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Naofumi HONDA and Takahiro KAWAI

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京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

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§ 1. Introduction

The purpose of this article is to call forth the interest of specialists in microlocal analysis in the computer-assisted study of the Landau-Nakanishi geometry by showing concrete examples which we have encountered in making the effort with Henry P. Stapp to elucidate the concrete contents of Sato's postulate ([2]) on the analytic structure of the *S-matrix* near the *3-particle threshold.* For the convenience of the reader we first recall the definition of a *Feynman graph G* and the *Landau-Nakanishi variety* (hereafter abbreviated as $\mathscr{L}\mathscr{N}$ variety) $\mathscr{L}(G)$ associated with G.

Definition 1.1. A Feynman graph G is a graph that consists of finitely many points $V_1, V_2, \ldots, V_{n'}$ (called vertices), finitely many line segments $L_1, L_2, \ldots, L_N$ (called internal lines) and finitely many half-lines $L_1^e, L_2^e, \ldots, L_n^e$ (called external lines), where each of the end-points W_ℓ^+ and W_ℓ^- of L_ℓ ($\ell = 1, 2, \ldots, N$) coincides with some V_j ($j = 1, 2, \ldots, n'$) satisfying the condition

$$W_\ell^+ \neq W_\ell^-, \tag{1.1}$$

and the (unique) end-point of L_r^e ($r = 1, 2, \ldots, n$) coincides with some V_j ($j = 1, 2, \ldots, n'$).

In this article we assume that each internal line and each external line are oriented (and specified with an arrow like "$\rightarrow\!\!-$" if necessary). Using this orientation we define the *incidence number* $[j : \ell]$ for a pair of a vertex V_j and an internal line L_ℓ by the following rule:

$$[j : \ell] = \begin{cases} +1 & \text{when the internal line } L_\ell \text{ ends at the vertex } V_j, \\ -1 & \text{when } L_\ell \text{ starts from } V_j, \\ 0 & \text{neither of the end-points of } L_\ell \text{ coincides with } V_j. \end{cases} \tag{1.2}$$

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*Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo, 060-0810, Japan.
**Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606-8502, Japan.

The incidence number $[j : r]$ for a pair of a vertex V_j and an external line L^e_r is defined in a similar manner.

We also assume that a ν-dimensional real (or complex if so specified) vector $p_r = (p_{r,0}, \dots, p_{r,\nu-1})$ $(r = 1, 2, \dots, n)$ is assigned to each external line L^e_r and a strictly positive number m_ℓ $(\ell = 1, 2, \dots, N)$ is assigned to each internal line L_ℓ.

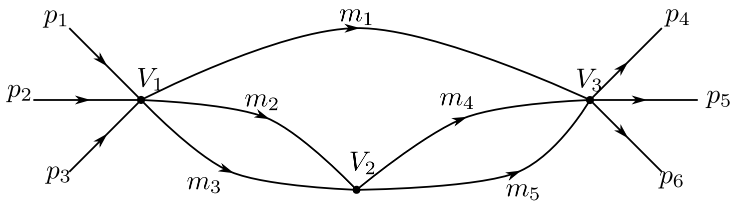

Figure 1. An example of a Feynman graph.

Remark 1.2. In this article we assume, for the sake of simplicity, that all constants m_ℓ are the same and we denote it by the number m. That is, we consider only the so-called equal mass case.

Remark 1.3. Unless otherwise stated, we assume $\nu = 2$ in what follows.

Remark 1.4. In this article we do **not** assume

$$p_r^2 (= p_{r,0}^2 - p_{r,1}^2) = m^2. \tag{1.3}$$

In passing we note that, here and in what follows, for ν-dimensional vector $k = (k_0, k_1, \dots, k_{\nu-1})$ the scalar k^2 stands for $k_0^2 - \sum_{\rho=1}^{\nu-1} k_\rho^2$.

In order to write down the defining equation of the $\mathscr{L}\mathscr{N}$ variety, we introduce the following numbers $j^\pm(\ell)$ and $j(r)$ for an internal line L_ℓ and an external line L^e_r:

$$[j^\pm(\ell) : \ell] = \pm 1, \tag{1.4}$$

$$[j(r) : r] \neq 0. \tag{1.5}$$

Definition 1.5. (i) The Landau-Nakanishi variety $\mathscr{L}(G)$ associated with a Feynman graph G is, by definition, the totality of $(p, \sqrt{-1}u)$ in $\mathbb{R}^{\nu n} \times (\sqrt{-1}\mathbb{R}^{\nu n})$ that satisfies the following equations for some $(\alpha_1, \dots, \alpha_N; k_1, \dots, k_N; v_1, \dots, v_{n'}; a) \in \mathbb{R}^N \times \mathbb{R}^{\nu N} \times$

$\mathbb{R}^{\nu n'} \times \mathbb{R}^{\nu}$:

$$\begin{cases} \sum_{r=1}^{n}[j:r]p_r + \sum_{\ell=1}^{N}[j:\ell]k_\ell = 0 & (j=1,2,\dots,n'), \\ \alpha_\ell(k_\ell^2 - m^2) = 0, \ \ k_{\ell,0} > 0 & (\ell = 1,2,\dots,N), \\ v_{j^+(\ell)} - v_{j^-(\ell)} = \alpha_\ell k_\ell & (\ell = 1,2,\dots,N), \\ u_r = -[j(r):r](v_{j(r)} + a) & (r = 1,2,\dots,n). \end{cases} \tag{1.6}$$

(ii) If $\alpha_\ell \geq 0$ ($\ell = 1,2,\dots,N$) in (1.6), $\mathscr{L}(G)$ is designated as $\mathscr{L}^+(G)$ and called the positive-α $\mathscr{LN}$ variety associated with G.
(iii) If $\alpha_\ell > 0$ ($\ell = 1,2,\dots,N$), then $\mathscr{L}^+(G)$ is designated as $\mathscr{L}^\oplus(G)$.

Remark 1.6. (i) If we formally define the Feynman integral $F_G(p)$ associated with G by

$$\int \cdots \int \frac{\prod_{j=1}^{n'} \delta^\nu\big(\sum_{r=1}^{n}[j:r]p_r + \sum_{\ell=1}^{N}[j:\ell]k_\ell\big)}{\prod_{\ell=1}^{N}(k_\ell^2 - m^2 + \sqrt{-1}0)} \prod_{\ell=1}^{N} d^\nu k_\ell, \tag{1.7}$$

then it is known ([2]) that under some moderate conditions $F_G(p)$ is well-defined as a microfunction and that it is supported by $\mathscr{L}^+(G)$. Thus $\mathscr{L}^+(G)$ is a variety in $\sqrt{-1}S^*\mathbb{R}^{\nu n}$. Denoting by π the canonical projection map from $\sqrt{-1}S^*\mathbb{R}^{\nu n}$ to $\mathbb{R}^{\nu n}$, we denote $\pi(\mathscr{L}^+(G))$ by $L^+(G)$. It is also called the positive-α $\mathscr{LN}$ variety. When we want to emphasize that we are dealing with the object projected down to the base manifold, we sometimes use somewhat loose expression "*(positive-α) LN surface*". As we will show in Section 2 and Section 3, some higher codimensional component of an LN "surface" is of particular interest.
(ii) When $F_G(p)$ is well-defined, it has the form

$$f_G(p)\delta^\nu\big(\sum_{j,r}[j:r]p_r\big). \tag{1.8}$$

The vector a in the last equation of (1.6) is a counterpart of the factor $\delta^\nu(\sum[j:r]p_r)$. The factor $f_G(p)$ is called a *Feynman amplitude* (or *function*).

Concerning the concrete figure of $L^+(G)$ the book of Eden et al. ([1]) is a good introduction. Thanks to the progress of computers, mathematicians can now make the figures in [1] much more precise so that they may give a fresh impetus to study the

Landau-Nakanishi geometry, if they put sufficiently enough energy and time into the study of the subject. Actually, as we show in Section 2, the detailed description of $L^+(G)$ gives rise to interesting mathematical problems even for a very simple graph G. Section 3 is devoted to showing what kind of anomalies is observed when G contains what we call the *non-external vertices*. The study of such graphs is not only challenging but also important in our future study of the analytic structure of the S-matrix near the 3-particle threshold, which will make essential use of the *Borel resummation*.

§ 2. LN surface $L(G)$ and its positive-α part $L^+(G)$ when G is an ice-cream cone graph

As one of the most basic graph that is relevant to the 3-particle threshold we consider the so-called *ice-cream cone graph*, that is,

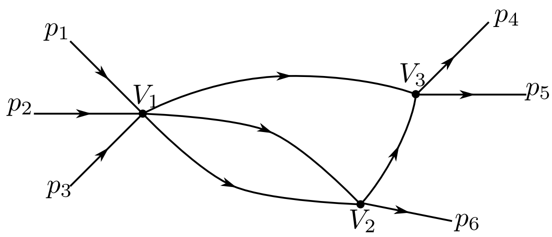

Figure 2. The ice-cream cone graph G_1.

The reason of our interest in $L^+(G_1)$ is twofold. First, $L^+(G_1)$ touches the 3-particle threshold $3PT$, and we know ([2], [3])

$$f_{G_1}(p)|_{3PT} = a(p)f_{G_0}(p) + b(p) \tag{2.1}$$

holds at a generic point of $3PT$, where $a(p)$ and $b(p)$ are holomorphic functions and the graph G_0 is described in the figure below:

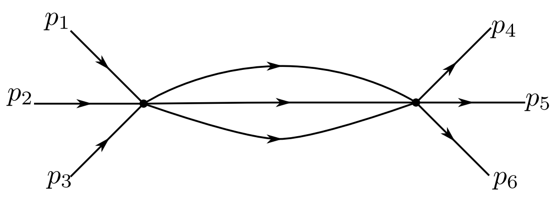

Figure 3. The Feynman graph G_0.

Second, if we consider a point p where the following configuration of Fig. 4 is realized, that is, if all internal lines are parallel keeping each vertex distinct, then we find

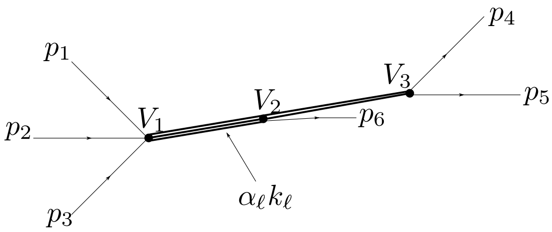

Figure 4. The configuration of vectors v_j's and $\alpha_\ell k_\ell$'s.

$$p_4 + p_5 = 2p_6, \tag{2.2}$$

$$p_6^2 = m^2. \tag{2.3}$$

The totality N_- of such points covers only a tiny portion of $L^+(G_1)$, but as Fig. 5 shows [1], N_- is a crucially important part of the singularity that $L^+(G_1)$ presents; the singularity is commonly known as "Whitney's umbrella", and N_- belongs to its most singular part. Thus explicitly writing down the holonomic system that $f_{G_1}(p)$ satisfies near N_- is a charming problem in microlocal analysis.

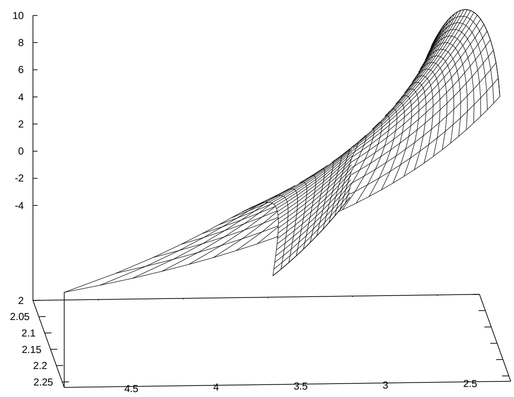

Figure 5. The "non-zero α" LN surface of G_1 with $\nu = 2$ and $m = 1$.

[1]The surface appearing in the figure is analytically isomorphic to the one defined by the following equations of parameters $s > 0$ and $t > 0$: $x = s + \dfrac{1}{s}$, $y = \dfrac{s^2t + 3s}{st - 1}$ and $z = t$. It has only one pinch point singularity N_- and also has a self-intersection curve corresponding to a shank of an umbrella.

§ 3. Truss-bridge graphs

As our eventual purpose is to understand the analytic structure of the S-matrix near the 3-particle threshold, it is natural to try to study the concrete figure of the positive-α LN surface $L^+(G)$ associated with Feynman graph G when it touches 3-particle threshold. One such a graph is G_1 studied in Section 2. One can readily note that $L^+(T_2)$ contains $L^+(G_1)$ and also note that $L^+(T_1)$ touches 3-particle threshold, where the truss-bridge graph T_1 (resp. T_2) is given in Fig. 6 (resp. Fig. 7) below.

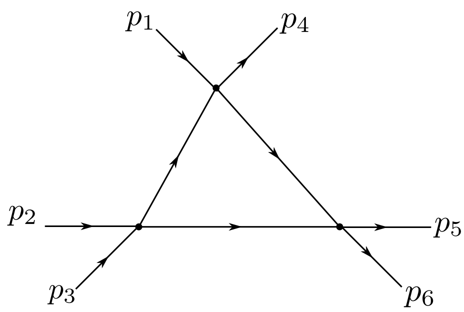

Figure 6. The truss-bridge graph T_1.

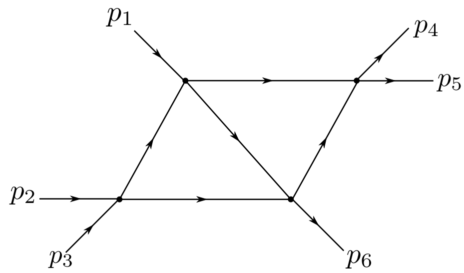

Figure 7. The truss-bridge graph T_2.

Thus it is natural to study $L^+(T_3)$, as the next target, where

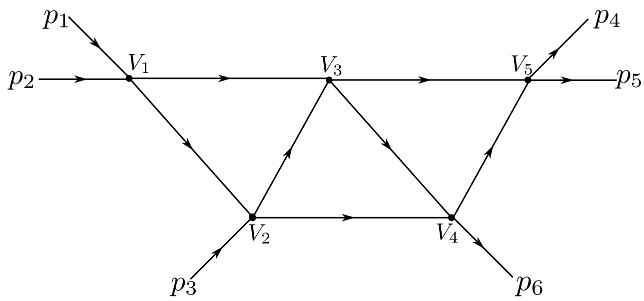

Figure 8. The truss-bridge graph T_3.

Interestingly enough, there is no reference which concretely describes $L^+(T_3)$, as far as we know. And, the actual figure shown in Fig. 9 is highly intriguing; the LN surface in the figure consists of two irreducible components. One is isomorphic to the surface defined by the following equations of parameters $s > 0$ and $t > 0$:

$$
\begin{aligned}
&x = s + 1/s, \\
&y = -\frac{\left(\left(b^2 - ab\right)s^2 + (a-b)\,s + 1\right)t^2 + \left((a-2b)\,s^2 + s\right)t + s^2}{\left(\left(b^2 - ab\right)s - b\right)t^2 + \left((a-2b)\,s + 1\right)t + s}, \qquad (3.1)\\
&z = bt^2/(bt-1),
\end{aligned}
$$

where a and b are some positive constants. This surface has two pinch point singularities and two self-intersection curves which form a combination of two umbrellas. Another component is the curve, i.e., the higher codimensional component, defined by equations of $s > 0$:

$$x = s + 1/s, \quad y = -\frac{as^2 - 3s}{as + 1}, \quad z = -b/(s^2 - bs). \tag{3.2}$$

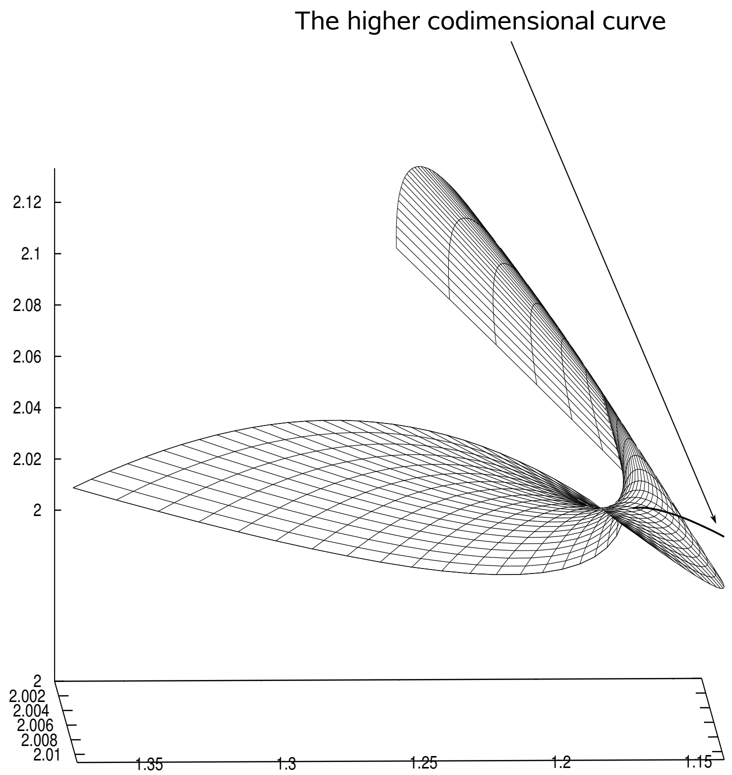

Figure 9. A generic slice of the "non-zero α" LN surface of T_3 in a transversally intersecting 3-dimensional space ($\nu = 2$ and $m = 1$).

Among other things, the existence of a higher codimensional component of the LN surface that corresponds to the configuration described in Fig. 10 was what we had not anticipated before the actual computation.

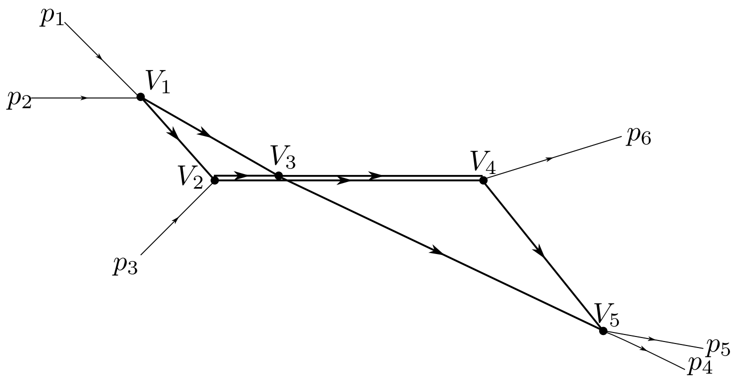

Figure 10. The configuration of vectors v_j's.

Note that the vertex V_3 may move freely from V_2 to V_4 in the configuration of Fig. 10 even if (p, k) is fixed. This flexibility of the configuration is tied up with the higher codimensionality of the component in question.

We believe that several intriguing features of $L^+(T_3)$ should be tied up with the existence of non-external vertex V_3. Here, and in what follows, we say that a vertex is non-external if no external line is incident upon the vertex. It is probably worth noting the following fact.

Let us consider the following graph $\widetilde{T_3}$:

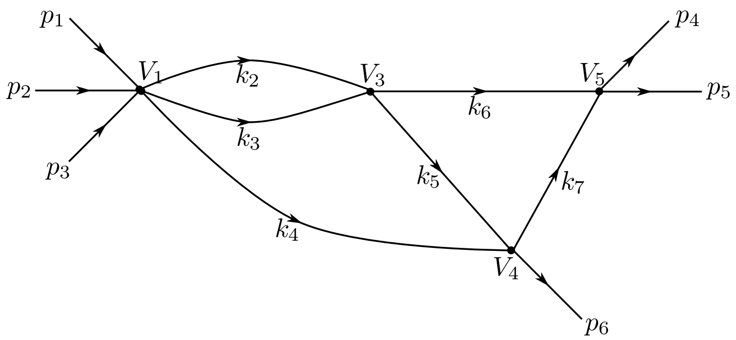

Figure 11. The Feynman graph $\widetilde{T_3}$.

Then, for any point p in $L^{\oplus}(\widetilde{T_3})$ $(\subset L^+(T_3))$, we find

$$p_6^2 = m^2; \tag{3.3}$$

otherwise stated, although the external line p_6 is originally assumed not necessarily to be on-shell, the current configuration forces it to be on-shell. We note that we encountered a similar situation in Section 2; at some particular points of $L^{\oplus}(G_1)$, p_6 lies on mass-shell. But this time at all points in $L^{\oplus}(\widetilde{T_3})$, p_6 obeys the mass-shell constraint. The confirmation of (3.3) is straightforward. First we note that the energy-momentum conservation at V_3 (i.e., the first equation of (1.6) with $j = 3$)

$$k_5 = k_6 = k_2 = k_3, \tag{3.4}$$

because $\nu = 2$ and $\alpha_\ell \gneqq 0$ ($\ell = 2, 3, 5, 6$). Then it follows from the third equation of (1.6) that

$$\alpha_4 k_4 = \alpha_3 k_3 + \alpha_5 k_5 = (\alpha_3 + \alpha_5) k_3, \tag{3.5}$$

and hence

$$k_4 = k_3. \tag{3.6}$$

Similarly the third equation of (1.6) applied to the triangle formed by V_3, V_4 and V_5 entails

$$\alpha_6 k_6 = \alpha_5 k_5 + \alpha_7 k_7. \tag{3.7}$$

Hence (3.4) guarantees

$$k_7 = k_5 = k_3. \tag{3.8}$$

Thus the energy-momentum conservation at V_4 implies

$$p_6 = k_3, \tag{3.9}$$

proving (3.3). In passing, we note that in the course of the above reasoning we have also confirmed

$$p_4 + p_5 = 2p_6. \tag{3.10}$$

The degeneration of this sort is a universal one, and we can confirm that at a point p in $L^{\oplus}(T_n)$ ($n \geq 4$) where T_n is the truss-bridge graph given in Fig. 12 below, all the internal lines become parallel, and hence we find (in the labeling of external energy-momentum vectors as in Fig. 12)

$$p_4 + p_5 = 2p_6, \;\; p_6^2 = m^2 \qquad \text{if } n \text{ is odd}, \tag{3.11}$$

and

$$p_5 + p_6 = 2p_4, \;\; p_4^2 = m^2 \qquad \text{if } n \text{ is even}. \tag{3.12}$$

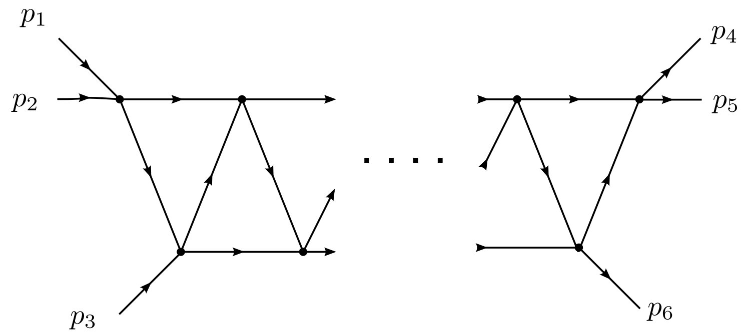

Figure 12. The truss-bridge graph T_n consisting of n-trusses.

We also note

$$p_1 + p_2 = 2p_3, \;\; p_3^2 = m^2 \tag{3.13}$$

holds. Hence, by setting

$$N = N_+ \cup N_-, \tag{3.14}$$

where

$$N_+ = \bigcup_{p_3^2=m^2} \{(p_1, p_2, p_3); p_1 + p_2 = 2p_3\} \tag{3.15}$$

and

$$N_- = \bigcup_{p_6^2=m^2} \{(p_4, p_5, p_6); p_4 + p_5 = 2p_6\}, \tag{3.16}$$

we find

$$L^{\oplus}(T_n) \subset N \qquad (n \geq 4) \tag{3.17}$$

with some change of labeling of (p_4, p_5, p_6) if necessary. Thus the micro-analytic structure of the S-matrix near N should be formidably difficult to study, but we believe the analysis of individual Feynman integrals $F_{T_n}(p)$ should be within reach of us.

§ 4. Concluding remarks and future problems

Having in mind the study of micro-analytic structure of the S-matrix near the 3-particle threshold, we have made a detailed study of the LN surfaces associated with an ice-cream cone graph and a truss-bridge graph T_n with $n = 3$ near the 3-particle threshold. Thanks to the power of recent computers our results are precise enough to stimulate the interest of mathematicians in the geometry of LN surfaces near the 3-particle threshold. Among other things we note that a central role is played by the set N given by (3.14) (or N_- for the configuration of Fig. 4). Although the singularity structure of the S-matrix near N should be too complicated to analyze, we believe the study of the holonomic structure of individual Feynman integrals near N is an interesting problem in microlocal analysis. Another interesting feature of our results is that the existence of non-external vertices in a Feynman graph normally gives strong constraint on the shape of the associated $\mathscr{LN}$ variety. (See [4] and [5] for some related topics.) The study of the holonomic structure of a Feynman integral associated with a Feynman graph containing non-external vertices is an important and challenging problem in microlocal analysis. One natural way to approach this problem is to introduce fictitiously an external vector p_j at a non-external vertex V_j and then set it to be 0. As one immediately realizes, this procedure normally leads to the restriction of a holonomic system to a submanifold which contains characteristic points. We believe concrete studies of Feynman integrals of this sort should contribute much to the progress of the theory of holonomic systems.

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