T \\
&\quad +4676\beta_1 - 595\beta_2 + 14\beta_3 \\
h_{11} &= 504\alpha_1 T^5 + (4200\alpha_1 - 588\alpha_2)T^3 + (72\alpha_3 - 756\alpha_2 - 6552\alpha_1)T + 7560\alpha_1\beta_1 - 588\alpha_2\beta_1 - 84\beta_2\alpha_1 \\
h_{10} &= 126T^9 - 6552T^7 + 1764\beta_1 T^6 + 25956T^5 - (21\beta_2 + 29400\beta_1)T^4 + (3780\beta_1^2 - 52920 + 3780\alpha_1^2)T^3 \\
&\quad +(42\beta_2 - 16716\beta_1 - 84\beta_3)T^2 + (20580\alpha_1^2 + 1176\beta_2\beta_1 - 198450 - 7980\beta_1^2 - 3192\alpha_2\alpha_1)T \\
&\quad +3780\alpha_1^2\beta_1 - 4641\beta_2 + 84\beta_3 - 1260\beta_1^3 + 10416\beta_1 \\
h_9 &= 1680\alpha_1 T^7 + (4620\alpha_1 - 1638\alpha_2)T^5 + (200\alpha_3 - 4900\alpha_2 - 145600\alpha_1)T^3 + (1260\alpha_2\beta_1 + 8820\beta_2\alpha_1 \\
&\quad +46200\alpha_1\beta_1)T^2 + (9800\alpha_1^3 + 45570\alpha_2 - 577500\alpha_1 - 600\alpha_3 - 29400\alpha_1\beta_1^2)T + 200\alpha_3\beta_1 \\
&\quad +168\alpha_2\beta_2 + 54880\alpha_1\beta_1 - 11620\alpha_2\beta_1 - 280\alpha_1\beta_3 + 5180\beta_2\alpha_1
\end{aligned}
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\begin{aligned}
h_8 &= 126T^{11} - 7770T^9 + 1890\beta_1 T^8 + 71820T^7 + (1323\beta_2 - 84420\beta_1)T^6 + (9450\alpha_1^2 + 9450\beta_1^2 \\
&\quad -532980)T^5 + (29295\beta_2 - 630\beta_3 + 312480\beta_1)T^4 + (220500\alpha_1^2 - 233100\beta_1^2 - 614250 \\
&\quad -5040\beta_2\beta_1)T^3 + (31500\beta_1^3 + 1260\beta_3 - 94500\alpha_1^2\beta_1 - 73395\beta_2 - 202860\beta_1)T^2 + (2976750 \\
&\quad +412650\beta_1^2 - 22680\beta_2\beta_1 - 7560\alpha_2\alpha_1 - 164430\alpha_1^2 + 1638\alpha_2^2 + 126\beta_2^2 - 720\alpha_1\alpha_3)T + 6300\beta_1^2\beta_2 \\
&\quad -682290\beta_1 - 67095\beta_2 + 1890\beta_3 + 6300\alpha_1^2\beta_2 - 31500\alpha_1^2\beta_1 - 90300\beta_1^3 \\
h_7 &= 2520\alpha_1 T^9 - (1512\alpha_2 + 15120\alpha_1)T^7 + (32760\alpha_2 + 573552\alpha_1 - 432\alpha_3)T^5 + (22680\beta_2\alpha_1 \\
&\quad -1033200\alpha_1\beta_1 - 2520\alpha_2\beta_1)T^4 + (346920\alpha_2 + 151200\alpha_1\beta_1^2 - 6240\alpha_3 - 50400\alpha_1^3 - 6076560\alpha_1)T^3 \\
&\quad +(2880\alpha_3\beta_1 + 15120\beta_2\alpha_1 - 6048\alpha_2\beta_2 + 2005920\alpha_1\beta_1 - 136080\alpha_2\beta_1)T^2 + (20160\alpha_1^2\alpha_2 \\
&\quad +20160\alpha_2\beta_1^2 + 6480\alpha_3 + 2320920\alpha_1 - 50400\alpha_1^3 - 52920\alpha_2 - 50400\alpha_1\beta_1^2)T - 50400\alpha_1^3\beta_1 \\
&\quad -50400\alpha_1\beta_1^3 + 168\alpha_2\beta_3 + 1512\alpha_2\beta_1 - 512400\alpha_1\beta_1 + 480\alpha_3\beta_1 - 77952\beta_2\alpha_1 - 1680\alpha_1\beta_3 \\
&\quad -168\alpha_3\beta_2 + 2520\alpha_2\beta_2 \\
h_6 &= 84T^{13} - 4872T^{11} + 924\beta_1 T^{10} + 71820T^9 + (2457\beta_2 - 110880\beta_1)T^8 + (297360 + 12600\beta_1^2 \\
&\quad +12600\alpha_1^2)T^7 + (54684\beta_2 - 1294776\beta_1 - 504\beta_3)T^6 + (370440\beta_1^2 - 21168\beta_2\beta_1 - 546840\alpha_1^2 \\
&\quad +21168\alpha_2\alpha_1 - 7276500)T^5 + (7114800\beta_1 + 88200\alpha_1^2\beta_1 + 21000\beta_3 - 29400\beta_1^3 - 986370\beta_2)T^4 \\
&\quad +(43659000 - 3528\alpha_2^2 - 6720\beta_1\beta_3 + 88200\alpha_1^2 + 1522920\beta_1^2 + 10584\beta_2^2 + 94080\beta_2\beta_1)T^3 \\
&\quad -(17966340\beta_1 + 176400\alpha_1^2\beta_1 + 1117200\beta_1^3 + 7560\beta_3 + 291060\beta_2)T^2 + (117600\beta_1^4 + 22344\alpha_2^2 \\
&\quad -336\beta_2\beta_3 + 3251640\alpha_1^2 + 3360\alpha_1\alpha_3 - 392784\beta_2\beta_1 + 15288\beta_2^2 + 4121880\beta_1^2 + 6720\beta_1\beta_3 - 336\alpha_2\alpha_3 \\
&\quad -733824\alpha_2\alpha_1 + 14685300 - 117600\alpha_1^4)T - 4480560\beta_1 + 11025\beta_2 + 32760\beta_3 - 664440\beta_1^3 \\
&\quad +129360\beta_1^2\beta_2 + 840\beta_1^2\beta_3 - 7056\beta_1\beta_2^2 - 3528\alpha_2^2\beta_1 + 840\alpha_1^2\beta_3 - 11760\alpha_1^2\beta_2 - 664440\alpha_1^2\beta_1 \\
&\quad -3528\alpha_1\alpha_2\beta_2 + 141120\alpha_1\alpha_2\beta_1 \\
h_5 &= 2016\alpha_1 T^{11} + (42\alpha_2 - 37380\alpha_1)T^9 + (125496\alpha_2 - 1465632\alpha_1 - 1008\alpha_3)T^7 + (3528\beta_2\alpha_1 \\
&\quad -317520\alpha_1\beta_1 - 31752\alpha_2\beta_1)T^6 + (21168\alpha_3 - 418068\alpha_2 + 105840\alpha_1\beta_1^2 - 11716488\alpha_1 - 35280\alpha_1^3)T^5 \\
&\quad +(11501280\alpha_1\beta_1 - 5040\alpha_3\beta_1 - 52920\beta_2\alpha_1 - 5040\alpha_1\beta_3 + 21168\alpha_2\beta_2 - 264600\alpha_2\beta_1)T^4 \\
&\quad +(23919840\alpha_1 + 70560\alpha_1^2\alpha_2 - 3880800\alpha_1\beta_1^2 + 70560\alpha_2\beta_1^2 - 1528800\alpha_1^3 + 45360\alpha_3 - 3545640\alpha_2)T^3 \\
&\quad +(352800\alpha_1^3\beta_1 + 1512\alpha_2\beta_3 + 352800\alpha_1\beta_1^3 - 1512\alpha_3\beta_2 + 10584\alpha_2\beta_2 + 1132488\alpha_2\beta_1 - 12947760\alpha_1\beta_1 \\
&\quad -5040\alpha_1\beta_3 - 56448\beta_2\alpha_1)T^2 + (5040\alpha_3\beta_1^2 + 6271020\alpha_1 - 105840\alpha_1^2\alpha_2 + 105840\alpha_2\beta_1^2 \\
&\quad -21168\alpha_2\beta_1\beta_2 + 992250\alpha_2 + 136080\alpha_3 + 2540160\alpha_1^3 + 5040\alpha_1^2\alpha_3 - 211680\alpha_1\beta_1\beta_2 + 7056\alpha_1\beta_2^2 \\
&\quad +2540160\alpha_1\beta_1^2 - 14112\alpha_1\alpha_2^2)T - 588000\alpha_1\beta_1^3 - 1512\alpha_2\beta_3 - 23520\alpha_2\beta_1^3 - 23520\alpha_1^3\beta_2 - 15120\alpha_3\beta_1 \\
&\quad -402192\beta_2\alpha_1 - 74088\alpha_2\beta_1 - 4021920\alpha_1\beta_1 - 588000\alpha_1^3\beta_1 - 31752\alpha_2\beta_2 + 30240\alpha_1\beta_3 \\
&\quad +1512\alpha_3\beta_2 + 70560\alpha_1^2\alpha_2\beta_1 + 70560\alpha_1\beta_1^2\beta_2 \\
h_4 &= 36T^{15} - 1260T^{13} + 13860T^{11} + (1617\beta_2 - 70980\beta_1)T^{10} + (9450\alpha_1^2 - 585900 + 9450\beta_1^2)T^9 \\
&\quad +(450\beta_3 - 40005\beta_2 - 1115100\beta_1)T^8 + (642600\beta_1^2 - 32829300 + 10080\alpha_2\alpha_1 - 970200\alpha_1^2)T^7 \\
&\quad +(264600\alpha_1^2\beta_1 - 1295070\beta_2 - 4200\beta_3 - 88200\beta_1^3 + 39660600\beta_1)T^6 + (107559900 + 343980\alpha_1^2 \\
&\quad +599760\beta_2\beta_1 - 5292\beta_2^2 - 35280\alpha_2\alpha_1 - 10080\alpha_1\alpha_3 - 12744900\beta_1^2 + 15876\alpha_2^2)T^5 + (5799150\beta_2 \\
&\quad -441000\alpha_1^2\beta_1 + 1323000\beta_1^3 - 88200\alpha_1^2\beta_2 - 88200\beta_1^2\beta_2 - 88464600\beta_1 - 144900\beta_3)T^4 + (67200\beta_1\beta_3 \\
&\quad +77395500 + 17640\beta_2^2 + 33600\alpha_1\alpha_3 + 88200\alpha_2^2 + 32075400\alpha_1^2 - 3998400\alpha_2\alpha_1 - 4821600\beta_2\beta_1 \\
&\quad +39013800\beta_1^2)T^3 + (52920\alpha_1\alpha_2\beta_2 - 7673400\beta_1^3 - 14729400\alpha_1^2\beta_1 - 352800\alpha_1^2\beta_2 - 189000\beta_3 \\
&\quad -12600\beta_1^2\beta_3 - 33075\beta_2 - 12600\alpha_1^2\beta_3 + 1058400\beta_1^2\beta_2 + 1411200\alpha_1\alpha_2\beta_1 - 52920\alpha_2^2\beta_1 \\
&\quad -23152500\beta_1)T^2 + (176400\alpha_1^2\beta_1\beta_2 + 58800\alpha_1^3\alpha_2 - 58800\beta_1^3\beta_2 + 8953770\beta_1^2 + 87360\beta_1\beta_3 \\
&\quad +53581500 - 91140\alpha_2^2 + 120\alpha_3^2 + 588000\beta_1^4 + 120\beta_3^2 - 176400\alpha_1\alpha_2\beta_1^2 + 173460\beta_2^2 - 1470000\beta_2\beta_1 \\
&\quad -4717230\alpha_1^2 + 213360\alpha_1\alpha_3 - 8400\alpha_2\alpha_3 - 588000\alpha_1^4 - 8400\beta_2\beta_3 - 3763200\alpha_2\alpha_1)T - 1680\alpha_1\alpha_2\beta_3 \\
&\quad -52920\alpha_1\alpha_2\beta_2 + 564480\alpha_1\alpha_2\beta_1 + 1680\alpha_1\alpha_3\beta_2 - 33600\alpha_1\alpha_3\beta_1 - 8599500\beta_1 + 429975\beta_2 + 47250\beta_3 \\
&\quad +1764\beta_2^3 - 1087800\beta_1^3 + 323400\beta_1^2\beta_2 - 4200\beta_1^2\beta_3 - 35280\beta_1\beta_2^2 + 1764\alpha_2^2\beta_2 + 17640\alpha_2^2\beta_1 \\
&\quad +29400\alpha_1^2\beta_3 - 241080\alpha_1^2\beta_2 - 1087800\alpha_1^2\beta_1
\end{aligned}
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$$
\begin{aligned}
h_3 \;=\;& 840\alpha_1T^{13} + (756\alpha_2 - 29400\alpha_1)T^{11} + (72380\alpha_2 - 3911320\alpha_1 - 280\alpha_3)T^9 + (1751400\alpha_1\beta_1 \\
& -3780\beta_2\alpha_1 - 26460\alpha_2\beta_1)T^8 + (7195440\alpha_1 - 1501080\alpha_2 + 3360\alpha_3 + 84000\alpha_1^3 - 252000\alpha_1\beta_1^2)T^7 \\
& +(388080\alpha_2\beta_1 + 682080\alpha_1\beta_1 - 58800\beta_2\alpha_1 + 6720\alpha_1\beta_3 - 14112\alpha_2\beta_2)T^6 + (108291960\alpha_1 \\
& -3175200\alpha_1\beta_1^2 - 6650280\alpha_2 - 2234400\alpha_1^3 - 95760\alpha_3)T^5 + (117600\alpha_3\beta_1 + 2005080\alpha_2\beta_1 \\
& -101194800\alpha_1\beta_1 + 1317120\beta_2\alpha_1 + 588000\alpha_1\beta_1^3 - 2520\alpha_3\beta_2 - 123480\alpha_2\beta_2 - 58800\alpha_1\beta_3 \\
& +588000\alpha_1^3\beta_1 + 2520\alpha_2\beta_3)T^4 + (70560\alpha_2\beta_1\beta_2 - 16800\alpha_3\beta_1^2 + 588000\alpha_1^2\alpha_2 - 117600\alpha_2\beta_1^2 \\
& -70560\alpha_1\beta_2^2 + 22108800\alpha_1\beta_1^2 + 352800\alpha_3 - 13009500\alpha_2 + 705600\alpha_1\beta_1\beta_2 - 16800\alpha_1^2\alpha_3 \\
& -6115200\alpha_1^3 - 100459800\alpha_1)T^3 + (5040\alpha_3\beta_2 + 4339440\alpha_2\beta_1 + 740880\alpha_2\beta_2 - 1176000\alpha_1\beta_1^3 \\
& +35632800\alpha_1\beta_1 + 4304160\beta_2\alpha_1 - 100800\alpha_3\beta_1 - 1176000\alpha_1^3\beta_1 - 5040\alpha_2\beta_3 - 252000\alpha_1\beta_3)T^2 \\
& +(134400\alpha_1\beta_1\beta_3 - 6720\alpha_1\beta_2\beta_3 + 3307500\alpha_2 - 6720\alpha_1\alpha_2\alpha_3 + 117600\alpha_1^2\alpha_3 - 3622080\alpha_1\beta_1\beta_2 \\
& -264600\alpha_3 + 23520\alpha_1\alpha_2^2 - 211680\alpha_2\beta_1\beta_2 + 11054400\alpha_1\beta_1^2 + 235200\alpha_1\beta_2^2 + 11054400\alpha_1^3 \\
& +7056\alpha_2\beta_2^2 + 106104600\alpha_1 - 117600\alpha_2\beta_1^2 + 7056\alpha_2^3 - 16800\alpha_3\beta_1^2 - 3739680\alpha_1^2\alpha_2)T \\
& +588000\alpha_1^2\alpha_2\beta_1 + 39200\alpha_2\beta_1^3 - 313600\alpha_1^3\beta_2 - 25200\alpha_1\beta_3 - 1724800\alpha_1^3\beta_1 - 1724800\alpha_1\beta_1^3 \\
& -7560\alpha_2\beta_3 - 1261260\beta_2\alpha_1 + 114660\alpha_2\beta_1 - 13318200\alpha_1\beta_1 + 50400\alpha_3\beta_1 - 52920\alpha_2\beta_2 \\
& -16800\alpha_1^2\alpha_3\beta_1 - 16800\alpha_1\beta_1^2\beta_3 + 235200\alpha_1\beta_1^2\beta_2 + 7560\alpha_3\beta_2 + 5600\alpha_3\beta_1^3 + 5600\alpha_1^3\beta_3 \\
h_2 \;=\;& 9T^{17} + 120T^{15} - 180\beta_1T^{14} - 18900T^{13} + (273\beta_2 - 19320\beta_1)T^{12} + (3780\alpha_1^2 + 3780\beta_1^2 \\
& -4014360)T^{11} + (396\beta_3 + 2693124\beta_1 - 66150\beta_2)T^{10} + (371700\alpha_1^2 + 15960\beta_2\beta_1 - 5880\alpha_2\alpha_1 \\
& -30778650 - 611100\beta_1^2)T^9 + (38727360\beta_1 + 44100\beta_1^3 + 2340\beta_3 - 143325\beta_2 - 132300\alpha_1^2\beta_1)T^8 \\
& +(181440\beta_2\beta_1 - 18650520\beta_1^2 + 504\alpha_2^2 + 60480\alpha_2\alpha_1 + 208542600 - 7572600\alpha_1^2 - 2880\beta_1\beta_3 \\
& +6552\beta_2^2)T^7 + (1940400\alpha_1^2\beta_1 - 98280\beta_3 - 338608620\beta_1 + 4292400\beta_1^3 - 70560\alpha_1^2\beta_2 - 70560\beta_1^2\beta_2 \\
& +15267420\beta_2)T^6 + (352800\alpha_1^4 + 17640\beta_2^2 + 1008\beta_2\beta_3 + 175641480\beta_1^2 - 338688\alpha_2\alpha_1 \\
& -37802520\alpha_1^2 + 1008\alpha_2\alpha_3 - 88200\alpha_2^2 - 446512500 - 7994448\beta_2\beta_1 - 10080\alpha_1\alpha_3 - 352800\beta_1^4 \\
& -60480\beta_1\beta_3)T^5 + (12600\alpha_1^2\beta_3 + 12600\beta_1^2\beta_3 + 1587600\beta_1^2\beta_2 - 529200\alpha_1^2\beta_2 + 2116800\alpha_1\alpha_2\beta_1 \\
& +340540200\beta_1 - 52920\alpha_1\alpha_2\beta_2 + 7673400\alpha_1^2\beta_1 + 47528775\beta_2 - 1474200\beta_3 + 17640\alpha_2^2\beta_1 \\
& -35280\beta_1\beta_2^2 - 34662600\beta_1^3)T^4 + (880320\beta_1\beta_3 + 2352000\beta_1^4 + 426720\alpha_1\alpha_3 - 18533760\alpha_2\alpha_1 \\
& -117600\beta_1^3\beta_2 - 352800\alpha_1\alpha_2\beta_1^2 + 240\alpha_3^2 - 35468160\beta_2\beta_1 + 440559000 + 558600\beta_2^2 - 47040\beta_2\beta_3 \\
& -47040\alpha_2\alpha_3 + 1511160\alpha_2^2 + 117600\alpha_1^3\alpha_2 + 352800\alpha_1^2\beta_1\beta_2 - 32190060\beta_1^2 + 240\beta_3^2 - 2352000\alpha_1^4 \\
& +32901540\alpha_1^2)T^3 + (510300\beta_3 - 126000\beta_1^2\beta_3 + 8608320\beta_1^2\beta_2 + 10080\beta_1\beta_2\beta_3 - 25200\alpha_1^2\beta_3 \\
& +6138720\alpha_1\alpha_2\beta_1 + 2469600\alpha_1^2\beta_2 + 10080\alpha_2\alpha_3\beta_1 - 595350\beta_2 - 458640\alpha_2^2\beta_1 - 165507300\beta_1 \\
& -100800\alpha_1\alpha_3\beta_1 + 317520\alpha_1\alpha_2\beta_2 - 11466000\beta_1^3 - 11466000\alpha_1^2\beta_1 - 141120\beta_1\beta_2^2 - 10584\beta_2^3 \\
& -10584\alpha_2^2\beta_2)T^2 + (35280\beta_1^2\beta_2^2 - 221760\beta_1\beta_3 - 511560\beta_2^2 + 1764000\beta_1^4 - 1058400\alpha_1^3\alpha_2 \\
& -120558375 - 720\beta_3^2 + 35280\alpha_1^2\beta_2^2 + 35280\beta_2\beta_3 + 1464120\beta_2\beta_1 + 784980\alpha_1^2 - 1058400\alpha_1^2\beta_1\beta_2 \\
& +35280\alpha_1^2\alpha_2^2 + 35280\alpha_2\alpha_3 + 784980\beta_1^2 + 1464120\alpha_2\alpha_1 - 1058400\alpha_1\alpha_2\beta_1^2 + 1764000\alpha_1^4 - 720\alpha_3^2 \\
& -221760\alpha_1\alpha_3 + 35280\alpha_2^2\beta_1^2 - 511560\alpha_2^2 - 1058400\beta_1^3\beta_2 + 3528000\alpha_1^2\beta_1^2)T - 10080\alpha_1\alpha_2\beta_3 \\
& -102312\alpha_1\alpha_2\beta_2 + 235200\alpha_1\alpha_2\beta_1 - 1008\alpha_2\alpha_3\beta_2 - 6720\alpha_2\alpha_3\beta_1 + 10080\alpha_1\alpha_3\beta_2 + 23520\alpha_1\alpha_3\beta_1 \\
& -6720\beta_1\beta_2\beta_3 + 117600\alpha_1\alpha_2\beta_1^3 + 117600\alpha_1^3\alpha_2\beta_1 + 14288400\beta_1 + 4465125\beta_2 + 170100\beta_3 + 17640\beta_2^3 \\
& +1855140\beta_1^3 + 552720\beta_1^2\beta_2 + 36120\beta_1^2\beta_3 - 30576\beta_1\beta_2^2 + 240\beta_1\beta_3^2 - 504\beta_2^2\beta_3 + 240\alpha_3^2\beta_1 \\
& +504\alpha_2^2\beta_3 + 17640\alpha_2^2\beta_2 + 71736\alpha_2^2\beta_1 + 12600\alpha_1^2\beta_3 + 317520\alpha_1^2\beta_2 + 1855140\alpha_1^2\beta_1 \\
& -58800\alpha_1^4\beta_2 + 58800\beta_1^4\beta_2
\end{aligned}
$$

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\begin{aligned}
h_1 \quad = \quad & 144\alpha_1 T^{15} + (294\alpha_2 - 7980\alpha_1)T^{13} + (202272\alpha_1 + 168\alpha_3 - 16212\alpha_2)T^{11} + (924\alpha_2\beta_1 + 6468\beta_2\alpha_1 \\
& -293160\alpha_1\beta_1)T^{10} + (37800\alpha_1\beta_1^2 + 4200\alpha_3 - 878430\alpha_2 + 6058500\alpha_1 - 12600\alpha_1^3)T^9 + (1800\alpha_1\beta_3 \\
& +1512\alpha_2\beta_2 + 349020\alpha_2\beta_1 - 7721280\alpha_1\beta_1 - 2520\alpha_3\beta_1 - 175140\beta_2\alpha_1)T^8 + (3074400\alpha_1\beta_1^2 \\
& -50400\alpha_2\beta_1^2 - 50400\alpha_1^2\alpha_2 + 3535560\alpha_2 + 1663200\alpha_1^3 - 146160\alpha_3 - 153372240\alpha_1)T^7 \\
& +(6720\alpha_3\beta_1 - 352800\alpha_1\beta_1^3 - 352800\alpha_1^3\beta_1 - 1176\alpha_3\beta_2 - 28560\alpha_1\beta_3 + 1176\alpha_2\beta_3 - 389256\alpha_2\beta_1 \\
& +161923440\alpha_1\beta_1 - 5531904\beta_2\alpha_1 + 45864\alpha_2\beta_2)T^6 + (2610720\alpha_1\beta_1\beta_2 - 42336\alpha_1\alpha_2^2 - 17216640\alpha_1^3 \\
& +57616650\alpha_2 - 599760\alpha_2\beta_1^2 - 21168\alpha_2\beta_1\beta_2 - 1617840\alpha_3 + 1296540\alpha_1 + 5040\alpha_3\beta_1^2 + 5040\alpha_1^2\alpha_3 \\
& -45440640\alpha_1\beta_1^2 + 2010960\alpha_1^2\alpha_2 - 21168\alpha_1\beta_2^2)T^5 + (4116000\alpha_1\beta_1^3 + 12600\alpha_2\beta_3 - 352800\alpha_1\beta_1^2\beta_2 \\
& -12600\alpha_3\beta_2 + 117600\alpha_2\beta_1^3 + 30164400\beta_2\alpha_1 - 352800\alpha_1^2\alpha_2\beta_1 + 117600\alpha_1^3\beta_2 - 44893800\alpha_2\beta_1 \\
& +4116000\alpha_1^3\beta_1 + 5644800\alpha_1\beta_1 + 982800\alpha_3\beta_1 - 582120\alpha_2\beta_2 - 705600\alpha_1\beta_3)T^4 + (35750400\alpha_1^3 \\
& +211680\alpha_2\beta_1\beta_2 + 261424800\alpha_1 - 7056\alpha_2^3 + 352800\alpha_1\alpha_2^2 - 7996800\alpha_1^2\alpha_2 + 8599500\alpha_2 \\
& -22014720\alpha_1\beta_1\beta_2 + 117600\alpha_1^2\alpha_3 + 6720\alpha_3\beta_1\beta_2 + 141120\alpha_1\beta_2^2 + 336000\alpha_1\beta_1\beta_3 + 35750400\alpha_1\beta_1^2 \\
& -6720\alpha_2\beta_1\beta_3 - 7056\alpha_2\beta_2^2 + 14017920\alpha_2\beta_1^2 - 218400\alpha_3\beta_1^2 - 415800\alpha_3)T^3 + (2469600\alpha_1^2\alpha_2\beta_1 \\
& +16800\alpha_3\beta_1^3 - 12230400\alpha_1\beta_1^3 - 87582600\alpha_1\beta_1 - 1999200\alpha_2\beta_1^3 + 83160\alpha_3\beta_2 - 12230400\alpha_1^3\beta_1 \\
& -50400\alpha_1\beta_1^2\beta_3 - 50400\alpha_1^2\alpha_3\beta_1 - 235200\alpha_1^3\beta_2 - 6112260\beta_2\alpha_1 - 502740\alpha_2\beta_1 - 83160\alpha_2\beta_3 \\
& +75600\alpha_1\beta_3 + 16800\alpha_1^3\beta_3 + 4233600\alpha_1\beta_1^2\beta_2 - 158760\alpha_2\beta_2)T^2 + (15677550\alpha_2 + 793800\alpha_3 \\
& +6356280\alpha_1^3 + 3751440\alpha_1^2\alpha_2 + 72240\alpha_1^2\alpha_3 - 378672\alpha_1\alpha_2^2 - 13440\alpha_1\alpha_2\alpha_3 + 480\alpha_1\alpha_3^2 + 56448\alpha_2^3 \\
& -1008\alpha_2^2\alpha_3 - 2016\alpha_2\beta_2\beta_3 - 40320\alpha_3\beta_1\beta_2 + 40320\alpha_2\beta_1\beta_3 - 416304\alpha_2\beta_1\beta_2 - 13440\alpha_1\beta_2\beta_3 \\
& -53760\alpha_1\beta_1\beta_3 + 1952160\alpha_1\beta_1\beta_2 + 6356280\alpha_1\beta_1^2 + 37632\alpha_1\beta_2^2 + 480\alpha_1\beta_3^2 + 1799280\alpha_2\beta_1^2 \\
& +56448\alpha_2\beta_2^2 + 126000\alpha_3\beta_1^2 + 1008\alpha_3\beta_2^2 + 117600\alpha_2\beta_1^4 + 1176000\alpha_1^5 - 117600\alpha_1^4\alpha_2 + 2352000\alpha_1^3\beta_1^2 \\
& -235200\alpha_1^3\beta_1\beta_2 + 1176000\alpha_1\beta_1^4 - 235200\alpha_1\beta_1^3\beta_2 - 14685300\alpha_1)T - 70560\alpha_1\beta_1\beta_2^2 + 70560\alpha_2\beta_1^2\beta_2 \\
& +3360\alpha_1^2\alpha_3\beta_2 + 70560\alpha_1\alpha_2^2\beta_1 + 7056\alpha_1\alpha_2^2\beta_2 - 70560\alpha_1^2\alpha_2\beta_2 - 3360\alpha_1^2\alpha_2\beta_3 - 7056\alpha_2\beta_1\beta_2^2 \\
& +3360\alpha_3\beta_1^2\beta_2 - 3360\alpha_2\beta_1^2\beta_3 + 113400\alpha_1\beta_3 + 7560\alpha_2\beta_3 + 2407860\beta_2\alpha_1 - 2407860\alpha_2\beta_1 \\
& -113400\alpha_3\beta_1 - 16800\alpha_1^2\alpha_3\beta_1 + 16800\alpha_1\beta_1^2\beta_3 + 282240\alpha_1\beta_1^2\beta_2 - 7560\alpha_3\beta_2 - 16800\alpha_3\beta_1^3 \\
& +16800\alpha_1^3\beta_3 + 7056\alpha_1\beta_2^3 - 7056\alpha_2^3\beta_1 - 282240\alpha_1^2\alpha_2\beta_1 - 282240\alpha_2\beta_1^3 + 282240\alpha_1^3\beta_2
\end{aligned}
$$

$$
\begin{aligned}
h_0 &= T^{19}+93T^{17}-51\beta_1T^{16}-8604T^{15}-(1020\beta_1+63\beta_2)T^{14}+(630\beta_1^2+630\alpha_1^2-701820)T^{13}\\
&+(244216\beta_1-26\beta_3-119\beta_2)T^{12}+(2016\alpha_2\alpha_1+1008\beta_2\beta_1-4810050-49980\alpha_1^2-13020\beta_1^2)T^{11}\\
&+(4620\alpha_1^2\beta_1-1540\beta_1^3-765996\beta_1+330897\beta_2-1284\beta_3)T^{10}+(966\alpha_2^2+560\alpha_1\alpha_3+640\beta_1\beta_3\\
&+1888810\beta_1^2+798\beta_2^2-122360\beta_2\beta_1+167090\alpha_1^2-54290250-75880\alpha_2\alpha_1)T^9+(99645210\beta_1\\
&-476700\beta_1^3+11340\beta_1^2\beta_2-2529135\beta_2+48690\beta_3+11340\alpha_1^2\beta_2-182700\alpha_1^2\beta_1)T^8+(672\alpha_2\alpha_3\\
&-5030760\alpha_1^2-48337800\beta_1^2-33600\alpha_1^4+603911700+1178688\alpha_2\alpha_1-65352\alpha_2^2-9600\beta_1\beta_3\\
&+672\beta_2\beta_3-13440\alpha_1\alpha_3+2778048\beta_2\beta_1-73416\beta_2^2+33600\beta_1^4)T^7+(14112\beta_1\beta_2^2-658560\beta_1^2\beta_2\\
&+28874475\beta_2-22680\beta_3-840\alpha_1^2\beta_3+3528\alpha_1\alpha_2\beta_2+10584\alpha_2^2\beta_1+188160\alpha_1^2\beta_2-840\beta_1^2\beta_3\\
&-846720\alpha_1\alpha_2\beta_1+2075640\alpha_1^2\beta_1+9131640\beta_1^3-699258420\beta_1)T^6+(35280\beta_1^3\beta_2+501321450\beta_1^2\\
&+228554130\alpha_1^2+394800\alpha_1\alpha_3+1006068\beta_2^2+120\beta_3^2-105840\alpha_1^2\beta_1\beta_2-35280\alpha_1^3\alpha_2-22512\alpha_2\alpha_3\\
&-588000\beta_1^4-22512\beta_2\beta_3-35995008\alpha_2\alpha_1-44074128\beta_2\beta_1+105840\alpha_1\alpha_2\beta_1^2+329280\beta_1\beta_3\\
&+588000\alpha_1^4+120\alpha_3^2+868476\alpha_2^2+1294290900)T^5+(1764\alpha_2^2\beta_2+18892440\beta_1^2\beta_2+147000\alpha_1^2\beta_3\\
&+614250\beta_3-382200\alpha_2^2\beta_1-5074440\alpha_1^2\beta_2-204418200\beta_1^3+1764\beta_2^3-162082200\alpha_1^2\beta_1\\
&+23966880\alpha_1\alpha_2\beta_1-907578000\beta_1+3360\alpha_2\alpha_3\beta_1-34894125\beta_2+3360\beta_1\beta_2\beta_3-268800\alpha_1\alpha_3\beta_1\\
&-5040\alpha_1\alpha_2\beta_3+5040\alpha_1\alpha_3\beta_2-52920\alpha_1\alpha_2\beta_2-435120\beta_1\beta_2^2-121800\beta_1^2\beta_3)T^4+(6162240\alpha_2\alpha_1\\
&+770280\alpha_2^2+96229140\alpha_1^2-278880\alpha_1\alpha_3-47040\alpha_2\alpha_3+240\alpha_3^2+12387200\alpha_1^4-1136800\alpha_1^3\alpha_2\\
&-11200\alpha_1^3\alpha_3+35280\alpha_1^2\alpha_2^2+33600\alpha_1\alpha_3\beta_1^2-5056800\alpha_1\alpha_2\beta_1^2-33600\alpha_1^2\beta_1\beta_3+823200\alpha_1^2\beta_1\beta_2\\
&+35280\beta_1^2\beta_2^2+11200\beta_1^3\beta_3-3096800\beta_1^3\beta_2+16922640\beta_2\beta_1+241080\beta_2^2+248815140\beta_1^2+240\beta_3^2\\
&-26880\beta_1\beta_3-47040\beta_2\beta_3+44060800\beta_1^4+56448000\alpha_1^2\beta_1^2+35280\alpha_1^2\beta_2^2+35280\alpha_2^2\beta_1^2+93767625)T^3\\
&+(31355100\beta_1-19547325\beta_2-1304100\beta_3-38808\beta_2^3-4704000\beta_1^5-22493940\beta_1^3-9408000\alpha_1^2\beta_1^3\\
&-4704000\alpha_1^4\beta_1-6938400\beta_1^2\beta_2-187320\beta_1^2\beta_3+630336\beta_1\beta_2^2-240\beta_1\beta_3^2+504\beta_2^2\beta_3-240\alpha_3^2\beta_1\\
&-504\alpha_2^2\beta_3-38808\alpha_2^2\beta_2-318696\alpha_2^2\beta_1-264600\alpha_1^2\beta_3-3104640\alpha_1^2\beta_2-22493940\alpha_1^2\beta_1-176400\alpha_1^4\beta_2\\
&+176400\beta_1^4\beta_2+20160\alpha_1\alpha_2\beta_3+949032\alpha_1\alpha_2\beta_2-3833760\alpha_1\alpha_2\beta_1+1008\alpha_2\alpha_3\beta_2+16800\alpha_2\alpha_3\beta_1\\
&-20160\alpha_1\alpha_3\beta_2+77280\alpha_1\alpha_3\beta_1+16800\beta_1\beta_2\beta_3+352800\alpha_1\alpha_2\beta_1^3+352800\alpha_1^3\alpha_2\beta_1)T^2+(504000\alpha_1\alpha_3\\
&+35280\alpha_2\alpha_3-1800\alpha_3^2-1999200\alpha_1^4+1540560\alpha_1^3\alpha_2+67200\alpha_1^3\alpha_3-199920\alpha_1^2\alpha_2^2-3360\alpha_1^2\alpha_2\alpha_3\\
&+7056\alpha_1\alpha_2^3-3360\alpha_2\alpha_3\beta_1^2+7056\alpha_2^2\beta_1\beta_2+67200\alpha_1\alpha_3\beta_1^2+7056\alpha_1\alpha_2\beta_2^2+1540560\alpha_1\alpha_2\beta_1^2\\
&-3360\alpha_1^2\beta_2\beta_3+67200\alpha_1^2\beta_1\beta_3+1540560\alpha_1^2\beta_1\beta_2+7056\beta_1\beta_2^3-199920\beta_1^2\beta_2^2+67200\beta_1^3\beta_3\\
&+1540560\beta_1^3\beta_2+7285320\beta_2\beta_1-22050\beta_2^2-31509450\beta_1^2-1800\beta_3^2+504000\beta_1\beta_3+35280\beta_2\beta_3\\
&-1999200\beta_1^4-3360\beta_1^2\beta_2\beta_3-564480\alpha_1\alpha_2\beta_1\beta_2-3998400\alpha_1^2\beta_1^2+82320\alpha_1^2\beta_2^2+82320\alpha_2^2\beta_1^2+196000\alpha_1^6\\
&+588000\alpha_1^4\beta_1^2+588000\alpha_1^2\beta_1^4+196000\beta_1^6-66976875+7285320\alpha_2\alpha_1-22050\alpha_2^2-31509450\alpha_1^2)T\\
&+5040\alpha_1\alpha_2\beta_3+3528\alpha_1\alpha_2\beta_2-1317120\alpha_1\alpha_2\beta_1-1008\alpha_2\alpha_3\beta_2-6720\alpha_2\alpha_3\beta_1-5040\alpha_1\alpha_3\beta_2\\
&-77280\alpha_1\alpha_3\beta_1-6720\beta_1\beta_2\beta_3+35280\alpha_1^2\alpha_2^2\beta_1-23520\alpha_1^3\alpha_2\beta_2-11200\alpha_1^3\alpha_3\beta_1-35280\alpha_1^2\beta_1\beta_2^2\\
&-196000\alpha_1\alpha_2\beta_1^3-196000\alpha_1^3\alpha_2\beta_1-11200\alpha_1\alpha_3\beta_1^3+8831025\beta_1+1091475\beta_2+28350\beta_3+1764\beta_2^3\\
&+313600\beta_1^5+3442740\beta_1^3+627200\alpha_1^2\beta_1^3+313600\alpha_1^4\beta_1-373380\beta_1^2\beta_2-39480\beta_1^2\beta_3-30576\beta_1\beta_2^2\\
&+240\beta_1\beta_3^2-504\beta_2^2\beta_3+240\alpha_3^2\beta_1+504\alpha_2^2\beta_3+1764\alpha_2^2\beta_2-34104\alpha_2^2\beta_1+37800\alpha_1^2\beta_3+943740\alpha_1^2\beta_2\\
&+3442740\alpha_1^2\beta_1+98000\alpha_1^4\beta_2-11760\alpha_2^2\beta_1^3-5600\beta_1^4\beta_3+5600\alpha_1^4\beta_3-98000\beta_1^4\beta_2+11760\beta_1^3\beta_2^2\\
&+70560\alpha_1\alpha_2\beta_1^2\beta_2\\
q_{15} &= -60\alpha_1T^2+21\alpha_2-270\alpha_1\\
q_{14} &= -540T^4+300\beta_1T^3+360T^2+(600\beta_1-105\beta_2)T+4020+225\alpha_1^2+225\beta_1^2\\
q_{13} &= (1050\alpha_1-105\alpha_2)T^2-1540\alpha_1+245\alpha_2-10\alpha_3\\
q_{12} &= -1260T^6+840\beta_1T^5+1260T^4+(2100\beta_1-315\beta_2)T^3+(28140+1575\alpha_1^2+1575\beta_1^2)T^2+(70\beta_3\\
&-875\beta_2+2380\beta_1)T+106260+5775\alpha_1^2-420\alpha_2\alpha_1-840\beta_2\beta_1+9975\beta_1^2\\
q_{11} &= 420\alpha_1T^6+(13650\alpha_1-735\alpha_2+)T^4+(180\alpha_3+34020\alpha_1-1890\alpha_2)T^2+(21000\alpha_1\beta_1-420\beta_2\alpha_1\\
&-2940\alpha_2\beta_1)T+131670\alpha_1-700\alpha_1^3-12915\alpha_2+180\alpha_3+2100\alpha_1\beta_1^2\\
q_{10} &= -1260T^8+1260\beta_1T^7+15120T^6-(21\beta_2+1680\beta_1)T^5+(4725\alpha_1^2+178920+4725\beta_1^2)T^4\\
&-(53060\beta_1+4130\beta_2+140\beta_3)T^3+(750960+2940\beta_2\beta_1+51450\alpha_1^2+55650\beta_1^2-7980\alpha_2\alpha_1)T^2\\
&+(58275\beta_2+18900\alpha_1^2\beta_1-269640\beta_1-1260\beta_3-6300\beta_1^3)T+411012-27160\alpha_2\alpha_1+399\alpha_2^2\\
&+245105\alpha_1^2+320\alpha_1\alpha_3-16520\beta_2\beta_1+77245\beta_1^2+280\beta_1\beta_3+483\beta_2^2
\end{aligned}
$$

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\begin{aligned}
q_9 \;=\;& 1050\alpha_1T^8+(43050\alpha_1-1365\alpha_2)T^6+(250\alpha_3-35525\alpha_2+112000\alpha_1)T^4+(14700\beta_2\alpha_1+273000\alpha_1\beta_1\\
&+2100\alpha_2\beta_1)T^3+(24500\alpha_1^3-1500\alpha_3-62475\alpha_2+2084250\alpha_1-73500\alpha_1\beta_1^2)T^2+(1000\alpha_3\beta_1\\
&-1400\alpha_1\beta_3+840\alpha_2\beta_2+84700\beta_2\alpha_1+700\alpha_2\beta_1-901600\alpha_1\beta_1)T-645750\alpha_1-121275\alpha_2\\
&+2250\alpha_3+87500\alpha_1^3-6300\alpha_1^2\alpha_2-6300\alpha_2\beta_1^2-10500\alpha_1\beta_1^2\\
q_8 \;=\;& 1050\beta_1T^9+63000T^8+(945\beta_2-18900\beta_1)T^7+(7875\beta_1^2+7875\alpha_1^2-142800)T^6+(602280\beta_1\\
&-630\beta_3+60795\beta_2)T^5+(5846400-6300\beta_2\beta_1+590625\alpha_1^2-291375\beta_1^2)T^4+(688275\beta_2\\
&-157500\alpha_1^2\beta_1+52500\beta_1^3-6300\beta_3-5978700\beta_1)T^3+(1283625\beta_1^2-1800\alpha_1\alpha_3-18900\alpha_2\alpha_1\\
&-233100\beta_2\beta_1+315\beta_2^2+4095\alpha_2^2+1478925\alpha_1^2-7867440)T^2+(94500\beta_1^3-15750\beta_3+527625\beta_2\\
&+31500\beta_1^2\beta_2+31500\alpha_1^2\beta_2+3102750\beta_1-787500\alpha_1^2\beta_1)T+4358760-274680\alpha_2\alpha_1+18795\alpha_2^2\\
&-32025\alpha_1^2+2400\alpha_1\alpha_3-420\alpha_2\alpha_3+21000\alpha_1^4-21000\beta_1^4-420\beta_2\beta_3-186480\beta_2\beta_1+15015\beta_2^2\\
&+339675\beta_1^2+4200\beta_1\beta_3\\
q_7 \;=\;& 1260\alpha_1T^{10}+(66150\alpha_1-945\alpha_2)T^8+(2930760\alpha_1-6300\alpha_2-360\alpha_3+)T^6+(22680\beta_2\alpha_1\\
&-1335600\alpha_1\beta_1-2520\alpha_2\beta_1)T^5+(9708300\alpha_1+853650\alpha_2+189000\alpha_1\beta_1^2-19800\alpha_3-63000\alpha_1^3)T^4\\
&+(327600\beta_2\alpha_1+4800\alpha_3\beta_1-10080\alpha_2\beta_2-394800\alpha_2\beta_1-10096800\alpha_1\beta_1)T^3+(50400\alpha_2\beta_1^2-55800\alpha_3\\
&+50400\alpha_1^2\alpha_2+875700\alpha_2-11333700\alpha_1-126000\alpha_1^3+2898000\alpha_1\beta_1^2)T^2+(26400\alpha_3\beta_1+3150000\alpha_1\beta_1\\
&-252000\alpha_1^3\beta_1-252000\alpha_1\beta_1^3+1021440\beta_2\alpha_1+840\alpha_2\beta_3-328440\alpha_2\beta_1-8400\alpha_1\beta_3-37800\alpha_2\beta_2\\
&-840\alpha_3\beta_2)T+330750\alpha_1-23625\alpha_2+27000\alpha_3-218400\alpha_1^3-130200\alpha_1^2\alpha_2-600\alpha_1^2\alpha_3+10080\alpha_1\alpha_2^2\\
&+2520\alpha_2\beta_1\beta_2-176400\alpha_1\beta_1\beta_2-218400\alpha_1\beta_1^2+7560\alpha_1\beta_2^2+46200\alpha_2\beta_1^2-600\alpha_3\beta_1^2\\
q_6 \;=\;& 1260T^{12}+420\beta_1T^{11}+118440T^{10}+(1365\beta_2-37800\beta_1+)T^9+(7875\alpha_1^2+4109700+7875\beta_1^2)T^8\\
&+(114660\beta_2-360\beta_3-2890440\beta_1)T^7+(17640\alpha_2\alpha_1+661500\beta_1^2-220500\alpha_1^2+40681200\\
&-17640\beta_2\beta_1)T^6+(88200\alpha_1^2\beta_1+930510\beta_2-29400\beta_1^3-49674240\beta_1+17640\beta_3)T^5+(352800\alpha_2\alpha_1\\
&+21307650\beta_1^2-235200\beta_2\beta_1+13230\beta_2^2-3417750\alpha_1^2-4410\alpha_2^2-8400\beta_1\beta_3-38698380)T^4\\
&+(882000\alpha_1^2\beta_1-3822000\beta_1^3+88200\beta_3+26151300\beta_1+2866500\beta_2)T^3+(214620\beta_2^2+55860\alpha_2^2\\
&+294000\beta_1^4-2628360\beta_2\beta_1-67200\beta_1\beta_3-294000\alpha_1^4-840\beta_2\beta_3-2454900\alpha_1^2-867300\beta_1^2\\
&+282240\alpha_2\alpha_1+32744040-840\alpha_2\alpha_3+8400\alpha_1\alpha_3)T^2+(294000\alpha_1^2\beta_2-17640\alpha_2^2\beta_1+4200\alpha_1^2\beta_3\\
&+294000\beta_1^2\beta_2+4200\beta_1^2\beta_3-35280\beta_1\beta_2^2+205800\beta_1^3-189000\beta_3+205800\alpha_1^2\beta_1-826875\beta_2\\
&-1323000\beta_1-17640\alpha_1\alpha_2\beta_2)T+32744220-736960\alpha_2\alpha_1+112210\alpha_2^2+4418575\alpha_1^2+60200\alpha_1\alpha_3\\
&-7840\alpha_2\alpha_3+100\alpha_3^2-29400\alpha_1^3\alpha_2+88200\alpha_1\alpha_2\beta_1^2-88200\alpha_1^2\beta_1\beta_2+29400\beta_1^3\beta_2-560560\beta_2\beta_1\\
&+138670\beta_2^2+4462675\beta_1^2+100\beta_3^2+47600\beta_1\beta_3-7840\beta_2\beta_3\\
q_5 \;=\;& 840\alpha_1T^{12}+(55230\alpha_1+21\alpha_2)T^{10}+(1225980\alpha_1+91035\alpha_2-630\alpha_3)T^8+(2520\beta_2\alpha_1-327600\alpha_1\beta_1\\
&-22680\alpha_2\beta_1)T^7+(88200\alpha_1\beta_1^2-18513180\alpha_1-2520\alpha_3+2403450\alpha_2-29400\alpha_1^3)T^6+(21168\alpha_2\beta_2\\
&-1040760\alpha_2\beta_1+17640\beta_2\alpha_1+18557280\alpha_1\beta_1-5040\alpha_1\beta_3-5040\alpha_3\beta_1)T^5+(88200\alpha_1^2\alpha_2+9856350\alpha_2\\
&+3439800\alpha_1+88200\alpha_2\beta_1^2-3087000\alpha_1^3+56700\alpha_3-4851000\alpha_1\beta_1^2)T^4+(1940400\alpha_1\beta_1-75600\alpha_1\beta_3\\
&+2257920\beta_2\alpha_1-7990920\alpha_2\beta_1+588000\alpha_1\beta_1^3+370440\alpha_2\beta_2+2520\alpha_2\beta_3+588000\alpha_1^3\beta_1-100800\alpha_3\beta_1\\
&-2520\alpha_3\beta_2)T^3+(1675800\alpha_2\beta_1^2-1940400\alpha_1\beta_1\beta_2+17640\alpha_1\beta_2^2-35280\alpha_1\alpha_2^2-705600\alpha_1^3\\
&+12600\alpha_3\beta_1^2-567000\alpha_3+6350400\alpha_1\beta_1^2+34728750\alpha_1-264600\alpha_1^2\alpha_2-694575\alpha_2+12600\alpha_1^2\alpha_3\\
&-52920\alpha_2\beta_1\beta_2)T^2+(1764000\alpha_1\beta_1+2504880\beta_2\alpha_1+264600\alpha_2\beta_2-117600\alpha_1^3\beta_2+226800\alpha_3\beta_1\\
&-588000\alpha_1^3\beta_1-1005480\alpha_2\beta_1+352800\alpha_1\beta_1^2\beta_2-117600\alpha_2\beta_1^3+22680\alpha_2\beta_3-22680\alpha_3\beta_2\\
&+352800\alpha_1^2\alpha_2\beta_1-352800\alpha_1\beta_3-588000\alpha_1\beta_1^3)T+6232800\alpha_1^3-1517040\alpha_1^2\alpha_2+46200\alpha_1^2\alpha_3\\
&+135240\alpha_1\alpha_2^2-3360\alpha_1\alpha_2\alpha_3-1764\alpha_2^3+5040\alpha_3\beta_1\beta_2-5040\alpha_2\beta_1\beta_3-17640\alpha_2\beta_1\beta_2-3360\alpha_1\beta_2\beta_3\\
&+84000\alpha_1\beta_1\beta_3-1622880\alpha_1\beta_1\beta_2+6232800\alpha_1\beta_1^2+152880\alpha_1\beta_2^2+105840\alpha_2\beta_1^2-1764\alpha_2\beta_2^2\\
&-37800\alpha_3\beta_1^2+49612500\alpha_1-99225\alpha_2-141750\alpha_3
\end{aligned}
$$

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\begin{aligned}
q_4 \quad = \quad & 1260T^{14} + 114660T^{12} + (735\beta_2 - 35700\beta_1)T^{11} + (4725\beta_1^2 + 4960620 + 4725\alpha_1^2)T^{10} + (250\beta_3 \\
& -3174500\beta_1 + 31675\beta_2)T^9 + (38448900 + 6300\alpha_2\alpha_1 - 417375\alpha_1^2 + 779625\beta_1^2)T^8 + (9000\beta_3 - 63000\beta_1^3 \\
& +189000\alpha_1^2\beta_1 - 24885000\beta_1 - 513450\beta_2)T^7 + (617400\beta_2\beta_1 - 8400\alpha_1\alpha_3 + 13230\alpha_2^2 + 175627620 \\
& +5843250\beta_1^2 + 205800\alpha_2\alpha_1 - 10885350\alpha_1^2 - 4410\beta_2^2)T^6 + (4851000\alpha_1^2\beta_1 + 6945750\beta_2 - 88200\alpha_1^2\beta_2 \\
& -88200\beta_1^2\beta_2 - 94500\beta_3 - 199773000\beta_1 - 441000\beta_1^3)T^5 + (90515250\beta_1^2 - 6174000\alpha_2\alpha_1 + 463050\alpha_2^2 \\
& -66150\beta_2^2 + 126000\beta_1\beta_3 - 3087000\beta_2\beta_1 - 22325940 - 126000\alpha_1\alpha_3 - 7827750\alpha_1^2)T^4 + (18301500\beta_1 \\
& +441000\beta_3 - 18669000\beta_1^3 + 3528000\alpha_1\alpha_2\beta_1 - 2352000\alpha_1^2\beta_2 - 88200\alpha_2^2\beta_1 + 1176000\beta_1^2\beta_2 \\
& +16611000\alpha_1^2\beta_1 - 21000\alpha_1^2\beta_3 + 88200\alpha_1\alpha_2\beta_2 + 18466875\beta_2 - 21000\beta_1^2\beta_3)T^3 + (300\alpha_3^2 + 147000\alpha_1^3\alpha_2 \\
& -33600\beta_1\beta_3 + 300\beta_3^2 + 1470000\beta_1^4 - 21000\alpha_2\alpha_3 - 474600\alpha_1\alpha_3 + 1888950\alpha_2^2 + 962850\beta_2^2 - 1470000\alpha_1^4 \\
& +71114925\alpha_1^2 - 16023000\beta_2\beta_1 - 5880000\alpha_2\alpha_1 + 36496425\beta_1^2 + 441000\alpha_1^2\beta_1\beta_2 - 441000\alpha_1\alpha_2\beta_1^2 \\
& +312558660 - 21000\beta_2\beta_3 - 147000\beta_1^3\beta_2)T^2 + (4086600\alpha_1^2\beta_2 - 617400\alpha_2^2\beta_1 - 105000\alpha_1^2\beta_3 + 992250\beta_3 \\
& +8400\alpha_1\alpha_3\beta_2 + 1646400\alpha_1\alpha_2\beta_1 - 8400\alpha_1\alpha_2\beta_3 + 8820\beta_2^3 - 529200\beta_1\beta_2^2 + 88200\alpha_1\alpha_2\beta_2 \\
& -18963000\alpha_1^2\beta_1 + 8820\alpha_2^2\beta_2 - 144868500\beta_1 + 5457375\beta_2 + 5733000\beta_1^2\beta_2 + 168000\alpha_1\alpha_3\beta_1 \\
& +63000\beta_1^2\beta_3 - 18963000\beta_1^3)T + 27907020 + 1837500\alpha_2\alpha_1 + 169050\alpha_2^2 + 53695425\alpha_1^2 - 222600\alpha_1\alpha_3 \\
& -21000\alpha_2\alpha_3 + 300\alpha_3^2 + 6566000\alpha_1^4 - 1225000\alpha_1^3\alpha_2 + 14000\alpha_1^3\alpha_3 + 44100\alpha_1^2\alpha_2^2 - 42000\alpha_1\alpha_3\beta_1^2 \\
& +147000\alpha_1\alpha_2\beta_1^2 + 42000\alpha_1^2\beta_1\beta_3 - 1911000\alpha_1^2\beta_1\beta_2 + 44100\beta_1^2\beta_2^2 - 14000\beta_1^3\beta_3 - 539000\beta_1^3\beta_2 \\
& -1029000\beta_2\beta_1 + 36750\beta_2^2 + 20399925\beta_1^2 + 300\beta_3^2 - 159600\beta_1\beta_3 - 21000\beta_2\beta_3 + 2254000\beta_1^4 \\
& +8820000\alpha_1^2\beta_1^2 + 44100\alpha_1^2\beta_2^2 + 44100\alpha_2^2\beta_1^2 \\
q_3 \quad = \quad & 300\alpha_1T^{14} + (315\alpha_2 + 24150\alpha_1)T^{12} + (69790\alpha_2 - 140\alpha_3 - 1233260\alpha_1)T^{10} + (1113000\alpha_1\beta_1 - 2100\beta_2\alpha_1 \\
& -14700\alpha_2\beta_1)T^9 + (1926225\alpha_2 - 157500\alpha_1\beta_1^2 - 6300\alpha_3 + 52500\alpha_1^3 - 52236450\alpha_1)T^8 + (4800\alpha_1\beta_3 \\
& -10080\alpha_2\beta_2 + 44839200\alpha_1\beta_1 - 428400\alpha_2\beta_1 - 142800\beta_2\alpha_1)T^7 + (19624500\alpha_2 - 263497500\alpha_1 \\
& -315000\alpha_3 - 9702000\alpha_1\beta_1^2 - 294000\alpha_1^3)T^6 + (2520\alpha_2\beta_3 - 4110120\alpha_2\beta_1 + 588000\alpha_1^3\beta_1 + 588000\alpha_1\beta_1^3 \\
& +151200\alpha_3\beta_1 + 75600\alpha_1\beta_3 - 1975680\beta_2\alpha_1 - 2520\alpha_3\beta_2 + 291589200\alpha_1\beta_1 - 476280\alpha_2\beta_2)T^5 \\
& +(226343250\alpha_1 + 116038125\alpha_2 - 147000\alpha_2\beta_1^2 - 21000\alpha_3\beta_1^2 - 21000\alpha_1^2\alpha_3 - 78204000\alpha_1\beta_1^2 \\
& -1827000\alpha_3 - 7644000\alpha_1^3 + 882000\alpha_1\beta_1\beta_2 + 88200\alpha_2\beta_1\beta_2 + 735000\alpha_1^2\alpha_2 - 88200\alpha_1\beta_2^2)T^4 \\
& +(5880000\alpha_1^3\beta_1 + 5880000\alpha_1\beta_1^3 - 1587600\alpha_2\beta_2 + 49039200\beta_2\alpha_1 + 25200\alpha_2\beta_3 - 78145200\alpha_2\beta_1 \\
& +1512000\alpha_3\beta_1 - 756000\alpha_1\beta_3 - 121716000\alpha_1\beta_1 - 25200\alpha_3\beta_2)T^3 + (17640\alpha_2^3 + 98196000\alpha_1\beta_1^2 \\
& -16800\alpha_1\alpha_2\alpha_3 + 176400\alpha_2\beta_1\beta_2 - 9349200\alpha_1^2\alpha_2 + 577489500\alpha_1 - 378000\alpha_3\beta_1^2 + 25578000\alpha_2\beta_1^2 \\
& +850500\alpha_3 + 98196000\alpha_1^3 + 17640\alpha_2\beta_2^2 + 672000\alpha_1\beta_1\beta_3 - 16800\alpha_1\beta_2\beta_3 + 50604750\alpha_2 + 58800\alpha_1\alpha_2^2 \\
& +294000\alpha_1^2\alpha_3 - 34927200\alpha_1\beta_1\beta_2 - 117600\alpha_1\beta_2^2)T^2 + (8232000\alpha_1\beta_1^2\beta_2 + 882000\alpha_1\beta_3 - 32144000\alpha_1\beta_1^3 \\
& -252000\alpha_3\beta_1 + 63000\alpha_2\beta_3 - 1323000\alpha_2\beta_2 - 63000\alpha_3\beta_2 - 4189500\beta_2\alpha_1 - 84000\alpha_1^2\alpha_3\beta_1 \\
& +2940000\alpha_1^2\alpha_2\beta_1 + 784000\alpha_1^3\beta_2 + 28000\alpha_3\beta_1^3 - 19183500\alpha_2\beta_1 + 28000\alpha_1^3\beta_3 - 4508000\alpha_2\beta_1^3 \\
& -84000\alpha_1\beta_1^2\beta_3 - 32144000\alpha_1^3\beta_1 - 227115000\alpha_1\beta_1)T + 294000\alpha_2\beta_1^4 + 2940000\alpha_1^5 - 294000\alpha_1^4\alpha_2 \\
& +5880000\alpha_1^3\beta_1^2 - 588000\alpha_1^3\beta_1\beta_2 + 2940000\alpha_1\beta_1^4 - 588000\alpha_1\beta_1^3\beta_2 + 992250\alpha_1 - 6449625\alpha_2 \\
& -283500\alpha_3 + 21457100\alpha_1^3 + 1636600\alpha_1^2\alpha_2 - 144200\alpha_1^2\alpha_3 + 374360\alpha_1\alpha_2^2 - 5600\alpha_1\alpha_2\alpha_3 - 400\alpha_1\alpha_3^2 \\
& -11760\alpha_2^3 + 840\alpha_2^2\alpha_3 + 1680\alpha_2\beta_2\beta_3 + 16800\alpha_3\beta_1\beta_2 - 16800\alpha_2\beta_1\beta_3 + 346920\alpha_2\beta_1\beta_2 - 5600\alpha_1\beta_2\beta_3 \\
& -123200\alpha_1\beta_1\beta_3 + 842800\alpha_1\beta_1\beta_2 + 21457100\alpha_1\beta_1^2 + 27440\alpha_1\beta_2^2 - 400\alpha_1\beta_3^2 + 793800\alpha_2\beta_1^2 \\
& -11760\alpha_2\beta_2^2 - 21000\alpha_3\beta_1^2 - 840\alpha_3\beta_2^2
\end{aligned}
$$

$$\begin{aligned}
q_2 \quad = \quad & 540T^{16} - 60\beta_1 T^{15} + 56160T^{14} + (105\beta_2 - 16800\beta_1)T^{13} + (1105440 + 1575\beta_1^2 + 1575\alpha_1^2)T^{12} \\
& +(172620\beta_1 + 180\beta_3 - 22050\beta_2)T^{11} + (7980\beta_2\beta_1 - 196350\beta_1^2 + 353850\alpha_1^2 - 24877440 \\
& -2940\alpha_2\alpha_1)T^{10} + (15300\beta_3 - 1598625\beta_2 + 24500\beta_1^3 - 73500\alpha_1^2\beta_1 + 58300200\beta_1)T^9 + (9883125\alpha_1^2 \\
& -1800\beta_1\beta_3 + 315\alpha_2^2 + 642600\beta_2\beta_1 - 27910575\beta_1^2 + 4095\beta_2^2 - 138600\alpha_2\alpha_1 - 56062440)T^8 + (433800\beta_3 \\
& +424676700\beta_1 - 2142000\alpha_1^2\beta_1 - 50400\beta_1^2\beta_2 + 4914000\beta_1^3 - 20418300\beta_2 - 50400\alpha_1^2\beta_2)T^7 + (294000\alpha_1^4 \\
& -73500\alpha_2^2 - 294000\beta_1^4 + 5803560\beta_2\beta_1 - 134400\beta_1\beta_3 + 102561900\alpha_1^2 - 8400\alpha_1\alpha_3 + 2426051040 \\
& +840\alpha_2\alpha_3 + 840\beta_2\beta_3 - 5221440\alpha_2\alpha_1 - 242300100\beta_1^2 + 191100\beta_2^2)T^6 + (12600\beta_1^2\beta_3 - 24078600\alpha_1^2\beta_1 \\
& -35280\beta_1\beta_2^2 + 17640\alpha_2^2\beta_1 + 12600\alpha_1^2\beta_3 + 529200\beta_1^2\beta_2 + 1701000\beta_3 - 14387625\beta_2 + 46481400\beta_1^3 \\
& +2822400\alpha_1\alpha_2\beta_1 - 2293200\alpha_1^2\beta_2 - 52920\alpha_1\alpha_2\beta_2 - 2182950000\beta_1)T^5 + (2940000\alpha_1^4 - 159600\beta_1\beta_3 \\
& +533400\alpha_1\alpha_3 + 782006925\alpha_1^2 - 441000\alpha_1\alpha_2\beta_1^2 - 58800\alpha_2\alpha_3 + 300\alpha_3^2 + 3795356160 + 300\beta_3^2 \\
& -2940000\beta_1^4 + 3344250\beta_2^2 - 70795200\alpha_2\alpha_1 - 147000\beta_1^3\beta_2 + 1888950\alpha_2^2 + 1273942425\beta_1^2 \\
& -63739200\beta_2\beta_1 + 441000\alpha_1^2\beta_1\beta_2 + 147000\alpha_1^3\alpha_2 - 58800\beta_2\beta_3)T^4 + (16800\beta_1\beta_2\beta_3 - 2712811500\beta_1 \\
& -17640\alpha_2^2\beta_2 + 36691200\beta_1^2\beta_2 - 8820000\alpha_1^2\beta_2 + 5386500\beta_3 - 940800\beta_1\beta_2^2 - 42000\beta_1^2\beta_3 \\
& -168000\alpha_1\alpha_3\beta_1 - 176400\alpha_1\alpha_2\beta_2 - 764400\alpha_2^2\beta_1 + 45511200\alpha_1\alpha_2\beta_1 - 504798000\alpha_1^2\beta_1 + 16800\alpha_2\alpha_3\beta_1 \\
& -179597250\beta_2 + 126000\alpha_1^2\beta_3 - 17640\beta_2^3 - 504798000\beta_1^3)T^3 + (88200\alpha_1^2\alpha_2^2 + 35412300\alpha_2\alpha_1 \\
& +644940450\beta_1^2 + 88200\alpha_2^2\beta_1^2 + 98916300\beta_2\beta_1 - 554400\alpha_1\alpha_3 + 88200\beta_2\beta_3 + 110250000\beta_1^4 \\
& -2646000\alpha_1^3\alpha_2 - 2822400\beta_1\beta_3 - 1278900\alpha_2^2 - 1800\beta_3^2 + 882000\alpha_1^2\beta_1\beta_2 + 88200\alpha_1^2\beta_2^2 - 480000960 \\
& +3483900\beta_2^2 + 39690000\alpha_1^4 + 88200\beta_1^2\beta_2^2 + 160722450\alpha_1^2 + 149940000\alpha_1^2\beta_1^2 + 88200\alpha_2\alpha_3 \\
& -6174000\beta_1^3\beta_2 - 1800\alpha_3^2 - 9702000\alpha_1\alpha_2\beta_1^2)T^2 + (67473000\beta_1 + 42170625\beta_2 + 1984500\beta_3 + 88200\beta_2^3 \\
& -11760000\beta_1^5 - 71868300\beta_1^3 - 23520000\alpha_1^2\beta_1^3 - 11760000\alpha_1^4\beta_1 - 9584400\beta_1^2\beta_2 + 684600\beta_1^2\beta_3 \\
& -2269680\beta_1\beta_2^2 + 1200\beta_1\beta_3^2 - 2520\beta_2^2\beta_3 + 1200\alpha_3^2\beta_1 + 2520\alpha_2^2\beta_3 + 88200\alpha_2^2\beta_2 + 358680\alpha_2^2\beta_1 \\
& +567000\alpha_1^2\beta_3 - 176400\alpha_1^2\beta_2 - 71868300\alpha_1^2\beta_1 - 294000\alpha_1^4\beta_2 + 294000\beta_1^4\beta_2 - 50400\alpha_1\alpha_2\beta_3 \\
& -2628360\alpha_1\alpha_2\beta_2 - 9408000\alpha_1\alpha_2\beta_1 - 5040\alpha_2\alpha_3\beta_2 - 33600\alpha_2\alpha_3\beta_1 + 50400\alpha_1\alpha_3\beta_2 + 117600\alpha_1\alpha_3\beta_1 \\
& -33600\beta_1\beta_2\beta_3 + 588000\alpha_1\alpha_2\beta_1^3 + 588000\alpha_1^3\alpha_2\beta_1)T + 55814060 - 9261000\alpha_2\alpha_1 + 165375\alpha_2^2 \\
& +16570575\alpha_1^2 - 491400\alpha_1\alpha_3 + 2700\alpha_3^2 + 4704000\alpha_1^4 - 617400\alpha_1^3\alpha_2 - 84000\alpha_1^3\alpha_3 + 323400\alpha_1^2\alpha_2^2 \\
& +8400\alpha_1^2\alpha_2\alpha_3 - 17640\alpha_1\alpha_2^3 + 8400\alpha_2\alpha_3\beta_1^2 - 17640\alpha_2^2\beta_1\beta_2 - 84000\alpha_1\alpha_3\beta_1^2 - 17640\alpha_1\alpha_2\beta_2^2 \\
& -617400\alpha_1\alpha_2\beta_1^2 + 8400\alpha_1^2\beta_2\beta_3 - 84000\alpha_1^2\beta_1\beta_3 - 617400\alpha_1^2\beta_1\beta_2 - 17640\beta_1\beta_2^3 + 323400\beta_1^2\beta_2^2 \\
& -84000\beta_1^3\beta_3 - 617400\beta_1^3\beta_2 - 9261000\beta_2\beta_1 + 165375\beta_2^2 + 16570575\beta_1^2 + 2700\beta_3^2 - 491400\beta_1\beta_3 \\
& +4704000\beta_1^4 + 8400\beta_1^2\beta_2\beta_3 + 705600\alpha_1\alpha_2\beta_1\beta_2 + 9408000\alpha_1^2\beta_1^2 - 29400\alpha_1^2\beta_2^2 - 29400\alpha_2^2\beta_1^2 \\
& +490000\alpha_1^6 + 1470000\alpha_1^4\beta_1^2 + 1470000\alpha_1^2\beta_1^4 + 490000\beta_1^6
\end{aligned}$$

$$
\begin{aligned}
q_1 \;=\;& 45\alpha_1 T^{16} + (105\alpha_2 + 4350\alpha_1)T^{14} + (5845\alpha_2 + 70\alpha_3 + 395080\alpha_1)T^{12} + (420\alpha_2\beta_1 + 2940\beta_2\alpha_1 \\
& -147000\alpha_1\beta_1)T^{11} + (22308930\alpha_1 + 8820\alpha_3 + 18900\alpha_1\beta_1^2 - 6300\alpha_1^3 - 326655\alpha_2)T^{10} + (1000\alpha_1\beta_3 \\
& +840\alpha_2\beta_2 + 118300\beta_2\alpha_1 + 252700\alpha_2\beta_1 - 1400\alpha_3\beta_1 - 15097600\alpha_1\beta_1)T^9 + (264275550\alpha_1 - 31500\alpha_2\beta_1^2 \\
& +3433500\alpha_1\beta_1^2 + 261450\alpha_3 - 31500\alpha_1^2\alpha_2 + 787500\alpha_1^3 - 13426875\alpha_2)T^8 + (840\alpha_2\beta_3 - 252000\alpha_1^3\beta_1 \\
& -67200\alpha_3\beta_1 + 83160\alpha_2\beta_2 - 2808960\beta_2\alpha_1 - 189428400\alpha_1\beta_1 + 27600\alpha_1\beta_3 + 9600360\alpha_2\beta_1 - 840\alpha_3\beta_2 \\
& -252000\alpha_1\beta_1^3)T^7 + (2646000\alpha_1\beta_1\beta_2 + 6820800\alpha_1^3 + 735000\alpha_1^2\alpha_2 + 42100800\alpha_1\beta_1^2 - 17640\alpha_1\beta_2^2 \\
& -1911000\alpha_2\beta_1^2 + 4200\alpha_1^2\alpha_3 + 812873250\alpha_1 + 4200\alpha_3\beta_1^2 + 1071000\alpha_3 - 20782125\alpha_2 - 17640\alpha_2\beta_1\beta_2 \\
& -35280\alpha_1\alpha_2^2)T^6 + (687960\alpha_2\beta_2 - 2940000\alpha_1^3\beta_1 - 370440\alpha_2\beta_1 + 24801840\beta_2\alpha_1 + 117600\alpha_2\beta_1^3 \\
& -352800\alpha_1^2\alpha_2\beta_1 - 806400\alpha_1\beta_3 + 478800\alpha_3\beta_1 + 117600\alpha_1^3\beta_2 - 352800\alpha_1\beta_1^2\beta_2 - 2940000\alpha_1\beta_1^3 \\
& -42840\alpha_3\beta_2 - 838958400\alpha_1\beta_1 + 42840\alpha_2\beta_3)T^5 + (10466400\alpha_2\beta_1^2 + 282240000\alpha_1\beta_1^2 - 8820\alpha_2^3 \\
& +588000\alpha_1\beta_1\beta_3 + 70560000\alpha_1^3 - 264600\alpha_1\alpha_2^2 - 146853000\alpha_1 - 176400\alpha_1\beta_2^2 + 10064250\alpha_3 \\
& -85829625\alpha_2 - 1764000\alpha_1^2\alpha_2 - 12230400\alpha_1\beta_1\beta_2 - 8820\alpha_2\beta_2^2 + 315000\alpha_1^2\alpha_3 - 88200\alpha_2\beta_1\beta_2 \\
& -8400\alpha_2\beta_1\beta_3 + 8400\alpha_3\beta_1\beta_2 - 273000\alpha_3\beta_1^2)T^4 + (28665000\alpha_1\beta_1 + 37800\alpha_3\beta_2 - 84000\alpha_1^2\alpha_3\beta_1 \\
& -43904000\alpha_1\beta_1^3 + 64077300\alpha_2\beta_1 + 9261000\alpha_2\beta_2 + 2142000\alpha_1\beta_3 - 43904000\alpha_1^3\beta_1 - 22182300\beta_2\alpha_1 \\
& -84000\alpha_1\beta_1^2\beta_3 + 28000\alpha_1^3\beta_3 + 28000\alpha_3\beta_1^3 - 6552000\alpha_3\beta_1 + 1960000\alpha_1^3\beta_2 + 4704000\alpha_1\beta_1^2\beta_2 \\
& -3332000\alpha_2\beta_1^3 - 588000\alpha_1^2\alpha_2\beta_1 - 37800\alpha_2\beta_3)T^3 + (94759875\alpha_2 + 6520500\alpha_3 - 40557300\alpha_1^3 \\
& +16434600\alpha_1^2\alpha_2 + 1188600\alpha_1^2\alpha_3 - 5180280\alpha_1\alpha_2^2 - 33600\alpha_1\alpha_2\alpha_3 + 1200\alpha_1\alpha_3^2 + 141120\alpha_2^3 - 2520\alpha_2^2\alpha_3 \\
& -5040\alpha_2\beta_2\beta_3 - 100800\alpha_3\beta_1\beta_2 + 100800\alpha_2\beta_1\beta_3 - 7391160\alpha_2\beta_1\beta_2 - 33600\alpha_1\beta_2\beta_3 - 1142400\alpha_1\beta_1\beta_3 \\
& +18992400\alpha_1\beta_1\beta_2 - 40557300\alpha_1\beta_1^2 + 2210880\alpha_1\beta_2^2 + 1200\alpha_1\beta_3^2 - 2557800\alpha_2\beta_1^2 + 141120\alpha_2\beta_2^2 \\
& +2331000\alpha_3\beta_1^2 + 2520\alpha_3\beta_2^2 + 294000\alpha_2\beta_1^4 + 2940000\alpha_1^5 - 294000\alpha_1^4\alpha_2 + 5880000\alpha_1^3\beta_1^2 - 588000\alpha_1^3\beta_1\beta_2 \\
& +2940000\alpha_1\beta_1^4 - 588000\alpha_1\beta_1^3\beta_2 - 290729250\alpha_1)T^2 + (3763200\alpha_1^3\beta_2 - 3763200\alpha_2\beta_1^3 - 3763200\alpha_1^2\alpha_2\beta_1 \\
& +3763200\alpha_1\beta_1^2\beta_2 - 420000\alpha_3\beta_1^3 + 420000\alpha_1^3\beta_3 - 35280\alpha_2^3\beta_1 + 35280\alpha_1\beta_2^3 + 39028500\beta_2\alpha_1 \\
& -39028500\alpha_2\beta_1 - 189000\alpha_3\beta_2 + 189000\alpha_2\beta_3 - 420000\alpha_1^2\alpha_3\beta_1 + 420000\alpha_1\beta_1^2\beta_3 - 1764000\alpha_1\beta_1\beta_2^2 \\
& +1764000\alpha_2\beta_1^2\beta_2 - 16800\alpha_2\beta_1^2\beta_3 - 35280\alpha_2\beta_1\beta_2^2 + 16800\alpha_3\beta_1^2\beta_2 - 1764000\alpha_1^2\alpha_2\beta_2 - 16800\alpha_1^2\alpha_2\beta_3 \\
& +16800\alpha_1^2\alpha_3\beta_2 + 1764000\alpha_1\alpha_2^2\beta_1 + 35280\alpha_1\alpha_2^2\beta_2 + 2079000\alpha_1\beta_3 - 2079000\alpha_3\beta_1)T + 1568000\alpha_1^5 \\
& -490000\alpha_1^4\alpha_2 - 28000\alpha_1^4\alpha_3 + 58800\alpha_1^3\alpha_2^2 + 3136000\alpha_1^3\beta_1^2 - 980000\alpha_1^3\beta_1\beta_2 - 56000\alpha_1^3\beta_1\beta_3 \\
& -58800\alpha_1^3\beta_2^2 + 352800\alpha_1^2\alpha_2\beta_1\beta_2 - 176400\alpha_1\alpha_2^2\beta_1^2 + 1568000\alpha_1\beta_1^4 - 980000\alpha_1\beta_1^3\beta_2 - 56000\alpha_1\beta_1^3\beta_3 \\
& +176400\alpha_1\beta_1^2\beta_2^2 + 28000\alpha_3\beta_1^4 + 44155125\alpha_1 + 5457375\alpha_2 + 141750\alpha_3 + 17213700\alpha_1^3 - 1866900\alpha_1^2\alpha_2 \\
& -197400\alpha_1^2\alpha_3 - 152880\alpha_1\alpha_2^2 - 33600\alpha_1\alpha_2\alpha_3 + 1200\alpha_1\alpha_3^2 + 8820\alpha_2^3 - 2520\alpha_2^2\alpha_3 - 5040\alpha_2\beta_2\beta_3 \\
& +25200\alpha_3\beta_1\beta_2 - 25200\alpha_2\beta_1\beta_3 + 17640\alpha_2\beta_1\beta_2 - 33600\alpha_1\beta_2\beta_3 - 386400\alpha_1\beta_1\beta_3 - 6585600\alpha_1\beta_1\beta_2 \\
& +17213700\alpha_1\beta_1^2 - 170520\alpha_1\beta_2^2 + 1200\alpha_1\beta_3^2 + 4718700\alpha_2\beta_1^2 + 8820\alpha_2\beta_2^2 + 189000\alpha_3\beta_1^2 + 2520\alpha_3\beta_2^2 \\
& +490000\alpha_2\beta_1^4 - 117600\alpha_2\beta_1^3\beta_2
\end{aligned}
$$

$$
\begin{aligned}
q_0 \;=\;& 90T^{18} - 15\beta_1 T^{17} + 11070T^{16} - (3180\beta_1 + 21\beta_2)T^{15} + (225\beta_1^2 + 225\alpha_1^2 + 552120)T^{14} - (3115\beta_2 \\
&+10\beta_3 + 201040\beta_1)T^{13} + (15698760 + 840\alpha_2\alpha_1 + 22575\beta_1^2 + 420\beta_2\beta_1 + 5775\alpha_1^2)T^{12} + (2100\alpha_1^2\beta_1 \\
&-700\beta_1^3 - 32025\beta_2 - 1740\beta_3 - 9056460\beta_1)T^{11} + (24745\alpha_1^2 + 2439605\beta_1^2 + 35980\alpha_2\alpha_1 + 483\alpha_2^2 \\
&+173346012 + 280\alpha_1\alpha_3 - 29260\beta_2\beta_1 + 399\beta_2^2 + 320\beta_1\beta_3)T^{10} + (6300\alpha_1^2\beta_2 + 52500\alpha_1^2\beta_1 + 6300\beta_1^2\beta_2 \\
&-833175\beta_2 - 353500\beta_1^3 - 58950\beta_3 - 112130550\beta_1)T^9 + (1912680\alpha_2\alpha_1 + 420\beta_2\beta_3 - 6542025\alpha_1^2 \\
&+1660282260 + 420\alpha_2\alpha_3 + 400680\beta_2\beta_1 - 21000\alpha_1^4 - 26985\beta_2^2 + 33696075\beta_1^2 - 15645\alpha_2^2 + 21000\beta_1^4 \\
&+8400\alpha_1\alpha_3 + 13800\beta_1\beta_3)T^8 + (487200\alpha_1^2\beta_2 + 7560\alpha_2^2\beta_1 + 53006625\beta_2 - 218400\beta_1^2\beta_2 - 2315470500\beta_1 \\
&+2520\alpha_1\alpha_2\beta_2 - 600\alpha_1^2\beta_3 - 5069400\beta_1^3 - 600\beta_1^2\beta_3 - 1233000\beta_3 + 1986600\alpha_1^2\beta_1 - 705600\alpha_1\alpha_2\beta_1 \\
&+10080\beta_1\beta_2^2)T^7 + (325113775\alpha_1^2 - 5320\alpha_2\alpha_3 + 665000\alpha_1\alpha_3 - 294000\alpha_1^4 + 1567190275\beta_1^2 \\
&-24162880\alpha_2\alpha_1 + 394450\alpha_2^2 - 29400\alpha_1^3\alpha_2 + 100\alpha_3^2 + 294000\beta_1^4 + 728000\beta_1\beta_3 - 5320\beta_2\beta_3 + 262150\beta_2^2 \\
&+1086513720 + 88200\alpha_1\alpha_2\beta_1^2 + 29400\beta_1^3\beta_2 + 100\beta_3^2 - 88200\alpha_1^2\beta_1\beta_2 - 61824280\beta_2\beta_1)T^6 + (1764\beta_2^3 \\
&-570271800\beta_1^3 - 336000\alpha_1\alpha_3\beta_1 + 5040\alpha_1\alpha_3\beta_2 + 24890040\beta_1^2\beta_2 + 2099160\alpha_1^2\beta_2 - 358591800\alpha_1^2\beta_1 \\
&-5040\alpha_1\alpha_2\beta_3 + 3360\alpha_2\alpha_3\beta_1 - 223440\beta_1\beta_2^2 + 17640\alpha_1\alpha_2\beta_2 + 163800\alpha_1^2\beta_3 + 1764\alpha_2^2\beta_2 \\
&+22790880\alpha_1\alpha_2\beta_1 - 241080\alpha_2^2\beta_1 + 3360\beta_1\beta_2\beta_3 - 691929000\beta_1 + 120822975\beta_2 - 172200\beta_1^2\beta_3 \\
&-7701750\beta_3)T^5 + (7078050\alpha_2^2 + 510159825\alpha_1^2 + 5430600\alpha_1\alpha_3 - 226800\alpha_2\alpha_3 + 2700\alpha_3^2 + 27244000\alpha_1^4 \\
&-2597000\alpha_1^3\alpha_2 - 14000\alpha_1^3\alpha_3 + 44100\alpha_1^2\alpha_2^2 + 42000\alpha_1\alpha_3\beta_1^2 - 6321000\alpha_1\alpha_2\beta_1^2 - 42000\alpha_1^2\beta_1\beta_3 \\
&-735000\alpha_1^2\beta_1\beta_2 + 44100\beta_1^2\beta_2^2 + 14000\beta_1^3\beta_3 - 4459000\beta_1^3\beta_2 - 172298700\beta_2\beta_1 + 1389150\beta_2^2 \\
&+613133325\beta_1^2 + 2700\beta_3^2 + 8139600\beta_1\beta_3 - 226800\beta_2\beta_3 + 113876000\beta_1^4 + 141120000\alpha_1^2\beta_1^2 + 44100\alpha_1^2\beta_2^2 \\
&+44100\alpha_2^2\beta_1^2 + 3510704520 - 141649200\alpha_2\alpha_1)T^4 + (1984500\beta_3 - 64680\beta_2^3 - 11760000\beta_1^5 \\
&-318553900\beta_1^3 - 23520000\alpha_1^2\beta_1^3 - 11760000\alpha_1^4\beta_1 + 68756800\beta_1^2\beta_2 - 3168200\beta_1^2\beta_3 + 815360\beta_1\beta_2^2 \\
&-400\beta_1\beta_3^2 + 840\beta_2^2\beta_3 - 400\alpha_3^2\beta_1 - 840\alpha_2^2\beta_3 - 64680\alpha_2^2\beta_2 - 4294360\alpha_2^2\beta_1 + 735000\alpha_1^2\beta_3 \\
&-27165600\alpha_1^2\beta_2 - 318553900\alpha_1^2\beta_1 - 294000\alpha_1^4\beta_2 + 294000\beta_1^4\beta_2 - 33600\alpha_1\alpha_2\beta_3 + 5109720\alpha_1\alpha_2\beta_2 \\
&+95922400\alpha_1\alpha_2\beta_1 + 1680\alpha_2\alpha_3\beta_2 + 95200\alpha_2\alpha_3\beta_1 + 33600\alpha_1\alpha_3\beta_2 - 3903200\alpha_1\alpha_3\beta_1 + 95200\beta_1\beta_2\beta_3 \\
&+588000\alpha_1\alpha_2\beta_1^3 + 588000\alpha_1^3\alpha_2\beta_1 - 2343694500\beta_1 - 118573875\beta_2)T^3 + (9040500\alpha_2\alpha_1 + 2767275\alpha_2^2 \\
&+323594775\alpha_1^2 - 352800\alpha_1\alpha_3 - 315000\alpha_2\alpha_3 + 9900\alpha_3^2 + 30282000\alpha_1^4 - 3204600\alpha_1^3\alpha_2 + 168000\alpha_1^3\alpha_3 \\
&-499800\alpha_1^2\alpha_2^2 - 8400\alpha_1^2\alpha_2\alpha_3 + 17640\alpha_1\alpha_2^3 - 8400\alpha_2\alpha_3\beta_1^2 + 17640\alpha_2^2\beta_1\beta_2 + 840000\alpha_1\alpha_3\beta_1^2 \\
&+17640\alpha_1\alpha_2\beta_2^2 - 19668600\alpha_1\alpha_2\beta_1^2 - 8400\alpha_1^2\beta_2\beta_3 - 168000\alpha_1^2\beta_1\beta_3 + 5027400\alpha_1^2\beta_1\beta_2 + 17640\beta_1\beta_2^3 \\
&-499800\beta_1^2\beta_2^2 + 504000\beta_1^3\beta_3 - 11436600\beta_1^3\beta_2 + 43438500\beta_2\beta_1 + 4354875\beta_2^2 + 723140775\beta_1^2 + 9900\beta_3^2 \\
&-1108800\beta_1\beta_3 - 315000\beta_2\beta_3 + 82026000\beta_1^4 - 8400\beta_1^2\beta_2\beta_3 - 2822400\alpha_1\alpha_2\beta_1\beta_2 + 112308000\alpha_1^2\beta_1^2 \\
&+911400\alpha_1^2\beta_2^2 + 911400\alpha_2^2\beta_1^2 + 490000\alpha_1^6 + 1470000\alpha_1^4\beta_1^2 + 1470000\alpha_1^2\beta_1^4 + 490000\beta_1^6 + 725582810)T^2 \\
&+(88200\beta_1^2\beta_3 - 599760\beta_1\beta_2^2 - 3600\beta_1\beta_3^2 + 7560\beta_2^2\beta_3 - 3600\alpha_3^2\beta_1 - 7560\alpha_2^2\beta_3 - 132300\alpha_2^2\beta_2 \\
&-546840\alpha_2^2\beta_1 - 63000\alpha_1^2\beta_3 - 7629300\alpha_1^2\beta_2 - 102797100\alpha_1^2\beta_1 - 686000\alpha_1^4\beta_2 - 58800\alpha_2^2\beta_1^3 - 28000\beta_1^4\beta_3 \\
&+28000\alpha_1^4\beta_3 + 686000\beta_1^4\beta_2 + 58800\beta_1^3\beta_2^2 + 352800\alpha_1\alpha_2\beta_1^2\beta_2 + 25200\alpha_1\alpha_2\beta_3 - 52920\alpha_1\alpha_2\beta_2 \\
&-3528000\alpha_1\alpha_2\beta_1 + 15120\alpha_2\alpha_3\beta_2 + 100800\alpha_2\alpha_3\beta_1 - 25200\alpha_1\alpha_3\beta_2 + 151200\alpha_1\alpha_3\beta_1 + 100800\beta_1\beta_2\beta_3 \\
&+176400\alpha_1^2\alpha_2^2\beta_1 - 117600\alpha_1^3\alpha_2\beta_2 - 56000\alpha_1^3\alpha_3\beta_1 - 176400\alpha_1^2\beta_1\beta_2^2 + 1372000\alpha_1\alpha_2\beta_1^3 + 1372000\alpha_1^3\alpha_2\beta_1 \\
&-56000\alpha_1\alpha_3\beta_1^3 - 190015875\beta_1 - 32248125\beta_2 - 992250\beta_3 - 132300\beta_2^3 - 10192000\beta_1^5 - 102797100\beta_1^3 \\
&-20384000\alpha_1^2\beta_1^3 - 10192000\alpha_1^4\beta_1 - 11157300\beta_1^2\beta_2)T + 490000\beta_1^6 + 5581406 + 3528000\alpha_2\alpha_1 \\
&+474075\alpha_2^2 + 14564025\alpha_1^2 + 88200\alpha_1\alpha_3 + 31500\alpha_2\alpha_3 + 900\alpha_3^2 + 5517400\alpha_1^4 + 891800\alpha_1^3\alpha_2 - 2800\alpha_1^3\alpha_3 \\
&+98980\alpha_1^2\alpha_2^2 - 2800\alpha_1^2\alpha_2\alpha_3 + 400\alpha_1^2\alpha_3^2 + 5880\alpha_1\alpha_2^3 - 1680\alpha_1\alpha_2^2\alpha_3 + 1764\alpha_2^4 - 2800\alpha_2\alpha_3\beta_1^2 \\
&+1680\alpha_2^2\beta_1\beta_3 + 5880\alpha_2^2\beta_1\beta_2 + 1680\alpha_1\alpha_3\beta_2^2 - 2800\alpha_1\alpha_3\beta_1^2 + 5880\alpha_1\alpha_2\beta_2^2 \\
&+891800\alpha_1\alpha_2\beta_1^2 - 2800\alpha_1^2\beta_2\beta_3 - 2800\alpha_1^2\beta_1\beta_3 + 891800\alpha_1^2\beta_1\beta_2 + 5880\beta_1\beta_2^3 + 400\beta_1^2\beta_3^2 + 98980\beta_1^2\beta_2^2 \\
&-2800\beta_1^3\beta_3 + 891800\beta_1^3\beta_2 + 3528000\beta_2\beta_1 + 474075\beta_2^2 + 14564025\beta_1^2 + 900\beta_3^2 + 88200\beta_1\beta_3 + 31500\beta_2\beta_3 \\
&+5517400\beta_1^4 + 1764\beta_2^4 - 2800\beta_1^2\beta_2\beta_3 - 1680\beta_1\beta_2^2\beta_3 + 23520\alpha_1\alpha_2\beta_1\beta_2 + 11034800\alpha_1^2\beta_1^2 + 87220\alpha_1^2\beta_2^2 \\
&+400\alpha_1^2\beta_3^2 + 87220\alpha_2^2\beta_1^2 + 3528\alpha_2^2\beta_2^2 + 400\alpha_3^2\beta_1^2 - 3360\alpha_1\alpha_2\beta_2\beta_3 - 3360\alpha_2\alpha_3\beta_1\beta_2 + 490000\alpha_1^6 \\
&+1470000\alpha_1^4\beta_1^2 + 1470000\alpha_1^2\beta_1^4.
\end{aligned}
$$

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