A Min-Max Theorem for k-submodular Functions and Extreme Points of the Associated Polyhedra

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Abstract

A. Huber and V. Kolmogorov (ISCO 2012) introduced a concept of k-submodular function as a generalization of ordinary submodular (set) functions and bisubmodular functions. They presented a min-max relation for the k-submodular function minimization by considering ℓ_1 norm, which requires a non-convex set of feasible solutions associated with the k-submodular function. Our approach overcomes the trouble incurred by the non-convexity by means of a new norm composed of ℓ_1 and ℓ_∞ norms. We show another min-max relation that characterizes the minimum of a k-submodular function in terms of the maximum of the negative of the norm values over the associated convex set of feasible solutions. The min-max relation given in the present paper is simpler than that of Huber and Kolmogorov.

We also show a counterexample to a characterization, given by Huber and Kolmogorov, of extreme points of the k-submodular polyhedron in their sense and make it a correct one by fixing a flaw therein.

1 Introduction

A. Huber and V. Kolmogorov [6] introduced a concept of k-submodular function, which is a generalization of ordinary submodular (set) functions and bisubmodular functions (see, e.g., [3, 4, 5]). Motivated by [8], Huber and Kolgomolov introduced convex polyhedra, what they call k-submodular polyhedra, associated with k-submodular functions. Earlier than Huber and Kolmogorov [6] A. Bouchet [2] also considered a class of k-submodular functions to define multimatroids as a generalization of delta-matroids [1, 3].

Kolmogorov [7] also considered a concept of tree-submodularity, which is more general than k-submodularity. It was shown in [7] that polynomial solvability of the k-submodular function minimization implies that of the tree-submodular function minimization for all trees.

Thapper and Zivny [9] showed a dichotomy theorem that classifies the polynomial-time solvability of the minimization problems of functions on finite domains in terms of binary fractional polymorphisms (see [9, 10, 11] for the details). One of the important applications of this result is the tractability of the k-submodular function minimization problem in the valued CSP model since its complexity was not known before. It, however, remains an open problem whether k-submodular functions can be minimized in polynomial time in the value oracle model.

In this paper we consider general k-submodular functions and their associated k-submodular polyhedra. We introduce a new norm that is composed of ℓ_1 and ℓ_∞ norms and show a min-max relation that the minimum of a k-submodular function is equal to the maximum of the negative of the norm values over the associated k-submodular polyhedron (see [4] about such a min-max relation for bisubmodular functions by means of ℓ_1 norm).

Huber and Kolmogorov [6] presented a min-max relation that characterizes the minimum of a k-submodular function in terms of ℓ_1 norm, which requires a non-convex set of feasible solutions associated with the k-submodular function for $k \geq 3$. Our approach with the new norm overcomes the trouble incurred by the non-convexity.

In the present paper we first give definitions and some preliminaries in Section 2. Section 3 shows a min-max relation that characterizes the minimum of a k-submodular function in terms of a new norm composed of ℓ_1 and ℓ_∞ norms. In Section 4 we give a counterexample to a characterization, presented in [6], of extreme points of k-submodular polyhedron in the sense of Huber and Kolmogorov, and then show a correct one. Finally, Section 5 gives concluding remarks.

2 Definitions and Preliminaries

Let V be a nonempty finite set and $\mathcal{U} \equiv \{U_1, U_2, \cdots, U_n\}$ be a partition of V. A subset $T \subseteq V$ is called a *subtransversal* (or *partial transversal*) of $\mathcal{U}$ if $|T \cap U| \leq 1$ for all $U \in \mathcal{U}$. Denote by $\mathcal{T}$ the set of all subtransversals of $\mathcal{U}$.

For any $T, T' \in \mathcal{T}$ define binary operations $\sqcup$ and $\sqcap$ on $\mathcal{T}$ by

$$T \sqcup T' = (T \cup T') \setminus \bigcup \{U \in \mathcal{U} \mid |U \cap (T \cup T')| = 2\}, \qquad T \sqcap T' = T \cap T'. \tag{1}$$

Let $k = \max\{|U| \mid U \in \mathcal{U}\}$. A function $f : \mathcal{T} \to \mathbb{R}$ is called *k-submodular* if

$$f(T) + f(T') \geq f(T \sqcup T') + f(T \sqcap T') \qquad (\forall T,\ T' \in \mathcal{T}). \tag{2}$$

This definition of a k-submodular function is equivalent to that given in [6]. We assume $f(\emptyset) = 0$. We call $(\mathcal{U}, f)$ a *k-submodular system* on V. Define a polyhedron

$$\mathrm{P}(f) = \{x \in \mathbb{R}^V \mid \forall T \in \mathcal{T} :\ x(T) \leq f(T)\}, \tag{3}$$

where we define $x(T) = \sum_{v \in T} x(v)$. We call $\mathrm{P}(f)$ the *k-submodular polyhedron* associated with the k-submodular system $(\mathcal{U}, f)$.

Bouchet [2] considered k-submodular functions that were monotone nondecreasing and had the unit-increase property to define a set system called a *multimatroid*, a generalization of delta-matroids [1]. General k-submodular functions were considered by Huber and Kolmogorov [6], where they assumed that $|U| = k$ for all $U \in \mathcal{U}$. They defined a polyhedron in a way slightly different from our $\mathrm{P}(f)$ in (3) by adding the following inequalities to those in (3).

$$\forall U \in \mathcal{U},\ \ \forall X \in \binom{U}{2} :\ \ x(X) \leq 0, \tag{4}$$

where $\binom{U}{2}$ is the set of all two-element subsets of U. We denote the 'k-submodular polyhedron' in the sense of Huber and Kolmogorov by $\mathrm{P}_2(f)$, i.e.,

$$\mathrm{P}_2(f) = \{x \in \mathbb{R}^V \mid \forall T \in \mathcal{T} :\ x(T) \leq f(T),\ \forall U \in \mathcal{U},\ \forall X \in \binom{U}{2} :\ x(X) \leq 0\}. \tag{5}$$

Note that we have

$$\mathrm{P}(f) \cap \mathbb{R}^V_{\leq 0} \ = \ \mathrm{P}_2(f) \cap \mathbb{R}^V_{\leq 0} \ \subseteq \ \mathrm{P}_2(f) \ \subseteq \ \mathrm{P}(f), \tag{6}$$

where $\mathbb{R}^V_{\leq 0}$ is the set of all nonpositive vectors in $\mathbb{R}^V$.

For any $x \in \mathrm{P}(f)$ (or $x \in \mathrm{P}_2(f)$) and $T \in \mathcal{T}$ we say T is *x-tight* if $x(T) = f(T)$. We can easily show the following (see [6]).

Lemma 2.1. *For any $x \in \mathrm{P}_2(f)$ and $X, Y \in \mathcal{T}$, if X and Y are x-tight, then $X \sqcup Y$ and $X \sqcap Y$ are also x-tight.* □

In the present paper the collection of vectors $x \in \mathrm{P}(f)$ with $x \leq \mathbf{0}$ plays an important rôle in showing our main theorem about a min-max relation for k-submodular functions. Note that such vectors belong to $\mathrm{P}_2(f)$.

For any $u \in V$ and $x \in \mathrm{P}(f)$ define

$$\hat{\mathrm{c}}(x, u) = \max\{\alpha \in \mathbb{R} \mid x + \alpha\chi_u \in \mathrm{P}(f)\}, \tag{7}$$

where χ_u is the unit vector in $\mathbb{R}^V$ with $\chi_u(u) = 1$ and $\chi_u(v) = 0$ for all $v \in V \setminus \{u\}$. Note that $\hat{\mathrm{c}}(x, u)$ can be expressed as

$$\hat{\mathrm{c}}(x, u) = \min\{f(X) - x(X) \mid u \in X \in \mathcal{T}\}. \tag{8}$$

We call $\hat{c}(x,u)$ the *saturation capacity* associated with x and u. If $\hat{c}(x,u) = 0$, we call u *saturated*, and otherwise ($\hat{c}(x,u) > 0$), *non-saturated*. Define $\mathrm{sat}(x)$ to be the set of saturated elements associated with x. We see that u is saturated if and only if there exists at least one x-tight set X such that $u \in X$. Let us denote by $\mathcal{T}(x)$ the collection of x-tight sets.

For any $x \in \mathrm{P}(f)$ and any saturated $u \in V$ define the *dependence function*

$$\mathrm{dep}(x,u) = \{v \in V \mid \exists \beta > 0 : \ x + \beta(\chi_u - \chi_v) \in \mathrm{P}(f)\}. \tag{9}$$

This can be rewritten as

$$\mathrm{dep}(x,u) = \bigcap \{X \mid u \in X \in \mathcal{T}(x)\}. \tag{10}$$

Here, it should be noted that we have $\mathrm{dep}(x,u) \in \mathcal{T}(x)$ if $x \in \mathrm{P}_2(f)$ (due to Lemma 2.1) but not necessarily otherwise. If $x \in \mathrm{P}_2(f)$, then $\mathrm{dep}(x,u)$ is the unique minimal x-tight set containing u.

Furthermore, for any $v \in \mathrm{dep}(x,u) \setminus \{u\}$ define

$$\tilde{c}(x,u,v) = \max\{\beta \in \mathbb{R} \mid x + \beta(\chi_u - \chi_v) \in \mathrm{P}(f)\} \ > 0, \tag{11}$$

which is called the *exchange capacity* for u and $v \in \mathrm{dep}(x,u) \setminus \{u\}$ associated with x. This can also be rewritten as

$$\tilde{c}(x,u,v) = \min\{f(X) - x(X) \mid X \in \mathcal{T}, \ u \in X, \ v \notin X\}. \tag{12}$$

The concepts of sat, $\hat{c}$, dep, and $\tilde{c}$ generalize those defined for ordinary submodular polyhedra (see [5]).

For any nonempty $W \subseteq V$ and $x \in \mathbb{R}^V$ we define $x^W \in \mathbb{R}^W$ by $x^W(v) = x(v)$ for all $v \in W$. Also define $(\mathcal{U}^W, f^W)$ to be the restriction of the k-submodular system $(\mathcal{U}, f)$ on V to W as follows. Let $\mathcal{U}^W = \{U \cap W \mid U \in \mathcal{U}, \ U \cap W \neq \emptyset\}$, $\mathcal{T}^W = \{T \cap W \mid T \in \mathcal{T}\}$ and $f^W(T) = f(T)$ for all $T \in \mathcal{T}^W$. For $k' = \max\{|U| \mid U \in \mathcal{U}^W\}$, $(\mathcal{U}^W, f^W)$ is a k'-submodular system on W. For any nonempty $T \in \mathcal{T}$, f^T is an ordinary submodular function on 2^T, which defines the associated *base polyhedron*

$$\mathrm{B}(f^T) = \{x \in \mathbb{R}^T \mid \forall X \subset T : x(X) \leq f(X), \ x(T) = f(T)\}. \tag{13}$$

(See [5].)

For any $x \in \mathbb{R}^V$ define

$$||x||_{1,\infty} = \sum_{i=1}^{n} \max_{u \in U_i} |x(u)|. \tag{14}$$

This defines a norm on $\mathbb{R}^V$, which is a composition of ℓ_1 and ℓ_∞ norms. Our main result is a min-max theorem based on the new norm $|| \cdot ||_{1,\infty}$ on $\mathbb{R}^V$.

3 A Min-Max Theorem

We show the following min-max theorem.

Theorem 3.1. *For a k-submodular system $(\mathcal{U}, f)$ on V with $f(\emptyset) = 0$ we have*

$$\min\{f(T) \mid T \in \mathcal{T}\} = \max\{-||x||_{1,\infty} \mid x \in \mathrm{P}(f)\}. \tag{15}$$

Moreover, if f is integer-valued, there exists an integral x that attains the maximum of the right-hand side. □

Remarks: It should be noted that Theorem 3.1 follows from the the min-max theorem shown by Huber and Kolmogorov [6]. We shall give a direct and simple proof of Theorem 3.1 in the following. □

In order to prove Theorem 3.1 we will show some lemmas. For simplicity we write $|| \cdot ||_{1,\infty}$ as $|| \cdot ||$.

Lemma 3.2. *For any $x \in \mathrm{P}(f)$ and $T \in \mathcal{T}$ we have*

$$f(T) \geq x(T) \geq -||x||. \tag{16}$$

(Proof) This easily follows from the definitions of $\mathrm{P}(f)$ and $||x||$. □

Let x^* be a maximizer of the right-hand side of (15). Because of the definition of $\mathrm{P}(f)$ we can assume that $x^* \leq \mathbf{0}$. Recall that $u \in V$ is saturated if for every $\alpha > 0$ we have $x^* + \alpha\chi_u \notin \mathrm{P}(f)$, and non-saturated otherwise. If $x^*(u) < 0$ for some non-saturated u, then we can make u saturated or $x^*(u) = 0$ without increasing the norm $||x^*||$. Hence we further assume that u is saturated for every $u \in V$ with $x^*(u) < 0$.

We fix such a maximizer x^* in the following argument.

Recall that $\mathcal{T}(x^*)$ is the collection of x^*-tight sets. It is a crucial fact that since $x^* \leq \mathbf{0}$, $\mathcal{T}(x^*)$ is closed with respect to binary operations $\sqcup$ and $\sqcap$, due to Lemma 2.1.

Lemma 3.3. *For every $u \in V$ with $x^*(u) < 0$ we have $\mathrm{dep}(x^*, u) \in \mathcal{T}(x^*)$.*

(Proof) By the assumption u is saturated and $x^* \leq \mathbf{0}$. It follows from Lemma 2.1 that $\mathrm{dep}(x^*, u) \in \mathcal{T}(x^*)$. □

We write $\mathrm{dep}(x^*, u)$ as $D(u)$ for simplicity in the sequel. For any $v \in V$ let $U(v)$ be the unique set $U \in \mathcal{U}$ such that $v \in U$.

Lemma 3.4. *Suppose that $u \in V$ and $x^*(u) < 0$. Then for $v \in V$ with $D(u) \cap U(v) = \emptyset$ we have $x^*(v) = 0$ or*

$$|(D(u) \cup D(v)) \cap U_i| \neq 2 \qquad (\forall i = 1, \cdots, n). \tag{17}$$

(Proof) If $x^*(v) < 0$ and some U_i violates (17), then $v \in (D(u) \sqcup D(v)) \sqcap D(v) \subset D(v)$, which contradicts the minimality of $D(v)$. □

Let u be an element of V such that $x^*(u) < 0$. Then, if for every $w \in D(u)$ we have $x^*(w) = \min\{x^*(v) \mid v \in U(w)\}$, we call u *legitimate*. Also, if for some $w \in D(u)$ we have $x^*(w) > \min\{x^*(v) \mid v \in U(w)\}$, we say u *is not legitimate with* w.

The following is a key lemma.

Lemma 3.5. *For any $U \in \mathcal{U}$ with $\min\{x^*(v) \mid v \in U\} < 0$ let W be the set of all the minimizers of $\min\{x^*(v) \mid v \in U\}$. Then there exists a legitimate $w \in W$.*

(Proof) Suppose on the contrary that no element in W is legitimate. Then, $|D(w)| > 1$ for all $w \in W$. For each $w \in W$ let w^- be an element of $D(w) \setminus \{w\}$ such that $x^*(w^-) > \min\{x^*(v) \mid v \in U(w^-)\}$. Put $z_{w^-} = x^*(w^-) - \min\{x^*(v) \mid v \in U(w^-)\}$.

Now, for each $w \in W$ there exists some (sufficiently small) $\alpha_w > 0$ such that $y_w \equiv x^* + \alpha_w(\chi_w - \chi_{w^-}) \in \mathrm{P}(f)$ and $\alpha_w \leq \min\{z_{w^-}, -x^*(w)\}$. It follows that a convex combination y^* of y_w $(w \in W)$ with positive coefficients has a norm $||y^*||$ smaller than $||x^*||$, a contradiction. $\square$

Now, for given x^*, we find a minimizer $T \in \mathcal{T}$ of f by the following procedure.

Procedure $\mathsf{Find\_Min}$

Step 1: $\tilde{\mathcal{U}} \leftarrow \{U \in \mathcal{U} \mid \exists u \in U : x^*(u) < 0\}$,
$T \leftarrow \emptyset$.

Step 2: While $\tilde{\mathcal{U}} \neq \emptyset$, do the following:

(1) Choose $U \in \tilde{\mathcal{U}}$ and let $\hat{u}$ be a legitimate element of U.

(2) $T \leftarrow T \cup D(\hat{u})$,
$\tilde{\mathcal{U}} \leftarrow \tilde{\mathcal{U}} \setminus \{U(v) \mid v \in D(\hat{u})\}$.

Step 3: Return T.

The following lemma completes the proof of the min-max relation in Theorem 3.1.

Lemma 3.6. Procedure $\mathsf{Find\_Min}$ *finds $T \in \mathcal{T}$ such that $-||x^*|| = f(T)$.*

(Proof) It follows from Lemma 3.5 we can find a legitimate $\hat{u}$ in Step 2. Furthermore, Lemma 3.4 validates $T \in \mathcal{T}$ and T being x^*-tight. The finally obtained T satisfies that $T \cap U \neq \emptyset$ for all $U \in \mathcal{U}$ with $\min\{x^*(v) \mid v \in U\} < 0$ and that for all $u \in T$ we have $x^*(u) = \min\{x^*(v) \mid v \in U(u)\}$. Hence, $-||x^*|| = x^*(T) = f(T)$. $\square$

Now we show the latter half of Theorem 3.1, the integrality property. Note that by definition $\mathrm{P}(f)$ is hereditary, i.e., closed downward, so that there exists an integral x in $\mathrm{P}(f)$.

Consider the following procedure.

Procedure $\mathsf{Find\_Max}$

Step 0: Let x be an integral non-positive vector in $\mathrm{P}(f)$.

Step 1: While there exists a non-saturated $v \in V$ with $x(v) < 0$, do the following:

$\alpha \leftarrow \min\{-x(v), \hat{\mathrm{c}}(x, v)\}$,

$x \leftarrow x + \alpha\chi_v$.

Step 2: $\tilde{\mathcal{U}} \leftarrow \{U \in \mathcal{U} \mid \exists u \in U : x(u) < 0\}$,

$T \leftarrow \emptyset$.

Step 3: While $\tilde{\mathcal{U}} \neq \emptyset$, do the following:

(1) Choose $U \in \tilde{\mathcal{U}}$.

(2) Define $W = \{u \in U \mid x(u) = \min\{x(v) \mid v \in U\}\}$.

(3) Choose $u \in W$.

(3-1) If u is not legitimate with $w \in D(u) \setminus \{u\}$, then

(a) $\beta \leftarrow \min\{-x(u), \tilde{\mathrm{c}}(x, u, w), x(w) - \min\{x(v) \mid v \in U(w)\}\}$,

(b) $x \leftarrow x + \beta(\chi_u - \chi_w)$,

(c) If $\exists v \in U : x(v) < 0$, then go to (2); else remove U from $\tilde{\mathcal{U}}$.

(3-2) If u is legitimate, then

$T \leftarrow T \cup D(u)$,

$\tilde{\mathcal{U}} \leftarrow \tilde{\mathcal{U}} \setminus \{U(v) \mid v \in D(u)\}$.

Step 4: Return x.

Lemma 3.7. *Suppose f is integer-valued. Starting with an integral $x \in \mathrm{P}(f)$ with $x \leq \mathbf{0}$,* Procedure $\mathsf{Find\_Max}$ *finds an integral maximizer for the min-max relation in Theorem* 3.1.

(Proof) During the execution of Procedure $\mathsf{Fin\_Max}$ x remains integral. If u in (3) of Step 3 is not legitimate, $x(u)$ becomes larger, and when $|W| \geq 2$, W becomes smaller. Hence, repeating (2), (3), and (4) in Step 3, we find a legitimate u or we get x with $x(v) = 0$ for all $v \in U$. It follows that Procedure $\mathsf{Find\_Max}$ terminates after a finite number of iterations and the finally obtained integral x and subtransversal T give max and min solutions, similarly as in the proof of Lemma 3.6. □

This completes the proof of Theorem 3.1.

4 Extreme Points of $\mathrm{P}_2(f)$

Huber and Kolmogorov [6] presented a characterization of extreme points of $\mathrm{P}_2(f)$ for a k-submodular function f. In particular, as a necessary condition, they state that if $x \in \mathbb{R}^V$

is a nonzero extreme point of $\mathrm{P}_2(f)$ then there is a nontrivial[1] chain $\emptyset = T_0 \subset T_1 \subset \cdots \subset T_k$ of elements in $\mathcal{T}$ such that

(i) $|T_i \setminus T_{i-1}| = 1$ for $1 \leq i \leq k$ and

(ii) T_i is x-tight for $0 \leq i \leq k$.

We give a counterexample to this claim by showing the existence of a nonzero extreme point that does not satisfy (i).

Let $U_1 = \{v_1, v_2, v_3\}$ and $U_2 = \{u_1, u_2, u_3\}$. Let $V = U_1 \cup U_2$, $\mathcal{U} = \{U_1, U_2\}$, and M be any integer greater than 5. Define $f : \mathcal{T} \to \mathbb{R}$ by

$$
\begin{aligned}
&f(\emptyset) = 0, \\
&f(\{v_1\}) = -1, \ \ f(\{u_1\}) = 1, \\
&f(\{v_i\}) = f(\{u_i\}) = M \ \text{ for } \ i = 2, 3, \\
&f(\{u_1, v_1\}) = -2, \\
&f(\{u_i, v_j\}) = f(\{u_i\}) + f(\{v_j\}) \ \text{ for } \ i, j = 1, 2, 3 \ \text{ with } \ (i, j) \neq (1, 1).
\end{aligned}
$$

Lemma 4.1. *f is k-submodular for $k = 3$.*

(Proof) Take any $T, T' \in \mathcal{T}$ and let us check $f(T) + f(T') \geq f(T \sqcup T') + f(T \sqcap T')$. We may assume $T \not\subset T'$ and $T' \not\subset T$. We shall use the fact that $f(\{u_i\}) + f(\{u_j\}) \geq 0$ and $f(\{v_i\}) + f(\{v_j\}) \geq 0$ for any distinct i, j.

1. If $|T| = 1$ and $|T'| = 1$, denote $T = \{x\}$ and $T' = \{y\}$.
   - If $U(x) = U(y)$, then $T \sqcup T' = \emptyset$ and $T \sqcap T' = \emptyset$. Thus $f(T) + f(T') = f(\{x\}) + f(\{y\}) \geq 0 = f(T \sqcup T') + f(T \sqcap T')$.
   - Otherwise, $T \sqcup T' = \{x, y\}$ and $T \sqcap T' = \emptyset$.
     If $\{x, y\} = \{v_1, u_1\}$, then $f(T) + f(T') = 0 > -2 = f(T \sqcup T') + f(T \sqcap T')$.
     If $\{x, y\} \neq \{v_1, u_1\}$, then $f(T) + f(T') = f(\{x\}) + f(\{y\}) = f(T \sqcup T') + f(T \sqcap T')$.
2. If $|T| = 2$ and $|T'| = 1$, denote $T = \{x, y\}$ and $T' = \{z\}$. We may assume that $U(y) = U(z)$. Then $T \sqcup T' = \{x\}$ and $T \sqcap T' = \emptyset$. Hence,
   - If $\{x, y\} = \{v_1, u_1\}$, then $f(T) + f(T') = -2 + M \geq \max\{f(\{v_1\}), f(\{u_1\})\} \geq f(\{x\}) = f(T \sqcup T') + f(T \sqcap T')$.
   - Otherwise, $f(T) + f(T') = f(\{x\}) + f(\{y\}) + f(\{z\}) \geq f(\{x\}) = f(T \sqcup T') + f(T \sqcap T')$.

[1] By a nontrivial chain, we mean $k \geq 1$.

3. If $|T| = 2$ and $|T'| = 2$, denote $T = \{x, y\}$ and $T' = \{z, w\}$.

   - If $\{x, y\} = \{v_1, u_1\}$, then $f(T \sqcup T') \leq 1$ and $f(T \sqcap T') \leq 1$. Therefore $f(T)+f(T') = -2+f(\{z\})+f(\{w\}) \geq -3+M \geq f(T\sqcup T')+f(T\sqcap T')$.
   - Otherwise, we may assume $\{z, w\} \neq \{v_1, u_1\}$. If $y = w$, then $T \sqcup T' = \{y\}$ and $T \sqcap T = \{y\}$, and hence $f(T) + f(T') = f(\{x\}) + f(\{z\}) + 2f(\{y\}) \geq 2f(\{y\}) = f(T \sqcup T') + f(T \sqcap T')$. If $y \neq w$, we may assume $T \cap T' = \emptyset$. Then $T \sqcup T' = \emptyset$ and $T \sqcap T' = \emptyset$, and hence $f(T) + f(T') = f(\{x\}) + f(\{y\}) + f(\{z\}) + f(\{w\}) \geq 0 = f(T \sqcup T') + f(T \sqcap T')$. $\square$

Now consider the nonzero $x^* \in \mathbb{R}^V$ given by

$$x^*(v_1) = -2, \;\; x^*(v_2) = 2, \;\; x^*(v_3) = -2,$$
$$x^*(u_1) = 0, \;\; x^*(u_2) = 0, \;\; x^*(u_3) = 0.$$

We can see by exhaustive checking that $x^* \in \mathrm{P}_2(f)$ and the following equations hold.

$$\begin{aligned} &x^*(\{v_1, u_1\}) = f(\{v_1, u_1\}), \\ &x^*(\{v_1, v_2\}) = x^*(\{v_2, v_3\}) = 0, \\ &x^*(\{u_1, u_2\}) = x^*(\{u_2, u_3\}) = x^*(\{u_3, u_1\}) = 0. \end{aligned} \tag{18}$$

Since the system of six equations in (18) uniquely determines the solution x^*, x^* is an extreme point of $\mathrm{P}_2(f)$.

Note that for any chain of elements in $\mathcal{T}$ satisfying Condition (i), Condition (ii) is violated for $x = x^*$, since $x^*(v_i) < f(\{v_i\})$ for any v_i and $x^*(u_i) < f(\{u_i\})$ for any u_i. Hence x^* cannot be any extreme point of $\mathrm{P}_2(f)$ that corresponds to the conditions given by Huber and Kolmogorov [6].

We have shown that the conditions provided in [6] do not give an exact characterization of extreme points of $\mathrm{P}_2(f)$. We will give a correct characterization of extreme points of $\mathrm{P}_2(f)$. Let $(\mathcal{U}, f)$ be a k-submodular system on V.

We first show some lemmas.

Lemma 4.2. *For a nonempty $T \in \mathcal{T}$ let x be a vector in $\mathbb{R}^V$ satisfying*

(A) $x^T \in \mathrm{B}(f^T)$*,*

(B) *For each $u \in T$,*
(B1) *if $x(u) \geq 0$, then $x(v) = -x(u)$ for all $v \in U(u) \setminus \{u\}$;*
(B2) *otherwise,*

  (1) *$x(v) = x(u)$ for all $v \in U(u) \setminus \{u\}$ but one v with $x(v) = -x(u)$ or*

  (2) *$x(v) = 0$ for all $v \in U(u) \setminus \{u\}$.*

Then we have $x^Z \in \mathrm{P}_2(f^Z)$ *for* $Z = \cup\{U(u) \mid u \in T\}$.

(Proof) For any $X \in \mathcal{T}$ such that $X \subseteq Z$ we have

$$\begin{aligned} x(X) &= x(X) + x(T) - f(T) \\ &\leq x(X \sqcup T) + x(X \sqcap T) - f(T) \\ &\leq f(X \sqcup T) + f(X \sqcap T) - f(T) \\ &\leq f(X). \end{aligned} \tag{19}$$

Because of the way of defining x by (B) it follows from (19) that $x^Z \in \mathrm{P}_2(f^Z)$. □

We also have

Lemma 4.3. *For a given* $x \in \mathrm{P}_2(f)$ *and a nonempty* $T \in \mathcal{T}$ *suppose that* $x^T \in \mathrm{B}(f^T)$. *Let* $Z = \bigcup\{U(u) \mid u \in T\}$. *For an element* $u \in T$ *define* $y \in \mathbb{R}^Z$ *by* $y(v) = x(v)$ *for all* $v \in Z \setminus (U(u) \setminus \{u\})$ *and* $y(v)$ *for all* $v \in (U(u) \setminus \{u\})$ *according to* (B1) *and* (B2), *replacing* x *by* y, *in Lemma* 4.2. *Then we have* $y \in \mathrm{P}_2(f^Z)$.

(Proof) Since $x \in \mathrm{P}_2(f)$, similarly as in (19) we can show that $y \in \mathrm{P}_2(f^Z)$. □

For $U \in \mathcal{U}$ consider the system of linear inequalities

$$x(u) + x(v) \leq 0 \qquad (\forall \{u, v\} \in \binom{U}{2}). \tag{20}$$

Denote by C_2^U the cone of feasible solutions of (20). We call $\{u, v\}$ *a tight pair* for a feasible solution x^* if the inequality of (20) for the pair $\{u, v\}$ holds with equality for $x = x^*$.

Lemma 4.4. *Suppose* $|U| \geq 3$. *The cone* C_2^U *is pointed and its extreme rays are given by* $x(u) = \alpha$ *and* $x(v) = -\alpha$ *for all* $v \in U \setminus \{u\}$ *with a parameter* $\alpha \geq 0$, *for all* $u \in U$. *Every component-wise maximal solution* x^* *of* (20) *lies on an extreme ray of* C_2^U *and if* $x^* \neq \mathbf{0}$, *the set of the tight pairs for* x^* *forms a star with center* u *such that* $x^*(u) > 0$.

(Proof) Since $|U| \geq 3$, if we replace all the inequalities of (20) by equations, it gives the unique solution $x = \mathbf{0}$. Hence C_2^U is pointed. Moreover, for any component-wise maximal feasible solution x^*, if $x^* \neq \mathbf{0}$, there exists only one $u \in U$ such that $x^*(u) > 0$. Since x^* is component-wise maximal, we must have $x^*(v) = -x^*(u)$ for all $v \in U \setminus \{u\}$. Hence x^* lies on an extreme ray of C_2^U and the tight pairs form a star with center u. □

Note that every extreme vector (lying on an extreme ray) of C_2^U is component-wise maximal.

For any subset $\mathcal{E} \subseteq \binom{U}{2}$ we regard $\mathcal{E}$ as the edge set of an undirected graph $G = (U, \mathcal{E})$ with vertex set U.

Lemma 4.5. *For any subset $\mathcal{E} \subseteq \binom{U}{2}$ the system of equations*

$$x(u) + x(v) = 0 \qquad (\forall \{u, v\} \in \mathcal{E}) \tag{21}$$

uniquely determines the solution $x = \mathbf{0}$ if and only if every connected component of the graph $G = (U, \mathcal{E})$ contains at least one odd cycle.

(Proof) Suppose that every connected component of the graph $G = (U, \mathcal{E})$ contains at least one odd cycle. Since equations (21) for an odd cycle determine $x(v) = 0$ for elements (vertices) v on the cycle, which then determines $x(v) = 0$ for other elements v in the same connected component.

Conversely, suppose that (21) determines the unique solution $x = \mathbf{0}$. Then we must have $\bigcup \mathcal{E} = U$. If some connected component having at least two vertices does not contain odd cycles, then it forms a bipartite graph. Hence the values $x(v)$ for vertices v in the connected component are not uniquely determined. (For, if $x(v_0)$ for a vertex v_0 of the bipartite graph is increased by α, then increasing $x(v)$ for every v at an even distance from v_0 by α and decreasing $x(v)$ for every v at an odd distance from v_0 by α keep x satisfy (21) for any $\alpha \in \mathbb{R}$.) Hence every connected component has at least one odd cycle. $\square$

For $x^T \in \mathbf{B}(f^T)$ define a directed graph $G_x^T = (T, A_x)$ with the vertex set T and the arc set A_x given by

$$A_x = \{(u, v) \mid u \in T,\ v \in \mathrm{dep}(x, u) \setminus \{u\}\}. \tag{22}$$

Let $H_x^i = (S_x^i, B_x^i)$ $(i \in I)$ be the strongly connected components of G_x^T. Choose any $w^i \in S_x^i$ for each $i \in I$. Then we call the set $W = \{w^i \mid i \in I\}$ a *covering set* of G_x^T.

It is known ([5]) that for any maximal chain of tight sets in $\mathcal{T}(x) \cap 2^T$

$$\emptyset = T_0 \subset T_1 \subset \cdots \subset T_p = T \tag{23}$$

the collection of the difference sets $T_j \setminus T_{j-1}$ $(j = 1, \cdots, p)$ is exactly the collection of vertex sets S_x^i $(i \in I)$ of the strongly connected components of G_x^T; in particular, $p = |I|$.

Lemma 4.6. *For any $x \in \mathrm{P}_2(f)$ and nonempty $T \in \mathcal{T}$ suppose that the following three statements hold*:

(1) *For every tight set $T' \in \mathcal{T}(x)$ we have $T' \subseteq \bigcup \{U(u) \mid u \in T\}$.*

(2) $x^T \in \mathrm{B}(f^T)$.

(3) (B) *in Lemma* 4.2 *is satisfied.*

If for some $i_0 \in I$

(a) *we have $|S_x^{i_0}| \geq 2$ and*

(b) *for some distinct* $u, v \in S_x^{i_0}$ *we have* $x(u) \neq 0$ *and* $x(v) \neq 0$, *and letting* $\mathcal{E}_u$ *and* $\mathcal{E}_v$ *be, respectively, the sets of all tight pairs for* $U(u)$ *and* $U(v)$, *the connected component of graph* $(U(u), \mathcal{E}_u)$ *containing* u *and that of* $(U(v), \mathcal{E}_v)$ *containing* v *are both bipartite* (*more specifically, stars*),

then x *is not an extreme point of* $\mathrm{P}_2(f)$.

(Proof) Under the assumption of the present lemma let u and v be those appearing in (b). Define

$$\alpha_1 = \min\{|x(u)|, |x(v)|\},$$
$$\alpha_2 = \min\{f(T') - x(T') \mid T' \in \mathcal{T},\ |T' \cap \{u, v\}| = 1\}.$$

By the assumption we have $\alpha_1 > 0$. Also, since $u, v \in S_x^{i_0}$, we have $v \in \mathrm{dep}(x, u)$ and $u \in \mathrm{dep}(x, v)$, so that $\alpha_2 > 0$. Then, for a real number α such that $0 < \alpha < \min\{\alpha_1, \alpha_2\}$, put $x(u) \leftarrow x(u) \pm \alpha$ and $x(v) \leftarrow x(v) \mp \alpha$ and modify $x(z)$ for $z \in U(u) \cup U(v)$ according to (B) in Lemma 4.2. (The modification of $x(w)$ for $w \in (U(u) \setminus \{u\}) \cup (U(v) \setminus \{v\})$ according to (B) can be made because the relevant components are stars. This includes the case where the relevant component is an isolated vertex in Case (B2)(2).) Let x^+ and x^- be the obtained new points. Since $\alpha_2 \leq \min\{\tilde{c}(x, u, v), \tilde{c}(x, v, u)\}$ and since $x^\pm$ satisfy the assumption of Lemma 4.2 because of the choice of α, we have $x^\pm \in \mathrm{P}_2(f)$ and $x = \frac{1}{2}(x^+ + x^-)$. This completes the proof of this lemma. □

We now show the following.

Theorem 4.7. *For a given* $x \in \mathrm{P}_2(f)$, x *is an extreme point of* $\mathrm{P}_2(f)$ *if and only if there exists a* $T \in \mathcal{T}$ *such that the following* (a)–(e) *hold*:

(a) *For every tight set* $T' \in \mathcal{T}(x)$ *we have* $T' \subseteq \bigcup\{U(u) \mid u \in T\}$.

(b) $x^T \in \mathrm{B}(f^T)$.

(c) *For each* $u \in T$,
(c1) *if* $x(u) \geq 0$, *then* $x(v) = -x(u)$ *for all* $v \in U(u) \setminus \{u\}$;
(c2) *otherwise*,

(1) $x(v) = x(u)$ *for all* $v \in U(u) \setminus \{u\}$ *but one* v' *with* $x(v') = -x(u)$ *or*

(2) $x(v) = 0$ *for all* $v \in U(u) \setminus \{u\}$.

(d) *For some covering set* $W = \{w^i \mid i \in I\}$ *of* G_x^T *with strongly connected components having vertex sets* $S_x^i \subseteq T$ $(i \in I)$ *we have* $x(v) = 0$ *for all* $v \in T \setminus W$. *Moreover, for each* $i \in I$ *and* $v \in S_x^i \setminus \{w^i\}$ *we have* $|U(v)| \geq 3$, *and if values of* $x(v)$ *are determined by* (2) *of* (c2), *we have* $|U(w^i)| \geq 4$.

(e) *For all* $v \in U \in \mathcal{U}$ *with* $U \cap T = \emptyset$ *we have* $x(v) = 0$. *Moreover,* $|U| \geq 3$ *for all* $U \in \mathcal{U}$ *such that* $U \cap T = \emptyset$.

Here Conditions (b), (c), *and* (d) *are void if* $T = \emptyset$.

(Proof) If (a)–(e) are satisfied for $x \in \mathrm{P}_2(f)$, then we have tight equations given as follows.

$$x(T_i) = f(T_i) \text{ for a maximal chain of tight sets for } x^T, \tag{24}$$

$$x(u) + x(v) = 0 \quad (\forall u \in T,\ \forall v \in U(u) \setminus \{u\} \text{ in Case (c1)}), \tag{25}$$

$$x(v) + x(z) = 0\ (\forall u \in T,\ \forall \{v, z\} \in \binom{U(u)}{2} \text{ in Case (c1) with } x(u){=}0), \tag{26}$$

$$x(v') + x(v) = 0\ \ (\forall u \in T,\ \forall v \in U(u) \setminus \{v'\} \text{ in Case (c2)(1)}), \tag{27}$$

$$x(v) + x(z) = 0\ \ (\forall i \in I,\ \forall \{v, z\} \in \binom{U(w^i) \setminus \{w^i\}}{2} \text{ in Case (c2)(2)}), \tag{28}$$

$$x(v) + x(z) = 0 \quad (\forall U \in \mathcal{U} \text{ with } U \cap T = \emptyset,\ \forall \{v, z\} \in \binom{U}{2}). \tag{29}$$

We can see that the system of equations (24)–(29) uniquely determines the solution x, due to Lemma 4.5, so that x is an extreme point of $\mathrm{P}_2(f)$.

Conversely, suppose that $x \in \mathrm{P}_2(f)$ is an extreme point. Then for each $u \in V$ there must exist a tight equation of type

(I) $x(T) = f(T)$ for some $T \in \mathcal{T}$ with $u \in T$ or

(II) $x(X) = 0$ for some $X \in \binom{U}{2}$ with $u \in X$ and $U \in \mathcal{U}$.

Denote by $\mathcal{T}(x)$ the collection of tight sets T of type (I) (as before) and define $W = \cup\{T \mid T \in \mathcal{T}(x)\}$.

Since $x \in \mathrm{P}_2(f)$, we have $\mathrm{dep}(x, u) \in \mathcal{T}(x)$ for all $u \in W$. Moreover, for any $u \in W$ and any $v \in W \setminus \cup\{U \in \mathcal{U} \mid U \cap \mathrm{dep}(x, u) \neq \emptyset\}$ we have $\mathrm{dep}(x, u) \cup \mathrm{dep}(x, v) \in \mathcal{T}(x)$. Hence, similarly as in the proof of Theorem 3.1, there exists $T \in \mathcal{T}(x)$ such that $T \cap U(u) \neq \emptyset$ for all $u \in W$. Let us show that for such T, Conditions (a)–(e) are satisfied.

Firstly, (a), (b), and (e) follow from the choice of T and Lemma 4.5.

Secondly, we show (c). Fixing the values of $x(u)$ for all $u \in T$ and discarding the constraints $x(T') \leq f(T')$ for all $T' \in \mathcal{T} \setminus 2^T$, component-wise maximal vectors x satisfying (20) are exactly those determined by (c), due to Lemmas 4.4 and 4.5. Hence, if x does not satisfy (c), then defining $Z = \bigcup\{U(v) \mid v \in T\}$, there exist $u \in T$ and $y \in \mathbb{R}^Z$, defined appropriately as in Lemma 4.3, such that (i) $x^Z \leq y$ and (ii) $x(\hat{w}) < y(\hat{w})$ for $\hat{w}$ with $\{\hat{w}\} = U(u) \cap T'$ for a tight set $T' \in \mathcal{T}(x)$. Since all the tight sets $T'' \in \mathcal{T}(x)$ for x are included in Z and we have $x \in \mathrm{P}_2(f)$ and $y \in \mathrm{P}_2(f^Z)$ because of Lemma 4.3, defining $y^* \in \mathbb{R}^V$ by $y^*(v) = y(v)$ for all $v \in Z$ and $y^*(v) = 0$ for all $V \setminus Z$, we have for a sufficiently small positive $\epsilon > 0$

$$z_\epsilon \equiv \epsilon x + (1-\epsilon)y^* \in \mathrm{P}_2(f). \tag{30}$$

Then we have $z_\epsilon(\hat{w}) > x(\hat{w})$, which implies $z_\epsilon(T') > f(T')$, a contradiction. Hence (c) is satisfied.

Finally, (d) follows from Lemma 4.6. $\square$

In the counterexample given above, T appearing in Theorem 4.7 is $T = \{v_1, u_1\}$, graph $G^T_{x^*}$ is strongly connected, and a covering set is $W = \{v_1\}$.

It should be noted that we have assumed the membership $x \in \mathrm{P}_2(f)$ in the characterization of extreme points, so that it is not well characterized so as to obtain extreme points efficiently.

5 Concluding Remarks

We have shown a min-max relation for k-submodular functions in terms of a new norm composed of ℓ_1 and ℓ_∞ norms, which is simpler and easier to understand than the min-max relation shown by Huber and Kolmogorov in [6] by using ℓ_1 norm alone.

We have also shown a characterization of extreme points of $\mathrm{P}_2(f)$, a k-submodular polyhedron in the sense of Huber and Kolmogorov, which fixes a flaw in [6].

Devising a polynomial-time algorithm for minimizing k-submodular functions is left open. As pointed out in [6] and discussed here in Section 4 as well, we need a good characterization of extreme points of $\mathrm{P}_2(f)$. A key to the good characterization is to develop a polynomial-time algorithm for linear optimization over $\mathrm{P}_2(f)$. Main difficulty in linear optimization over $\mathrm{P}_2(f)$ is that a polynomial-time algorithm for it requires an efficient membership algorithm for discerning whether $\mathbf{0} \in \mathrm{P}(f)$.

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