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Abstract

A. Huber and V. Kolmogorov (ISCO 2012) introduced a concept of k-submodular function as a generalization of ordinary submodular (set) functions and bisubmodular functions and obtained a min-max theorem for minimization of k-submodular functions. Also F. Kuivinen (2011) considered submodular functions on (product lattices of) diamonds and showed a min-max theorem for minimization of submodular functions on diamonds.

In the present paper we consider a common generalization of k-submodular functions and submodular functions on diamonds, which we call a transversal submodular function (or a t-submodular function, for short). We show a min-max theorem for minimization of t-submodular functions in terms of a new norm composed of ℓ_1 and ℓ_∞ norms. This reveals a relationship between the obtained min-max theorem and that for minimization of ordinary submodular set functions due to J. Edmonds (1970). We also show how our min-max theorem for t-submodular functions can be used to prove the min-max theorem for k-submodular functions by Huber and Kolmogorov and that for submodular functions on diamonds by Kuivinen. Moreover, we show a counterexample to a characterization, given by Huber and Kolmogorov (ISCO 2012), of extreme points of the k-submodular polyhedron and make it a correct one by fixing a flaw therein.

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1 Introduction

A. Huber and V. Kolmogorov [7] introduced a concept of k-submodular function, which is a generalization of ordinary submodular (set) functions and bisubmodular functions (see, e.g., [3, 5, 6]). Motivated by [10], Huber and Kolmogorov introduced convex polyhedra, what they call k-submodular polyhedra, associated with k-submodular functions. Earlier than Huber and Kolmogorov [7] A. Bouchet [2] also considered a class of k-submodular functions to define multimatroids as a generalization of delta-matroids [1, 3]. Kolmogorov [8] also considered a concept of tree-submodularity, which is more general than k-submodularity. It was shown in [8] that polynomial solvability of the k-submodular function minimization implies that of the tree-submodular function minimization for all trees.

Huber and Kolmogorov [7] presented a min-max theorem that characterizes the minimum of a k-submodular function in terms of ℓ_1 norm. Also F. Kuivinen [10] considered submodular functions on (product lattices of) diamonds and showed a min-max theorem for minimization of submodular functions on diamonds.

Thapper and Zivny [11] showed a dichotomy theorem that classifies the polynomial-time solvability of the minimization problems of functions on finite domains in terms of binary fractional polymorphisms (see [11, 12, 13] for the details). One of the important applications of this result is the tractability of the k-submodular function minimization problem and the minimization problem of submodular functions on lattices in the valued CSP model since its complexity was not known before. It, however, remains an open problem whether those functions can be minimized in polynomial time in the value oracle model.

In the present paper we consider a common generalization of k-submodular functions and submodular functions on diamonds, which we call a *transversal submodular function* (or a *t-submodular function*, for short). We show a min-max theorem for minimization of t-submodular functions in terms of a new norm composed of ℓ_1 and ℓ_∞ norms. This reveals a relationship between the obtained min-max theorem and that for minimization of ordinary submodular set functions due to J. Edmonds [4]. We also show how our min-max theorem for t-submodular functions can be used to prove the min-max theorem for k-submodular functions by Huber and Kolmogorov [7] and that for submodular functions on diamonds by Kuivinen [10]. Moreover, we show a counterexample to a characterization, given by Huber and Kolmogorov [7], of extreme points of the k-submodular polyhedron and make it a correct one by fixing a flaw therein.

The present paper is organized as follows. In Section 2 we introduce the concept of transversal submodular (t-submodular) function and show a min-max theorem that characterizes the minimum of a t-submodular function in terms of a new norm composed of ℓ_1 and ℓ_∞ norms. As special cases of t-submodular functions we consider k-submodular functions in Section 3 and submodular functions on lattices and, in particular, diamonds

in Section 4 in detail.

2 A Min-Max Theorem for Transversal Submodular Functions

Let V be a nonempty finite set and $\mathcal{U} \equiv \{U_1, U_2, \cdots, U_n\}$ be a partition of V. A subset $T \subseteq V$ is called a *subtransversal* (or *partial transversal*) of $\mathcal{U}$ if $|T \cap U| \leq 1$ for all $U \in \mathcal{U}$. Denote by $\mathcal{T}$ the set of all subtransversals of $\mathcal{U}$. Define for any $T \in \mathcal{T}$

$$\mathcal{U}(T) = \{U \in \mathcal{U} \mid U \cap T \neq \emptyset\}. \tag{1}$$

For any $v \in V$ define $U(v)$ to be the unique $U \in \mathcal{U}$ that contains v. Note that $\mathcal{U}(T) = \{U(v) \mid v \in T\}$ for $T \in \mathcal{T}$.

We consider two binary operations $\bigtriangledown$ and $\bigtriangleup$ on $\mathcal{T}$ satisfying the condition that for all $T_1,\ T_2 \in \mathcal{T}$

$$T_1 \bigtriangledown T_2 \in \mathcal{T}, \qquad \mathcal{U}(T_1 \bigtriangledown T_2) \subseteq \mathcal{U}(T_1) \cup \mathcal{U}(T_2), \tag{2}$$

$$T_1 \bigtriangleup T_2 \in \mathcal{T}, \qquad \mathcal{U}(T_1 \bigtriangleup T_2) \subseteq \mathcal{U}(T_1) \cap \mathcal{U}(T_2). \tag{3}$$

Define a function $f : \mathcal{T} \to \mathbb{R}$ with $f(\emptyset) = 0$ satisfying

$$f(T_1) + f(T_2) \geq f(T_1 \bigtriangledown T_2) + f(T_1 \bigtriangleup T_2) \qquad (\forall T_1, T_2 \in \mathcal{T}). \tag{4}$$

We call f a *transversal submodular function* or a *t-submodular function*, for short.

For any $x \in \mathbb{R}^V$ define

$$||x||_{1,\infty} = \sum_{i=1}^{n} \max_{u \in U_i} |x(u)|. \tag{5}$$

This defines a norm on $\mathbb{R}^V$, which is a composition of ℓ_1 and ℓ_∞ norms. Our main result is a min-max theorem based on the new norm $||\cdot||_{1,\infty}$ on $\mathbb{R}^V$.

Our general framework reveals a relationship between the general min-max relation for t-submodular functions and that for ordinary submodular set functions due to J. Edmonds [4].

Define a function $F : 2^{\mathcal{U}} \to \mathbb{R}$ as follows.

$$F(\mathcal{X}) = \min\{f(T) \mid T \in \mathcal{T},\ \mathcal{U}(T) \subseteq \mathcal{X}\} \qquad (\forall \mathcal{X} \subseteq \mathcal{U}). \tag{6}$$

Lemma 2.1. *$F : 2^{\mathcal{U}} \to \mathbb{R}$ is a submodular function with $F(\emptyset) = 0$.*

(Proof) For any $\mathcal{X},\ \mathcal{Y} \subseteq \mathcal{U}$ there exist $T_{\mathcal{X}},\ T_{\mathcal{Y}} \in \mathcal{T}$ such that

$$\mathcal{U}(T_{\mathcal{X}}) \subseteq \mathcal{X}, \quad \mathcal{U}(T_{\mathcal{Y}}) \subseteq \mathcal{Y}, \quad F(\mathcal{X}) = f(T_{\mathcal{X}}), \quad F(\mathcal{Y}) = f(T_{\mathcal{Y}}). \tag{7}$$

Hence from (2)–(7) we have

$$F(\mathcal{X})+F(\mathcal{Y}) = f(T_{\mathcal{X}})+f(T_{\mathcal{Y}}) \geq f(T_{\mathcal{X}} \bigtriangledown T_{\mathcal{Y}})+f(T_{\mathcal{X}} \bigtriangleup T_{\mathcal{Y}}) \geq F(\mathcal{X} \cup \mathcal{Y})+F(\mathcal{X} \cap \mathcal{Y}).$$

We also have $F(\emptyset) = f(\emptyset) = 0$. □

We can easily see that

$$\min\{f(T) \mid T \in \mathcal{T}\} = \min\{F(\mathcal{X}) \mid \mathcal{X} \subseteq \mathcal{U}\}, \tag{8}$$

which is equal to $F(\mathcal{U})$ since F is monotone non-increasing.

Hence we have the following.

Lemma 2.2.

$$\min\{f(T) \mid T \in \mathcal{T}\} = \max\{x(\mathcal{U}) \mid x \leq \mathbf{0},\ x \in \mathrm{P}(F)\}, \tag{9}$$

where $\mathrm{P}(F) = \{x \in \mathbb{R}^{\mathcal{U}} \mid \forall \mathcal{X} \subseteq \mathcal{U} : x(\mathcal{X}) \leq F(\mathcal{X})\}$*, the submodular polyhedron associated with submodular function* F *(see* [6]*), and* $x(\mathcal{X}) = \sum_{U \in \mathcal{X}} x(U)$.

(Proof) This follows from (8) and Edmonds' min-max theorem for submodular function minimization [4] (see [6, Corollary 3.5]). □

It should be noted that since F is monotone non-increasing, every $x \in \mathrm{P}(F)$ is non-positive, so that we may suppress the condition $x \leq \mathbf{0}$ appearing in (9).

For any $x \in \mathbb{R}^{\mathcal{U}}$ define $z_x \in \mathbb{R}^V$ by

$$z_x(v) = x(U(v)) \qquad (\forall v \in V). \tag{10}$$

Here it should be noted that $x(U(v))$ is the value of $x \in \mathbb{R}^{\mathcal{U}}$ for the coordinate $U(v) \in \mathcal{U}$.

Lemma 2.3. *Suppose we are given a nonpositive* $x \in \mathbb{R}^{\mathcal{U}}$*, i.e.,* $x \leq \mathbf{0}$*. Then, we have* $x \in \mathrm{P}(F)$ *if and only if* $z_x \in \mathrm{P}(f)$*, where*

$$\mathrm{P}(f) = \{z \in \mathbb{R}^V \mid \forall T \in \mathcal{T} : z(T)(\equiv \sum_{v \in T} z(v)) \leq f(T)\}. \tag{11}$$

(Proof) Suppose $x \in \mathrm{P}(F)$. Then, for any $T \in \mathcal{T}$

$$z_x(T) = x(\mathcal{U}(T)) \leq F(\mathcal{U}(T)) \leq f(T). \tag{12}$$

Hence $z_x \in \mathrm{P}(f)$. Conversely, suppose $z_x \in \mathrm{P}(f)$ for $x \in \mathbb{R}^{\mathcal{U}}$ with $x \leq \mathbf{0}$. Then, for any $\mathcal{X} \subseteq \mathcal{U}$ and any $T \in \mathcal{T}$ such that $\mathcal{U}(T) \subseteq \mathcal{X}$ we have

$$x(\mathcal{X}) \leq x(\mathcal{U}(T)) = z_x(T) \leq f(T), \tag{13}$$

where the first inequality holds since $x \leq \mathbf{0}$. This implies

$$x(\mathcal{X}) \leq \min\{f(T) \mid T \in \mathcal{T},\ \mathcal{U}(T) \subseteq \mathcal{X}\} = F(\mathcal{X}). \tag{14}$$

Hence $x \in \mathrm{P}(F)$. □

We are now ready to show the following.

Theorem 2.4. *For any t-submodular function f with $f(\emptyset)$ we have the following min-max relation.*

$$\min\{f(T) \mid T \in \mathcal{T}\} = \max\{-||z||_{1,\infty} \mid z \in \mathrm{P}(f)\}. \tag{15}$$

Moreover, if f is integer-valued, there exists an integral vector z that attains the maximum on the right-hand side of (15)*.*

(Proof) Denote the right-hand side of (15) by RHS. It follows from Lemmas 2.2 and 2.3 that

$$\begin{aligned}
\mathrm{RHS} &= \max\{-||z||_{1,\infty} \mid z \leq \mathbf{0},\ z \in \mathrm{P}(f)\} \\
&= \max\{-||z_x||_{1,\infty} \mid x \in \mathbb{R}^{\mathcal{U}}, x \leq \mathbf{0},\ z_x \in \mathrm{P}(f)\} \\
&= \max\{x(\mathcal{U}) \mid x \leq \mathbf{0},\ x \in \mathrm{P}(F)\} \\
&= \min\{F(\mathcal{X}) \mid \mathcal{X} \subseteq \mathcal{U}\} \\
&= \min\{f(T) \mid T \in \mathcal{T}\},
\end{aligned}$$

where the first and second equalities are due to the hereditary property of polyhedron $\mathrm{P}(f)$ and the definition of norm $||\cdot||_{1,\infty}$.

Moreover, if f is integer-valued, then so is the corresponding submodular function $F : 2^{\mathcal{U}} \to \mathbb{R}$. Therefore, there exists an integral $x \in \mathbb{R}^{\mathcal{U}}$ that attains the maximum of the right-hand side of (9) (see [4], [6, Corollary 3.5]). Then $z_x \in \mathbb{R}^V$ defined by (10) is an integral maximizer of the right-hand side of (15), due to Lemmas 2.2 and 2.3. □

It should be noted that the proof of Theorem 2.4 shows that the maximum on the right-hand side of (15) is attained by a nonpositive $z_x \in \mathbb{R}^V$ defined by (10), i.e., the maximizer z_x takes on the same nonpositive value on each $U \in \mathcal{U}$. Because of this property the general min-max relation, Theorem 2.4, implies the min-max relations shown in [7] and [10]. We will give detailed arguments in the sequel.

We consider k-submodular functions in Section 3 and submodular functions on lattices and in particular diamonds in Section 4 as special cases of t-submodular functions.

3 k-submodular functions

As an example of t-submodular functions we consider k-submodular functions due to Huber and Kolmogorov [7] and give a constructive proof of a min-max theorem. We also consider a characterization of extreme points of k-submodular polyhedron in the sense of Huber and Kolmogorov.

Let $V, \mathcal{U}$, and $\mathcal{T}$ be those appearing in Section 2.

3.1 Min-max theorems

For any $T, T' \in \mathcal{T}$ define binary operations $\sqcup$ and $\sqcap$ on $\mathcal{T}$ by

$$T \sqcup T' = (T \cup T') \setminus \bigcup \{U \in \mathcal{U} \mid |U \cap (T \cup T')| = 2\}, \qquad T \sqcap T' = T \cap T'. \quad (16)$$

Let $k = \max\{|U| \mid U \in \mathcal{U}\}$. A function $f : \mathcal{T} \to \mathbb{R}$ is called *k-submodular* if

$$f(T) + f(T') \geq f(T \sqcup T') + f(T \sqcap T') \qquad (\forall T,\ T' \in \mathcal{T}). \quad (17)$$

This definition of a k-submodular function is equivalent to that given in [7] except that $|U| = k$ for all $U \in \mathcal{U}$ there. We assume $f(\emptyset) = 0$. We can easily see that k-submodular functions are t-submodular functions with binary operations $\sqcup$ and $\sqcap$.

We call $(\mathcal{U}, f)$ a *k-submodular system* on V. Define a polyhedron

$$\mathrm{P}(f) = \{x \in \mathbb{R}^V \mid \forall T \in \mathcal{T} :\ x(T) \leq f(T)\}. \quad (18)$$

We call $\mathrm{P}(f)$ the *k-submodular polyhedron* associated with the k-submodular system $(\mathcal{U}, f)$.

Bouchet [2] considered k-submodular functions that were monotone nondecreasing and had the unit-increase property to define a set system called a *multimatroid*, a generalization of delta-matroids [1]. General k-submodular functions were considered by Huber and Kolmogorov [7]. They defined a polyhedron in a way slightly different from our $\mathrm{P}(f)$ in (18) by adding the following inequalities to those in (18).

$$\forall U \in \mathcal{U},\ \ \forall X \in \binom{U}{2} :\ \ x(X) \leq 0, \quad (19)$$

where $\binom{U}{2}$ is the set of all two-element subsets of U. We denote the "k-submodular polyhedron" in the sense of Huber and Kolmogorov by $\mathrm{P}_2(f)$, i.e.,

$$\mathrm{P}_2(f) = \{x \in \mathbb{R}^V \mid \forall T \in \mathcal{T} :\ x(T) \leq f(T),\ \forall U \in \mathcal{U},\ \forall X \in \binom{U}{2} :\ x(X) \leq 0\}. \quad (20)$$

Note that we have

$$\mathrm{P}(f) \cap \mathbb{R}^V_{\leq 0} \ = \ \mathrm{P}_2(f) \cap \mathbb{R}^V_{\leq 0} \ \subseteq \ \mathrm{P}_2(f) \ \subseteq \ \mathrm{P}(f), \quad (21)$$

where $\mathbb{R}^V_{\leq 0}$ is the set of all nonpositive vectors in $\mathbb{R}^V$.

As a corollary of Theorem 2.4 we get

Corollary 3.1. *For any k-submodular function $f : \mathcal{T} \to \mathbb{R}$ with $f(\emptyset) = 0$*

$$\min\{f(T) \mid T \in \mathcal{U}\} = \max\{-||x||_{1,\infty} \mid x \in \mathrm{P}(f)\}. \quad (22)$$

Moreover, if f is integer-valued, then there exists an integral x that attains the maximum on the right-hand side of (22)*.*

It should be noted that Corollary 3.1 also follows from the min-max relation for k-submodular functions shown by Huber and Kolmogorov [7].

For polyhedron $\mathrm{P}_2(f)$ considered by Huber and Kolmogorov [7] we have the following.

Theorem 3.2. *For any k-submodular function $f : \mathcal{T} \to \mathbb{R}$ with $f(\emptyset) = 0$*

$$\min\{f(T) \mid T \in \mathcal{U}\} = \max\{-||x||_{1,\infty} \mid x \in \mathrm{P}_2(f)\}. \tag{23}$$

Moreover, if f is integer-valued, then there exists an integral x that attains the maximum on the right-hand side of (23)*.*

(Proof) By the proof of Theorem 2.4 there exists a maximizer $x^* \in \mathrm{P}(f)$ of (22) such that for each $U \in \mathcal{U}$, $x^*(v) = -\gamma_U$ for some $\gamma_U \geq 0$ for all $v \in U$. Since $\mathrm{P}_2(f) \subseteq \mathrm{P}(f)$ and x^* also belongs to $\mathrm{P}_2(f)$, the maximum value in (23) is equal to that in (22). Similarly, the latter integrality property follows. □

3.2 Constructive proof of Corollary 3.1

In the following we give another constructive proof of Corollary 3.1, which reveals fundamental properties of k-submodular polyhedra and is interesting in its own right.

For any $x \in \mathrm{P}(f)$ (or $x \in \mathrm{P}_2(f)$) and $T \in \mathcal{T}$ we say T is *x-tight* if $x(T) = f(T)$. We can easily show the following (see [7]).

Lemma 3.3. *For any $x \in \mathrm{P}_2(f)$ and $X, Y \in \mathcal{T}$, if X and Y are x-tight, then $X \sqcup Y$ and $X \sqcap Y$ are also x-tight.* □

It should be noted that the collection of vectors $x \in \mathrm{P}(f)$ with $x \leq \mathbf{0}$ plays an important rôle in our arguments and that such vectors belong to $\mathrm{P}_2(f)$.

For any $u \in V$ and $x \in \mathrm{P}(f)$ define

$$\hat{\mathrm{c}}(x, u) = \max\{\alpha \in \mathbb{R} \mid x + \alpha\chi_u \in \mathrm{P}(f)\}, \tag{24}$$

where χ_u is the unit vector in $\mathbb{R}^V$ with $\chi_u(u) = 1$ and $\chi_u(v) = 0$ for all $v \in V \setminus \{u\}$. Note that $\hat{\mathrm{c}}(x, u)$ can be expressed as

$$\hat{\mathrm{c}}(x, u) = \min\{f(X) - x(X) \mid u \in X \in \mathcal{T}\}. \tag{25}$$

We call $\hat{\mathrm{c}}(x, u)$ the *saturation capacity* associated with x and u. If $\hat{\mathrm{c}}(x, u) = 0$, we call u *saturated*, and otherwise ($\hat{\mathrm{c}}(x, u) > 0$), *non-saturated*. Define $\mathrm{sat}(x)$ to be the set of saturated elements associated with x. We see that u is saturated if and only if there exists at least one x-tight set X such that $u \in X$. Let us denote by $\mathcal{T}(x)$ the collection of x-tight sets.

For any $x \in \mathrm{P}(f)$ and any saturated $u \in V$ define the *dependence function*

$$\mathrm{dep}(x, u) = \{v \in V \mid \exists \beta > 0 : \ x + \beta(\chi_u - \chi_v) \in \mathrm{P}(f)\}. \tag{26}$$

This can be rewritten as

$$\mathrm{dep}(x, u) = \bigcap \{X \mid u \in X \in \mathcal{T}(x)\}. \tag{27}$$

Here, it should be noted that we have $\mathrm{dep}(x, u) \in \mathcal{T}(x)$ if $x \in \mathrm{P}_2(f)$ (due to Lemma 3.3) but not necessarily otherwise. If $x \in \mathrm{P}_2(f)$, then $\mathrm{dep}(x, u)$ is the unique minimal x-tight set containing u.

Furthermore, for any $v \in \mathrm{dep}(x, u) \setminus \{u\}$ define

$$\tilde{\mathrm{c}}(x, u, v) = \max\{\beta \in \mathbb{R} \mid x + \beta(\chi_u - \chi_v) \in \mathrm{P}(f)\} \ > 0, \tag{28}$$

which is called the *exchange capacity* for u and $v \in \mathrm{dep}(x, u) \setminus \{u\}$ associated with x. This can also be rewritten as

$$\tilde{\mathrm{c}}(x, u, v) = \min\{f(X) - x(X) \mid X \in \mathcal{T}, \ u \in X, \ v \notin X\}. \tag{29}$$

The concepts of sat, $\hat{\mathrm{c}}$, dep, and $\tilde{\mathrm{c}}$ generalize those defined for ordinary submodular polyhedra (see [6]).

For any nonempty $W \subseteq V$ and $x \in \mathbb{R}^V$ we define $x^W \in \mathbb{R}^W$ by $x^W(v) = x(v)$ for all $v \in W$. Also define $(\mathcal{U}^W, f^W)$ to be the restriction of the k-submodular system $(\mathcal{U}, f)$ on V to W as follows. Let $\mathcal{U}^W = \{U \cap W \mid U \in \mathcal{U}, \ U \cap W \neq \emptyset\}$, $\mathcal{T}^W = \{T \cap W \mid T \in \mathcal{T}\}$ and $f^W(T) = f(T)$ for all $T \in \mathcal{T}^W$. For $k' = \max\{|U| \mid U \in \mathcal{U}^W\}$, $(\mathcal{U}^W, f^W)$ is a k'-submodular system on W. For any nonempty $T \in \mathcal{T}$, f^T is an ordinary submodular function on 2^T, which defines the associated *base polyhedron*

$$\mathrm{B}(f^T) = \{x \in \mathbb{R}^T \mid \forall X \subset T : x(X) \leq f(X), \ x(T) = f(T)\}. \tag{30}$$

(See [6].)

In order to prove Corollary 3.1 we will show some lemmas.

Lemma 3.4. *For any $x \in \mathrm{P}(f)$ and $T \in \mathcal{T}$ we have*

$$f(T) \geq x(T) \geq -||x||_{1,\infty}. \tag{31}$$

(Proof) This easily follows from the definitions of $\mathrm{P}(f)$ and $||x||_{1,\infty}$. □

Let x^* be a maximizer of the right-hand side of (22). Because of the definition of $\mathrm{P}(f)$ we can assume that $x^* \leq \mathbf{0}$. Recall that $u \in V$ is saturated if for every $\alpha > 0$ we have $x^* + \alpha\chi_u \notin \mathrm{P}(f)$, and non-saturated otherwise. If $x^*(u) < 0$ for some non-saturated u, then we can make u saturated or $x^*(u) = 0$ without increasing the norm $||x^*||$. Hence we further assume that u is saturated for every $u \in V$ with $x^*(u) < 0$.

We fix such a maximizer x^* in the following argument.

Recall that $\mathcal{T}(x^*)$ is the collection of x^*-tight sets. It is a crucial fact that since $x^* \leq \mathbf{0}$, $\mathcal{T}(x^*)$ is closed with respect to binary operations $\sqcup$ and $\sqcap$, due to Lemma 3.3.

Lemma 3.5. *For every $u \in V$ with $x^*(u) < 0$ we have* $\mathrm{dep}(x^*, u) \in \mathcal{T}(x^*)$.

(Proof) By the assumption u is saturated and $x^* \leq \mathbf{0}$. It follows from Lemma 3.3 that $\mathrm{dep}(x^*, u) \in \mathcal{T}(x^*)$. □

We write $\mathrm{dep}(x^*, u)$ as $D(u)$ for simplicity in the sequel. For any $v \in V$ let $U(v)$ be the unique set $U \in \mathcal{U}$ such that $v \in U$.

Lemma 3.6. *Suppose that $u \in V$ and $x^*(u) < 0$. Then for $v \in V$ with $D(u) \cap U(v) = \emptyset$ we have $x^*(v) = 0$ or*

$$|(D(u) \cup D(v)) \cap U_i| \neq 2 \qquad (\forall i = 1, \cdots, n). \tag{32}$$

(Proof) If $x^*(v) < 0$ and some U_i violates (32), then $v \in (D(u) \sqcup D(v)) \sqcap D(v) \subset D(v)$, which contradicts the minimality of $D(v)$. □

Let u be an element of V such that $x^*(u) < 0$. Then, if for every $w \in D(u)$ we have $x^*(w) = \min\{x^*(v) \mid v \in U(w)\}$, we call u *legitimate*. Also, if for some $w \in D(u)$ we have $x^*(w) > \min\{x^*(v) \mid v \in U(w)\}$, we say u *is not legitimate with* w.

The following is a key lemma.

Lemma 3.7. *For any $U \in \mathcal{U}$ with $\min\{x^*(v) \mid v \in U\} < 0$ let W be the set of all the minimizers of $\min\{x^*(v) \mid v \in U\}$. Then there exists a legitimate $w \in W$.*

(Proof) Suppose on the contrary that no element in W is legitimate. Then, $|D(w)| > 1$ for all $w \in W$. For each $w \in W$ let w^- be an element of $D(w) \setminus \{w\}$ such that $x^*(w^-) > \min\{x^*(v) \mid v \in U(w^-)\}$. Put $z_{w^-} = x^*(w^-) - \min\{x^*(v) \mid v \in U(w^-)\}$.

Now, for each $w \in W$ there exists some (sufficiently small) $\alpha_w > 0$ such that $y_w \equiv x^* + \alpha_w(\chi_w - \chi_{w^-}) \in \mathrm{P}(f)$ and $\alpha_w \leq \min\{z_{w^-}, -x^*(w)\}$. It follows that a convex combination y^* of y_w $(w \in W)$ with positive coefficients has a norm $||y^*||_{1,\infty}$ smaller than $||x^*||_{1,\infty}$, a contradiction. □

Now, for given x^*, we find a minimizer $T \in \mathcal{T}$ of f by the following procedure.

Procedure Find_Min

Step 1: $\tilde{\mathcal{U}} \leftarrow \{U \in \mathcal{U} \mid \exists u \in U : x^*(u) < 0\}$,
$T \leftarrow \emptyset$.

Step 2: While $\tilde{\mathcal{U}} \neq \emptyset$, do the following:

(1) Choose $U \in \tilde{\mathcal{U}}$ and let $\hat{u}$ be a legitimate element of U.

(2) $T \leftarrow T \cup D(\hat{u})$,
$\tilde{\mathcal{U}} \leftarrow \tilde{\mathcal{U}} \setminus \{U(v) \mid v \in D(\hat{u})\}$.

Step 3: Return T.

The following lemma completes the proof of the min-max relation in Corollary 3.1.

Lemma 3.8. Procedure Find_Min *finds* $T \in \mathcal{T}$ *such that* $-||x^*||_{1,\infty} = f(T)$.

(Proof) It follows from Lemma 3.7 we can find a legitimate $\hat{u}$ in Step 2. Furthermore, Lemma 3.6 validates $T \in \mathcal{T}$ and T being x^*-tight. The finally obtained T satisfies that $T \cap U \neq \emptyset$ for all $U \in \mathcal{U}$ with $\min\{x^*(v) \mid v \in U\} < 0$ and that for all $u \in T$ we have $x^*(u) = \min\{x^*(v) \mid v \in U(u)\}$. Hence, $-||x^*||_{1,\infty} = x^*(T) = f(T)$. □

Now we show the latter half of Corollary 3.1, the integrality property. Note that by definition $\mathrm{P}(f)$ is hereditary, i.e., closed downward, so that there exists an integral x in $\mathrm{P}(f)$.

Consider the following procedure.

Procedure Find_Max

Step 0: Let x be an integral non-positive vector in $\mathrm{P}(f)$.

Step 1: While there exists a non-saturated $v \in V$ with $x(v) < 0$, do the following:
$\alpha \leftarrow \min\{-x(v), \hat{\mathrm{c}}(x, v)\}$,
$x \leftarrow x + \alpha\chi_v$.

Step 2: $\tilde{\mathcal{U}} \leftarrow \{U \in \mathcal{U} \mid \exists u \in U : x(u) < 0\}$,
$T \leftarrow \emptyset$.

Step 3: While $\tilde{\mathcal{U}} \neq \emptyset$, do the following:
(1) Choose $U \in \tilde{\mathcal{U}}$.
(2) Define $W = \{u \in U \mid x(u) = \min\{x(v) \mid v \in U\}\}$.
(3) Choose $u \in W$.
(3-1) If u is not legitimate with $w \in D(u) \setminus \{u\}$, then
(a) $\beta \leftarrow \min\{-x(u), \tilde{\mathrm{c}}(x, u, w), x(w) - \min\{x(v) \mid v \in U(w)\}\}$,
(b) $x \leftarrow x + \beta(\chi_u - \chi_w)$,
(c) If $\exists v \in U : x(v) < 0$, then go to (2); else remove U from $\tilde{\mathcal{U}}$.
(3-2) If u is legitimate, then
$T \leftarrow T \cup D(u)$,
$\tilde{\mathcal{U}} \leftarrow \tilde{\mathcal{U}} \setminus \{U(v) \mid v \in D(u)\}$.

Step 4: Return x.

Lemma 3.9. *Suppose f is integer-valued. Starting with an integral $x \in \mathrm{P}(f)$ with $x \leq \mathbf{0}$,* Procedure Find_Max *finds an integral maximizer for the min-max relation in Corollary* 3.1.

(Proof) During the execution of Procedure Fin_Max x remains integral. If u in (3) of Step 3 is not legitimate, $x(u)$ becomes larger, and when $|W| \geq 2$, W becomes smaller. Hence, repeating (2), (3), and (4) in Step 3, we find a legitimate u or we get x with

$x(v) = 0$ for all $v \in U$. It follows that Procedure Find_Max terminates after a finite number of iterations and the finally obtained integral x and subtransversal T give max and min solutions, similarly as in the proof of Lemma 3.8. □

This completes a constructive proof of Corollary 3.1.

3.3 Extreme Points of $\mathrm{P}_2(f)$

Huber and Kolmogorov [7] presented a characterization of extreme points of $\mathrm{P}_2(f)$ for a k-submodular function f. In particular, as a necessary condition, they state that if $x \in \mathbb{R}^V$ is a nonzero extreme point of $\mathrm{P}_2(f)$ then there is a nontrivial chain $\emptyset = T_0 \subset T_1 \subset \cdots \subset T_k$ of elements in $\mathcal{T}$ such that

(i) $|T_i \setminus T_{i-1}| = 1$ for $1 \leq i \leq k$ and

(ii) T_i is x-tight for $0 \leq i \leq k$.

We give a counterexample to this claim by showing the existence of a nonzero extreme point that does not satisfy (i).

Let $U_1 = \{v_1, v_2, v_3\}$ and $U_2 = \{u_1, u_2, u_3\}$. Let $V = U_1 \cup U_2$, $\mathcal{U} = \{U_1, U_2\}$, and M be any integer greater than 5. Define $f : \mathcal{T} \to \mathbb{R}$ by

$$
\begin{aligned}
&f(\emptyset) = 0,\\
&f(\{v_1\}) = -1, \;\; f(\{u_1\}) = 1,\\
&f(\{v_i\}) = f(\{u_i\}) = M \;\text{ for }\; i = 2, 3,\\
&f(\{u_1, v_1\}) = -2,\\
&f(\{u_i, v_j\}) = f(\{u_i\}) + f(\{v_j\}) \;\text{ for }\; i, j = 1, 2, 3 \;\text{ with }\; (i, j) \neq (1, 1).
\end{aligned}
$$

Lemma 3.10. *f is k-submodular for $k = 3$.*

(Proof) Take any $T, T' \in \mathcal{T}$ and let us check $f(T) + f(T') \geq f(T \sqcup T') + f(T \sqcap T')$. We may assume $T \not\subset T'$ and $T' \not\subset T$. We shall use the fact that $f(\{u_i\}) + f(\{u_j\}) \geq 0$ and $f(\{v_i\}) + f(\{v_j\}) \geq 0$ for any distinct i, j.

1. If $|T| = 1$ and $|T'| = 1$, denote $T = \{x\}$ and $T' = \{y\}$.

   - If $U(x) = U(y)$, then $T \sqcup T' = \emptyset$ and $T \sqcap T' = \emptyset$. Thus $f(T) + f(T') = f(\{x\}) + f(\{y\}) \geq 0 = f(T \sqcup T') + f(T \sqcap T')$.
   - Otherwise, $T \sqcup T' = \{x, y\}$ and $T \sqcap T' = \emptyset$.
     If $\{x, y\} = \{v_1, u_1\}$, then $f(T) + f(T') = 0 > -2 = f(T \sqcup T') + f(T \sqcap T')$.
     If $\{x, y\} \neq \{v_1, u_1\}$, then $f(T) + f(T') = f(\{x\}) + f(\{y\}) = f(T \sqcup T') + f(T \sqcap T')$.

By a nontrivial chain, we mean $k \geq 1$.

2. If $|T| = 2$ and $|T'| = 1$, denote $T = \{x, y\}$ and $T' = \{z\}$. We may assume that $U(y) = U(z)$. Then $T \sqcup T' = \{x\}$ and $T \sqcap T' = \emptyset$. Hence,

   - If $\{x, y\} = \{v_1, u_1\}$, then $f(T)+f(T') = -2+M \geq \max\{f(\{v_1\}), f(\{u_1\})\} \geq f(\{x\}) = f(T \sqcup T') + f(T \sqcap T')$.
   - Otherwise, $f(T) + f(T') = f(\{x\}) + f(\{y\}) + f(\{z\}) \geq f(\{x\}) = f(T \sqcup T') + f(T \sqcap T')$.

3. If $|T| = 2$ and $|T'| = 2$, denote $T = \{x, y\}$ and $T' = \{z, w\}$.

   - If $\{x, y\} = \{v_1, u_1\}$, then $f(T \sqcup T') \leq 1$ and $f(T \sqcap T') \leq 1$. Therefore $f(T)+f(T') = -2+f(\{z\})+f(\{w\}) \geq -3+M \geq f(T \sqcup T')+f(T \sqcap T')$.
   - Otherwise, we may assume $\{z, w\} \neq \{v_1, u_1\}$. If $y = w$, then $T \sqcup T' = \{y\}$ and $T \sqcap T = \{y\}$, and hence $f(T) + f(T') = f(\{x\}) + f(\{z\}) + 2f(\{y\}) \geq 2f(\{y\}) = f(T \sqcup T') + f(T \sqcap T')$. If $y \neq w$, we may assume $T \cap T' = \emptyset$. Then $T \sqcup T' = \emptyset$ and $T \sqcap T' = \emptyset$, and hence $f(T) + f(T') = f(\{x\}) + f(\{y\}) + f(\{z\}) + f(\{w\}) \geq 0 = f(T \sqcup T') + f(T \sqcap T')$. □

Now consider the nonzero $x^* \in \mathbb{R}^V$ given by

$$x^*(v_1) = -2, \;\; x^*(v_2) = 2, \;\; x^*(v_3) = -2,$$
$$x^*(u_1) = 0, \;\; x^*(u_2) = 0, \;\; x^*(u_3) = 0.$$

We can see by exhaustive checking that $x^* \in \mathrm{P}_2(f)$ and the following equations hold.

$$\begin{aligned} &x^*(\{v_1, u_1\}) = f(\{v_1, u_1\}), \\ &x^*(\{v_1, v_2\}) = x^*(\{v_2, v_3\}) = 0, \\ &x^*(\{u_1, u_2\}) = x^*(\{u_2, u_3\}) = x^*(\{u_3, u_1\}) = 0. \end{aligned} \tag{33}$$

Since the system of six equations in (33) uniquely determines the solution x^*, x^* is an extreme point of $\mathrm{P}_2(f)$.

Note that for any chain of elements in $\mathcal{T}$ satisfying Condition (i), Condition (ii) is violated for $x = x^*$, since $x^*(v_i) < f(\{v_i\})$ for any v_i and $x^*(u_i) < f(\{u_i\})$ for any u_i. Hence x^* cannot be any extreme point of $\mathrm{P}_2(f)$ that corresponds to the conditions given by Huber and Kolmogorov [7].

We have shown that the conditions provided in [7] do not give an exact characterization of extreme points of $\mathrm{P}_2(f)$. We will give a correct characterization of extreme points of $\mathrm{P}_2(f)$. Let $(\mathcal{U}, f)$ be a k-submodular system on V.

We first show some lemmas.

Lemma 3.11. *For a nonempty $T \in \mathcal{T}$ let x be a vector in $\mathbb{R}^V$ satisfying*

(A) $x^T \in \mathrm{B}(f^T)$,

(B) *For each* $u \in T$,
(B1) *if* $x(u) \geq 0$, *then* $x(v) = -x(u)$ *for all* $v \in U(u) \setminus \{u\}$;
(B2) *otherwise,*
(1) $x(v) = x(u)$ *for all* $v \in U(u) \setminus \{u\}$ *but one* v *with* $x(v) = -x(u)$ *or*
(2) $x(v) = 0$ *for all* $v \in U(u) \setminus \{u\}$.

Then we have $x^Z \in \mathrm{P}_2(f^Z)$ *for* $Z = \cup\{U(u) \mid u \in T\}$.

(Proof) For any $X \in \mathcal{T}$ such that $X \subseteq Z$ we have

$$\begin{aligned} x(X) &= x(X) + x(T) - f(T) \\ &\leq x(X \sqcup T) + x(X \sqcap T) - f(T) \\ &\leq f(X \sqcup T) + f(X \sqcap T) - f(T) \\ &\leq f(X). \end{aligned} \tag{34}$$

Because of the way of defining x by (B) it follows from (34) that $x^Z \in \mathrm{P}_2(f^Z)$. □

We also have

Lemma 3.12. *For a given* $x \in \mathrm{P}_2(f)$ *and a nonempty* $T \in \mathcal{T}$ *suppose that* $x^T \in \mathrm{B}(f^T)$. *Let* $Z = \bigcup\{U(u) \mid u \in T\}$. *For an element* $u \in T$ *define* $y \in \mathbb{R}^Z$ *by* $y(v) = x(v)$ *for all* $v \in Z \setminus (U(u) \setminus \{u\})$ *and* $y(v)$ *for all* $v \in (U(u) \setminus \{u\})$ *according to* (B1) *and* (B2), *replacing* x *by* y, *in Lemma* 3.11. *Then we have* $y \in \mathrm{P}_2(f^Z)$.

(Proof) Since $x \in \mathrm{P}_2(f)$, similarly as in (34) we can show that $y \in \mathrm{P}_2(f^Z)$. □

For $U \in \mathcal{U}$ consider the system of linear inequalities

$$x(u) + x(v) \leq 0 \qquad (\forall \{u, v\} \in \binom{U}{2}). \tag{35}$$

Denote by C_2^U the cone of feasible solutions of (35). We call $\{u, v\}$ *a tight pair* for a feasible solution x^* if the inequality of (35) for the pair $\{u, v\}$ holds with equality for $x = x^*$.

Lemma 3.13. *Suppose* $|U| \geq 3$. *The cone* C_2^U *is pointed and its extreme rays are given by* $x(u) = \alpha$ *and* $x(v) = -\alpha$ *for all* $v \in U \setminus \{u\}$ *with a parameter* $\alpha \geq 0$, *for all* $u \in U$. *Every component-wise maximal solution* x^* *of* (35) *lies on an extreme ray of* C_2^U *and if* $x^* \neq \mathbf{0}$, *the set of the tight pairs for* x^* *forms a star with center* u *such that* $x^*(u) > 0$.

(Proof) Since $|U| \geq 3$, if we replace all the inequalities of (35) by equations, it gives the unique solution $x = \mathbf{0}$. Hence C_2^U is pointed. Moreover, for any component-wise maximal feasible solution x^*, if $x^* \neq \mathbf{0}$, there exists only one $u \in U$ such that $x^*(u) > 0$. Since x^* is component-wise maximal, we must have $x^*(v) = -x^*(u)$ for all $v \in U \setminus \{u\}$. Hence x^* lies on an extreme ray of C_2^U and the tight pairs form a star with center u. □

Note that every extreme vector (lying on an extreme ray) of C_2^U is component-wise maximal.

For any subset $\mathcal{E} \subseteq \binom{U}{2}$ we regard $\mathcal{E}$ as the edge set of an undirected graph $G = (U, \mathcal{E})$ with vertex set U.

Lemma 3.14. *For any subset $\mathcal{E} \subseteq \binom{U}{2}$ the system of equations*

$$x(u) + x(v) = 0 \qquad (\forall \{u, v\} \in \mathcal{E}) \tag{36}$$

uniquely determines the solution $x = \mathbf{0}$ if and only if every connected component of the graph $G = (U, \mathcal{E})$ contains at least one odd cycle.

(Proof) Suppose that every connected component of the graph $G = (U, \mathcal{E})$ contains at least one odd cycle. Since equations (36) for an odd cycle determine $x(v) = 0$ for elements (vertices) v on the cycle, which then determines $x(v) = 0$ for other elements v in the same connected component.

Conversely, suppose that (36) determines the unique solution $x = \mathbf{0}$. Then we must have $\bigcup \mathcal{E} = U$. If some connected component having at least two vertices does not contain odd cycles, then it forms a bipartite graph. Hence the values $x(v)$ for vertices v in the connected component are not uniquely determined. (For, if $x(v_0)$ for a vertex v_0 of the bipartite graph is increased by α, then increasing $x(v)$ for every v at an even distance from v_0 by α and decreasing $x(v)$ for every v at an odd distance from v_0 by α keep x satisfy (36) for any $\alpha \in \mathbb{R}$.) Hence every connected component has at least one odd cycle. □

For $x^T \in \mathbf{B}(f^T)$ define a directed graph $G_x^T = (T, A_x)$ with the vertex set T and the arc set A_x given by

$$A_x = \{(u, v) \mid u \in T,\ v \in \mathrm{dep}(x, u) \setminus \{u\}\}. \tag{37}$$

Let $H_x^i = (S_x^i, B_x^i)$ $(i \in I)$ be the strongly connected components of G_x^T. Choose any $w^i \in S_x^i$ for each $i \in I$. Then we call the set $W = \{w^i \mid i \in I\}$ a *covering set* of G_x^T.

It is known ([6]) that for any maximal chain of tight sets in $\mathcal{T}(x) \cap 2^T$

$$\emptyset = T_0 \subset T_1 \subset \cdots \subset T_p = T \tag{38}$$

the collection of the difference sets $T_j \setminus T_{j-1}$ $(j = 1, \cdots, p)$ is exactly the collection of vertex sets S_x^i $(i \in I)$ of the strongly connected components of G_x^T; in particular, $p = |I|$.

Lemma 3.15. *For any $x \in \mathrm{P}_2(f)$ and nonempty $T \in \mathcal{T}$ suppose that the following three statements hold*:

(1) *For every tight set $T' \in \mathcal{T}(x)$ we have $T' \subseteq \bigcup \{U(u) \mid u \in T\}$.*

(2) $x^T \in \mathrm{B}(f^T)$.

(3) (B) *in Lemma* 3.11 *is satisfied.*

If for some $i_0 \in I$

(a) *we have* $|S_x^{i_0}| \geq 2$ *and*

(b) *for some distinct* $u, v \in S_x^{i_0}$ *we have* $x(u) \neq 0$ *and* $x(v) \neq 0$*, and letting* $\mathcal{E}_u$ *and* $\mathcal{E}_v$ *be, respectively, the sets of all tight pairs for* $U(u)$ *and* $U(v)$*, the connected component of graph* $(U(u), \mathcal{E}_u)$ *containing* u *and that of* $(U(v), \mathcal{E}_v)$ *containing* v *are both bipartite* (*more specifically, stars*)*,*

then x *is not an extreme point of* $\mathrm{P}_2(f)$.

(Proof) Under the assumption of the present lemma let u and v be those appearing in (b). Define

$$\alpha_1 = \min\{|x(u)|, |x(v)|\},$$
$$\alpha_2 = \min\{f(T') - x(T') \mid T' \in \mathcal{T},\ |T' \cap \{u, v\}| = 1\}.$$

By the assumption we have $\alpha_1 > 0$. Also, since $u, v \in S_x^{i_0}$, we have $v \in \mathrm{dep}(x, u)$ and $u \in \mathrm{dep}(x, v)$, so that $\alpha_2 > 0$. Then, for a real number α such that $0 < \alpha < \min\{\alpha_1, \alpha_2\}$, put $x(u) \leftarrow x(u) \pm \alpha$ and $x(v) \leftarrow x(v) \mp \alpha$ and modify $x(z)$ for $z \in U(u) \cup U(v)$ according to (B) in Lemma 3.11. (The modification of $x(w)$ for $w \in (U(u) \setminus \{u\}) \cup (U(v) \setminus \{v\})$ according to (B) can be made because the relevant components are stars. This includes the case where the relevant component is an isolated vertex in Case (B2)(2).) Let x^+ and x^- be the obtained new points. Since $\alpha_2 \leq \min\{\tilde{\mathrm{c}}(x, u, v), \tilde{\mathrm{c}}(x, v, u)\}$ and since $x^\pm$ satisfy the assumption of Lemma 3.11 because of the choice of α, we have $x^\pm \in \mathrm{P}_2(f)$ and $x = \frac{1}{2}(x^+ + x^-)$. This completes the proof of this lemma. □

We now show the following.

Theorem 3.16. *For a given* $x \in \mathrm{P}_2(f)$*,* x *is an extreme point of* $\mathrm{P}_2(f)$ *if and only if there exists a* $T \in \mathcal{T}$ *such that the following* (a)–(e) *hold*:

(a) *For every tight set* $T' \in \mathcal{T}(x)$ *we have* $T' \subseteq \bigcup\{U(u) \mid u \in T\}$.

(b) $x^T \in \mathrm{B}(f^T)$.

(c) *For each* $u \in T$*,*
(c1) *if* $x(u) \geq 0$*, then* $x(v) = -x(u)$ *for all* $v \in U(u) \setminus \{u\}$;
(c2) *otherwise,*
(1) $x(v) = x(u)$ *for all* $v \in U(u) \setminus \{u\}$ *but one* v' *with* $x(v') = -x(u)$ *or*
(2) $x(v) = 0$ *for all* $v \in U(u) \setminus \{u\}$.

(d) *For some covering set* $W = \{w^i \mid i \in I\}$ *of* G_x^T *with strongly connected components having vertex sets* $S_x^i \subseteq T$ $(i \in I)$ *we have* $x(v) = 0$ *for all* $v \in T \setminus W$. *Moreover, for each* $i \in I$ *and* $v \in S_x^i \setminus \{w^i\}$ *we have* $|U(v)| \geq 3$, *and if values of* $x(v)$ *are determined by* (2) *of* (c2), *we have* $|U(w^i)| \geq 4$.

(e) *For all* $v \in U \in \mathcal{U}$ *with* $U \cap T = \emptyset$ *we have* $x(v) = 0$. *Moreover,* $|U| \geq 3$ *for all* $U \in \mathcal{U}$ *such that* $U \cap T = \emptyset$.

Here Conditions (b), (c), *and* (d) *are void if* $T = \emptyset$.

(Proof) If (a)–(e) are satisfied for $x \in \mathrm{P}_2(f)$, then we have tight equations given as follows.

$$x(T_i) = f(T_i) \text{ for a maximal chain of tight sets for } x^T, \tag{39}$$
$$x(u) + x(v) = 0 \quad (\forall u \in T,\ \forall v \in U(u) \setminus \{u\} \text{ in Case (c1)}), \tag{40}$$
$$x(v) + x(z) = 0\ (\forall u \in T,\ \forall \{v, z\} \in \binom{U(u)}{2} \text{ in Case (c1) with } x(u){=}0), \tag{41}$$
$$x(v') + x(v) = 0\ \ (\forall u \in T,\ \forall v \in U(u) \setminus \{v'\} \text{ in Case (c2)(1)}), \tag{42}$$
$$x(v) + x(z) = 0\ \ (\forall i \in I,\ \forall \{v, z\} \in \binom{U(w^i) \setminus \{w^i\}}{2} \text{ in Case (c2)(2)}), \tag{43}$$
$$x(v) + x(z) = 0 \quad (\forall U \in \mathcal{U} \text{ with } U \cap T = \emptyset,\ \forall \{v, z\} \in \binom{U}{2}). \tag{44}$$

We can see that the system of equations (39)–(44) uniquely determines the solution x, due to Lemma 3.14, so that x is an extreme point of $\mathrm{P}_2(f)$.

Conversely, suppose that $x \in \mathrm{P}_2(f)$ is an extreme point. Then for each $u \in V$ there must exist a tight equation of type

(I) $x(T) = f(T)$ for some $T \in \mathcal{T}$ with $u \in T$ or

(II) $x(X) = 0$ for some $X \in \binom{U}{2}$ with $u \in X$ and $U \in \mathcal{U}$.

Denote by $\mathcal{T}(x)$ the collection of tight sets T of type (I) (as before) and define $W = \cup\{T \mid T \in \mathcal{T}(x)\}$.

Since $x \in \mathrm{P}_2(f)$, we have $\mathrm{dep}(x, u) \in \mathcal{T}(x)$ for all $u \in W$. Moreover, for any $u \in W$ and any $v \in W \setminus \cup\{U \in \mathcal{U} \mid U \cap \mathrm{dep}(x, u) \neq \emptyset\}$ we have $\mathrm{dep}(x, u) \cup \mathrm{dep}(x, v) \in \mathcal{T}(x)$. Hence, similarly as in the constructive proof of Corollary 3.1, there exists $T \in \mathcal{T}(x)$ such that $T \cap U(u) \neq \emptyset$ for all $u \in W$. Let us show that for such T, Conditions (a)–(e) are satisfied.

Firstly, (a), (b), and (e) follow from the choice of T and Lemma 3.14.

Secondly, we show (c). Fixing the values of $x(u)$ for all $u \in T$ and discarding the constraints $x(T') \leq f(T')$ for all $T' \in \mathcal{T} \setminus 2^T$, component-wise maximal vectors x

satisfying (35) are exactly those determined by (c), due to Lemmas 3.13 and 3.14. Hence, if x does not satisfy (c), then defining $Z = \bigcup\{U(v) \mid v \in T\}$, there exist $u \in T$ and $y \in \mathbb{R}^Z$, defined appropriately as in Lemma 3.12, such that (i) $x^Z \leq y$ and (ii) $x(\hat{w}) < y(\hat{w})$ for $\hat{w}$ with $\{\hat{w}\} = U(u) \cap T'$ for a tight set $T' \in \mathcal{T}(x)$. Since all the tight sets $T'' \in \mathcal{T}(x)$ for x are included in Z and we have $x \in \mathrm{P}_2(f)$ and $y \in \mathrm{P}_2(f^Z)$ because of Lemma 3.12, defining $y^* \in \mathbb{R}^V$ by $y^*(v) = y(v)$ for all $v \in Z$ and $y^*(v) = 0$ for all $V \setminus Z$, we have for a sufficiently small positive $\epsilon > 0$

$$z_\epsilon \equiv \epsilon x + (1-\epsilon)y^* \in \mathrm{P}_2(f). \tag{45}$$

Then we have $z_\epsilon(\hat{w}) > x(\hat{w})$, which implies $z_\epsilon(T') > f(T')$, a contradiction. Hence (c) is satisfied.

Finally, (d) follows from Lemma 3.15. □

In the counterexample given above, T appearing in Theorem 3.16 is $T = \{v_1, u_1\}$, graph $G^T_{x^*}$ is strongly connected, and a covering set is $W = \{v_1\}$.

It should be noted that we have assumed the membership $x \in \mathrm{P}_2(f)$ in the characterization of extreme points, so that it is not well characterized so as to obtain extreme points efficiently.

4 Submodular functions on lattices

As another example of t-submodular functions we consider submodular functions on lattices and, in particular, diamonds.

Let $V, \mathcal{U}$, and $\mathcal{T}$ be those appearing in Section 2.

4.1 Min-max theorems

For each $U \in \mathcal{U}$ let 0_U be a new element and put $\hat{U} = U \cup \{0_U\}$. Suppose that for each $U \in \mathcal{U}$ we are given an arbitrary lattice $\mathcal{L}_U = (\hat{U}, \vee_U, \wedge_U)$ with lattice operations, join $\vee_U$ and meet $\wedge_U$, where 0_U is the minimum element of $\mathcal{L}_U$. Denote by 1_U the maximum element of $\mathcal{L}_U$.

Let $\mathcal{L} = \otimes_{U\in\mathcal{U}}\mathcal{L}_U (= (\otimes_{U\in\mathcal{U}}\hat{U}, \vee, \wedge))$ be the product of lattices $\mathcal{L}_U = (\hat{U}, \vee_U, \wedge_U)$ for $U \in \mathcal{U}$. A function $f : \otimes_{U\in\mathcal{U}}\hat{U} \to \mathbb{R}$ is called a *submodular function* on $\mathcal{L}$ if

$$f(\hat{T}) + f(\hat{T}') \geq f(\hat{T} \vee \hat{T}') + f(\hat{T} \wedge \hat{T}')$$

for all $\hat{T}, \hat{T}' \in \otimes_{U\in\mathcal{U}}\hat{U}$.

This function can be seen as a special case of t-submodular functions as follows. Note that every subtransversal $T \in \mathcal{T}$ is identified with the unique $\hat{T} \in \otimes_{U\in\mathcal{U}}\hat{U}$ satisfying

$$\hat{T} \cap \hat{U} = \begin{cases} T \cap U & \text{if } T \cap U \neq \emptyset \\ \{0_U\} & \text{if } T \cap U = \emptyset \end{cases} \qquad (\forall U \in \mathcal{U}). \tag{46}$$

Also define for any $T, T' \in \mathcal{T}$

$$T \vee_0 T' = (\hat{T} \vee \hat{T}') \cap V, \qquad T \wedge_0 T' = (\hat{T} \wedge \hat{T}') \cap V. \tag{47}$$

For a submodular function f on $\mathcal{L}$ we can identify f with a function $\bar{f}$ on $\mathcal{T}$ defined by

$$\bar{f}(T) = f(\hat{T}) \qquad (\forall T \in \mathcal{T}). \tag{48}$$

Hence we have function $\bar{f}$ satisfying

$$\bar{f}(T) + \bar{f}(T') \geq \bar{f}(T \vee_0 T') + \bar{f}(T \wedge_0 T') \qquad (\forall T,\ T' \in \mathcal{T}). \tag{49}$$

We can easily see that $\bar{f}$ is a t-submodular function with respect to binary operations $\vee_0$ and $\wedge_0$ (i.e., (2) and (3) are satisfied for $\bigtriangledown = \vee_0$ and $\bigtriangleup = \wedge_0$).

Define

$$\mathrm{P}(\bar{f}) = \{x \in \mathbb{R}^V \mid \forall T \in \mathcal{T} : x(T) \leq \bar{f}(T)\}. \tag{50}$$

Here we assume $\bar{f}(\emptyset) = 0$.

As a corollary of Theorem 2.4 we obtain the following.

Corollary 4.1. *For any submodular function f on the product of lattices with $\bar{f}(\emptyset) = 0$*

$$\min\{\bar{f}(T) \mid T \in \mathcal{U}\} = \max\{-||x||_{1,\infty} \mid x \in \mathrm{P}(\bar{f})\}. \tag{51}$$

Moreover, if $\bar{f}$ is integer-valued, then there exists an integral x that attains the maximum on the right-hand side of (51)*.*

It should be noted that because of the proof of Theorem 2.4 there exists a maximizer x of the right-hand side of (51) such that $x(u) = x(v)$ for all $u, v \in U$ and all $U \in \mathcal{U}$ and that such a maximizer x can be integral if $\bar{f}$ is integer-valued.

Motivated by a result by Kuivinen [10] (which will be examined in the next subsection), we consider the following additional constraint:

(K1′) For each $U \in \mathcal{U}$, $x(u) + x(v) \leq x(u \vee_U v) + x(u \wedge_U v)$ for all $\{u, v\} \in \binom{\hat{U}}{2}$, where $x(0_U) = 0$ for all $U \in \mathcal{U}$.

We define an associated polyhedron $\mathrm{P}'(\bar{f})$ by

$$\mathrm{P}'(\bar{f}) = \{x \mid x \in \mathrm{P}(\bar{f}),\ \text{(K1}')\}. \tag{52}$$

Since the maximizer x to be used in the proof of Corollary 4.1 as a specialization of the proof of Theorem 2.4 satisfies (K1′), we also get

Corollary 4.2. *For any submodular function f on the product of lattices with $\bar{f}(\emptyset) = 0$*

$$\min\{\bar{f}(T) \mid T \in \mathcal{U}\} = \max\{-||x||_{1,\infty} \mid x \in \mathrm{P}'(\bar{f})\}. \tag{53}$$

Moreover, if $\bar{f}$ is integer-valued, then there exists an integral x that attains the maximum on the right-hand side of (53)*.*

For any $x \in \mathrm{P}(\bar{f})$ we call $T \in \mathcal{T}$ *x-tight* if $x(T) = \bar{f}(T)$. The following lemma will frequently be used later.

Lemma 4.3. *Suppose we are given a vector $x \in \mathbb{R}^V$ satisfying* (K1′)*. Then, x regarded as a function on $\mathcal{T}$ is supermodular, i.e., $x(T) + x(T') \leq x(T \vee_0 T') + x(T \wedge_0 T')$ for all $T, T' \in \mathcal{T}$.*

Moreover, if $x \in \mathrm{P}'(\bar{f})$ and $T, T' \in \mathcal{T}$ are x-tight, then $T \vee_0 T'$ and $T \wedge_0 T'$ are also x-tight, and for each $U \in \mathcal{U}$, $x(u)+x(v) = x(u \wedge_U v)+x(u \vee_U v)$ holds for all $u \in \hat{T} \cap \hat{U}$ and $v \in \hat{T}' \cap \hat{U}$, where $x(0_U) = 0$.

(Proof) Since x satisfies (K1′), x is supermodular on $\mathcal{T}$. It then follows that $\bar{f} - x$ is submodular and nonnegative when $x \in \mathrm{P}'(\bar{f})$. Hence, the latter part of this lemma holds, where modularity follows from submodularity of $\bar{f}$ and supermodularity of x. □

4.2 Submodular functions on diamonds

Corollary 4.2 looks like a straightforward consequence of Theorem 2.4, but it turns out that it already gives a good characterization in the case when each $\mathcal{L}_U$ is a diamond. Before showing it (in Section 4.2.2) let us first examine its connection with a result by Kuivinen [10].

4.2.1 Kuivinen's min-max theorem

We assume that $|U| \geq 3$ and all the elements in $U \setminus \{1_U\}$ are incomparable in $\mathcal{L}_U$ for each $U \in \mathcal{U}$. Then lattice $\mathcal{L}_U$ on $\hat{U} = U \cup \{0_U\}$ is called a *diamond*. We assume that for each $U \in \mathcal{U}$ $\mathcal{L}_U$ is a diamond.

Corollary 4.2 gives a min-max formula for a submodular function on the product lattice of diamonds. Note that in this special case (K1′) is simplified to

(K1′) For each $U \in \mathcal{U}$, $x(u) + x(v) \leq x(1_U)$ for all $\{u, v\} \in \binom{\bar{U}}{2}$, where $\bar{U} = U \setminus \{1_U\}$.

Kuivinen [10] considered the following stronger constraints:

(K1) For each $U \in \mathcal{U}$ $x(1_U) = \max\{x(u) + x(v) \mid \{u, v\} \in \binom{\bar{U}}{2}\}$.

(K2) For each $U \in \mathcal{U}$ there exists $p \in \bar{U}$ such that $x(p) \geq x(v)$ for all $v \in \bar{U}$ and $x(u) = x(v)$ for all $u, v \in \bar{U} \setminus \{p\}$. (Such an x is called *unified* in [10].)

Note that (K1) implies (K1′).

Kuivinen [10] showed the following theorem. We will prove it by using Corollary 4.2 and the property of a maximizer appearing in the proof of Theorem 2.4.

Theorem 4.4.

$$\min\{\bar{f}(T) \mid T \in \mathcal{U}\} = \max\{\sum_{U\in\mathcal{U}} x(1_U) \mid x \in \mathrm{P}(\bar{f}),\ x \le \mathbf{0},\ \text{(K1)},\ \text{(K2)}\}. \tag{54}$$

Moreover, if $\bar{f}$ is integer-valued, then there exists an integral x that attains the maximum on the right-hand side of (54)*.*

In order to prove this theorem we need one more term and a lemma. We call $v \in V$ *saturated* for $x \in \mathrm{P}(\bar{f})$ if there is an x-tight set that contains v. Lemma 4.3 is specialized to the product lattice of diamonds as follows.

Lemma 4.5. *Let $\{u, v\} \in \binom{\bar{U}}{2}$ for some $U \in \mathcal{U}$. Suppose that u and v are saturated for $x \in \mathrm{P}'(\bar{f})$. Then $x(u) + x(v) = x(1_U)$ holds.*

(Proof of Theorem 4.4) We can easily see the weak duality, i.e., for any $T \in \mathcal{T}$ and any $x \in \mathrm{P}(\bar{f})$ such that $x \le \mathbf{0}$ and (K1) and (K2) are satisfied, we have the inequality

$$\bar{f}(T) \ge \sum_{U\in\mathcal{U}} x(1_U). \tag{55}$$

Let x^* be a maximizer of the right-hand side of (53). As we remarked before, such a maximizer x^* can be taken so that $x^* \le \mathbf{0}$ and $x^*(u) = x^*(v)$ for $u, v \in U$ and all $U \in \mathcal{U}$.

Put $y \leftarrow x^*$. For each $U \in \mathcal{U}$, let us choose any element from $\bar{U} = U \setminus \{1_U\}$ and denoted it by p_U. We increase the value of $y(p_U)$ as much as possible while keeping $y \le \mathbf{0}$, (K1′), and $y \in \mathrm{P}(\bar{f})$. Denote the resulting y by y^*.

Note that $x^*(p_U) \le y^*(p_U)$ and $x^*(v) = y^*(v)$ for all $v \in U \setminus \{p_U\}$, and hence (K2) is satisfied for y^*. We will modify y^* so that (K1) becomes satisfied while keeping (K2), by doing the following for each $U \in \mathcal{U}$.

Initially, for each $U \in \mathcal{U}$ one of the following three holds: (i) $y^*(p_U) = 0$, (ii) $y^*(p_U) + y^*(v) = y^*(1_U)$ for some $v \in U \setminus \{1_U, p_U\}$, and (iii) p_U is saturated for y^*. If (i) or (ii) holds, (K1) and (K2) are satisfied for U.

Hence, suppose that p_U is saturated for y^* and $y^*(p_U) < 0$. Then increase the values of $y^*(v)$ for all $v \in U \setminus \{1_U, p_U\}$ by the same value as much as possible while keeping $y^*(v) \le y^*(p_U)$, (K1′), (K2), and $y^* \in \mathrm{P}(\bar{f})$. If some $v \in U \setminus \{1_U, p_u\}$ is saturated for the new y^*, we have $y^*(p_U) + y^*(v) = y^*(1_U)$ by Lemma 4.5 and thus y^* satisfies (K1) and (K2) for U. Therefore, suppose that $y^*(p_U) + y^*(v) < y^*(1_U)$ for all $v \in U \setminus \{1_U, p_U\}$. Then we have $y^*(v) = y^*(p_U)$ for all $v \in U \setminus \{1_U, p_U\}$. We pick any element in $U \setminus \{1_U, p_U\}$ and denote it by q_U. We then increase the value of $y^*(q_U)$ as

much as possible while keeping (K1′) and $y^* \in \mathrm{P}(\bar{f})$. Note that for the resulting y^* we have $y^*(q_U) \leq 0$ since $y^*(q_U) \leq y^*(1_U) - y^*(p_U) \leq x^*(1_U) - x^*(p_U) = 0$. Hence, $y^*(p_U) + y^*(q_U) = y^*(1_U)$ holds or q_U is saturated for y^*, where the latter case also implies $y^*(p_U) + y^*(q_U) = y^*(1_U)$ by Lemma 4.5. Therefore, the resulting y^* satisfies (K1) and (K2) for U. □

4.2.2 A good characterization

Kuivinen showed that Theorem 4.4 gives a good characterization of the minimization problem of $\bar{f}$. Here we show that Corollary 4.2 also gives a good characterization in the case when $\mathcal{L}$ is the product lattice of diamonds. This implies that only (K1′) is essential.

We should remark that the proofs of the following two theorems given here are almost direct adaptation of Kuivinen's technique [10].

We define a partial order $\preceq$ on $\mathcal{T}$ in such a way that $T \preceq T'$ if for each $U \in \mathcal{U}$ $u \preceq_U v$ holds for all $u, v \in \hat{U}$ such that $\hat{T} \cap \hat{U} = \{u\}$ and $\hat{T}' \cap \hat{U} = \{v\}$, where $\preceq_U$ denotes the partial order in $\mathcal{L}_U$.

Theorem 4.6. *Let f be a submodular function on the product lattice of diamonds with $f(\emptyset) = 0$. If $x \in \mathbb{R}^V$ is an extreme point of $\mathrm{P}'(\bar{f})$, then there exists a chain of x-tight sets $\emptyset = T_0 \prec T_1 \prec \cdots \prec T_k$ satisfying*

(i) *for each $U \in \mathcal{U}$, $T_k \cap U \neq \emptyset$, i.e., T_k is a transversal of $\mathcal{U}$, and*

(ii) *for each $i \in \{1, \cdots, k\}$, defining $\mathcal{U}_i = \{U \mid T_{i-1} \cap U \neq T_i \cap U\}$, there exists at most one $U \in \mathcal{U}_i$ such that $T_{i-1} \cap U = \emptyset$ and $T_i \cap U = \{1_U\}$.*

(Proof) Let $\mathcal{C} : \emptyset = T_0 \prec T_1 \prec \cdots \prec T_k$ be a maximal chain of x-tight sets. It should be noted that T_k is a unique maximal x-tight set with respect to partial order $\preceq$ on $\mathcal{T}$. We show that chain $\mathcal{C}$ satisfies (i) and (ii).

Since x is an extreme point, there is a vector $c \in \mathbb{R}^V$ such that x is a unique maximizer of LP: $\max\{\langle c, x\rangle \mid x \in \mathrm{P}'(f)\}$. Since the LP is bounded, we may assume that for each $U \in \mathcal{U}$, $c(v) > 0$ for all $v \in U$ and $\sum_{u \in \bar{U}} c(u) + 2c(1_U) > 0$.

For any $\alpha \in \mathbb{R}$ and $U \in \mathcal{U}$, define $\chi_{U,\alpha} = \sum_{v \in \bar{U}} \alpha\chi_v + 2\alpha\chi_{1_U}$, where χ_v denotes the characteristic vector of $\{v\}$ in $\mathbb{R}^V$. Note that $x + \chi_{U,\alpha}$ satisfies (K1′) for any U and α.

Observe that for each $U \in \mathcal{U}$ there is an x-tight set T_U with $T_U \cap U \neq \emptyset$, since otherwise, setting $x' = x + \chi_{U,\alpha}$ for some small $\alpha > 0$, we have $x' \in \mathrm{P}'(\bar{f})$ and $\langle x', c\rangle > \langle x, c\rangle$, which is a contradiction. Since we have an x-tight set $\bigvee_0\{T_U \mid U \in \mathcal{U}\} \preceq T_k$, chain $\mathcal{C}$ satisfies (i).

To prove (ii), suppose to the contrary that there exist some index $i \in \{1, \cdots, k\}$ and distinct $U_1, U_2 \in \mathcal{U}$ such that $T_{i-1} \cap U_j = \emptyset$ and $T_i \cap U_j = \{1_{U_j}\}$ for $j = 1, 2$. Since c can be chosen generically, we may assume $\sum_{u \in \bar{U_1}} c(u) + 2c(1_{U_1}) > \sum_{u \in \bar{U_2}} c(u) + 2c(1_{U_2})$

without loss of generality. Then, putting $x' = x + \chi_{U_1,\alpha} - \chi_{U_2,\alpha}$, we have $\langle c, x' \rangle > \langle c, x \rangle$ for any $\alpha > 0$. This implies that there is an x-tight set T for which $T \cap U_1 \neq \emptyset$ and $T \cap U_2 \neq \{1_{U_2}\}$. Then define $T' = (T \wedge_0 T_i) \vee_0 T_{i-1}$, which is an x-tight set satisfying $T_{i-1} \prec T' \prec T_i$. This contradicts the maximality of chain $\mathcal{C}$. Hence (ii) is satisfied. □

Theorem 4.7. *Let f be a submodular function on the product lattice of diamonds. Given a vector $x \in \mathbb{R}^V$ and a chain of sets $\emptyset = T_0 \prec T_1 \prec \cdots \prec T_k$ in $\mathcal{T}$ satisfying $x(T_i) = \bar{f}(T_i)$ $(i = 0, 1, \cdots, k)$ and* (i) *and* (ii) *of Theorem* 4.6*, one can check whether $x \in \mathrm{P}'(\bar{f})$ or not in polynomial time.*

(Proof) Define $T_{k+1} = \{1_U \mid U \in \mathcal{U}\}$. We first prove the following.

Claim: Suppose that x satisfies (K1′). Then, $x \in \mathrm{P}'(\bar{f})$ holds if and only if x satisfies

(**) for each $i = 1, \cdots, k+1$, $x(T) \leq \bar{f}(T)$ for all $T \in \mathcal{T}$ with $T_{i-1} \preceq T \preceq T_i$.

(Proof) The "only if" part is immediate. Hence, assume that x satisfies (**). We prove

(†) for each $i = 0, 1, \cdots, k+1$, $x(T) \leq \bar{f}(T)$ for all $T \in \mathcal{T}$ with $T \preceq T_i$,

by induction on i from $i = 0$ through $i = k+1$.

The base case when $i = 0$ is clear.

Suppose that (†) holds for some $i \in \{0, \cdots, k\}$ and put $i \leftarrow i+1$. Let us consider any $T \in \mathcal{T}$ with $T \preceq T_i$. Since $T_{i-1} \preceq T \vee_0 T_{i-1} \preceq T_i$ and $T \wedge_0 T_{i-1} \preceq T_{i-1}$, we have $x(T \vee_0 T_{i-1}) \leq \bar{f}(T \vee_0 T_{i-1})$ and $x(T \wedge_0 T_{i-1}) \leq \bar{f}(T \wedge_0 T_{i-1})$. Therefore, because of the supermodularity of x (due to (K1′)) and the submodularity of $\bar{f}$ we get

$$\begin{aligned} \bar{f}(T) + \bar{f}(T_{i-1}) &\geq \bar{f}(T \vee_0 T_{i-1}) + \bar{f}(T \wedge_0 T_{i-1}) \\ &\geq x(T \vee_0 T_{i-1}) + x(T \wedge_0 T_{i-1}) \\ &\geq x(T) + x(T_{i-1}). \end{aligned}$$

This implies $x(T) \leq \bar{f}(T)$ since T_{i-1} is x-tight. □

Obviously one can check (K1′) in polynomial time. Hence assume that x satisfies (K1′). Then, in order to prove the present theorem it suffices to show that we can check whether (**) holds in polynomial time.

Suppose that $1 \leq i < k+1$. Also suppose that there is $U^* \in \mathcal{U}(T_i)$ such that $U^* \cap T_{i-1} = \emptyset$ and $U^* \cap T_i = \{1_{U^*}\}$, where recall the notation (1). Define

$$T^* = \{v \in V \mid U \in \mathcal{U}, T_{i-1} \cap U \neq T_i \cap U, T_i \cap U = \{v\}\} \setminus U^*.$$

It follows from (ii) of Theorem 4.6 that for each $v \in U^*$ the set, denoted by $\mathcal{T}^*_v$, of all $T \in \mathcal{T}$ with $T_{i-1} \preceq T \preceq T_i$ and $T \cap U^* = \{v\}$ is equal to $\{(T_{i-1} \vee_0 S) \cup \{v\} \mid S \subseteq T^*\}$. Hence for each $v \in U^*$ we can regard the function $\bar{f}$ restricted to $\mathcal{T}^*_v$ as an ordinary submodular (set) function on 2^{T^*}. Consequently, by using any existing submodular function minimization algorithm, we can check whether $x(T) \leq \bar{f}(T)$ for all $T \in \mathcal{T}$ with $T_{i-1} \preceq T \preceq T_i$ in polynomial time.

If there is no $U^* \in \mathcal{U}(T_i)$ such that $U^* \cap T_{i-1} = \emptyset$ and $U^* \cap T_i = \{1_{U^*}\}$, then the set $\mathcal{T}^*$ of all $T \in \mathcal{T}$ with $T_{i-1} \preceq T \preceq T_i$ is equal to $\{T_{i-1} \vee_0 S \mid S \subseteq T^*\}$, and the function $\bar{f}$ restricted to $\mathcal{T}^*$ can be regarded as an ordinary submodular function on 2^{T^*}. Thus we can check whether $x(T) \leq \bar{f}(T)$ for all $T \in \mathcal{T}$ with $T_{i-1} \preceq T \preceq T_i$ in polynomial time.

Finally, if $i = k+1$, then condition (i) of Theorem 4.6 implies that there is no $U^* \in \mathcal{U}$ such that $U^* \cap T_k = \emptyset$. Therefore, the same argument can be applied again to check whether $x(T) \leq \bar{f}(T)$ for all $T \in \mathcal{T}$ with $T_k \preceq T (\preceq T_{k+1})$ in polynomial time. □

Theorem 4.6 and Theorem 4.7 imply that Corollary 4.2 is a good characterization of the minimization problem of f. Indeed, suppose that one wants to decide whether $T \in \mathcal{T}$ is a minimizer of $\bar{f}$. Then, if the answer is no, then there is a certificate T' with $\bar{f}(T') < \bar{f}(T)$. On the other hand, if the answer is yes, then from Theorem 4.6 (and fundamental facts on polyhedra) there is a certificate which consists of $y \in \mathbb{R}^V$, $z_i \in \mathbb{R}^V$ $(i = 1, \cdots, m)$, $T_0^i, T_1^i, \ldots, T_{k_i}^i \in \mathcal{T}$ $(i = 1, \cdots, m)$, and $\lambda_i \in (0, 1]$ $(i = 1, \cdots, m)$ for some m with $1 \leq m \leq |V| + |\mathcal{U}|$ such that

- y is a point in the characteristic cone of $\mathrm{P}'(\bar{f})$;
- $\sum_{i=1}^m \lambda_i = 1$;
- $\forall i = 1, \cdots, m$: $\emptyset = T_0^i \prec T_1^i \prec \cdots \prec T_{k_i}^i$ and $z_i(T_j^i) = \bar{f}(T_j^i)$ $(\forall j = 1, \cdots, k_i)$;
- $\forall i = 1, \cdots, m$: z_i and $T_0^i \prec \cdots \prec T_{k_i}^i$ satisfy conditions (i) and (ii) of Theorem 4.6;
- $\forall i = 1, \cdots, m$: $z_i \in \mathrm{P}'(\bar{f})$;
- $-\|x\|_{1,\infty} = \hat{f}(T)$, where $x = y + \sum_{i=1}^m \lambda_i z_i$.

Since these conditions can be checked in polynomial time by Theorem 4.7, we can conclude that T is a minimizer of $\bar{f}$ by Corollary 4.2.

In case of k-submodular functions, condition (ii) of Theorem 4.6 may not hold, which would become an obstacle when adopting the same approach to get a good characterization for minimization of k-submodular functions.

5 Concluding Remarks

We have shown a min-max theorem for t-submodular functions in terms of a new norm composed of ℓ_1 and ℓ_∞ norms, which reveals its relation to the min-max theorem for ordinary submodular set functions due to Edmonds [4]. The obtained min-max relation looks nice but it is not clear whether the min-max relation gives us a good characterization in general. In particular, a good characterization for minimizing k-submodular functions is still open. In order to get a good characterization we have some degrees of freedom in

choosing polyhedra appearing on the side of maximization. Finding a right polyhedron that leads us to a good characterization is left for future research.

Devising a polynomial-time algorithm for minimizing k-submodular functions is also left open. As pointed out in [7] and discussed here as well, we need a good characterization of extreme points of $\mathrm{P}_2(f)$. A key to the good characterization is to develop a polynomial-time algorithm for linear optimization over $\mathrm{P}_2(f)$. Main difficulty in linear optimization over $\mathrm{P}_2(f)$ is that a polynomial-time algorithm for it requires an efficient membership algorithm for discerning whether $\mathbf{0} \in \mathrm{P}(f)$.

We have also shown a min-max relation for submodular functions on product lattices of general lattices. When each component lattice is a diamond, it gives a good characterization, whose proof technique can also be adapted to show that our min-max relation gives a good characterization when each component lattice is a pentagon. It was shown by Krokhin and Larose [9] that submodular functions on pentagons can be minimized in polynomial time in the value oracle model, while the tractability on diamonds is still open [10]. It is an interesting open problem to find nontrivial lattices other than diamonds and those considered in [9] for which we have a good characterization.

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