

RIMS-1787

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and Extreme Points of the Associated Polyhedra**

By

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September 2013



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A Min-Max Theorem for k -submodular Functions and Extreme Points of the Associated Polyhedra

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September 2, 2013

Abstract

A. Huber and V. Kolmogorov (ISCO 2012) introduced a concept of k -submodular function as a generalization of ordinary submodular (set) functions and bisubmodular functions. They presented a min-max relation for the k -submodular function minimization by considering ℓ_1 norm, which requires a non-convex set of feasible solutions associated with the k -submodular function. Our approach overcomes the trouble incurred by the non-convexity by means of a new norm composed of ℓ_1 and ℓ_∞ norms. We show another min-max relation that characterizes the minimum of a k -submodular function in terms of the maximum of the negative of the norm values over the associated convex set of feasible solutions. The min-max relation given in the present paper is simpler than that of Huber and Kolmogorov.

We also show a counterexample to a characterization, given by Huber and Kolmogorov, of extreme points of the k -submodular polyhedron in their sense and make it a correct one by fixing a flaw therein.

1 Introduction

A. Huber and V. Kolmogorov [6] introduced a concept of k -submodular function, which is a generalization of ordinary submodular (set) functions and bisubmodular functions (see, e.g., [3, 4, 5]). Motivated by [8], Huber and Kolgomolov introduced convex polyhedra, what they call k -submodular polyhedra, associated with k -submodular functions. Earlier than Huber and Kolmogorov [6] A. Bouchet [2] also considered a class of k -submodular functions to define multimatroids as a generalization of delta-matroids [1, 3].

Kolmogorov [7] also considered a concept of tree-submodularity, which is more general than k -submodularity. It was shown in [7] that polynomial solvability of the k -submodular function minimization implies that of the tree-submodular function minimization for all trees.

Thapper and Zivny [9] showed a dichotomy theorem that classifies the polynomial-time solvability of the minimization problems of functions on finite domains in terms of binary fractional polymorphisms (see [9, 10, 11] for the details). One of the important applications of this result is the tractability of the k -submodular function minimization problem in the valued CSP model since its complexity was not known before. It, however, remains an open problem whether k -submodular functions can be minimized in polynomial time in the value oracle model.

In this paper we consider general k -submodular functions and their associated k -submodular polyhedra. We introduce a new norm that is composed of ℓ_1 and ℓ_∞ norms and show a min-max relation that the minimum of a k -submodular function is equal to the maximum of the negative of the norm values over the associated k -submodular polyhedron (see [4] about such a min-max relation for bisubmodular functions by means of ℓ_1 norm).

Huber and Kolmogorov [6] presented a min-max relation that characterizes the minimum of a k -submodular function in terms of ℓ_1 norm, which requires a non-convex set of feasible solutions associated with the k -submodular function for $k \geq 3$. Our approach with the new norm overcomes the trouble incurred by the non-convexity.

In the present paper we first give definitions and some preliminaries in Section 2. Section 3 shows a min-max relation that characterizes the minimum of a k -submodular function in terms of a new norm composed of ℓ_1 and ℓ_∞ norms. In Section 4 we give a counterexample to a characterization, presented in [6], of extreme points of k -submodular polyhedron in the sense of Huber and Kolmogorov, and then show a correct one. Finally, Section 5 gives concluding remarks.

2 Definitions and Preliminaries

Let V be a nonempty finite set and $\mathcal{U} \equiv \{U_1, U_2, \dots, U_n\}$ be a partition of V . A subset $T \subseteq V$ is called a *subtransversal* (or *partial transversal*) of \mathcal{U} if $|T \cap U| \leq 1$ for all $U \in \mathcal{U}$. Denote by \mathcal{T} the set of all subtransversals of \mathcal{U} .

For any $T, T' \in \mathcal{T}$ define binary operations \sqcup and \sqcap on \mathcal{T} by

$$T \sqcup T' = (T \cup T') \setminus \bigcup \{U \in \mathcal{U} \mid |U \cap (T \cup T')| = 2\}, \quad T \sqcap T' = T \cap T'. \quad (1)$$

Let $k = \max\{|U| \mid U \in \mathcal{U}\}$. A function $f : \mathcal{T} \rightarrow \mathbb{R}$ is called *k -submodular* if

$$f(T) + f(T') \geq f(T \sqcup T') + f(T \sqcap T') \quad (\forall T, T' \in \mathcal{T}). \quad (2)$$

This definition of a k -submodular function is equivalent to that given in [6]. We assume $f(\emptyset) = 0$. We call (\mathcal{U}, f) a k -submodular system on V . Define a polyhedron

$$P(f) = \{x \in \mathbb{R}^V \mid \forall T \in \mathcal{T} : x(T) \leq f(T)\}, \quad (3)$$

where we define $x(T) = \sum_{v \in T} x(v)$. We call $P(f)$ the k -submodular polyhedron associated with the k -submodular system (\mathcal{U}, f) .

Bouchet [2] considered k -submodular functions that were monotone nondecreasing and had the unit-increase property to define a set system called a *multimatroid*, a generalization of delta-matroids [1]. General k -submodular functions were considered by Huber and Kolmogorov [6], where they assumed that $|U| = k$ for all $U \in \mathcal{U}$. They defined a polyhedron in a way slightly different from our $P(f)$ in (3) by adding the following inequalities to those in (3).

$$\forall U \in \mathcal{U}, \forall X \in \binom{U}{2} : x(X) \leq 0, \quad (4)$$

where $\binom{U}{2}$ is the set of all two-element subsets of U . We denote the ' k -submodular polyhedron' in the sense of Huber and Kolmogorov by $P_2(f)$, i.e.,

$$P_2(f) = \{x \in \mathbb{R}^V \mid \forall T \in \mathcal{T} : x(T) \leq f(T), \forall U \in \mathcal{U}, \forall X \in \binom{U}{2} : x(X) \leq 0\}. \quad (5)$$

Note that we have

$$P(f) \cap \mathbb{R}_{\leq 0}^V = P_2(f) \cap \mathbb{R}_{\leq 0}^V \subseteq P_2(f) \subseteq P(f), \quad (6)$$

where $\mathbb{R}_{\leq 0}^V$ is the set of all nonpositive vectors in \mathbb{R}^V .

For any $x \in P(f)$ (or $x \in P_2(f)$) and $T \in \mathcal{T}$ we say T is x -tight if $x(T) = f(T)$. We can easily show the following (see [6]).

Lemma 2.1. *For any $x \in P_2(f)$ and $X, Y \in \mathcal{T}$, if X and Y are x -tight, then $X \sqcup Y$ and $X \cap Y$ are also x -tight. \square*

In the present paper the collection of vectors $x \in P(f)$ with $x \leq \mathbf{0}$ plays an important rôle in showing our main theorem about a min-max relation for k -submodular functions. Note that such vectors belong to $P_2(f)$.

For any $u \in V$ and $x \in P(f)$ define

$$\hat{c}(x, u) = \max\{\alpha \in \mathbb{R} \mid x + \alpha \chi_u \in P(f)\}, \quad (7)$$

where χ_u is the unit vector in \mathbb{R}^V with $\chi_u(u) = 1$ and $\chi_u(v) = 0$ for all $v \in V \setminus \{u\}$. Note that $\hat{c}(x, u)$ can be expressed as

$$\hat{c}(x, u) = \min\{f(X) - x(X) \mid u \in X \in \mathcal{T}\}. \quad (8)$$

We call $\hat{c}(x, u)$ the *saturation capacity* associated with x and u . If $\hat{c}(x, u) = 0$, we call u *saturated*, and otherwise ($\hat{c}(x, u) > 0$), *non-saturated*. Define $\text{sat}(x)$ to be the set of saturated elements associated with x . We see that u is saturated if and only if there exists at least one x -tight set X such that $u \in X$. Let us denote by $\mathcal{T}(x)$ the collection of x -tight sets.

For any $x \in P(f)$ and any saturated $u \in V$ define the *dependence function*

$$\text{dep}(x, u) = \{v \in V \mid \exists \beta > 0 : x + \beta(\chi_u - \chi_v) \in P(f)\}. \quad (9)$$

This can be rewritten as

$$\text{dep}(x, u) = \bigcap \{X \mid u \in X \in \mathcal{T}(x)\}. \quad (10)$$

Here, it should be noted that we have $\text{dep}(x, u) \in \mathcal{T}(x)$ if $x \in P_2(f)$ (due to Lemma 2.1) but not necessarily otherwise. If $x \in P_2(f)$, then $\text{dep}(x, u)$ is the unique minimal x -tight set containing u .

Furthermore, for any $v \in \text{dep}(x, u) \setminus \{u\}$ define

$$\tilde{c}(x, u, v) = \max\{\beta \in \mathbb{R} \mid x + \beta(\chi_u - \chi_v) \in P(f)\} > 0, \quad (11)$$

which is called the *exchange capacity* for u and $v \in \text{dep}(x, u) \setminus \{u\}$ associated with x . This can also be rewritten as

$$\tilde{c}(x, u, v) = \min\{f(X) - x(X) \mid X \in \mathcal{T}, u \in X, v \notin X\}. \quad (12)$$

The concepts of sat , \hat{c} , dep , and \tilde{c} generalize those defined for ordinary submodular polyhedra (see [5]).

For any nonempty $W \subseteq V$ and $x \in \mathbb{R}^V$ we define $x^W \in \mathbb{R}^W$ by $x^W(v) = x(v)$ for all $v \in W$. Also define (\mathcal{U}^W, f^W) to be the restriction of the k -submodular system (\mathcal{U}, f) on V to W as follows. Let $\mathcal{U}^W = \{U \cap W \mid U \in \mathcal{U}, U \cap W \neq \emptyset\}$, $\mathcal{T}^W = \{T \cap W \mid T \in \mathcal{T}\}$ and $f^W(T) = f(T)$ for all $T \in \mathcal{T}^W$. For $k' = \max\{|U| \mid U \in \mathcal{U}^W\}$, (\mathcal{U}^W, f^W) is a k' -submodular system on W . For any nonempty $T \in \mathcal{T}$, f^T is an ordinary submodular function on 2^T , which defines the associated *base polyhedron*

$$B(f^T) = \{x \in \mathbb{R}^T \mid \forall X \subset T : x(X) \leq f(X), x(T) = f(T)\}. \quad (13)$$

(See [5].)

For any $x \in \mathbb{R}^V$ define

$$\|x\|_{1,\infty} = \sum_{i=1}^n \max_{u \in U_i} |x(u)|. \quad (14)$$

This defines a norm on \mathbb{R}^V , which is a composition of ℓ_1 and ℓ_∞ norms. Our main result is a min-max theorem based on the new norm $\|\cdot\|_{1,\infty}$ on \mathbb{R}^V .

3 A Min-Max Theorem

We show the following min-max theorem.

Theorem 3.1. *For a k -submodular system (\mathcal{U}, f) on V with $f(\emptyset) = 0$ we have*

$$\min\{f(T) \mid T \in \mathcal{T}\} = \max\{-\|x\|_{1,\infty} \mid x \in P(f)\}. \quad (15)$$

Moreover, if f is integer-valued, there exists an integral x that attains the maximum of the right-hand side. \square

In order to prove Theorem 3.1 we will show some lemmas. For simplicity we write $\|\cdot\|_{1,\infty}$ as $\|\cdot\|$.

Lemma 3.2. *For any $x \in P(f)$ and $T \in \mathcal{T}$ we have*

$$f(T) \geq x(T) \geq -\|x\|. \quad (16)$$

(Proof) This easily follows from the definitions of $P(f)$ and $\|x\|$. \square

Let x^* be a maximizer of the right-hand side of (15). Because of the definition of $P(f)$ we can assume that $x^* \leq \mathbf{0}$. Recall that $u \in V$ is saturated if for every $\alpha > 0$ we have $x^* + \alpha\chi_u \notin P(f)$, and non-saturated otherwise. If $x^*(u) < 0$ for some non-saturated u , then we can make u saturated or $x^*(u) = 0$ without increasing the norm $\|x^*\|$. Hence we further assume that u is saturated for every $u \in V$ with $x^*(u) < 0$.

We fix such a maximizer x^* in the following argument.

Recall that $\mathcal{T}(x^*)$ is the collection of x^* -tight sets. It is a crucial fact that since $x^* \leq \mathbf{0}$, $\mathcal{T}(x^*)$ is closed with respect to binary operations \sqcup and \sqcap , due to Lemma 2.1.

Lemma 3.3. *For every $u \in V$ with $x^*(u) < 0$ we have $\text{dep}(x^*, u) \in \mathcal{T}(x^*)$.*

(Proof) By the assumption u is saturated and $x^* \leq \mathbf{0}$. It follows from Lemma 2.1 that $\text{dep}(x^*, u) \in \mathcal{T}(x^*)$. \square

We write $\text{dep}(x^*, u)$ as $D(u)$ for simplicity in the sequel. For any $v \in V$ let $U(v)$ be the unique set $U \in \mathcal{U}$ such that $v \in U$.

Lemma 3.4. *Suppose that $u \in V$ and $x^*(u) < 0$. Then for $v \in V$ with $D(u) \cap U(v) = \emptyset$ we have $x^*(v) = 0$ or*

$$|(D(u) \cup D(v)) \cap U_i| \neq 2 \quad (\forall i = 1, \dots, n). \quad (17)$$

(Proof) If $x^*(v) < 0$ and some U_i violates (17), then $v \in (D(u) \sqcup D(v)) \sqcap D(v) \subset D(v)$, which contradicts the minimality of $D(v)$. \square

Let u be an element of V such that $x^*(u) < 0$. Then, if for every $w \in D(u)$ we have $x^*(w) = \min\{x^*(v) \mid v \in U(w)\}$, we call u *legitimate*. Also, if for some $w \in D(u)$ we have $x^*(w) > \min\{x^*(v) \mid v \in U(w)\}$, we say u is *not legitimate with w* .

The following is a key lemma.

Lemma 3.5. *For any $U \in \mathcal{U}$ with $\min\{x^*(v) \mid v \in U\} < 0$ let W be the set of all the minimizers of $\min\{x^*(v) \mid v \in U\}$. Then there exists a legitimate $w \in W$.*

(Proof) Suppose on the contrary that no element in W is legitimate. Then, $|D(w)| > 1$ for all $w \in W$. For each $w \in W$ let w^- be an element of $D(w) \setminus \{w\}$ such that $x^*(w^-) > \min\{x^*(v) \mid v \in U(w^-)\}$. Put $z_{w^-} = x^*(w^-) - \min\{x^*(v) \mid v \in U(w^-)\}$.

Now, for each $w \in W$ there exists some (sufficiently small) $\alpha_w > 0$ such that $y_w \equiv x^* + \alpha_w(\chi_w - \chi_{w^-}) \in P(f)$ and $\alpha_w \leq \min\{z_{w^-}, -x^*(w)\}$. It follows that a convex combination y^* of y_w ($w \in W$) with positive coefficients has a norm $\|y^*\|$ smaller than $\|x^*\|$, a contradiction. \square

Now, for given x^* , we find a minimizer $T \in \mathcal{T}$ of f by the following procedure.

Procedure Find_Min

Step 1: $\tilde{U} \leftarrow \{U \in \mathcal{U} \mid \exists u \in U : x^*(u) < 0\}$,
 $T \leftarrow \emptyset$.

Step 2: While $\tilde{U} \neq \emptyset$, do the following:

- (1) Choose $U \in \tilde{U}$ and let \hat{u} be a legitimate element of U .
- (2) $T \leftarrow T \cup D(\hat{u})$,
 $\tilde{U} \leftarrow \tilde{U} \setminus \{U(v) \mid v \in D(\hat{u})\}$.

Step 3: Return T .

The following lemma completes the proof of the min-max relation in Theorem 3.1.

Lemma 3.6. *Procedure Find_Min finds $T \in \mathcal{T}$ such that $-\|x^*\| = f(T)$.*

(Proof) It follows from Lemma 3.5 we can find a legitimate \hat{u} in Step 2. Furthermore, Lemma 3.4 validates $T \in \mathcal{T}$ and T being x^* -tight. The finally obtained T satisfies that $T \cap U \neq \emptyset$ for all $U \in \mathcal{U}$ with $\min\{x^*(v) \mid v \in U\} < 0$ and that for all $u \in T$ we have $x^*(u) = \min\{x^*(v) \mid v \in U(u)\}$. Hence, $-\|x^*\| = x^*(T) = f(T)$. \square

Now we show the latter half of Theorem 3.1, the integrality property. Note that by definition $P(f)$ is hereditary, i.e., closed downward, so that there exists an integral x in $P(f)$.

Consider the following procedure.

Procedure Find_Max

Step 0: Let x be an integral non-positive vector in $P(f)$.

Step 1: While there exists a non-saturated $v \in V$ with $x(v) < 0$, do the following:

$$\alpha \leftarrow \min\{-x(v), \hat{c}(x, v)\},$$

$$x \leftarrow x + \alpha \chi_v.$$

Step 2: $\tilde{\mathcal{U}} \leftarrow \{U \in \mathcal{U} \mid \exists u \in U : x(u) < 0\}$,

$$T \leftarrow \emptyset.$$

Step 3: While $\tilde{\mathcal{U}} \neq \emptyset$, do the following:

(1) Choose $U \in \tilde{\mathcal{U}}$.

(2) Define $W = \{u \in U \mid x(u) = \min\{x(v) \mid v \in U\}\}$.

(3) Choose $u \in W$.

(3-1) If u is not legitimate with $w \in D(u) \setminus \{u\}$, then

(a) $\beta \leftarrow \min\{-x(u), \tilde{c}(x, u, w), x(w) - \min\{x(v) \mid v \in U(w)\}\}$,

(b) $x \leftarrow x + \beta(\chi_u - \chi_w)$,

(c) If $\exists v \in U : x(v) < 0$, then go to (2); else remove U from $\tilde{\mathcal{U}}$.

(3-2) If u is legitimate, then

$$T \leftarrow T \cup D(u),$$

$$\tilde{\mathcal{U}} \leftarrow \tilde{\mathcal{U}} \setminus \{U(v) \mid v \in D(u)\}.$$

Step 4: Return x .

Lemma 3.7. *Suppose f is integer-valued. Starting with an integral $x \in P(f)$ with $x \leq 0$, Procedure Find_Max finds an integral maximizer for the min-max relation in Theorem 3.1.*

(Proof) During the execution of Procedure Find_Max x remains integral. If u in (3) of Step 3 is not legitimate, $x(u)$ becomes larger, and when $|W| \geq 2$, W becomes smaller. Hence, repeating (2), (3), and (4) in Step 3, we find a legitimate u or we get x with $x(v) = 0$ for all $v \in U$. It follows that Procedure Find_Max terminates after a finite number of iterations and the finally obtained integral x and subtransversal T give max and min solutions, similarly as in the proof of Lemma 3.6. \square

This completes the proof of Theorem 3.1.

4 Extreme Points of $P_2(f)$

Huber and Kolmogorov [6] presented a characterization of extreme points of $P_2(f)$ for a k -submodular function f . In particular, as a necessary condition, they state that if $x \in \mathbb{R}^V$

is a nonzero extreme point of $P_2(f)$ then there is a nontrivial¹ chain $\emptyset = T_0 \subset T_1 \subset \cdots \subset T_k$ of elements in \mathcal{T} such that

- (i) $|T_i \setminus T_{i-1}| = 1$ for $1 \leq i \leq k$ and
- (ii) T_i is x -tight for $0 \leq i \leq k$.

We give a counterexample to this claim by showing the existence of a nonzero extreme point that does not satisfy (i).

Let $U_1 = \{v_1, v_2, v_3\}$ and $U_2 = \{u_1, u_2, u_3\}$. Let $V = U_1 \cup U_2$, $\mathcal{U} = \{U_1, U_2\}$, and M be any integer greater than 5. Define $f : \mathcal{T} \rightarrow \mathbb{R}$ by

$$\begin{aligned} f(\emptyset) &= 0, \\ f(\{v_1\}) &= -1, \quad f(\{u_1\}) = 1, \\ f(\{v_i\}) &= f(\{u_i\}) = M \quad \text{for } i = 2, 3, \\ f(\{u_1, v_1\}) &= -2, \\ f(\{u_i, v_j\}) &= f(\{u_i\}) + f(\{v_j\}) \quad \text{for } i, j = 1, 2, 3 \text{ with } (i, j) \neq (1, 1). \end{aligned}$$

Lemma 4.1. *f is k -submodular for $k = 3$.*

(Proof) Take any $T, T' \in \mathcal{T}$ and let us check $f(T) + f(T') \geq f(T \sqcup T') + f(T \cap T')$. We may assume $T \not\subseteq T'$ and $T' \not\subseteq T$. We shall use the fact that $f(\{u_i\}) + f(\{u_j\}) \geq 0$ and $f(\{v_i\}) + f(\{v_j\}) \geq 0$ for any distinct i, j .

1. If $|T| = 1$ and $|T'| = 1$, denote $T = \{x\}$ and $T' = \{y\}$.
 - If $U(x) = U(y)$, then $T \sqcup T' = \emptyset$ and $T \cap T' = \emptyset$. Thus $f(T) + f(T') = f(\{x\}) + f(\{y\}) \geq 0 = f(T \sqcup T') + f(T \cap T')$.
 - Otherwise, $T \sqcup T' = \{x, y\}$ and $T \cap T' = \emptyset$.
If $\{x, y\} = \{v_1, u_1\}$, then $f(T) + f(T') = 0 > -2 = f(T \sqcup T') + f(T \cap T')$.
If $\{x, y\} \neq \{v_1, u_1\}$, then $f(T) + f(T') = f(\{x\}) + f(\{y\}) = f(T \sqcup T') + f(T \cap T')$.
2. If $|T| = 2$ and $|T'| = 1$, denote $T = \{x, y\}$ and $T' = \{z\}$. We may assume that $U(y) = U(z)$. Then $T \sqcup T' = \{x\}$ and $T \cap T' = \emptyset$. Hence,
 - If $\{x, y\} = \{v_1, u_1\}$, then $f(T) + f(T') = -2 + M \geq \max\{f(\{v_1\}), f(\{u_1\})\} \geq f(\{x\}) = f(T \sqcup T') + f(T \cap T')$.
 - Otherwise, $f(T) + f(T') = f(\{x\}) + f(\{y\}) + f(\{z\}) \geq f(\{x\}) = f(T \sqcup T') + f(T \cap T')$.

¹By a nontrivial chain, we mean $k \geq 1$.

3. If $|T| = 2$ and $|T'| = 2$, denote $T = \{x, y\}$ and $T' = \{z, w\}$.

- If $\{x, y\} = \{v_1, u_1\}$, then $f(T \sqcup T') \leq 1$ and $f(T \cap T') \leq 1$. Therefore $f(T) + f(T') = -2 + f(\{z\}) + f(\{w\}) \geq -3 + M \geq f(T \sqcup T') + f(T \cap T')$.
- Otherwise, we may assume $\{z, w\} \neq \{v_1, u_1\}$. If $y = w$, then $T \sqcup T' = \{y\}$ and $T \cap T' = \{y\}$, and hence $f(T) + f(T') = f(\{x\}) + f(\{z\}) + 2f(\{y\}) \geq 2f(\{y\}) = f(T \sqcup T') + f(T \cap T')$. If $y \neq w$, we may assume $T \cap T' = \emptyset$. Then $T \sqcup T' = \emptyset$ and $T \cap T' = \emptyset$, and hence $f(T) + f(T') = f(\{x\}) + f(\{y\}) + f(\{z\}) + f(\{w\}) \geq 0 = f(T \sqcup T') + f(T \cap T')$. \square

Now consider the nonzero $x^* \in \mathbb{R}^V$ given by

$$\begin{aligned} x^*(v_1) &= -2, & x^*(v_2) &= 2, & x^*(v_3) &= -2, \\ x^*(u_1) &= 0, & x^*(u_2) &= 0, & x^*(u_3) &= 0. \end{aligned}$$

We can see by exhaustive checking that $x^* \in P_2(f)$ and the following equations hold.

$$\begin{aligned} x^*(\{v_1, u_1\}) &= f(\{v_1, u_1\}), \\ x^*(\{v_1, v_2\}) &= x^*(\{v_2, v_3\}) = 0, \\ x^*(\{u_1, u_2\}) &= x^*(\{u_2, u_3\}) = x^*(\{u_3, u_1\}) = 0. \end{aligned} \tag{18}$$

Since the system of six equations in (18) uniquely determines the solution x^* , x^* is an extreme point of $P_2(f)$.

Note that for any chain of elements in \mathcal{T} satisfying Condition (i), Condition (ii) is violated for $x = x^*$, since $x^*(v_i) < f(\{v_i\})$ for any v_i and $x^*(u_i) < f(\{u_i\})$ for any u_i . Hence x^* cannot be any extreme point of $P_2(f)$ that corresponds to the conditions given by Huber and Kolmogorov [6].

We have shown that the conditions provided in [6] do not give an exact characterization of extreme points of $P_2(f)$. We will give a correct characterization of extreme points of $P_2(f)$. Let (\mathcal{U}, f) be a k -submodular system on V .

We first show some lemmas.

Lemma 4.2. *For a nonempty $T \in \mathcal{T}$ let x be a vector in \mathbb{R}^V satisfying*

(A) $x^T \in B(f^T)$,

(B) *For each $u \in T$,*

(B1) *if $x(u) \geq 0$, then $x(v) = -x(u)$ for all $v \in U(u) \setminus \{u\}$;*

(B2) *otherwise,*

(1) $x(v) = x(u)$ for all $v \in U(u) \setminus \{u\}$ but one v with $x(v) = -x(u)$ or

(2) $x(v) = 0$ for all $v \in U(u) \setminus \{u\}$.

Then we have $x^Z \in P_2(f^Z)$ for $Z = \cup\{U(u) \mid u \in T\}$.

(Proof) For any $X \in \mathcal{T}$ such that $X \subseteq Z$ we have

$$\begin{aligned} x(X) &= x(X) + x(T) - f(T) \\ &\leq x(X \sqcup T) + x(X \cap T) - f(T) \\ &\leq f(X \sqcup T) + f(X \cap T) - f(T) \\ &\leq f(X). \end{aligned} \tag{19}$$

Because of the way of defining x by (B) it follows from (19) that $x^Z \in P_2(f^Z)$. \square

We also have

Lemma 4.3. *For a given $x \in P_2(f)$ and a nonempty $T \in \mathcal{T}$ suppose that $x^T \in B(f^T)$. Let $Z = \cup\{U(u) \mid u \in T\}$. For an element $u \in T$ define $y \in \mathbb{R}^Z$ by $y(v) = x(v)$ for all $v \in Z \setminus (U(u) \setminus \{u\})$ and $y(v)$ for all $v \in (U(u) \setminus \{u\})$ according to (B1) and (B2), replacing x by y , in Lemma 4.2. Then we have $y \in P_2(f^Z)$.*

(Proof) Since $x \in P_2(f)$, similarly as in (19) we can show that $y \in P_2(f^Z)$. \square

For $U \in \mathcal{U}$ consider the system of linear inequalities

$$x(u) + x(v) \leq 0 \quad (\forall \{u, v\} \in \binom{U}{2}). \tag{20}$$

Denote by C_2^U the cone of feasible solutions of (20). We call $\{u, v\}$ a *tight pair* for a feasible solution x^* if the inequality of (20) for the pair $\{u, v\}$ holds with equality for $x = x^*$.

Lemma 4.4. *Suppose $|U| \geq 3$. The cone C_2^U is pointed and its extreme rays are given by $x(u) = \alpha$ and $x(v) = -\alpha$ for all $v \in U \setminus \{u\}$ with a parameter $\alpha \geq 0$, for all $u \in U$. Every component-wise maximal solution x^* of (20) lies on an extreme ray of C_2^U and if $x^* \neq \mathbf{0}$, the set of the tight pairs for x^* forms a star with center u such that $x^*(u) > 0$.*

(Proof) Since $|U| \geq 3$, if we replace all the inequalities of (20) by equations, it gives the unique solution $x = \mathbf{0}$. Hence C_2^U is pointed. Moreover, for any component-wise maximal feasible solution x^* , if $x^* \neq \mathbf{0}$, there exists only one $u \in U$ such that $x^*(u) > 0$. Since x^* is component-wise maximal, we must have $x^*(v) = -x^*(u)$ for all $v \in U \setminus \{u\}$. Hence x^* lies on an extreme ray of C_2^U and the tight pairs form a star with center u . \square

Note that every extreme vector (lying on an extreme ray) of C_2^U is component-wise maximal.

For any subset $\mathcal{E} \subseteq \binom{U}{2}$ we regard \mathcal{E} as the edge set of an undirected graph $G = (U, \mathcal{E})$ with vertex set U .

Lemma 4.5. For any subset $\mathcal{E} \subseteq \binom{U}{2}$ the system of equations

$$x(u) + x(v) = 0 \quad (\forall \{u, v\} \in \mathcal{E}) \quad (21)$$

uniquely determines the solution $x = \mathbf{0}$ if and only if every connected component of the graph $G = (U, \mathcal{E})$ contains at least one odd cycle.

(Proof) Suppose that every connected component of the graph $G = (U, \mathcal{E})$ contains at least one odd cycle. Since equations (21) for an odd cycle determine $x(v) = 0$ for elements (vertices) v on the cycle, which then determines $x(v) = 0$ for other elements v in the same connected component.

Conversely, suppose that (21) determines the unique solution $x = \mathbf{0}$. Then we must have $\bigcup \mathcal{E} = U$. If some connected component having at least two vertices does not contain odd cycles, then it forms a bipartite graph. Hence the values $x(v)$ for vertices v in the connected component are not uniquely determined. (For, if $x(v_0)$ for a vertex v_0 of the bipartite graph is increased by α , then increasing $x(v)$ for every v at an even distance from v_0 by α and decreasing $x(v)$ for every v at an odd distance from v_0 by α keep x satisfy (21) for any $\alpha \in \mathbb{R}$.) Hence every connected component has at least one odd cycle. \square

For $x^T \in \mathbf{B}(f^T)$ define a directed graph $G_x^T = (T, A_x)$ with the vertex set T and the arc set A_x given by

$$A_x = \{(u, v) \mid u \in T, v \in \text{dep}(x, u) \setminus \{u\}\}. \quad (22)$$

Let $H_x^i = (S_x^i, B_x^i)$ ($i \in I$) be the strongly connected components of G_x^T . Choose any $w^i \in S_x^i$ for each $i \in I$. Then we call the set $W = \{w^i \mid i \in I\}$ a *covering set* of G_x^T .

It is known ([5]) that for any maximal chain of tight sets in $\mathcal{T}(x) \cap 2^T$

$$\emptyset = T_0 \subset T_1 \subset \cdots \subset T_p = T \quad (23)$$

the collection of the difference sets $T_j \setminus T_{j-1}$ ($j = 1, \dots, p$) is exactly the collection of vertex sets S_x^i ($i \in I$) of the strongly connected components of G_x^T ; in particular, $p = |I|$.

Lemma 4.6. For any $x \in \mathbf{P}_2(f)$ and nonempty $T \in \mathcal{T}$ suppose that the following three statements hold:

- (1) For every tight set $T' \in \mathcal{T}(x)$ we have $T' \subseteq \bigcup \{U(u) \mid u \in T\}$.
- (2) $x^T \in \mathbf{B}(f^T)$.
- (3) (B) in Lemma 4.2 is satisfied.

If for some $i_0 \in I$

- (a) we have $|S_x^{i_0}| \geq 2$ and

- (b) for some distinct $u, v \in S_x^{i_0}$ we have $x(u) \neq 0$ and $x(v) \neq 0$, and letting \mathcal{E}_u and \mathcal{E}_v be, respectively, the sets of all tight pairs for $U(u)$ and $U(v)$, the connected component of graph $(U(u), \mathcal{E}_u)$ containing u and that of $(U(v), \mathcal{E}_v)$ containing v are both bipartite (more specifically, stars),

then x is not an extreme point of $P_2(f)$.

(Proof) Under the assumption of the present lemma let u and v be those appearing in (b). Define

$$\begin{aligned}\alpha_1 &= \min\{|x(u)|, |x(v)|\}, \\ \alpha_2 &= \min\{f(T') - x(T') \mid T' \in \mathcal{T}, |T' \cap \{u, v\}| = 1\}.\end{aligned}$$

By the assumption we have $\alpha_1 > 0$. Also, since $u, v \in S_x^{i_0}$, we have $v \in \text{dep}(x, u)$ and $u \in \text{dep}(x, v)$, so that $\alpha_2 > 0$. Then, for a real number α such that $0 < \alpha < \min\{\alpha_1, \alpha_2\}$, put $x(u) \leftarrow x(u) \pm \alpha$ and $x(v) \leftarrow x(v) \mp \alpha$ and modify $x(z)$ for $z \in U(u) \cup U(v)$ according to (B) in Lemma 4.2. (The modification of $x(w)$ for $w \in (U(u) \setminus \{u\}) \cup (U(v) \setminus \{v\})$ according to (B) can be made because the relevant components are stars. This includes the case where the relevant component is an isolated vertex in Case (B2)(2).) Let x^+ and x^- be the obtained new points. Since $\alpha_2 \leq \min\{\tilde{c}(x, u, v), \tilde{c}(x, v, u)\}$ and since x^\pm satisfy the assumption of Lemma 4.2 because of the choice of α , we have $x^\pm \in P_2(f)$ and $x = \frac{1}{2}(x^+ + x^-)$. This completes the proof of this lemma. \square

We now show the following.

Theorem 4.7. For a given $x \in P_2(f)$, x is an extreme point of $P_2(f)$ if and only if there exists a $T \in \mathcal{T}$ such that the following (a)–(e) hold:

- (a) For every tight set $T' \in \mathcal{T}(x)$ we have $T' \subseteq \bigcup\{U(u) \mid u \in T\}$.
- (b) $x^T \in B(f^T)$.
- (c) For each $u \in T$,
- (c1) if $x(u) \geq 0$, then $x(v) = -x(u)$ for all $v \in U(u) \setminus \{u\}$;
- (c2) otherwise,
- (1) $x(v) = x(u)$ for all $v \in U(u) \setminus \{u\}$ but one v' with $x(v') = -x(u)$ or
- (2) $x(v) = 0$ for all $v \in U(u) \setminus \{u\}$.
- (d) For some covering set $W = \{w^i \mid i \in I\}$ of G_x^T with strongly connected components having vertex sets $S_x^i \subseteq T$ ($i \in I$) we have $x(v) = 0$ for all $v \in T \setminus W$. Moreover, for each $i \in I$ and $v \in S_x^i \setminus \{w^i\}$ we have $|U(v)| \geq 3$, and if values of $x(v)$ are determined by (2) of (c2), we have $|U(w^i)| \geq 4$.

(e) For all $v \in U \in \mathcal{U}$ with $U \cap T = \emptyset$ we have $x(v) = 0$. Moreover, $|U| \geq 3$ for all $U \in \mathcal{U}$ such that $U \cap T = \emptyset$.

Here Conditions (b), (c), and (d) are void if $T = \emptyset$.

(Proof) If (a)–(e) are satisfied for $x \in P_2(f)$, then we have tight equations given as follows.

$$x(T_i) = f(T_i) \text{ for a maximal chain of tight sets for } x^T, \quad (24)$$

$$x(u) + x(v) = 0 \quad (\forall u \in T, \forall v \in U(u) \setminus \{u\} \text{ in Case (c1)}), \quad (25)$$

$$x(v) + x(z) = 0 \quad (\forall u \in T, \forall \{v, z\} \in \binom{U(u)}{2} \text{ in Case (c1) with } x(u)=0), \quad (26)$$

$$x(v') + x(v) = 0 \quad (\forall u \in T, \forall v \in U(u) \setminus \{v'\} \text{ in Case (c2)(1)}), \quad (27)$$

$$x(v) + x(z) = 0 \quad (\forall i \in I, \forall \{v, z\} \in \binom{U(w^i) \setminus \{w^i\}}{2} \text{ in Case (c2)(2)}), \quad (28)$$

$$x(v) + x(z) = 0 \quad (\forall U \in \mathcal{U} \text{ with } U \cap T = \emptyset, \forall \{v, z\} \in \binom{U}{2}). \quad (29)$$

We can see that the system of equations (24)–(29) uniquely determines the solution x , due to Lemma 4.5, so that x is an extreme point of $P_2(f)$.

Conversely, suppose that $x \in P_2(f)$ is an extreme point. Then for each $u \in V$ there must exist a tight equation of type

(I) $x(T) = f(T)$ for some $T \in \mathcal{T}$ with $u \in T$ or

(II) $x(X) = 0$ for some $X \in \binom{U}{2}$ with $u \in X$ and $U \in \mathcal{U}$.

Denote by $\mathcal{T}(x)$ the collection of tight sets T of type (I) (as before) and define $W = \cup\{T \mid T \in \mathcal{T}(x)\}$.

Since $x \in P_2(f)$, we have $\text{dep}(x, u) \in \mathcal{T}(x)$ for all $u \in W$. Moreover, for any $u \in W$ and any $v \in W \setminus \cup\{U \in \mathcal{U} \mid U \cap \text{dep}(x, u) \neq \emptyset\}$ we have $\text{dep}(x, u) \cup \text{dep}(x, v) \in \mathcal{T}(x)$. Hence, similarly as in the proof of Theorem 3.1, there exists $T \in \mathcal{T}(x)$ such that $T \cap U(u) \neq \emptyset$ for all $u \in W$. Let us show that for such T , Conditions (a)–(e) are satisfied.

Firstly, (a), (b), and (e) follow from the choice of T and Lemma 4.5.

Secondly, we show (c). Fixing the values of $x(u)$ for all $u \in T$ and discarding the constraints $x(T') \leq f(T')$ for all $T' \in \mathcal{T} \setminus 2^T$, component-wise maximal vectors x satisfying (20) are exactly those determined by (c), due to Lemmas 4.4 and 4.5. Hence, if x does not satisfy (c), then defining $Z = \cup\{U(v) \mid v \in T\}$, there exist $u \in T$ and $y \in \mathbb{R}^Z$, defined appropriately as in Lemma 4.3, such that (i) $x^Z \leq y$ and (ii) $x(\hat{w}) < y(\hat{w})$ for \hat{w} with $\{\hat{w}\} = U(u) \cap T'$ for a tight set $T' \in \mathcal{T}(x)$. Since all the tight sets $T'' \in \mathcal{T}(x)$ for x are included in Z and we have $x \in P_2(f)$ and $y \in P_2(f^Z)$ because of Lemma 4.3,

defining $y^* \in \mathbb{R}^V$ by $y^*(v) = y(v)$ for all $v \in Z$ and $y^*(v) = 0$ for all $V \setminus Z$, we have for a sufficiently small positive $\epsilon > 0$

$$z_\epsilon \equiv \epsilon x + (1 - \epsilon)y^* \in P_2(f). \quad (30)$$

Then we have $z_\epsilon(\hat{w}) > x(\hat{w})$, which implies $z_\epsilon(T') > f(T')$, a contradiction. Hence (c) is satisfied.

Finally, (d) follows from Lemma 4.6. \square

In the counterexample given above, T appearing in Theorem 4.7 is $T = \{v_1, u_1\}$, graph $G_{x^*}^T$ is strongly connected, and a covering set is $W = \{v_1\}$.

It should be noted that we have assumed the membership $x \in P_2(f)$ in the characterization of extreme points, so that it is not well characterized so as to obtain extreme points efficiently.

5 Concluding Remarks

We have shown a min-max relation for k -submodular functions in terms of a new norm composed of ℓ_1 and ℓ_∞ norms, which is simpler and easier to understand than the min-max relation shown by Huber and Kolmogorov in [6] by using ℓ_1 norm alone.

We have also shown a characterization of extreme points of $P_2(f)$, a k -submodular polyhedron in the sense of Huber and Kolmogorov, which fixes a flaw in [6].

Devising a polynomial-time algorithm for minimizing k -submodular functions is left open. As pointed out in [6] and discussed here in Section 4 as well, we need a good characterization of extreme points of $P_2(f)$. A key to the good characterization is to develop a polynomial-time algorithm for linear optimization over $P_2(f)$. Main difficulty in linear optimization over $P_2(f)$ is that a polynomial-time algorithm for it requires an efficient membership algorithm for discerning whether $\mathbf{0} \in P(f)$.

Acknowledgments

The present work was supported by JSPS Grant-in-Aid for Scientific Research (B) 25280004.

References

- [1] A. Bouchet: Greedy algorithm and symmetric matroids. *Mathematical Programming* **38** (1987) 147–159.
- [2] A. Bouchet: Multimatroids I—Coverings by independent sets. *SIAM Journal on Discrete Mathematics* **10** (1997) 626–646.

- [3] A. Bouchet and W. H. Cunningham: Delta-matroids, jump systems, and bisubmodular polyhedra. *SIAM Journal on Discrete Mathematics* **8** (1995) 17–32.
- [4] S. Fujishige: A min-max theorem for bisubmodular polyhedra. *SIAM Journal on Discrete Mathematics* **10** (1997) 294–308.
- [5] S. Fujishige: *Submodular Functions and Optimization*, Second Ed., Elsevier, 2005.
- [6] A. Huber and V. Kolmogorov: Towards minimizing k -submodular functions. Proceedings of ISCO 2012, LNCS **7422** (2012) 451–462.
- [7] V. Kolmogorov: Submodularity on a tree: unifying L^{\natural} -convex and bisubmodular functions. Proceedings of MFCS 2011, LNCS **6907** (2011) 400–411.
- [8] F. Kuivinen: On the complexity of submodular function minimisation on diamonds. *Discrete Optimization* **8** (2011) 459–477.
- [9] J. Thapper and S. Živný: The power of linear programming for valued CSPs. Proceedings of FOCS 2012, IEEE (2012) 669–678.
- [10] J. Thapper and S. Živný: The complexity of finite valued CSPs. Proceedings of STOC 2013.
- [11] S. Živný: *The Complexity of Valued Constraint Satisfaction Problems*, Springer, 2012.