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On the exact WKB analysis of singularly perturbed ordinary differential equations at an irregular singular point

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Abstract

We announce the results of [K]. We present decomposition of WKB solutions to monomially summable series at an irregular singular point of singularly perturbed ordinary differential equations when the equations satisfy some stability conditions (Assumption I and II).

1 Introduction

The purpose of this report is to announce the results of [K]. The main object studied there is the following singularly perturbed ordinary differential equation:

$$\left(\varepsilon^n \frac{d^n}{dx^n} + a_{n-1}(x,\varepsilon)\varepsilon^{n-1}\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_0(x,\varepsilon)\right)\psi = 0, \tag{1}$$

where $a_k(x,\varepsilon) \in \mathbb{C}\{x,\varepsilon\}[x^{-1}]$. WKB solutions of (1) are formal solutions of the following form:

$$\psi(x,\varepsilon) = \exp\left[\int^x S(x,\varepsilon)dx\right], \tag{2}$$

where $S(x,\varepsilon) = \varepsilon^{-1}S_{-1}(x) + S_0(x) + \cdots$ is a formal power series solution (in ε-variable) of the Riccati equation associated with (1). As is well known, $S(x,\varepsilon)$ is a divergent series in general. To give analytical meaning to such a divergent series, we employ Borel resummation method. (See [KT] for details.) Therefore, it is indespensable to know where the solutions are Borel summable, especially when we discuss the Stokes phenomena for such solutions.

Before discussing the general situation, let us first consider the following Schrödinger equation

$$\left(\varepsilon^2 \frac{d^2}{dx^2} - R(x)\right)\psi = 0, \tag{3}$$

where $R(x)$ is a rational function. The Borel summability of solutions of the Riccati equation associated with (3) except on the Stokes curves is verified in [KoS]. (See also [CDK], [DLS] and [KKo].) The proof is given by solving the Borel transformed Riccati equation along its characteristic curve. However, it is difficult to apply their method directly to the case of higher order equations because of the complexity of their Stokes geometry. (Cf. [AKT] and [H].) And, the global aspects of summability structure of WKB solutions are not well-known.

On the other hand, let us consider the following Schrödinger equation

$$\left(\varepsilon^2 \frac{d^2}{dx^2} - (x - \varepsilon^2 x^2)\right)\psi = 0. \tag{4}$$

It is known that WKB solutions of (4) are $(4, 1)$-summable. (See [Su] and [SuT].) Therefore, the Borel resummation method is not applicable to such multi-summable solutions; we have to modify the resummation method. (Cf. [B1].) Then, analysis of (1) will become complicated. Judging by current circumstances of our study, it seems important to know the condition that guarantees that the Borel resummation method works appropriately in the analysis of (1).

In this article, we focus our attention on an irregular singular point of (1) since its structure seems important when we determine the multi-summability type of WKB solutions of (1). Related to the problem, in [BM], summability structure of formal power series solutions of inhomogeneous linear singularly perturbed system of ordinary differential equations was studied. Further, in [CMS], monomial summability of formal power series solutions for nonlinear cases was discussed. The aim of this study is to apply their theories to the case where the Newton polygon of the symbol of (1) has several line segments; we construct the decomposition of WKB solutions to monomially summable series (Theorem 1) when the Newton polygon satisfies some stability conditions under the perturbation (Assumption I and II). As an application of it, we discuss the Borel summability (in ε-variable) of formal power series solutions of the Riccati equation associated with (1).

2 Main results

In this section, we explain the core results of [K]. Let

$$P(x,\varepsilon,\xi)=\xi^n+a_{n-1}(x,\varepsilon)\xi^{n-1}+\cdots+a_0(x,\varepsilon) \tag{5}$$

be the symbol of (1) and let $a_l(x,0)$ $(l=0,1,\cdots,n-1)$ behave as

$$a_l(x,0)=c_l x^{\nu_l}+O(x^{\nu_l+1}) \tag{6}$$

at $x=0$, where $c_l(\neq 0)\in\mathbb{C}$ and $\nu_l\in\mathbb{Z}$. (When $a_l(x,0)\equiv 0$, we regard ν_l as $+\infty$.) We set $c_n=1$ and $\nu_n=0$. Then, the Newton polygon $\mathcal{N}_0$ of $P(x,0,\xi)$ is defined by the convex hull of the set

$$\bigcup_{0\le l\le n}\bigcup_{j\in\mathbb{N}}\{(l,\nu_l+j)\}. \tag{7}$$

Let α_p $(p=1,2,\cdots)$ be the slopes of the line segments of $\mathcal{N}_0$ in decreasing order and let j_p $(p=1,2,\cdots)$ be the corresponding lengths of the line segments projected onto the l-axis. Therefore, they satisfy $\alpha_p=\left(\nu_{n-|\vec{j}|_{p-1}}-\nu_{n-|\vec{j}|_p}\right)/j_p$, where

$$|\vec{j}|_p=\sum_{i=1}^{p} j_i. \tag{8}$$

We assume that m (≥ 1) of the slopes are strictly greater than 1, i.e., $\alpha_1>\alpha_2>\cdots>\alpha_m>1\ge\alpha_{m+1}>\cdots$. (When all the slopes are strictly greater than 1, we regard $\alpha_{m+1}=1$ and $j_{m+1}=0$.) Then, (1) has an irregular singular point at $x=0$.

Now, we assume the following conditions:

Assumption I. Line segments of $\mathcal{N}_0$ corresponding to the slopes α_p

$(p = 1, \cdots, m)$ are stable near $\varepsilon = 0$, i.e., $x^{\sigma_l} a_l(x, \varepsilon)$ $(l = 0, \cdots, n-1)$ are bounded at $x = \varepsilon = 0$, where

(9)

$$\sigma_l = \begin{cases} \displaystyle\sum_{i=1}^{p-1} (\alpha_i - \alpha_p) j_i + \alpha_p (n - l) & \text{when} \ \ n - |\vec{j}|_p \leq l \leq n - |\vec{j}|_{p-1}, \\ \displaystyle\sum_{i=1}^{m} (\alpha_i - \alpha_m) j_i + \alpha_m (n - l) & \text{when} \ \ l \leq n - |\vec{j}|_m. \end{cases}$$

Assumption II. These line segments of $\mathcal{N}_0$ are non-degenerate, i.e., the discriminant of

$$D_p(\beta) = \sum_l c_l \beta^{l-n+|\vec{j}|_p} \tag{10}$$

does not vanish, where the sum is taken over $\{l \mid n - |\vec{j}|_p \leq l \leq n - |\vec{j}|_{p-1}, \nu_l = \sigma_l\}$.

Remark 1. The equation (4) has an irregular singular point at $x = \infty$. It violates Assumption I there.

Then, roots $\xi_p^{(j)}(x)$ $(j = 1, \cdots, j_p)$ of $P(x, 0, \xi) = 0$ corresponding to the line segment with the slope α_p of $\mathcal{N}_0$ behave as

$$\xi_p^{(j)}(x) = \beta_p^{(j)} x^{-\alpha_p} + o(x^{-\alpha_p}) \tag{11}$$

at $x = 0$, where $\beta_p^{(j)} (\neq 0)$ $(j = 1, \cdots, j_p)$ are the distinct roots of $D_p(\beta) = 0$. Applying a ramified coordinate transformation, we may assume that α_p $(p = 1, \cdots, m)$ are positive integers strictly greater than 1.

Let us first consider the case where $\mathcal{N}_0$ has only one line segment and Assumption I and II are satisfied. Since (1) can be rewritten in the form

$$-\varepsilon\frac{d}{dx}\Psi=\begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 0 \\ 0 & \cdots & \cdots & 0 & -1 \\ a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \end{pmatrix}\Psi, \tag{12}$$

employing a splitting lemma (cf. [B2]), we find that (1) has WKB solutions $\psi^{(j)}(x,\varepsilon)$ ($j=1,\cdots,n$) of the following form:

$$\psi^{(j)}(x,\varepsilon)=T^{(j)}(x,\varepsilon)\exp[\varepsilon^{-1}\int^x \Xi^{(j)}(\tilde{x},\varepsilon)d\tilde{x}], \tag{13}$$

where $T^{(j)}(x,\varepsilon)$ and $\Xi^{(j)}(x,\varepsilon)$ ($j=1,\cdots,n$) are formal series in $\mathbb{C}[[x,\varepsilon]][x^{-1}]$ and $\Xi^{(j)}(x,\varepsilon)$ satisfies

$$\Xi^{(j)}(x,0)=\xi_1^{(j)}(x). \tag{14}$$

As a consequence of [CMS], we find that $T^{(j)}(x,\varepsilon)$ and $\Xi^{(j)}(x,\varepsilon)$ can be written by linear combinations of 1-summable series in $x^{r_1}\varepsilon$ ($r_1=\alpha_1-1$) with the coefficients in $\mathbb{C}[x,x^{-1}]$. Here, the summability with respect to the monomial $x^{r_1}\varepsilon$ is a kind of the summability property of the formal series

$$\varinjlim_{R\to 0}\varprojlim_{N\to\infty}\mathscr{O}_R/(x^{r_1}\varepsilon)^N\mathscr{O}_R, \tag{15}$$

where $\mathscr{O}_R$ is the space of holomorphic functions on $\{(x,\varepsilon)\in\mathbb{C}^2 \mid |x|,|\varepsilon| < R\}$. See [CMS] for details. (See also [M] for the notion of strong asymptotic developability.) Further, the singular directions of $T^{(j)}(x,\varepsilon)$ and $\Xi^{(j)}(x,\varepsilon)$ are estimated as

$$\arg\left(x^{r_1}\varepsilon\right)=\arg\left(\beta_1^{(i)}-\beta_1^{(j)}\right)\qquad(i\neq j), \tag{16}$$

i.e., they are 1-summable in $x^{r_1}\varepsilon$ except for the directions (16) at least.

Therefore, we find that there exist non-negative real continuous functions $d^{(j)}(\theta)$ $(j = 1, \cdots, j_1)$ on S^1 that do not vanish on

$$S^1 \setminus \bigcup_{i \neq j} \{ (\beta_1^{(i)} - \beta_1^{(j)}) / |\beta_1^{(i)} - \beta_1^{(j)}| \} \tag{17}$$

and $T^{(j)}(x, \varepsilon)$ and $\Xi^{(j)}(x, \varepsilon)$ are 1-summable in ε-variable in a direction $\arg \varepsilon = 0$ on

$$\{ x \in \mathbb{C} \setminus \{0\} \mid |x| < d^{(j)}\big((x/|x|)^{r_1}\big) \}. \tag{18}$$

Remark 2. We distinguish the words "1-summable in a direction $\arg \varepsilon = 0$" and "Borel summable"; we say a formal power series is 1-summable in a direction $\arg \varepsilon = 0$ (resp., Borel summable) when its formal 1-Borel transform converges and analytically extends to a sectorial region (resp., strip-shaped region) containing the positive real axis of the Borel plane and its exponential size there is at most 1. (Compare the definition in [B1] and [KT].) Hence, 1-summability in a direction $\arg \varepsilon = 0$ implies Borel summability.

Now, let us consider the case where $\mathcal{N}_0$ has several line segments. In such a case, following a similar discussion in [B2], we obtain

Theorem 1 ([K]). *Suppose that* (1) *satisfies Assumption I and II. Then, there exist a transformation* $T(x, \varepsilon) = T_1(x, \varepsilon) \cdots T_m(x, \varepsilon) \in \mathrm{GL}\big(n, \mathbb{C}[[x, \varepsilon]][x^{-1}]\big)$ *such that* (12) *is transformed by* $\Psi = T\Phi$ *to the following system :*

$$\varepsilon \frac{d}{dx} \Phi = \begin{pmatrix} \Xi_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \Xi_m & 0 \\ 0 & \cdots & 0 & \widetilde{\Xi} \end{pmatrix} \Phi, \tag{19}$$

where elements of

(20)

$$\Xi_p(x,\varepsilon) = \begin{pmatrix} \Xi_p^{(1)} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Xi_p^{(j_p)} \end{pmatrix} \quad \Big(\in \mathrm{GL}\big(j_p, \mathbb{C}[[x,\varepsilon]][x^{-1}]\big)\Big)$$

and $T_p(x,\varepsilon)$ $(p = 1, \cdots, m)$ *are linear combinations of* 1*-summable series in* $x^{r_p}\varepsilon$ $(r_p = \alpha_p - 1)$ *with the coefficients in* $\mathbb{C}[x, x^{-1}]$*. Further,* $\Xi_p^{(j)}(x,\varepsilon)$ *satisfies*

$$\Xi_p^{(j)}(x,0) = \xi_p^{(j)}(x). \tag{21}$$

Remark 3. The element $\widetilde{\Xi}$ originate from the roots of $P(x,0,\xi) = 0$ corresponding to the line segments with the slopes α_p $(p \geq m+1)$. We also find that the elements of $\widetilde{\Xi}$ are written by linear combinations of 1-summable series in $x^{r_m}\varepsilon$ with the coefficients in $\mathbb{C}[x, x^{-1}]$.

Remark 4. The proof of Theorem 1 proceeds by the induction on the number of line segments with the slope $\alpha_p > 1$. When we reduce the number, we use a transformation of the equation to a meromorphic form in the category of monomially summable series. Similar transformations are also discussed by M. Canalis-Durand, J. Mozo-Fernández and R. Schäfke ([S]).

Therefore, we find WKB solutions $\psi_p^{(j)}(x,\varepsilon)$ of (1) of the following form:

$$\psi_p^{(j)}(x,\varepsilon) = \widetilde{T}_p^{(j)}(x,\varepsilon)\exp[\varepsilon^{-1}\int^x \Xi_p^{(j)}(\tilde{x},\varepsilon)d\tilde{x}], \tag{22}$$

where $\widetilde{T}_p^{(j)}(x,\varepsilon)$ is the $(1, j+|\vec{j}|_{p-1})$-element of $T(x,\varepsilon)$. More precisely, we can estimate the singular directions of $\Xi_p^{(j)}(x,\varepsilon)$ and $\widetilde{T}_p^{(j)}(x,\varepsilon)$ as

follows: $\Xi_p^{(j)}(x,\varepsilon)$ is 1-summable in $x^{r_p}\varepsilon$ on
(23)
$$S^1 \setminus \bigcup_{i\neq j}\big\{\big(\beta_p^{(i)}-\beta_p^{(j)}\big)/\big|\beta_p^{(i)}-\beta_p^{(j)}\big|\big\}\setminus\big\{-\beta_p^{(j)}/\big|\beta_p^{(j)}\big| \;\big|\; \text{if}\;\; j_{p+1}\neq 0\big\},$$

where $\big\{-\beta_p^{(j)}/\big|\beta_p^{(j)}\big| \;\big|\; \text{if}\;\; j_{p+1}\neq 0\big\} = \big\{-\beta_p^{(j)}/\big|\beta_p^{(j)}\big|\big\}$ if $j_{p+1}\neq 0$, otherwise we regard it as the empty set. Components of $\widetilde{T}_p^{(j)}(x,\varepsilon)$ originating from $T_q(x,\varepsilon)$ are 1-summable in $x^{r_q}\varepsilon$ on

$$S^1 \setminus \bigcup_{1\leq i\leq j_q}\{\beta_q^{(i)}/|\beta_q^{(i)}|\} \tag{24}$$

when $q>p$, 1-summable in $x^{r_p}\varepsilon$ on (23) when $q=p$ and convergent when $q<p$. Therefore, we find that there exist non-negative real continuous functions $d_{p,q}^{(j)}(\theta)$ $(q=1,\cdots,p)$ on S^1 that do not vanish on (23) when $q=p$ and on (24) when $q<p$ such that $\Xi_p^{(j)}(x,\varepsilon)$ and $\widetilde{T}_p^{(j)}(x,\varepsilon)$ are 1-summable in ε-variable in a direction $\arg\varepsilon=0$ on

$$\big\{x\in\mathbb{C}\setminus\{0\} \;\big|\; |x|<d_p^{(j)}\big(x/|x|\big)\big\}, \tag{25}$$

where

$$d_p^{(j)}\big(x/|x|\big) = \min_{1\leq q\leq p}\big\{d_{p,q}^{(j)}\big((x/|x|)^{r_q}\big)\big\}. \tag{26}$$

Now, let us define $S_p^{(j)}(x,\varepsilon)$ by

$$S_p^{(j)}(x,\varepsilon) = \frac{d}{dx}\log\big(\psi_p^{(j)}(x,\varepsilon)\big). \tag{27}$$

Then, $S_p^{(j)}(x,\varepsilon)$ is a formal series solution in ε-variable with the coefficients $\mathbb{C}\{x\}[x^{-1}]$ of the Riccati equation associated with (1) and, from the construction of $\Xi_p^{(j)}(x,\varepsilon)$ and $\widetilde{T}_p^{(j)}(x,\varepsilon)$, we find $S_p^{(j)}(x,\varepsilon)$ satisfies

$$S_p^{(j)}(x,\varepsilon) = \varepsilon^{-1}\xi_p^{(j)}(x) + O(\varepsilon^0). \tag{28}$$

Here we note that such formal series solutions of the Riccati equation are uniquely determined. Further, as a consequence of the above discussion, we obtain

Theorem 2 ([K])**.** *Let* $S_p^{(j)}(x,\varepsilon)$ *be a formal series solution in* ε*-variable of the Riccati equation associated with* (1) *in the form of* (28)*. Then, it is* 1*-summable in* ε*-variable in a direction* $\arg\varepsilon = 0$ *on* (25)*.*

Remark 5. The singular directions of $\widetilde{T}_p^{(j)}(x,\varepsilon)$ are corresponding to the directions $\operatorname{Im}\,\omega_{p,q}^{(i,j)} = 0$ and $\operatorname{Re}\,\omega_{p,q}^{(i,j)} > 0$ $\big((q,i) \neq (p,j)\big)$, where

$$\omega_{p,q}^{(i,j)} = \big(\xi_q^{(i)}(x) - \xi_p^{(j)}(x)\big)dx. \tag{29}$$

Here we note that $\omega_{p,q}^{(i,j)}$ $\big((q,i) \neq (p,j)\big)$ play a central role when we determine Stokes geometry for (1). (Cf. [H].)

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