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On the Kashaev invariant and the twisted Reidemeister torsion of two-bridge knots

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Abstract

It is conjectured that, in the asymptotic expansion of the Kashaev invariant of a hyperbolic knot, the first coefficient is presented by the complex volume of the knot complement, and the second coefficient is presented by a constant multiple of the square root of the twisted Reidemeister torsion associated with the holonomy representation of the hyperbolic structure of the knot complement. In particular, this conjecture has rigorously been proved for some simple hyperbolic knots, and the second coefficient is presented by a modification of the square root of the Hessian of the potential function of the hyperbolic structure of the knot complement.

In this paper, we define an invariant of a parametrized knot diagram to be a modification of the Hessian of the potential function obtained from the parametrized knot diagram. Further, we show that this invariant is equal (up to sign) to a constant multiple of the twisted Reidemeister torsion for any two-bridge knot.

1 Introduction

In [10, 11], Kashaev defined the Kashaev invariant $\langle L \rangle_N \in \mathbb{C}$ of a link L for $N = 2, 3, \cdots$ by using the quantum dilogarithm at $q = e^{2\pi\sqrt{-1}/N}$. In [12], he conjectured that, for any hyperbolic link L, $\frac{2\pi}{N} \log \langle L \rangle_N$ goes to the hyperbolic volume of $S^3 - L$ as $N \to \infty$, and verified the conjecture for some simple knots, by formal calculations. In [14], H. Murakami and J. Murakami proved that the Kashaev invariant $\langle L \rangle_N$ of any link L is equal to the N-colored Jones polynomial $J_N(L; e^{2\pi\sqrt{-1}/N})$ of L evaluated at $q = e^{2\pi\sqrt{-1}/N}$. Further, as an extension of Kashaev's conjecture, they conjectured that, for any knot K, $\frac{2\pi}{N} \log |J_N(K; e^{2\pi\sqrt{-1}/N})|$ goes to the (normalized) simplicial volume of $S^3 - K$. This is called *the volume conjecture*. As a complexification of the volume conjecture, it is conjectured in [15] that, for a hyperbolic link L, $J_N(L; e^{2\pi\sqrt{-1}/N}) \sim e^{N\varsigma(L)}$ as $N \to \infty$, where we put

$$\varsigma(L) \;=\; \frac{1}{2\pi\sqrt{-1}} \big(\mathrm{cs}(S^3 - L) + \sqrt{-1}\,\mathrm{vol}(S^3 - L)\big),$$

and "cs" and "vol" denote the Chern-Simons invariant and the hyperbolic volume; we call it the *complex hyperbolic volume* (which is the $\mathrm{SL}_2\mathbb{C}$ Chern-Simons invariant). Furthermore, it is conjectured in [8] (see also [3, 9, 26]) from the viewpoint of the $\mathrm{SL}_2\mathbb{C}$ Chern-Simons theory that the asymptotic expansion of $J_N(K; e^{2\pi\sqrt{-1}/k})$ of a hyperbolic knot K as $N, k \to \infty$ fixing $u = N/k$ is presented by the following form,

$$J_N(K; e^{2\pi\sqrt{-1}/k}) \underset{\substack{N,k\to\infty \\ u=N/k:\ \text{fixed}}}{\sim} e^{N\varsigma} N^{3/2}\, \omega \cdot \Big(1 + \sum_{i=1}^{\infty} \kappa_i \cdot \big(\frac{2\pi\sqrt{-1}}{N}\big)^i\Big) \qquad (1)$$

for some scalars $\varsigma, \omega, \kappa_i$ depending on K and u, though they do not discuss the Jones polynomial in the Chern-Simons theory in the case of vanishing quantum dimension, which is discussed in [22]. We note that the colored Jones polynomial is defined at generic q, while the Kashaev invariant is defined only at $q = e^{2\pi\sqrt{-1}/N}$. The semi-classical approximation (*i.e.*, the "$e^{N\varsigma} N^{3/2}\, \omega$" part) of the above expansion is proved for the figure-eight knot in [1] at $q = e^{2\pi\sqrt{-1}/N}$ and in [13] at generic q around $e^{2\pi\sqrt{-1}/N}$. As for rigorous proofs for other hyperbolic knots, it is shown in [16, 18, 17] that, for any hyperbolic knot K with up to 7 crossings, the asymptotic expansions of the Kashaev invariant of K is presented by the following form,

$$\langle K \rangle_{N} \; = \; e^{N\varsigma(K)} N^{3/2}\, \omega(K) \cdot \Big(1 + \sum_{i=1}^{d} \kappa_i(K) \cdot \big(\frac{2\pi\sqrt{-1}}{N} \big)^i \; + O\big(\frac{1}{N^{d+1}} \big) \Big), \tag{2}$$

for any d, where $\omega(K)$ and $\kappa_i(K)$'s are some scalars. By another approach to this problem, in [2], motivated by the above mentioned conjectures, a formal power series is constructed as an invariant of a hyperbolic knot by using the canonical simplicial decomposition of the hyperbolic knot complement; it is conjectured that this power series is equal to the expansion (2).

We consider the second coefficient of the semi-classical approximation (*i.e.*, the "ω" part) of the above expansions. As explained in [21], such a coefficient of the semi-classical approximation of the Chern-Simons path integral is calculated as the regularized determinant of the Laplacian, and it is presented by the square root of the Ray-Singer torsion at a flat connection, which is equal to the twisted Reidemeister torsion. Further, by similar arguments, it is conjectured in [8, 9, 13] that the ω of (1) is a scalar multiple of the square root of (the Ray-Singer torsion at a flat connection or) the twisted Reidemeister torsion of the cochain complex of the knot complement with the $\mathfrak{sl}_2$ coefficient twisted by the adjoint action of the holonomy representation of the hyperbolic structure of the knot complement; this conjecture is confirmed for the figure-eight knot in [1, 13], and numerically checked for some knots in [4]. Furthermore, the "ω" part of the power series of [2] is conjectured (and confirmed in many cases) to be a constant multiple of the square root of the twisted Reidemeister torsion. Hence, we conjecture that $\omega(K)$ of (2) is equal to a constant multiple of the square root of the twisted Reidemeister torsion. In the proof of (2) in [16, 18, 17], we use the Poisson summation formula and the saddle point method (see Section 4.2 and [16, 18, 17]), and we must check many technical concrete inequalities to calculate such procedures. Because of such technical difficulties, it is difficult at the present stage to prove (2) rigorously for general knots. However, by assuming such inequalities of the assumption of the saddle point method, we can guess the resulting form of (2). In particular, by formal calculation assuming such assumption of the saddle point method, $\omega(K)^{-2}$ is presented by a modification of the Hessian of the potential function obtained from a knot diagram parameterized by hyperbolicity parameters.

In this paper, we formulate $\omega_2(D)$ of a parameterized diagram D of a knot K such that $\omega_2(D) = \pm\, \omega(K)^2$, *i.e.*, we define $\omega_2(D)^{-1}$ to be a modification of the Hessian of the potential function obtained from D (Definition 4.2). Further, from a parameterized knot diagram, we construct a monodromy representation of a knot group into $\mathrm{PGL}_2\mathbb{C}$ (Section 3.1), and we can consider the twisted Reidemeister torsion associated with such a monodromy representation. The following theorem is the main theorem of this paper, which confirm the above mentioned conjecture of $\omega(K)$ for any two-bridge knot assuming the above mentioned technical assumptions of the Poisson summation formula and the saddle point method.

Theorem 1.1. *Let K be any two-bridge knot, and let D be an appropriate parameterized diagram of K. Then,*

$$\omega_2(D) \;=\; \pm\frac{\tau(K)}{2\sqrt{-1}}\,,$$

where $\tau(K)$ is the twisted Reidemeister torsion associated with the monodromy representation obtained from the parameterization of D.

For example, as shown in Examples 3.1, 3.2, 4.5 and 4.6, for the $\overline{5_2}$ knot and the $\overline{6_1}$ knot with the holonomy representations of the hyperbolic structures, the values of $\omega(K)$ and $\tau(K)$ are numerically given by

$$\begin{aligned}
\omega(\overline{5_2}) &= 0.09019057740... + \sqrt{-1}\cdot 0.6499757866...\,,\\
\tau(\overline{5_2}) &= -0.2344867659... - \sqrt{-1}\cdot 0.8286683659...\,,\\
\omega(\overline{6_1}) &= -0.5213883634... + \sqrt{-1}\cdot 0.07173228265...\,,\\
\tau(\overline{6_1}) &= 0.1496015098... + \sqrt{-1}\cdot 0.5334006103...\,,
\end{aligned}$$

where we can confirm that the values of $\omega(\overline{5_2})$ and $\omega(\overline{6_1})$ are equal to the values given in [16, 18], and the values of $\tau(\overline{5_2})$ and $\tau(\overline{6_1})$ are equal to the values obtained from [20] (see Examples 3.1 and 3.2). Hence, we can numerically verify the theorem as

$$\begin{aligned}
\omega(\overline{5_2})^2 &= -0.4143341829... + \sqrt{-1}\cdot 0.1172433829... = \frac{\tau(\overline{5_2})}{2\sqrt{-1}}\,,\\
\omega(\overline{6_1})^2 &= 0.2667003051... - \sqrt{-1}\cdot 0.07480075491... = \frac{\tau(\overline{6_1})}{2\sqrt{-1}}\,.
\end{aligned}$$

Further, by results in [16, 18, 17], the theorem means that the above mentioned conjecture of $\omega(K)$ is confirmed as

$$\omega(K)^2 \;=\; \pm\frac{\tau(K)}{2\sqrt{-1}}$$

for any hyperbolic knot with up to 7 crossings, since they are two-bridge knots.

The theorem means that the Hessian of the potential function is related to the twisted Reidemeister torsion. We explain how they are related, roughly speaking, as follows. As mentioned above, the twisted Reidemeister torsion of the problem is the Reidemeister torsion of the cochain complex of the knot complement with the $\mathfrak{sl}_2$ coefficient twisted by the adjoint action of the holonomy representation of the hyperbolic structure of the knot complement. This Reidemeister torsion is determined by the alternating product of the determinants of the coboundary maps of this cochain complex; in particular, its essential factor is the determinant of the coboundary map $d_1 : C^1 \to C^2$ with respect to an appropriate basis. Further, it is well known that H^1 of this cochain complex is naturally isomorphic to the tangent space of the space of conjugacy classes of $\mathrm{PGL}_2\mathbb{C}$ representations of the knot group. Hence, roughly speaking, the twisted Reidemeister torsion is given by the determinant of the matrix whose entries are the coefficients of the defining equations of the tangent space of the representation space. On the other hand, we can reconstruct the representation space by using an ideal tetrahedral decomposition of the knot complement. The shape of an ideal tetrahedron is parameterized by the cross-ratio of the coordinates of its four vertices, and the representation space is parameterized by solutions of hyperbolicity equations of such parameters. Further, the hyperbolicity equations are given by differentials of the potential function. Hence, the tangent space of the representation space is presented by the Hesse matrix of the potential function, and its determinant (*i.e.*, the Hessian of the potential function) is expected to be related to the twisted Reidemeister torsion, as mentioned above.

We explain an outline of the proof of the theorem. We consider a parameterized knot diagram of an open two-bridge knot, where an open knot is a 1-tangle whose closure is a knot. We decompose such a knot diagram into elementary tangle diagrams. Further, we reformulate $\tau(K)$ and $\omega_2(D)$ as compositions of operator invariants of such elementary diagrams. In other words, regarding an open two-bridge knot as a plat closure of a 3-braid, we reformulate $\tau(K)$ and $\omega_2(D)$ in terms of "representations" of parameterized 3-braids. Further, we show the theorem by comparing recursive formulas of both sides of the required formula of the theorem.

The paper is organized as follows. In Section 2, we review some basic facts used in this paper, such as the definition of the Kashaev invariant and a parameterization of a knot diagram by hyperbolicity parameters. In Section 3, we explain how we calculate the twisted Reidemeister torsion for two-bridge knots. We construct a monodromy representation of a knot group into $\mathrm{PGL}_2\mathbb{C}$ from a parameterized knot diagram, and calculate the twisted Reidemeister torsion associated with this monodromy representation, by decomposing a two-bridge knot diagram into elementary tangle diagrams. In Section 4, we define $\omega_2(D)$ for an oriented parameterized open knot diagram D, and show a relation of it to the Kashaev invariant, and calculate it for two-bridge knots. In Section 5, we show a proof of Theorem 1.1, by comparing recursive formulas of both sides of the required formula of the theorem.

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2 Preliminaries

In this section, we review some basic facts used in this paper. In Section 2.1, we review the definition of the Kashaev invariant. In Section 2.2, we review a parameterization of a knot diagram by hyperbolicity parameters.

2.1 Kashaev invariant

In this section, we review the definition of the Kashaev invariant following [25], and review some related formulas.

Let N be an integer ≥ 2. We put $q=\exp(2\pi\sqrt{-1}/N)$, and put

$$(x)_n \;=\; (1-x)(1-x^2)\cdots(1-x^n)$$

for $n\geq 0$. It is known [14] (see also [16]) that for any n,m with $n\leq m$,

$$(q)_n(\overline{q})_{N-n-1}=N, \tag{3}$$

$$\sum_{n\leq k\leq m}\frac{1}{(q)_{m-k}(\overline{q})_{k-n}}=1. \tag{4}$$

Following Faddeev [6], we define a holomorphic function $\varphi(t)$ on $\{t\in\mathbb{C}\mid 0<\operatorname{Re}t<1\}$ by

$$\varphi(t)\;=\;\int_{-\infty}^{\infty}\frac{e^{(2t-1)x}dx}{4x\,\sinh x\,\sinh(x/N)}\,,$$

noting that this integrand has poles at $n\pi\sqrt{-1}$ ($n\in\mathbb{Z}$), where, to avoid the pole at 0, we choose the following contour of the integral,

$$(-\infty,-1\,]\;\cup\;\big\{z\in\mathbb{C}\;\big|\;|z|=1,\ \operatorname{Im}z\geq 0\big\}\;\cup\;[\,1,\infty).$$

It is known [7, 23] that

$$\begin{aligned}(q)_n\;&=\;\exp\Big(\varphi\big(\frac{1}{2N}\big)-\varphi\big(\frac{2n+1}{2N}\big)\Big),\\ (\overline{q})_n\;&=\;\exp\Big(\varphi\big(1-\frac{2n+1}{2N}\big)-\varphi\big(1-\frac{1}{2N}\big)\Big).\end{aligned} \tag{5}$$

Further, it is known [7, 23] (see also [16]) that

$$\begin{aligned}\frac{1}{N}\,\varphi(t)\;&=\;\frac{1}{2\pi\sqrt{-1}}\operatorname{Li}_2\big(e^{2\pi\sqrt{-1}\,t}\big)\;+O\big(\frac{1}{N^2}\big),\\ \frac{1}{N}\,\varphi'(t)\;&=\;-\log\big(1-e^{2\pi\sqrt{-1}\,t}\big)\;+O\big(\frac{1}{N^2}\big).\end{aligned} \tag{6}$$

Furthermore, it is known (due to Kashaev, see [16]) that

$$\begin{aligned}\varphi\big(\frac{1}{2N}\big)\;&=\;\frac{N}{2\pi\sqrt{-1}}\,\frac{\pi^2}{6}+\frac{1}{2}\log N+\frac{\pi\sqrt{-1}}{4}-\frac{\pi\sqrt{-1}}{12N}\,,\\ \varphi\big(1-\frac{1}{2N}\big)\;&=\;\frac{N}{2\pi\sqrt{-1}}\,\frac{\pi^2}{6}-\frac{1}{2}\log N+\frac{\pi\sqrt{-1}}{4}-\frac{\pi\sqrt{-1}}{12N}\,.\end{aligned} \tag{7}$$

Following Yokota [25],[1] we review the definition of the Kashaev invariant. We put

$$\mathcal{N}\;=\;\{0,1,\cdots,N-1\}.$$

For $i,j,k,l\in\mathcal{N}$, we put

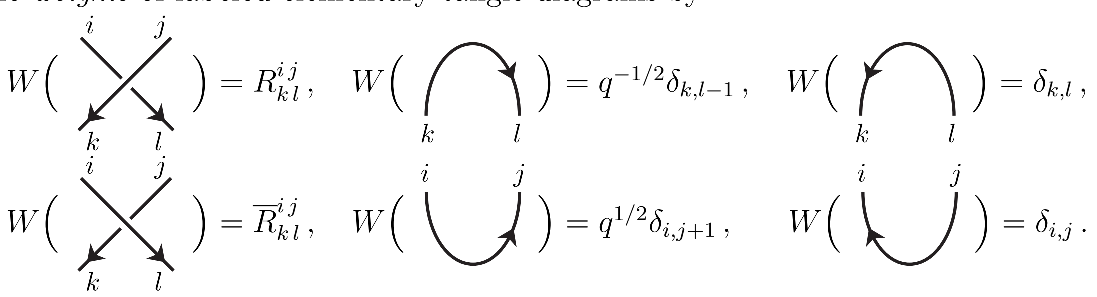
$$R^{ij}_{kl}\;=\;\frac{N\,q^{-\frac{1}{2}+i-k}\,\theta^{ij}_{kl}}{(q)_{[i-j]}(\overline{q})_{[j-l]}(q)_{[l-k-1]}(\overline{q})_{[k-i]}}\,,\qquad \overline{R}^{ij}_{kl}\;=\;\frac{N\,q^{\frac{1}{2}+j-l}\,\theta^{ij}_{kl}}{(\overline{q})_{[i-j]}(q)_{[j-l]}(\overline{q})_{[l-k-1]}(q)_{[k-i]}}\,,$$

[1] We make a minor modification of the definition of weights of critical points from the definition in [25], in order to make $\langle\, K\,\rangle_N$ invariant under Reidemeister moves.

where $[m] \in \mathcal{N}$ denotes the residue of m modulo N, and we put

$$\theta^{i\,j}_{k\,l} \;=\; \begin{cases} 1 & \text{if } \; [i-j]+[j-l]+[l-k-1]+[k-i]=N-1, \\ 0 & \text{otherwise.} \end{cases}$$

Let K be an oriented knot. We consider a 1-tangle whose closure is isotopic to K such that its string is oriented downward at its end points; abusing the notation, we also denote this 1-tangle by K, and call such a 1-tangle an *open knot*. Let D be a diagram of this 1-tangle. We present D by a union of elementary tangle diagrams shown in (8). We decompose the string of D into edges by cutting it at crossings and critical points with respect to the height function of $\mathbb{R}^2$. A *labeling* is an assignment of an element of $\mathcal{N}$ to each edge. Here, we assign 0 to the two edges adjacent to the end points of D. We define the *weights* of labeled elementary tangle diagrams by



(8)

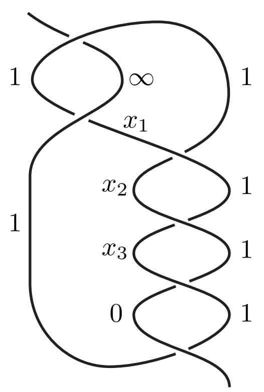
Then, *the Kashaev invariant* $\langle K \rangle_N$ of K is defined by

$$\langle K \rangle_N \;=\; \sum_{\text{labelings}} \prod_{\substack{\text{crossings}\\ \text{of } D}} W(\text{crossings}) \prod_{\substack{\text{critical}\\ \text{points of } D}} W(\text{critical points}) \;\in \mathbb{C}.$$

2.2 Knot diagrams parameterized by hyperbolicity parameters

In this section, we review a parameterization of an open knot diagram by hyperbolicity parameters, following [24]. Further, we review a potential function of a parameterized open knot diagram.

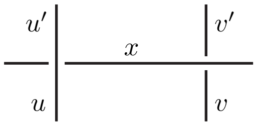
We parameterize edges of an open knot diagram by parameters in $\mathbb{C}\cup\{\infty\}$, for example, as follows.



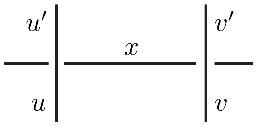
(9)

We parameterize edges adjacent to unbounded regions by 1. We parameterize edges next to the terminal edges by 0 or ∞ as shown above; we parameterize such an edge by ∞

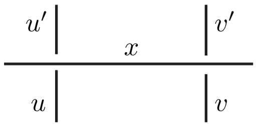
(resp. 0) if it is connected to the terminal edge by an under-path (resp. an over-path). We parameterize the other edges in such a way that the parameters satisfy the *hyperbolicity equations*, which are given as follows.



$$\Big(1-\frac{x}{u}\Big)\Big(1-\frac{v'}{x}\Big) \;=\; \Big(1-\frac{x}{u'}\Big)\Big(1-\frac{v}{x}\Big)$$



$$\Big(1-\frac{x}{u}\Big)\Big(1-\frac{x}{v'}\Big) \;=\; \Big(1-\frac{x}{u'}\Big)\Big(1-\frac{x}{v}\Big)$$



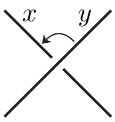
$$\Big(1-\frac{u}{x}\Big)\Big(1-\frac{v'}{x}\Big) \;=\; \Big(1-\frac{u'}{x}\Big)\Big(1-\frac{v}{x}\Big)$$

We call such parameters *hyperbolicity parameters*. For example, for the knot diagram (9), the hyperbolicity equations are given by

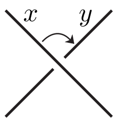
$$\begin{aligned} 1-\frac{x_2}{x_1} \;&=\; (1-x_1)\Big(1-\frac{1}{x_1}\Big), \\ \Big(1-\frac{x_2}{x_1}\Big)\Big(1-\frac{1}{x_2}\Big) \;&=\; (1-x_2)\Big(1-\frac{x_3}{x_2}\Big), \\ \Big(1-\frac{x_3}{x_2}\Big)\Big(1-\frac{1}{x_3}\Big) \;&=\; 1-x_3\,. \end{aligned}$$

As we explain in Section 3.1, such a parameterization gives a monodromy representation of the knot group into $\mathrm{PGL}_2\mathbb{C}$. Hence, in many cases (including all two-bridge knots), each solution of hyperbolicity equations is isolated (*i.e.*, 0-dimensional).

We consider an open knot diagram parameterized by hyperbolicity parameters. We consider an angle consisting of two adjacent edges at a crossing. We associate such an angle with the following value,

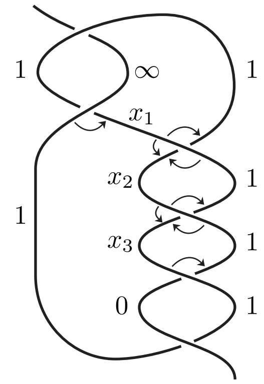


$$\rightsquigarrow \quad \mathrm{Li}_2\Big(\frac{x}{y}\Big) - \mathrm{Li}_2(1)$$



$$\rightsquigarrow \quad \mathrm{Li}_2(1) - \mathrm{Li}_2\Big(\frac{y}{x}\Big)$$

where we consider the orientation of an angle from the over-path to the under-path, and the left case is the case where this orientation is counter-clockwise, and the right case is the case where this orientation is clockwise. We recall that $\mathrm{Li}_2(1) = \frac{\pi^2}{6}$. For a parameterized open knot diagram, we put the *potential function* V to be the sum of such values for all angles except for the constant terms, regarding V as a function of hyperbolicity parameters.



For example, for the above knot diagram, the potential function V is given by

$$\begin{aligned} V(x_1,x_2,x_3) &= \mathrm{Li}_2(x_1) - \mathrm{Li}_2\big(\frac{1}{x_1}\big) + \mathrm{Li}_2\big(\frac{x_2}{x_1}\big) - \mathrm{Li}_2(x_2) \\ &\quad -\mathrm{Li}_2\big(\frac{1}{x_2}\big) + \mathrm{Li}_2\big(\frac{x_3}{x_2}\big) - \mathrm{Li}_2(x_3) - \mathrm{Li}_2\big(\frac{1}{x_3}\big) + 2\,\mathrm{Li}_2(1). \end{aligned} \tag{10}$$

We note that

$$x\frac{\partial}{\partial x}\mathrm{Li}_2\big(\frac{x}{y}\big) = -\log\big(1-\frac{x}{y}\big), \qquad y\frac{\partial}{\partial y}\mathrm{Li}_2\big(\frac{x}{y}\big) = \log\big(1-\frac{x}{y}\big). \tag{11}$$

We also note that the hyperbolicity equations are given by

$$\frac{\partial}{\partial x_i}V = 0 \quad \text{for all } i,$$

and, hence, a solution of the hyperbolicity equations gives a critical point of V.

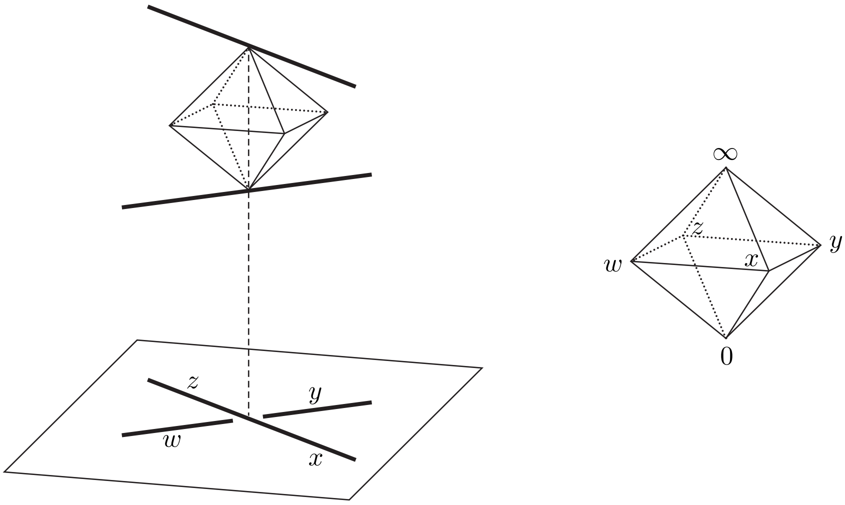
3 Calculation of the twisted Reidemeister torsion

In this section, we explain how we calculate the twisted Reidemeister torsion for two-bridge knots. In Section 3.1, we explain how we calculate the monodromy representation of a knot group into $\mathrm{PGL}_2\mathbb{C}$ when a knot diagram is parameterized by hyperbolicity parameters. In Section 3.2, we explain how we calculate the twisted Reidemeister torsion for the $\overline{5_2}$ knot, as the simplest example among two-bridge knots; the calculation is reduced to the calculations of $\det\left(\boxed{\hat{E}_2}\ \boxed{\hat{D}_1\hat{E}_1}\right)$ and $\det\big(\check{D}_1\check{E}_1\big)$. In Section 3.3, we decompose open two-bridge knot diagrams into elementary tangle diagrams, to formulate such calculations for any two-bridge knot. In Sections 3.4 and 3.5, we calculate $\det\left(\boxed{\hat{E}_2}\ \boxed{\hat{D}_1\hat{E}_1}\right)$ and $\det\big(\check{D}_1\check{E}_1\big)$ respectively for any two-bridge knot. By using them, we calculate the twisted Reidemeister torsion for any two-bridge knot in Section 3.6. See also [5, 20] for the calculation of the Reidemeister torsion for twist knots.

3.1 The monodromy representation

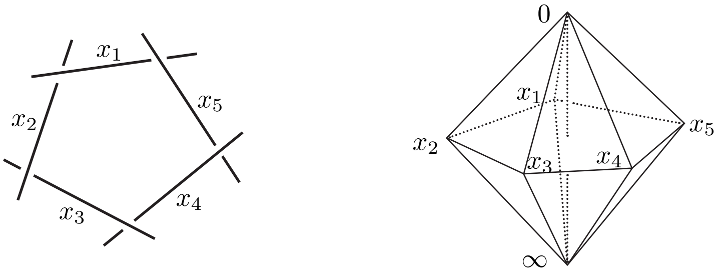
In this section, we explain how we calculate the monodromy representation of a knot group into $\mathrm{PGL}_2\mathbb{C}$ from a parameterized knot diagram.

We review how to make an ideal tetrahedral decomposition of $S^3 - K$ from a knot diagram, following [19, 24]. There are four tetrahedra at each crossing of the knot diagram, and, by making an octahedron as the union of such four tetrahedra at each crossing, we obtain an octahedral decomposition of $S^3 - K$. As in [24], we associate a complex parameter to each edge of the knot diagram, and consider the hyperbolicity equations with respect to the parameters. Then, the shape of an ideal octahedron at each crossing is determined, as follows.



(12)

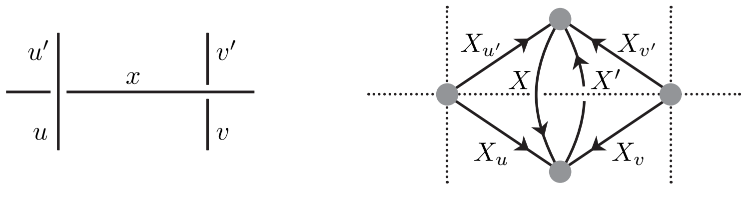
We can glue ideal tetrahedra at each face of a knot diagram. For example, we can make the polyhedron of the following right picture by gluing 5 tetrahedra at the face of the left picture.



(13)

Here, we note that the edge $\overline{x_1x_2}$ of the tetrahedron "$\infty 0 x_1 x_2$" at the crossing of the edges of x_1 and x_2 in the left picture corresponds to the edge $\overline{\infty 0}$ of the tetrahedron "$0 \infty x_1 x_2$" of the right picture.

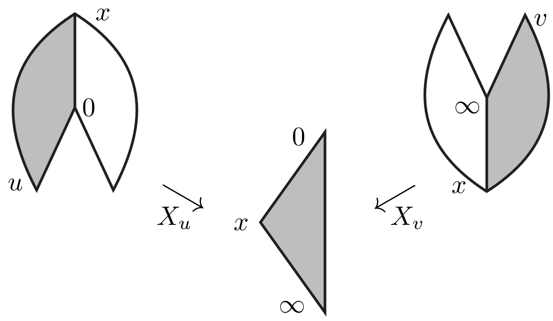
We consider the following left picture as a part of a knot diagram.



As mentioned in Section 2.2, the hyperbolicity equation of these parameters is

$$\big(1-\frac{x}{u}\big)\big(1-\frac{v'}{x}\big) \;=\; \big(1-\frac{x}{u'}\big)\big(1-\frac{v}{x}\big).$$

We consider tetrahedra at each crossing as in (12), and consider tetrahedra at each face as in (13). Further, we consider maps taking such tetrahedra to each other as in the right picture; for example, the map X_u in the right picture takes a tetrahedron at the left crossing placed as in (12) to a tetrahedron at the lower face placed as in (13). Such maps take vertices of the tetrahedra, as follows.



Hence,

$$\begin{aligned} &X_u(x)=0, && X_u(0)=x, && X_u(u)=\infty,\\ &X_v(v)=0, && X_v(\infty)=x, && X_v(x)=\infty, \end{aligned}$$

where $\mathrm{PGL}_2\mathbb{C}$ acts on $\mathbb{C}\cup\{\infty\}$ by the Möbius transformation. It follows that

$$X_u \;\sim\; \begin{pmatrix} 1 & -x \\ 1/u & -1 \end{pmatrix}, \qquad X_v \;\sim\; \begin{pmatrix} 1 & -v \\ 1/x & -1 \end{pmatrix},$$

where "$\sim$" means the equality in $\mathrm{PGL}_2\mathbb{C}$. Similarly, we have that

$$X_{u'} \;\sim\; \begin{pmatrix} 1 & -x \\ 1/u' & -1 \end{pmatrix}, \qquad X_{v'} \;\sim\; \begin{pmatrix} 1 & -v' \\ 1/x & -1 \end{pmatrix}.$$

Therefore,

$$X \;=\; X_u X_{u'}^{-1} \;\sim\; \begin{pmatrix} \frac{x}{u'}-1 & 0 \\ \frac{1}{u'}-\frac{1}{u} & \frac{x}{u}-1 \end{pmatrix}, \qquad X' \;=\; X_{v'} X_v^{-1} \;\sim\; \begin{pmatrix} \frac{v'}{x}-1 & v-v' \\ 0 & \frac{v}{x}-1 \end{pmatrix}.$$

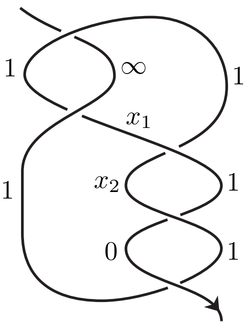
We note that, from the construction, X fixes 0 and x, and X' fixes ∞ and x by the Möbius transformation.

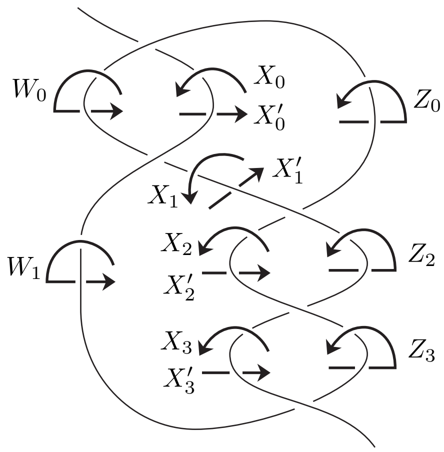
By using such matrices, we can calculate the monodromy representation $\pi_1(S^3-K)\to\mathrm{PGL}_2\mathbb{C}$ from a knot diagram with parameters.

3.2 Calculation of the twisted Reidemeister torsion for the $\overline{5_2}$ knot

In this section, we explain how we calculate the twisted Reidemeister torsion for the $\overline{5_2}$ knot, before we explain the calculation for any two-bridge knot later.

The $\overline{5_2}$ *knot* is the knot presented by the following picture; it is the mirror image of the 5_2 knot.





As in [24], the parameters of the knot diagram is given as in the left picture. The hyperbolicity equations are

$$(1-x_1)\big(1-\frac{1}{x_1}\big) \;=\; 1-\frac{x_2}{x_1}, \qquad \big(1-\frac{x_2}{x_1}\big)\big(1-\frac{1}{x_2}\big) \;=\; 1-x_2\,.$$

Hence,

$$x_2 \;=\; x_1^2-x_1+1, \qquad x_2+1-\frac{x_2}{x_1} \;=\; 0.$$

We calculate X_i and X'_i by the way of Section 3.1; for example,

$$X'_0 \sim \begin{pmatrix} 1 & x_1-1 \\ 0 & 1 \end{pmatrix}, \quad X_1 \sim \begin{pmatrix} 1 & 0 \\ 1 & 1-x_1 \end{pmatrix}, \quad X'_1 \sim \begin{pmatrix} \frac{1}{x_1}-1 & x_2-1 \\ 0 & \frac{x_2}{x_1}-1 \end{pmatrix},$$

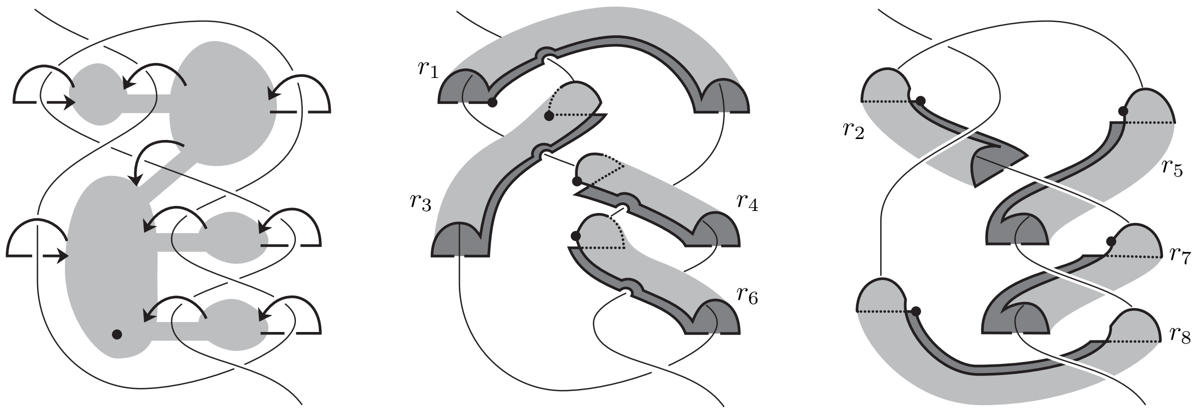
$$\cdots, \quad X_3 \sim \begin{pmatrix} 1 & 0 \\ \frac{1}{x_2}-1 & 1 \end{pmatrix}.$$

By using them, we can calculate the other matrices; for example,

$$X_0 \sim X'_3 \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad W_i \;\sim\; \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \quad Z_i \;\sim\; \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$$

for each i.

We consider a cell decomposition of the knot complement, as follows, The (large) 0-cell is a shaded region of the following left picture. The 1-cells are the arrows of the following left picture. The 2-cells are given as in the right two pictures.



Here, the base points of the 0-cell and the 2-cells are depicted by dots in the pictures, and the base points of the 1-cells are the tops of the arrows.

We consider the cochain complex C^* of this cell decomposition with the $\mathfrak{sl}_2$ coefficient twisted by the monodromy representation of Section 3.1. The relator given by the 2-cell r_1 is presented by

$$W_0\, X_0\, Z_0\, X_0^{-1}.$$

Its perturbation is given by

$$(1+\varepsilon\, e_{W_0})\, W_0 \cdot (1+\varepsilon\, e_{X_0})\, X_0 \cdot (1+\varepsilon\, e_{Z_0})\, Z_0 \cdot X_0^{-1}\, (1-\varepsilon\, e_{X_0}) \ + \ O(\varepsilon^2)$$

for e_{W_0}, e_{Z_0}, $e_{Z_0} \in \mathfrak{sl}_2$. Its coefficient of ε is presented by

$$e_{r_1} \ = \ e_{W_0} + (\mathcal{W}_0 - 1)\, e_{X_0} + \mathcal{W}_0\, e_{Z_0}\,,$$

where we put $\mathcal{W}_i = \mathrm{ad}(W_i)$, $\mathcal{X}_i = \mathrm{ad}(X_i)$, $\mathcal{Z}_i = \mathrm{ad}(Z_i)$, $\cdots$. Similarly, from the relator $W_0\, X_0'^{-1} X_1^{-1} X_1'^{-1} X_0'$ of the 2-cell r_2, we obtain

$$e_{r_2} \ = \ e_{W_0} - \mathcal{X}_0'^{-1} \mathcal{X}_1'\, e_{X_1}\,.$$

Further, from the relator $X_0\, X_1^{-1} W_1^{-1}\, X_1\, X_0'$ of the 2-cell r_3, we obtain

$$e_{r_3} \ = \ e_{X_0} - \mathcal{X}_0'^{-1} \mathcal{X}_1^{-1}\, e_{W_1} + \mathcal{X}_1^{-1} (\mathcal{W}_1^{-1} - 1)\, e_{X_1}\,.$$

By calculating similarly, the coboundary map $D_1 : C^1 \to C^2$ is presented by

$$D_1 = \begin{pmatrix} 1 & \mathcal{W}_0 - 1 & \mathcal{W}_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -\mathcal{X}_0'^{-1}\mathcal{X}_1' & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\mathcal{X}_0'^{-1}\mathcal{X}_1^{-1} & \mathcal{X}_1^{-1}(\mathcal{W}_1^{-1} - 1) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \mathcal{X}_1\mathcal{X}_1' - 1 & -\mathcal{X}_2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\mathcal{X}_1' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \mathcal{X}_2\mathcal{X}_2' - 1 & -\mathcal{X}_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\mathcal{X}_2' & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \mathcal{Z}_3^{-1} \end{pmatrix},$$

with respect to the basis $(e_{W_0}, e_{X_0}, e_{Z_0}, e_{W_1}, e_{X_1}, e_{X_2}, e_{Z_2}, e_{X_3}, e_{Z_3})$ of C^1 and the basis $(e_{r_1}, e_{r_2}, e_{r_3}, \cdots, e_{r_8})$ of C^2. Further, the coboundary map $D_0 : C^0 \to C^1$ is presented by a matrix of the following form,

$$D_0 = \begin{pmatrix} (\mathcal{W}_0-1)\mathcal{X}_0'^{-1}\mathcal{X}_1' \\ (\mathcal{X}_0-\mathcal{X}_0'^{-1})\mathcal{X}_1' \\ (\mathcal{Z}_0-1)\mathcal{X}_1' \\ \mathcal{W}_1-1 \\ \mathcal{X}_1\mathcal{X}_1'-1 \\ \mathcal{X}_2\mathcal{X}_2'-1 \\ (\mathcal{Z}_2-1)\mathcal{X}_2' \\ \mathcal{X}_3\mathcal{X}_3'-1 \\ (\mathcal{Z}_3-1)\,\mathcal{X}_3' \end{pmatrix} = \begin{pmatrix} \vdots \\ \vdots \\ \vdots \\ \mathcal{X}_3-1 \\ \mathcal{Z}_3-1 \end{pmatrix},$$

with respect to the basis $(e_{W_0}, e_{X_0}, e_{Z_0}, e_{W_1}, e_{X_1}, e_{X_2}, e_{Z_2}, e_{X_3}, e_{Z_3})$ of C^1.

We consider a subcomplex $\hat{C}^*$ of C^*, as follows. Recalling that X_i and X_i' have fixed points mentioned in Section 3.1, we modify D_1 by multiplying

$$\Big(\mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \quad \mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \quad 1 \quad \mathrm{ad}\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \quad \mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1}$$
$$\mathrm{ad}\begin{pmatrix} x_2 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \quad \mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \quad \mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \Big)$$

from the left, and multiplying

$$\Big(\mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad 1 \quad \mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathrm{ad}\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\mathrm{ad}\begin{pmatrix} x_2 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathrm{ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \Big)^T$$

from the right. Then, the modified D_1 has entries of the following form,

$$\mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \cdot (\mathcal{W}_0 - 1) = \left(\begin{array}{cc|c} -1 & 0 & -1 \\ 1 & 2 & -1 \\ \hline 0 & 0 & 0 \end{array}\right),$$

$$\mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \cdot \mathcal{W}_0 \cdot \mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \left(\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 1 \\ \hline 0 & 0 & 1 \end{array}\right),$$

$$\mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \cdot (-\mathcal{X}_0'^{-1}\mathcal{X}_1') \cdot \mathrm{ad}\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix} = \left(\begin{array}{cc|c} x_1-1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & \frac{1}{x_1-1} \end{array}\right),$$

$\cdots,$

and we can verify that any entry of the modified D_1 is of the following form,

$$\left(\begin{array}{cc|c} * & * & * \\ * & * & * \\ \hline 0 & 0 & * \end{array}\right).$$

Further, we modify D_0 by multiplying

$$\Bigg(\mathrm{ad}\begin{pmatrix}1&1\\1&0\end{pmatrix}^{-1} \quad 1 \quad \mathrm{ad}\begin{pmatrix}1&1\\1&0\end{pmatrix}^{-1} \quad \mathrm{ad}\begin{pmatrix}1&1\\1&0\end{pmatrix}^{-1} \quad \mathrm{ad}\begin{pmatrix}x_1&1\\1&0\end{pmatrix}^{-1}$$
$$\mathrm{ad}\begin{pmatrix}x_2&1\\1&0\end{pmatrix}^{-1} \quad \mathrm{ad}\begin{pmatrix}1&1\\1&0\end{pmatrix}^{-1} \quad \mathrm{ad}\begin{pmatrix}0&1\\1&0\end{pmatrix}^{-1} \quad \mathrm{ad}\begin{pmatrix}1&1\\1&0\end{pmatrix}^{-1} \Bigg)$$

from the left. Then, the modified D_0 has entries of the following form,

$$\begin{aligned} &\cdots, \\ &\mathrm{ad}\begin{pmatrix}0&1\\1&0\end{pmatrix}^{-1}\cdot(\mathcal{X}_3-1) \;=\; \left(\begin{array}{ccc} * & * & 0 \\ \frac{1}{x_2}-1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \\ &\mathrm{ad}\begin{pmatrix}1&1\\1&0\end{pmatrix}^{-1}\cdot(\mathcal{Z}_3-1) \;=\; \left(\begin{array}{ccc} -1 & -4 & 3 \\ -1 & -2 & 1 \\ \hline 0 & 0 & 0 \end{array}\right), \end{aligned}$$

and we can verify that any entry of the modified D_0 is of the following form,

$$\left(\begin{array}{ccc} * & * & * \\ * & * & * \\ \hline 0 & 0 & 0 \end{array}\right).$$

We put $\hat{C}^1$ to be the vector subspace of C^1 consisting of vectors of the form

$$\big(* * 0 \mid * * 0 \mid \ \cdots \ \mid \ \cdots \ \mid * * 0 \big)^T.$$

We put $\hat{C}^2$ to be the vector subspace of C^2 consisting of vectors of the form

$$\big(* * 0 \mid * * 0 \mid \ \cdots \ \mid * * 0 \big)^T.$$

We put $\hat{C}^0 = C^0$. Since the modified D_0 and D_1 preserve these subspaces, $\hat{C}^*$ forms a subcomplex of C^* by these modified D_0 and D_1. We put $\hat{D}_0$ and $\hat{D}_1$ to be the restrictions of these modified D_0 and D_1 to $\hat{C}^*$.

We put $\check{C}^* = C^*/\hat{C}^*$. By definition, $\check{C}^0 = 0$. We put $\check{D}_1$ to be the map on $\check{C}^1$ induced by the modified D_1.

$$\begin{array}{ccccc} \hat{C}^2 & \xleftarrow{\hat{D}_1} & \hat{C}^1 & \xleftarrow{\hat{D}_0} & \hat{C}^0 \\ \downarrow & & \downarrow & & \downarrow \\ C^2 & \longleftarrow & C^1 & \longleftarrow & C^0 \\ \downarrow & & \downarrow & & \downarrow \\ \check{C}^2 & \xleftarrow{\check{D}_1} & \check{C}^1 & \longleftarrow & 0 \end{array}$$

The calculation of the Reidemeister torsion of C^* is reduced to the calculations of the Reidemeister torsions of $\hat{C}^*$ and $\check{C}^*$,

$$\tau(C^*) \;=\; \tau(\hat{C}^*)\,\tau(\check{C}^*).$$

We can verify that $H^2(C^*) \cong H^2(\hat{C}^*) \cong \mathbb{C}$ and $H^1(C^*) \cong H^1(\check{C}^*) \cong \mathbb{C}$ and the other cohomology groups of these cochain complexes vanish.

We calculate the Reidemeister torsion of $\hat{C}^*$, as follows. We define the map $\hat{D}_2 : \hat{C}^2 \to \mathbb{C}$ to be the map evaluating 2-cochains by the cohomology class $[\partial E_K]$ of the boundary of the knot exterior E_K, where we choose the base point of ∂E_K to be the base point of the 2-cell r_3. Then, the following complex forms an acyclic complex,

$$0 \longleftarrow \mathbb{C} \xleftarrow{\hat{D}_2} \hat{C}^2 \xleftarrow{\hat{D}_1} \hat{C}^1 \xleftarrow{\hat{D}_0} \hat{C}^0 \longleftarrow 0.$$

The Reidemeister torsion of $\hat{C}^*$ is presented by

$$\tau(\hat{C}^*) \;=\; \frac{\det\left(\boxed{\;\hat{E}_1\;}\;\boxed{\;\hat{D}_0\;}\right)\,\det\big(\hat{D}_2\hat{E}_2\big)}{\det\left(\boxed{\hat{E}_2}\;\boxed{\;\hat{D}_1\hat{E}_1\;}\right)},$$

where we put

$$\hat{E}_2 \;=\; \begin{pmatrix} 0\\0\\0\\0\\1\\0\\ \vdots\\0 \end{pmatrix}, \qquad \hat{E}_1 \;=\; \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1\\ & & & & 0\\ & & & & 0\\ & & & & 0 \end{pmatrix}.$$

By definition, we have that

$$\det\left(\boxed{\;\hat{E}_1\;}\;\boxed{\;\hat{D}_0\;}\right) \doteq \det\left(\begin{array}{c}\text{the lowest three}\\ \text{rows of } \hat{D}_0\end{array}\right) = \det\begin{pmatrix} \frac{1}{x_2}-1 & 0 & 0\\ -1 & -4 & 3\\ -1 & -2 & 1\end{pmatrix} \doteq 2\,\big(1-\frac{1}{x_2}\big),$$

where "$\doteq$" means that the left-hand side is equal to either of ± 1 multiple of the right-hand side. For a general two-bridge knot, this value becomes $2\left(1-\frac{1}{x_{m-1}}\right)$.

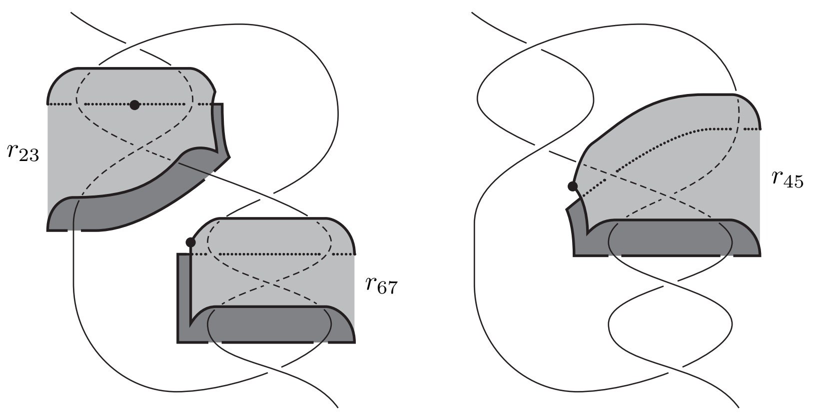
Further, we calculate $\det\left(\hat{D}_2\hat{E}_2\right)$, as follows. As mentioned above, $\hat{D}_2$ is the map evaluating 2-cochains by $[\partial E_K]$ of the boundary ∂E_K of the knot exterior E_K. We regard K as a 1-tangle in a 3-ball B^3. Then, ∂E_K consists of the boundary $\partial N(K)$ of a tubular neighbourhood of K and a 2-holed ∂B^3. Since $\partial N(K)$ is obtained by connecting 2-cells $r_3, r_8, r_6, \cdots$ in the form of a tube along the monodromy, the contribution of $\partial N(K)$ to $\hat{D}_2$ is given by

$$\begin{aligned} & e_{r_3} + \mathcal{X}_0'^{-1}\mathcal{X}_1{}^{-1}e_{r_8} + \mathcal{X}_0'^{-1}\mathcal{X}_1{}^{-1}\mathcal{Z}_3{}^{-1}\mathcal{X}_3{}^{-1}e_{r_6} + \mathcal{X}_0'^{-1}\mathcal{X}_1{}^{-1}\mathcal{Z}_3{}^{-1}\mathcal{X}_3{}^{-1}\mathcal{X}_1'^{-1}e_{r_5} \\ & - \mathcal{X}_0'^{-1}\mathcal{X}_1{}^{-1}\mathcal{Z}_3{}^{-1}\mathcal{X}_3{}^{-1}\mathcal{X}_1'^{-1}\mathcal{W}_0{}^{-1}e_{r_1} + \mathcal{X}_0'^{-1}\mathcal{X}_1{}^{-1}\mathcal{Z}_3{}^{-1}\mathcal{X}_3{}^{-1}\mathcal{X}_1'^{-1}\mathcal{W}_0{}^{-1}e_{r_2} \\ & + \mathcal{X}_0'^{-1}\mathcal{X}_1{}^{-1}\mathcal{Z}_3{}^{-1}\mathcal{X}_3{}^{-1}\mathcal{X}_1'^{-1}\mathcal{W}_0{}^{-1}\mathcal{X}_0'^{-1}\mathcal{X}_1'\,e_{r_4} \\ & + \mathcal{X}_0'^{-1}\mathcal{X}_1{}^{-1}\mathcal{Z}_3{}^{-1}\mathcal{X}_3{}^{-1}\mathcal{X}_1'^{-1}\mathcal{W}_0{}^{-1}\mathcal{X}_0'^{-1}\,\mathcal{X}_1'\,\mathcal{X}_2\,e_{r_7}\,. \end{aligned}$$

Further, the contribution of a 2-holed ∂B^3 to $\hat{D}_2$ is given by

$$-e_{r_1} + (e_{r_2} + \mathcal{W}_0\,e_{r_3}) + \mathcal{W}_0\,(\mathcal{X}_1{}^{-1}e_{r_4} + e_{r_5}) + \mathcal{W}_0\,\mathcal{X}_1{}^{-1}(e_{r_6} + \mathcal{X}_2\,e_{r_7}) + \mathcal{X}_0'^{-1}\mathcal{X}_1'\,e_{r_8}\,.$$

(*How to obtain this formula*: We consider the following 2-chains r_{23}, r_{45}, r_{67}.



The relator around r_{23} is $\mathcal{W}_0\mathcal{X}_0\mathcal{X}_1^{-1}\mathcal{W}_1^{-1}\mathcal{X}_1'^{-1}\mathcal{X}_0'$, and its differential is given by

$$e_{r_{23}} = e_{W_0} + \mathcal{W}_0\,e_{X_0} - \mathcal{X}_0'^{-1}\mathcal{X}_1'\,e_{W_1} - \mathcal{X}_0'^{-1}\mathcal{X}_1'\,\mathcal{W}_1\,e_{X_1} = e_{r_2} + \mathcal{W}_0\,e_{r_3}\,.$$

Similarly, we can show that $e_{r_{45}} = e_{r_4} + \mathcal{X}_1\,e_{r_5}$ and $e_{r_{67}} = e_{r_6} + \mathcal{X}_2\,e_{r_7}$. The constibution of the 2-holed ∂B^3 is obtained by connecting them along the monodromy,

$$-e_{r_1} + e_{r_{23}} + \mathcal{W}_0\,\mathcal{X}_1{}^{-1}e_{r_{45}} + \mathcal{W}_0\,\mathcal{X}_1{}^{-1}e_{r_{67}} + \mathcal{X}_0'^{-1}\mathcal{X}_1'\,e_{r_8}\,,$$

and this gives the above mentioned formula.) Hence, $D_2 : C^2 \to \mathbb{C}$ is presented by

$$D_2 = (0\ \ 0\ \ 1)\Bigg(\Big(-\mathcal{X}_0'^{-1}\mathcal{X}_1{}^{-1}\mathcal{Z}_3{}^{-1}\mathcal{X}_3{}^{-1}\mathcal{X}_1'^{-1}\mathcal{W}_0{}^{-1},\ \mathcal{X}_0'^{-1}\mathcal{X}_1{}^{-1}\mathcal{Z}_3{}^{-1}\mathcal{X}_3{}^{-1}\mathcal{X}_1'^{-1}\mathcal{W}_0{}^{-1},\ 1,\ \cdots\Big) - \Big(-1,\ 1,\ \mathcal{W}_0,\ \cdots\Big)\Bigg),$$

with respect to the basis $e_{r_1}, e_{r_2}, e_{r_3}, \cdots$. Further, $\hat{D}_2$ is the restriction of the modified D_2 to $\hat{C}^2$. Moreover, we can see from the definition of $\hat{E}_2$ that only the part of e_{r_3} contributes to $\hat{D}_2\hat{E}_2$. Hence,

$$\hat{D}_2\hat{E}_2 \;=\; \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \left(1 - \mathcal{W}_0\right) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \;=\; \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3 & -4 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \;=\; 1.$$

Therefore,

$$\det\left(\hat{D}_2\hat{E}_2\right) \;=\; 1.$$

We note that this holds for any two-bridge knot, since only the top 2-cell r_3 contributes to the resulting value, independently of the other part of the knot, as shown above.

The Reidemeister torsion of $\check{C}^*$ is presented by

$$\tau(\check{C}^*) \;=\; \frac{\det\left(\boxed{h_1}\;\boxed{\quad \check{E}_1 \quad}\right)}{\det\left(\check{D}_1\check{E}_1\right)},$$

where we put

$$h_1 \;=\; \begin{pmatrix} * \\ 1 \\ * \\ * \\ \vdots \\ * \end{pmatrix}, \qquad \check{E}_1 \;=\; \begin{pmatrix} 1 & & & & \\ 0 & 0 & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Here, h_1 presents a cohomology class, which evaluates e_{X_1} to be 1. By definition, we have that

$$\det\left(\boxed{h_1}\;\boxed{\quad \check{E}_1 \quad}\right) \;=\; 1.$$

Therefore,

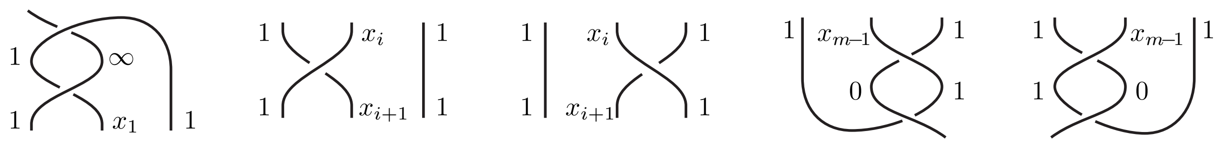
$$\frac{2}{\tau(K)} \;=\; \frac{1}{1 - \frac{1}{x_{m-1}}} \det\left(\check{D}_1\check{E}_1\right) \det\left(\boxed{\hat{E}_2}\;\boxed{\hat{D}_1\hat{E}_1}\right). \tag{14}$$

It is a problem to calculate the latter two factors in the right-hand side. We calculate them for any two-bridge knot in Sections 3.4 and 3.5.

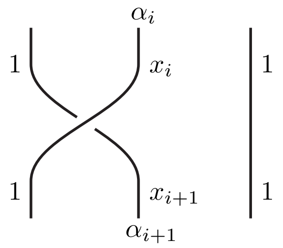
3.3 Decomposition of two-bridge knot diagrams into elementary diagrams

In this section, we decompose open two-bridge knot diagrams into elementary diagrams, and explain how we describe the hyperbolicity equations among parameters of such knot diagrams.

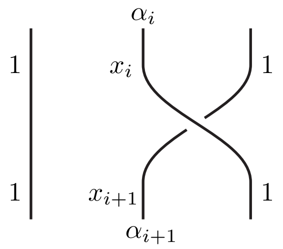
Any open two-bridge knot can be presented by a plat closure of a 3-braid of a product of copies of σ_1 and σ_2^{-1}, *i.e.*, any open two-bridge knot diagram (or its mirror image) can be obtained by gluing copies of the following tangle diagrams, which we call *elementary diagrams*.



To describe the hyperbolicity equations among these parameters, we consider the parameters α_i and α_{i+1} at the ends of middle strands of σ_1 and σ_2^{-1}, as follows,

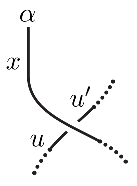


$$\alpha_i = \frac{1 - \frac{1}{x_i}}{1 - \frac{x_{i+1}}{x_i}}, \qquad \alpha_{i+1} = \frac{1 - x_{i+1}}{1 - \frac{x_{i+1}}{x_i}},$$

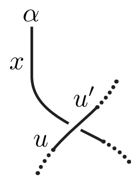


$$\alpha_i = \frac{1 - \frac{x_{i+1}}{x_i}}{1 - \frac{1}{x_i}}, \qquad \alpha_{i+1} = \frac{1 - \frac{x_{i+1}}{x_i}}{1 - x_{i+1}}.$$

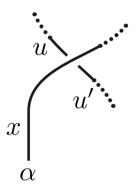
In general, for a parameterized tangle, we consider the parameter α at the end of a strand of the tangle diagram, as follows,



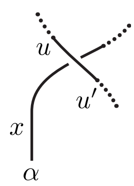
$$\alpha = \frac{1 - \frac{u}{x}}{1 - \frac{u'}{x}},$$



$$\alpha = \frac{1 - \frac{x}{u}}{1 - \frac{x}{u'}},$$



$$\alpha = \frac{1 - \frac{u}{x}}{1 - \frac{u'}{x}},$$



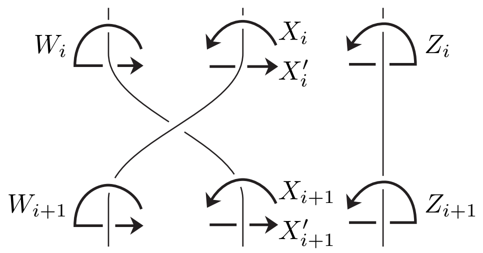
$$\alpha = \frac{1 - \frac{x}{u}}{1 - \frac{x}{u'}}.$$

When we glue two tangle diagrams, it is required that these parameters coincide at each connecting point, which implies the hyperbolicity equation among parameters of the resulting tangle diagram.

3.4 Calculation of $\det\left(\hat{E}_2 \,\middle|\, \hat{D}_1\hat{E}_1\right)$

In this section, we calculate $\det\left(\boxed{\hat{E}_2}\;\boxed{\hat{D}_1\hat{E}_1}\right)$ for any two-bridge knot.

We consider the contribution of σ_1 to $\det\left(\boxed{\hat{E}_2}\;\boxed{\hat{D}_1\hat{E}_1}\right)$.



As explained in Section 3.1, we have that

$$X'_i \sim \begin{pmatrix} \frac{x_{i+1}}{x_i}-1 & 1-x_{i+1} \\ 0 & \frac{1}{x_i}-1 \end{pmatrix}, \qquad X_{i+1} \sim \begin{pmatrix} \frac{x_{i+1}}{x_i}-1 & 0 \\ \frac{1}{x_i}-1 & x_{i+1}-1 \end{pmatrix}.$$

The relators among these matrices are given by

$$\begin{aligned} & W_i\, X_i'^{-1} X_{i+1}^{-1} X_{i+1}'^{\,-1} X_i', \\ & X_i\, X_{i+1}^{-1} W_{i+1}^{-1} X_{i+1}\, X_i', \\ & Z_i\, Z_{i+1}^{-1}, \end{aligned}$$

and, as explained in Section 3.2, their differentials are given by

$$\begin{aligned} & e_{W_i} - \mathcal{X}_i'^{-1}\mathcal{X}_{i+1}' e_{X_{i+1}}, \\ & e_{X_i} - \mathcal{X}_i'^{-1}\mathcal{X}_{i+1}^{-1} e_{W_{i+1}} + \mathcal{X}_i'^{-1}\mathcal{X}_{i+1}^{-1}(1-\mathcal{W}_{i+1}) e_{X_{i+1}}, \\ & e_{Z_i} - e_{Z_{i+1}}. \end{aligned}$$

Hence, the corresponding part of D_1 is presented by

$$D_1 = \left(\begin{array}{ccc|ccc} \ddots & \ddots & \ddots & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & -\mathcal{X}_i'^{-1}\mathcal{X}_{i+1}' & 0 \\ 0 & 1 & 0 & -\mathcal{X}_i'^{-1}\mathcal{X}_{i+1}^{-1} & \mathcal{X}_i'^{-1}\mathcal{X}_{i+1}^{-1}(1-\mathcal{W}_{i+1}) & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & \ddots & \ddots & \ddots \end{array}\right)$$

with respect to the basis $(e_{W_i}, e_{X_i}, e_{Z_i}, e_{W_{i+1}}, e_{X_{i+1}}, e_{Z_{i+1}})$. It is necessary to calculate the determinant of the following matrices,

$$\left(\begin{array}{ccc|ccc} A_i & B_i & C_i & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & -\mathcal{X}_i'^{-1}\mathcal{X}_{i+1}' & 0 \\ 0 & 1 & 0 & -\mathcal{X}_i'^{-1}\mathcal{X}_{i+1}^{-1} & \mathcal{X}_i'^{-1}\mathcal{X}_{i+1}^{-1}(1-\mathcal{W}_{i+1}) & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & \ddots & \ddots & \ddots \end{array}\right)$$

$$
\approx \left(\begin{array}{ccc|ccc}
1 & 0 & 0 & 0 & -\mathcal{X}_i'^{-1}\mathcal{X}_{i+1}' & 0 \\
0 & 1 & 0 & -\mathcal{X}_i'^{-1}\mathcal{X}_{i+1}^{-1} & \mathcal{X}_i'^{-1}\mathcal{X}_{i+1}^{-1}(1-\mathcal{W}_{i+1}) & 0 \\
0 & 0 & 1 & 0 & 0 & -1 \\
\hline
A_i & B_i & C_i & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \ddots & \ddots & \ddots
\end{array}\right)
$$

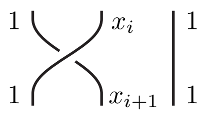
$$
\approx \left(\begin{array}{ccc|ccc}
1 & 0 & 0 & 0 & -\mathcal{X}_i'^{-1}\mathcal{X}_{i+1}' & 0 \\
0 & 1 & 0 & -\mathcal{X}_i'^{-1}\mathcal{X}_{i+1}^{-1} & \mathcal{X}_i'^{-1}\mathcal{X}_{i+1}^{-1}(1-\mathcal{W}_{i+1}) & 0 \\
0 & 0 & 1 & 0 & 0 & -1 \\
\hline
0 & 0 & 0 & A_{i+1} & B_{i+1} & C_{i+1} \\
\hline
0 & 0 & 0 & \ddots & \ddots & \ddots
\end{array}\right)
$$

$$
\approx \left(\begin{array}{ccc}
A_{i+1} & B_{i+1} & C_{i+1} \\
\hline
\ddots & \ddots & \ddots
\end{array}\right),
$$

where '≈" means that the matrices of both sides are related by elementary transformations (hence, they have equal determinants), and we put

$$
\begin{pmatrix} A_{i+1} & B_{i+1} & C_{i+1} \end{pmatrix} = \begin{pmatrix} A_i & B_i & C_i \end{pmatrix} \begin{pmatrix} 0 & \mathcal{X}_i'^{-1}\mathcal{X}_{i+1}' & 0 \\ \mathcal{X}_i'^{-1}\mathcal{X}_{i+1}^{-1} & \mathcal{X}_i'^{-1}\mathcal{X}_{i+1}^{-1}(\mathcal{W}_{i+1}-1) & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

Hence, σ_1 is taken by the "representation" by



$$
\longmapsto \begin{pmatrix} 0 & \mathcal{X}_i'^{-1}\mathcal{X}_{i+1}' & 0 \\ \mathcal{X}_i'^{-1}\mathcal{X}_{i+1}^{-1} & \mathcal{X}_i'^{-1}\mathcal{X}_{i+1}^{-1}(\mathcal{W}_{i+1}-1) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{15}
$$

We consider the subcomplex $\hat{C}^*$ as in Section 3.2, and consider the corresponding matrix to the matrix of (15). To the matrix of (15), we multiply

$$
\left(\mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \quad \mathrm{ad}\begin{pmatrix} x_i & 1 \\ 1 & 0 \end{pmatrix}^{-1} \quad \mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \right)
$$

from the left, and multiply

$$
\left(\mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathrm{ad}\begin{pmatrix} x_{i+1} & 1 \\ 1 & 0 \end{pmatrix} \quad \mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right)^T
$$

from the right. Then, the entries of the resulting matrix are presented by

$$
\mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \cdot \mathcal{X}_i'^{-1}\mathcal{X}_{i+1}' \cdot \mathrm{ad}\begin{pmatrix} x_{i+1} & 1 \\ 1 & 0 \end{pmatrix} = \left(\begin{array}{cc|c} -\frac{x_i(x_{i+1}-1)}{x_i-1} & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & -\frac{x_i-1}{x_i(x_{i+1}-1)} \end{array}\right),
$$

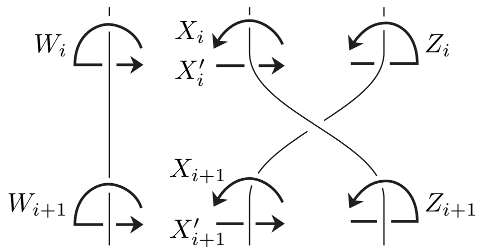
$$
\mathrm{ad}\begin{pmatrix} x_i & 1 \\ 1 & 0 \end{pmatrix}^{-1} \cdot \mathcal{X}_i'^{-1}\mathcal{X}_{i+1}^{-1} \cdot \mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \left(\begin{array}{cc|c} -\frac{x_{i+1}-1}{(x_i-1)x_i} & \frac{2}{x_i} & * \\ 0 & 1 & * \\ \hline 0 & 0 & -\frac{(x_i-1)x_i}{x_{i+1}-1} \end{array}\right),
$$

$$\mathrm{ad}\begin{pmatrix} x_i & 1\\ 1 & 0\end{pmatrix}^{-1}\cdot \mathcal{X}_i'^{-1}\mathcal{X}_{i+1}^{-1}(\mathcal{W}_{i+1}-1)\cdot \mathrm{ad}\begin{pmatrix} x_{i+1} & 1\\ 1 & 0\end{pmatrix}$$
$$= \left(\begin{array}{cc|c} -\frac{(x_{i+1}-1)^2(2x_i+x_{i+1}-1)}{(x_i-1)x_i} & \frac{2(x_{i+1}-1)(2x_i+x_{i+1}-2)}{(x_i-1)x_i} & * \\ -(x_{i+1}-1)^2 & 2(x_{i+1}-1) & * \\ \hline 0 & 0 & 0\end{array}\right).$$

Hence, the restriction of the matrix of (15) to the subcomplex $\hat{C}^*$ is presented by

$$\left(\begin{array}{cc|cc|cc} 0 & 0 & -\frac{x_i(x_{i+1}-1)}{x_i-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline -\frac{x_{i+1}-1}{(x_i-1)x_i} & \frac{2}{x_i} & -\frac{(x_{i+1}-1)^2(2x_i+x_{i+1}-1)}{(x_i-1)x_i} & \frac{2(x_{i+1}-1)(2x_i+x_{i+1}-2)}{(x_i-1)x_i} & 0 & 0 \\ 0 & 1 & -(x_{i+1}-1)^2 & 2(x_{i+1}-1) & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right). \qquad (16)$$

We consider the contribution of σ_2^{-1} to $\det\left(\boxed{\hat{E}_2}\ \boxed{\hat{D}_1\hat{E}_1}\right)$.



As explained in Section 3.1, we have that

$$X_i' \sim \begin{pmatrix} \frac{1}{x_i}-1 & x_{i+1}-1 \\ 0 & \frac{x_{i+1}}{x_i}-1\end{pmatrix}, \qquad X_{i+1} \sim \begin{pmatrix} x_{i+1}-1 & 0 \\ 1-\frac{1}{x_i} & \frac{x_{i+1}}{x_i}-1\end{pmatrix}.$$

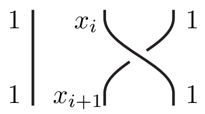
The relators among these matrices are given by

$$\begin{aligned} &W_i\,W_{i+1}^{-1}\,,\\ &X_i\,X_i'\,X_{i+1}\,Z_{i+1}^{-1}X_{i+1}^{-1}\,,\\ &Z_i\,X_i'\,X_{i+1}'^{-1}X_{i+1}^{-1}X_i'^{-1}. \end{aligned}$$

Hence, similarly as the case of σ_1, the corresponding part of D_1 is presented by

$$D_1 = \left(\begin{array}{ccc|ccc} \ddots & \ddots & \ddots & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \mathcal{X}_i\mathcal{X}_i'-1 & -\mathcal{X}_{i+1} \\ 0 & 0 & 1 & 0 & -\mathcal{X}_i' & 0 \\ \hline 0 & 0 & 0 & \ddots & \ddots & \ddots \end{array}\right)$$

with respect to the basis $(e_{W_i}, e_{X_i}, e_{Z_i}, e_{W_{i+1}}, e_{X_{i+1}}, e_{Z_{i+1}})$. Further, σ_2^{-1} is taken by the "representation" by



$$\longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\mathcal{X}_i\mathcal{X}_i' & \mathcal{X}_{i+1} \\ 0 & \mathcal{X}_i' & 0 \end{pmatrix}. \tag{17}$$

We consider its restriction to the subcomplex $\hat{C}^*$, similarly as the case of σ_1. To the matrix of (17), we multiply

$$\left(\mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \quad \mathrm{ad}\begin{pmatrix} x_i & 1 \\ 1 & 0 \end{pmatrix}^{-1} \quad \mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \right)$$

from the left, and multiply

$$\left(\mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathrm{ad}\begin{pmatrix} x_{i+1} & 1 \\ 1 & 0 \end{pmatrix} \quad \mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right)^T$$

from the right. Then, the entries of the resulting matrix are presented by

$$\begin{aligned}
&\mathrm{ad}\begin{pmatrix} x_i & 1 \\ 1 & 0 \end{pmatrix}^{-1} \cdot (1-\mathcal{X}_i\mathcal{X}_i') \cdot \mathrm{ad}\begin{pmatrix} x_{i+1} & 1 \\ 1 & 0 \end{pmatrix} \\
&\qquad = \left(\begin{array}{cc|c} \frac{(2x_i-x_{i+1}+1)(x_{i+1}-1)}{x_i^2} & \frac{2(x_i-x_{i+1}+1)(x_{i+1}-1)}{x_i^2(x_i-x_{i+1})} & * \\ \frac{(x_i-x_{i+1})(x_{i+1}-1)}{x_i} & \frac{2(x_{i+1}-1)}{x_i} & * \\ \hline 0 & 0 & 0 \end{array} \right),
\end{aligned}$$

$$\mathrm{ad}\begin{pmatrix} x_i & 1 \\ 1 & 0 \end{pmatrix}^{-1} \cdot \mathcal{X}_{i+1} \cdot \mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \left(\begin{array}{cc|c} -\frac{x_{i+1}-1}{x_i(x_i-x_{i+1})} & \frac{2(x_i-1)}{x_i(x_i-x_{i+1})} & * \\ 0 & 1 & * \\ \hline 0 & 0 & \frac{x_i(x_{i+1}-x_i)}{x_{i+1}-1} \end{array} \right),$$

$$\mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \cdot \mathcal{X}_i' \cdot \mathrm{ad}\begin{pmatrix} x_{i+1} & 1 \\ 1 & 0 \end{pmatrix} = \left(\begin{array}{cc|c} \frac{x_i-x_{i+1}}{x_i-1} & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & \frac{x_i-1}{x_i-x_{i+1}} \end{array} \right).$$

Hence, the restriction of the matrix of (17) to the subcomplex $\hat{C}^*$ is presented by

$$\left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{(2x_i-x_{i+1}+1)(x_{i+1}-1)}{x_i^2} & \frac{2(x_i-x_{i+1}+1)(x_{i+1}-1)}{x_i^2(x_i-x_{i+1})} & -\frac{x_{i+1}-1}{x_i(x_i-x_{i+1})} & \frac{2(x_i-1)}{x_i(x_i-x_{i+1})} \\ 0 & 0 & \frac{(x_i-x_{i+1})(x_{i+1}-1)}{x_i} & \frac{2(x_{i+1}-1)}{x_i} & 0 & 1 \\ \hline 0 & 0 & \frac{x_i-x_{i+1}}{x_i-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right). \tag{18}$$

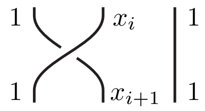
The matrices (16) and (18) give a 6-dimensional "representation" of parameterized 3-braids. In fact, only a 3-dimensional subspace contributes to the calculation of the required value. A basis of this 3-dimensional subspace is given by

$$e_1 = \begin{pmatrix} 0 & 0 & -\alpha_i(x_i-1)x_i & 1-\alpha_i+x_i+\alpha_i x_i & -(\alpha_i-1)(x_i-1) & (\alpha_i-1)(x_i-3) \end{pmatrix},$$

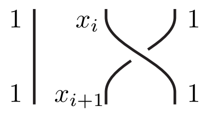
$$e_2 = \begin{pmatrix} 0 & -\alpha_i & \alpha_i(x_i-1)x_i & -1+\alpha_i-2\alpha_i x_i & 0 & -1 \end{pmatrix},$$
$$e_3 = \begin{pmatrix} (\alpha_i-1)(x_i-1) & (\alpha_i-1)(x_i+1) & -\alpha_i(x_i-1)x_i & 1-\alpha_i-x_i+3\alpha_i x_i & 0 & 0 \end{pmatrix}.$$

(*How to obtain this basis*: $(\sigma_1\sigma_2)^3$ is a central element of the 3-braid group. For any open two-bridge knot, when we insert $(\sigma_1\sigma_2)^3$ at any place of a 3-braid, the knot type of the open two-bridge knot does not change. Further, the values of x_i and α_i are invariant under the action of $(\sigma_1\sigma_2)^3$. Hence, only 1-eigenspace with respect to the action of $(\sigma_1\sigma_2)^3$ contributes to the required value. This 1-eigenspace is the above mentioned 3-dimensional subspace. In this 1-eigenspace, we obtain e_3 as an eigenvector of the action of σ_1, and obtain e_1 as an eigenvector of the action of σ_2, and obtain e_2 as an eigenvector of the action of $\sigma_1\sigma_2\sigma_1$. We note that these "eigenvectors" are eigenvectors with respect to the action of multiplying matrices to row vectors from the right.)

With respect to the basis (e_1, e_2, e_3), the matrices (16) and (18) are rewritten as the following matrices, by which we define a "representation" Φ_2 of parameterized 3-braids,

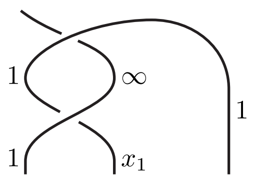
$\Phi_2\Bigg($  $\Bigg)$

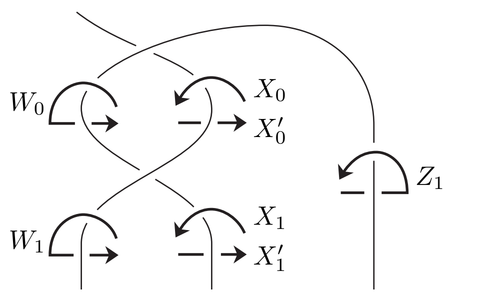
$$= -\frac{1}{x_{i+1}}\begin{pmatrix} 1 & 2\,x_{i+1} & 1 \\ 0 & -x_{i+1} & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

$\Phi_2\Bigg($  $\Bigg)$

$$= \frac{1}{1-\alpha_{i+1}}\begin{pmatrix} 1 & 0 & 0 \\ -1 & -x_{i+1} & 0 \\ 1 & 2\,x_{i+1} & 1 \end{pmatrix}. \tag{19}$$

We define Φ_2 of the top part of an open two-bridge knot, as follows.





As explained in Section 3.1, we have that

$$X_0 \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_0' \sim \begin{pmatrix} 1 & x_1-1 \\ 0 & 1 \end{pmatrix}, \quad X_1 \sim \begin{pmatrix} 1 & 0 \\ 1 & 1-x_1 \end{pmatrix}, \quad X_1' \sim \begin{pmatrix} \frac{1}{x_1} & -x_1 \\ 0 & \frac{1}{x_1}-1 \end{pmatrix}.$$

As explained in Section 3.2, the corresponding part of D_1 is presented by

$$\left(\begin{array}{cc|ccc} 1 & \mathcal{W}_0-1 & 0 & 0 & \mathcal{Z}_1^{-1} \\ \hline 1 & 0 & 0 & -\mathcal{X}_0'^{-1}\mathcal{X}_1' & 0 \\ 0 & 1 & -\mathcal{X}_0'^{-1}\mathcal{X}_1^{-1} & \mathcal{X}_0'^{-1}\mathcal{X}_1^{-1}(1-\mathcal{W}_1) & 0 \\ \hline & & \ddots & \ddots & \ddots \end{array}\right) \tag{20}$$

with respect to the basis $(e_{W_0}, e_{X_0}, e_{W_1}, e_{X_1}, e_{Z_1})$. To the above matrix, we multiply

$$\begin{pmatrix} \mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} & \mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} & 1 \end{pmatrix}$$

from the left, and multiply

$$\left(\mathrm{ad}\begin{pmatrix}1&1\\1&0\end{pmatrix}\quad 1\quad \mathrm{ad}\begin{pmatrix}1&1\\1&0\end{pmatrix}\quad \mathrm{ad}\begin{pmatrix}x_1&1\\1&0\end{pmatrix}\quad \mathrm{ad}\begin{pmatrix}1&1\\1&0\end{pmatrix}\right)^T$$

from the right. Then, the entries of the resulting matrix are presented by

$$\begin{aligned}
\mathrm{ad}\begin{pmatrix}1&1\\1&0\end{pmatrix}^{-1}\cdot(\mathcal{W}_0-1) &= \left(\begin{array}{cc|c}-1&0&-1\\1&2&-1\\\hline 0&0&0\end{array}\right),\\
\mathrm{ad}\begin{pmatrix}1&1\\1&0\end{pmatrix}^{-1}\cdot\mathcal{Z}_1^{-1}\cdot\mathrm{ad}\begin{pmatrix}1&1\\1&0\end{pmatrix} &= \left(\begin{array}{cc|c}1&-2&-1\\0&1&1\\\hline 0&0&1\end{array}\right),\\
\mathrm{ad}\begin{pmatrix}1&1\\1&0\end{pmatrix}^{-1}\cdot(-\mathcal{X}_0'^{-1}\mathcal{X}_1')\cdot\mathrm{ad}\begin{pmatrix}x_1&1\\1&0\end{pmatrix} &= \left(\begin{array}{cc|c}x_1-1&0&0\\0&-1&0\\\hline 0&0&\frac{1}{x_1-1}\end{array}\right),\\
(-\mathcal{X}_0'^{-1}\mathcal{X}_1^{-1})\cdot\mathrm{ad}\begin{pmatrix}1&1\\1&0\end{pmatrix} &= \left(\begin{array}{cc|c}1-x_1&0&0\\0&-1&0\\\hline 0&0&\frac{1}{1-x_1}\end{array}\right),\\
\mathcal{X}_0'^{-1}\mathcal{X}_1^{-1}(1-\mathcal{W}_1)\cdot\mathrm{ad}\begin{pmatrix}x_1&1\\1&0\end{pmatrix} &= \left(\begin{array}{cc|c}*&*&x_1-1\\(x_1-1)^2&2(1-x_1)&-1\\\hline 0&0&0\end{array}\right).
\end{aligned}$$

Hence, the restriction of the matrix (20) to the subcomplex $\hat{C}^*$ is presented by

$$\left(\begin{array}{cc|cc|cc|cc|cc}
1&0&-1&0&0&0&0&0&1&-2\\
0&1&1&2&0&0&0&0&0&1\\\hline
1&0&0&0&0&0&x_1-1&0&0&0\\
0&1&0&0&0&0&0&-1&0&0\\\hline
0&0&1&0&1-x_1&0&*&*&0&0\\
0&0&0&1&0&-1&(x_1-1)^2&2(1-x_1)&0&0\\\hline
&&&&\ddots&\ddots&\ddots&\ddots&\ddots&\ddots
\end{array}\right).$$

When we calculate $\det\left(\boxed{\hat{E}_2}\ \boxed{\hat{D}_1\hat{E}_1}\right)$, we remove the fifth row of $\hat{D}_1$ from the definition of $\hat{E}_2$. The matrix obtained from the above matrix by removing the fifth row is equivalent to the following matrix by elementary transformations,

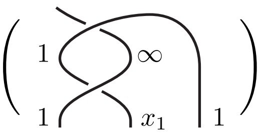
$$\left(\begin{array}{cc|cc|cc}
0&-2&2x_1^2-3x_1+1&-4x_1+3&-1&1\\\hline
\ddots&\ddots&\ddots&\ddots&\ddots&\ddots
\end{array}\right).$$

Further, the vector

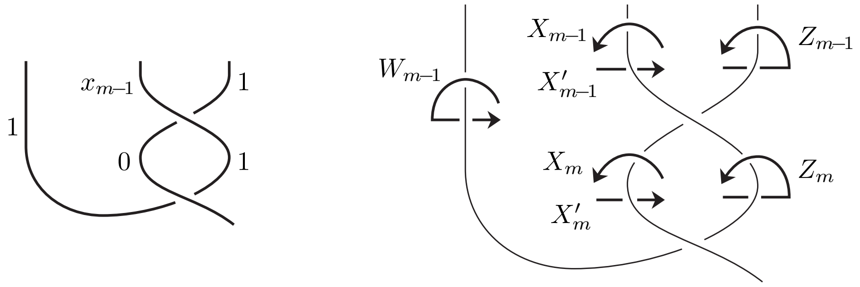
$$\begin{pmatrix}0&-2&2x_1^2-3x_1+1&-4x_1+3&-1&1\end{pmatrix}$$

is rewritten as the following vector with respect to the basis (e_1, e_2, e_3), by which we define Φ_2 of the top part of an open two-bridge knot,

$$\Phi_2\Big(\ \ \ \ \ \Big) \;=\; -\frac{1}{x_1(x_1-1)}\begin{pmatrix}1 & 2x_1 & 0\end{pmatrix}. \tag{21}$$



We define Φ_2 of a bottom part of an open two-bridge knot, as follows.



As explained in Section 3.1, we have that

$$X_{m-1}\sim\begin{pmatrix}1-\frac{1}{x_{m-1}} & 1\\ 0 & 1\end{pmatrix},\quad X'_{m-1}\sim\begin{pmatrix}x_{m-1}-1 & 0\\ -\frac{1}{x_{m-1}} & x_{m-1}\end{pmatrix},$$

$$X_m\sim\begin{pmatrix}1 & 0\\ \frac{1}{x_{m-1}}-1 & 1\end{pmatrix},\quad X'_m\sim\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

noting that $x_m=0$. By calculating D_1 at the bottom of an open two-bridge knot similarly as above, the corresponding part of D_1 is presented by

$$\left(\begin{array}{ccc|cc} \ddots & \ddots & \ddots & & \\ \hline 1 & 0 & 0 & 0 & \mathcal{Z}_m^{-1}\\ 0 & 1 & 0 & \mathcal{X}_{m-1}\mathcal{X}'_{m-1}-1 & -\mathcal{X}_m\\ 0 & 0 & 1 & -\mathcal{X}'_{m-1} & 0 \end{array}\right) \tag{22}$$

with respect to the basis $(e_{W_{m-1}}, e_{X_{m-1}}, e_{Z_{m-1}}, e_{X_m}, e_{Z_m})$. To this matrix (22), we multiply

$$\left(\operatorname{ad}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^{-1}\ \operatorname{ad}\begin{pmatrix}x_{m-1} & 1\\ 1 & 0\end{pmatrix}^{-1}\ \operatorname{ad}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^{-1}\right)$$

from the left, and multiply

$$\left(\operatorname{ad}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\ \operatorname{ad}\begin{pmatrix}x_{m-1} & 1\\ 1 & 0\end{pmatrix}\ \operatorname{ad}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\ \operatorname{ad}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\ \operatorname{ad}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\right)^T$$

from the right. Then, the entries of the resulting matrix are presented by

$$\operatorname{ad}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^{-1}\cdot\mathcal{Z}_m^{-1}\cdot\operatorname{ad}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} = \left(\begin{array}{cc|c} 1 & -2 & -1\\ 0 & 1 & 1\\ \hline 0 & 0 & 1\end{array}\right),$$

$$\operatorname{ad}\begin{pmatrix} x_{m-1} & 1 \\ 1 & 0 \end{pmatrix}^{-1} \cdot (\mathcal{X}_{m-1}\mathcal{X}'_{m-1} - 1) \cdot \operatorname{ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \left(\begin{array}{cc|c} \frac{2x_{m-1}+1}{x_{m-1}^2} & * & * \\ 1 & * & * \\ \hline 0 & 0 & 0 \end{array}\right),$$

$$\operatorname{ad}\begin{pmatrix} x_{m-1} & 1 \\ 1 & 0 \end{pmatrix}^{-1} \cdot (-\mathcal{X}_m) \cdot \operatorname{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \left(\begin{array}{cc|c} -\frac{1}{x_{m-1}^2} & * & * \\ 0 & -1 & * \\ \hline 0 & 0 & -x_{m-1}^2 \end{array}\right),$$

$$\operatorname{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \cdot (-\mathcal{X}'_{m-1}) \cdot \operatorname{ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \left(\begin{array}{cc|c} \frac{x_{m-1}}{1-x_{m-1}} & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & \frac{1-x_{m-1}}{x_{m-1}} \end{array}\right).$$

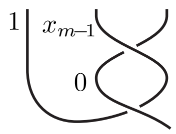
Hence, the restriction of the matrix (22) to the subcomplex $\hat{C}^*$ is presented by

$$\left(\begin{array}{cc|cc|cc|cc|cc} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & \frac{2x_{m-1}+1}{x_{m-1}^2} & * & * & * \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & * & * & * \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & \frac{x_{m-1}}{1-x_{m-1}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \end{array}\right).$$

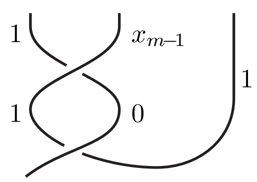
When we calculate $\det\left(\boxed{\hat{E}_2}\ \boxed{\hat{D}_1\hat{E}_1}\right)$, we remove the rightmost three columns of $\hat{D}_1$ from the definition of $\hat{E}_1$. We remove the rightmost three columns from the above matrix, and insert each of e_1, e_2, e_3 into the first row. Then, putting $\alpha_{m-1} = x_{m-1}/(x_{m-1}-1)$, their determinants are equal to

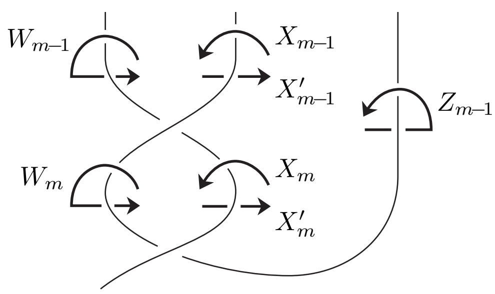
$$\frac{x_{m-1}}{x_{m-1}-1}, \quad -\frac{x_{m-1}}{x_{m-1}-1}, \quad 2\cdot\frac{x_{m-1}}{x_{m-1}-1}$$

respectively. Hence, we define Φ_2 of the bottom part of an open two-bridge knot by

$$\Phi_2\Bigg(\quad$$ 
$$\Bigg) = \frac{x_{m-1}}{x_{m-1}-1}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}. \tag{23}$$

We define Φ_2 of the other bottom part of an open two-bridge knot, as follows.





As explained in Section 3.1, we have that

$$X_{m-1} \sim \begin{pmatrix} x_{m-1} & 0 \\ \frac{1}{x_{m-1}} & x_{m-1}{-}1 \end{pmatrix}, \quad X'_{m-1} \sim \begin{pmatrix} -1 & 1 \\ 0 & \frac{1}{x_{m-1}}{-}1 \end{pmatrix},$$
$$X_m \sim \begin{pmatrix} 1 & 0 \\ 1{-}\frac{1}{x_{m-1}} & 1 \end{pmatrix}, \quad X'_m \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

noting that $x_m = 0$. By calculating D_1 at the bottom of an open two-bridge knot similarly as above, the corresponding part of D_1 is presented by

$$\left(\begin{array}{ccc|cc} \ddots & \ddots & \ddots & & \\ \hline 1 & 0 & 0 & -\mathcal{X}'^{\,-1}_{m-1} & 0 \\ 0 & 1 & 0 & \mathcal{X}'^{\,-1}_{m-1}\mathcal{X}_m^{-1}(1{-}\mathcal{W}_m) & -\mathcal{X}'^{\,-1}_{m-1}\mathcal{X}_m^{-1} \\ 0 & 0 & 1 & 0 & \mathcal{Z}_{m-1} \end{array}\right) \tag{24}$$

with respect to the basis $(e_{W_{m-1}}, e_{X_{m-1}}, e_{Z_{m-1}}, e_{X_m}, e_{W_m})$. To this matrix (24), we multiply

$$\left(\mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \ \mathrm{ad}\begin{pmatrix} x_{m-1} & 1 \\ 1 & 0 \end{pmatrix}^{-1} \ \mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1}\right)$$

from the left, and multiply

$$\left(\mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \ \mathrm{ad}\begin{pmatrix} x_{m-1} & 1 \\ 1 & 0 \end{pmatrix} \ \mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \ \mathrm{ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ \mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\right)^T$$

from the right. Then, the entries of the resulting matrix are presented by

$$\mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \cdot (-\mathcal{X}'^{\,-1}_{m-1}) \cdot \mathrm{ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \left(\begin{array}{cc|c} \frac{x_{m-1}}{1-x_{m-1}} & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & \frac{1-x_{m-1}}{x_{m-1}} \end{array}\right),$$

$$\mathrm{ad}\begin{pmatrix} x_{m-1} & 1 \\ 1 & 0 \end{pmatrix}^{-1} \cdot (-\mathcal{X}'^{\,-1}_{m-1}\mathcal{X}_m^{-1}) \cdot \mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \left(\begin{array}{cc|c} * & -\frac{2}{x_{m-1}} & * \\ 0 & -1 & * \\ \hline 0 & 0 & x_{m-1}(1{-}x_{m-1}) \end{array}\right),$$

$$\mathrm{ad}\begin{pmatrix} x_{m-1} & 1 \\ 1 & 0 \end{pmatrix}^{-1} \cdot \mathcal{X}'^{\,-1}_{m-1}\mathcal{X}_m^{-1}(1{-}\mathcal{W}_m) \cdot \mathrm{ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \left(\begin{array}{cc|c} \frac{2x_{m-1}-1}{x_{m-1}(x_{m-1}-1)} & \frac{4}{x_{m-1}} & * \\ 1 & 2 & -1 \\ \hline 0 & 0 & 0 \end{array}\right),$$

$$\mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \cdot \mathcal{Z}_{m-1} \cdot \mathrm{ad}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \left(\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 1 & -1 \\ \hline 0 & 0 & 1 \end{array}\right).$$

Hence, the restriction of the matrix (24) to the subcomplex $\hat{C}^*$ is presented by

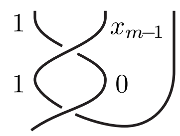
$$\left(\begin{array}{cc|cc|cc|cc|cc}
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & \frac{x_{m-1}}{1-x_{m-1}} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
\hline
0 & 0 & 1 & 0 & 0 & 0 & \frac{2x_{m-1}-1}{x_{m-1}(x_{m-1}-1)} & * & * & * \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & * & * & * \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right).$$

Similarly as the above case, we remove the rightmost three columns from the above matrix, and insert each of e_1, e_2, e_3 into the first row. Then, putting $\alpha_{m-1} = 1 - \frac{1}{x_{m-1}}$, their determinants are equal to

$$-2, \qquad 1, \qquad -1$$

respectively. Hence, we define Φ_2 of this bottom part of an open two-bridge knot by

$$\Phi_2\Big(\quad\text{[diagram]}\quad\Big) = -\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}. \tag{25}$$



For a diagram D of any open two-bridge knot, by decomposing D into a union of elementary tangle diagrams, we define $\Phi_2(D)$ to be the composition of Φ_2 of such elementary diagrams, whose values are given in (19), (21), (23), (25). Then, by the above arguments, we have that

$$\det\left(\boxed{\hat{E}_2}\ \boxed{\hat{D}_1\hat{E}_1}\right) = \Phi_2(D). \tag{26}$$

3.5 Calculation of $\det(\check{D}_1\check{E}_1)$

In this section, we calculate $\det\left(\check{D}_1\check{E}_1\right)$ for any two-bridge knot.

Similarly as in Section 3.4, we can see that, at the top part of an open two-bridge knot diagram, $\check{D}_1$ is presented by

$$\left(\begin{array}{cc|ccc}
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & \frac{1}{x_1-1} & 0 \\
0 & 1 & \frac{1}{1-x_1} & 0 & 0 \\
\hline
0 & 0 & \ddots & \ddots & \ddots
\end{array}\right)$$

with respect to the basis $e_{W_0}, e_{X_0}, e_{W_1}, e_{X_1}, e_{Z_1}$. Further, at the part of σ_1, $\check{D}_1$ is presented by

$$\left(\begin{array}{ccc|ccc} \ddots & \ddots & \ddots & & & \\ \hline 1 & 0 & 0 & 0 & \frac{x_i-1}{x_i(x_{i+1}-1)} & 0 \\ 0 & 1 & 0 & \frac{x_i(x_i-1)}{x_{i+1}-1} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ \hline & & & \ddots & \ddots & \ddots \end{array}\right)$$

with respect to the basis $e_{W_i}, e_{X_i}, e_{Z_i}, e_{W_{i+1}}, e_{X_{i+1}}, e_{Z_{i+1}}$. At the part of σ_2^{-1}, $\check{D}_1$ is presented by

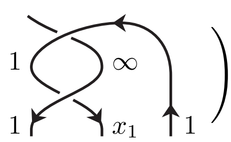
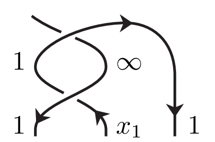
$$\left(\begin{array}{ccc|ccc} \ddots & \ddots & \ddots & & & \\ \hline 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{x_i(x_i-x_{i+1})}{x_{i+1}-1} \\ 0 & 0 & 1 & 0 & \frac{x_i-1}{x_{i+1}-x_i} & 0 \\ \hline & & & \ddots & \ddots & \ddots \end{array}\right)$$

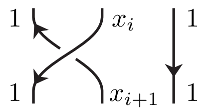
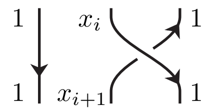
with respect to the basis $e_{W_i}, e_{X_i}, e_{Z_i}, e_{W_{i+1}}, e_{X_{i+1}}, e_{Z_{i+1}}$. At the bottom parts of an open two-bridge knot diagram, $\check{D}_1$ are presented by

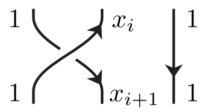
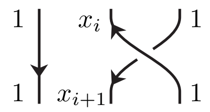
$$\left(\begin{array}{ccc|cc} \ddots & \ddots & \ddots & & \\ \hline 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -x_{m-1}^2 \\ 0 & 0 & 1 & \frac{1-x_{m-1}}{x_{m-1}} & 0 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{ccc|cc} \ddots & \ddots & \ddots & & \\ \hline 1 & 0 & 0 & \frac{1-x_{m-1}}{x_{m-1}} & 0 \\ 0 & 1 & 0 & 0 & x_{m-1}(1-x_{m-1}) \\ 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

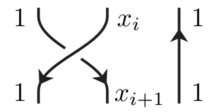
respectively, with respect to the basis $e_{W_{m-1}}, e_{X_{m-1}}, e_{Z_{m-1}}, e_{X_m}, e_{Z_m}$ and the basis $e_{W_{m-1}}$, $e_{X_{m-1}}, e_{Z_{m-1}}, e_{X_m}, e_{W_m}$. The matrix of $\check{D}_1$ is a union of copies of the above mentioned matrices.

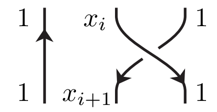
From the definition of $\check{E}_1$, the matrix of $\check{D}_1\check{E}_1$ is the matrix obtained from $\check{D}_1$ by removing the second column. Its determinant is equal to the product of some entries of $\check{D}_1$, since most of the entries of $\check{D}_1$ are equal to 0. The choice of entries which contribute to the determinant depends on the orientations of strands; more concretely, we choose the following values depending on the orientations of strands, whose product presents the value of the required determinant,

$\Phi_1\Big($  $\Big) = \dfrac{-1}{(x_1-1)^2}$, $\Phi_1\Big($  $\Big) = \dfrac{1}{1-x_1}$,

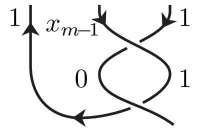
$\Phi_1\Big($  $\Big) = \dfrac{x_i(x_i-1)}{x_{i+1}-1}$, $\Phi_1\Big($  $\Big) = \dfrac{x_i(x_i-x_{i+1})}{x_{i+1}-1}$,

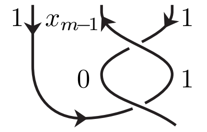
$\Phi_1\Big($  $\Big) = \dfrac{x_i-1}{x_i(x_{i+1}-1)}$, $\Phi_1\Big($  $\Big) = \dfrac{x_i-1}{x_{i+1}-x_i}$,

$\Phi_1\Bigg($ 

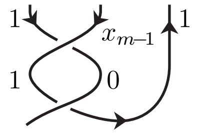
$\Bigg) = \dfrac{(x_i-1)^2}{(x_{i+1}-1)^2}, \quad \Phi_1\Bigg($ 

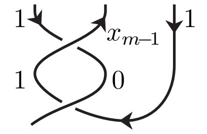
$\Bigg) = -\dfrac{x_i(x_i-1)}{x_{i+1}-1},$

$\Phi_1\Bigg($ 

$\Bigg) = x_{m-1}(x_{m-1}-1), \quad \Phi_1\Bigg($ 

$\Bigg) = \dfrac{1-x_{m-1}}{x_{m-1}},$

$\Phi_1\Bigg($ 

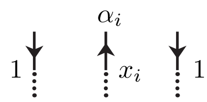
$\Bigg) = (x_{m-1}-1)^2, \quad \Phi_1\Bigg($ 

$\Bigg) = \dfrac{1-x_{m-1}}{x_{m-1}}.$

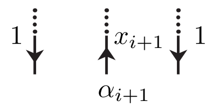
For a diagram D of any open two-bridge knot, by decomposing D into a union of elementary diagrams, we define $\Phi_1(D)$ to be the composition of Φ_1 of such elementary diagrams, whose values are given above. Then, by the above arguments, we have that

$$\det\big(\check{D}_1\check{E}_1\big) \ \doteq \ \Phi_1(D).$$

For an elementary tangle diagram T, we define $\hat{\Phi}_1(T)$ from $\Phi_1(T)$ by multiplying

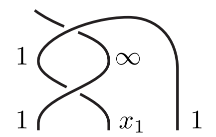
$\dfrac{x_i}{\frac{1}{\alpha_i}-1}$ when the top of T is parameterized by 

and multiplying

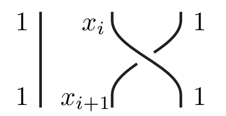
$\dfrac{\frac{1}{\alpha_{i+1}}-1}{x_{i+1}}$ when the bottom of T is parameterized by 

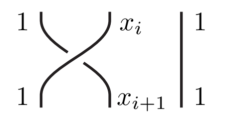
.

Then, we can verify that $\hat{\Phi}_1(T)$ does not depend on the orientation of T ignoring the difference of sign; more concretely, its values are given by

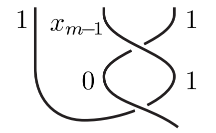
$\hat{\Phi}_1\Bigg($ 

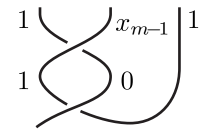
$\Bigg) = \dfrac{1}{(x_1-1)^2},$

$\hat{\Phi}_1\Bigg($ 

$\Bigg) = \dfrac{x_i(x_i-1)}{x_{i+1}-1}, \quad \hat{\Phi}_1\Bigg($ 

$\Bigg) = \dfrac{(x_i-1)^2}{(x_{i+1}-1)^2},$

$\hat{\Phi}_1\Bigg($ 

$\Bigg) = x_{m-1}(x_{m-1}-1), \quad \hat{\Phi}_1\Bigg($ 

$\Bigg) = (x_{m-1}-1)^2.$

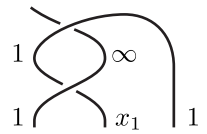
By the above construction, $\Phi_1(D) \doteq \hat{\Phi}_1(D)$ for a diagram D of any open two-bridge knot. Hence, we have that

$$\det\big(\check{D}_1\check{E}_1\big) \ \doteq \ \hat{\Phi}_1(D). \tag{27}$$

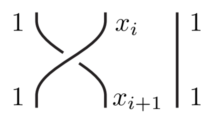
3.6 Calculation of the twisted Reidemeister torsion for any two-bridge knot

In this section, we calculate the twisted Reidemeister torsion $\tau(K)$ for any two-bridge knot K, by applying (27) and (26) to (14).

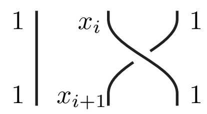
We define $\Phi(\,\cdot\,)$ by $\Phi(\,\cdot\,) = \hat{\Phi}_1(\,\cdot\,)\,\Phi_2(\,\cdot\,)$. Its concrete values are given by

$\Phi\Big($  $\Big)$

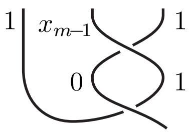
$$= \frac{1}{x_1(x_1-1)^3}\begin{pmatrix}1 & 2x_1 & 0\end{pmatrix},$$

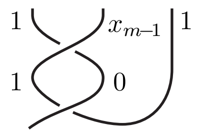
$\Phi\Big($  $\Big)$

$$= \frac{(x_i-1)^2}{x_{i+1}(x_{i+1}-1)^2}\begin{pmatrix}1 & 2\,x_{i+1} & 1\\ 0 & -x_{i+1} & -1\\ 0 & 0 & 1\end{pmatrix},$$

$\Phi\Big($  $\Big)$

$$= \frac{x_i^2}{x_{i+1}}\begin{pmatrix}1 & 0 & 0\\ -1 & -x_{i+1} & 0\\ 1 & 2\,x_{i+1} & 1\end{pmatrix},$$

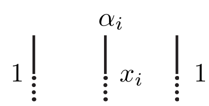
$\Phi\Big($  $\Big)$

$= x_{m-1}^2\begin{pmatrix}1\\ -1\\ 2\end{pmatrix},\quad$ $\Phi\Big($  $\Big)$

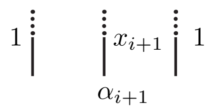
$= (x_{m-1}-1)^2\begin{pmatrix}2\\ -1\\ 1\end{pmatrix},$

ignoring the difference of sign.

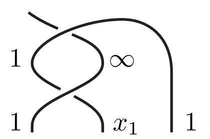
For an elementary tangle diagram T, we define $\hat{\Phi}(T)$ from $\Phi(T)$ by multiplying

$$\frac{1}{(\alpha_i-1)^2(x_i-1)^4}$$ when the top of T is parameterized by 

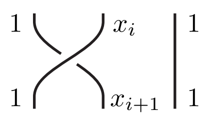
and multiplying

$(\alpha_{i+1}-1)^2(x_{i+1}-1)^4$ when the bottom of T is parameterized by 

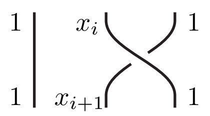
and dividing the value of the bottom part by $1-\frac{1}{x_{m-1}}$. Its concrete values are given by

$\hat{\Phi}\Big($  $\Big)$

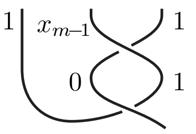
$$= x_1(x_1-1)\begin{pmatrix}1 & 2x_1 & 0\end{pmatrix},$$

$\hat{\Phi}\Big($  $\Big)$

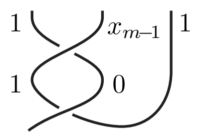
$$= x_{i+1}\begin{pmatrix}1 & 2\,x_{i+1} & 1\\ 0 & -x_{i+1} & -1\\ 0 & 0 & 1\end{pmatrix},$$

$\hat{\Phi}\Big($  $\Big)$

$$= x_{i+1}\begin{pmatrix}1 & 0 & 0\\ -1 & -x_{i+1} & 0\\ 1 & 2\,x_{i+1} & 1\end{pmatrix},$$

$\hat{\Phi}\Big($  $\Big)$

$$= \frac{x_{m-1}^3}{(x_{m-1}-1)^3}\begin{pmatrix}1\\ -1\\ 2\end{pmatrix},$$



$$\hat{\Phi}\left(\cdots \right) \;=\; \frac{x_{m-1}^3}{(x_{m-1}-1)^3}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

By the above construction, $\Phi(D)/(1-\frac{1}{x_{m-1}}) \doteq \hat{\Phi}(D)$ for a diagram D of any open two-bridge knot.

Hence, for a diagram D of any open two-bridge knot K, we have that

$$\frac{2}{\tau(K)} \;=\; \hat{\Phi}(D)\,. \tag{28}$$

Example 3.1. We numerically calculate the twisted Reidemeister torsion for the $\overline{5_2}$ knot, which is the knot shown in Section 3.2. As shown in Section 3.2, the hyperbolicity equations are presented by

$$x_2 \;=\; x_1^2 - x_1 + 1, \qquad x_2 + 1 - \frac{x_2}{x_1} \;=\; 0.$$

Hence,

$$x_1^3 - 2\,x_1^2 + 3\,x_1 - 1 \;=\; 0.$$

Corresponding to the holonomy representation of the hyperbolic structure of the knot complement, we choose a solution

$$x_1 = 0.784920145... + \sqrt{-1}\cdot 1.307141278...\ ,$$

which gives the complex hyperbolic volume by

$$\varsigma(\overline{5_2}) \;=\; \frac{1}{2\pi\sqrt{-1}}\,V(x_1,x_2) \;=\; 0.450109610... - \sqrt{-1}\cdot 0.4813049796...\ .$$

Therefore, by (28),

$$\begin{aligned} \frac{2}{\tau(\overline{5_2})} \;&=\; x_1(x_1-1)\begin{pmatrix} 1 & 2x_1 & 0 \end{pmatrix}\cdot x_2 \begin{pmatrix} 1 & 0 & 0 \\ -1 & -x_2 & 0 \\ 1 & 2x_2 & 1 \end{pmatrix}\cdot \frac{x_2^3}{(x_2-1)^3}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \\ &=\; -0.6323164993... + \sqrt{-1}\cdot 2.2345852998...\ , \end{aligned}$$

and, hence, the value of the twisted Reidemeister torsion of the $\overline{5_2}$ knot is give by

$$\tau(\overline{5_2}) \;=\; -0.2344867659... - \sqrt{-1}\cdot 0.8286683659...\ .$$

We can confirm that the above value is also obtained from [20], by transforming the Reidemeister torsion associated with the longitude (of [20]) to the Reidemeister torsion associated with the meridian (the above value) as mentioned in [13].

Example 3.2. We numerically calculate the twisted Reidemeister torsion for the $\overline{6_1}$ knot, which is the knot shown in Section 2.2. As shown in Section 2.2, the hyperbolicity equations are presented by

$$x_2 = x_1^2 - x_1 + 1, \qquad x_3 = x_2 + 1 - \frac{x_2}{x_1} = 0, \qquad x_3 + 1 - \frac{x_3}{x_2} = 0.$$

Hence,

$$x_1^4 - 3\,x_1^3 + 6\,x_1^2 - 5\,x_1 + 2 = 0.$$

Corresponding to the holonomy representation of the hyperbolic structure of the knot complement, we choose a solution

$$x_1 = 0.8951233822... + \sqrt{-1} \cdot 1.5524918200... ,$$

which gives the complex hyperbolic volume by

$$\varsigma(\overline{6_1}) = \frac{1}{2\pi\sqrt{-1}} V(x_1, x_2, x_3) = 0.5035603876... - \sqrt{-1} \cdot 1.0807800768... .$$

Therefore, by (28),

$$\begin{aligned} \frac{2}{\tau(\overline{6_1})} &= x_1(x_1-1) \begin{pmatrix} 1 & 2x_1 & 0 \end{pmatrix} \cdot x_2 \begin{pmatrix} 1 & 0 & 0 \\ -1 & -x_2 & 0 \\ 1 & 2x_2 & 1 \end{pmatrix} \cdot x_3 \begin{pmatrix} 1 & 0 & 0 \\ -1 & -x_3 & 0 \\ 1 & 2x_3 & 1 \end{pmatrix} \cdot \frac{x_3^3}{(x_3-1)^3} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \\ &= 0.9749303264... - \sqrt{-1} \cdot 3.4760907942... , \end{aligned}$$

and, hence, the value of the twisted Reidemeister torsion of the $\overline{6_1}$ knot is give by

$$\tau(\overline{6_1}) = 0.1496015098... + \sqrt{-1} \cdot 0.5334006103... .$$

We can confirm that the above value is also obtained from [20], by transforming the Reidemeister torsion associated with the longitude (of [20]) to the Reidemeister torsion associated with the meridian (the above value) as mentioned in [13].

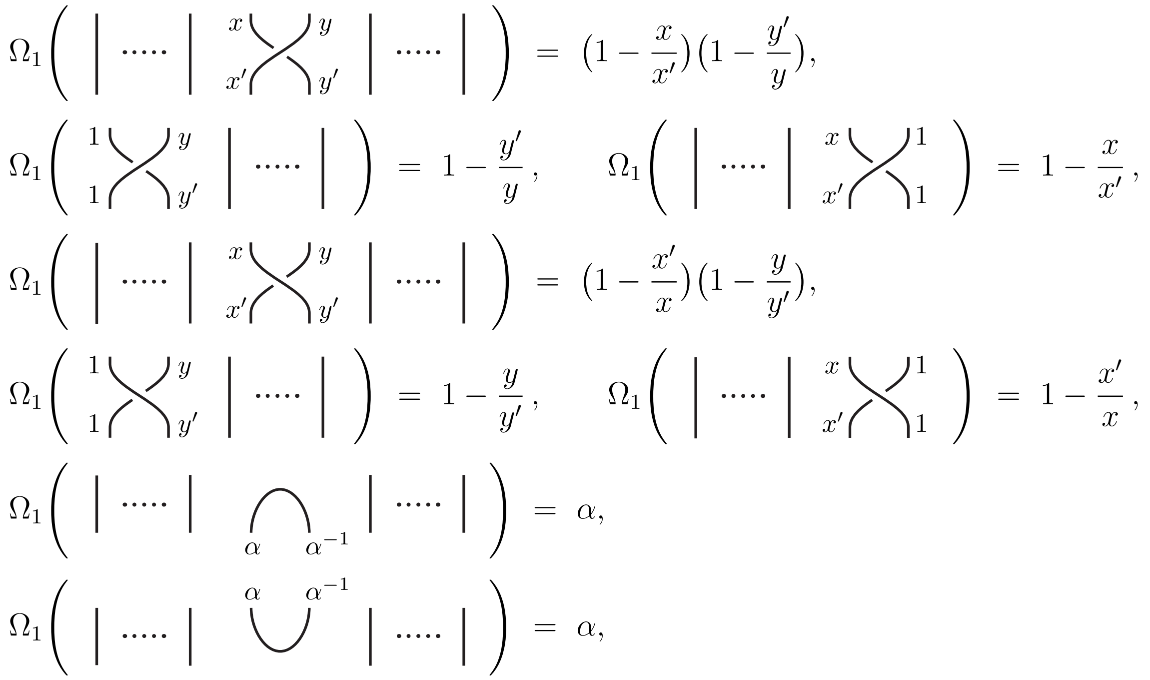
4 Definition and calculation of ω_2

In this section, we define $\omega_2(D)$ for an oriented parameterized open knot diagram D in Section 4.1, and show that it is (formally, in general) equal (up to sign) to the square of $\omega(K)$ of the asymptotic expansion (2) of the Kashaev invariant in Section 4.2. Further, we calculate $\omega_2(D)$ for open two-bridge knot diagrams in Section 4.3.

4.1 Definition of ω_2

In this section, we define $\omega_2(D)$ for an oriented parameterized open knot diagram D in Definition 4.2, motivated by the square of $\omega(K)$ of the asymptotic expansion (2) of the Kashaev invariant. We show that $\omega_2(D)$ is invariant under the RII and RIII moves under a certain assumption on the values of hyperbolicity parameters in Proposition 4.3.

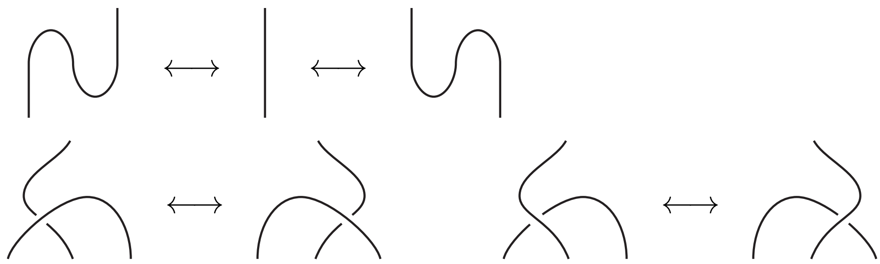
For a parameterized knot diagram D, we slice D by horizontal lines in such a way that each region has a crossing or a critical point, and we put Ω_1 of each region as follows, and we put $\Omega_1(D)$ to be the product of them.



where the parameter α at an end of a strand is defined as in Section 3.3.

Lemma 4.1. *For a parameterized knot diagram D, the value of $\Omega_1(D)$ is determined independently of the way of slicing D.*

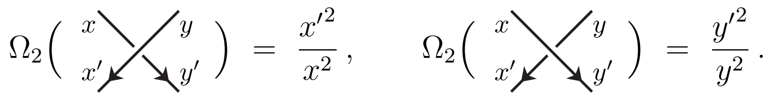
Proof. It is sufficient to show that $\Omega_1(D)$ is invariant under the following moves.



We obtain the invariance under the moves of the first line from the definition of Ω_1.

We obtain the invariance under the moves of the second line from the definition of Ω_1 and hyperbolicity equations among parameters. □

For an oriented parameterized knot diagram D, we put Ω_2 of each crossing as follows, and put $\Omega_2(D)$ to be the product of them.



For a parameterized open knot diagram D, we recall that the potential function V is defined as in Section 2.2, which is a function of hyperbolicity parameters x_i's. We also recall that a solution of hyperbolicity equations gives a critical point of V. We define the Hesse matrix at a critical point of V by

$$H \;=\; \Big(\big(x_i \frac{\partial}{\partial x_i}\big)\big(x_j \frac{\partial}{\partial x_j}\big) V \Big)_{i,j}.$$

We note that

$$\big(x\frac{\partial}{\partial x}\big)^2 \operatorname{Li}_2\big(\frac{x}{y}\big) \;=\; \big(y\frac{\partial}{\partial y}\big)^2 \operatorname{Li}_2\big(\frac{x}{y}\big) \;=\; \frac{x}{y-x}, \qquad \big(x\frac{\partial}{\partial x}\big)\big(y\frac{\partial}{\partial y}\big) \operatorname{Li}_2\big(\frac{x}{y}\big) \;=\; -\frac{x}{y-x},$$

which are obtained from (11). Hence, for example, the Hesse matrix of the potential function of (10) is given by

$$H \;=\; \begin{pmatrix} \frac{x_1}{1-x_1} - \frac{1}{x_1-1} + \frac{x_2}{x_1-x_2} & -\frac{x_2}{x_1-x_2} & 0 \\ -\frac{x_2}{x_1-x_2} & \frac{x_2}{x_1-x_2} - \frac{x_2}{1-x_2} - \frac{1}{x_2-1} + \frac{x_3}{x_2-x_3} & -\frac{x_3}{x_2-x_3} \\ 0 & -\frac{x_3}{x_2-x_3} & \frac{x_3}{x_2-x_3} - \frac{x_3}{1-x_3} - \frac{1}{x_3-1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1+x_1}{1-x_1} + \frac{x_2}{x_1-x_2} & -\frac{x_2}{x_1-x_2} & 0 \\ -\frac{x_2}{x_1-x_2} & \frac{x_2}{x_1-x_2} + 1 + \frac{x_3}{x_2-x_3} & -\frac{x_3}{x_2-x_3} \\ 0 & -\frac{x_3}{x_2-x_3} & \frac{x_3}{x_2-x_3} + 1 \end{pmatrix}.$$

Definition 4.2. For an oriented parameterized open knot diagram D, we define $\omega_2(D)$ by

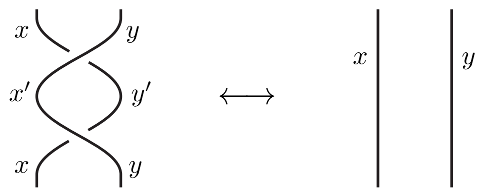
$$\omega_2(D) \;=\; \frac{1}{\sqrt{-1}\;\Omega_1(D)\,\Omega_2(D)\,\det H}.$$

We expect that this gives an invariant of an oriented parameterized open knot. The following proposition is a partial evidence of this expectation.

Proposition 4.3. *For an oriented parameterized open knot diagram D, $\omega_2(D)$ is invariant under the RII and RIII moves, if the values of the hyperbolicity parameters at the moves are generic.*

Here, "generic" means that both sides of the hyperbolicity equations of the knot diagrams appearing in the RII and RIII moves are always non-zero.

Proof. We show the invariance under the RII move, as follows. (The following proof works when $x' \neq x \neq y \neq y'$.)



We calculate ω_2 of the left-hand side. By definition,

$$\Omega_1(\text{LHS}) \;=\; \big(1-\frac{x}{x'}\big)^2\big(1-\frac{y'}{y}\big)^2.$$

Further, we can verify by definition that

$$\Omega_2\big(\mathrm{LHS}\big) \;=\; \frac{x'^2 y^2}{x^2\, y'^2}\,,$$

independently of a choice of orientations of the strands. The Hesse matrix for the left-hand side is given by the following form,

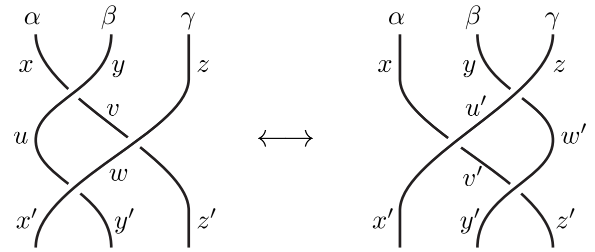
$$\left(\begin{array}{cc|cc|cc}
\frac{x}{y-x}-\frac{x}{x'-x}+a_1 & -\frac{x}{y-x}+a_2 & \frac{x}{x'-x} & 0 & c_1 & c_2 \\
-\frac{x}{y-x}+a_2 & \frac{x}{y-x}-\frac{y'}{y-y'}+a_3 & 0 & \frac{y'}{y-y'} & c_3 & c_4 \\
\hline
\frac{x}{x'-x} & 0 & 0 & 0 & -\frac{x}{x'-x} & 0 \\
0 & \frac{y'}{y-y'} & 0 & 0 & 0 & -\frac{y'}{y-y'} \\
\hline
c_1 & c_3 & -\frac{x}{x'-x} & 0 & \frac{x}{x'-x}-\frac{x}{y-x}+b_1 & \frac{x}{y-x}+b_2 \\
c_2 & c_4 & 0 & -\frac{y'}{y-y'} & \frac{x}{y-x}+b_2 & \frac{y'}{y-y'}-\frac{x}{y-x}+b_3
\end{array}\right).$$

This matrix can be transformed into the following form by elementary transformation,

$$\left(\begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix} + \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} + \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix}\right) \oplus \left(\begin{array}{cc|cc}
0 & 0 & \frac{x}{x'-x} & 0 \\
0 & 0 & 0 & \frac{y'}{y-y'} \\
\hline
\frac{x}{x'-x} & 0 & 0 & 0 \\
0 & \frac{y'}{y-y'} & 0 & 0
\end{array}\right).$$

The first direct summand gives the Hesse matrix of the right-hand side. The determinant of the second direct summand is the error term, and it cancels with $\Omega_1\big(\mathrm{LHS}\big)\,\Omega_2\big(\mathrm{LHS}\big)$. Hence, $\omega_2(D)$ is invariant under the RII move.

We show the invariance under the RIII moves, as follows. (The following proof works when both sides of hyperbolicity equations appearing in the knot diagrams in the proof are always non-zero.)



When we give values of α, β, γ, x, y, z, the values of the other parameters are determined by

$$\begin{aligned}
u &= \frac{xy}{\alpha x + y - \alpha y}, \qquad v = \frac{x-y+\beta y}{\beta}, \\
w &= \frac{xz}{\alpha x + y - \alpha y - \beta y + \alpha\beta y + \beta z - \alpha\beta z}, \\
u' &= \frac{yz}{\beta y + z - \beta z}, \qquad w' = \frac{y-z+\gamma z}{\gamma},
\end{aligned}$$

$$\begin{aligned} v' &= \frac{\beta xy + xz - \beta xz - \gamma xz + \beta\gamma xz - yz + \gamma yz}{\beta\gamma y}, \\ x' &= \frac{xyz}{\alpha\beta xy + \alpha xz - \alpha\beta xz + yz - \alpha yz}, \\ y' &= \frac{x\,(y - z + \gamma z)}{\gamma\,(\alpha x + y - \alpha y)}, \qquad z' = \frac{x - y + \beta y - \beta z + \beta\gamma z}{\beta\gamma}, \end{aligned}$$

noting that the values of x', y', z' do not change as functions of α, β, γ, x, y, z under the RIII move. By definition,

$$\begin{aligned} \Omega_1\big(\mathrm{LHS}\big) &= \big(1 - \frac{x}{u}\big)\big(1 - \frac{v}{y}\big)\big(1 - \frac{v}{w}\big)\big(1 - \frac{z'}{z}\big)\big(1 - \frac{u}{x'}\big)\big(1 - \frac{y'}{w}\big), \\ \Omega_1\big(\mathrm{RHS}\big) &= \big(1 - \frac{y}{u'}\big)\big(1 - \frac{w'}{z}\big)\big(1 - \frac{x}{x'}\big)\big(1 - \frac{v'}{u'}\big)\big(1 - \frac{v'}{y'}\big)\big(1 - \frac{z'}{w'}\big). \end{aligned}$$

Further, we can verify by definition that

$$\frac{\Omega_2\big(\mathrm{LHS}\big)}{\Omega_2\big(\mathrm{RHS}\big)} = \frac{y^2\,w^2\,v'^2}{v^2\,y'^2 u'^2},$$

independently of a choice of orientations of the strands. The Hesse matrix of the left-hand side is given by the following form,

$$\left(\begin{array}{c|c|c} \cdots + A_1 & B_1 & C_1 + \cdots \\ \hline B_1^T & D_1 & E_1 \\ \hline \cdots + C_1^T & E_1^T & F_1 + \cdots \end{array}\right),$$

where

$$A_1 = \begin{pmatrix} \frac{x}{y-x} - \frac{x}{u-x} & -\frac{x}{y-x} & 0 \\ -\frac{x}{y-x} & \frac{x}{y-x} - \frac{v}{y-v} & 0 \\ 0 & 0 & \frac{v}{z-v} - \frac{z'}{z-z'} \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \frac{x}{u-x} & 0 & 0 \\ 0 & \frac{v}{y-v} & 0 \\ 0 & -\frac{v}{z-v} & 0 \end{pmatrix}, \qquad C_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{z'}{z-z'} \end{pmatrix},$$

$$D_1 = \begin{pmatrix} \frac{v}{u-v} - \frac{x}{u-x} + \frac{u}{w-u} - \frac{u}{x'-u} & -\frac{v}{u-v} & -\frac{u}{w-u} \\ -\frac{v}{u-v} & \frac{v}{u-v} - \frac{v}{y-v} + \frac{v}{z-v} - \frac{v}{w-v} & \frac{v}{w-v} \\ -\frac{u}{w-u} & \frac{v}{w-v} & \frac{z'}{w-z'} - \frac{v}{w-v} + \frac{u}{w-u} - \frac{y'}{w-y'} \end{pmatrix},$$

$$E_1 = \begin{pmatrix} \frac{u}{x'-u} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{y'}{w-y'} & -\frac{z'}{w-z'} \end{pmatrix}, \quad F_1 = \begin{pmatrix} \frac{y'}{x'-y'} - \frac{u}{x'-u} & -\frac{y'}{x'-y'} & 0 \\ -\frac{y'}{x'-y'} & \frac{y'}{x'-y'} - \frac{y'}{w-y'} & 0 \\ 0 & 0 & \frac{z'}{w-z'} - \frac{z'}{z-z'} \end{pmatrix}.$$

Further, the Hesse matrix of the right-hand side is given by the following form,

$$\left(\begin{array}{c|c|c} \cdots + A_2 & B_2 & C_2 + \cdots \\ \hline B_2^T & D_2 & E_2 \\ \hline \cdots + C_2^T & E_2^T & F_2 + \cdots \end{array}\right),$$

where

$$A_2 = \begin{pmatrix} \frac{x}{u'-x} - \frac{x}{x'-x} & 0 & 0 \\ 0 & \frac{y}{z-y} - \frac{y}{u'-y} & -\frac{y}{z-y} \\ 0 & -\frac{y}{z-y} & \frac{y}{z-y} - \frac{w'}{z-w'} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} -\frac{x}{u'-x} & 0 & 0 \\ \frac{y}{u'-y} & 0 & 0 \\ 0 & 0 & \frac{w'}{z-w'} \end{pmatrix}, \qquad C_2 = \begin{pmatrix} \frac{x}{x'-x} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} \frac{w'}{u'-w'} - \frac{y}{u'-y} + \frac{x}{u'-x} - \frac{v'}{u'-v'} & \frac{v'}{u'-v'} & -\frac{w'}{u'-w'} \\ \frac{v'}{u'-v'} & \frac{v'}{x'-v'} - \frac{v'}{u'-v'} + \frac{v'}{w'-v'} - \frac{v'}{y'-v'} & -\frac{v'}{w'-v'} \\ -\frac{w'}{u'-w'} & -\frac{v'}{w'-v'} & \frac{w'}{u'-w'} - \frac{w'}{z-w'} + \frac{v'}{w'-v'} - \frac{z'}{w'-z'} \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{v'}{x'-v'} & \frac{v'}{y'-v'} & 0 \\ 0 & 0 & \frac{z'}{w'-z'} \end{pmatrix}, \quad F_2 = \begin{pmatrix} \frac{v'}{x'-v'} - \frac{x}{x'-x} & 0 & 0 \\ 0 & \frac{z'}{y'-z'} - \frac{v'}{y'-v'} & -\frac{z'}{y'-z'} \\ 0 & -\frac{z'}{y'-z'} & \frac{z'}{y'-z'} - \frac{z'}{w'-z'} \end{pmatrix}.$$

These two Hesse matrices can be transformed into the following form ($i = 1, 2$) by elementary transformations,

$$\left(\begin{array}{c|c|c} \cdots + A_i - B_i D_i^{-1} B_i^T & 0 & C_i - B_i D_i^{-1} E_i + \cdots \\ \hline 0 & D_i & 0 \\ \hline \cdots + C_i^T - E_i^T D_i^{-1} B_i^T & 0 & F_i - E_i^T D_i^{-i} E_i + \cdots \end{array}\right).$$

Here, the parts of "$\cdots$" are the contributions from the outside of the RIII move, and they are invariant under the RIII move. Further, we can verify by direct calculation that

$$\begin{aligned} A_1 - B_1 D_1^{-1} B_1^T &= A_2 - B_2 D_2^{-1} B_2^T, \\ C_1 - B_1 D_1^{-1} E_1 &= C_2 - B_2 D_2^{-1} E_2, \\ F_1 - E_1^T D_1^{-1} E_1 &= F_2 - E_2^T D_2^{-1} E_2. \end{aligned}$$

(We can verify the first two formulas by direct calculations. Then, the third formula can be obtained from the first formula by the symmetry of π rotation of the RIII move.) Hence, the change of the determinants of the Hesse matrices is equal to the ratio of $\det(D_1)$ and $\det(D_2)$. Since we can verify by direct calculation that

$$\det(D_1)\, \Omega_1\big(\text{LHS}\big)\, \Omega_2\big(\text{LHS}\big) = \det(D_2)\, \Omega_1\big(\text{RHS}\big)\, \Omega_2\big(\text{RHS}\big),$$

it is shown that ω_2 is invariant under the RIII move. □

Remark 4.4. Definition 4.2 gives an appropriate definition of $\omega_2(D)$ which (formally, in general) presents (± 1) times the square of $\omega(K)$ of the asymptotic expansion (2) of the Kashaev invariant, when a critical point of V is isolated and its Hesse matrix is non-degenerate. It mainly tends to hold, say, for alternating knot diagrams. However, when D is redundant as a knot diagram (say, when D has a loop of the RI move), Definition 4.2 does not work well in the above sense. It might be difficult to show that $\omega_2(D)$ gives a knot invariant by showing its invariance under the Reidemeister moves.

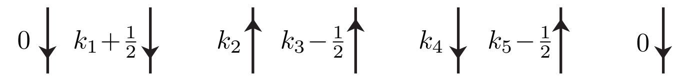
4.2 Relation to the Kashaev invariant

In this section, we explain that $\omega_2(D)$ is (formally, in general) equal (up to sign) to the square of $\omega(K)$ of the asymptotic expansion (2) of the Kashaev invariant, which we show in (30) and (31) at the end of this section.

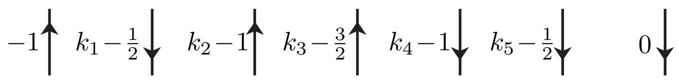
We consider an oriented open knot diagram whose ends are downward oriented. We slice such a knot diagram by horizontal lines in such a way that each region has a crossing or a critical point. We consider a section of a knot diagram by such a horizontal line, and associate the ith strand on the horizontal line with the following color,

$$k_{i-1} - \frac{n-i}{2} + \begin{pmatrix} \text{the number of upward-oriented strands} \\ \text{in the right of the } i\text{th strand} \end{pmatrix}$$

where n is the number of strands on the horizontal line. Here, we put $k_0 = k_{n-1} = 0$. For example, strands are colored by

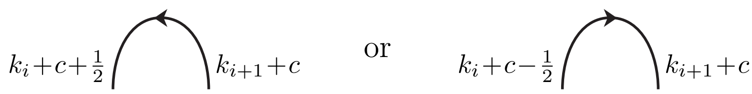


and

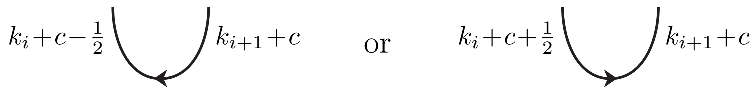


depending on the orientations of the strands. We regard k_i as an integer parameter for even i, and regard k_i as a half-integer parameter for odd i.

Around a maximal point, strands are colored by

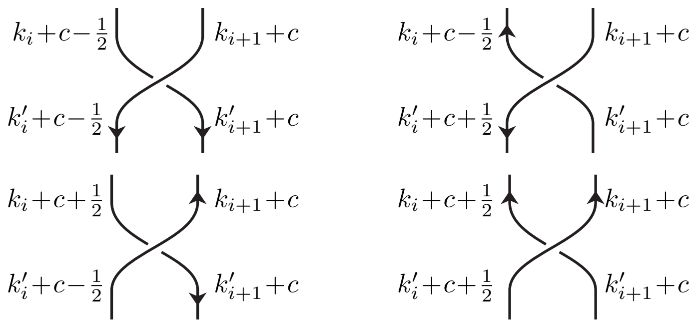


and, in any case, $k_i - \frac{1}{2} = k_{i+1}$. Further, around a minimal point, strands are colored by

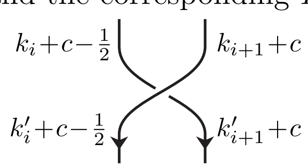


and, in any case, $k_i + \frac{1}{2} = k_{i+1}$. These error terms of $\frac{1}{2}$ correspond to the values of $\Omega_1(\,\cdot\,)^{-1/2}$ of critical points defined in Section 4.1 putting $q^{k_j} = x_j$.

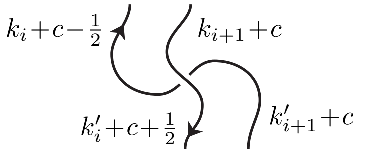
Around a positive crossing, strands are colored by



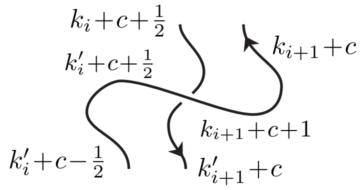
and the corresponding R matrices are given by



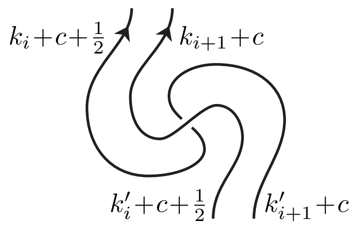
$$R^{k_i+c-\frac{1}{2}\ \ k_{i+1}+c}_{k'_i+c-\frac{1}{2}\ \ k'_{i+1}+c} \sim \frac{N\, q^{k_i-k'_i}}{(q)_{k_i-k_{i+1}-\frac{1}{2}}(\overline{q})_{k_{i+1}-k'_{i+1}}(q)_{k'_{i+1}-k'_i-\frac{1}{2}}(\overline{q})_{k'_i-k_i}},$$



$$\overline{R}^{k_{i+1}+c\;\, k'_{i+1}+c}_{k_i+c-\frac12\;\, k'_i+c+\frac12} \sim \frac{N\, q^{k'_{i+1}-k'_i}}{(\overline{q})_{k_{i+1}-k'_{i+1}}(q)_{k'_{i+1}-k'_i-\frac12}(\overline{q})_{k'_i-k_i}(q)_{k_i-k_{i+1}-\frac12}},$$



$$\overline{R}^{k'_i+c+\frac12\;\, k_i+c+\frac12}_{k'_{i+1}+c\;\, k_{i+1}+c+1} \sim \frac{N\, q^{k_i-k_{i+1}}}{(\overline{q})_{k'_i-k_i}(q)_{k_i-k_{i+1}-\frac12}(\overline{q})_{k_{i+1}-k'_{i+1}}(q)_{k'_{i+1}-k'_i-\frac12}},$$



$$R^{k'_{i+1}+c\;\, k'_i+c+\frac12}_{k_{i+1}+c\;\, k_i+c+\frac12} \sim \frac{N\, q^{k'_{i+1}-k_{i+1}}}{(q)_{k'_{i+1}-k'_i-\frac12}(\overline{q})_{k'_i-k_i}(q)_{k_i-k_{i+1}-\frac12}(\overline{q})_{k_{i+1}-k'_{i+1}}}.$$

The contributions of the "$q^{\cdots}$" part of the numerators of the right-hand sides to the Kashaev invariant are presented by $\Omega_2(\,\cdot\,)^{-1/2}$ defined in Section 4.1, putting $q^{k_j}=x_j$ and $q^{k'_j}=x'_j$. The contributions of the denominators of the right-hand sides to the Kashaev invariant are equal, and their contributions to the Kashaev invariant are presented by

$$\exp\Big(\cdots+\varphi(t_i-t_{i+1})\,-\varphi\big(1-t_{i+1}+t'_{i+1}-\frac{1}{2N}\big)+\varphi(t'_{i+1}-t'_i)\,-\varphi\big(1-t'_i+t_i-\frac{1}{2N}\big)\,+\cdots\Big),$$

putting $t_j=\frac{k_j}{N}$ and $t'_j=\frac{k'_j}{N}$. Further, since

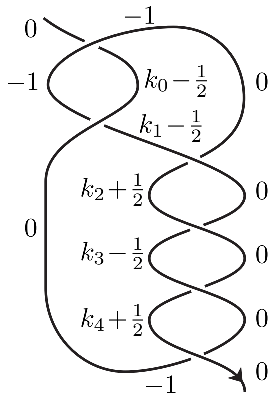
$$\begin{aligned}
\varphi\big(1-t_{i+1}+t'_{i+1}-\frac{1}{2N}\big) &= \varphi\big(1-t_{i+1}+t'_{i+1}\big)-\frac{1}{2N}\,\varphi'\big(1-t_{i+1}+t'_{i+1}\big)+O\big(\frac{1}{N^2}\big)\\
&= \varphi\big(1-t_{i+1}+t'_{i+1}\big)+\frac12\log\big(1-\frac{x'_{i+1}}{x_{i+1}}\big)+O\big(\frac{1}{N^2}\big),\\
\varphi\big(1-t'_i+t_i-\frac{1}{2N}\big) &= \varphi\big(1-t'_i+t_i\big)+\frac12\log\big(1-\frac{x_i}{x'_i}\big)+O\big(\frac{1}{N^2}\big),
\end{aligned}$$

the contributions from the "$\varphi(\cdots-\frac{1}{2N})$" parts to the Kashaev invariant are the multiples of $\big(1-\frac{x_i}{x'_i}\big)^{-1/2}$ and $\big(1-\frac{x'_{i+1}}{x_{i+1}}\big)^{-1/2}$, putting $x_j=e^{2\pi\sqrt{-1}\,t_j}$ and $x'_j=e^{2\pi\sqrt{-1}\,t'_j}$, and they are presented by $\Omega_1(\,\cdot\,)^{-1/2}$ of a crossing defined in Section 4.1. By using the remaining part, we put $\check{V}$ to be the sum of the following form,

$$\begin{aligned}
\check{V} &= \frac1N\Big(\cdots+\varphi(t_i-t_{i+1})\,-\varphi\big(1-t_{i+1}+t'_{i+1}\big)\,+\varphi(t'_{i+1}-t'_i)\,-\varphi\big(1-t'_i+t_i\big)\,+\cdots\Big)\\
&= \frac{1}{2\pi\sqrt{-1}}\Big(\cdots+\mathrm{Li}_2\big(\frac{x_i}{x_{i+1}}\big)\,-\mathrm{Li}_2\big(\frac{x'_{i+1}}{x_{i+1}}\big)\,+\mathrm{Li}_2\big(\frac{x'_{i+1}}{x'_i}\big)\,-\mathrm{Li}_2\big(\frac{x_i}{x'_i}\big)\,+\cdots\Big)\,+O\big(\frac{1}{N^2}\big),
\end{aligned}$$

where we obtain the second equality by (6). By using this $\check{V}$, we can calculate the asymptotic expansion of the Kashaev invariant, as we explain below.

We explain how we calculate the asymptotic expansion of the Kashaev invariant by using the following example, which is the mirror image $\overline{6_1}$ of the 6_1 knot.



By definition, the Kashaev invariant of the $\overline{6_1}$ knot is presented by the following form, ignoring the q^{constant} terms,

$$\begin{aligned}
\langle\, \overline{6_1}\,\rangle_N \ &\sim \sum_{k_0,\cdots,k_4} \overline{R}^{\,0\quad 0}_{k_0-\frac12\ \ 0}\ \overline{R}^{\,k_0-\frac12\ \ k_1-\frac12}_{-1\qquad 0}\ R^{\,0\qquad 0}_{k_1-\frac12\ \ k_2+\frac12}\ R^{\,k_3+\frac12\ \ k_2+\frac12}_{0\qquad 1}\ R^{\,0\qquad 0}_{k_3-\frac12\ \ k_4+\frac12}\ R^{\,0\ \ k_4+\frac12}_{0\qquad 1} \\
&\sim \sum_{k_0,\cdots,k_4} \frac{N}{(\overline{q})_{N-k_0-\frac12}(q)_{k_0-\frac12}} \times \frac{N\,q^{k_1}}{(\overline{q})_{k_0-k_1}(q)_{k_1-\frac12}(q)_{N-k_0-\frac12}} \times \frac{N\,q^{-k_1}}{(\overline{q})_{N-k_2-\frac12}(q)_{k_2-k_1}(\overline{q})_{k_1-\frac12}} \\
&\qquad \times \frac{N\,q^{k_3}}{(q)_{k_3-k_2}(\overline{q})_{k_2-\frac12}(\overline{q})_{N-k_3-\frac12}} \times \frac{N\,q^{-k_3}}{(\overline{q})_{N-k_4-\frac12}(q)_{k_4-k_3}(\overline{q})_{k_3-\frac12}} \times \frac{N}{(q)_{N-k_4-\frac12}(\overline{q})_{k_4-\frac12}} \\
&= \sum_{k_1,k_2,k_3} \frac{N^4\cdot q^{k_1}\cdot q^{-k_1}\cdot q^{k_3}\cdot q^{-k_3}}{(q)_{k_1-\frac12}(\overline{q})_{k_1-\frac12}(q)_{k_2-k_1}(\overline{q})_{N-k_2-\frac12}(\overline{q})_{k_2-\frac12}(q)_{k_3-k_2}(\overline{q})_{N-k_3-\frac12}(\overline{q})_{k_3-\frac12}}\,,
\end{aligned}$$

where we obtain the last equality by (3) and (4). Hence, by (5),

$$\langle\, \overline{6_1}\,\rangle_N \ \sim\ N^4 \sum_{k_1,k_2,k_3} q^{k_1}\cdot q^{-k_1}\cdot q^{k_3}\cdot q^{-k_3}\cdot \exp\Big(N\cdot \hat{V}\big(\frac{k_1}{N},\frac{k_2}{N},\frac{k_3}{N}\big)\Big),$$

where we put

$$\begin{aligned}
\hat{V}(t_1,t_2,t_3) \ =\ \frac{1}{N}\Big(&\varphi(t_1)-\varphi(1-t_1)+\varphi\big(t_2-t_1+\frac{1}{2N}\big)-\varphi(t_2)-\varphi(1-t_2) \\
&+\varphi\big(t_3-t_2+\frac{1}{2N}\big)-\varphi(t_3)-\varphi(1-t_3)-3\,\varphi\big(\frac{1}{2N}\big)+5\,\varphi\big(1-\frac{1}{2N}\big)\Big).
\end{aligned}$$

Further, the "$\varphi\big(\cdots+\frac{1}{2N}\big)$" parts are calculated as

$$\begin{aligned}
\varphi\big(t_2-t_1+\frac{1}{2N}\big) \ &=\ \varphi(t_2-t_1)\ +\frac{\varphi'(t_2-t_1)}{2N}\ +O\big(\frac{1}{N^2}\big) \\
&=\ \varphi(t_2-t_1)\ -\frac{1}{2}\log\big(1-\frac{x_2}{x_1}\big)\ +O\big(\frac{1}{N^2}\big), \\
\varphi\big(t_3-t_2+\frac{1}{2N}\big) \ &=\ \varphi(t_3-t_2)\ -\frac{1}{2}\log\big(1-\frac{x_3}{x_2}\big)\ +O\big(\frac{1}{N^2}\big).
\end{aligned}$$

Hence,

$$\langle\, \overline{6_1}\,\rangle_{N} \;\sim\; e^{\pi\sqrt{-1}/2} \sum_{k_1,k_2,k_3} x_1\cdot x_1^{-1}\cdot x_3\cdot x_3^{-1}\Big(\big(1-\frac{x_2}{x_1}\big)\big(1-\frac{x_3}{x_2}\big)\Big)^{-1/2} \exp\Big(N\cdot \check{V}\big(\frac{k_1}{N},\frac{k_2}{N},\frac{k_3}{N}\big)\Big),$$

where we put

$$\begin{aligned}\check{V}(t_1,t_2,t_3) \;&=\; \frac{1}{N}\Big(\varphi(t_1)-\varphi(1-t_1)+\varphi\big(t_2-t_1\big)-\varphi(t_2)\\ &\qquad -\varphi(1-t_2)+\varphi\big(t_3-t_2\big)-\varphi(t_3)-\varphi(1-t_3)\Big)+2\cdot\frac{1}{2\pi\sqrt{-1}}\cdot\frac{\pi^2}{6}\\ &=\; \frac{1}{2\pi\sqrt{-1}}\,V(x_1,x_2,x_3)\;+O\big(\frac{1}{N^2}\big),\end{aligned}$$

putting $x_i = e^{2\pi\sqrt{-1}\,t_i}$. As shown in [16, 18, 17], for hyperbolic knots with up to 7 crossings, we can calculate the asymptotic expansion of the sum of the above form as

$$\begin{aligned}\langle\, \overline{6_1}\,\rangle_{N} \;&\sim\; e^{\pi\sqrt{-1}/2}\,N^3\int \Omega_1(D)^{-1/2}\Omega_2(D)^{-1/2}\exp\big(N\cdot\check{V}(t_1,t_2,t_3)\big)\,dt_1\,dt_2\,dt_3\\ &\sim\; e^{\pi\sqrt{-1}/2}\,N^3\,\Omega_1(D)^{-1/2}\Omega_2(D)^{-1/2}e^{N\,\varsigma(\overline{6_1})}\,\frac{(2\pi)^{3/2}}{N^{3/2}}\,\big(\det(-\check{H})\big)^{-1/2},\end{aligned}$$

where D is a diagram of the $\overline{6_1}$ knot mentioned above, and the first approximation is an approximation of a sum by an integral which is shown by the Poisson summation formula, and the second approximation is obtained by the saddle point method at an appropriate critical point $(t_{1;c},t_{2;c},t_{3;c})$ of $\check{V}$, putting $\check{H}$ to be the Hesse matrix at this critical point,

$$\check{H} \;=\; \Big(\frac{\partial^2}{\partial t_i\,\partial t_j}\,\check{V}\Big)_{i,j}.$$

In fact, in a formal sense, such approximations are standard method (and we can guess the form of the resulting formula by formal calculation), but, to be precise, we must check some technical inequalities to prove such approximations; see [16, 18, 17] for details. For the $\overline{6_1}$ knot, we can rigorously obtain that

$$\langle\, \overline{6_1}\,\rangle_{N} \;\sim\; e^{N\,\varsigma(\overline{6_1})}\cdot N^{3/2}\cdot\omega(\overline{6_1}),$$

where

$$\begin{aligned}\varsigma(\overline{6_1}) \;&=\; \check{V}(t_{1;c},t_{2;c},t_{3;c}),\\ \omega(\overline{6_1}) \;&=\; e^{\pi\sqrt{-1}/2}\,\Omega_1(D)^{-1/2}\Omega_2(D)^{-1/2}(2\pi)^{3/2}\big(\det(-\check{H})\big)^{-1/2}.\end{aligned}$$

We note that $\varsigma(\overline{6_1})$ presents the complex hyperbolic volume of the complement of the $\overline{6_1}$ knot. By the above formula, we have that

$$\frac{1}{\omega(\overline{6_1})^2} \;=\; -\,\Omega_1(D)\,\Omega_2(D)\,\frac{1}{(2\pi)^3}\,\det(-\check{H})$$

$$\doteq\ \sqrt{-1}\ \Omega_1(D)\,\Omega_2(D)\ \det H,$$

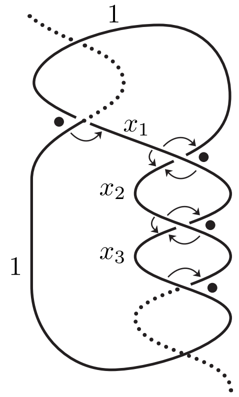
where we obtain the second equality, since $x_i\,\frac{\partial}{\partial x_i}=\frac{1}{2\pi\sqrt{-1}}\,\frac{\partial}{\partial t_i}$ and hence

$$\check{H}\ \sim\ 2\pi\sqrt{-1}\ H.$$

Therefore, from the definition of ω_2,

$$\omega(\overline{6_1})^2\ =\ \omega_2(D).$$

Generalizing the above argument, we explain how we calculate the asymptotic expansion of the Kashaev invariant (formally, in general), as follows. We consider an oriented open knot K, and consider a parameterized diagram D of K.



We let n_1 be the number of counter-clockwise angles, let n_2 be clockwise angles, let n be the number of hyperbolicity parameters, let n_c be the number of crossings (ignoring dotted lines), and let n_0 be the number of edges parameterized by 1 (ignoring dotted lines). Since n_0 is equal to the number of angles marked by dots in the above picture, we have that

$$n_1+n_2+n_0\ =\ 4\,n_c-4,\qquad 2(n_0+n)\ =\ 4\,n_c-2. \tag{29}$$

Similarly as the case of the $\overline{6_1}$ knot, we can obtain that

$$\langle\,K\,\rangle_N\ \sim\ e^{(n_2-n_1)\pi\sqrt{-1}/4}N^{n_c+n-(n_1+n_2)/2}\times\int\Omega_1(D)^{-1/2}\Omega_2(D)^{-1/2}\exp\big(N\cdot\check{V}(t_1,\cdots,t_n)\big)\,dt_1\cdots dt_n\,,$$

where

$$\check{V}(t_1,\cdots,t_n)\ =\ \frac{1}{2\pi\sqrt{-1}}\,V(x_1,\cdots,x_n)\,+O\big(\frac{1}{N^2}\big),$$

putting $x_i=e^{2\pi\sqrt{-1}\,t_i}$. We note that $n_c+n-\frac{n_1+n_2}{2}=\frac{n+3}{2}$ by (29). Hence, we obtain the following approximations (formally, in general),

$$\langle\,K\,\rangle_N\ \sim\ e^{(n_2-n_1)\pi\sqrt{-1}/4}\,N^{(n+3)/2}\int\Omega_1(D)^{-1/2}\Omega_2(D)^{-1/2}\exp\big(N\cdot\check{V}(t_1,\cdots,t_n)\big)\,dt_1\cdots dt_n$$

$$
\begin{aligned}
&\sim\ e^{(n_2-n_1)\pi\sqrt{-1}/4}\, N^{(n+3)/2}\, \Omega_1(D)^{-1/2}\Omega_2(D)^{-1/2} e^{N\varsigma(K)}\, \frac{(2\pi)^{n/2}}{N^{n/2}} \big(\det(-\check{H})\big)^{-1/2} \\
&\sim\ e^{(n_2-n_1)\pi\sqrt{-1}/4}\, N^{3/2}\, \Omega_1(D)^{-1/2}\Omega_2(D)^{-1/2} e^{N\varsigma(K)}\, (2\pi)^{n/2} \big(\det(-\check{H})\big)^{-1/2}.
\end{aligned}
$$

Therefore,

$$
\langle\, K\,\rangle_{N} \ \sim\ e^{N\varsigma(K)} \cdot N^{3/2} \cdot \omega(K), \tag{30}
$$

where

$$
\begin{aligned}
\varsigma(K) &= \check{V}(t_{1;c}, \cdots, t_{n;c}), \\
\omega(K) &= e^{(n_2-n_1)\pi\sqrt{-1}/4}\, \Omega_1(D)^{-1/2}\Omega_2(D)^{-1/2}(2\pi)^{n/2}\big(\det(-\check{H})\big)^{-1/2}.
\end{aligned}
$$

Further,

$$
\begin{aligned}
\frac{1}{\omega(K)^2} &= e^{(n_1-n_2)\pi\sqrt{-1}/2}\, \Omega_1(D)\, \Omega_2(D)\, \frac{1}{(2\pi)^n}\, \det(-\check{H}) \\
&= e^{(n_1-n_2-n)\pi\sqrt{-1}/2}\, \Omega_1(D)\, \Omega_2(D)\, \det H \\
&\doteq \sqrt{-1}\ \Omega_1(D)\, \Omega_2(D)\, \det H,
\end{aligned}
$$

since $n_1 - n_2 - n$ is odd by (29). Hence, from the definition of ω_2, we obtain that

$$
\omega(K)^2 \ \doteq\ \omega_2(D), \tag{31}
$$

as required.

Example 4.5. We numerically verify (31) for the $\overline{5_2}$ knot, which is the knot shown in Section 3.2. As shown in Example 3.1, we obtain the values of hyperbolicity parameters and the complex volume. Further, from the definition of ω_2, we have that

$$
\begin{aligned}
&H = \begin{pmatrix} \frac{1+x_1}{1-x_1} + \frac{x_2}{x_1-x_2} & -\frac{x_2}{x_1-x_2} \\ -\frac{x_2}{x_1-x_2} & \frac{x_2}{x_1-x_2} + 1 \end{pmatrix}, \qquad \Omega_1(D) = 1 - \frac{x_2}{x_1}, \\
&\omega_2(D) \ =\ \frac{1}{\sqrt{-1}\ \Omega_1(D)\ \det H} \ =\ -0.4143341829... + \sqrt{-1}\cdot 0.117243382...\ ,
\end{aligned}
$$

where D is the $\overline{5_2}$ knot diagram shown in Section 3.2. Hence,

$$
\omega(\overline{5_2}) \ =\ \omega_2(D)^{1/2} \ =\ 0.09019057740... + \sqrt{-1}\cdot 0.6499757866...\ ,
$$

where we choose the sign of the square root depending the orientation of the domain of the integral of the saddle point method; for details, see [16]. Further, from the definition of the Kashaev invariant, we have that

$$
\langle\, \overline{5_2}\,\rangle_{N} \ =\ \sum_{0\le i\le j<N} \frac{N^3\, q^{-1}}{(q)_i\, (\overline{q})_i\, (q)_{j-i}\, (\overline{q})_j\, (\overline{q})_{N-j-1}}\,,
$$

see [16]. By calculating this sum concretely as shown in the following table, we can numerically observe that the limit of $\langle\, \overline{5_2}\, \rangle_N \, e^{-N\varsigma(\overline{5_2})} N^{-3/2}$ tends to the above mentioned value of $\omega(\overline{5_2})$, noting that $q \to 1$ as $N \to \infty$.

N	$q \langle\, \overline{5_2}\, \rangle_N \, e^{-N\varsigma(\overline{5_2})}\, N^{-3/2}$
50	$0.09574104848... + \sqrt{-1} \cdot 0.6581517399...$
100	$0.09297541546... + \sqrt{-1} \cdot 0.6540225631...$
200	$0.09158517383... + \sqrt{-1} \cdot 0.6519891312...$

Example 4.6. We numerically verify (31) for the $\overline{6_1}$ knot, which is the knot shown in Section 2.2. As shown in Example 3.2, we obtain the values of hyperbolicity parameters and the complex volume. Further, by using the Hesse matrix shown in Section 4.1, we have that

$$\omega_2(D) \;=\; \frac{1}{\sqrt{-1}\;\Omega_1\;\det H} \;=\; -0.2667003051... + \sqrt{-1} \cdot 0.07480075491...\;,$$

where D is the $\overline{6_1}$ knot diagram shown in Section 2.2. Hence,

$$\omega(\overline{6_1}) \;=\; \big(-\omega_2(D)\big)^{1/2} \;=\; -0.5213883634... + \sqrt{-1} \cdot 0.07173228265...\;,$$

where we choose the sign of $\omega_2(D)$ depending the sign of (31), and choose the sign of the square root depending the orientation of the domain of the integral of the saddle point method; for details, see [18]. Further, from the definition of the Kashaev invariant, we have that

$$\langle\, \overline{6_1}\, \rangle_N \;=\; \sum_{0 \le i \le j \le k < N} \frac{N^4\, q^{-1}}{(q)_i\, (\overline{q})_i\, (q)_{j-i}\, (\overline{q})_j\, (\overline{q})_{N-j-1}\, (q)_{k-j}\, (\overline{q})_k\, (\overline{q})_{N-k-1}}\,,$$

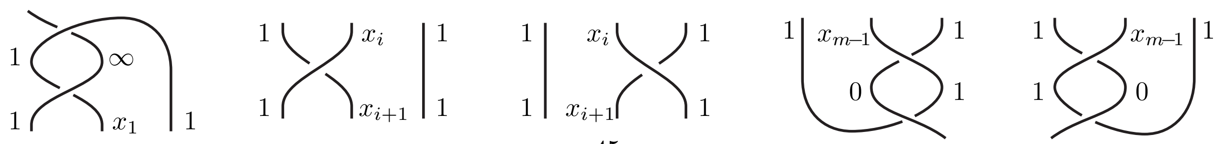
as shown before in this section. By calculating this sum concretely as shown in the following table, we can numerically observe that the limit of $\langle\, \overline{6_1}\, \rangle_N \, e^{-N\varsigma(\overline{6_1})} N^{-3/2}$ tends to the above mentioned value of $\omega(\overline{6_1})$.

N	$\langle\, \overline{6_1}\, \rangle_N \, e^{-N\varsigma(\overline{6_1})}\, N^{-3/2}$
50	$-0.5121772692... + \sqrt{-1} \cdot 0.1473909514...$
100	$-0.5181425383... + \sqrt{-1} \cdot 0.1096254180...$
200	$-0.5201050838... + \sqrt{-1} \cdot 0.09068263776...$

4.3 Calculation of ω_2 for open two-bridge knot diagrams

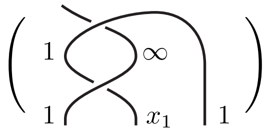
In this section, we calculate ω_2 for open two-bridge knot diagrams. To calculate it, we introduce an operator invariant Ψ, and present ω_2 in terms of Ψ.

As we explain in Section 3.3, any open two-bridge knot diagram (or its mirror image) can be obtained by gluing copies of the following elementary tangle diagrams.

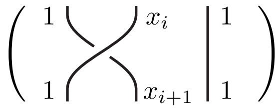


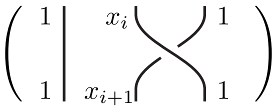
Let D be an open two-bridge knot diagram obtained by gluing copies of the above elementary diagrams.

From the definition of Ω_1, $\Omega_1(D)$ is equal to the product of Ω_1 of such elementary diagrams, whose values are given as follows,

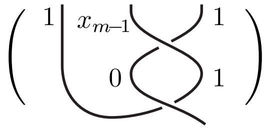
Ω_1 

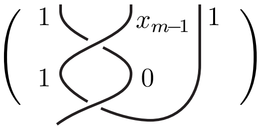
$= 1,$

Ω_1 

$= 1 - \dfrac{x_{i+1}}{x_i}$, Ω_1 

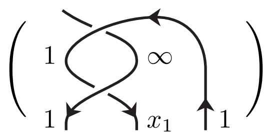
$= 1 - \dfrac{x_{i+1}}{x_i}$,

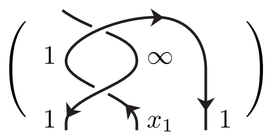
Ω_1 

$= 1,$ Ω_1 

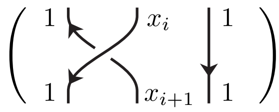
$= 1.$

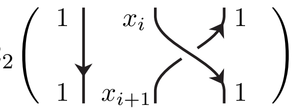
Further, from the definition of Ω_2, $\Omega_2(D)$ is equal to the product of Ω_2 of elementary diagrams, whose values are given as follows,

Ω_2 

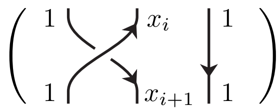
$= 1,$ Ω_2 

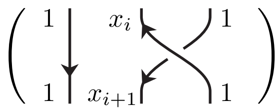
$= \dfrac{1}{x_1^2}$,

Ω_2 

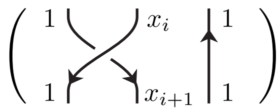
$= \dfrac{1}{x_{i+1}^2}$, Ω_2 

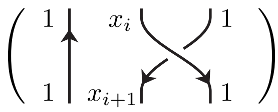
$= \dfrac{1}{x_{i+1}^2}$,

Ω_2 

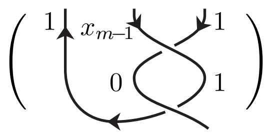
$= x_i^2$, Ω_2 

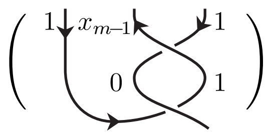
$= x_i^2$,

Ω_2 

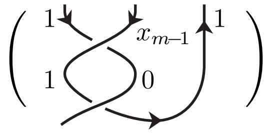
$= 1,$ Ω_2 

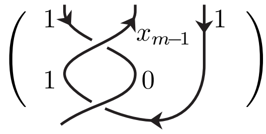
$= 1,$

Ω_2 

$= 1,$ Ω_2 

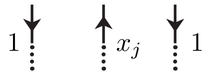
$= x_{m-1}^2$,

Ω_2 

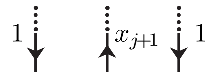
$= 1,$ Ω_2 

$= x_{m-1}^2$.

For an elementary tangle diagram T, we define $\hat{\Omega}_2(T)$ from $\Omega_2(T)$ by multiplying

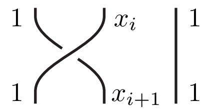
$\dfrac{1}{x_j^2}$ when the top of T is parameterized by 

and multiplying

x_{j+1}^2 when the bottom of T is parameterized by  .

Then, we can verify that $\hat{\Omega}_2(T) = 1$ for each elementary diagram T. Hence, $\Omega_2(D) = 1$.

We calculate the contribution of each elementary diagram to the Hesse matrix. The contribution of the diagram



to the potential function is given by

$$\cdots + \mathrm{Li}_2\big(\frac{1}{x_i}\big) - \mathrm{Li}_2\big(\frac{x_{i+1}}{x_i}\big) + \mathrm{Li}_2(x_{i+1}) + \cdots .$$

Hence, its contribution to the Hesse matrix is given by

$$\begin{pmatrix} \ddots & \ddots & & \\ \ddots & \cdots + \frac{1}{x_i-1} - \frac{x_{i+1}}{x_i-x_{i+1}} & \frac{x_{i+1}}{x_i-x_{i+1}} & \\ & \frac{x_{i+1}}{x_i-x_{i+1}} & -\frac{x_{i+1}}{x_i-x_{i+1}} + \frac{x_{i+1}}{1-x_{i+1}} + \cdots & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

We calculate the determinant of a matrix of the above form recursively, as follows. For an indeterminant y, we put

$$\det\begin{pmatrix} \ddots & \ddots \\ \ddots & \cdots + y \end{pmatrix} = A_i\, y + B_i \,,$$

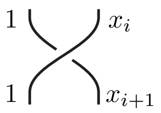
$$\det\begin{pmatrix} \ddots & \ddots & \\ \ddots & \cdots + \frac{1}{x_i-1} - \frac{x_{i+1}}{x_i-x_{i+1}} & \frac{x_{i+1}}{x_i-x_{i+1}} \\ & \frac{x_{i+1}}{x_i-x_{i+1}} & -\frac{x_{i+1}}{x_i-x_{i+1}} + \frac{x_{i+1}}{1-x_{i+1}} + y \end{pmatrix} = A_{i+1}\, y + B_{i+1} \,.$$

Then, we have that

$$\big(A_{i+1} \;\; B_{i+1}\big) = \big(A_i \;\; B_i\big) \begin{pmatrix} \frac{1}{x_i-1} - \frac{x_{i+1}}{x_i-x_{i+1}} & \big(\frac{1}{x_i-1} - \frac{x_{i+1}}{x_i-x_{i+1}}\big)\big(-\frac{x_{i+1}}{x_i-x_{i+1}} + \frac{x_{i+1}}{1-x_{i+1}}\big) - \big(\frac{x_{i+1}}{x_i-x_{i+1}}\big)^2 \\ 1 & -\frac{x_{i+1}}{x_i-x_{i+1}} + \frac{x_{i+1}}{1-x_{i+1}} \end{pmatrix}.$$

Including the contribution of Ω_1, we put

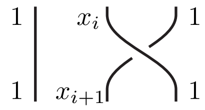
$$\Psi\Bigg(\ \Bigg)$$



$$= \big(1 - \frac{x_{i+1}}{x_i}\big) \begin{pmatrix} \frac{1}{x_i-1} - \frac{x_{i+1}}{x_i-x_{i+1}} & \big(\frac{1}{x_i-1} - \frac{x_{i+1}}{x_i-x_{i+1}}\big)\big(-\frac{x_{i+1}}{x_i-x_{i+1}} + \frac{x_{i+1}}{1-x_{i+1}}\big) - \big(\frac{x_{i+1}}{x_i-x_{i+1}}\big)^2 \\ 1 & -\frac{x_{i+1}}{x_i-x_{i+1}} + \frac{x_{i+1}}{1-x_{i+1}} \end{pmatrix}$$

$$= \frac{x_{i+1}}{x_i}\begin{pmatrix} -\frac{x_i(x_{i+1}-1)}{(x_i-1)x_{i+1}} & 1 \\ \frac{x_i-x_{i+1}}{x_{i+1}} & -\frac{x_i-1}{x_{i+1}-1}\end{pmatrix}.$$

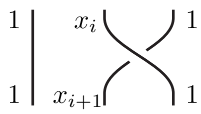
Similarly, the contribution from the diagram



to the Hesse matrix is given by

$$\begin{pmatrix} \ddots & & \ddots & & \\ \ddots & \cdots - \frac{1}{x_i-1} + \frac{x_{i+1}}{x_i-x_{i+1}} & & -\frac{x_{i+1}}{x_i-x_{i+1}} & \\ & -\frac{x_{i+1}}{x_i-x_{i+1}} & & \frac{x_{i+1}}{x_i-x_{i+1}} - \frac{x_{i+1}}{1-x_{i+1}} + \cdots & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$$

Hence, similarly as above, we put

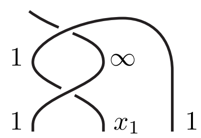
$\Psi\Big($ 

$\Big)$

$$= \Big(1 - \frac{x_{i+1}}{x_i}\Big)\begin{pmatrix} -\frac{1}{x_i-1} + \frac{x_{i+1}}{x_i-x_{i+1}} & \Big(\frac{1}{x_i-1} - \frac{x_{i+1}}{x_i-x_{i+1}}\Big)\Big(-\frac{x_{i+1}}{x_i-x_{i+1}} + \frac{x_{i+1}}{1-x_{i+1}}\Big) - \Big(\frac{x_{i+1}}{x_i-x_{i+1}}\Big)^2 \\ 1 & \frac{x_{i+1}}{x_i-x_{i+1}} - \frac{x_{i+1}}{1-x_{i+1}} \end{pmatrix}$$

$$= \frac{x_{i+1}}{x_i}\begin{pmatrix} \frac{x_i(x_{i+1}-1)}{(x_i-1)x_{i+1}} & 1 \\ \frac{x_i-x_{i+1}}{x_{i+1}} & \frac{x_i-1}{x_{i+1}-1}\end{pmatrix}.$$

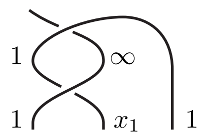
Further, the contribution from the diagram



to the Hesse matrix is given by

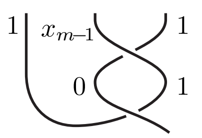
$$\begin{pmatrix} \frac{x_1}{1-x_1} + \cdots & \ddots \\ \ddots & \ddots \end{pmatrix}.$$

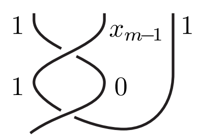
Hence, we put

$\Psi\Big($ 

$\Big) = \begin{pmatrix} 1 & \frac{x_1}{1-x_1} \end{pmatrix}.$

Furthermore, by similar arguments as above, we put

$\Psi\Big($ 

$\Big) = \begin{pmatrix} \frac{1}{1-x_{m-1}} \\ 1 \end{pmatrix}, \quad \Psi\Big($ 

$\Big) = \begin{pmatrix} \frac{1}{x_{m-1}-1} \\ 1 \end{pmatrix}.$

For an open two-bridge knot diagram D obtained by gluing copies of elementary diagrams, we define $\Psi(D)$ to be the product of Ψ of such elementary diagrams.

By the above construction of Ψ, we have that

$$\Psi(D) \;=\; \Omega_1(D)\,\Omega_2(D)\,\det H.$$

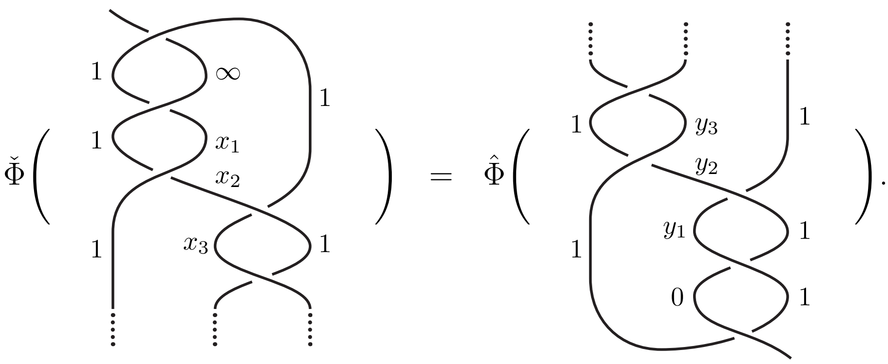
Hence, from the definition of ω_2, we have that

$$\frac{1}{\sqrt{-1}\;\omega_2(D)} \;=\; \Psi(D). \tag{32}$$

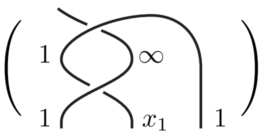
5 Proof of Theorem 1.1

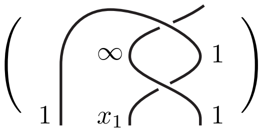
In this section, we prove Theorem 1.1. We introduce $\check{\Phi}$, $\check{\Psi}$ and ϕ_m, ψ_m modifying $\hat{\Phi}$, Ψ, and reduce the proof the the theorem to Proposition 5.1.

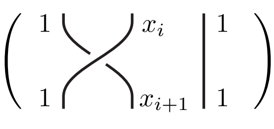
For an open two-bridge knot diagram D, we define $\check{\Phi}(D)$ to be $\hat{\Phi}$ of the diagram obtained from D by π rotation and by exchanging the positive and negative crossings,

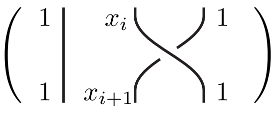


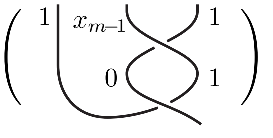
In other words, $\check{\Phi}$ is defined by the following formulas,

$\check{\Phi}$  $= \dfrac{1}{(x_1-1)^3}\begin{pmatrix}1 & -1 & 2\end{pmatrix},$

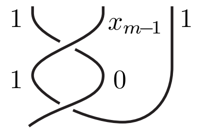
$\check{\Phi}$  $= \dfrac{1}{(x_1-1)^3}\begin{pmatrix}2 & -1 & 1\end{pmatrix},$

$\check{\Phi}$  $= \dfrac{1}{x_i}\begin{pmatrix}1 & -1 & 1\\ 0 & -\frac{1}{x_i} & \frac{2}{x_i}\\ 0 & 0 & 1\end{pmatrix},$

$\check{\Phi}$  $= \dfrac{1}{x_i}\begin{pmatrix}1 & 0 & 0\\ \frac{2}{x_i} & -\frac{1}{x_i} & 0\\ 1 & -1 & 1\end{pmatrix},$

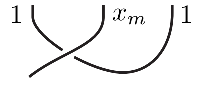
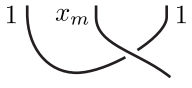
$\check{\Phi}$  $= \dfrac{x_{m-1}-1}{x_{m-1}^2}\begin{pmatrix}1\\ \frac{2}{x_{m-1}}\\ 0\end{pmatrix}.$

Further, similarly as the calculation in Section 3, we can show that

$\check{\Phi}\Bigg($  $\Bigg)$

$$= \frac{1-x_{m-1}}{x_{m-1}^2}\begin{pmatrix} 0 \\ \frac{2}{x_{m-1}} \\ 1 \end{pmatrix}.$$

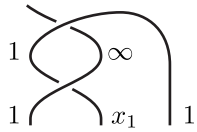
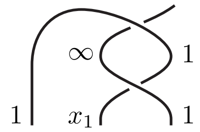
Without assuming that $x_m = 0$, we can put

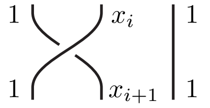
$\check{\Phi}\Bigg($  $\Bigg) = \check{\Phi}\Bigg($  $\Bigg)$

$$= \frac{x_{m-1}-1}{x_{m-1}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

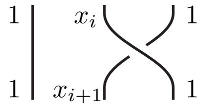
consistently with the above definition.

We define $\check{\Psi}$ by the following formulas,

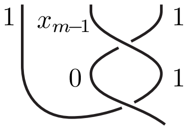
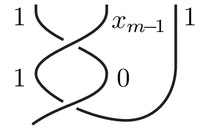
$\check{\Psi}\Bigg($  $\Bigg) = \begin{pmatrix} 1 & \frac{x_1}{1-x_1} \end{pmatrix}$, $\quad\check{\Psi}\Bigg($  $\Bigg) = \begin{pmatrix} 1 & \frac{x_1}{x_1-1} \end{pmatrix}$,

$\check{\Psi}\Bigg($  $\Bigg)$

$$= \frac{x_{i+1}}{x_i}\begin{pmatrix} -\frac{x_i(x_{i+1}-1)}{(x_i-1)x_{i+1}} & 1 \\ \frac{x_i-x_{i+1}}{x_{i+1}} & -\frac{x_i-1}{x_{i+1}-1} \end{pmatrix},$$

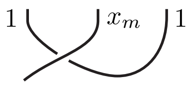
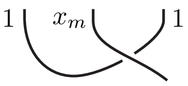
$\check{\Psi}\Bigg($  $\Bigg)$

$$= -\frac{x_{i+1}}{x_i}\begin{pmatrix} \frac{x_i(x_{i+1}-1)}{(x_i-1)x_{i+1}} & 1 \\ \frac{x_i-x_{i+1}}{x_{i+1}} & \frac{x_i-1}{x_{i+1}-1} \end{pmatrix},$$

$\check{\Psi}\Bigg($  $\Bigg) = \begin{pmatrix} \frac{1-x_m}{1-x_{m-1}} \\ 1 \end{pmatrix}$, $\quad\check{\Psi}\Bigg($  $\Bigg) = \begin{pmatrix} \frac{x_m-1}{1-x_{m-1}} \\ 1 \end{pmatrix}$,

without assuming that $x_m = 0$. When $x_m = 0$, this definition is equal to the definition of Ψ except for the sign of $\check{\Psi}(\sigma_2^{-1})$. Hence, $\Psi(D) \doteq \check{\Psi}(D)$ when $x_m = 0$.

Without assuming that $x_m = 0$, we can put

$\check{\Psi}\Bigg($  $\Bigg) = \check{\Psi}\Bigg($  $\Bigg)$

$$= -\frac{x_{m-1}(x_m-1)}{x_m(x_{m-1}-1)}\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

consistently with the above definition.

We recall that our diagram of an open two-bridge knot is a plat closure of a product of copies of σ_1 and σ_2^{-1}. By the hyperbolicity equations, the values of x_i are recursively determined by

$$x_{i+1} = \begin{cases} x_i + 1 - \dfrac{x_i}{x_{i-1}} & \text{if the strand of } x_i \text{ is between } \sigma_1 \text{ and } \sigma_1 \\ & \qquad\text{or between } \sigma_2^{-1} \text{ and } \sigma_2^{-1}, \\ x_i + \dfrac{(x_i-1)^2}{1-\frac{x_i}{x_{i-1}}} & \text{otherwise.} \end{cases}$$

Putting $x_1 = x$ (and $x_0 = \infty$), we can regard x_i as a rational function of x; we put it to be $f_i(x)$. The hyperbolicity equation of the knot is given by $f_m(x) = 0$.

Without assuming that $x_m = 0$, we put

$$\phi_m(x) \;=\; (x_m-1)^2\ \check{\Phi}(D), \qquad \psi_m(x) \;=\; \frac{1-\frac{x_m}{x_{m-1}}}{1-x_m}\ \check{\Psi}(D),$$

as rational functions of x.

Proof of Theorem 1.1. The required formula of the theorem is rewritten as

$$\frac{2}{\tau(K)} \;=\; \frac{1}{\sqrt{-1}\ \omega_2(D)}$$

for a diagram D of an open two-bridge knot K. By (28), the left-hand side is equal to $\hat{\Phi}(D)$. By (32), the right-hand side is equal to $\Psi(D)$. They are equal to $\check{\Phi}(D)$ and $\check{\Psi}(D)$ respectively, as we explained above. Further, they are equal to $\phi_m(c)$ and $\psi_m(c)$ respectively, for a root $x=c$ of $f_m(x) = 0$. Since they are equal by Proposition 5.1 below, we obtain the required formula of the theorem. □

As mentioned above, the proof of the theorem is reduced to the following proposition.

Proposition 5.1.

$$\phi_m(x) \;=\; \psi_m(x).$$

Proof. We recall that our diagram of an open two-bridge knot is a plat closure of a product of copies of σ_1 and σ_2^{-1}. We put the end of this product to be $\cdots b_3 b_2 b_1 b_0$, where $b_3, b_2, b_1, b_0 = \sigma_1$ or σ_2^{-1}. We note that the knot type does not depend on the choice of b_0. Further, by the symmetry of Lemma 5.2 below, we can assume that $b_1 = \sigma_2^{-1}$. In the following of this proof, we prove the proposition by induction on m, in the four cases of the choices of b_3 and b_2. The initial cases of the induction (the cases where $m \le 3$) hold by Example 5.3 below. In the following proof, we show the required formula of the case of m, assuming the case of m' for $m' < m$.

We note that $f_k(x)$ is not equal to 1. (Because, the equation $f_k(x) = 0$ is the hyperbolicity equation of some two-bridge knot, which is a hyperbolic knot or the $(2, n)$ torus knot. In any case, $f_k(x)$ is a non-trivial rational function of x. In particular, it is not equal to 1.)

We further note that $f_k(x)$ and $f_{k+1}(x)$ are not equal. (Because, if $f_k(x)$ and $f_{k+1}(x)$ were equal, we can show by the recursive formula of x_i that $f_j(x) = 1$ for some j, which contradicts the above claim.)

For simplicity, we denote $\phi_i(x)$, $\psi_i(x)$, $f_i(x)$ by ϕ_i, ψ_i, f_i. By definition, $f_i = x_i$, without assuming that $x_m = 0$. We put

$$P_i \;=\; \frac{f_{i+1}}{f_i}\begin{pmatrix} -\frac{f_i(f_{i+1}-1)}{(f_i-1)f_{i+1}} & 1 \\ \frac{f_i-f_{i+1}}{f_{i+1}} & -\frac{f_i-1}{f_{i+1}-1}\end{pmatrix}, \quad Q_i \;=\; -\frac{f_{i+1}}{f_i}\begin{pmatrix} \frac{f_i(f_{i+1}-1)}{(f_i-1)f_{i+1}} & 1 \\ \frac{f_i-f_{i+1}}{f_{i+1}} & \frac{f_i-1}{f_{i+1}-1}\end{pmatrix},$$

$$v_m \;=\; \frac{f_{m-1}-f_m}{f_m(f_{m-1}-1)}\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$P'_i \;=\; \frac{1}{f_i}\begin{pmatrix} 1 & -1 & 1 \\ 0 & -\frac{1}{f_i} & \frac{2}{f_i} \\ 0 & 0 & 1 \end{pmatrix}, \quad Q'_i \;=\; \frac{1}{f_i}\begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{f_i} & -\frac{1}{f_i} & 0 \\ 1 & -1 & 1 \end{pmatrix},$$

$$v'_m \;=\; \frac{(f_{m-1}-1)\,(f_m-1)^2}{f_{m-1}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

By definition, f_i satisfies the same recursive formula as x_i,

$$f_{i+1} \;=\; \begin{cases} f_i + 1 - \dfrac{f_i}{f_{i-1}} & \text{if the strand of } x_i \text{ is between } \sigma_1 \text{ and } \sigma_1 \\ & \qquad \text{or between } \sigma_2^{-1} \text{ and } \sigma_2^{-1}, \\ f_i + \dfrac{(f_i-1)^2}{1-\frac{f_i}{f_{i-1}}} & \text{otherwise.} \end{cases}$$

For simplicity, in the following of this proof, we write formulas of the case of $m = 10$. (We can easily obtain the formulas of general m from them by replacing 10, 9, 8, 7 with m, $m-1$, $m-2$, $m-3$.)

Case 1: *the case where* $b_3 = b_2 = \sigma_2^{-1}$.
In this case, we have that

$$f_{10} = f_9 + 1 - \frac{f_9}{f_8}, \qquad f_9 = f_8 + 1 - \frac{f_8}{f_7}. \tag{33}$$

In the definition of $\check{\Psi}$, the differences among ψ_8, ψ_9, ψ_{10} are presented by

$$v_8, \quad Q_8\,v_9, \quad Q_8\,Q_9\,v_{10}.$$

These vectors are linearly dependent. By calculating their coefficient concretely, we have that

$$\frac{(f_7-1)\,f_9}{(f_7-f_8)(f_9-1)}\cdot v_8 - \frac{f_8+f_9-f_8f_9-f_8f_{10}}{(f_8-f_9)(f_{10}-1)}\cdot Q_8v_9 + \frac{f_9}{f_9-f_{10}}\cdot Q_8Q_9v_{10} \;=\; 0.$$

This is rewritten as the following linear relation among ψ_8, ψ_9, ψ_{10},

$$\frac{(f_7-1)\,f_9}{(f_7-f_8)(f_9-1)}\cdot \psi_8 - \frac{f_8+f_9-f_8f_9-f_8f_{10}}{(f_8-f_9)(f_{10}-1)}\cdot \psi_9 + \frac{f_9}{f_9-f_{10}}\cdot \psi_{10} \;=\; 0.$$

By (33), this is rewritten as

$$\frac{f_9}{f_8}\,\psi_8 - 2\,f_8\,\psi_9 + f_8\,f_9\,\psi_{10} \;=\; 0. \tag{34}$$

Similarly as above, we can show the following linear relation among ψ_7, ψ_8, ψ_9,

$$-\frac{(f_6-1)(f_7-f_8)\,f_8}{(f_6-f_7)(f_8-1)}\,\psi_7 - 2\,f_7\,\psi_8 + f_7\,f_8\,\psi_9 \;=\; 0, \tag{35}$$

noting that, in this case, we can not use $f_8 = f_7 + 1 - \frac{f_7}{f_6}$ at the present stage.

Similarly as the case of $\check{\Phi}$, we can show the following linear relation among ϕ_7, ϕ_8, ϕ_9, ϕ_{10},

$$\begin{aligned} &\frac{f_6\,(f_9-1)}{(f_6-1)(f_7-1)^2\,f_7}\,\phi_7 + \frac{f_7\,(f_9-2)}{(f_7-1)(f_8-1)}\,\phi_8 \\ &- \frac{f_8^2\,(2f_8-1)}{(f_8-1)(f_9-1)}\,\phi_9 + \frac{f_8\,f_9^3\,(f_8-1)}{(f_9-1)(f_{10}-1)^2}\,\phi_{10} \;=\; 0. \end{aligned}$$

By (33), this is rewritten as

$$\frac{f_6\,f_8\,(f_8-1)}{(f_6-1)\,f_7^3}\,\phi_7 + (f_9-2)\,\phi_8 - f_8\,(2f_8-1)\,\phi_9 + f_8^2\,f_9\,\phi_{10} \;=\; 0. \tag{36}$$

Since $\psi_7 = \phi_7,\ \psi_8 = \phi_8,\ \psi_9 = \phi_9$ by the assumption of the induction, we can eliminate ϕ_7, ψ_8, ϕ_8, ψ_9, ϕ_9 by using (34), (35), (36). Then, we obtain

$$\Big(\frac{(f_6-1)(f_7-f_8)\,f_8}{(f_6-f_7)\,f_7\,(f_8-1)} + \frac{f_6\,f_8\,(f_8-1)}{(f_6-1)\,f_7^3}\Big)\,\psi_7 \;=\; f_8^2\,f_9\,(\psi_{10}-\phi_{10}).$$

To show the proposition, it is sufficient to show that $\psi_{10} = \phi_{10}$. Hence, it is sufficient to show that

$$\frac{(f_6-1)(f_7-f_8)\,f_8}{(f_6-f_7)\,f_7\,(f_8-1)} + \frac{f_6\,f_8\,(f_8-1)}{(f_6-1)\,f_7^3} \;=\; 0.$$

This is rewritten as

$$\Big(-f_8+f_7+1-\frac{f_7}{f_6}\Big)\Big(-f_8+f_7+\frac{(f_7-1)^2}{1-\frac{f_7}{f_6}}\Big) \;=\; 0, \tag{37}$$

which holds by the recursive formula of f_i. Therefore, we obtain the proposition in this case.

Case 2: *the case where* $b_3 = \sigma_1$ *and* $b_2 = \sigma_2^{-1}$.
In this case, we have that

$$f_{10} = f_9 + 1 - \frac{f_9}{f_8}, \qquad f_9 = f_8 + \frac{(f_8-1)^2}{1-\frac{f_8}{f_7}}. \tag{38}$$

In the definition of $\check{\Psi}$, the differences among ψ_8, ψ_9, ψ_{10} are presented by

$$v_8, \quad Q_8\,v_9, \quad Q_8\,Q_9\,v_{10}.$$

These vectors are linearly dependent. By calculating their coefficient concretely, similarly as in Case 1, we have that

$$\frac{(f_7-1)\,f_9}{(f_7-f_8)(f_9-1)}\,\psi_8 - \frac{f_8+f_9-f_8f_9-f_8f_{10}}{(f_8-f_9)(f_{10}-1)}\,\psi_9 + \frac{f_9}{f_9-f_{10}}\,\psi_{10} \;=\; 0.$$

By (38), this is rewritten as

$$\frac{f_7\,(f_8-1)}{f_8\,(f_7-f_8)}\,\psi_8-\frac{2\,f_8}{f_9}\,\psi_9+f_8\,\psi_{10} \;=\; 0. \tag{39}$$

Similarly, by calculating the coefficients of the linear dependence among

$$v_7,\quad P_7\,v_8,\quad P_7\,Q_8\,v_9,$$

we have that

$$-\frac{(f_6-1)\,f_8}{(f_6-f_7)(f_8-1)}\,\psi_7+\frac{f_7-f_8+f_7f_8-f_7f_9}{(f_7-f_8)(f_9-1)}\,\psi_8+\frac{f_8}{f_8-f_9}\,\psi_9 \;=\; 0.$$

By (38), this is rewritten as

$$\frac{f_6-1}{f_6-f_7}\,\psi_7+\frac{f_7f_8-2f_7+f_8}{(f_7-f_8)\,f_8}\,\psi_8+\frac{f_7-f_8}{f_7\,(f_8-1)}\,\psi_9 \;=\; 0. \tag{40}$$

Further, in the definition of $\check{\Phi}$, by calculating the coefficients of the linear dependence among

$$v_7',\quad P_7'\,v_8',\quad P_7'\,Q_8'\,v_9',\quad P_7'\,Q_8'\,Q_9'\,v_{10}',$$

we have that

$$\begin{aligned}\frac{f_6\,(f_9-1)}{f_7\,(f_6-1)(f_7-1)^2}\,\phi_7-\frac{f_7\,(1-2f_8+f_8f_9)}{(f_7-1)(f_8-1)^2}\,\phi_8&\\ +\frac{f_8^2\,(2f_8-1)}{(f_8-1)(f_9-1)}\,\phi_9-\frac{f_8\,f_9^3\,(f_8-1)}{(f_9-1)(f_{10}-1)^2}\,\phi_{10} &\;=\; 0.\end{aligned}$$

By (38), this is rewritten as

$$\frac{f_6\,(f_8-1)^3}{f_7\,(f_6-1)(f_7-f_8)^2}\,\phi_7-\frac{f_7\,(1-2f_8+f_8f_9)}{f_8\,(f_7-f_8)}\,\phi_8+(2f_8-1)\,\phi_9-f_8\,f_9\,\phi_{10} \;=\; 0. \tag{41}$$

Since $\psi_7=\phi_7,\;\;\psi_8=\phi_8,\;\;\psi_9=\phi_9$ by the assumption of the induction, we obtain the following relation from (39) and (41),

$$\frac{f_6\,(f_8-1)^3}{f_7\,(f_6-1)(f_7-f_8)^2}\,\psi_7-\frac{f_7\,(1-2f_8+f_9)}{f_8\,(f_7-f_8)}\,\psi_8-\psi_9 \;=\; f_8\,f_9\,(\phi_{10}-\psi_{10}).$$

To show the proposition, it is sufficient to show that $\psi_{10}=\phi_{10}$. Hence, it is sufficient to show that

$$\frac{f_6\,(f_8-1)^3}{f_7\,(f_6-1)(f_7-f_8)^2}\,\psi_7-\frac{f_7\,(1-2f_8+f_9)}{f_8\,(f_7-f_8)}\,\psi_8-\psi_9 \;=\; 0.$$

By using (38), this is rewritten as

$$\frac{f_6\,(f_8-1)^3}{f_7\,(f_6-1)(f_7-f_8)^2}\,\psi_7-\frac{f_7\,(f_8-1)(f_7f_8-2f_7-f_8)}{f_8\,(f_7-f_8)^2}\,\psi_8-\psi_9 \;=\; 0.$$

Further, by using (40), we can eliminate ψ_8 and ψ_9. Then, we obtain

$$\Big(\frac{f_6\,(f_8-1)^3}{f_7\,(f_6-1)(f_7-f_8)^2} + \frac{(f_6-1)\,f_7\,(f_8-1)}{(f_6-f_7)(f_7-f_8)} \Big)\,\psi_7 \;=\; 0.$$

Since the coefficient is ψ_7 is rewritten as (37), this formula holds similarly as in Case 1. Therefore, we obtain the proposition in this case.

Case 3: *the case where* $b_3 = b_2 = \sigma_1$.
In this case, we have that

$$f_{10} = f_9 + \frac{(f_9-1)^2}{1-\frac{f_9}{f_8}}\,, \qquad f_9 = f_8 + 1 - \frac{f_8}{f_7}\,. \tag{42}$$

In the definition of $\check{\Psi}$, the differences among ψ_8, ψ_9, ψ_{10} are presented by

$$v_8, \quad P_8\,v_9, \quad P_8\,Q_9\,v_{10}.$$

These vectors are linearly dependent. By calculating their coefficient concretely, similarly as in Case 1, we have that

$$-\frac{(f_7-1)\,f_9}{(f_7-f_8)(f_9-1)}\,\psi_8 - \frac{f_8-f_9+f_8f_9-f_8f_{10}}{(f_8-f_9)(f_{10}-1)}\,\psi_9 + \frac{f_9}{f_9-f_{10}}\,\psi_{10} \;=\; 0.$$

By (42), this is rewritten as

$$-\frac{f_9}{f_8}\,\psi_8 + \frac{f_8f_9-2f_8+f_9}{f_9-1}\,\psi_9 + \frac{f_9\,(f_8-f_9)^2}{f_8\,(f_9-1)^2}\,\psi_{10} \;=\; 0. \tag{43}$$

Similarly, by calculating the coefficients of the linear dependence among

$$v_7, \quad P_7\,v_8, \quad P_7\,P_8\,v_9,$$

we have that

$$\frac{(f_6-1)\,f_8}{(f_6-f_7)(f_8-1)}\,\psi_7 + \frac{-f_7-f_8+f_7f_8+f_7f_9}{(f_7-f_8)(f_9-1)}\,\psi_8 + \frac{f_8}{f_8-f_9}\,\psi_9 \;=\; 0.$$

By (42), this is rewritten as

$$\frac{(f_6-1)(f_7-f_8)}{(f_6-f_7)\,f_7\,(f_8-1)}\,\psi_7 + \frac{2}{f_8}\,\psi_8 - \psi_9 \;=\; 0. \tag{44}$$

Further, in the definition of $\check{\Phi}$, by calculating the coefficients of the linear dependence among

$$v'_7, \quad P'_7\,v'_8, \quad P'_7\,P'_8\,v'_9, \quad P'_7\,P'_8\,Q'_9\,v'_{10},$$

we have that

$$\begin{aligned} &-\frac{f_6\,(f_9-1)}{(f_6-1)\,f_7\,(f_7-1)^2(f_8-1)}\,\phi_7 - \frac{f_7\,(f_9-2)}{(f_7-1)(f_8-1)^2}\,\phi_8 \\ &\quad + \frac{f_8^2\,(1-2f_8+f_8f_9)}{(f_8-1)^2(f_9-1)^2}\,\phi_9 + \frac{f_8\,f_9^3}{(f_9-1)(f_{10}-1)^2}\,\phi_{10} \;=\; 0. \end{aligned}$$

By (42), this is rewritten as

$$-\frac{f_6\,(f_8-1)}{f_7^3\,(f_6-1)}\,\phi_7-\frac{f_9-2}{f_8}\,\phi_8+\frac{1-2f_8+f_8f_9}{f_9-1}\,\phi_9+\frac{f_9\,(f_8-f_9)^2}{f_8\,(f_9-1)^2}\,\phi_{10} \;=\; 0. \tag{45}$$

Since $\psi_7=\phi_7,\ \ \psi_8=\phi_8,\ \ \psi_9=\phi_9$ by the assumption of the induction, we obtain the following relation from (43) and (45),

$$\frac{f_6\,(f_8-1)}{f_7^3\,(f_6-1)}\,\psi_7-\frac{2}{f_8}\,\psi_8+\psi_9 \;=\; \frac{f_9\,(f_8-f_9)^2}{f_8\,(f_9-1)^2}\,(\phi_{10}-\psi_{10}).$$

To show the proposition, it is sufficient to show that $\psi_{10}=\phi_{10}$. Hence, it is sufficient to show that

$$\frac{f_6\,(f_8-1)}{f_7^3\,(f_6-1)}\,\psi_7-\frac{2}{f_8}\,\psi_8+\psi_9 \;=\; 0.$$

By this formula and (44), we can eliminate ψ_8 and ψ_9. Then, we obtain

$$\Big(\frac{f_6\,(f_8-1)}{f_7^3\,(f_6-1)}+\frac{(f_6-1)(f_7-f_8)}{(f_6-f_7)\,f_7\,(f_8-1)}\Big)\,\psi_7 \;=\; 0.$$

Since the coefficient of ψ_7 is rewritten as (37), this formula holds similarly as in Case 1. Therefore, we obtain the proposition in this case.

Case 4: *the case where* $b_3=\sigma_2^{-1}$ *and* $b_2=\sigma_1$.
In this case, we have that

$$f_{10}=f_9+\frac{(f_9-1)^2}{1-\frac{f_9}{f_8}}\,, \qquad f_9=f_8+\frac{(f_8-1)^2}{1-\frac{f_8}{f_7}}\,. \tag{46}$$

We put

$$\psi'_k=\frac{\psi_k}{(x_{k-1}-1)(x_k-1)^2}\,, \qquad \phi'_k=\frac{\phi_k}{(x_{k-1}-1)(x_k-1)^2}$$

for $k=7,8,9,10$.

In the definition of $\check{\Psi}$, the differences among ψ_8, ψ_9, ψ_{10} are the same as in Case 3, and we have that

$$-\frac{(f_7-1)\,f_9}{(f_7-f_8)(f_9-1)}\,\psi_8-\frac{f_8-f_9+f_8f_9-f_8f_{10}}{(f_8-f_9)(f_{10}-1)}\,\psi_9+\frac{f_9}{f_9-f_{10}}\,\psi_{10} \;=\; 0.$$

By replacing ψ_k with ψ'_k and by using (46), this is rewritten as

$$f_7\,f_9\,(f_8-1)\,\psi'_8-f_8^2\,(-2f_8+f_9+f_8f_9)\,\psi'_9-f_8\,f_9^3\,(f_8-1)\,\psi'_{10} \;=\; 0. \tag{47}$$

Similarly, by calculating the coefficients of the linear dependence among

$$v_7,\quad Q_7\,v_8,\quad Q_7\,P_8\,v_9,$$

we have that

$$-\frac{(f_6-1)\,f_8\,(f_9-1)}{(f_6-f_7)(f_8-1)}\,\psi_7+\frac{f_7-f_8+f_7f_8-f_7f_9}{f_7-f_8}\,\psi_8+\frac{f_8\,(f_9-1)}{f_8-f_9}\,\psi_9 \;=\; 0.$$

By replacing ψ_k with ψ'_k and by using (46), this is rewritten as

$$\frac{(f_6-1)^2(f_7-1)^2\,f_8^2}{(f_6-f_7)(f_8-1)(f_9-1)}\,\psi'_7+(-2f_7+f_8+f_7f_8)\,\psi'_8+\frac{f_8^2\,(f_7-f_8)(f_9-1)}{f_7\,(f_8-1)}\,\psi'_9\;=\;0. \quad (48)$$

Further, in the definition of $\check{\Phi}$, by calculating the coefficients of the linear dependence among

$$v'_7,\quad Q'_7\,v'_8,\quad Q'_7\,P'_8\,v'_9,\quad Q'_7\,P'_8\,Q'_9\,v'_{10},$$

we have that

$$\frac{f_6\,(f_9-1)}{f_7\,(f_6-1)(f_7-1)^2}\,\phi_7-\frac{f_7\,(1-2f_8+f_8f_9)}{(f_7-1)(f_8-1)^2}\,\phi_8 \\ +\frac{f_8^2\,(1-2f_8+f_8f_9)}{(f_8-1)(f_9-1)^2}\,\phi_9+\frac{f_8\,f_9^3\,(f_8-1)}{(f_9-1)(f_{10}-1)^2}\,\phi_{10}\;=\;0.$$

By replacing ϕ_k with ϕ'_k and by using (46), this is rewritten as

$$\frac{f_6\,(f_9-1)}{f_7}\,\phi'_7-f_7\,(1-2f_8+f_8f_9)\,\phi'_8+f_8^2\,(1-2f_8+f_8f_9)\,\phi'_9+f_8\,f_9^3\,(f_8-1)\,\phi'_{10}\;=\;0. \quad (49)$$

Since $\psi'_7=\phi'_7,\ \psi'_8=\phi'_8,\ \psi'_9=\phi'_9$ by the assumption of the induction, we obtain the following relation from (47) and (49),

$$\frac{f_6\,(f_9-1)}{f_7}\,\psi'_7-f_7\,(1-2f_8+f_9)\,\psi'_8-f_8^2\,(f_9-1)\,\psi'_9\;=\;f_8\,f_9^3\,(f_8-1)\,(\psi'_{10}-\phi'_{10}).$$

To show the proposition, it is sufficient to show that $\psi_{10}=\phi_{10}$. Hence, it is sufficient to show that

$$\frac{f_6\,(f_9-1)}{f_7}\,\psi'_7-f_7\,(1-2f_8+f_9)\,\psi'_8-f_8^2\,(f_9-1)\,\psi'_9\;=\;0.$$

By (46), this is rewritten as

$$\frac{f_6\,(f_7-f_8)(f_9-1)}{f_7^2\,(f_8-1)}\,\psi'_7-(-2f_7+f_8+f_7f_8)\,\psi'_8-\frac{f_8^2\,(f_7-f_8)(f_9-1)}{f_7\,(f_8-1)}\,\psi'_9\;=\;0.$$

By this formula and (48), we can eliminate ψ'_8 and ψ'_9. Then, we obtain

$$\Big(\frac{(f_6-1)^2(f_7-1)^2\,f_8^2}{(f_6-f_7)(f_8-1)(f_9-1)}+\frac{f_6\,(f_7-f_8)(f_9-1)}{f_7^2\,(f_8-1)}\Big)\,\psi'_7\;=\;0.$$

Since the coefficient of ψ_7 is rewritten as (37), this formula holds similarly as in Case 1. Therefore, we obtain the proposition in this case. □

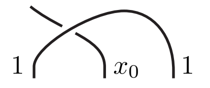
The following lemma is used in the proof of the above proposition.

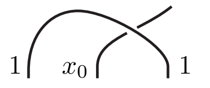
Lemma 5.2. *By the reflection of an open two-bridge knot diagram with respect to a vertical line, σ_1 and σ_2^{-1} are exchanged, and the values of ϕ_m and ψ_m become (-1)-multiple of the original values.*

Proof. In the definition of $\check{\Phi}$, $\check{\Phi}(\sigma_1)$ and $\check{\Phi}(\sigma_2^{-1})$ are conjugate by $M_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Further, the value of $\check{\Phi}$ of the top part of a two-bridge knot diagram becomes M_1-multiple of the original vector. Furthermore, the value of $\check{\Phi}$ of the bottom part of a two-bridge knot diagram becomes $(-M_1)$-multiple of the original vector. Hence, the value of ϕ_m becomes (-1)-multiple of the original value.

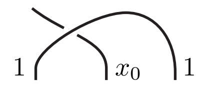
In the definition of $\check{\Psi}$, $\check{\Psi}(\sigma_1)$ and $\check{\Phi}(\sigma_2^{-1})$ are conjugate by $M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Further, the value of $\check{\Psi}$ of the top part of a two-bridge knot diagram becomes M_2-multiple of the original vector. Furthermore, the value of $\check{\Psi}$ of the bottom part of a two-bridge knot diagram becomes $(-M_2)$-multiple of the original vector. Hence, the value of ψ_m becomes (-1)-multiple of the original value, as required. $\square$

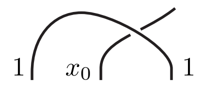
Without assuming that $x_0 = \infty$, we can put

$\check{\Phi}\Big($ 

$\Big) = \check{\Phi}\Big($ 

$\Big) = \dfrac{x_0}{(x_1-1)^3}\begin{pmatrix} 1 & 0 & 1 \end{pmatrix},$

$\check{\Psi}\Big($ 

$\Big) = \check{\Psi}\Big($ 

$\Big) = \dfrac{x_0}{1-x_1}\begin{pmatrix} 1 & 0 \end{pmatrix},$

consistently with the definitions of $\check{\Phi}$ and $\check{\Psi}$. Indeed, we can show Lemma 5.2 without using these formulas, but these formulas are helpful to understand the symmetry of Lemma 5.2.

The following example shows the initial cases of the induction of the proof of Proposition 5.1.

Example 5.3. Proposition 5.1 holds for $m \le 3$.

Proof. By the symmetry of Lemma 5.2, it is sufficient to show the formula of Proposition 5.1 for the plat closures of $\sigma_1^2 \cdot b \cdot b_0$ for $b = 1$, σ_1, σ_2^{-1}, σ_1^2, $\sigma_1\sigma_2^{-1}$, $\sigma_2^{-1}\sigma_1$, σ_2^{-2}, recalling that the knot type does not depend on the choice of $b_0 = \sigma_1$ or σ_2^{-1}. We note that some of them are not knots, but 2-component links. We calculate both sides of the formula concretely for these cases.

For the plat closure of $\sigma_1^2 \cdot b_0$,

$$\phi_1(x) \;=\; \psi_1(x) \;=\; -\frac{1}{x-1}\,.$$

For the plat closure of $\sigma_1^2 \cdot \sigma_1 \cdot b_0$,

$$\phi_2(x) \;=\; \psi_2(x) \;=\; -\frac{2}{x\,(x-1)}\,.$$

For the plat closure of $\sigma_1^2 \cdot \sigma_2^{-1} \cdot b_0$,

$$\phi_2(x) \;=\; \psi_2(x) \;=\; \frac{x-2}{x}\,.$$

For the plat closure of $\sigma_1^2 \cdot \sigma_1^2 \cdot b_0$,

$$\phi_3(x) \;=\; \psi_3(x) \;=\; -\,\frac{3x-1}{x^2\,(x-1)(x+1)}\,.$$

For the plat closure of $\sigma_1^2 \cdot \sigma_1\sigma_2^{-1} \cdot b_0$,

$$\phi_3(x) \;=\; \psi_3(x) \;=\; \frac{x(x^2-x+2)}{(x-1)(x+1)}\,.$$

For the plat closure of $\sigma_1^2 \cdot \sigma_2^{-1}\sigma_1 \cdot b_0$,

$$\phi_3(x) \;=\; \psi_3(x) \;=\; -\,\frac{2\,x\,(x+1)}{x^2-x+1}\,.$$

For the plat closure of $\sigma_1^2 \cdot \sigma_2^{-2} \cdot b_0$,

$$\phi_3(x) \;=\; \psi_3(x) \;=\; \frac{3x^2-5x+1}{x^2\,(x^2-x+1)}\,.$$

Hence, Proposition 5.1 holds for $m \leq 3$. □

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