RIMS-1786

SPIN NETWORKS, EHRHART QUASI-POLYNOMIALS, AND COMBINATORICS OF DORMANT INDIGENOUS BUNDLES

By

Yasuhiro WAKABAYASHI

August 2013

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

SPIN NETWORKS, EHRHART QUASI-POLYNOMIALS, AND COMBINATORICS OF DORMANT INDIGENOUS BUNDLES

YASUHIRO WAKABAYASHI

Abstract. It follows from work of S. Mochizuki, F. Liu, and B. Osserman that there is a relationship between Ehrhart's theory concerning rational polytopes and the geometry of the moduli stack classifying dormant indigenous bundles on a proper hyperbolic curve in positive characteristic. This relationship was established by considering the (finite) cardinality of the set consisting of certain colorings on a 3-regular graph called spin networks. In the present paper, we recall the correspondences between spin networks, lattice points of rational polytopes, and dormant indigenous bundles and present some identities and explicit computations of invariants associated with the objects involved.

Contents

1. Introduction 1
2. Spin networks 7
3. Liu-Osserman polytopes 9
4. Sub-quasi-graphs 11
5. Reconstruction of graphs (the proof of Theorem A (i)) 13
6. Independence of Ehrhart quasi-polynomials (the proof of Theorem A (ii)) 16
7. Ehrhart-Macdonald reciprocity (and the proof of Theorem A (iii)) 19
8. Dormant indigenous bundles (and the proof of Theorem A (iv)) 22
9. Appendix 30
References 33

1. Introduction

S. Mochizuki, F. Liu, and B. Osserman established a relationship (cf. [19]; [14]) between

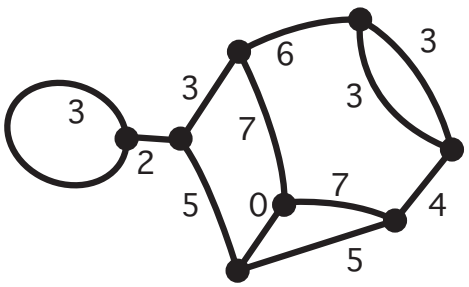
- certain combinatorial invariants, such as the number of *spin networks* and the *Ehrhart quasi-polynomials* of certain rational polytopes, and

- the degree over the moduli stack of curves $\overline{\mathcal{M}}_{g,\mathbb{F}_p}$ of the moduli stack classifying *dormant indigenous bundles*, which we denote by $\overline{\mathcal{M}}^{\mathrm{Zzz...}}_{g,r,\mathbb{F}_p}$.

More precisely, the work of S. Mochizuki reduces the computation of the p-curvature of an indigenous bundle to an entirely combinatorial issue (cf. [19], Introduction, § 1.2, p. 41, Theorem 1.3; [19], Chap. IV, p. 211, Theorem 2.3; [19], Chap. IV, p. 221, Theorem 3.2; [19], Chap. V, § 1). This reduction allows one to perform various explicit computations (cf. [19], Chap. V, § 1, p. 237, Corollary 1.3; [19], Chap. V, § 3.2, p. 267, Corollary 3.7). Moreover, by applying this reduction, F. Liu and B. Osserman conclude that the number of dormant indigenous bundles on a general curve may be expressed as a polynomial with respect to the characteristic of the base field (cf. [14], Theorem 2.1).

In the present paper, we explore further this nontrivial interaction between combinatorics and algebraic geometry in positive characteristic that appears in the work of earlier authors. In particular, we recall the correspondences between the various objects indicated above (i.e., *spin networks*, *Ehrhart quasi-polynomials*, and *dormant indigenous bundles*) and present some identities and explicit computations of invariants associated with them.

1.1. A *spin network* is a type of diagram which is used, in physics, to represent states and interactions between particles and fields in quantum mechanics (cf. [30]). The history of spin networks in this context dates back to the early seventies to the work of R. Penrose, which arose from attempts to build up space-time and quantum mechanics simultaneous from *combinatorial* principles (cf. [23]; [24]). Penrose posited a system consisting of a number of "units", each of which has a total angular momentum (cf. [26], § 1, 2). These units interact in ways that conserve total angular momentum. The system is then described by an arbitrary 3-regular graph whose edges are labeled by integers, corresponding to twice the total angular momentum. The nodes describe interactions at which the units meet. The only condition imposed is that at the nodes, the conservation of angular momentum must be satisfied. Such a combinatorial description makes it possible to consider the cardinality of certain naturally defined finite sets of such systems, i.e., *spin networks*, with, say, a given fixed underlying 3-regular graph. One natural question in this context is the following:

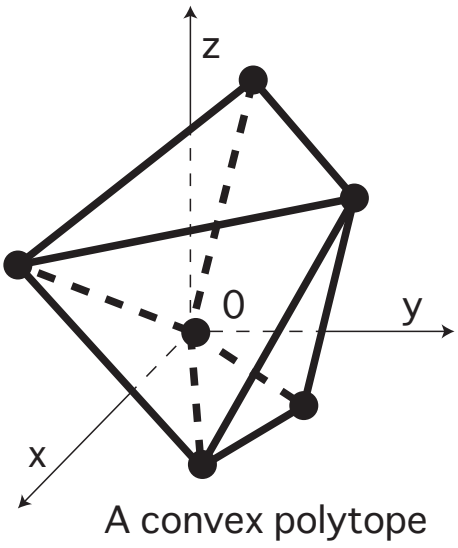


A spin network

> *Can one calculate explicitly the cardinality of such finite sets of spin networks?*

1.2. The *Ehrhart quasi-polynomial* associated to a rational convex polytype in a finite-dimensional vector space over the field of real numbers $\mathbb{R}$ is a periodic sequence of polynomials (with coefficients in the field of rational numbers $\mathbb{Q}$)

that encodes the relationship between the volume of the given polytope and the number of lattice points inside this polytope, as we shall explain below. (For definitions and basic properties concerning polytopes, we shall refer to [4].)



Let n be a nonnegative integer, V an n-dimensional $\mathbb{R}$-vector space, and L a lattice in V (i.e., a finitely generated submodule of V such that the natural map $L \otimes_{\mathbb{Z}} \mathbb{R} \to V$ is an isomorphism). Write $\mathbb{Q} \cdot L$ for the image of the natural map $L \otimes_{\mathbb{Z}} \mathbb{Q} \to V$. A *rational convex polytope* in V is a (necessarily *compact*) subset of V that may be obtained as the *convex hull* (cf. [4], Ch. I, 1.4 Definition) of a finite set of points in $\mathbb{Q} \cdot L \subseteq V$. Let us fix a rational convex polytope $\mathcal{P}$ in V. The *dimension* of $\mathcal{P}$ is defined to be the dimension d of the smallest $\mathbb{R}$-subspace of V containing $\mathcal{P}$. For m a nonnegative integer, we shall denote by $m\mathcal{P}$ the polytope obtained as the image of $\mathcal{P}$ via the map $V \to V$ given by multiplication by m. The smallest positive integer $k \in \mathbb{Z}_{>0}$ for which the *vertices* (cf. [4], Ch. I, 4.3 Definition, (a)) of $k\mathcal{P}$ belong to L is called the *denominator* of $\mathcal{P}$. Let us denote by

$$i_{\mathcal{P}} : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$$

the lattice-point counting function, i.e., the function which to any nonnegative integer m assigns the cardinality of the set $m\mathcal{P} \cap L$:

$$i_{\mathcal{P}}(m) = \sharp(m\mathcal{P} \cap L).$$

E. Ehrhart proved (cf. [1]; [2]; [3]) that the function $i_{\mathcal{P}}$ is a quasi-polynomial function of degree d and period k with coefficients in $\mathbb{Q}$, i.e., that there is a (unique) sequence of polynomials

$$f^{\mathcal{P}}(t) := (f_i^{\mathcal{P}}(t))_{i \in \mathbb{Z}},$$

where $f_i^{\mathcal{P}}(t)$ denotes a polynomial of degree d with coefficients in $\mathbb{Q}$, such that

$$i_{\mathcal{P}}(m) = f_i^{\mathcal{P}}(m) \quad \text{for } m \equiv i \pmod{k}.$$

We shall write $f^{\mathcal{P}}(m) := f_m^{\mathcal{P}}(m)$ $(m \in \mathbb{Z})$. The sequence $f^{\mathcal{P}}(t) = (f_i^{\mathcal{P}}(t))_{i \in \mathbb{Z}}$ is called the *Ehrhart quasi-polynomial of* $\mathcal{P}$.

The number of lattice points inside a rational convex polytope has been studied intensively in combinatorics, algebraic geometry, number theory, and differential geometry. The determination of the Ehrhart quasi-polynomial $f^{\mathcal{P}}(t)$ of $\mathcal{P}$ has significant implications for various areas of mathematics. Certain of the coefficients of the various constituent polynomials $f_i^{\mathcal{P}}(t) \in \mathbb{Q}[t]$ of $f^{\mathcal{P}}(t)$ are easy

to understand (cf. [31]). For example, Ehrhart showed (cf. [1]) that

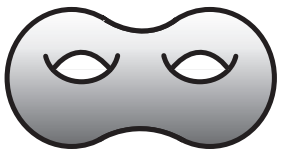
$$\text{“ the leading coefficient of } f_i^{\mathcal{P}}(t)\text{”} = \mathrm{Vol}(\mathcal{P}),$$

where we observe that any basis of V determines a standard Euclidean measure on V which, when considered up to positive real multiples, is independent of the choice of basis; if the dimension of $\mathcal{P}$ is equal to n, then we write $\mathrm{Vol}(\mathcal{P})$ for the volume of $\mathcal{P}$ with respect to the measure given by the positive real multiple of such a standard Euclidean measure on V for which that any fundamental domain of L has volume 1; if the dimension of $\mathcal{P}$ is $< n = \dim(V)$, then we compute its volume with respect to the lattice obtained by intersecting its affine hull with L.

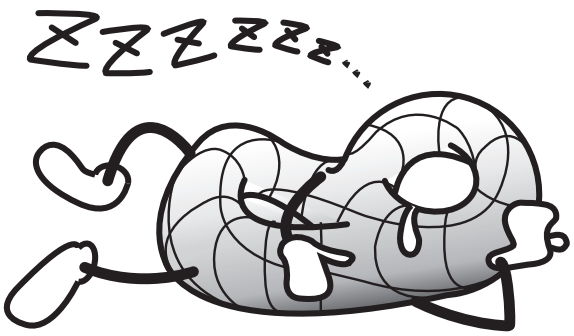
To the knowledge of the author, no simple general procedure is known for determining arbitrary coefficients of the constituent polynomials of the Ehrhart quasi-polynomial. In this context, it is of interest to consider the following question:

> *Do there exist classes of rational convex polytopes for which one can explicitly determine the associated Ehrhart quasi-polynomial?*

1.3. An *indigenous bundle* is a $\mathbb{P}^1$-bundle on an algebraic curve, together with a connection, that satisfies certain properties (cf. Definition 8.1). The notion of an indigenous bundle was originally introduced and studied by Gunning in the context of compact hyperbolic Riemann surfaces (cf. [6]). One may think of an indigenous bundle as an *algebraic* object that encodes the (*analytic*, i.e., *non-algebraic*) uniformization data for a Riemann surface. Various equivalent formulations, involving such diverse types of mathematical objects as differential operators, atlases of coordinate charts, and kernel functions, have been studied by many mathematicians.



A hyperbolic curve



A hyperbolic curve equipped with a “dormant” indigenous bundle

Just as in the case of the theory over $\mathbb{C}$, one may define the notion of an indigenous bundle and the moduli space classifying indigenous bundles in *positive characteristic*. Various properties of such objects were firstly discussed in the context of the p-adic Teichmüller theory developed by S. Mochizuki (cf. [18], [19]). A *dormant torally indigenous bundle* is an indigenous bundle satisfying certain conditions, including a condition peculiar to positive characteristic, i.e., the condition that its p-curvature vanish identically (cf. Definition 8.2; Definition 8.3). If the underlying curve is a proper hyperbolic curve, then a dormant torally indigenous bundle corresponds,

in a certain sense, to a certain type of rank 2 semistable bundle (cf., e.g., [21], Proposition 4.2) whose pull-back by Frobenius is unstable. Such semistable bundles have been studied in a different context (cf. [22], [10]).

Let

$$\overline{\mathcal{M}}_{g,r,\mathbb{F}_p}^{\mathrm{Zzz...}}$$

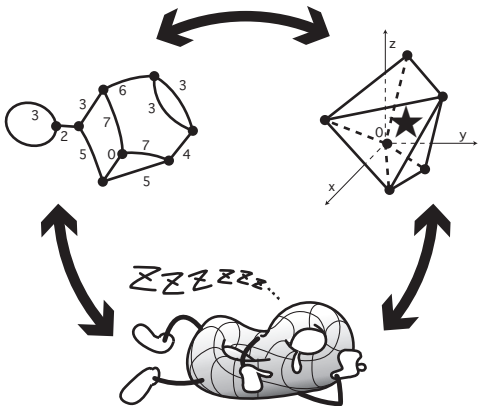
be the moduli stack classifying pointed stable curves of type (g,r) (with $2g-2+r>0$) over $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ equipped with a *dormant* torally indigenous bundle (cf. the notation "Zzz..."!). It is known (cf. Theorem 8.4) that $\overline{\mathcal{M}}_{g,r,\mathbb{F}_p}^{\mathrm{Zzz...}}$ is represented by a proper, smooth Deligne-Mumford stack over $\mathbb{F}_p$ of dimension $3g-3+r$. Moreover, if we denote by $\overline{\mathcal{M}}_{g,r,\mathbb{F}_p}$ the moduli stack classifying pointed stable curves of type (g,r) over $\mathbb{F}_p$, then the natural projection $\overline{\mathcal{M}}_{g,r,\mathbb{F}_p}^{\mathrm{Zzz...}} \to \overline{\mathcal{M}}_{g,r,\mathbb{F}_p}$ is finite, faithfully flat, and generically étale.

One natural question concerning the geometry of $\overline{\mathcal{M}}_{g,r,\mathbb{F}_p}^{\mathrm{Zzz...}}$ is the following:

> *Can one calculate explicitly the degree* $\deg_{\overline{\mathcal{M}}_{g,r,\mathbb{F}_p}}(\overline{\mathcal{M}}_{g,r,\mathbb{F}_p}^{\mathrm{Zzz...}})$ *of* $\overline{\mathcal{M}}_{g,r,\mathbb{F}_p}^{\mathrm{Zzz...}}$ *over* $\overline{\mathcal{M}}_{g,r,\mathbb{F}_p}$ *?*

The generic étaleness of $\overline{\mathcal{M}}_{g,r,\mathbb{F}_p}^{\mathrm{Zzz...}}$ over $\overline{\mathcal{M}}_{g,r,\mathbb{F}_p}$ implies that if X is a sufficiently generic stable curve of type (g,r) over an algebraically closed field of characteristic p, then the number of dormant torally indigenous bundles on X is exactly $\deg_{\overline{\mathcal{M}}_{g,r,\mathbb{F}_p}}(\overline{\mathcal{M}}_{g,r,\mathbb{F}_p}^{\mathrm{Zzz...}})$.

1.4. In the present paper, we discuss a certain convex polytope arising from a 3-regular *quasi-graph* G (cf. Definition 2.1), which we call the *Liu-Osserman polytope* of G and denote by $\mathcal{P}_G$ (cf. Definition 3.1). $\mathcal{P}_G$ is a rational convex polytope, embedded in the space of real-valued functions on the edges of G, satisfying certain triangle inequalities and other constraints. One verifies easily from the definitions a fairly straightforward relationship between the set of spin networks with a given fixed underlying 3-regular graph G and the lattice points inside $\mathcal{P}_G$. In fact, the only substantive discrepancy is a factor that arises from the number of 2-regular sub-quasi-graphs of G (cf. Proposition 4.3 or [14], Lemma 3.3). Moreover, it follows from [19], Introduction, § 1.2, Theorem 1.3; [19], Chap. V, § 1 (cf. Corollary 8.10) that, for every sufficiently large prime number p, there is a bijective correspondence between the lattice points inside $(p-2)\mathcal{P}_G$ and the set of isomorphism classes of dormant torally indigenous bundles on a totally degenerate curve whose dual quasi-graph is isomorphic to G (cf. Definition 8.7). In particular, this circle of ideas allows one to conclude that the three questions displayed in italics above are, in essence, *equivalent*. In the present paper, by



applying these ideas, we obtain some identities and explicit computations which answer these questions, as we explain in the statements of the following results.

Theorem A.

Let $G = (V, E, I)$ be a connected 3-regular quasi-graph (cf. Definition 2.1, 2.2, 2.3). Write $\mathcal{P}_G$ for the Liu-Osserman polytope of G (cf. Definition 3.1) and $f^G(t) := (f_i^G(t))_{i\in\mathbb{Z}}$ for the Ehrhart quasi-polynomial of $\mathcal{P}_G$. Then the following hold:

(i) *The isomorphism class of the quasi-graph G may be reconstructed from the isomorphism class of the polytope $\mathcal{P}_G$ (see Proposition 5.3 for a precise statement).*

(ii) *The quasi-polynomial $f^G(t)$ depends only on the pair of integers $(\sharp V, \sharp E)$, where for a set S we denote by $\sharp S$ the cardinality of S (see Corollary 6.4 and Remark 2.5 for a precise statement).*

(iii) *If, moreover, $\sharp E$ is even (resp., odd), then, for $i \in \mathbb{Z}$, the polynomial $f_i^G(t) \in \mathbb{Q}[t]$ may be expressed in the following form:*

$$f_i^G(t) = \mathrm{Vol}(\mathcal{P}_G) \cdot \prod_{j=1}^{\frac{1}{2}\cdot\sharp E} (t^2 + 4t + a_i^j),$$

$$\Big(\textit{resp.,}\ f_i^G(t) = \mathrm{Vol}(\mathcal{P}_G) \cdot (t+2) \prod_{j=1}^{\frac{1}{2}\cdot(\sharp E-1)} (t^2 + 4t + a_i^j),\Big)$$

where the a_i^j's are complex numbers such that

$$\prod_{j=1}^{\frac{1}{2}\cdot\sharp E} a_i^j = \mathrm{Vol}(\mathcal{P}_G)^{-1} \quad \Big(\textit{resp.,}\ 2 \cdot \prod_{j=1}^{\frac{1}{2}\cdot(\sharp E-1)} a_i^j = \mathrm{Vol}(\mathcal{P}_G)^{-1}\Big).$$

(iv) *If, moreover, G is a graph, then for each odd $i \in \mathbb{Z}$, the polynomial $f_i^G(t) \in \mathbb{Q}[t]$ may be expressed as follows:*

$$f_i^G(t) = -\frac{(t+2)^g}{2^{2g-1}} \cdot \mathrm{Res}_{x=0}\Big[\frac{\cot((t+2)x)}{\sin^{2g-2}(x)}dx\Big],$$

where $g := 1 - \sharp V + \sharp E$ and $\mathrm{Res}_{x=0}(f)$ denotes the residue of f at $x = 0$. In particular, we have

$$\mathrm{Vol}(\mathcal{P}_G) = \frac{(-1)^g \cdot B_{2g-2}}{2 \cdot (2g-2)!},$$

where B_{2g-2} denotes the $(2g-2)$-nd Bernoulli number (cf. Remark 8.5).

Also, we conclude the following result.

Theorem B (= Corollary 8.10)**.**

Let $G = (V, E, I)$ be a 3-regular graph. Write $f^G(t)$ for the Ehrhart quasi-polynomial of the Liu-Osserman polytope of G, $\mathrm{Spin}_G(m)$ ($m \in \mathbb{Z}_{\geq 0}$) for the set of m-colored spin networks on G (cf. Definition 2.6), and N_G for the set of

2-regular sub-quasi-graphs of G (cf. Definition 4.1). Then, for p an odd prime with $p > 2(g-1)$, we have equalities

$$f^G(p-2) = \frac{\sharp \mathrm{Spin}_G(p-2)}{\sharp N_G} = \deg_{\overline{\mathcal{M}}_{g,0,\mathbb{F}_p}}(\overline{\mathcal{M}}^{\mathrm{Zzz...}}_{g,0,\mathbb{F}_p})$$
$$= -\frac{p^g}{2^{2g-1}} \cdot \mathrm{Res}_{x=0}\Big[\frac{\cot(px)}{\sin^{2g-2}(x)}dx\Big],$$

where $g := 1 - \sharp V + \sharp E$.

We shall remark on results of the present paper, displayed in Theorem A, B.

Remark 1.4.1
Our discussions and results in the present paper follow, to a substantial extent, the idea discussed in [14]. The two first equalities of the display in Theorem B are implicit in [14] (cf. the proofs of [14], Lemma 3.3, Theorem 3.9). Moreover, the latter assertion of Theorem A (ii) is derived (cf. the proof of Lemma 7.2) as a natural consequence of the discussion in the proof of [14], Theorem 2.1.

Remark 1.4.2
Theorem A (ii) contains the content of [14], Theorem 2.4, which asserts that the odd values of the Ehrhart quasi-polynomial depend only on the type (g, r). Liu and Ossermann conjectured, in the context of this result, that the even values also depend only on the type (g, r) (cf. [14], Conjecture 4.2). Thus, Theorem A (ii) yields an affirmative answer to the conjecture.

Remark 1.4.3
One key ingredient in the proof of Theorem A (iv), as well as in the proof of Theorem B, is an explicit calculation of the degree $\deg_{\overline{\mathcal{M}}_{g,0,\mathbb{F}_p}}(\overline{\mathcal{M}}^{\mathrm{Zzz...}}_{g,0,\mathbb{F}_p})$ (for $p > 2(g-1)$) obtained by the author in [29] (cf. Theorem 8.4 (ii); [29], Theorem A). One special case of this calculation, which was verified by S. Mochizuki (cf. [19], Chap. V, Corollary 3.7), H. Lange-C. Pauly (cf. [13], Theorem 2), and B. Osserman (cf. [22], Theorem 1.2) (by applying different methods) is the following equality:

$$\deg_{\overline{\mathcal{M}}_{2,0,\mathbb{F}_p}}(\overline{\mathcal{M}}^{\mathrm{Zzz...}}_{2,0,\mathbb{F}_p}) = \frac{1}{24} \cdot (p^3 - p).$$

2. Spin networks

We start by recalling the notion of a 3-regular graph and a spin network. We shall follow the definitions of a spin network discussed in [24], p.241. For notations and conventions concerning multisets, we shall refer to [25]. The following definition of a *quasi-graph* is essentially the same as [14], Definition 2.2. Also, the following definition of a *graph* is the same as the definition of a multigraph in [32]. Unlike some definitions of the notion of a "graph", in this definition, we do not consider the *distinct branches* of an edge; in particular, for

us, an automorphism of a graph (cf. Definition 5.1) that induces the identity automorphisms on the sets of vertices and edges, but which may *permute* the branches of an edge will be regarded as the identity automorphism.

Definition 2.1.
A *(finite) quasi-graph* is a triple

$$G = (V, E, I)$$

consisting of

- a finite set V, whose elements are called *vertices*,
- a finite set E, whose elements are called *edges*, and
- a map $I : E \to V^{[1]} \sqcup V^{[2]}$, called an *incidence relation*, where $V^{[i]}$ $(i = 1, 2)$ denotes the set of multisets over V with cardinality i (cf. [25], Definition 1).

If $v \in_+ I(e)$ for $v \in V$, $e \in E$ (cf. [25], Definition 2), then we shall say that e is *incident to* v. A *loop* is an edge incident to the same vertex. Elements of $E^{\mathrm{free}} := I^{-1}(V^{[1]})$ are called *free*, and elements of $E^{\mathrm{fix}} := I^{-1}(V^{[2]})$ are called *fixed* (hence, $E = E^{\mathrm{free}} \sqcup E^{\mathrm{fix}}$). A *(finite) graph* is a quasi-graph $G = (V, E, I)$ satisfying the condition $E^{\mathrm{free}} = \emptyset$.

Let us fix a quasi-graph $G = (V, E, I)$.

Definition 2.2.
We shall say that G is *connected* if for any two of its vertices u, v there exists a sequence $e_1, e_2, \cdots, e_l$ of edges of G such that $u \in_+ I(e_1)$, $v \in_+ I(e_l)$ and $I(e_j) \cap I(e_{j+1}) \neq \emptyset$ for $j = 1, \cdots, l-1$.

For a vertex $v \in V$, we shall denote by

$$A_G(v)$$

the multiset consisting of edges incident to v, where any loop incident to v occurs twice in $A_G(v)$.

Definition 2.3.
For $m \in \mathbb{Z}_{>0}$, we shall say that G is *m-regular* if for any vertex $v \in V$, the cardinality of the multiset $A_G(v)$ is exactly m.

Definition 2.4.
Let g, r be nonnegative integers, and assume that G is connected (cf. Definition 2.2). We shall say that G is *of type* (g, r) if

$$g = 1 - \sharp V + \sharp E^{\mathrm{fix}}, \text{and } r = \sharp E^{\mathrm{free}}.$$

We shall refer to the integer g as the *genus* of G.

Remark 2.5.
If a quasi-graph $G_0 = (V_0, E_0, I_0)$ of type (g, r) is connected and 3-regular, then one verifies from the incidence relation that

$$3 \cdot \sharp V_0 + \sharp E_0^{\mathrm{free}} = 2 \cdot \sharp E_0.$$

Hence, we have

$$\sharp V_0 = 2g - 2 + r, \text{ and } \sharp E_0 = 3g - 3 + 2r.$$

In particular, the number of vertices and edges in a connected 3-regular quasi-graph (resp., a connected 3-regular graph) is completely determined by the type of the quasi-graph (resp., graph).

Definition 2.6.
Let $[a, b, c]$ be a multiset with cardinality three over the set $\mathbb{Z}_{\geq 0}$. We shall say that $[a, b, c]$ is *preadmissible* if any of the three integers $a + b - c$, $a - b + c$, and $-a + b + c$ is nonnegative. We shall say that $[a, b, c]$ is *admissible* if it is preadmissible and $a + b + c$ is even. For $n \in \frac{1}{2} \cdot \mathbb{Z}_{\geq 0}$, we shall say that $[a, b, c]$ is *n-colored* if $a + b + c \leq 2n$.

Definition 2.7.
A *spin network* (resp., An *n-colored spin network* ($n \in \mathbb{Z}_{\geq 0}$)) on G is a collection $(\lambda_e)_{e \in E}$ of nonnegative integers indexed by E such that for each vertex $v \in V$ the multiset $\biguplus_{e \in_+ A_G(v)} [\lambda_e]$ (cf. [25], Definition 7.3) is admissible (resp., admissible and n-colored).

Denote by

$$\mathrm{Spin}_G \quad (\text{resp., } \mathrm{Spin}_G(n))$$

the set of spin networks on G (resp., the set of n-colored spin networks on G). Observe that if $(\lambda_e)_{e \in E}$ is a spin network on G, then for each $m \in \mathbb{Z}_{\geq 0}$ the collection $(\lambda_e + 2m)_{e \in E}$ forms also a spin network on G. It follows that Spin_G has infinitely many elements. On the other hand, each $\mathrm{Spin}_G(n)$ ($n \in \mathbb{Z}_{\geq 0}$) is evidently a finite subset of Spin_G, and

$$\mathrm{Spin}_G = \bigcup_{n \geq 0} \mathrm{Spin}_G(n).$$

In particular, we may discuss the cardinality $\sharp \mathrm{Spin}_G(n)$ of the various subsets $\mathrm{Spin}_G(n)$ ($n \in \mathbb{Z}_{\geq 0}$) of Spin_G (cf. § 1.1).

3. Liu-Osserman polytopes

Next, we define a certain rational convex polytope constructed from a 3-regular quasi-graph, for which we call the *Liu-Ossermann polytope* of the quasi-graph.

Fix a pair of nonnegative integers (g, r) with $2g-2+r>0$, and a connected 3-regular quasi-graph $G=(V,E,I)$ of type (g,r) (cf. Definition 2.4). Denote by

$$\mathbb{R}^E$$

the space of real valued functions $v : E \to \mathbb{R}$ on E. The space $\mathbb{R}^E$ is an $\sharp E(=3g-3+2r)$-dimensional $\mathbb{R}$-vector space (cf. Remark 2.5), and the subset

$$\mathbb{Z}^E$$

of $\mathbb{R}^E$ consisting of integer valued functions $v : E \to \mathbb{Z}(\subseteq \mathbb{R})$ forms a lattice of $\mathbb{R}^E$.

Definition 3.1. (cf. [14], Definition 2.3)
Let us define a subset

$$\mathcal{P}_G$$

of $\mathbb{R}^E$ to be the set of real-valued functions $w : E \to \mathbb{R}$ on E satisfying the following inequalities:

(i) for each $e \in E$, $w(e) \geq 0$,
(ii) for each $v \in V$, $\sum_{e \in_+ A_G(v)} w(e) \leq 1$, and
(iii) for each $v \in V$ and $e \in_+ A_G(v)$, $w(e) \leq \sum_{e' \in_+ A_G(v)-[e]} w(e')$

(cf. [25], Definition 8). One verifies that $\mathcal{P}_G$ is a connected $\sharp E$-dimensional (namely, full dimensional) convex polytope in the space $\mathbb{R}^E$. We shall call $\mathcal{P}_G(\subseteq \mathbb{R}^E)$ the *Liu-Osserman polytope* of G.

A vertex v of $\mathcal{P}_G$ must satisfy all of the inequalities listed in Definition 3.1 (i), (ii), and (iii). Moreover, by replacing the inequalities with equalities, one obtains a collection of linear constraints, and the vertex v must satisfy some $\sharp E$ independent constraints among these equalities. By rationality of coefficients in such linear equalities, $\mathcal{P}_G$ is a rational convex polytope with respect to the lattice $\mathbb{Z}^E(\subseteq \mathbb{R}^E)$.

Let m be a nonnegative integer. It follows from the definition of $\mathcal{P}_G$ that the lattice-points $m\mathcal{P}_G \cap \mathbb{Z}^E$ inside $m\mathcal{P}_G$ (cf. § 1.2) corresponds bijectively to the set of $\mathbb{Z}_{\geq 0}$-valued functions $w : E \to \mathbb{Z}_{\geq 0}$ such that for each $v \in V$ the following condition $(\mathrm{iv})_{G,v,m}$ holds:

$$(\mathrm{iv})_{G,v,m} \quad \text{the multiset } A_G(v) \text{ is preadmissible and } \frac{m}{2}\text{-colored.}$$

As we explained in § 1.2, the lattice-point counting function

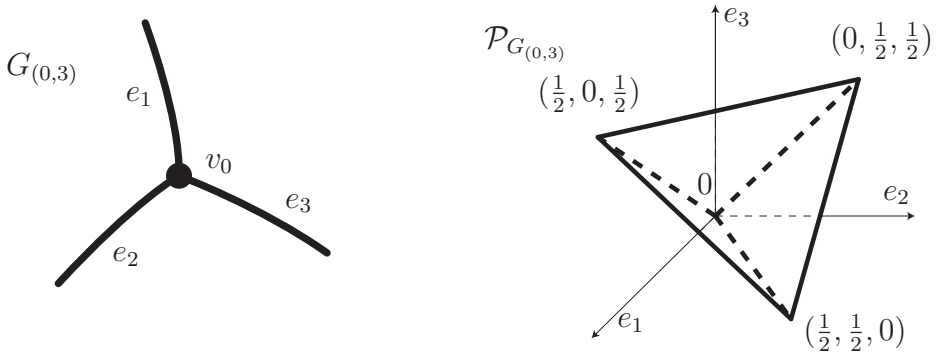
$$i_{\mathcal{P}_G}; m \mapsto i_{\mathcal{P}_G}(m) := \sharp(m\mathcal{P}_G \cap \mathbb{Z}^E)$$

is a quasi-polynomial function of degree $\sharp E$. We denote this quasi-polynomial by

$$f^G(t) = (f_i^G(t))_{i \in \mathbb{Z}},$$

where $f_i^G(t) \in \mathbb{Q}[t]$.

Example 3.2.
Let $G_{(0,3)} = (V_{(0,3)} = \{v_0\}, E_{(0,3)} = \{e_1, e_2, e_3\}, I_{(0,3)})$ be a connected 3-regular quasi-graph of type $(0, 3)$, which is uniquely determined up to isomorphism (cf. Definition 5.1). $G_{(0,3)}$ and the Liu-Osserman polytope $\mathcal{P}_{G_{(0,3)}}$ of $G_{(0,3)}$ may be illustrated as follows:



It follows from a straightforward calculation that the lattice-point counting function $i_{\mathcal{P}_G}$ may be expressed as follows:

$$i_{\mathcal{P}_G}(m) = \begin{cases} \frac{1}{24}(m+2)(m^2+4m+3) & \text{if } m \text{ is odd,} \\ \frac{1}{24}(m+2)(m^2+4m+12) & \text{if } m \text{ is even.} \end{cases}$$

4. Sub-quasi-graphs

In this section, we recall a relationship between the set of colored spin networks on a given 3-regular quasi-graph and the lattice points inside the Liu-Osserman polytope of the quasi-graph. This relationship (= Proposition 4.3) is implicit in the proof of [14], Lemma 3.3.

Fix a connected 3-regular quasi-graph $G = (V, E, I)$.

Definition 4.1. (cf. [14], Definition 3.2)
A *sub-quasi-graph* of G is a quasi-graph $H = (V_H, E_H, I_H)$ satisfying the following conditions

- V_H and E_H are nonempty subsets of V and E respectively,
- $\bigcup_{e \in E_H} I(e)^* \subseteq V_H$ (cf, [25], § 1 for the definition of the support of a multiset M, denoted by M^*), and
- $I|_{E_H} = I_H$ as functions from E_H to $V_H^{[1]} \sqcup V_H^{[2]} (\subseteq V^{[1]} \sqcup V^{[2]})$.

Denote by

$$N_G$$

the set of (not necessarily connected) 2-regular sub-quasi-graphs of G. It is evident that $\sharp N_G > 0$.

Remark 4.2.
There exists a natural bijective correspondence between N_G and $2\mathcal{P}_G \cap \mathbb{Z}^E$. Indeed, if $H = (V_H, E_H, I_H)$ is a 2-regular sub-quasi-graph of G, then the function $w(H) : E \to \mathbb{Z}$ defined by

$$w(H)(e) = \begin{cases} 0 & \text{if } e \notin E_H, \\ 1 & \text{if } e \in E_H. \end{cases}$$

is an element of $2\mathcal{P}_G \cap \mathbb{Z}^E$. Conversely, let $w : E \to \mathbb{Z}$ be an element of $2\mathcal{P}_G \cap \mathbb{Z}^E$. Set

$$E(w) := \{e \in E \mid w(e) = 1\}, \quad V(w) := \bigcup_{e \in E(w)} I(e)^*.$$

The image of $E(w)$ via the incidence relation $I : E \to V^{[1]} \sqcup V^{[2]}$ is contained in $V(w)^{[1]} \sqcup V(w)^{[2]}$; write $I(w)$ for $I|_{E(w)}$, as a map from $E(w)$ to $V(w)^{[1]} \sqcup V(w)^{[2]}$. Then the triple $G(w) = (V(w), E(w), I(w))$ forms a 2-regular sub-quasi-graph of G. One verifies that the assignments $H \mapsto w(H)$, $w \mapsto G(w)$ determines a bijective correspondence $N_G \xrightarrow{\sim} 2\mathcal{P}_G \cap \mathbb{Z}^E$. In particular, we have

$$\sharp N_G = i_{\mathcal{P}_G}(2).$$

Proposition 4.3. *(cf.* [14]*, Lemma 3.3)*
For an odd $n \in \mathbb{Z}_{\geq 0}$, there exists a natural bijection

$$\mathrm{Spin}_G(n) \xrightarrow{\sim} (n\mathcal{P}_G \cap \mathbb{Z}^E) \times N_G.$$

In particular, we have

$$i_{\mathcal{P}_G}(n) (= f^G(n)) = \frac{\sharp \mathrm{Spin}_G(n)}{\sharp N_G}.$$

Proof. First, we shall construct a map $(n\mathcal{P}_G \cap \mathbb{Z}^E) \times N_G \to \mathrm{Spin}_G(n)$. Let $(w : E \to \mathbb{Z}, H = (V_H, E_H, I_H))$ be an element of $(n\mathcal{P}_G \cap \mathbb{Z}^E) \times N_G$. To the pair (w, H), we associate a function $w_H : E \to \mathbb{Z}$ defined by

$$w_H(e) = \begin{cases} 2 \cdot w(e) & \text{if } e \notin E_H, \\ n - 2 \cdot w(n) & \text{if } e \in E_H. \end{cases}$$

Since n is odd, an element e of E lies in E_H if and only if $w_H(e)$ is odd. Here, for $m \in \mathbb{Z}_{\geq 0}$, one may verify easily the following fact:

$(A)_m$ a multiset $[a, b, c]$ with cardinality three is admissible and m-colored if and only if the multiset $[a, m - b, m - c]$ is admissible and m-colored.

The collection $(2 \cdot w(e))_{e \in E}$ forms an n-colored spin network on G, it follows from the definitions of w_H and the fact $(A)_n$ that the collection $(w_H(e))_{e \in E}$ forms also an n-colored spin network on G. Hence, the assignment $(w, H) \mapsto (w_H(e))_{e \in E}$ determines a map

$$\alpha : (n\mathcal{P}_G \cap \mathbb{Z}^E) \times N_G \to \mathrm{Spin}_G(n).$$

Next, we consider the bijectivity of α. Let $\lambda = (\lambda_e)_{e \in E}$ be an element of $\mathrm{Spin}_G(n)$. Set $E_\lambda := \{e \in E |\ \lambda_e \text{ is odd}\}$, $V_\lambda := \bigcup_{e \in E_\lambda} I(e)^*$ (cf. [25], § 1). The image of E_λ via the incidence relation $I : E \to V^{[1]} \sqcup V^{[2]}$ is contained in $V_\lambda^{[1]} \sqcup V_\lambda^{[2]}$; write I_λ for $I|_{E_\lambda}$ as a map from E_λ to $V_\lambda^{[1]} \sqcup V_\lambda^{[2]}$. Then the triple $H_\lambda := (V_\lambda, E_\lambda, I_\lambda)$ forms a 2-regular sub-quasi-graph of G. Moreover, denote by $w_\lambda : E \to \mathbb{R}$ the function defined by

$$w_\lambda(e) = \begin{cases} \frac{1}{2} \cdot \lambda_e & \text{if } e \notin E_\lambda, \\ \frac{1}{2} \cdot (n - \lambda_e) & \text{if } e \in E_\lambda. \end{cases}$$

It follows from the fact $(A)_n$ and the definition of E_λ that w_λ is an element of $n\mathcal{P}_G \cap \mathbb{Z}^E$. One verifies easily that the assignment $\lambda \mapsto (w_\lambda, H_\lambda)$ determines an inverse to α, and hence completes the proof of Proposition 4.3. □

5. Reconstruction of graphs (the proof of Theorem A (i))

In this section, we make and prove a precise statement of Theorem A (i). To do this, we start by defining the notion of an isomorphism of graphs, as well as of polytopes.

Definition 5.1.

Let $G = (V, E, I)$, $G' = (V', E', I')$ be quasi-graphs. An *isomorphism* from G to G' is a pair $\xi = (\xi^{\mathrm{ver}}, \xi^{\mathrm{edg}})$ consisting of bijections $\xi^{\mathrm{ver}} : V \xrightarrow{\sim} V'$, $\xi^{\mathrm{edg}} : E \xrightarrow{\sim} E'$ that are compatible, in the evident sense, with the respective incidence relations I, I'. We shall say that G is *isomorphic to* G' if there exists an isomorphism from G to G'. Denote by

$$\mathrm{Isom}(G, G')$$

the set of isomorphisms from G to G'. If $G = G'$, then the set $\mathrm{Isom}(G, G)$ (i.e., the set of automorphisms of G) forms a group under composition of morphisms.

Definition 5.2.

Let $\mathcal{P} \subseteq \mathbb{R}^n$, $\mathcal{P}' \subseteq \mathbb{R}^{n'}$ be polytopes embedded in a finite-dimensional $\mathbb{R}$-vector space. An *isomorphism* from $\mathcal{P}(\subseteq \mathbb{R}^n)$ to $\mathcal{P}'(\subseteq \mathbb{R}^{n'})$ is an $\mathbb{R}$-linear bijection $L : \mathbb{R}^n \to \mathbb{R}^{n'}$ that induces, by restricting, a bijection $L|_{\mathcal{P}} : \mathcal{P} \xrightarrow{\sim} \mathcal{P}'$ from $\mathcal{P}$ to $\mathcal{P}'$. We shall say that $\mathcal{P}$ is *isomorphic to* $\mathcal{P}'$ if there exists an isomorphism from $\mathcal{P}$ to $\mathcal{P}'$. Denote by

$$\mathrm{Isom}(\mathcal{P}, \mathcal{P}')$$

the set of isomorphisms from $\mathcal{P}$ to $\mathcal{P}'$. If $\mathcal{P} = \mathcal{P}'$, then the set $\mathrm{Isom}(\mathcal{P}, \mathcal{P})$ (i.e., the set of automorphisms of $\mathcal{P}$) forms a group under composition of morphisms.

By the above definitions, it makes sense to speak of the isomorphism class of a quasi-graph and the isomorphism class of a polytope.

Let $G = (V, E, I)$, $G' = (V', E', I')$ be connected 3-regular quasi-graphs and $\xi = (\xi^{\mathrm{ver}}, \xi^{\mathrm{edg}}) : G \to G'$ an isomorphism from G to G'. The bijection $\xi^{\mathrm{edg}} : E \xrightarrow{\sim} E'$ induces naturally an $\mathbb{R}$-linear bijection $L_\xi : \mathbb{R}^E \xrightarrow{\sim} \mathbb{R}^{E'}$. Moreover, it follows from the definition of an isomorphism of quasi-praphs that L_ξ is an isomorphism between the Liu-Osserman polytopes $\mathcal{P}_G (\subseteq \mathbb{R}^E)$ and $\mathcal{P}_{G'} (\subseteq \mathbb{R}^{E'})$. Thus, the assignment $\xi \mapsto L_\xi$ determines a map

$$\Xi_{G,G'} : \mathrm{Isom}(G, G') \to \mathrm{Isom}(\mathcal{P}_G, \mathcal{P}_{G'}).$$

If $G = G'$, then $\Xi_{G,G}$ is a homomorphism between the respective automorphism groups.

Now we formulate precisely the statement of Theorem A (i) as follows:

Proposition 5.3.

Let $G_{(2,0)} = (V_{(2,0)}, E_{(2,0)}, I_{(2,0)})$ be the connected 3-regular graph of type $(2, 0)$ (i.e., $\sharp V_{(2,0)} = 2$ and $\sharp E_{(2,0)} = 3$ by Remark 2.5) determined uniquely by the following condition : $V_{(2,0)} = \{u, v\}$, $E_{(2,0)} = \{e_1, e_2, e_3\}$, and $I_{(2,0)} : e_i \mapsto [u, v]$ $(i = 1, 2, 3)$.

(i) *Let $G = (V, E, I)$ be a connected 3-regular quasi-graph which is isomorphic to $G_{(2,0)}$. Write $\mathfrak{S}_V$, $\mathfrak{S}_E$ for the symmetric groups on V, E respectively. Also, write*

$$\xi_G : \mathrm{Isom}(G, G) \to \mathfrak{S}_V \times \mathfrak{S}_E$$

for the natural injection and

$$\xi_{\mathcal{P}_G} : \mathfrak{S}_E \to \mathrm{Isom}(\mathcal{P}_G, \mathcal{P}_G)$$

for the map that sends each $\sigma \in \mathfrak{S}_E$ to the automorphism $\mathbb{R}^E \xrightarrow{\sim} \mathbb{R}^E$ of $\mathcal{P}_G$ determined by assigning $v \mapsto v \circ \sigma$ $(v \in \mathbb{R}^E)$. Then the maps ξ_G, $\xi_{\mathcal{P}_G}$ are isomorphisms of groups. Moreover, the map $\Xi_{G,G} : \mathrm{Isom}(G, G) \to \mathrm{Isom}(\mathcal{P}_G, \mathcal{P}_G)$ may be identified, via these isomorphisms, with the second projection $\mathfrak{S}_V \times \mathfrak{S}_E \to \mathfrak{S}_E$.

(ii) *Let $G = (V, E, I)$, $G' = (V', E', I')$ be connected 3-regular quasi-graphs neither of which is isomorphic to $G_{(2,0)}$. Then, the map $\Xi_{G,G'}$ is bijective.*

Proof. Assertion (i) is straightforward. We consider assertion (ii). First, we shall verify that $\Xi_{G,G'}$ is injective. Let $\xi_1 = (\xi_1^{\mathrm{ver}}, \xi_1^{\mathrm{edg}})$, $\xi_2 = (\xi_2^{\mathrm{ver}}, \xi_2^{\mathrm{edg}})$ be elements of $\mathrm{Isom}(G, G')$ satisfying that $(L_{\xi_1} :=) \Xi_{G,G'}(\xi_1) = \Xi_{G,G'}(\xi_2) (=: L_{\xi_2})$. The equality $L_{\xi_1} = L_{\xi_2} : \mathbb{R}^E \xrightarrow{\sim} \mathbb{R}^{E'}$ of $\mathbb{R}$-linear bijections implies that $\xi_1^{\mathrm{edg}} = \xi_2^{\mathrm{edg}} : E \xrightarrow{\sim} E'$. Moreover, we claim that $\xi_1^{\mathrm{ver}} = \xi_2^{\mathrm{ver}}$. Indeed, let us assume that there is an element v of V such that $\xi_1^{\mathrm{ver}}(v) \neq \xi_2^{\mathrm{ver}}(v)$. The fact $\xi_1^{\mathrm{edg}} = \xi_2^{\mathrm{edg}} : E \xrightarrow{\sim} E'$ implies that $\xi_1^{\mathrm{edg}}(A_G(v)) = \xi_2^{\mathrm{edg}}(A_G(v))$. But G' is connected, so it occurs only when G' is isomorphic to $G_{(2,0)}$. It contradicts to the hypothesis, and hence, concludes that $\xi_1^{\mathrm{ver}} = \xi_2^{\mathrm{ver}}$. Thus we obtain that $\xi_1^{\mathrm{ver}} = \xi_2^{\mathrm{ver}}$ and $\xi_1^{\mathrm{edg}} = \xi_2^{\mathrm{edg}}$, i.e., $\xi_1 = \xi_2$. This completes the injectivity of $\Xi_{G,G'}$.

Next, we verify the surjectivity of $\Xi_{G,G'}$. Toward the following discussion, we set, for $v \in V$ and a sufficiently small $\epsilon \in \mathbb{R}_{\geq 0}$, a hyperplane

$$H_v^\epsilon := \{f \in \mathbb{R}^E | \sum_{e \in_+ A_G(v)} f(e) = 1 - \epsilon\}$$

in $\mathbb{R}^E$ and a function $f_v^\epsilon : E \to \mathbb{R}$ defined by

$$f_v^\epsilon(e) = \begin{cases} \frac{1}{3} & \text{if } e \in_+ A(v), \text{ and} \\ \frac{1}{3} - \epsilon & \text{if } e \notin_+ A(v). \end{cases}$$

For a sufficiently small $\epsilon \geq 0$, the element f_v^ϵ of $\mathbb{R}^E$ lies in $\mathcal{P}_G$. Denote by F_G the set of facets of $\mathcal{P}_G$ (cf. [4], Ch. I, 4.3 Definition, (c)) in which the origin of $\mathbb{R}^E$ does not lie. It follows from the definition of $\mathcal{P}_G$ that any facet h belonging to F_G is a subset of the hyperplane $H_{v_h}^0$ for some vertex $v_h \in V$. Since the various hyperplanes H_v^0 ($v \in V$) are not parallel to each other, the assignment $h \mapsto v_h$ is well-defined and determines an injection

$$\alpha_G : F_G \to V.$$

We claim that α_G is, moreover, surjective (i.e., α_G is bijective). Indeed, for each $v_0 \in V$, the point $f_{v_0}^\epsilon$ in $\mathcal{P}_G$ satisfies that $f_{v_0}^\epsilon \in H_{v_0}^0$, and $f_{v_0}^\epsilon \notin H_v^0$ for $v \neq v_0$. Hence, $H_{v_0}^0 \cap \mathcal{P}_G$ is a nonempty facet of $\mathcal{P}_G$, and moreover, an element of F_G that is sent, by construction, to v_0 via α_G. This implies that α_G is surjective (i.e., α_G is bijective).

Next, for v, $u \in V$, we consider a set $N_{v,u}$ defined by

$$N_{v,u} := \{n \in \mathbb{R}_{\geq 0} | f_v^\epsilon \in H_u^{(3-n)\cdot\epsilon}\}.$$

Since G is 3-regular, $N_{v,u}$ includes exactly one nonnegative integer $n_{v,u}$, which coincides with the number of edges e with $I(e) = [v, u]$. The pair (E, I) may be determined completely by the map

$$\beta_G : V \times V \to \mathbb{Z}_{\geq 0}$$

$$(v, u) \mapsto N_{v,u}.$$

(Note that if e_1 and e_2 are distinct edges such that $I(e_1) = I(e_2) = \{v, u\}$ for some v, $u \in V$, then we may not distinguish, by considering the map β_G, e_1 with e_2. But this ambiguity will not confuse the isomorphism class of $\mathcal{P}_G$.) In particular, if f is an element of $\mathrm{Isom}(\mathcal{P}_G, \mathcal{P}_{G'})$, then by applying the affine geometry of $\mathcal{P}_G (\subseteq \mathbb{R}^E)$, $\mathcal{P}_{G'} (\subseteq \mathbb{R}^{E'})$ and the maps α_G, β_G, $\alpha_{G'}$, $\beta_{G'}$ that there exists an isomorphism $G \xrightarrow{\sim} G'$ which is mapped to f via $\Xi_{G,G'}$, i.e., $\Xi_{G,G'}$ is surjective. This completes the proof of assertion (ii). □

Also, we conclude the following result.

Corollary 5.4.

Let $G = (V, E, I)$, $G' = (V', E', I')$ be connoted 3-regular graphs, and suppose that the Liu-Osserman polytope $\mathcal{P}_G$ of G is isomorphic to the Liu-Osserman polytope $\mathcal{P}_{G'}$ of G'. Then, G is isomorphic to G'.

Proof. Recall (cf. Remark 2.5) that the genus of a graph is determined by the number of its vertices. By considering the sets F_G, $F_{G'}$ and the bijections $\alpha_G : F_G \xrightarrow{\sim} V$, $\alpha_{G'} : F_{G'} \xrightarrow{\sim} V'$ discussed in the proof of Proposition 5.3, the assumption $\mathcal{P}_G \cong \mathcal{P}_{G'}$ implies that $\sharp V = \sharp V'$. Hence, it satisfies either (i) both of G and G' are of genus > 2, or (ii) both of G and G' are of genus 2. In the case of (i), the assertion follows from Proposition 5.3 (ii). Now, let us consider the case (ii). Let $G_{(2,0)}$ be as in Proposition 5.3, and $G'_{(2,0)}$ a connected 3-regular graph of genus 2 not isomorphic to $G_{(2,0)}$, which is uniquely determined up to isomorphism. Since $\mathcal{P}_{G_{(2,0)}}$ is not isomorphic to $\mathcal{P}_{G'_{(2,0)}}$, the assumption $\mathcal{P}_G \cong \mathcal{P}_{G'}$ implies that either $G \cong G' \cong G_{(2,0)}$ or $G \cong G' \cong G_{(2,0)}$. This completes the proof of Corollary 5.4. □

6. Independence of Ehrhart quasi-polynomials (the proof of Theorem A (ii))

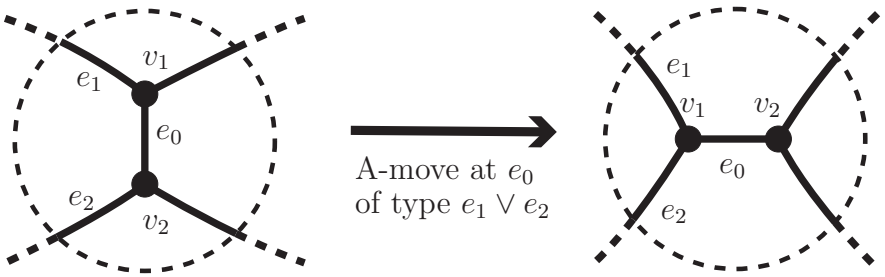
In this section, we make and prove a precise statement of Theorem A (ii), which yields an affirmative answer to a conjecture proposed in [14] (cf. [14], Conjecture 4.2).

Definition 6.1.

Let $G = (V, E, I)$ be a 3-regular quasi-graph, v_1, v_2 distinct vertices of G, and e_0, e_1, e_2 edges of G satisfying that $e_0 \neq e_1$, $e_0 \neq e_2$, and $[e_0, e_i] \subseteq A_G(v_i)$ $(i = 1, 2)$. The *A-move of G at e_0 of type $e_1 \vee e_2$* is the 3-regular quasi-graph $G' = (V', E', I')$ determined uniquely by the following conditions:

(1) $V' = V$, $E' = E$,
(2) $I'|_{E'\setminus\{e_1,e_2\}} = I|_{E\setminus\{e_1,e_2\}}$, and
(3) $A_{G'}(v_1) = [e_0, e_1, e_2]$, $A_G(v_1) \uplus A_G(v_2) = A_{G'}(v_1) \uplus A_{G'}(v_2)$.

For simplicity, we often refer to G' as the *A-move of G at e_0*.



Proposition 6.2.

Let (g, r) be a pair of nonnegative integers with $2g - 2 + r > 0$. Then any two

connected 3-regular quasi-graphs of type (g, r) can be obtained from one another by a finite sequence of A-moves.

Proof. Let Σ be an oriented topological surface that has genus g and r boundary components. Recall that each pants decomposition of Σ associates naturally a 3-regular graph of type (g, r) (cf. [8], APPENDIX). This association turns any elementary move between pants decompositions, in the sense of [17], § 2.2, into either an operation of taking an A-move of a quasi-graph or the identity map. Recall (cf. [7], § 2, Theorem 2) that any two pants decompositions of Σ may be obtained from one another by a finite sequence of elementary moves. On the other hand, for a 3-regular quasi-graph G of type (g, r), there exists at least one pants decomposition Σ_G of Σ that induces G by this association. Hence, by passing to this association, the proof of Proposition 6.2 is completed. □

Proposition 6.3.
Let $G = (V, E, I)$, e_0, e_1, e_2, v_1, v_2, and $G' = (V', E', I')$ be as in Definition 6.1. Then, for each $n \in \mathbb{Z}_{\geq 0}$, there exists a natural bijection

$$n\mathcal{P}_G \cap \mathbb{Z}^E \xrightarrow{\sim} n\mathcal{P}_{G'} \cap \mathbb{Z}^{E'}.$$

In particular, we have an equality $i_{\mathcal{P}_G} = i_{\mathcal{P}_{G'}} : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ of functions.

Proof. Denote by e_3, e_4 the edges satisfying that $A_G(v_1) = [e_0, e_1, e_3]$, and $A_G(v_2) = [e_0, e_2, e_4]$.

We shall construct a bijection $n\mathcal{P}_G \cap \mathbb{Z}^E \xrightarrow{\sim} n\mathcal{P}_{G'} \cap \mathbb{Z}^{E'}$. Let w be an element of $\mathbb{Z}^E (= \mathbb{Z}^{E'})$. Write $a := w(e_1)$, $b := w(e_3)$, $c := w(e_2)$, $d := w(e_4)$ and $e := w(e_0)$. The function w satisfies the conditions both $(\mathrm{iv})_{G,v_1,n}$ and $(\mathrm{iv})_{G,v_2,n}$ (cf. the discussion following Definition 3.1) if and only if the following two conditions hold:

(i) $e \leq \min\{a + b,\ n - a - b,\ c + d,\ n - c - d\}$,
(ii) $e \geq \max\{a - b,\ b - a,\ c - d,\ d - c\}$.

On the other hand, w satisfies the condition $(\mathrm{iv})_{G',v_1,n}$ and $(\mathrm{iv})_{G',v_2,n}$ if and only if the following two conditions hold:

(iii) $e \leq \min\{a + c,\ n - a - c,\ b + d,\ n - b - d\}$,
(iv) $e \geq \max\{a - c,\ c - a,\ b - d,\ d - b\}$.

Here, we set

$$P := \frac{1}{2} \cdot (n - |n - a - b - c - d|),$$
$$Q := \frac{1}{2} \cdot |a + b - c - d|,$$
$$R := \frac{1}{2} \cdot |a - b + c - d|,$$
$$S := \frac{1}{2} \cdot |a - b - c + d|.$$

Since equalities

$$\min\{A, B\} = \frac{A + B - |A - B|}{2}, \quad \max\{A, B\} = \frac{A + B + |A - B|}{2}$$

(A, $B \in \mathbb{R}$) hold, the conditions (i), (ii), (iii) and (iv) are, respectively, equivalent to the following conditions:

$$\begin{aligned}
\text{(i)} &\Leftrightarrow e \leq \min\bigl\{\min\{a+b,\ n-c-d\},\ \min\{n-a-b,\ c+d\}\bigr\} \\
&\Leftrightarrow e \leq \min\bigl\{P + \frac{1}{2}\cdot(a+b-c-d),\ P - \frac{1}{2}\cdot(a+b-c-d)\bigr\} \\
&\Leftrightarrow e \leq P - Q, \\
\text{(ii)} &\Leftrightarrow e \geq \max\bigl\{\max\{a-b,\ c-d\},\ \max\{b-a,\ d-c\}\bigr\} \\
&\Leftrightarrow e \geq \max\bigl\{S + \frac{1}{2}\cdot(a-b+c-d),\ S - \frac{1}{2}\cdot(a-b+c-d)\bigr\} \\
&\Leftrightarrow e \geq S + R, \\
\text{(iii)} &\Leftrightarrow e \leq \min\bigl\{\min\{a+c,\ n-b-d\},\ \min\{n-a-c,\ b+d\}\bigr\} \\
&\Leftrightarrow e \leq \min\bigl\{P + \frac{1}{2}\cdot(a-b+c-d),\ P - \frac{1}{2}\cdot(a-b+c-d)\bigr\} \\
&\Leftrightarrow e \leq P - R, \\
\text{(iv)} &\Leftrightarrow e \geq \max\bigl\{\max\{a-c,\ b-d\},\ \max\{c-a,\ d-b\}\bigr\} \\
&\Leftrightarrow e \geq \max\bigl\{S + \frac{1}{2}\cdot(a+b-c-d),\ S - \frac{1}{2}\cdot(a+b-c-d)\bigr\} \\
&\Leftrightarrow e \geq S + Q.
\end{aligned}$$

Now, we define a function $w' : E'(= E) \to \mathbb{R}$ as follows:

$$w'(e') = \begin{cases} w(e') & \text{if } e \neq e_0, \text{ and} \\ w(e') + Q - R & \text{if } e' = e_0, \end{cases}$$

where $Q - R$ may be verified to be an integer. If w lies in $n\mathcal{P}_G \cap \mathbb{Z}^E$, then it follows from the above observation concerning the conditions (i)-(iv) that w' lies on $n\mathcal{P}_G \cap \mathbb{Z}^E$, and vice versa. Thus, the assignment $w \mapsto w'$ determines a bijection $n\mathcal{P}_G \cap \mathbb{Z}^E \xrightarrow{\sim} n\mathcal{P}_{G'} \cap \mathbb{Z}^E$. This completes the proof of Proposition 6.3. □

Now we shall formulate precisely the statement of Theorem A (ii), which follows by applying Proposition 6.2, Proposition 6.3 (and the fact concerning the leading coefficient of the Ehrhart quasi-polynomial (cf. § 1.2)).

Corollary 6.4.
Let (g, r) be a pair of nonnegative integers such that $2g - 2 + r > 0$, and $G = (V, E, I)$, $G' = (V', E', I')$ connoted 3-regular quasi-graphs of type (g, r). Then, we have an equality

$$i_{\mathcal{P}_G} = i_{\mathcal{P}_{G'}} : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$$

of functions. In particular,

$$\mathrm{Vol}(\mathcal{P}_G) = \mathrm{Vol}(\mathcal{P}_{G'}), \quad f^G(t) = f^{G'}(t)$$

(cf. the discussion preceding Example 3.2), and the respective minimum periods of $i_{\mathcal{P}_G}$, $i_{\mathcal{P}_{G'}}$ coincide.

7. Ehrhart-Macdonald reciprocity (and the proof of Theorem A (iii))

Next, we consider the proof of Theorem A (iii). The discussion of this section stems from the discussion in the proof of [14], Theorem 2,1, applied the Ehrhart-Macdonald reciprocity (cf. the proof of Lemma 7.2). Here, recall the Ehrhart-Macdonald reciprocity as follows.

Let $\mathcal{Q}$ be a rational convex polytope embedded in an n-dimensional $\mathbb{R}$-vector space $\mathbb{R}^n$. Denote by $\mathcal{Q}^\circ$ the relative interior of $\mathcal{Q}$ (cf. [4], 1.7 Definition) and $i^\circ_{\mathcal{Q}} : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ the function which, to any $m \in \mathbb{Z}_{\geq 0}$, assigns the number of lattice points inside $(m\mathcal{Q})^\circ$, i.e.,

$$i^\circ_{\mathcal{Q}}(m) = \sharp((m\mathcal{Q})^\circ \cap \mathbb{Z}^n).$$

If $f^{\mathcal{Q}}(t)$ denotes the Ehrhart quasi-polynomial of $\mathcal{Q}$, then the Ehrhart-Macdonald reciplocity (cf. [27]; [17]) asserts that

$$i^\circ_{\mathcal{Q}}(m) = (-1)^{\dim(\mathcal{Q})} \cdot f^{\mathcal{Q}}(-m)$$

for $m \in \mathbb{Z}_{\geq 0}$.

Now, let us fix a connected 3-regular quasi-graph $G = (V, E, I)$. We recall the following

Proposition 7.1.
The minimal period of the Ehrhart quasi-polynomial $f^G(t) = (f^G_i(t))_{i\in\mathbb{Z}}$ of $\mathcal{P}_G$ divides 4, i.e., $f^G_i(t) = f^G_{i+4}(t)$ for $i \in \mathbb{Z}$.

Proof. Recall (cf. [14], Lemma 3.4) that for any vertex v of the polytope $\mathcal{P}_G$, each of the coordinates of v is equal to 0, $\frac{1}{2}$ or $\frac{1}{4}$. Therefore, the assertion follows from a well-known fact of Ehrhart's theory explained in § 1.2 (or see [1]: [2]; [3]). □

Next, we determine a somewhat specific form of the polynomial $f^G_i(t)$ by applying the Ehrhart-Macdonald reciprocity and Proposition 7.1.

Lemma 7.2. *(cf. the proof of Lemma 7.2)*
For each $i \in \mathbb{Z}$, we have

$$f^G_i(t-4) = (-1)^{g-1} \cdot f^G_i(-t).$$

Proof. Fix an integer m with $m \geq 4$. A lattice point inside the interior $(m\mathcal{P}_G)^\circ$ of $m\mathcal{P}_G$ corresponds, by definition, to an integer-valued function $w : E \to \mathbb{Z}$ satisfying the following inequalities:

(i)° for each $e \in E$, $w(e) > 0$,
(ii)° for each $v \in V$, $\sum_{e\in_+ A(v)} w(e) < m$,

(iii)$^\circ$ for each $v \in V$ and $e \in_+ A(v)$, $w(e) < \sum_{e' \in_+ A(v)-[e]} w(e')$.

One verifies that an element $w : E \to \mathbb{Z}$ of $\mathbb{Z}^E$ lies in $(m\mathcal{P}_G)^\circ$ if and only if the function $w' := w - 1$ (i.e., $w'(e) = w(e) - 1$ for $e \in E$) lies in $(m-4)\mathcal{P}_G$. That is, the assignment $w \mapsto w'$ determines a bijection

$$(m\mathcal{P}_G)^\circ \cap \mathbb{Z}^E \xrightarrow{\sim} ((m-4)\mathcal{P}_G) \cap \mathbb{Z}^E.$$

Thus, we have

$$i^\circ_{\mathcal{P}_G}(m) = i_{\mathcal{P}_G}(m-4)\big(= f^G_{m-4}(m-4)\big),$$

and hence, by applying the Ehrhart-Macdonald reciplocity,

$$f^G_{m-4}(m-4) = (-1)^{3g-3+2r} \cdot f^G_{-m}(-m) \ \big(= (-1)^{g-1} \cdot f^G_{-m}(-m)\big)$$

(cf. Remark 2.5). On the other hand, it follows from Proposition 7.1 and [14], Theorem 2.4 that $f^G_{m-4}(t) = f^G_m(t)$ and $f^G_{-m}(t) = f^G_m(t)$ for each $m \in \mathbb{Z}$. Hence, we conclude that

$$f^G_i(m-4) = (-1)^{g-1} \cdot f^G_i(-m)$$

for various pairs of integers (i, m) satisfying that $m \geq 4$ and $i \equiv m \bmod 4$. Thus, the required equality of polynomials is satisfied. □

Lemma 7.3.

Let $f(t) \in \mathbb{C}[t]$ be a monic polynomial of degree $N > 0$ satisfying that

$$f(t-s) = (-1)^{N'} f(-t)$$

for $s \in \mathbb{C}$ and $N' \in \mathbb{Z}_{\geq 0}$. If N is even (resp., odd), then $f(t)$ may be expressed as the following form:

$$f(t) = \prod_{j=1}^{\frac{N}{2}} (t^2 + st + a^j),$$

$$\big(\textit{resp.,}\ f(t) = (t + \frac{s}{2}) \cdot \prod_{j=1}^{\frac{N}{2}-1} (t^2 + st + a^j).\big)$$

where a^j's are complex numbers satisfying that $\prod_{j=1}^{\frac{N}{2}} a^j = 1$ (resp., $\frac{s}{2} \cdot \prod_{j=1}^{\frac{N}{2}-1} a^j = 1$).

Proof. The equality $f(t-s) = (-1)^N f(-t)$ implies that if r is a root of $f(t)$, then $-r - s$ is also a root of $f(t)$. Thus, we may express $f(t)$ as

$$f(t) = \prod_k (t - r_k)^{m(r_k)} (t + r_k + s)^{m(r_k)^\vee}$$

(where $r_k \in \mathbb{C}$ and $m(r_k)$, $m(r_k)^\vee \in \mathbb{Z}_{>0}$ for $k = 1, 2, \cdots$)

$$\begin{aligned}
&= \prod_{r_k = -r_k - s} (t - r_k)^{m(r_k)} (t + r_k + s)^{m(r_k)^\vee} \\
&\quad \cdot \prod_{r_k \neq -r_k - s} (t - r_k)^{m(r_k)} (t + r_k + s)^{m(r_k)^\vee} \\
&= (t + \frac{s}{2})^m \cdot \prod_{r_k \neq -\frac{s}{2}} (t - r_k)^{m(r_k)} (t + r_k + s)^{m(r_k)^\vee} \qquad (*)
\end{aligned}$$

for some $m \in \mathbb{Z}_{\geq 0}$.

Here, for a polynomial $h \in \mathbb{C}[t]$, we shall denote by $h^{(j)}$ the j-th derivative of h with respect to t. By taking the j-th derivatives ($j = 0, 1, 2, \cdots$) on both sides of the equality $f(t - s) = (-1)^N f(-t)$, we obtain an equality

$$f^{(j)}(t - s) = (-1)^{N+j} f^{(j)}(-t).$$

Similar to the case $j = 0$, if r is a root of $f^{(j)}(t)$, then $-r - s$ is also a root of $f^{(j)}(t)$. By taking account of this observation and the expression $(*)$ of $f(t)$, it follows from a routine argument that $m(r_k) = m(r_k)^\vee$ for all r_k with $r_k \neq -\frac{s}{2}$. Thus, $f(t)$ may be expressed as follows:

$$\begin{aligned}
f(t) &= (t + \frac{s}{2})^m \cdot \prod_{r_k \neq -\frac{s}{2}} ((t - r_k)(t + r_k + s))^{m(r_k)} \\
&= (t + \frac{s}{2})^\delta \cdot \Big((t^2 + st + \frac{s^2}{4})^{\lfloor \frac{m}{2} \rfloor} \cdot \prod_{r_k \neq -\frac{s}{2}} (t^2 + st - r_k^2 + s r_k)^{m(r_k)} \Big),
\end{aligned}$$

where $\delta = 0$ if m is even (equivalently, N is even), and $\delta = 1$ if m is odd (equivalently, N is odd). This completes the proof of Lemma 7.3.

□

By combining Lemma 7.2 and Lemma 7.3 and the fact concerning the leading coefficient of $f^G(t)$ (cf. § 1.2), we may conclude Theorem A (iii) as follows:

Corollary 7.4. *(= Theorem A (iii))*
If $\sharp E$ is even (resp., odd), then for $i \in \mathbb{Z}$ the polynomial $f_i^G(t) \in \mathbb{Q}[t]$ may be expressed as the following form.

$$f_i^G(t) = \mathrm{Vol}(\mathcal{P}_G) \cdot \prod_{j=1}^{\frac{\sharp E}{2}} (t^2 + 4t + a_i^j),$$

$$\Big(\textit{resp.,}\ f_i^G(t) = \mathrm{Vol}(\mathcal{P}_G) \cdot (t + 2) \cdot \prod_{j=1}^{\frac{\sharp E - 1}{2}} (t^2 + 4t + a_i^j), \Big)$$

where a_i^j's are complex numbers satisfying that

$$\prod_{j=1}^{\frac{\sharp E}{2}} a_i^j = \mathrm{Vol}(\mathcal{P}_G)^{-1} \quad (\textit{resp.},\ 2 \cdot \prod_{j=1}^{\frac{\sharp E-1}{2}} a_i^j = \mathrm{Vol}(\mathcal{P}_G)^{-1}).$$

8. Dormant indigenous bundles (and the proof of Theorem A (iv))

In this section, we recall a relationship between the lattice points inside $n\mathcal{P}_G$ ($n \in \mathbb{Z}_{\geq 0}$) and the set of isomorphism classes of certain $\mathbb{P}^1$-bundles on a given proper hyperbolic curve in positive characteristic, which are called dormant torally indigenous bundles. To prove Theorem A (iv), we apply this relationship and reduces to counting such bundles, which was discussed in [29].

Fix a pair of nonnegative integers (g, r) with $2g - 2 + r > 0$. Let k be a field in which 2 is invertible, and denote by $\overline{\mathcal{M}}_{g,r,k}$ the moduli stack of pointed stable curves over k of genus g with r marked points (i.e., of type (g, r)), and $\zeta : \mathcal{C} \to \overline{\mathcal{M}}_{g,r,k}$ the tautological curve, with its r marked points $s_1, \cdots, s_r : \overline{\mathcal{M}}_{g,r} \to \mathcal{C}$. Recall that $\overline{\mathcal{M}}_{g,r,R}$ has a natural log structure given by the divisor at infinity, where we shall denote the resulting log stack by $\overline{\mathcal{M}}_{g,r,k}^{\log}$. Also, by taking the divisor which is union of the s_i and the pull-back of the divisor at infinity of $\overline{\mathcal{M}}_{g,r,k}$, we obtain a log structure on $\mathcal{C}$; we denote the resulting log stack by $\mathcal{C}^{\log}$. Also, $\zeta : \mathcal{C} \to \overline{\mathcal{M}}_{g,r,k}$ extends naturally to a morphism of log stack $\zeta^{\log} : \mathcal{C}^{\log} \to \overline{\mathcal{M}}_{g,r,k}^{\log}$.

Now, let S be a scheme over k and $(X/S, \{\sigma_i : S \to X\}_i)$ a pointed stable curve over S of type (g, r). It detemines its classifying morphism $S \to \overline{\mathcal{M}}_{g,r,k}$, that induces an isomorphism $X \xrightarrow{\sim} S \times_{\overline{\mathcal{M}}_{g,r}} \mathcal{C}$ over S. By pulling-back the log structures of $\overline{\mathcal{M}}_{g,r,k}^{\log}$ and $\mathcal{C}^{\log}$, we obtain log structures on S and X; we denote the resulting log stacks by $S^{\log}$, $X^{\log}$ respectively. Also, the structure morphism of X/S extends to a morphism $X^{\log} \to S^{\log}$ of log schemes, which is log smooth (cf. [11], § 3). In this way, we consider the pointed stable curve $(X/S, \{\sigma_i\}_i)$ as an object of log geometry.

First, we shall recall the definition of an indigenous bundle. Write PGL_2 for the projective linear group of a 1-dimensional projective space and B for a Borel subgroup of PGL_2. Also, denote by $\mathfrak{sl}_2$, $\mathfrak{b}$ the Lie algebras corresponding to PGL_2, B respectively, and $\iota : \mathfrak{b} \to \mathfrak{sl}_2$ the natural injection of Lie algebras induced by the inclusion $B \hookrightarrow \mathrm{PGL}_2$.

Suppose that $\pi : \mathcal{E} \to X$ is a PGL_2-torsor over X. By pulling-back the log structure of $X^{\log}$, we obtain a log structure on $\mathcal{E}$; we denote the resulting log stack by $\mathcal{E}^{\log}$. Write $\mathcal{T}_{X/S}$, $\mathcal{T}_{\mathcal{E}/S}$ (resp., $\mathcal{T}_{X/S}^{\log}$, $\mathcal{T}_{\mathcal{E}/S}^{\log}$) for the sheaves of derivations of X, $\mathcal{E}$ over S (resp., the sheaves of log derivations of $X^{\log}$, $\mathcal{E}^{\log}$ over $S^{\log}$)

respectively. We have natural morphisms

$$\iota_X : \mathcal{T}^{\log}_{X/S} \to \mathcal{T}_{X/S}, \quad \iota_{\mathcal{E}} : \mathcal{T}^{\log}_{\mathcal{E}/S} \to \mathcal{T}_{\mathcal{E}/S}.$$

Since $\mathcal{T}^{\log}_{X/S}$, $\mathcal{T}^{\log}_{\mathcal{E}/S}$ are locally free, and ι_X, $\iota_{\mathcal{E}}$ are isomorphisms over a scheme-theoretically dense open subscheme of X (i.e., the smooth locus of X over S), these morphisms are injective. The direct image $\pi_*\mathcal{T}^{\log}_{\mathcal{E}/S} \to \pi_*\mathcal{T}_{\mathcal{E}/S}$ of $\iota_{\mathcal{E}}$ is compatible with the respective natural PGL_2-actions. Hence, if we denote by $\widetilde{\mathcal{T}}_{\mathcal{E}/S}$, $\widetilde{\mathcal{T}}^{\log}_{\mathcal{E}/S}$ the subsheaves of G-invariant sections of $\pi_*\mathcal{T}_{\mathcal{E}/S}$, $\pi_*\mathcal{T}^{\log}_{\mathcal{E}/S}$ respectively (i.e., $\widetilde{\mathcal{T}}_{\mathcal{E}/S} := (\pi_*\mathcal{T}_{\mathcal{E}/S})^G$, $\widetilde{\mathcal{T}}^{\log}_{\mathcal{E}/S} := (\pi_*\mathcal{T}^{\log}_{\mathcal{E}/S})^G$), then $\iota_{\mathcal{E}}$ yields an injection

$$\widetilde{\iota}_{\mathcal{E}} : \widetilde{\mathcal{T}}^{\log}_{\mathcal{E}/S} \hookrightarrow \widetilde{\mathcal{T}}_{\mathcal{E}/S}.$$

Moreover, the PGL_2-torsor $\mathcal{E}$ induces naturally a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E} \times^{\mathrm{PGL}_2} \mathfrak{sl}_2 & \longrightarrow & \widetilde{\mathcal{T}}^{\log}_{\mathcal{E}/S} & \xrightarrow{\alpha^{\log}_{\mathcal{E}}} & \mathcal{T}^{\log}_{X/S} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \widetilde{\iota}_{\mathcal{E}}} & & \downarrow{\scriptstyle \iota_X} & & \\
0 & \longrightarrow & \mathcal{E} \times^{\mathrm{PGL}_2} \mathfrak{sl}_2 & \longrightarrow & \widetilde{\mathcal{T}}_{\mathcal{E}/S} & \xrightarrow{\alpha_{\mathcal{E}}} & \mathcal{T}_{X/S} & \longrightarrow & 0,
\end{array}$$

where $\mathcal{E} \times^{\mathrm{PGL}_2} \mathfrak{sl}_2 (\cong (\pi_*\mathcal{T}_{\mathcal{E}/X})^{\mathrm{PGL}_2})$ denotes the adjoint bundle associated to $\mathcal{E}$, and the upper and lower horizontal sequences are exact.

Recall that an *S-log connection* on $\mathcal{E}$ is a split injection

$$\nabla : \mathcal{T}^{\log}_{X/S} \to \widetilde{\mathcal{T}}^{\log}_{\mathcal{E}/S}$$

of the upper horizontal sequence in the above diagram (i.e., $\alpha^{\log}_{\mathcal{E}} \circ \nabla = \mathrm{id}_{\mathcal{T}^{\log}_{X/S}}$). Since $\mathcal{T}^{\log}_{X/S}$ is locally free of rank one, any S-log connection is necessarily integrable, i.e., compatible with the Lie bracket structures on $\mathcal{T}^{\log}_{X/S}$ and $\widetilde{\mathcal{T}}^{\log}_{\mathcal{E}/S} = (\pi_*\mathcal{T}^{\log}_{\mathcal{E}/S})^{\mathrm{PGL}_2}$.

Suppose that there exists a B-reduction $\mathcal{E}_B$ of a PGL_2-torsor $\mathcal{E}$, i.e., a B-torsor $\pi_B : \mathcal{E}_B \to X$ over X together with an isomorphism $\mathcal{E}_B \times^B \mathrm{PGL}_2 \xrightarrow{\sim} \mathcal{E}$. Then the natural morphism $\mathcal{E}_B \to \mathcal{E}$ induces an injection

$$i : ((\pi_{B*}\mathcal{T}^{\log}_{\mathcal{E}_B/S})^B =:)\widetilde{\mathcal{T}}^{\log}_{\mathcal{E}_B/S} \hookrightarrow \widetilde{\mathcal{T}}^{\log}_{\mathcal{E}/S}$$

of $\mathcal{O}_X$-modules. Hence, for an S-log connection ∇ on $\mathcal{E}$, we have the composite

$$\mathrm{KS}_{\mathcal{E},\mathcal{E}_B,\nabla} : \mathcal{T}^{\log}_{X/S} \xrightarrow{\nabla} \widetilde{\mathcal{T}}^{\log}_{\mathcal{E}/S} \longrightarrow \widetilde{\mathcal{T}}^{\log}_{\mathcal{E}/S}/i(\widetilde{\mathcal{T}}^{\log}_{\mathcal{E}_B/S}).$$

We shall refer to the morphism $\mathrm{KS}_{\mathcal{E},\mathcal{E}_B,\nabla}$ as the *Kodaira-Spencer map* associated to the triple $(\mathcal{E}, \mathcal{E}_B, \nabla)$.

Definition 8.1.

(i) An *indigenous bundle* on $(X/S, \{\sigma_i\}_i)$ is a triple

$$\mathcal{E}^{\circledast} = (\pi : \mathcal{E} \to X, \pi_B : \mathcal{E}_B \to X, \nabla : \mathcal{T}^{\log}_{X/S} \to \widetilde{\mathcal{T}}^{\log}_{\mathcal{E}/S})$$

consisting of a PGL_2-torsor $\pi : \mathcal{E} \to X$, a B-reduction $\pi_B : \mathcal{E}_B \to X$, and an S-log connection $\nabla : \mathcal{T}^{\log}_{X/S} \to \widetilde{\mathcal{T}}^{\log}_{\mathcal{E}/S}$ on $\mathcal{E}$ such that the Kodaira-Spencer map $\mathrm{KS}_{\mathcal{E},\mathcal{E}_B,\nabla}$ associated to the triple $(\mathcal{E}, \mathcal{E}_B, \nabla)$ is an isomorphism.

(ii) Let $\mathcal{E}^{\circledast}_1 = (\mathcal{E}_1, (\mathcal{E}_1)_B, \nabla_1)$, $\mathcal{E}^{\circledast}_2 = (\mathcal{E}_2, (\mathcal{E}_2)_B, \nabla_2)$ be indigenous bundles on $(X/S, \{\sigma_i\}_i)$. An *isomorphism* from $\mathcal{E}^{\circledast}_1$ to $\mathcal{E}^{\circledast}_2$ is an isomorphism $(\mathcal{E}_1)_B \xrightarrow{\sim} (\mathcal{E}_2)_B$ of B-torsors such that the induced isomorphism $\mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_2$ of PGL_2-torsors is compatible with the respective connections ∇_1, ∇_2.

Next, we recall the definition of a torally indigenous bundle. Let $\mathcal{E}^{\circledast} = (\mathcal{E}, \mathcal{E}_B, \nabla)$ be an indigenous bundle on $(X/S, \{\sigma_i\}_i)$. Consider, for $i \in \{1, \cdots, r\}$, the morphism of sequences of $\mathcal{O}_X$-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E} \times^{\mathrm{PGL}_2} \mathfrak{sl}_2 & \longrightarrow & \widetilde{\mathcal{T}}^{\log}_{\mathcal{E}/S} & \xrightarrow{\alpha^{\log}_{\mathcal{E}}} & \mathcal{T}^{\log}_{X/S} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sigma_{i*}\sigma_i^*(\mathcal{E} \times^{\mathrm{PGL}_2} \mathfrak{sl}_2) & \longrightarrow & \sigma_{i*}\sigma_i^*\widetilde{\mathcal{T}}_{\mathcal{E}/S} & \xrightarrow{\sigma_{i*}\sigma_i^*\alpha_{\mathcal{E}}} & \sigma_{i*}\sigma_i^*\mathcal{T}_{X/S} & \longrightarrow & 0,
\end{array}$$

obtained by composing the diagram discussed above with the adjunction morphism $\eta_{(-)} : (-) \to \sigma_{i*}\sigma_i^*(-)$. (The local triviality of the PGL_2-torsor $\mathcal{E}$ implies that the lower horizontal sequence is also exact.) In particular, the vertical arrows in the diagram are the composites

$$\eta_{\mathcal{E}\times^{\mathrm{PGL}_2}\mathfrak{sl}_2}, \quad \eta_{\widetilde{\mathcal{T}}_{\mathcal{E}/S}} \circ \widetilde{\iota}_{\mathcal{E}}, \quad \eta_{\mathcal{T}_{X/S}} \circ \iota_X$$

respectively. Since $\sigma_{i*}\sigma_i^*\mathcal{T}_{X/S} \cong \sigma_{i*}\sigma_i^*(\mathcal{T}_{X/S}/\iota_X(\mathcal{T}^{\log}_{X/S}))$, the composite $\eta_{\mathcal{T}_{X/S}} \circ \iota_X : \mathcal{T}^{\log}_{X/S} \to \sigma_{i*}\sigma_i^*\mathcal{T}_{X/S}$ (i.e., the right vertical arrow in the diagram) is the zero map. Thus, it follows that the composite

$$(\eta_{\widetilde{\mathcal{T}}_{\mathcal{E}/S}} \circ \widetilde{\iota}_{\mathcal{E}}) \circ \nabla : \mathcal{T}^{\log}_{X/S} \to \sigma_{i*}\sigma_i^*\widetilde{\mathcal{T}}_{\mathcal{E}/S}$$

factors through the injection $\sigma_{i*}\sigma_i^*(\mathcal{E} \times^{\mathrm{PGL}_2} \mathfrak{sl}_2) \hookrightarrow \sigma_{i*}\sigma_i^*\widetilde{\mathcal{T}}_{\mathcal{E}/S}$. The resulting morphism $\mathcal{T}^{\log}_{X/S} \to \sigma_{i*}\sigma_i^*(\mathcal{E} \times^{\mathrm{PGL}_2} \mathfrak{sl}_2)$ corresponds, via the adjunction relation "$\sigma_i^*(-) \dashv \sigma_{i*}(-)$", to a morphism

$$\sigma_i^*\mathcal{T}^{\log}_{X/S} \to \sigma_i^*(\mathcal{E} \times^{\mathrm{PGL}_2} \mathfrak{sl}_2).$$

Here, we observe that there exists a canonical isomorphism $\sigma_i^*\mathcal{T}^{\log}_{X/S} \xrightarrow{\sim} \mathcal{O}_S$ which maps any local section of the form $d\log(x)|_S \in \sigma_i^*\mathcal{T}^{\log}_{X/S}$ (for a local function x defining σ_i) to $1 \in \mathcal{O}_S$. Thus, we obtain a global section

$$\mu_i \in \Gamma(S, \sigma_i^*(\mathcal{E} \times^{\mathrm{PGL}_2} \mathfrak{sl}_2))$$

determined by the image of $1 \in \Gamma(S, \mathcal{O}_S)$ via the morphism $\mathcal{O}_S (\cong \sigma_i^*\mathcal{T}^{\log}_{X/S}) \to \sigma_i^*(\mathcal{E} \times^{\mathrm{PGL}_2} \mathfrak{sl}_2)$ just discussed. We shall refer to μ_i as the *monodromy operator* of $\mathcal{E}^{\circledast}$ at σ_i.

Denote by

$$\kappa_i^{\mathcal{E}^{\circledast}} : \sigma_i^*(\mathcal{E} \times^{\mathrm{PGL}_2} \mathfrak{sl}_2) \otimes_{\mathcal{O}_S} \sigma_i^*(\mathcal{E} \times^{\mathrm{PGL}_2} \mathfrak{sl}_2) \to \mathcal{O}_S$$

the nondegenerate bilinear form on $\sigma_i^*(\mathcal{E} \times^{\mathrm{PGL}_2} \mathfrak{sl}_2)$ induced by the Killing form $\kappa(-,-)$ on $\mathfrak{sl}_2$ (i.e., $\kappa(a,b) = \frac{1}{4} \cdot \mathrm{tr}(\mathrm{ad}(a)\mathrm{ad}(b))$ for a, $b \in \mathfrak{sl}_2$). Let ρ_i be an element of $\Gamma(S, \mathcal{O}_S)/\{\pm 1\}$ (i.e., equivalence classes of elements $\lambda \in \Gamma(S, \mathcal{O}_S)$, in which λ and $-\lambda$ are identified). We shall say that $\mathcal{E}^{\circledast}$ is *of radius* ρ_i at σ_i if the monodromy operator μ_i at σ_i satisfies the condition

$$\kappa_i^{\mathcal{E}^{\circledast}}(\mu_i, \mu_i) = 2\rho_i^2.$$

If $S = \mathrm{Spec}(k)$ for an algebraically closed field k in which 2 is invertible, then there exists a unique element $\rho_i \in \Gamma(S, \mathcal{O}_S)/\{\pm 1\}$ such that $\mathcal{E}^{\circledast}$ is of radius ρ_i at σ_i. Thus, it makes sense to speak of *the radius* of $\mathcal{E}^{\circledast}$ at σ_i. Now suppose that S is an *arbitrary* scheme on which 2 is invertible, and that $\mathcal{E}^{\circledast}$ is of radius ρ_i at σ_i for $\rho_i \in \Gamma(S, \mathcal{O}_S^{\times})/\{\pm 1\}$ $\big(\subseteq \Gamma(S, \mathcal{O}_S)/\{\pm 1\}\big)$. Then one verifies easily that ρ_i may be characterized as the *unique* element $\rho \in \Gamma(S, \mathcal{O}_S)/\{\pm 1\}$ such that $\kappa_i^{\mathcal{E}^{\circledast}}(\mu_i, \mu_i) = 2\rho^2$.

Definition 8.2 (cf. [19], Chap. I, § 4, Definition 4.1)**.**
We shall say that an indigenous bundle $\mathcal{E}^{\circledast} = (\mathcal{E}, \mathcal{E}_B, \nabla)$ on $(X/S, \{\sigma_i\}_i)$ is *torally indigenous* if there exists a set $\{\rho_i\}_{i=1}^r$ of elements in $\{0\} \cup \Gamma(S, \mathcal{O}_S^{\times})/\{\pm 1\}$ such that $\mathcal{E}^{\circledast}$ is of radius ρ_i at σ_i for all i.

Next, we recall the definition of a dormant indigenous bundle. In the following, let us assume that $k = \mathbb{Z}/p\mathbb{Z}(=: \mathbb{F}_p)$ for p an odd prime. Denote by $F_X : X \to X$ the absolute Frobenius of X. If ∂ is a log derivation corresponding to a local section of $\mathcal{T}_{X/S}^{\log}$ (respectively, $\widetilde{\mathcal{T}}_{\mathcal{E}/S}^{\log} = (\pi_* \mathcal{T}_{\mathcal{E}/S}^{\log})^{\mathrm{PGL}_2}$), then we shall denote by $\partial^{(p)}$ the p^{thm} symbolic power of ∂ (cf. [20], (1.2.1)), which is also a derivation corresponding to a local section of $\mathcal{T}_{X/S}^{\log}$ (respectively, $\widetilde{\mathcal{T}}_{\mathcal{E}/S}^{\log} = (\pi_* \mathcal{T}_{\mathcal{E}/S}^{\log})^{\mathrm{PGL}_2}$). Since $\alpha_{\mathcal{E}}^{\log}(\partial^{(p)}) = \alpha_{\mathcal{E}}^{\log}(\partial)^{(p)}$ for any local section of $\mathcal{T}_{X/S}^{\log}$, the image of the p-linear map from $\mathcal{T}_{X/S}^{\log}$ to $\widetilde{\mathcal{T}}_{\mathcal{E}/S}^{\log}$ defined by assigning $\partial \mapsto \nabla(\partial^{(p)}) - \nabla(\partial)^{(p)}$ is contained in $\mathcal{E} \times^{\mathrm{PGL}_2} \mathfrak{sl}_2 (= \ker(\alpha_{\mathcal{E}}^{\log}))$. Thus, we obtain an $\mathcal{O}_X$-linear morphism

$$\psi_{\mathcal{E}^{\circledast}} : \mathcal{T}_{X/S}^{\otimes p} \to \mathcal{E} \times^{\mathrm{PGL}_2} \mathfrak{sl}_2$$

determined by assigning

$$\partial^{\otimes p} \mapsto \nabla(\partial^{(p)}) - \nabla(\partial)^{(p)}.$$

We shall refer to the morphism $\psi_{\mathcal{E}^{\circledast}}$ as the *p-curvature map* of $\mathcal{E}^{\circledast}$.

Definition 8.3.
We shall say that an indigenous bundle $\mathcal{E}^{\circledast}$ on X/S is *dormant* if the p-curvature map $\psi_{\mathcal{E}^{\circledast}}$ vanishes identically on X.

We define an isomorphism of dormant torally indigenous bundles to be an isomorphism of indigenous bundles. Thus, it makes sense to speak of the isomorphism class of a dormant torally indigenous bundle. Write (Set) for the

category of (small) sets, and $(Sch)_{\overline{\mathcal{M}}_{g,r,\mathbb{F}_p}}$ for the category of (relative) schemes over $\overline{\mathcal{M}}_{g,r,\mathbb{F}_p}$. We set

$$\overline{\mathcal{M}}^{\mathrm{Zzz...}}_{g,r,\mathbb{F}_p} : (Sch)_{\overline{\mathcal{M}}_{g,r,\mathbb{F}_p}} \to (Set)$$

to be the (Set)-valued functor on $(Sch)_{\overline{\mathcal{M}}_{g,r,\mathbb{F}_p}}$ which, to any $\overline{\mathcal{M}}_{g,r,\mathbb{F}_p}$-scheme T, classifying a pointed stable curve Y/T, assigns the set of isomorphic classes of dormant torally indigenous bundles on Y/T that are of radius $\in \mathbb{F}_p/\{\pm 1\}(\subseteq \Gamma(T, \mathcal{O}_T)/\{\pm 1\})$ for all i. By forgetting the datum of a dormant torally indigenous bundle, we obtain a natural transformation

$$\overline{\mathcal{M}}^{\mathrm{Zzz...}}_{g,r,\mathbb{F}_p} \to \overline{\mathcal{M}}_{g,r,\mathbb{F}_p}$$

(cf. the figure following Acknowledgement !).

Theorem 8.4.

(i) *The functor $\overline{\mathcal{M}}^{\mathrm{Zzz...}}_{g,r,\mathbb{F}_p}$ is represented by a proper, smooth Deligne-Mumford stack over $\mathbb{F}_p$ of dimension $3g-3+r$. The morphism $\overline{\mathcal{M}}^{\mathrm{Zzz...}}_{g,r,\mathbb{F}_p} \to \overline{\mathcal{M}}_{g,r,\mathbb{F}_p}$ is finite, faithfully flat and unramified over the points of $\overline{\mathcal{M}}_{g,r,\mathbb{F}_p}$ classifying totally degenerate curves (cf. Definition 8.7). In particular, $\overline{\mathcal{M}}^{\mathrm{Zzz...}}_{g,r,\mathbb{F}_p}$ is generically étale over $\overline{\mathcal{M}}_{g,r,\mathbb{F}_p}$.*

(ii) *If, moreover, $r = 0$ and $p > 2(g-1)$, then the degree $\deg_{\overline{\mathcal{M}}_{g,0,\mathbb{F}_p}}(\overline{\mathcal{M}}^{\mathrm{Zzz...}}_{g,0,\mathbb{F}_p})$ of $\overline{\mathcal{M}}^{\mathrm{Zzz...}}_{g,0,\mathbb{F}_p}$ over $\overline{\mathcal{M}}_{g,0,\mathbb{F}_p}$ may be calculated as follows:*

$$\deg_{\overline{\mathcal{M}}_{g,0,\mathbb{F}_p}}(\overline{\mathcal{M}}^{\mathrm{Zzz...}}_{g,0,\mathbb{F}_p}) = \frac{p^{g-1}}{2^{2g-1}} \cdot \sum_{\theta=1}^{p-1} \frac{1}{\sin^{2g-2}(\frac{\pi\cdot\theta}{p})}.$$

Proof. Assertion (i) follows from [19], Introduction, § 1.2, Theorem 1.3 (ii); [19], Chap. II, Lemma 2.7; [19], Chap. II, §2.3, Theorem 2.8 (and its proof). Assertion (ii) follows from [29], Theorem A. □

Remark 8.5.

In [29], § 6.2 (ii), we observed that the value $\deg_{\overline{\mathcal{M}}_{g,0,\mathbb{F}_p}}(\overline{\mathcal{M}}^{\mathrm{Zzz...}}_{g,0,\mathbb{F}_p})$ $(p > 2(g-1))$ has another expression:

$$\deg_{\overline{\mathcal{M}}_{g,0,\mathbb{F}_p}}(\overline{\mathcal{M}}^{\mathrm{Zzz...}}_{g,0,\mathbb{F}_p}) = -\frac{p^g}{2^{2g-1}} \cdot \mathrm{Res}_{x=0}\Big[\frac{\cot(px)}{\sin^{2g-2}(x)}dx\Big],$$

where $\mathrm{Res}_{x=0}(f)$ denotes the residue of f at $x = 0$. Thus, $\deg_{\overline{\mathcal{M}}_{g,0,\mathbb{F}_p}}(\overline{\mathcal{M}}^{\mathrm{Zzz...}}_{g,0,\mathbb{F}_p})$ may be computed by considering the relation $\frac{1}{\sin^2(x)} = 1 + \cot^2(x)$ and the coefficient of the Laurent expansion (cf. [33], the proof of Theorem 1 (iii))

$$\cot(x) = \frac{1}{x} + \sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j} B_{2j}}{(2j)!} x^{2j-1},$$

where B_{2j} denotes the $(2j)$-th Bernoulli number, i.e.,

$$\frac{w}{e^w - 1} = 1 - \frac{w}{2} + \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} w^{2j}.$$

More precisely, if we denote by $F_g(t)$ the polynomial defined as the constant term (with respect to x) of the power series

$$-\frac{t^{g-1}}{2^{2g-1}} \cdot \Big(1 + \sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j} B_{2j}}{(2j)!} (tx)^{2j}\Big)\Big(1 + \frac{1}{x^2} \cdot \Big(1 + \sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j} B_{2j}}{(2j)!} x^{2j}\Big)^2\Big)^{g-1}$$

$$\left(= -\frac{t^g}{2^{2g-1}} \cdot \mathrm{Res}_{x=0}\Big[\frac{\cot(tx)}{\sin^{2g-2}(x)} dx\Big]\right),$$

then $\deg_{\overline{\mathcal{M}}_{g,0,\mathbb{F}_p}}(\overline{\mathcal{M}}^{\mathrm{Zzz...}}_{g,0,\mathbb{F}_p}) = F_g(p)$ for any odd prime $p > 2(g-1)$. In particular, it follows from an explicit computation that

$$\text{“the leading term of } F_g(t)\text{”} = \frac{(-1)^g \cdot B_{2g-2}}{2 \cdot (2g-2)!} \cdot t^{3g-3},$$

where it is well-known that $B_{2j} \neq 0$ for $j \in \mathbb{Z}_{>0}$.

We shall fix an algebraically closed field K of characteristic p. If $\mathbb{X} = (X/K, \{\sigma_i\}_i)$ is a pointed stable curve over K of type (g, r), then we denote by

$$M_{\mathbb{X}}^{\mathrm{Zzz...}}$$

the set of isomorphism classes of dormant torally indigenous bundles on $\mathbb{X}$. (Since $\mathrm{Spec}(K)$ is reduced and connected, for a dormant indigenous bundle $\mathcal{E}^{\circledast}$ the radius of $\mathcal{E}^{\circledast}$ at each σ_i is necessarily contained in $\mathbb{F}_p/\{\pm 1\} (\subseteq K/\{\pm 1\})$ (cf. [19], Chap.İI, p.128. Proposition 1.5).) The generic étalness of $\overline{\mathcal{M}}^{\mathrm{Zzz...}}_{g,r,\mathbb{F}_p}$ over $\overline{\mathcal{M}}_{g,r,\mathbb{F}_p}$ (cf. Theorem 8.4 (i)) implies that if $\mathbb{X}$ and $\mathbb{Y}$ are sufficiently general pointed stable curves over K of type (g, r), including the case of totally degenerate curves (cf. Definition 8.7), then we have

$$\sharp M_{\mathbb{X}}^{\mathrm{Zzz...}} = \sharp M_{\mathbb{Y}}^{\mathrm{Zzz...}} = \deg_{\overline{\mathcal{M}}_{g,r,\mathbb{F}_p}}(\overline{\mathcal{M}}^{\mathrm{Zzz...}}_{g,r,\mathbb{F}_p}).$$

In particular, if, moreover, $r = 0$, $p > 2(g-1)$, and $\mathbb{X}$ is totally degenerate, then the value $\sharp M_{\mathbb{X}}^{\mathrm{Zzz...}}$ may be calculated explicitly by the formula asserted in Theorem 8.4 (ii).

Definition 8.6 (cf. [14], Definition 3.8)**.**
Let $\mathbb{X} = (X/K, \{\sigma_i\}_i)$ be a pointed stable curve over K of type (g, r). The *dual quasi-graph* associated to $\mathbb{X}$ is the quasi-graph

$$G_{\mathbb{X}} = (V_{\mathbb{X}}, E_{\mathbb{X}}, I_{\mathbb{X}}),$$

where

(1) The vertices $V_{\mathbb{X}}$ are the components of the underlying stable curve X.

(2) The fixed edges $E^{\mathrm{fix}}_{\mathbb{X}}$ are the nodes of X, and the free edges $E^{\mathrm{free}}_{\mathbb{X}}$ are the marked points $\{\sigma_i\}_i$.
(3) The incidence relation $I_{\mathbb{X}} : (E^{\mathrm{fix}}_{\mathbb{X}} \sqcup E^{\mathrm{free}}_{\mathbb{X}} =:) E_{\mathbb{X}} \to V^{[1]}_{\mathbb{X}} \sqcup V^{[2]}_{\mathbb{X}}$ is defined as follows:
- If $e \in E^{\mathrm{fix}}_{\mathbb{X}}$, then $I_{\mathbb{X}}(e) = [v_1, v_2]$, where v_1 and v_2 are the component of X intersecting at the nodes,
- If $e \in E^{\mathrm{free}}_{\mathbb{X}}$, then $I_{\mathbb{X}}(e) = [v]$, where v is the component of X on which e lies.

Definition 8.7.
A pointed stable curve $\mathbb{X} = (X/K, \{\sigma_i\}_i)$ over K is called *totally degenerate* if each component of the normalization of X is isomorphic to the projective line and the dual quasi-graph associated to $\mathbb{X}$ is 3-regular.

One verifies that the assignment $\mathbb{X} \mapsto G_{\mathbb{X}}$ defines a bijective correspondence between the set of isomorphism classes of unpointed totally degenerate curves over K of genus g (resp., totally degenerate curves over K of type (g, r)) and the set of 3-regular graphs of genus g (resp., 3-regular quasi-graphs of type (g, r)).

In the following, we shall relate the set $M^{\mathrm{Zzz\cdots}}_{\mathbb{X}}$ for an unpointed totally degenerate curve $\mathbb{X}$ of genus ≥ 2 to the lattice points inside the polytope $(p-2)\mathcal{P}_{G_{\mathbb{X}}}$. As a first step, we consider a pointed stable curve

$$^{\mathrm{tri}}\mathbb{P} = (\mathbb{P}, \{\sigma_1, \sigma_2, \sigma_3\})$$

of type $(0, 3)$, which is uniquely determined up to isomorphism. (In particular, the underling curve $\mathbb{P}$ is isomorphic to the projective line.) Let $\mathcal{E}^{\circledast}$ be a dormant torally indigenous bundle on $\mathbb{X}$. If we denote by $\rho_i \in K/\{\pm 1\}$ $(i = 1, 2, 3)$ the radius of the monodromy operator of $\mathcal{E}^{\circledast}$ at σ_i, then the assignment $\mathcal{E}^{\circledast} \mapsto (2\rho_1, 2\rho_2, 2\rho_3)$ determines a map

$$\mathrm{rad} : M^{\mathrm{Zzz\cdots}}_{^{\mathrm{tri}}\mathbb{P}} \to (K/\{\pm 1\})^{\times 3}.$$

Next, let $G_{(0,3)} = (V_{(0,3)}, E_{(0,3)}, I_{(0,3)})$ be as in Example 3.2, which is isomorphic to the dual quasi-graph associated to $^{\mathrm{tri}}\mathbb{P}$. The set of lattice points $n\mathcal{P}_{G_{(0,3)}} \cap \mathbb{Z}^{E_{(0,3)}}$ corresponds bijectively (cf. the discussion following Definition 3.1) to the set

$$\Big\{(\lambda_i)_{i=1}^3 \in \mathbb{Z}^{\times 3}_{\geq 0} \Big| \sum_{i=1}^{3} \lambda_i \leq n,\ \lambda_1 + \lambda_2 \leq \lambda_3,\ \lambda_2 + \lambda_3 \leq \lambda_1,\ \lambda_3 + \lambda_1 \leq \lambda_2\Big\}.$$

Consider a map

$$\mathrm{inc} : (p-2)\mathcal{P}_{G_{(0,3)}} \cap \mathbb{Z}^{E_0} \to (K/\{\pm 1\})^{\times 3}.$$

defined by assigning $(\lambda_1, \lambda_2, \lambda_3) \mapsto (\overline{2\lambda_1 + 1}, \overline{2\lambda_2 + 1}, \overline{2\lambda_3 + 1})$, where for $a \in \mathbb{Z}$ we denote by $\overline{a}$ the image of a natural map $\mathbb{Z} \to K/\{\pm 1\}$. Since $2\lambda_i \leq \sum_{i=1}^3 \lambda_i \leq p - 2$, the map inc is injective.

Proposition 8.8.
The map rad *factors through the map* inc*, i.e., there exists a map*

$$\mathrm{rad}' : M_{\mathbb{P}}^{\mathrm{Zzz}\cdots} \to (p-2)\mathcal{P}_{G_0} \cap \mathbb{Z}^{E_0}$$

satisfying that $\mathrm{rad} = \mathrm{inc} \circ \mathrm{rad}'$*. Moreover, the resulting map* rad' *is bijective.*

Proof. See [19], Introduction, § 1.2, p.41, Theorem 1.3; [19], Chap. IV, p.211, Theorem 2.3; [19], Chap. V, § 1, and [14], Theorem 3.9. □

Next, let $\mathbb{X} = (X/K, \emptyset)$ be an unpointed totally degenerate curve over K of genus $g \geq 2$. Denote by $G_{\mathbb{X}} = (V_{\mathbb{X}}, E_{\mathbb{X}}, I_{\mathbb{X}})$ the dual (quasi-)graph associated to $\mathbb{X}$, and by $\widetilde{X} := \coprod_{v \in V_{\mathbb{X}}} \mathbb{P}_v$ the normalization of X (and hence, $\mathbb{P}_v \cong \mathbb{P}$ for $v \in V_{\mathbb{X}}$). By ordering suitably the elements in the multiset $A_{G_{\mathbb{X}}}(v)$, the collection of data

$$^{\mathrm{tri}}\mathbb{P}_v = (\mathbb{P}_v, A_{G_{\mathbb{X}}}(v))$$

forms a pointed stable curve of type $(0, 3)$. One verifies that the natural morphism $\mathbb{P}_v^{\log} \to X^{\log}$ is log étale. Hence, if $\mathcal{E}^{\circledast} = (\mathcal{E}, \mathcal{E}_B, \nabla)$ is a dormant torally indigenous bundle on $\mathbb{X}$, then for any $v \in V_{\mathbb{X}}$ the restriction $\mathcal{E}^{\circledast}|_{\mathbb{P}_v} := (\mathcal{E}|_{\mathbb{P}_v}, \mathcal{E}_B|_{\mathbb{P}_v}, \nabla|_{\mathbb{P}_v})$ forms a dormant torally indigenous bundle on $^{\mathrm{tri}}\mathbb{P}_v$. Moreover, for any node $e \in E_{\mathbb{X}}$ with $I_{\mathbb{X}}(e) = [v_1, v_2]$, the radius of the monodromy operators of $\mathcal{E}^{\circledast}|_{\mathbb{P}_{v_1}}$ and $\mathcal{E}^{\circledast}|_{\mathbb{P}_{v_2}}$ at e coincide. In particular, if we denote by $\rho_e \in K/\{\pm 1\}$ this radius, then we obtain a well-defined map

$$\mathrm{rad}_{\mathbb{X}} : M_{\mathbb{X}}^{\mathrm{Zzz}\cdots} \to (K/\{\pm 1\})^{E_{\mathbb{X}}}$$

defined by $\mathcal{E}^{\circledast} \mapsto (\rho_e)_{e \in E_{\mathbb{X}}}$. The assignment $\mathcal{E}^{\circledast} \mapsto (\mathcal{E}^{\circledast}|_{\mathbb{P}_v})_{v \in V_{\mathbb{X}}}$ defines a bijective correspondence between $M_{\mathbb{X}}^{\mathrm{Zzz}\cdots}$ and the set of collections $([\mathcal{E}_v^{\circledast}])_{v \in V_{\mathbb{X}}}$, indexed by $V_{\mathbb{X}}$, consisting of the isomorphism classes $[\mathcal{E}_v^{\circledast}]$ of a dormant torally indigenous bundle $\mathcal{E}_v^{\circledast}$ on $^{\mathrm{tri}}\mathbb{P}_v$ having radii which agree at any two marked points which are glued together (cf. [19], Chap. II, p.128, Proposition 1.5). Thus, by combining with Proposition 8.8, we have the following

Corollary 8.9.
Consider a map

$$\mathrm{inc}_X : (p-2)\mathcal{P}_G \cap \mathbb{Z}^{E_X} \to (K/\{\pm 1\})^{E_X}$$

defined by assigning

$$\mathrm{inc}_X : (\lambda_e)_{e \in E_X} \mapsto (\overline{2\lambda_e + 1})_{e \in E_X}.$$

Then inc_X *is injective, and the map* rad_X *factors through* inc_X*, i.e.,there exists a map*

$$\mathrm{rad}'_X : M_X^{\mathrm{Zzz}\cdots} \to (p-2)\mathcal{P}_G \cap \mathbb{Z}^{E_X}$$

satisfying that $\mathrm{rad}_X = \mathrm{inc} \circ \mathrm{rad}'$*. Moreover, the resulting map* $\mathrm{rad}'_X : M_X^{\mathrm{Zzz}\cdots} \to (p-2)\mathcal{P}_G \cap \mathbb{Z}^{E_X}$ *is bijective.*

By applying Theorem 8.4, Corollary 8.8, and the discussion following Remark 8.5, we obtain the following

Corollary 8.10 (= Theorem B)**.**
Let G be a 3-regular graph of genus $g > 1$. Write $f^G(t)$ for the Ehrhart quasi-polynomial of the Liu-Osserman polytope of G, $\mathrm{Spin}_G(m)$ ($m \in \mathbb{Z}_{\geq 0}$) for the set of m-colored spin networks on G (cf. Definition 2.6), and N_G for the set of 2-regular sub-quasi-graphs of G (cf. Definition 4.1). Then, for p an odd prime with $p > 2(g-1)$, we have equalities

$$\begin{aligned} f^G(p-2) =& \frac{\sharp \mathrm{Spin}_G(p-2)}{\sharp N_G} = \deg_{\overline{\mathcal{M}}_{g,0,\mathbb{F}_p}}(\overline{\mathcal{M}}^{\mathrm{Zzz...}}_{g,0,\mathbb{F}_p}) \\ =& -\frac{p^g}{2^{2g-1}} \cdot \mathrm{Res}_{x=0}\Big[\frac{\cot(px)}{\sin^{2g-2}(x)}dx\Big]. \end{aligned}$$

Corollary 8.11 (= Theorem A (iv))**.**
Let G be a connected 3-regular graph of genus g. Write $\mathcal{P}_G$ for the Liu-Osserman polytope of G and $f^G(t) = (f_i^G(t))_{i\in\mathbb{Z}}$ for the Ehrhart quasi-polynomial of $\mathcal{P}_G$. For i an odd integer, the polynomial $f_i^G(t) \in \mathbb{Q}[t]$ has the following expression:

$$f_i^G(t) = -\frac{(t+2)^g}{2^{2g-1}} \cdot \mathrm{Res}_{x=0}\Big[\frac{\cot((t+2)x)}{\sin^{2g-2}(x)}dx\Big],$$

where $\mathrm{Res}_{x=0}(f)$ denotes the residue of f at $x = 0$. In particular, we have that

$$\mathrm{Vol}(\mathcal{P}_G) = \frac{(-1)^g \cdot B_{2g-2}}{2 \cdot (2g-2)!}$$

Proof. If $F_g(t)$ is the polynomial defined in Remark 8.5, then it follows from Corollary 8.10 and the discussion following Remark 8.5 that $f^G(p-2) = F_g(p)$ for each odd prime p with $p > 2(g-1)$. Since the period of $i_{\mathcal{P}_G}$ divides 4 and there are infinitely many primes q such that $q \equiv 1 \bmod 4$ (resp., $q \equiv 3 \bmod 4$), we have an equality $f_i^G(t-2) = F_g(t)$ for $i \equiv 1 \bmod 4$ (resp., $i \equiv 3 \bmod 4$). Hence the former assertion follows from Theorem 8.3 (ii). The latter assertion follows from the fact concerning the leading coefficient of Ehrhart quasi-polynomials (cf. § 1.2). □

9. Appendix

Let G be a 3-regular graph of genus $g \geq 2$. By applying Corollary 8.11 and the discussion in Remark 8.5 to our calculations, we obtain an explicit expression for the constituent polynomial $f^g(t) := f_{2n+1}^G(t)$ (for an arbitrary $n \in \mathbb{Z}$) under consideration. We shall display some of these expressions as below. One may propose, from these expressions, various conjectures concerning the coefficients of $f^g(t)$, e.g., the conjecture that if we write $f^g(t) = (t+2)^{g-1} \cdot (\sum_{j=0}^{2g-2} c^j t^j)$ ($c^j \in \mathbb{Q}$), then $1 > c^j > 0$ for all j. Of course, one may ask more sharper

bounds for c^j. But at the present time it is not clear to the author whether such a conjecture is true or not.

$$
\begin{aligned}
f^2(t) =& \frac{1}{24} \cdot (t+2) \cdot (t^2+4t+3),\\
f^3(t) =& \frac{1}{1440} \cdot (t+2)^2 \cdot (t^4+8t^3+34t^2+72t+45),\\
f^4(t) =& \frac{1}{120960} \cdot (t+2)^3 \cdot (2t^6+24t^5+141t^4+488t^3+1152t^2+1728t\\
&+945),\\
f^5(t) =& \frac{1}{7257600} \cdot (t+2)^4 \cdot (3t^8+48t^7+376t^6+1824t^5+6054t^4\\
&+14128t^3+24192t^2+28800t+14175),\\
f^6(t) =& \frac{1}{191600640} \cdot (t+2)^5 \cdot (2t^{10}+40t^9+393t^8+2448t^7+10702t^6\\
&+34344t^5+82804t^4+150048t^3+206784t^2+207360t+93555),\\
f^7(t) =& \frac{1}{5230697472000} \cdot (t+2)^6\\
&\cdot (1382t^{12}+33168t^{11}+392148t^{10}+2978320t^9+\\
&16138719t^8+65701872t^7+206720900t^6+509218224t^5\\
&+985601016t^4+1485556416t^3+1740216960t^2+1524096000t\\
&+683512875)\\
f^8(t) =& \frac{1}{8966909952000} \cdot (t+2)^7\\
&\cdot (60t^{14}+1680t^{13}+23222t^{12}+207888t^{11}\\
&+1342188t^{10}+6603760t^9+25558779t^8+79242672t^7\\
&+198678320t^6+403621824t^5+661637376t^4+863640576t^3\\
&+889159680t^2+696729600t+273648375)\\
f^9(t) =& \frac{1}{64023737057280000} \cdot (t+2)^8\\
&\cdot (10851t^{16}+347232t^{15}+5494080t^{14}\\
&+56609280t^{13}+423792536t^{12}+2440848960t^{11}\\
&+11188170336t10+41678058880t^9+127806050886t^8\\
&+324826420320t^7+685561303520t^6+1197933257088t^5\\
&+1719267273216t^4+1995359754240t^3+1846399795200t^2\\
&+1316818944000t+488462349375)
\end{aligned}
$$

$$
\begin{aligned}
f^{10}(t) =& \frac{1}{102181884343418880000} \cdot (t+2)^9 \\
& \cdot (438670t^{18} + 15792120t^{17} + 281454687t^{16} \\
& + 3279274464t^{15} + 27908405100t^{14} + 183965922000t^{13} \\
& + 972990925932t^{12} + 4223486472480t^{11} + 15270421403142^{10} \\
& + 46421838764920t^9 + 119278057144932t^8 + 259467174492480t^7 \\
& + 476995459785120t^6 + 736884549092736t^5 + 947000531630592t^4 \\
& + 995440509388800t^3 + 841246647091200t^2 + 553063956480000t \\
& + 194896477400625).
\end{aligned}
$$

Acknowledgement

I cannot express enough my sincere and deep gratitude to Professors Shinichi Mochizuki and Yuichiro Hoshi for their helpful suggestions and heartfelt encouragements, as well as for reading preliminary versions of the present paper. But I alone, of course, am responsible for any errors and misconceptions in the present paper.

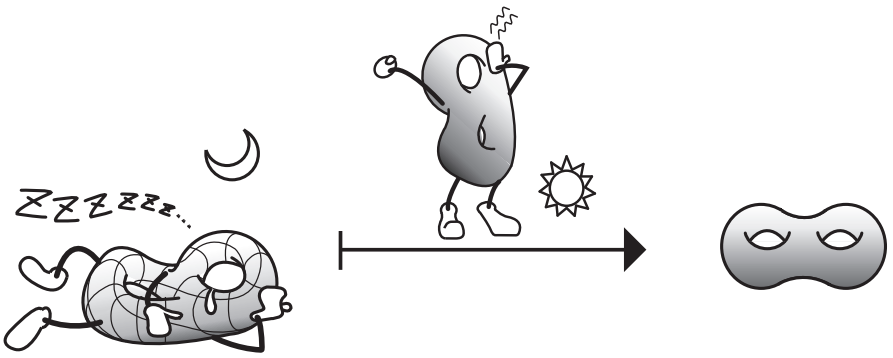
My sincere thanks go to Professors Kirti Joshi, Fu Liu, and Brian Osserman. As we mentioned in Introduction, the present paper is a continuation of earlier work of S. Mochizuki, F. Liu, and B. Osserman. Without their philosophies, theoretical insights and endless creativity, my study of mathematics would have remained "*dormant*" (cf. the figure following Acknowledgement !).

Conversations with many people were very helpful, the incomplete list includes Tatsuya Ohshita, Katsurou Takahashi, Kento Fujita, Kohei Iwaki, Nobuo Sato, Yu Iijima, Yasuhiro Ishitsuka, Tatsuyuki Hikita (and hyperbolic curves of positive characteristic!). I am very grateful to all these people.

Part of this work was written during my stay at Hausdorff Center for Mathematics in Bonn, Germany. The hospitality and the very warm atmosphere of this institution is gratefully acknowledged.

I was supported by the Grant-in-Aid for Scientific Research (KAKENHI No. 24-5691) and the Grant-in-Aid for JSPS fellows.

Finally, my special thanks go to the staff members at "*CAFE PROVERBS [15:17]*" in Kyoto, Japan, and the various individuals with whom I became acquainted there. I deeply appreciates the relaxed and comfortable environment that they provided for writing the present paper, as well as excellent coffee and food that they serve !



The forgetting map $\overline{\mathcal{M}}^{\text{Zzz...}}_{g,r,\mathbb{F}_p} \to \overline{\mathcal{M}}_{g,r,\mathbb{F}_p}$

References

[1] E. Ehrhart, Sur les polyèdres rationnels homothétiques à n dimensions. *C. R. Acad. Sci. Paris* **254** (1962), pp. 616-618.

[2] E. Ehrhart, Sur un problème de géométrie diophantienne linéaire. I. Polyèdres et réseaux. *J. Reine Angew. Math.* **226** (1967), pp. 1-29.

[3] E. Ehrhart, Sur un problème de géométrie diophantienne linéaire. II. Systèmes diophantiens linéaires. *J. Reine Angew. Math.* **227** (1967), pp. 25-49.

[4] G. Eward, *Combinatorial convexity and algebraic geometry.* Graduate Text in Mathematics, **168**, Springer-Verlag, New York, (1996).

[5] R. C. Gunning, Lectures on Riemann surfaces. *Princeton Mathematical Notes* **2** Princeton University Press, Princeton, NJ, (1966).

[6] R. C. Gunning, Special coordinate covering of Riemann surfaces. *Math. Ann.* **170** (1967), pp. 67-86.

[7] A. Hatcher, P. Lochak, L. Schneps, On the Teichmüller tower of mapping class groups. *J. Reine Angew. Math.* **521** (2000), pp. 1-24.

[8] A. Hatcher, W. Thurston, A presentation for the mapping class group of a closed orientable surface. *Topology* **19** (1980), pp. 221-237.

[9] K. Joshi, C. Pauly, Hitchin-Mochizuki morphism, opers and Frobenius-destabilized vector bundles over curves. *math. AG/0912.3602* (2009).

[10] K. Joshi, S. Ramanan, E. Z. Xia, and J. K. Yu, On vector bundles destabilized by Frobenius pull-back. *Compositio Mathematica* **142** (2006), pp. 616-630.

[11] K. Kato, Logarithmic structures of Fontaine-Illusie. *Proceedings of the First JAMI Conference,* Johns-Hopkins University Press (1990), pp. 191-224.

[12] N. M. Katz, Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin. *Inst. Hautes Etudes Sci. Publ. Math.* **39** (1970), pp. 175-232.

[13] H. Lange, C. Pauly, On Frobenius-destabilized rank-2 vector bundles over curves. *Comment. Math. Helv.* **83** (2008), pp. 179-209.

[14] F. Liu, B. Osserman, Mochizuki's indigenous bundles and Ehrhart polynomials. *Journal of Algebraic Combinatorics* **23**, (2006), pp. 125-136.

[15] Q. Liu, *Algebraic geometry and arithmetic curves.* Oxford Graduate Texts in Mathematics, vol 6, Oxford Univ. Press (2002).

[16] I. G. Macdonald, Polynomials associated with finite cell-complexes. *J. London Math. Soc.* **4** (1971), pp. 181-192.

[17] D. Margalit, Automorphisms of the pants complex. *Duke Math. J.* **121** (2004), pp. 457-479.

[18] S. Mochizuki, A theory of ordinary p-adic curves. *Publ. RIMS* **32** (1996), pp. 957-1151.
[19] S. Mochizuki, *Foundations of p-adic Teichmüller theory*. American Mathematical Society, (1999).
[20] A. Ogus, F-crystals, Griffiths transversality and the Hodge decomposition. *Astérisque* **221** (1994).
[21] B. Osserman, Mochizuki's crys-stable bundles: A lexicon and applications. *Publ. Res. Inst. Math. Sci.* **43** (2007), pp. 95-119.
[22] B. Osserman, Frobenius-unstable bundles and p-curvature. *Transactions of the Amer. Math. Soc.* **360** (2008), pp. 273-305.
[23] R. Penrose, Angular momentum: An approach to combinatorial space time. *Quantum Theory and Beyond (Ted Bastin, ed.), Research Notes in Math., Cambridge Univ. Press* (1971), pp. 151-180.
[24] R. Penrose, Applications of negative dimensional tensors. *Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969), Academic Press, London* (1971), pp. 221-244.
[25] D. Singh, A. M. Ibrahim, T. Yohanna, J. N. Singh, An overview of the applications of multisets. *Novi Sad J. Math.* **37** (2007), pp. 73-92.
[26] L. Smolin, The future of spin networks. *arXiv:gr-qc/9702030* (1997).
[27] R. P. Stanley, Combinatorial reciprocity theorems. *Advances in Math.* **14** (1974), pp. 194-253.
[28] W. P. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc. (N.S.)* **19** (1988), pp. 417-431.
[29] Y. Wakabayashi, An explicit formula for the generic number of dormant indigenous bundles. *RIMS Preprint* **1766** (2013).
[30] Wikipedia, Spin network. 26 February 2013 at 23:52, `http://en.wikipedia.org/wiki/Spin_network`.
[31] Wikipedia, Ehrhart polynomial. 12 March 2013 at 23:36, `http://en.wikipedia.org/wiki/Ehrhart_polynomial`.
[32] Wikipedia, Graph. 21 May 2013 at 07:12, `http://en.wikipedia.org/wiki/Graph_(mathematics)`.
[33] D. Zagier, Elementary aspects of the Verlinde formula and of the Harder-Narasimhan-Atiyah-Bott formula. *Israel Math. Conf. Proc.* **9** (1996), pp. 445-462.