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A PRO-l VERSION OF THE CONGRUENCE SUBGROUP PROBLEM FOR MAPPING CLASS GROUPS OF GENUS ONE

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Abstract. Let l be a prime number. In the present paper, we discuss a *pro-l version* of the congruence subgroup problem for mapping class groups of genus one. Our main result is that the pro-2 version has an *affirmative* answer, but the pro-l version for $l \geq 11$ has a *negative* answer. In order to give a negative answer to the problem in the case where $l \geq 11$, we also consider the issue of whether or not the image of the natural outer action of the absolute Galois group of a certain number field on the geometric pro-l fundamental group of a modular curve is a pro-l group.

Contents

Introduction

Let l be a prime number. In the present paper, we discuss a *pro-l version* of the congruence subgroup problem for mapping class groups of genus one.

Let us first recall the *congruence subgroup problem for mapping class groups* as follows (cf., e.g., [3], [16]): Let (g, r) be a pair of nonnegative integers such that $2g-2+r > 0$ and $\Sigma_{g,r}$ a topological surface of type (g, r), i.e., a topological space obtained by removing r distinct points from a connected orientable compact topological surface of genus g. Write $\pi_1^{\mathrm{top}}(\Sigma_{g,r})$ for the topological fundamental group of $\Sigma_{g,r}$ (which is well-defined up to conjugation) and $\mathrm{MCG}_{g,r}$ for the (pure) mapping class group of $\Sigma_{g,r}$, i.e., the group of isotopy classes of orientation-preserving automorphisms of $\Sigma_{g,r}$

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that fix each removed point. Then a classical result due to Dehn and Nielsen asserts that the natural homomorphism

$$\rho^{\mathrm{top}}_{g,r}\colon\ \mathrm{MCG}_{g,r}\ \longrightarrow\ \mathrm{Out}(\pi_1^{\mathrm{top}}(\Sigma_{g,r}))$$

is *injective*. Now we shall say that a subgroup $J \subseteq \mathrm{MCG}_{g,r}$ of $\mathrm{MCG}_{g,r}$ is a *congruence subgroup* if there exists a characteristic subgroup $H \subseteq \pi_1^{\mathrm{top}}(\Sigma_{g,r})$ of $\pi_1^{\mathrm{top}}(\Sigma_{g,r})$ of finite index such that the inclusion

$$\ker\bigl(\mathrm{MCG}_{g,r} \overset{\rho^{\mathrm{top}}_{g,r}}{\to} \mathrm{Out}(\pi_1^{\mathrm{top}}(\Sigma_{g,r})) \to \mathrm{Out}(\pi_1^{\mathrm{top}}(\Sigma_{g,r})/H)\bigr)\ \subseteq\ J$$

holds. Then the *congruence subgroup problem for the mapping class group of type* (g,r) may be stated as follows:

$(\mathrm{CSP})_{g,r}$: Is every subgroup of $\mathrm{MCG}_{g,r}$ of finite index a *congruence subgroup*?

If $g \leq 1$, then the problem $(\mathrm{CSP})_{g,r}$ was answered *affirmatively* in [2, Theorems 2, 3A, 5]. If $g = 2$, then it follows immediately from [4, Theorem 3.5], together with [12, Theorem B] (cf. also Proposition 1.3 of the present paper), that the problem $(\mathrm{CSP})_{g,r}$ has an *affirmative* answer. However, the problem $(\mathrm{CSP})_{g,r}$ in the case where $g \geq 3$ *remains unsolved.*

Now let us observe that since (as is well-known) $\pi_1^{\mathrm{top}}(\Sigma_{g,r})$ is *finitely generated*, if we write $\pi_1^{\wedge}(\Sigma_{g,r})$ for the profinite completion of the discrete group $\pi_1^{\mathrm{top}}(\Sigma_{g,r})$, then the outer automorphism group $\mathrm{Out}(\pi_1^{\wedge}(\Sigma_{g,r}))$ of $\pi_1^{\wedge}(\Sigma_{g,r})$ admits a *natural structure of profinite group.* In particular, if we write $\mathrm{MCG}^{\wedge}_{g,r}$ for the profinite completion of the discrete group $\mathrm{MCG}_{g,r}$, then the homomorphism $\rho^{\mathrm{top}}_{g,r}$ induces a continuous homomorphism

$$\rho^{\wedge}_{g,r}\colon\ \mathrm{MCG}^{\wedge}_{g,r}\ \longrightarrow\ \mathrm{Out}(\pi_1^{\wedge}(\Sigma_{g,r})).$$

Here, one verifies easily that the problem $(\mathrm{CSP})_{g,r}$ has an *affirmative* answer if and only if this continuous homomorphism $\rho^{\wedge}_{g,r}$ is *injective.*

Next, let us consider a *pro-l version* of the congruence subgroup problem for mapping class groups. Let us first recall that, for a characteristic subgroup $H \subseteq \pi_1^{\mathrm{top}}(\Sigma_{g,r})$ of $\pi_1^{\mathrm{top}}(\Sigma_{g,r})$ of *index a power of l*, the group $\mathrm{Out}(\pi_1^{\mathrm{top}}(\Sigma_{g,r})/H)$ is *not an l-group* in general; on the other hand, it is well-known that if we write $\Sigma^{\mathrm{cpt}}_{g,r}$ for the compactification of $\Sigma_{g,r}$ (so $\Sigma^{\mathrm{cpt}}_{g,r}$ is homeomorphic to "$\Sigma_{g,0}$") and

$$\mathrm{MCG}_{g,r}[l]\ :=\ \ker\bigl(\mathrm{MCG}_{g,r} \to \mathrm{Aut}(H_1(\Sigma^{\mathrm{cpt}}_{g,r}, \mathbb{F}_l))\bigr),$$

then the image of the composite

$$\mathrm{MCG}_{g,r}[l]\ \hookrightarrow\ \mathrm{MCG}_{g,r}\ \overset{\rho^{\mathrm{top}}_{g,r}}{\to}\ \mathrm{Out}(\pi_1^{\mathrm{top}}(\Sigma_{g,r}))\ \to\ \mathrm{Out}(\pi_1^{\mathrm{top}}(\Sigma_{g,r})/H)$$

is always an *l-group.* From this observation, we shall say that a subgroup $J \subseteq \mathrm{MCG}_{g,r}[l]$ of $\mathrm{MCG}_{g,r}[l]$ is an *l-congruence subgroup* if there exists a characteristic subgroup $H \subseteq \pi_1^{\mathrm{top}}(\Sigma_{g,r})$ of $\pi_1^{\mathrm{top}}(\Sigma_{g,r})$ of index a power of l such that the inclusion

$$\ker\bigl(\mathrm{MCG}_{g,r} \overset{\rho^{\mathrm{top}}_{g,r}}{\to} \mathrm{Out}(\pi_1^{\mathrm{top}}(\Sigma_{g,r})) \to \mathrm{Out}(\pi_1^{\mathrm{top}}(\Sigma_{g,r})/H)\bigr)\ \subseteq\ J$$

holds. Then the following problem may be regarded as a *pro-l version* of the congruence subgroup problem for mapping class groups:

$(\mathrm{CSP})^{\text{pro-}l}_{g,r}$: Is every subgroup of $\mathrm{MCG}_{g,r}[l]$ of index a power of l an *l-congruence subgroup*?

If $g = 0$, then the problem $(\mathrm{CSP})^{\text{pro-}l}_{g,r}$ was answered *affirmatively* in [2, Remark following the proof of Theorem 1].

Here, let us observe that, as in the profinite case, if we write $\pi_1^{\text{pro-}l}(\Sigma_{g,r})$, $\mathrm{MCG}_{g,r}[l]^{(l)}$ for the pro-l completions of the discrete groups $\pi_1^{\text{top}}(\Sigma_{g,r})$, $\mathrm{MCG}_{g,r}[l]$, respectively, then the homomorphism $\rho^{\text{top}}_{g,r}$ induces a continuous homomorphism

$$\rho^{\text{pro-}l}_{g,r} \colon \mathrm{MCG}_{g,r}[l]^{(l)} \longrightarrow \mathrm{Out}(\pi_1^{\text{pro-}l}(\Sigma_{g,r})),$$

and, moreover, it holds that the problem $(\mathrm{CSP})^{\text{pro-}l}_{g,r}$ has an *affirmative* answer if and only if this continuous homomorphism $\rho^{\text{pro-}l}_{g,r}$ is *injective*. We note that, in [7, Theorem 1, the discussion following Theorem 1], it was proved that if $g \geq 2$, then the natural continuous homomorphism from the pro-l completion of the *Torelli subgroup* of $\mathrm{MCG}_{g,r}$ (i.e., the subgroup of $\mathrm{MCG}_{g,r}$ obtained by forming the kernel of the natural homomorphism

$$\mathrm{MCG}_{g,r} \longrightarrow \mathrm{Aut}(H_1(\Sigma^{\text{cpt}}_{g,r}, \mathbb{Z})))$$

to $\mathrm{MCG}_{g,r}[l]^{(l)}$ is *not injective*. In particular, the continuous homomorphism induced by $\rho^{\text{top}}_{g,r}$ from the pro-l completion of (not $\mathrm{MCG}_{g,r}[l]$ but) the *Torelli subgroup* of $\mathrm{MCG}_{g,r}$ to $\mathrm{Out}(\pi_1^{\text{pro-}l}(\Sigma_{g,r}))$ is *not injective*.

In the present paper, we discuss the problem $(\mathrm{CSP})^{\text{pro-}l}_{g,r}$ in the case where $g = 1$, i.e., a pro-l version of the congruence subgroup problem for mapping class groups of genus one. The main result of the present paper is as follows (cf. Corollaries 2.3, 4.7):

Theorem A. *Let r be a positive integer. Then the following hold.*

(i) *The problem $(\mathrm{CSP})^{\text{pro-}2}_{1,r}$ has an affirmative answer.*

(ii) *If $l \geq 11$, then the problem $(\mathrm{CSP})^{\text{pro-}l}_{1,r}$ has a negative answer.*

Theorem A, (i), is proved by a similar argument to the argument applied in [2, Theorem 5], which gives rise to an *affirmative* answer to the problem $(\mathrm{CSP})_{g,r}$ in the case where $g = 1$. In order to prove Theorem A, (ii), we also prove the following result concerning the images of the pro-l outer Galois actions associated to modular curves (cf. Theorem 3.13):

Theorem B. *Let $\overline{\mathbb{Q}}$ be an algebraic closure of the field of rational numbers $\mathbb{Q}$. For a positive integer N, let $\zeta_N \in \overline{\mathbb{Q}}$ be a primitive N-th root of unity. Then, for a prime number l, the following conditions are equivalent:*

(P) *$l \leq 7$.*

(Y) *The pro-l outer Galois action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_l))$ associated to the modular curve $Y(l)$ (cf. "Fundamental groups" in "Notations and Conventions") parametrizing elliptic curves with $\Gamma(l)$-structures over $\mathbb{Q}(\zeta_l)$ (cf., e.g., [18]) factors through a pro-l quotient of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_l))$.*

The proof of Theorem A, (ii), in the case where $r = 1$ may be summarized as follows: Let us fix a prime number $l \geq 11$ and assume that the problem

$(\mathrm{CSP})^{\text{pro-}l}_{1,1}$ has an *affirmative* answer. Then it follows from the discussion following the statement of the problem $(\mathrm{CSP})^{\text{pro-}l}_{g,r}$ that the homomorphism $\rho^{\text{pro-}l}_{1,1}$ is *injective*. On the other hand, it follows immediately from the various definitions involved that we have a natural isomorphism of $\mathrm{MCG}_{1,1}[l]^{(l)}$ with the geometric pro-l fundamental group of the modular curve $Y(l)$ of Theorem B. Moreover, as an immediate consequence of a fact concerning the pro-l outer Galois action associated to a tripod (i.e., projective line minus three points) and the fact that *Oda's problem* has an *affirmative* answer (cf. [28, Theorem 0.5, (2)]), the *injectivity* of $\rho^{\text{pro-}l}_{1,1}$ implies that the image of the pro-l outer Galois action associated to $Y(l)$ *factors* through a pro-l quotient. But since $l \geq 11$, this *contradicts* Theorem B. This completes the outline of the proof. Here, it is of interest to observe that:

> The problem $(\mathrm{CSP})^{\text{pro-}l}_{g,r}$ (as well as the problem $(\mathrm{CSP})_{g,r}$) is stated and formulated by a purely *topological* and *combinatorial group-theoretic* setting. Nevertheless, our approach to the problem $(\mathrm{CSP})^{\text{pro-}l}_{g,r}$ is based on a highly *arithmetic* phenomenon concerning the outer Galois actions associated to modular curves.

Finally, we remark that one may think of the problem $(\mathrm{CSP})^{\text{pro-}l}_{g,r}$ as a sort of *geometric analogue of Ihara's problem* concerning the pro-l outer Galois action associated to a tripod (cf., e.g., [14, Lecture I, §2], [25, Introduction]). The conjecture due to *Rasmussen* and *Tamagawa* given in [25, Conjecture 1] was motivated by this *problem of Ihara* and asserts the finiteness of abelian varieties that satisfy certain conditions, one of which is a similar condition to the condition imposed on "$Y(l)$" in condition (Y) of Theorem B. On the other hand, to the knowledge of the authors, at least at the time of writing, it does not appear that any argument has been obtained for deriving an answer of *Ihara's problem* from the *conjecture of Rasmussen-Tamagawa*. In this context, it is of interest to observe that the problem $(\mathrm{CSP})^{\text{pro-}l}_{g,r}$ — which may be thought of as a sort of *geometric analogue of Ihara's problem* — directly relates, as discussed in the above outline of the proof of Theorem A, (ii), to the consideration of the issue of whether or not a modular curve satisfies a similar condition to the condition studied in the *conjecture of Rasmussen-Tamagawa*.

The present paper is organized as follows: In §1, we recall generalities on the *relative pro-l completions* of mapping class groups. In §2, we consider the pro-2 outer geometric monodromy action to prove Theorem A, (i). In §3, we discuss the issue of whether or not the pro-l outer Galois action associated to a modular curve factors through a pro-l quotient and, in particular, prove Theorem B. In §4, we prove Theorem A, (ii), by means of the results obtained in the previous sections.

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Notations and Conventions

Numbers: The notation $\mathbb{Z}$ will be used to denote the ring of rational integers. The notation $\mathbb{Q}$ will be used to denote the field of rational numbers. If l is a prime number, then the notation $\mathbb{F}_l$ will be used to denote the quotient $\mathbb{Z}/l$, and the notation $\mathbb{Z}_l$ (respectively, $\mathbb{Q}_l$) will be used to denote the l-adic completion of $\mathbb{Z}$ (respectively, $\mathbb{Q}$). If A is a ring, then the notation $A^\times$ will be used to denote the multiplicative group of A.

Profinite groups: If G is a profinite group, and $H \subseteq G$ is a closed subgroup of G, then we shall write G^{ab} for the *abelianization* of G (i.e., the quotient of G by the closure of the commutator subgroup of G), $|G : H|$ for the *index* of H in G, and $Z_G(H)$ for the *centralizer of H in G*, i.e.,

$$Z_G(H) := \{g \in G \mid g \cdot h \cdot g^{-1} = h \text{ for any } h \in H\} \subseteq G.$$

We shall say that a profinite group G is *torsion-free* if G has no nontrivial element of finite order. We shall say that a profinite group G is *center-free* if $Z_G(G) = \{1\}$. We shall say that a profinite group G is *slim* if for every open subgroup $H \subseteq G$, it holds that $Z_G(H) = \{1\}$.

If G is a profinite group, then we shall denote by $\mathrm{Aut}(G)$ the group of (continuous) automorphisms of the topological group G, by $\mathrm{Inn}(G)$ the group of inner automorphisms of G, and by $\mathrm{Out}(G)$ the quotient of $\mathrm{Aut}(G)$ with respect to the normal subgroup $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$. If, moreover, G is *topologically finitely generated*, then one verifies that the topology of G admits a basis of *characteristic open subgroups*, which thus induces a *profinite topology* on the group $\mathrm{Aut}(G)$, hence also a *profinite topology* on the group $\mathrm{Out}(G)$.

Let G be a profinite group, $N \subseteq G$ a normal open subgroup of G, $G \twoheadrightarrow Q$ a quotient of G, l a prime number, and N^l the maximal pro-l quotient of N. Then we shall say that Q is the *maximal almost pro-l quotient of G with respect to N* if the kernel of the surjection $G \twoheadrightarrow Q$ coincides with the kernel of $N \twoheadrightarrow N^l$, i.e., $Q = G/\ker(N \twoheadrightarrow N^l)$. (Note that since N is *normal* in G, and the kernel $\ker(N \twoheadrightarrow N^l)$ of the natural surjection $N \twoheadrightarrow N^l$ is *characteristic* in N, it holds that $\ker(N \twoheadrightarrow N^l)$ is *normal* in G.)

Fundamental groups: Let l be a prime number, k a perfect field, $\overline{k}$ an algebraic closure of k, and G_k the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$ of k. For a scheme X which is a geometrically connected and of finite type over k, we shall write Δ^l_X for the *pro-l geometric fundamental group* of X, i.e., the maximal pro-l quotient of the algebraic fundamental group $\pi_1(X \otimes_k \overline{k})$ of $X \otimes_k \overline{k}$, and Π^l_X for the *geometrically pro-l fundamental group* of X, i.e., the quotient of the algebraic fundamental group $\pi_1(X)$ of X by the kernel of the natural surjection $\pi_1(X \otimes_k \overline{k}) \to \Delta^l_X$. We shall write

$$\rho^l_X \colon G_k \longrightarrow \mathrm{Out}(\Delta^l_X)$$

for the outer action determined by the natural exact sequence

$$1 \longrightarrow \Delta_X^l \longrightarrow \Pi_X^l \xrightarrow{\mathrm{pr}_k} G_k \longrightarrow 1 .$$

We shall refer to ρ_X^l as the *pro-l outer Galois action associated to X*.

Curves: Let k be a field and (g, r) a pair of nonnegative integers. Then we shall say that a scheme X over k is a *curve of type* (g, r) over k if there exist a scheme X^{cpt} which is of dimension 1, smooth, proper, geometrically connected over k of genus g and a closed subscheme $D \subseteq X^{\mathrm{cpt}}$ which is finite and étale over k of degree r such that X is isomorphic to the complement of D in X^{cpt} over k. In this case, it follows from elementary algebraic geometry that these X^{cpt} and D are *uniquely determined* by X up to unique canonical isomorphism. We shall refer to X^{cpt} as the *smooth compactification* of X and D as the *divisor at infinity* of X. We shall say that a scheme X over k is a *hyperbolic curve* over k if there exists a pair (g, r) of nonnegative integers such that $2g-2+r > 0$, and, moreover, X is a curve of type (g, r) over k. As is well-known, for a curve X of type (g, r) over an algebraically closed field of characteristic zero, the isomorphism class of the algebraic fundamental group $\pi_1(X)$ of X (respectively, the pro-l geometric fundamental group of X) depends only on (g, r) (respectively, (g, r, l)). We shall write $\Delta_{g,r}$ (respectively, $\Delta_{g,r}^l$) for the algebraic fundamental group (respectively, the pro-l geometric fundamental group) of a curve of type (g, r) over an algebraically closed field of characteristic zero. If (g, r) is a pair of nonnegative integers such that $2g - 2 + r > 0$, then the notation $(\mathcal{M}_{g,r})_k$ will be used to denote the moduli stack of r-pointed smooth proper curves of genus g over k whose r marked points are equipped with an ordering.

Let n be a positive integer, (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$, and X a curve of type (g, r) over k. Suppose that the divisor at infinity D of X consists of r distinct k-rational points. Then we shall refer to the scheme obtained by pulling back the (representable) (1-)morphism $(\mathcal{M}_{g,r+n})_k \to (\mathcal{M}_{g,r})_k$ given by forgetting the last n marked points via the classifying (1-)morphism $\mathrm{Spec}(k) \to (\mathcal{M}_{g,r})_k$ of the r-pointed smooth proper curve of genus g over k obtained by equipping the r marked points of X with an ordering as the *n-th configuration space* of X. Note that one verifies immediately that the isomorphism class of this pull-back does not depend on the choice of the ordering of the r marked points of X.

1. The relative pro-l completions of mapping class groups

Throughout the present paper, let l be a prime number, k a field of characteristic zero, and $\overline{k}$ an algebraic closure of k. Write $G_k := \mathrm{Gal}(\overline{k}/k)$. In the present §1, we recall generalities on the *relative pro-l completions of mapping class groups*. Much of the content of the present §1 is contained in [7].

Definition 1.1 ([7, §3])**.** Let (g, r) be a pair of nonnegative integers such that $2g - 2 + r > 0$.

(i) We shall write

$$\Pi_{(\mathcal{M}_{g,r})_k}$$

for the algebraic fundamental group of $(\mathcal{M}_{g,r})_k$. Since the isomorphism class of the kernel of the homomorphism $\Pi_{(\mathcal{M}_{g,r})_k} \twoheadrightarrow G_k$ that arises from the structure (1-)morphism $(\mathcal{M}_{g,r})_k \to \mathrm{Spec}(k)$ does not depend on the choice of the field k of characteristic zero, we shall write

$$\Gamma_{g,r}$$

for the kernel of $\Pi_{(\mathcal{M}_{g,r})_k} \twoheadrightarrow G_k$. Note that $\Gamma_{g,r}$ is isomorphic to the algebraic fundamental group of $(\mathcal{M}_{g,r})_{\overline{k}}$. Thus, we have natural exact sequences of profinite groups

$$1 \longrightarrow \Gamma_{g,r} \longrightarrow \Pi_{(\mathcal{M}_{g,r})_k} \longrightarrow G_k \longrightarrow 1\ ,$$

$$1 \longrightarrow \Delta_{g,r} \longrightarrow \Pi_{(\mathcal{M}_{g,r+1})_k} \longrightarrow \Pi_{(\mathcal{M}_{g,r})_k} \longrightarrow 1$$

(cf. [23]).

(ii) We shall write

$$(\rho^{\text{puni-}l}_{g,r})_k \colon \Pi_{(\mathcal{M}_{g,r})_k} \longrightarrow \mathrm{Out}^{\mathrm{C}}(\Delta^l_{g,r})$$

for the outer action determined by the exact sequence of the final display of (i) and the natural surjection $\Delta_{g,r} \twoheadrightarrow \Delta^l_{g,r}$, where we refer to [21, Definition 1.1 (ii)] for the definition of $\mathrm{Out}^{\mathrm{C}}$. By regarding $\Delta^l_{g,r}$ as the pro-l geometric fundamental group of a curve X of type (g,r) over $\overline{k}$ (i.e., the geometric fiber of the (1-)morphism $(\mathcal{M}_{g,r+1})_k \to (\mathcal{M}_{g,r})_k$ at a $\overline{k}$-valued geometric point of $(\mathcal{M}_{g,r})_k$) and $\Delta^l_{g,0}$ as the pro-l geometric fundamental group of the smooth compactification of X, for a positive integer n, one obtains a natural homomorphism

$$\varphi^{l^n}_{g,r} \colon \mathrm{Out}^{\mathrm{C}}(\Delta^l_{g,r}) \longrightarrow \mathrm{Aut}((\Delta^l_{g,0})^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} (\mathbb{Z}/l^n)).$$

Note that $\varphi^{l^n}_{g,r}$ (respectively, $\ker(\varphi^{l^n}_{g,r} \circ (\rho^{\text{puni-}l}_{g,r})_k) \cap \Gamma_{g,r}$) does not depend on the choice of X (respectively, k). Let $(\Gamma_{g,r}[l])^l$ be the maximal pro-l quotient of $\Gamma_{g,r}[l] := \ker(\varphi^l_{g,r} \circ (\rho^{\text{puni-}l}_{g,r})_k) \cap \Gamma_{g,r}$. We shall write

$$\Gamma^{\text{rel-}l}_{g,r}$$

for the maximal almost pro-l quotient of $\Gamma_{g,r}$ with respect to $\Gamma_{g,r}[l]$, i.e., the quotient of $\Gamma_{g,r}$ with respect to the kernel of $\Gamma_{g,r}[l] \twoheadrightarrow (\Gamma_{g,r}[l])^l$, and refer to $\Gamma^{\text{rel-}l}_{g,r}$ as the *relative pro-l completion of the mapping class group* of type (g,r). Note that since $\Gamma_{g,r}[l]$ is *normal* in $\Pi_{(\mathcal{M}_{g,r})_k}$, and the kernel of $\Gamma_{g,r}[l] \twoheadrightarrow (\Gamma_{g,r}[l])^l$ is *characteristic* in $\Gamma_{g,r}[l]$, it holds that $\ker(\Gamma_{g,r}[l] \twoheadrightarrow (\Gamma_{g,r}[l])^l)$ is *normal* in $\Pi_{(\mathcal{M}_{g,r})_k}$. We shall write

$$\Pi^{\text{rel-}l}_{(\mathcal{M}_{g,r})_k}$$

for the quotient of $\Pi_{(\mathcal{M}_{g,r})_k}$ with respect to the kernel of $\Gamma_{g,r}[l] \twoheadrightarrow (\Gamma_{g,r}[l])^l$ and

$$(\rho^{\text{rel-}l}_{g,r})_k \colon G_k \longrightarrow \mathrm{Out}(\Gamma^{\text{rel-}l}_{g,r})$$

for the outer Galois action determined by the exact sequence

$$1 \longrightarrow \Gamma_{g,r}^{\text{rel-}l} \longrightarrow \Pi_{(\mathcal{M}_{g,r})_k}^{\text{rel-}l} \xrightarrow{\text{pr}_k} G_k \longrightarrow 1$$

that arises from the exact sequence of the third display of (i).

(iii) We shall write

$$\Gamma_{g,r}^{\text{geo-}l} \quad (\simeq (\rho_{g,r}^{\text{puni-}l})_k(\Gamma_{g,r}))$$

for the quotient of $\Gamma_{g,r}$ with respect to the kernel of the surjection $\Gamma_{g,r} \twoheadrightarrow (\rho_{g,r}^{\text{puni-}l})_k(\Gamma_{g,r})$. Note that the kernel of $\Gamma_{g,r} \twoheadrightarrow (\rho_{g,r}^{\text{puni-}l})_k(\Gamma_{g,r})$ is *normal* in $\Pi_{(\mathcal{M}_{g,r})_k}$. We shall write

$$\Pi_{(\mathcal{M}_{g,r})_k}^{\text{geo-}l}$$

for the quotient of $\Pi_{(\mathcal{M}_{g,r})_k}$ with respect to the kernel of $\Gamma_{g,r} \twoheadrightarrow (\rho_{g,r}^{\text{puni-}l})_k(\Gamma_{g,r})$ and

$$(\rho_{g,r}^{\text{geo-}l})_k \colon G_k \longrightarrow \text{Out}(\Gamma_{g,r}^{\text{geo-}l})$$

for the outer Galois action determined by the exact sequence

$$1 \longrightarrow \Gamma_{g,r}^{\text{geo-}l} \longrightarrow \Pi_{(\mathcal{M}_{g,r})_k}^{\text{geo-}l} \longrightarrow G_k \longrightarrow 1$$

that arises from the exact sequence of the third display of (i).

Proposition 1.2 (cf. [7, Proposition 3.1, (2)]). *Let n be a positive integer, (g,r) a pair of nonnegative integers such that $2g-2+r>0$, X a curve of type (g,r) over $\overline{k}$, and X_n the n-th configuration space of the curve X. Then the (1-)morphism $(\mathcal{M}_{g,r+n})_k \to (\mathcal{M}_{g,r})_k$ given by forgetting the last n point and the classifying (1-)morphism* $\text{Spec}(\overline{k}) \to (\mathcal{M}_{g,r})_{\overline{k}}$ *of X determine the following commutative diagram*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_{X_n}^{l} & \longrightarrow & \Pi_{(\mathcal{M}_{g,r+n})_k}^{\text{rel-}l} & \longrightarrow & \Pi_{(\mathcal{M}_{g,r})_k}^{\text{rel-}l} & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Delta_{X_n}^{l} & \longrightarrow & \Gamma_{g,r+n}^{\text{rel-}l} & \longrightarrow & \Gamma_{g,r}^{\text{rel-}l} & \longrightarrow & 1
\end{array}$$

where the horizontal sequences are exact, the vertical arrows are injective, and the left-hand vertical arrow is the identity morphism of $\Delta_{X_n}^{l}$.

In particular, by considering the case where $n=1$, we conclude that the homomorphism $(\rho_{g,r}^{\text{puni-}l})_k$ factors through $\Pi_{(\mathcal{M}_{g,r})_k}^{\text{rel-}l}$. We shall write

$$(\rho_{g,r}^{\text{univ-}l})_k \colon \Pi_{(\mathcal{M}_{g,r})_k}^{\text{rel-}l} \longrightarrow \text{Out}(\Delta_{g,r}^{l})$$

for the resulting homomorphism, whose restriction to $\Gamma_{g,r}^{\text{rel-}l} \subseteq \Pi_{(\mathcal{M}_{g,r})_k}^{\text{rel-}l}$ we denote by

$$\rho_{g,r}^{\text{univ-}l} \colon \Gamma_{g,r}^{\text{rel-}l} \longrightarrow \text{Out}(\Delta_{g,r}^{l}).$$

Proof. Let us first observe that it follows immediately from the exact sequence of the final display of Definition 1.1, (ii), that, to verify Proposition 1.2, it suffices to verify the exactness of the lower sequence of the commutative diagram in the statement of Proposition 1.2. Thus, we may assume without loss of generality, by replacing k by $\overline{k}$, that k is an algebraically closed field. Let Y be the curve of type $(g, r+1)$ over k obtained by removing a k-rational point from X and Y_{n-1} the $(n-1)$-st configuration space of Y. Then it follows from the (easily verified) *right exactness* of the functor of taking maximal pro-l quotient and [11, Lemma 15, (iv)] that we have the following commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta^l_{Y_{n-1}} & \longrightarrow & \Delta^l_{X_n} & \longrightarrow & \Delta^l_X & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
 & & \Delta^l_{Y_{n-1}} & \longrightarrow & \Gamma^{\text{rel-}l}_{g,r+n} & \longrightarrow & \Gamma^{\text{rel-}l}_{g,r+1} & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Gamma^{\text{rel-}l}_{g,r} & = & \Gamma^{\text{rel-}l}_{g,r} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

where the vertical and horizontal sequences are exact, the lower horizontal arrow is the identity morphism of $\Gamma^{\text{rel-}l}_{g,r}$, and the left-hand vertical arrow is the identity morphism of $\Delta^l_{Y_{n-1}}$. Thus, to verify Proposition 1.2, by *induction on n*, we may assume without loss of generality that $n = 1$. On the other hand, if $n = 1$, then the desired exactness follows from the proof of [7, Proposition 3.1, (2)]. □

Proposition 1.3. *Let (g, r) be a pair of nonnegative integers such that $2g - 2 + r > 0$. Then the homomorphism*

$$\rho^{\text{univ-}l}_{g,r} \colon \Gamma^{\text{rel-}l}_{g,r} \longrightarrow \operatorname{Out}(\Delta^l_{g,r})$$

is injective if and only if the homomorphism

$$\rho^{\text{univ-}l}_{g,r+1} \colon \Gamma^{\text{rel-}l}_{g,r+1} \longrightarrow \operatorname{Out}(\Delta^l_{g,r+1})$$

is injective.

Proof. Let us first observe that it follows immediately from the definition of the homomorphisms under consideration that, to verify Proposition 1.3, we may assume without loss of generality, by replacing k by $\overline{k}$, that k is an algebraically closed field. Let X be a curve of type (g, r) over k, X_2 the 2-nd configuration space of X, and Y the curve of type $(g, r+1)$ over k obtained by removing a k-rational point from X. Then it follows from Proposition 1.2 and [11, Lemma 15, (iv)] that we have the following commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Delta_Y^l & = & \Delta_Y^l & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Delta_{X_2}^l & \longrightarrow & \Gamma_{g,r+2}^{\text{rel-}l} & \longrightarrow & \Gamma_{g,r}^{\text{rel-}l} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Delta_X^l & \longrightarrow & \Gamma_{g,r+1}^{\text{rel-}l} & \longrightarrow & \Gamma_{g,r}^{\text{rel-}l} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

where the horizontal and vertical sequences are exact, the upper horizontal arrow is the identity morphism of Δ_Y^l, and the right-hand vertical arrow is the identity morphism of $\Gamma_{g,r}^{\text{rel-}l}$. Now let us observe that one verifies easily that the outer action $\Gamma_{g,r}^{\text{rel-}l} \to \text{Out}(\Delta_{X_2}^l)$ determined by the middle horizontal sequence of the above diagram factors through the closed subgroup

$$\text{Out}^{\text{FC}}(\Delta_{X_2}^l) \subseteq \text{Out}(\Delta_{X_2}^l)$$

where we refer to [21, Definition 1.1 (ii)] for the definition of Out^{FC}. Therefore, it follows from [11, Lemma 17, (ii)] and [2, Remark following the proof of Theorem 1] that we obtain the following commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_X^l & \longrightarrow & \Gamma_{g,r+1}^{\text{rel-}l} & \longrightarrow & \Gamma_{g,r}^{\text{rel-}l} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta_X^l & \longrightarrow & \Gamma_{g,r+1}^{\text{geo-}l} & \longrightarrow & \Gamma_{g,r}^{\text{geo-}l} & \longrightarrow & 1
\end{array}$$

where the horizontal sequences are exact, and the left-hand vertical arrow is the identity morphism of Δ_X^l. In particular, $\rho_{g,r}^{\text{univ-}l}$ is injective (i.e., the right-hand vertical arrow of this diagram is injective) if and only if $\rho_{g,r+1}^{\text{univ-}l}$ is injective (i.e., the middle vertical arrow of this diagram is injective). This completes the proof of Proposition 1.3. □

Remark 1.4. A similar result to Proposition 1.3 for the profinite case can be found in [4, Lemma 3.6].

2. A pro-2 version of the congruence subgroup problem for mapping class groups of genus one

In the present §2, we maintain the notation of the preceding §1. In the present §2, we consider the congruence subgroup problem for the relative pro-2 completions of mapping class groups. In particular, we prove that the quotient of the profinite completion of the mapping class group of genus one determined by the pro-2 outer geometric monodromy representation *coincides* with the relative pro-2 completion of the mapping class group of genus one.

Definition 2.1.

(i) Let $(\mathcal{M}_{\mathrm{Lgd}})_k$ be the affine algebraic surface over k defined by the equation
$$y^2 = x(x-1)(x-\lambda)$$
in $\mathrm{Spec}(k[x, y, \lambda])$, where x, y, and λ are indeterminates. Then one verifies easily that the projection
$$(\mathcal{M}_{\mathrm{Lgd}})_k \to \mathbb{P}^1_k \setminus \{0, 1, \infty\} \simeq (\mathcal{M}_{0,4})_k, \quad (x, y, \lambda) \mapsto \lambda$$
gives rise to a family of curves of type $(1, 1)$, which we shall refer to as the *Legendre family of elliptic curves.* We shall write $\Pi^l_{\mathrm{Lgd}_k}$ for the geometrically pro-l fundamental group of $(\mathcal{M}_{\mathrm{Lgd}})_k$ and Δ^l_{Lgd} for the pro-l geometric fundamental group of $(\mathcal{M}_{\mathrm{Lgd}})_k$. It is well-known that the classifying (1-)morphism $(\mathcal{M}_{0,4})_k \to (\mathcal{M}_{1,1})_k$ determined by $(\mathcal{M}_{\mathrm{Lgd}})_k \to (\mathcal{M}_{0,4})_k$ is a finite étale covering of $(\mathcal{M}_{1,1})_k$. In particular, $\Pi_{(\mathcal{M}_{0,4})_k}$ may be regarded as an open subgroup of $\Pi_{(\mathcal{M}_{1,1})_k}$. Moreover, let us observe that one verifies easily that $\Pi_{(\mathcal{M}_{0,4})_k}$ is contained in $\ker(\varphi^2_{1,1} \circ (\rho^{\text{puni-2}}_{1,1})_k)$. Thus, it follows from [9, Proposition 1.2] that we obtain a natural exact sequence
$$1 \longrightarrow \Delta^2_{1,1} \longrightarrow \Pi^2_{\mathrm{Lgd}_k} \longrightarrow \Pi^{\text{rel-2}}_{(\mathcal{M}_{0,4})_k} \longrightarrow 1\ .$$
We shall write
$$(\rho^2_{\mathrm{Lgd}})_k \colon \Pi^{\text{rel-2}}_{(\mathcal{M}_{0,4})_k} \longrightarrow \mathrm{Out}(\Delta^2_{1,1})$$
for the outer action determined by this exact sequence and
$$\rho^2_{\mathrm{Lgd}} : \Gamma^{\text{rel-2}}_{0,4} \longrightarrow \mathrm{Out}(\Delta^2_{1,1})$$
for the restriction of $(\rho^2_{\mathrm{Lgd}})_k$ to $\Gamma^{\text{rel-2}}_{0,4} \subseteq \Pi^{\text{rel-2}}_{(\mathcal{M}_{0,4})_k}$.

(ii) We shall write
$$[2] : (\mathcal{M}_{\mathrm{Lgd}\backslash\mathrm{Lgd}[2]})_k \longrightarrow (\mathcal{M}_{\mathrm{Lgd}})_k$$
for the finite étale covering over $(\mathcal{M}_{0,4})_k$ given by *multiplication by* 2 (i.e., relative to the operation on the family of elliptic curves given by the canonical relative compactification of $(\mathcal{M}_{\mathrm{Lgd}})_k$ over $(\mathcal{M}_{0,4})_k$), $\Pi^l_{\mathrm{Lgd}\backslash\mathrm{Lgd}[2]_k}$ for the geometrically pro-l fundamental group of the covering $(\mathcal{M}_{\mathrm{Lgd}\backslash\mathrm{Lgd}[2]})_k$, $\Delta^l_{\mathrm{Lgd}\backslash\mathrm{Lgd}[2]}$ for the pro-l geometric fundamental group of the covering $(\mathcal{M}_{\mathrm{Lgd}\backslash\mathrm{Lgd}[2]})_k$, and
$$\pi_1([2]) \colon \Pi^2_{\mathrm{Lgd}\backslash\mathrm{Lgd}[2]_k} \longrightarrow \Pi^2_{\mathrm{Lgd}_k}$$
for the outer injection induced by the above finite étale covering $(\mathcal{M}_{\mathrm{Lgd}\backslash\mathrm{Lgd}[2]})_k \overset{[2]}{\to} (\mathcal{M}_{\mathrm{Lgd}})_k$. Thus, one verifies easily that the composite $(\mathcal{M}_{\mathrm{Lgd}\backslash\mathrm{Lgd}[2]})_k \to (\mathcal{M}_{\mathrm{Lgd}})_k \to (\mathcal{M}_{0,4})_k$ is a family of curves

of type $(1,4)$, and, moreover, the exact sequence of the third display of (i) determines a natural exact sequence

$$1 \longrightarrow \Delta^2_{1,4} \longrightarrow \Pi^2_{\mathrm{Lgd}\backslash\mathrm{Lgd}[2]_k} \longrightarrow \Pi^{\text{rel-2}}_{(\mathcal{M}_{0,4})_k} \longrightarrow 1 \, .$$

We shall write

$$(\rho^2_{\mathrm{Lgd}\backslash\mathrm{Lgd}[2]})_k \colon \Pi^{\text{rel-2}}_{(\mathcal{M}_{0,4})_k} \longrightarrow \mathrm{Out}(\Delta^2_{1,4})$$

for the outer action determined by this exact sequence and

$$\rho^2_{\mathrm{Lgd}\backslash\mathrm{Lgd}[2]} \colon \Gamma^{\text{rel-2}}_{0,4} \longrightarrow \mathrm{Out}(\Delta^2_{1,4})$$

for the restriction of $(\rho^2_{\mathrm{Lgd}\backslash\mathrm{Lgd}[2]})_k$ to $\Gamma^{\text{rel-2}}_{0,4} \subseteq \Pi^{\text{rel-2}}_{(\mathcal{M}_{0,4})_k}$. Note that, as is well-known, the quotient of $(\mathcal{M}_{\mathrm{Lgd}\backslash\mathrm{Lgd}[2]})_k$ by the natural action of $\mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{M}_{1,2})_k) \simeq \{\pm 1\}$ is isomorphic to $(\mathcal{M}_{0,5})_k$ over $(\mathcal{M}_{0,4})_k$, and the resulting morphism $q \colon (\mathcal{M}_{\mathrm{Lgd}\backslash\mathrm{Lgd}[2]})_k \to (\mathcal{M}_{0,5})_k$ is a finite étale covering over $(\mathcal{M}_{0,4})_k$. We shall write

$$\pi_1(q) \colon \Pi^2_{\mathrm{Lgd}\backslash\mathrm{Lgd}[2]_k} \longrightarrow \Pi^{\text{rel-2}}_{(\mathcal{M}_{0,5})_k}$$

for the outer injection determined by the morphism q.

Theorem 2.2. *The homomorphism*

$$\rho^2_{\mathrm{Lgd}} \colon \Gamma^{\text{rel-2}}_{0,4} \longrightarrow \mathrm{Out}(\Delta^2_{1,1})$$

is injective.

Proof. Let us first observe that we have the following commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta^2_{1,1} & \longrightarrow & \Delta^2_{\mathrm{Lgd}} & \longrightarrow & \Gamma^{\text{rel-2}}_{0,4} & \longrightarrow & 1 \\
 & & \uparrow{\scriptstyle \pi_1([2])} & & \uparrow{\scriptstyle \pi_1([2])} & & \| & & \\
1 & \longrightarrow & \Delta^2_{1,4} & \longrightarrow & \Delta^2_{\mathrm{Lgd}\backslash\mathrm{Lgd}[2]} & \longrightarrow & \Gamma^{\text{rel-2}}_{0,4} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \pi_1(q)} & & \downarrow{\scriptstyle \pi_1(q)} & & \| & & \\
1 & \longrightarrow & \Delta^2_{0,4} & \longrightarrow & \Gamma^{\text{rel-2}}_{0,5} & \longrightarrow & \Gamma^{\text{rel-2}}_{0,4} & \longrightarrow & 1
\end{array}$$

where the horizontal sequences are exact, the vertical arrows are injective, and the right-hand vertical arrows are the identity morphisms of $\Gamma^{\text{rel-2}}_{0,4}$. By [11, Lemma 23, (i), (iii)], $\ker(\rho^2_{\mathrm{Lgd}\backslash\mathrm{Lgd}[2]})$ is an *open* subgroup of $\ker(\rho^2_{\mathrm{Lgd}})$ and a subgroup of $\ker(\rho^{\text{univ-2}}_{0,4})$. Thus, since $\ker(\rho^{\text{univ-2}}_{0,4})$ is *trivial* (cf. [2, Remark following the proof of Theorem 1]), $\ker(\rho^2_{\mathrm{Lgd}\backslash\mathrm{Lgd}[2]})$ is *trivial*. In particular, $\ker(\rho^2_{\mathrm{Lgd}})$ is a *finite* group. On the other hand, since $\Gamma^{\text{rel-2}}_{0,4} \simeq \Delta^2_{0,3}$ is *torsion-free* (cf., e.g., [22, Remark 1.2.2]), $\ker(\rho^2_{\mathrm{Lgd}})$ is *trivial*. This completes the proof of Theorem 2.2. □

Corollary 2.3. *Let r be a positive integer. Then the homomorphism*

$$\rho^{\text{univ-2}}_{1,r} \colon \Gamma^{\text{rel-2}}_{1,r} \longrightarrow \mathrm{Out}(\Delta^2_{1,r})$$

is injective.

In particular, the problem $(\mathrm{CSP})^{\text{pro-2}}_{1,r}$ *in the Introduction has an affirmative answer.*

Proof. Let us first observe that it follows from Proposition 1.3 that, to verify the first portion of Corollary 2.3, we may assume that $r = 1$. It is well-known that $\Gamma^{\text{rel-2}}_{0,4} \to \Gamma^{\text{rel-2}}_{1,1}$ determined by the classifying (1-)morphism $(\mathcal{M}_{0,4})_{\overline{k}} \to (\mathcal{M}_{1,1})_{\overline{k}}$ of the family $(\mathcal{M}_{\mathrm{Lgd}})_{\overline{k}} \to (\mathcal{M}_{0,4})_{\overline{k}}$ of curves of type $(1,1)$ is an *open* injective, and the kernel of the homomorphism

$$\varphi^4_{1,1} \circ \rho^{\text{univ-2}}_{1,1} \colon \Gamma^{\text{rel-2}}_{1,1} \longrightarrow \mathrm{Aut}(\Delta^2_{1,0} \otimes_{\mathbb{Z}_2} (\mathbb{Z}/4))$$

is *torsion-free* (cf., e.g., [17, §1.4], [22, Remark 1.2.2]). Therefore, it follows immediately from Theorem 2.2 that $\ker(\rho^{\text{univ-2}}_{1,1})$ is trivial. This completes the proof of the first portion of Corollary 2.3. Thus, the final portion of Corollary 2.3 follows immediately from the discussion following the statement of the problem $(\mathrm{CSP})^{\text{pro-}l}_{g,r}$ in the Introduction. This completes the proof of Corollary 2.3. □

Remark 2.4. The argument given in the proof of Corollary 2.3 is essentially the same as the argument applied in [2] to prove [2, Theorem 5].

Corollary 2.5. *The equality*

$$\ker((\rho^2_{\mathrm{Lgd}})_k) = \ker((\rho^{\text{univ-2}}_{0,4})_k)$$

holds.

Proof. Let us first observe that it follows from Theorem 2.2 and [2, Remark following the proof of Theorem 1] that, to verify Corollary 2.5, it suffices to prove that

$$\mathrm{im}(\ker((\rho^2_{\mathrm{Lgd}})_k) \overset{\mathrm{pr}_k}{\to} G_k) = \mathrm{im}(\ker((\rho^{\text{univ-2}}_{0,4})_k) \overset{\mathrm{pr}_k}{\to} G_k).$$

Since $(\mathcal{M}_{1,2})_k \to (\mathcal{M}_{1,1})_k$ is isomorphic to the universal curve of type $(1,1)$ over k, we obtain the following commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta^2_{1,1} & \longrightarrow & \Pi^2_{\mathrm{Lgd}_k} & \longrightarrow & \Pi^{\text{rel-2}}_{(\mathcal{M}_{0,4})_k} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta^2_{1,1} & \longrightarrow & \Pi^{\text{rel-2}}_{(\mathcal{M}_{1,2})_k} & \longrightarrow & \Pi^{\text{rel-2}}_{(\mathcal{M}_{1,1})_k} & \longrightarrow & 1
\end{array}$$

where the horizontal sequences are exact, the vertical arrows are injective, and the left-hand vertical arrow is the identity morphism of $\Delta^2_{1,1}$. Now let us observe that one verifies easily from the above commutative diagram that

$$\ker((\rho^2_{\mathrm{Lgd}})_k) \subseteq \ker((\rho^{\text{univ-2}}_{1,1})_k).$$

Moreover, it is well-known that

$$\ker(\varphi^4_{1,1} \circ (\rho^{\text{univ-2}}_{1,1})_k) \subseteq \Pi^{\text{rel-2}}_{(\mathcal{M}_{0,4})_k} \subseteq \Pi^{\text{rel-2}}_{(\mathcal{M}_{1,1})_k},$$

which thus implies that

$$\ker((\rho^{\text{univ-2}}_{1,1})_k) = \ker((\rho^{\text{univ-2}}_{1,1})_k) \cap \Pi^{\text{rel-2}}_{(\mathcal{M}_{0,4})_k} = \ker((\rho^2_{\mathrm{Lgd}})_k).$$

Thus, we conclude that

$$\mathrm{im}(\ker((\rho^2_{\mathrm{Lgd}})_k) \overset{\mathrm{pr}_k}{\to} G_k) = \mathrm{im}(\ker((\rho^{\text{univ-2}}_{1,1})_k) \overset{\mathrm{pr}_k}{\to} G_k).$$

On the other hand, since Oda's problem is answered in the *affirmative* (cf. [28, Theorem 0.5]), the equalities

$$\mathrm{im}(\ker((\rho^{\text{univ-2}}_{1,1})_k) \overset{\mathrm{pr}_k}{\to} G_k) = \ker(\rho^2_{\mathbb{P}^1_k \setminus \{0,1,\infty\}}) = \mathrm{im}(\ker((\rho^{\text{univ-2}}_{0,4})_k) \overset{\mathrm{pr}_k}{\to} G_k)$$

hold. Therefore, we obtain that

$$\mathrm{im}(\ker((\rho^2_{\mathrm{Lgd}})_k) \overset{\mathrm{pr}_k}{\to} G_k) = \mathrm{im}(\ker((\rho^{\text{univ-2}}_{0,4})_k) \overset{\mathrm{pr}_k}{\to} G_k).$$

This completes the proof of Corollary 2.5. □

Definition 2.6. Let (E, O) be an elliptic curve over k (i.e., a pair of a proper smooth curve E of genus one over k and a k-rational point O of E), $E[2]$ the 2-torsion subgroup of $E \otimes_k \overline{k}$, and $k(E[2]) \subseteq \overline{k}$ the field generated by $E[2]$ over k. Then, by a standard argument in algebraic geometry (cf., e.g., [8, Chapter IV, §4]), there exists $\lambda \in k(E[2]) \setminus \{0, 1\}$ such that, after possibly applying a suitable automorphism of $\mathbb{P}^1_{k(E[2])}$ over $k(E[2])$, the set of branch points of the finite morphism $f\colon E \otimes_k k(E[2]) \to \mathbb{P}^1_{k(E[2])}$ determined by the linear system $|2O|$ coincides with $\{0, 1, \lambda, \infty\}$. Moreover, one verifies easily that the set

$$\mathfrak{m}_E := \{\lambda, 1/\lambda, 1-\lambda, 1/(1-\lambda), \lambda/(\lambda-1), (\lambda-1)/\lambda\} \subseteq k(E[2])$$

is *uniquely determined* by the isomorphism class of $E \otimes_k k(E[2])$ over $k(E[2])$. We shall refer to $\mathfrak{m}_E$ as the *Legendre invariant set of E*.

Remark 2.7. Let (E, O) be an elliptic curve over k. Then it follows from the definition of $\mathfrak{m}_E$ that the isomorphism class of $E \otimes_k k(E[2]) \setminus \{O\}$ over $k(E[2])$ may be recovered from $\mathfrak{m}_E$ by considering the scheme obtained by pulling back the Legendre family of elliptic curves $(\mathcal{M}_{\mathrm{Lgd}})_{k(E[2])} \to \mathbb{P}^1_{k(E[2])} \setminus \{0, 1, \infty\}$ via the $k(E[2])$-rational point $[\lambda]\colon \operatorname{Spec}(k(E[2])) \to \mathbb{P}^1_{k(E[2])} \setminus \{0, 1, \infty\}$ determined by $\lambda \in \mathfrak{m}_E$.

Corollary 2.8. *Let (E, O) be an elliptic curve over k and $\lambda \in \mathfrak{m}_E$. Then it holds that*

$$\ker(\rho^2_{\mathbb{P}^1_{k(E[2])} \setminus \{0,1,\lambda,\infty\}}) = \ker(\rho^2_{E \otimes_k k(E[2]) \setminus \{O\}}),$$

$$|\,\mathrm{im}((\rho^{\text{univ-2}}_{0,4})_{k(E[2])})\colon \mathrm{im}(\rho^2_{\mathbb{P}^1_{k(E[2])} \setminus \{0,1,\lambda,\infty\}})|$$
$$= |\,\mathrm{im}((\rho^2_{\mathrm{Lgd}})_{k(E[2])})\colon \mathrm{im}(\rho^2_{E \otimes_k k(E[2]) \setminus \{O\}})|.$$

In particular, the following conditions are equivalent:

(i) $E \setminus \{O\}$ *is quasi-*$\{2\}$*-monodromically full (cf.* [10, Definition 2.2, (iii)]*) (respectively, the equality* $\mathrm{im}((\rho^2_{\mathrm{Lgd}})_{k(E[2])}) = \mathrm{im}(\rho^2_{E \otimes_k k(E[2]) \setminus \{O\}})$ *holds);*

(ii) $\mathbb{P}^1_{k(E[2])} \setminus \{0, 1, \lambda, \infty\}$ *is quasi-*$\{2\}$*-monodromically full (cf.* [10, Definition 2.2, (iii)]*) (respectively,* $\{2\}$*-monodromically full (cf.* [10, Definition 2.2, (i)]*)).*

Proof. Let us first observe that, to verify Corollary 2.8, we may assume without loss of generality, by replacing k by $k(E[2])$, that every 2-torsion point of E is k-rational. Thus, $\lambda \in \mathfrak{m}_E$ determines a k-rational point $[\lambda]\colon \operatorname{Spec}(k) \to \mathbb{P}^1_k \setminus \{0, 1, \infty\}$. Write $\pi_1([\lambda])\colon G_k \to \Pi^{\text{rel-2}}_{(\mathcal{M}_{0,4})_k}$ for the outer homomorphism determined by $[\lambda]$. Now it follows from the various definitions involved that the homomorphism $(\rho_{0,4}^{\text{univ-2}})_k \circ \pi_1([\lambda])\colon G_k \to \mathrm{Out}(\Delta_{0,4}^2)$ (respectively, $(\rho_{\mathrm{Lgd}}^2)_k \circ \pi_1([\lambda])\colon G_k \to \mathrm{Out}(\Delta_{1,1}^2)$) coincides with $\rho^2_{\mathbb{P}^1_k \setminus \{0,1,\lambda,\infty\}}$ (respectively, $\rho^2_{E\setminus\{O\}}$). Therefore, it follows from Corollary 2.5 that the two equalities in the statement of Corollary 2.8 hold. Finally, the equivalence at the final portion of the statement of Corollary 2.8 follows immediately from the various definitions involved. This completes the proof of Corollary 2.8. □

Corollary 2.9. *Let (E_1, O_1) and (E_2, O_2) be elliptic curves over k. Suppose that k is a finitely generated extension of $\mathbb{Q}$, that every 2-torsion point of E_i is k-rational, and that $\rho^2_{E_i\setminus\{O_i\}}(G_k) = (\rho^2_{\mathrm{Lgd}})_k(\Pi^{\text{rel-2}}_{(\mathcal{M}_{0,4})_k})$ for $i = 1, 2$. Then the following conditions are equivalent;*

(i) *$E_1 \setminus \{O_1\}$ is isomorphic to $E_2 \setminus \{O_2\}$ over k;*
(ii) *the kernel of $\rho^2_{E_1\setminus\{O_1\}}$ coincides with the kernel of $\rho^2_{E_2\setminus\{O_2\}}$.*

Proof. The implication

$$\text{(i)} \Longrightarrow \text{(ii)}$$

is immediate; thus, to verify Corollary 2.9, it suffices to show the implication

$$\text{(ii)} \Longrightarrow \text{(i)}.$$

Suppose that condition (ii) is satisfied. Let $\mathfrak{m}_{E_i} \subseteq k$ be the Legendre invariant set of E_i and λ_i an element of $\mathfrak{m}_{E_i}$ for $i = 1, 2$. Then it follows from Corollary 2.8 that $\mathbb{P}^1_k \setminus \{0, 1, \lambda_1, \infty\}$ and $\mathbb{P}^1_k \setminus \{0, 1, \lambda_2, \infty\}$ are $\{2\}$-monodromically full, and

$$\ker(\rho^2_{\mathbb{P}_k\setminus\{0,1,\lambda_1,\infty\}}) = \ker(\rho^2_{\mathbb{P}_k\setminus\{0,1,\lambda_2,\infty\}}).$$

Thus, it follows from [10, Theorem A] that $\mathbb{P}^1_k \setminus \{0, 1, \lambda_1, \infty\}$ is isomorphic to $\mathbb{P}^1_k \setminus \{0, 1, \lambda_2, \infty\}$ over k, which thus implies that

$$\mathfrak{m}_{E_1} = \mathfrak{m}_{E_2}.$$

Therefore, by Remark 2.7, $E_1 \setminus \{O_1\}$ is isomorphic to $E_2 \setminus \{O_2\}$ over k. This completes the proof of Corollary 2.9. □

3. The pro-l outer Galois actions associated to modular curves

In the present §3, we discuss the issue of whether or not the pro-l outer Galois action associated to a *modular curve* (cf. "Fundamental groups" in "Notations and Conventions") factor through a pro-l quotient of the absolute Galois group of a certain number field.

In the present §3, let $\overline{\mathbb{Q}}$ be an algebraic closure of the field of rational numbers $\mathbb{Q}$. For a positive integer N, let $\zeta_N \in \overline{\mathbb{Q}}$ be a *primitive N-th root of unity.* For a subfield F of $\overline{\mathbb{Q}}$, write $G_F := \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. If A is a ring, then we shall denote by $SL_2(A)$ the special linear group of degree 2 over A.

Definition 3.1. Let N be a positive integer. Then we shall write

$$\Gamma(N) \ := \ \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, a \equiv d \equiv 1,\ c \equiv b \equiv 0 \pmod{N} \right\};$$

$$\varGamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, a \equiv d \equiv 1,\, c \equiv 0 \pmod{N} \right\};$$
$$\varGamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod{N} \right\}.$$

Definition 3.2. Let N be a positive integer.

(i) We shall write
$$Y_1(N), \quad Y_0(N)$$
for the respective *modular curves* parametrizing elliptic curves with $\varGamma_1(N)$-, $\varGamma_0(N)$-structures over $\mathbb{Q}$ (cf., e.g., [18]);
$$X_1(N), \quad X_0(N)$$
for the respective *smooth compactifications* of $Y_1(N)$, $Y_0(N)$ over $\mathbb{Q}$;
$$J_1(N), \quad J_0(N)$$
for the respective *Jacobian varieties* of $X_1(N)$, $X_0(N)$.

(ii) We shall write
$$Y(N)$$
for the *modular curve* parametrizing elliptic curves with $\varGamma(N)$-structures over $\mathbb{Q}(\zeta_N)$ (cf., e.g., [18]);
$$X(N)$$
for the *smooth compactification* of $Y(N)$ over $\mathbb{Q}(\zeta_N)$;
$$J(N)$$
for the *Jacobian variety* of $X(N)$.

Lemma 3.3. *Let*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \varDelta_1 & \longrightarrow & \varPi_1 & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \| & & \\ 1 & \longrightarrow & \varDelta_2 & \longrightarrow & \varPi_2 & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

be a commutative diagram of profinite groups, where the horizontal sequences are exact, and the right-hand vertical arrow is the identity morphism of G. Write
$$\rho_1 \colon G \longrightarrow \mathrm{Out}(\varDelta_1), \quad \rho_2 \colon G \longrightarrow \mathrm{Out}(\varDelta_2)$$
for the outer actions of G on $\varDelta_1$, $\varDelta_2$ determined by the upper, lower horizontal sequences of the above diagram, respectively. Suppose that one of the following conditions is satisfied:

(a) *The homomorphism α is surjective.*
(b) *The profinite group $\varDelta_2$ is slim, and the homomorphism α is open.*
(c) *The profinite group $\varDelta_1$ is center-free, the homomorphism α is injective, and the image of the middle vertical arrow $\varPi_1 \to \varPi_2$ of the above diagram is normal.*

Then the following hold:

(i) *It holds that* $\ker(\rho_1) \subseteq \ker(\rho_2)$.
(ii) *Suppose, moreover, that $\varDelta_2$ is pro-l, and that the homomorphism α is an open injection. Then it holds that ρ_1 factors through a pro-l quotient of G if and only if ρ_2 factors through a pro-l quotient of G.*

Proof. First, we verify assertion (i). If condition (a) is satisfied, then assertion (i) follows immediately from the various definitions involved. Next, we verify assertion (i) in the case where condition (b) is satisfied. Now let us observe that it follows immediately from assertion (i) in the case where condition (a) is satisfied that we may assume without loss of generality — by replacing Δ_1 by the image of α — that α is an *open injection*. Thus, assertion (i) in the case where condition (b) is satisfied follows from a similar argument to the argument applied in the proof of [11, Lemma 23, (iii)].

Next, we verify assertion (i) in the case where condition (c) is satisfied. Let us observe that, to verify assertion (i) in the case where condition (c) is satisfied, it follows immediately from the various definitions involved that it suffices to verify the inclusion

$$Z_{\Pi_1}(\Delta_1) \subseteq Z_{\Pi_2}(\Delta_2).$$

Now since $\Delta_1 = \Pi_1 \cap \Delta_2$ is *normal* in Π_2 (cf. condition (c)), hence also in Δ_2, and *center-free*, it follows immediately from a similar argument to the argument applied in the proof of [9, Lemma 4.10] that, to verify the above inclusion $Z_{\Pi_1}(\Delta_1) \subseteq Z_{\Pi_2}(\Delta_2)$, it suffices to verify the following assertion:

> If $\alpha \in Z_{\Pi_1}(\Delta_1)$, then the automorphism of Δ_2/Δ_1 obtained by conjugation by $\alpha \in \Pi_1 \subseteq \Pi_2$ is the *identity automorphism.*

On the other hand, the commutative diagram of the statement of Lemma 3.3 determines an isomorphism $\Delta_2/\Delta_1 \xrightarrow{\sim} \Pi_2/\Pi_1$. Thus, since $\alpha \in Z_{\Pi_1}(\Delta_1) \subseteq \Pi_1$, it follows that the automorphism of Δ_2/Δ_1 obtained by conjugation by α is the *identity automorphism.* This completes the proof of assertion (i). Assertion (ii) follows immediately from assertion (i), together with a similar argument to the argument applied in the proof of [11, Lemma 23, (i)]. This completes the proof of Lemma 3.3. □

Lemma 3.4. *Let F be a field of characteristic zero that contains a primitive l-th root of unity, $\overline{F}$ an algebraic closure of F, and Y a hyperbolic curve over F. Write $G_F := \mathrm{Gal}(\overline{F}/F)$, X for the smooth compactification of Y over F, and J for the Jacobian variety of X. Then the following hold:*

(i) *It holds that*

$$\ker(\rho^l_Y) \;\subseteq\; \ker(\rho^l_X) \;\subseteq\; \ker(\rho^l_J).$$

Moreover, the quotient $\ker(\rho^l_J)/\ker(\rho^l_X)$ is pro-l.

(ii) *Suppose, moreover, that the natural action of G_F on the set of cusps of Y factors through a pro-l quotient of G_F. Then the quotient $\ker(\rho^l_X)/\ker(\rho^l_Y)$ (cf. (i)), hence also $\ker(\rho^l_J)/\ker(\rho^l_Y)$ (cf. (i)), is pro-l.*

(iii) *In the situation of (ii), if, moreover, X is of genus zero, then the outer Galois action ρ^l_Y of G_F on Δ^l_Y factors through a pro-l quotient of G_F.*

Proof. Assertion (i) follows immediately from Lemma 3.3, (i) (in the case where condition (a) is satisfied), and [1, Corollary 7], together with the fact that the natural morphism $X \otimes_F \overline{F} \to J \otimes_F \overline{F}$ determined by an $\overline{F}$-rational point of X induces an *isomorphism* $(\Delta^l_X)^{\mathrm{ab}} \xrightarrow{\sim} \Delta^l_J$.

Next, we verify assertion (ii). First, let us observe that since the natural action of G_F on the set of cusps of Y factors through a pro-l quotient of G_F, we may assume without loss of generality, by replacing F by a suitable finite extension of F, that Y is *split*, i.e., every cusp of Y is defined over F. Write $V_{Y/X}$ for the kernel of the natural surjection of *free* $\mathbb{Z}_l$*-modules of finite rank* $(\Delta^l_Y)^{\mathrm{ab}} \twoheadrightarrow (\Delta^l_X)^{\mathrm{ab}}$ (cf., e.g., [22, Remark 1.2.2]). Then one verifies immediately from [1, Corollary 7] that (since the module $\mathrm{Hom}_{\mathbb{Z}_l}((\Delta^l_X)^{\mathrm{ab}}, V_{Y/X})$ of $\mathbb{Z}_l$-linear homomorphisms from $(\Delta^l_X)^{\mathrm{ab}}$ to $V_{Y/X}$ is a *pro-l* group), to complete the verification of assertion (ii), it suffices to verify that the natural action of $\ker(\rho^l_X)$ on $V_{Y/X}$ factors through a *pro-l* quotient of $\ker(\rho^l_X)$. On the other hand, this follows immediately from the (easily verified) fact that $V_{Y/X}$ is isomorphic, as a $\mathbb{Z}_l$-module equipped with an action of G_F, to the direct sum of finitely many copies of the l-adic cyclotomic character of G_F (cf. our assumption that Y is *split*). This completes the proof of assertion (ii). Assertion (iii) follows immediately from assertion (ii), together with the (easily verified) fact that $\Delta^l_X = \{1\}$ (cf. our assumption that X is of *genus zero*). This completes the proof of Lemma 3.4. □

Lemma 3.5. *Suppose that* $l \notin \{2, 3, 5, 7, 13\}$*. Then the restriction of the action of* $G_{\mathbb{Q}}$ *on the l-adic Tate module* $T_l(J_0(l))$ *of* $J_0(l)$ *to* $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ *does not factor through any pro-l quotient of* $G_{\mathbb{Q}(\zeta_l)}$*.*

Proof. Write $\chi\colon G_{\mathbb{Q}} \to \mathbb{F}_l^{\times}$ for the character of $G_{\mathbb{Q}}$ determined by

$$\sigma(\zeta_l) = \zeta_l^{\chi(\sigma)} \ \ (\text{for } \sigma \in G_{\mathbb{Q}}),$$

$\mathrm{End}(J_0(l))$ for the ring of endomorphisms of the abelian variety $J_0(l)$, $\mathbb{T} \subseteq \mathrm{End}(J_0(l))$ for the *Hecke algebra* (where we refer to [19, p.90, Definition]), and

$$V := \Delta^l_{J_0(l)} \otimes_{\mathbb{Z}_l} \mathbb{F}_l.$$

Then the action of $G_{\mathbb{Q}}$ on $T_l(J_0(l))$ (respectively, the definition of the Hecke algebra) induces a homomorphism of groups (respectively, rings)

$$G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{\mathbb{F}_l}(V), \ \ (\text{respectively, } \mathbb{T} \longrightarrow \mathrm{End}_{\mathbb{F}_l}(V)).$$

Note that since the action of $\mathbb{T} \subseteq \mathrm{End}(J_0(l))$ on $J_0(l)$ is defined over $\mathrm{Spec}(\mathbb{Q})$, the action of $G_{\mathbb{Q}}$ on V commutes with the action of $\mathbb{T}$ on V. Write $\mathbb{T}[G_{\mathbb{Q}}] \subseteq \mathrm{End}_{\mathbb{F}_l}(V)$ for the subring of $\mathrm{End}_{\mathbb{F}_l}(V)$ generated by the images of $G_{\mathbb{Q}} \to \mathrm{Aut}_{\mathbb{F}_l}(V)$ and $\mathbb{T} \to \mathrm{End}_{\mathbb{F}_l}(V)$. Here, let us recall that since l is not contained in $\{2, 3, 5, 7, 13\}$, the dimension of V over $\mathbb{F}_l$ is > 0.

Assume that the restriction of the action of $G_{\mathbb{Q}}$ on the l-adic Tate module $T_l(J_0(l))$ of $J_0(l)$ to $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ does factor through some pro-l quotient of $G_{\mathbb{Q}(\zeta_l)}$. Then it follows from [25, Lemma 3] that there exists an integer i such that

$$V^{\chi^i} := \{v \in V \mid g \cdot v = \chi^i(g) \cdot v \, (\text{for all } g \in G_{\mathbb{Q}})\} \subseteq V$$

is a nontrivial subspace of V. Since the action of $G_{\mathbb{Q}}$ on V commutes with the action of $\mathbb{T}$ on V, for any $g \in G_{\mathbb{Q}}$, $t \in \mathbb{T}$, and $v \in V^{\chi^i}$, it holds that

$$g \cdot (t \cdot v) = t \cdot (g \cdot v) = t \cdot (\chi^i(g) \cdot v) = \chi^i(g) \cdot (t \cdot v).$$

Thus, V^{χ^i} is a $\mathbb{T}[G_{\mathbb{Q}}]$-submodule of V. Let W be a constituent of a $\mathbb{T}$-Jordan-Hölder filtration of V^{χ^i}. Then it follows from the definition of V^{χ^i} that the $\mathbb{T}$-module W is a $\mathbb{T}[G_{\mathbb{Q}}]$-subquotient of V^{χ^i} and, moreover, a constituent of a $\mathbb{T}[G_{\mathbb{Q}}]$-Jordan-Hölder filtration of V, i.e., W is a *constituent* of V in the sense of [19, p.112]. Thus, the annihilator $\mathfrak{M}$ in $\mathbb{T}$ concerning the action on W is a maximal ideal of $\mathbb{T}$, and the action of $\mathbb{T}$ on W induces an injection

$$\mathbb{T}/\mathfrak{M} \hookrightarrow \mathrm{End}_{\mathbb{F}_l}(W).$$

Since W is a simple $\mathbb{T}$-module by the definition of W, the dimension of W over $\mathbb{T}/\mathfrak{M}$ is equal to 1, i.e., the dimension of W is equal to 1 in the sense of [19, p.112]. Hence, it follows from [19, Chapter II, Proposition 14.1] that $\mathfrak{M}$ is an *Eisenstein prime* of $\mathbb{T}$, where we refer to [19, p.96, Definition]. Thus, by [19, Chapter II, Proposition 9.7], the characteristic of the field $\mathbb{T}/\mathfrak{M}$ is prime to l. On the other hand, one verifies easily that $\mathrm{End}_{\mathbb{F}_l}(W)$ is of order a power of l. Thus, we obtain a contradiction. This completes the proof of Lemma 3.5 □

Remark 3.6. The observation given in the proof of Lemma 3.5 was related to the authors by *Akio Tamagawa.*

Lemma 3.7. *Let m be a positive integer. Write $J \subseteq \Gamma_1(l)$ (respectively, $\subseteq \Gamma(l)$) (cf. Definition 3.1) for the normal subgroup obtained by forming the intersection of all $\Gamma_1(l)$- (respectively, $\Gamma(l)$-) conjugates of $\Gamma_1(l^m) \subseteq \Gamma_1(l)$ (respectively, $\Gamma(l^m) \subseteq \Gamma(l)$). Then the index $|\Gamma_1(l) : J|$ (respectively, $|\Gamma(l) : J|$) is a power of l. In particular, the natural finite étale covering $Y_1(l^m) \to Y_1(l)$ (respectively, $Y(l^m) \to Y(l)$) induces an outer open injection $\Pi^{\underline{l}}_{Y_1(l^m)} \hookrightarrow \Pi^{\underline{l}}_{Y_1(l)}$ (respectively, $\Pi^{\underline{l}}_{Y(l^m)} \hookrightarrow \Pi^{\underline{l}}_{Y(l)}$).*

Proof. Let us first observe that, to verify Lemma 3.7, it suffices to verify that the finite groups $\Gamma_1(l)/\Gamma(l^m)$ and $\Gamma(l)/\Gamma(l^m)$ are *l-groups.* On the other hand, one verifies easily that $\Gamma(l)/\Gamma(l^m) \simeq \ker(SL_2(\mathbb{Z}/l^m) \to SL_2(\mathbb{Z}/l))$ is an *l-group.* Thus, Lemma 3.7 follows from the easily verified fact that the index $|\Gamma_1(l) : \Gamma(l)|$ is *equal to l.* This completes the proof of Lemma 3.7. □

Remark 3.8. In the notation of Lemma 3.7, let p be a prime factor of $l-1$, p^ν the largest power of p that divides $l-1$, and $a \in (\mathbb{Z}/l^m)^\times$ an element of order p^ν. Then one verifies easily that the subgroup generated by the matrix

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in SL_2(\mathbb{Z}/l^m) \xleftarrow{\sim} SL_2(\mathbb{Z})/\Gamma(l^m)$$

is a *p-Sylow subgroup* of $\Gamma_0(l^m)/\Gamma(l^m) \subseteq SL_2(\mathbb{Z})/\Gamma(l^m)$. Thus, by considering the conjugate of the above matrix by the matrix

$$\begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix} \in \Gamma_0(l)/\Gamma(l^m) \subseteq SL_2(\mathbb{Z})/\Gamma(l^m),$$

one verifies immediately that

> if $l \geq 5$ and $m \geq 2$, then the assertion obtained by replacing "Γ_1" or "Γ" in Lemma 3.7 by "Γ_0" does *not hold.*

Lemma 3.9. *The following hold:*

(i) *The restriction of the action of* $G_{\mathbb{Q}}$ *on the* 13*-adic Tate module* $T_{13}(J_0(169))$ *of* $J_0(169)$ *to* $G_{\mathbb{Q}(\zeta_{13})} \subseteq G_{\mathbb{Q}}$ *does not factor through any pro-*13 *quotient of* $G_{\mathbb{Q}(\zeta_{13})}$.

(ii) *The restriction of the action of* $G_{\mathbb{Q}}$ *on the* 13*-adic Tate module* $T_{13}(J_1(13))$ *of* $J_1(13)$ *to* $G_{\mathbb{Q}(\zeta_{13})} \subseteq G_{\mathbb{Q}}$ *does not factor through any pro-*13 *quotient of* $G_{\mathbb{Q}(\zeta_{13})}$.

Proof. First, we verify assertion (i). Let us observe that it follows immediately from the *Eichler-Shimura relation* (cf., e.g., [19, p.89]) that the trace of the action of the arithmetic Frobenius element Frob_3 at 3 (respectively, Frob_{29} at 29) on $T_{13}(J_0(169))$ *coincides* with the trace of the action of the *Hecke operator* T_3 (respectively, T_{29}) (cf., e.g., [19, p.87]). Now we claim that

> the characteristic polynomial of the action of the Hecke operator T_3 (respectively, T_{29}) on $T_{13}(J_0(169))$ is
> $$(t-2)^2(t^3+2t^2-t-1)^2$$
> $$(\text{respectively, } (t-3)^2(t^3+t^2-44t+83)^2).$$
> In particular, the trace of the action of Frob_3 (respectively, Frob_{29}) on $T_{13}(J_0(169))$ is 0 (respectively, 4).

Indeed, the above claim follows immediately from [27, `http://modular.math.washington.edu/Tables/charpoly.html`].

On the other hand, it follows, by considering the *semi-simplification* of the action of $G_{\mathbb{Q}}$ on $T_{13}(J_0(169)) \otimes_{\mathbb{Z}_{13}} \mathbb{F}_{13}$, from [25, Lemma 3], together with *Class field theory*, that if the restriction of the action of $G_{\mathbb{Q}}$ on $T_{13}(J_0(169))$ to $G_{\mathbb{Q}(\zeta_{13})} \subseteq G_{\mathbb{Q}}$ *factors* through a pro-13 quotient of $G_{\mathbb{Q}(\zeta_{13})}$, then the traces of the actions of Frob_3, Frob_{29} on $T_{13}(J_0(169)) \otimes_{\mathbb{Z}_{13}} \mathbb{F}_{13}$ *coincide*. Thus, since $0 \not\equiv 4 \pmod{13}$, we conclude that the restriction of the action of $G_{\mathbb{Q}}$ on $T_{13}(J_0(169))$ to $G_{\mathbb{Q}(\zeta_{13})} \subseteq G_{\mathbb{Q}}$ does *not factor* through any pro-13 quotient of $G_{\mathbb{Q}(\zeta_{13})}$. This completes the proof of assertion (i).

Next, we verify assertion (ii). Now one verifies immediately from assertion (i), Lemma 3.4, (i), and Lemma 3.3, (i) (in the case where condition (b) is satisfied — cf., e.g., [22, Proposition 1.4]), that the restriction of the outer action of $G_{\mathbb{Q}}$ on $\Delta^{13}_{Y_1(169)}$ to $G_{\mathbb{Q}(\zeta_{13})} \subseteq G_{\mathbb{Q}}$ does *not factor* through a pro-13 quotient of $G_{\mathbb{Q}(\zeta_{13})}$. Thus, it follows immediately from Lemma 3.3, (ii) (in the case where condition (b) is satisfied — cf., e.g., [22, Proposition 1.4]), together with Lemma 3.8, that the restriction of the outer action of $G_{\mathbb{Q}}$ on $\Delta^{13}_{Y_1(13)}$ to $G_{\mathbb{Q}(\zeta_{13})} \subseteq G_{\mathbb{Q}}$ does *not factor* through a pro-13 quotient of $G_{\mathbb{Q}(\zeta_{13})}$. In particular, since the complement of $Y_1(13)$ in $X_1(13)$ consists of six $\mathbb{Q}$-rational points and a $\mathbb{Q}(\zeta_{13}+\zeta_{13}^{-1})$-rational point (cf., e.g., the discussion given in [20, §3]), one verifies from Lemma 3.4, (ii), that the restriction of the action of $G_{\mathbb{Q}}$ on $T_{13}(J_1(13))$ to $G_{\mathbb{Q}(\zeta_{13})} \subseteq G_{\mathbb{Q}}$ does *not factor* through any pro-13 quotient of $G_{\mathbb{Q}(\zeta_{13})}$. This completes the proof of assertion (ii), hence also of Lemma 3.9. □

Remark 3.10. The observation given in the proof of Lemma 3.9, (i), was related to the authors by *Akio Tamagawa*. The content of Lemma 3.9, (ii), was pointed out to the authors by *Seidai Yasuda*.

Lemma 3.11. *The following hold:*

(i) *The restriction to* $G_{\mathbb{Q}(\zeta_7)} \subseteq G_{\mathbb{Q}}$ *of the action of* $G_{\mathbb{Q}}$ *on the* 7*-adic Tate module* $T_7(J_0(49))$ *of* $J_0(49)$ *factors through a pro-*7 *quotient of* $G_{\mathbb{Q}(\zeta_7)}$.

(ii) *The action of* $G_{\mathbb{Q}(\zeta_7)}$ *on the* 7*-adic Tate module* $T_7(J(7))$ *of* $J(7)$ *factors through a pro-*7 *quotient of* $G_{\mathbb{Q}(\zeta_7)}$.

Proof. Assertion (i) follows immediately from [25, Table 1], together with the well-known fact that $X_0(49)$ admits a *structure of elliptic curve over* $\mathbb{Q}$ and is listed as "49a". Next, we verify assertion (ii). Let us first observe that the (easily verified) inclusion

$$\begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \cdot \Gamma(7) \cdot \begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix} \subseteq \Gamma_0(49)$$

(cf. Definition 3.1) implies the existence of a dominant morphism $X(7) \to X_0(49) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_7)$ over $\mathbb{Q}(\zeta_7)$, hence also a *surjection* $J(7) \twoheadrightarrow J_0(49) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_7) = X_0(49) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_7)$ (cf. the proof of assertion (i)) over $\mathbb{Q}(\zeta_7)$. Thus, it follows immediately from [24, Theorem 2] that we have a $G_{\mathbb{Q}(\zeta_7)}$*-equivariant* isomorphism

$$T_7(J(7)) \otimes_{\mathbb{Z}_7} \mathbb{Q}_7 \overset{\sim}{\to} \left(T_7(J_0(49))^{\oplus 3}\right) \otimes_{\mathbb{Z}_7} \mathbb{Q}_7 .$$

Thus, it follows immediately from assertion (i) that the restriction of the action of $G_{\mathbb{Q}}$ on $T_7(J(7))$ to $G_{\mathbb{Q}(\zeta_7)} \subseteq G_{\mathbb{Q}}$ *factors* through a pro-7 quotient of $G_{\mathbb{Q}(\zeta_7)}$. This completes the proof of Lemma 3.11. □

Remark 3.12. The content of Lemma 3.11 was pointed out to the authors by *Seidai Yasuda.*

Theorem 3.13. *Let* l *be a prime number. Consider the following conditions:*

(P) $l \in \{2, 3, 5, 7\}$.

(Q) $l \in \{2, 3, 5, 7, 13\}$.

(Y_1) *The restriction of the outer action of* $G_{\mathbb{Q}}$ *on* $\Delta^l_{Y_1(l)}$ *to* $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ *factors through a pro-*l *quotient of* $G_{\mathbb{Q}(\zeta_l)}$.

(X_1) *The restriction of the outer action of* $G_{\mathbb{Q}}$ *on* $\Delta^l_{X_1(l)}$ *to* $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ *factors through a pro-*l *quotient of* $G_{\mathbb{Q}(\zeta_l)}$.

(J_1) *The restriction of the action of* $G_{\mathbb{Q}}$ *on the* l*-adic Tate module* $T_l(J_1(l))$ *of* $J_1(l)$ *to* $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ *factors through a pro-*l *quotient of* $G_{\mathbb{Q}(\zeta_l)}$.

(Y_0) *The restriction of the outer action of* $G_{\mathbb{Q}}$ *on* $\Delta^l_{Y_0(l)}$ *to* $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ *factors through a pro-*l *quotient of* $G_{\mathbb{Q}(\zeta_l)}$.

(X_0) *The restriction of the outer action of* $G_{\mathbb{Q}}$ *on* $\Delta^l_{X_0(l)}$ *to* $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ *factors through a pro-*l *quotient of* $G_{\mathbb{Q}(\zeta_l)}$.

(J_0) *The restriction of the action of* $G_{\mathbb{Q}}$ *on the* l*-adic Tate module* $T_l(J_0(l))$ *of* $J_0(l)$ *to* $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ *factors through a pro-*l *quotient of* $G_{\mathbb{Q}(\zeta_l)}$.

(Y) *The outer action of* $G_{\mathbb{Q}(\zeta_l)}$ *on* $\Delta^l_{Y(l)}$ *factors through a pro-*l *quotient of* $G_{\mathbb{Q}(\zeta_l)}$.

(X) *The outer action of* $G_{\mathbb{Q}(\zeta_l)}$ *on* $\Delta^l_{X(l)}$ *factors through a pro-*l *quotient of* $G_{\mathbb{Q}(\zeta_l)}$.

(J) *The action of* $G_{\mathbb{Q}(\zeta_l)}$ *on the* l*-adic Tate module* $T_l(J(l))$ *of* $J(l)$ *factors through a pro-*l *quotient of* $G_{\mathbb{Q}(\zeta_l)}$.

Then the implications

$$\begin{aligned}&(\mathrm{P}) \Longleftrightarrow (\mathrm{Y}_1) \Longleftrightarrow (\mathrm{X}_1) \Longleftrightarrow (\mathrm{J}_1) \Longleftrightarrow (\mathrm{Y}) \Longleftrightarrow (\mathrm{X}) \Longleftrightarrow (\mathrm{J}) \\ &\Longrightarrow (\mathrm{Q}) \Longleftrightarrow (\mathrm{Y}_0) \Longleftrightarrow (\mathrm{X}_0) \Longleftrightarrow (\mathrm{J}_0)\end{aligned}$$

hold.

Proof. Let us first observe that we have an immediate implication

$$(\mathrm{P}) \Longrightarrow (\mathrm{Q}).$$

Next, let us observe that the implications

$$\begin{array}{ccccc} (\mathrm{Y}) & \Longrightarrow & (\mathrm{X}) & \Longrightarrow & (\mathrm{J}) \\ \Downarrow & & & & \\ (\mathrm{Y}_1) & \Longrightarrow & (\mathrm{X}_1) & \Longrightarrow & (\mathrm{J}_1) \\ \Downarrow & & & & \\ (\mathrm{Y}_0) & \Longrightarrow & (\mathrm{X}_0) & \Longrightarrow & (\mathrm{J}_0) \end{array}$$

follow immediately from Lemma 3.3, (i) (in the case where condition (b) is satisfied — cf., e.g., [22, Proposition 1.4]) and Lemma 3.4, (i). Next, let us observe that the implications

$$(\mathrm{J}_1) \Longrightarrow (\mathrm{Y}_1), \quad (\mathrm{J}_0) \Longrightarrow (\mathrm{Y}_0), \quad (\mathrm{J}) \Longrightarrow (\mathrm{Y})$$

follow immediately from Lemma 3.4, (ii), together with the fact that every cusp of $Y(l)$, hence also of $Y_0(l)$ and $Y_1(l)$, is defined over $\mathbb{Q}(\zeta_l)$ (cf. the discussion given in [17, §1.4]).

Next, we verify the implication

$$(\mathrm{Y}_0) \Longrightarrow (\mathrm{Q}).$$

Suppose that condition (Y_0) is satisfied. Then it follows from the implication $(\mathrm{Y}_0) \Rightarrow (\mathrm{J}_0)$ already verified that condition (J_0) is satisfied. Thus, it follows from Lemma 3.5 that condition (Q) is satisfied. This completes the proof of the implication $(\mathrm{Y}_0) \Rightarrow (\mathrm{Q})$.

Next, we verify the implication

$$(\mathrm{Y}_1) \Longrightarrow (\mathrm{P}).$$

Suppose that condition (Y_1) is satisfied. Then it follows from the implications $(\mathrm{Y}_1) \Rightarrow (\mathrm{Y}_0)$, $(\mathrm{Y}_0) \Rightarrow (\mathrm{Q})$ already verified that, to complete the verification of condition (P), it suffices to verify that $l \neq 13$. On the other hand, if $l = 13$, then it follows immediately from Lemma 3.9, (ii) that condition (J_1), hence also (cf. the implication $(\mathrm{Y}_1) \Rightarrow (\mathrm{J}_1)$ already verified) condition (Y_1), is *not satisfied*. This completes the proof of the implication $(\mathrm{Y}_1) \Rightarrow (\mathrm{P})$.

Next, we verify the implication

$$(\mathrm{P}) \Longrightarrow (\mathrm{Y}).$$

Suppose that condition (P) is satisfied. If $l \neq 7$, then since (as is well-known) $X(l)$ is of *genus zero*, and every cusp of $Y(l)$ is defined over $\mathbb{Q}(\zeta_l)$ (cf. the discussion given in [17, §1.4]), condition (Y) follows from Lemma 3.4, (iii). If $l = 7$, then it follows from Lemma 3.11, (ii), that condition (J), hence also (cf. the implication (J) $\Rightarrow$ (Y) already verified) condition (Y), is satisfied. This completes the proof of the implication (P) $\Rightarrow$ (Y).

Next, we verify the implication

$$(\mathrm{Q}) \Longrightarrow (\mathrm{Y}_0).$$

Suppose that condition (Q) is satisfied. If $l \neq 13$ (i.e., condition (P) is satisfied), then condition (Y_0) follows immediately from the implications (P) $\Rightarrow$ (Y), (Y) $\Rightarrow$ (Y_0) already verified. Thus, to complete the verification of condition (Y_0), we may assume without loss of generality that $l = 13$. Now let us recall that $X_0(13)$ is of *genus zero*. Thus, it follows from Lemma 3.4, (iii), that condition (X_0), hence also (cf. the implications (X_0) $\Rightarrow$ (J_0), (J_0) $\Rightarrow$ (Y_0) already verified) condition (Y_0), is satisfied. This completes the proof of the implication (Q) $\Rightarrow$ (Y_0), hence also of Theorem 3.13. □

Corollary 3.14. *Let l be a prime number and m a positive integer. Then the following conditions are equivalent:*

(P) $l \in \{2, 3, 5, 7\}$.

(Y_1) *The restriction of the outer action of $G_{\mathbb{Q}}$ on $\Delta^l_{Y_1(l^m)}$ to $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ factors through a pro-l quotient of $G_{\mathbb{Q}(\zeta_l)}$.*

(X_1) *The restriction of the outer action of $G_{\mathbb{Q}}$ on $\Delta^l_{X_1(l^m)}$ to $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ factors through a pro-l quotient of $G_{\mathbb{Q}(\zeta_l)}$.*

(J_1) *The restriction of the action of $G_{\mathbb{Q}}$ on the l-adic Tate module $T_l(J_1(l^m))$ of $J_1(l^m)$ to $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ factors through a pro-l quotient of $G_{\mathbb{Q}(\zeta_l)}$.*

(Y) *The outer action of $G_{\mathbb{Q}(\zeta_{l^m})}$ on $\Delta^l_{Y(l^m)}$ factors through a pro-l quotient of $G_{\mathbb{Q}(\zeta_{l^m})}$.*

(X) *The outer action of $G_{\mathbb{Q}(\zeta_{l^m})}$ on $\Delta^l_{X(l^m)}$ factors through a pro-l quotient of $G_{\mathbb{Q}(\zeta_{l^m})}$.*

(J) *The action of $G_{\mathbb{Q}(\zeta_{l^m})}$ on the l-adic Tate module $T_l(J(l^m))$ of $J(l^m)$ factors through a pro-l quotient of $G_{\mathbb{Q}(\zeta_{l^m})}$.*

Proof. Let us first observe that the implications

$$\begin{array}{ccccc} (\mathrm{Y}) & \Longleftrightarrow & (\mathrm{X}) & \Longleftrightarrow & (\mathrm{J}) \\ \Downarrow & & & & \\ (\mathrm{Y}_1) & \Longleftrightarrow & (\mathrm{X}_1) & \Longleftrightarrow & (\mathrm{J}_1) \end{array}$$

follow immediately from similar arguments to the arguments applied in the first paragraph of the proof of Theorem 3.13. Thus, it follows immediately from Theorem 3.13 that, to complete the verification of Corollary 3.14, it suffices to verify the following assertion:

> It holds that condition (Y_1) (respectively, (Y)) is satisfied if and only if condition (Y_1) (respectively, (Y)) in the case where we take "m" to be 1 is satisfied.

Now if condition (Y_1) (respectively, (Y)) is satisfied, then it follows from Lemma 3.3, (i) (in the case where condition (b) is satisfied — cf., e.g., [22, Proposition 1.4]), that condition (Y_1) (respectively, (Y)) in the case where we take "m" to be 1 is satisfied. On the other hand, if condition (Y_1)

(respectively, (Y)) in the case where we take "m" to be 1 is satisfied, then it follows immediately from Lemma 3.3, (ii) (in the case where condition (b) is satisfied — cf., e.g., [22, Proposition 1.4]), together with Lemma 3.7, that condition (Y_1) (respectively, (Y)) is satisfied. This completes the proof of Corollary 3.14. □

Corollary 3.15. *Let l be a prime number and m a positive integer. Then the following conditions are equivalent:*

- (Q′) $l \in \{2, 3, 5, 7, 13\}$*, and* $m = 1$ *if* $l = 13$.
- (Y_0) *The restriction of the outer action of* $G_{\mathbb{Q}}$ *on* $\Delta^l_{Y_0(l^m)}$ *to* $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ *factors through a pro-l quotient of* $G_{\mathbb{Q}(\zeta_l)}$.
- (X_0) *The restriction of the outer action of* $G_{\mathbb{Q}}$ *on* $\Delta^l_{X_0(l^m)}$ *to* $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ *factors through a pro-l quotient of* $G_{\mathbb{Q}(\zeta_l)}$.
- (J_0) *The restriction of the action of* $G_{\mathbb{Q}}$ *on the l-adic Tate module* $T_l(J_0(l^m))$ *of* $J_0(l^m)$ *to* $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ *factors through a pro-l quotient of* $G_{\mathbb{Q}(\zeta_l)}$.

Proof. Let us first observe that if $m = 1$, then Corollary 3.15 follows from Theorem 3.13. Thus, it suffices to verify Corollary 3.15 in the case where $m > 1$. Note that the implications

$$(Y_0) \Longleftrightarrow (X_0) \Longleftrightarrow (J_0)$$

follow immediately from similar arguments to the arguments applied in the first paragraph of the proof of Theorem 3.13. If $l \notin \{2, 3, 5, 7, 13\}$ (respectively, $l \in \{2, 3, 5, 7\}$; $l = 13$), then it follows immediately, in light of the equivalences in the above display, from Lemma 3.3, (i) (in the case where condition (b) is satisfied — cf., e.g., [22, Proposition 1.4]), together with the implication (Y_0) ⇒ (Q) of Theorem 3.13 (respectively, the implication (P) ⇒ (Y_1) of Corollary 3.14; Lemma 3.9, (i)), that the three conditions (Y_0), (X_0), and (J_0) are *not satisfied* (respectively, are *satisfied*; are *not satisfied*). This completes the proof of Corollary 3.15. □

4. A pro-l version of the congruence subgroup problem for mapping class groups of genus one: The general case

In the present §4, we maintain the notation of §1 and the preceding §3. In the present §4, we continue our sturdy of the congruence subgroup problem for the relative pro-l completions of mapping class groups. In particular, we prove that, if $l \neq 2, 3, 5, 7$, then the quotient of the profinite completion of the mapping class group of genus one determined by the pro-l outer geometric monodromy representation does *not coincide* with the relative pro-l completion of the mapping class group of genus one.

In the present §4, by means of an injection $\overline{\mathbb{Q}} \hookrightarrow \overline{k}$, let us regard $\overline{\mathbb{Q}}$ as a subfield of $\overline{k}$. Write $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For a subfield F of $\overline{k}$ which contains ζ_l, write $(Y(l))_F := Y(l) \otimes_{\mathbb{Q}(\zeta_l)} F$ and $(X(l))_F := X(l) \otimes_{\mathbb{Q}(\zeta_l)} F$. If A is a ring, then we shall denote by $PSL_2(A)$ the projective special linear group of degree 2 over A.

Lemma 4.1. *Let*

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathit{\Delta} & \xrightarrow{i^{\mathit{\Pi}_1}} & \mathit{\Pi}_1 & \longrightarrow & G_1 & \longrightarrow & 1 \\
 & & {\scriptstyle i^{\mathit{\Pi}_2}}\downarrow & & \downarrow{\scriptstyle i_{\mathit{\Pi}_1}} & & \| & & \\
1 & \longrightarrow & \mathit{\Pi}_2 & \xrightarrow[i_{\mathit{\Pi}_2}]{} & \mathit{\Pi}_3 & \longrightarrow & G_1 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & G_2 & = & G_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

be a commutative diagram of profinite groups, where the horizontal and vertical sequences are exact, the right-hand vertical arrow is the identity morphism of G_1*, and the lower horizontal arrow is the identity morphism of* G_2*. Write*

$$\rho_1 \colon G_1 \longrightarrow \mathrm{Out}(\mathit{\Delta}) \quad \textit{and} \quad \rho_2 \colon G_2 \longrightarrow \mathrm{Out}(\mathit{\Delta})$$

for the outer actions associated to the top horizontal and left-hand vertical exact sequences, respectively. Then any element of $\mathrm{im}(\rho_2)$ *commutes with any element of* $\mathrm{im}(\rho_1)$*.*

Proof. One verifies easily that the left-hand upper square in the diagram in the statement of Lemma 4.1 is *cartesian*, i.e., the equality $i_{\mathit{\Pi}_1} \circ i^{\mathit{\Pi}_1}(\mathit{\Delta}) = i_{\mathit{\Pi}_1}(\mathit{\Pi}_1) \cap i_{\mathit{\Pi}_2}(\mathit{\Pi}_2)$ holds. Thus, it follows that the commutator subgroup $[i_{\mathit{\Pi}_1}(\mathit{\Pi}_1), i_{\mathit{\Pi}_2}(\mathit{\Pi}_2)] \subseteq \mathit{\Pi}_3$ is contained in $i_{\mathit{\Pi}_1} \circ i^{\mathit{\Pi}_1}(\mathit{\Delta})$. In particular, one verifies immediately from the various definitions involved that any element of $\mathrm{im}(\rho_2)$ commutes with any element of $\mathrm{im}(\rho_1)$. This completes the proof of Lemma 4.1. $\square$

Proposition 4.2. *Suppose that* k *contains* ζ_l*, and that* $l > 2$*. Then the equality*

$$\ker(\rho^{l}_{(Y(l))_k}) = \ker((\rho^{\text{rel-}l}_{1,1})_k)$$

holds.

Proof. Let us first observe that, by the various definitions involved, we have the following commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathit{\Delta}^l_{(Y(l))_k} & \longrightarrow & \mathit{\Gamma}^{\text{rel-}l}_{1,1} & \longrightarrow & SL_2(\mathbb{F}_l) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathit{\Pi}^l_{(Y(l))_k} & \longrightarrow & \mathit{\Pi}^{\text{rel-}l}_{(\mathcal{M}_{1,1})_k} & \longrightarrow & SL_2(\mathbb{F}_l) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & G_k & = & G_k & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

where the vertical and horizontal sequences are exact, the lower horizontal arrow is the identity morphism of G_k, and the right-hand vertical arrow is the identity morphism of $SL_2(\mathbb{F}_l)$. In particular, since $\mathit{\Delta}^l_{(Y(l))_k}$ is center-free (cf., e.g., [17, §1.4], [22, Proposition 1.4]), it follows from Lemma 3.3, (i) (in the case where condition (c) is satisfied), that, to verify Proposition 4.2, it suffices to verify that

$$\ker(\rho^l_{(Y(l))_k}) \supseteq \ker((\rho^{\text{rel-}l}_{1,1})_k).$$

Write $\varphi^l_{SL} \colon SL_2(\mathbb{F}_l) \to \mathrm{Out}(\mathit{\Delta}^l_{(Y(l))_k})$ for the homomorphism determined by the upper horizontal sequence of the above commutative diagram. Let σ be an element of $\ker((\rho^{\text{rel-}l}_{1,1})_k)$. Note that, by [15, Lemma 2.2], $\rho^l_{(Y(l))_k}(\sigma)$ is contained in $\mathrm{im}(\varphi^l_{SL})$.

First, suppose that $l > 3$. Since (one verifies easily that)

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL_2(\mathbb{F}_l)$$

is contained in the image of the restriction to $Z_{\mathit{\Gamma}^{\text{rel-}l}_{1,1}}(\mathit{\Gamma}^{\text{rel-}l}_{1,1}) \subseteq \mathit{\Gamma}^{\text{rel-}l}_{1,1}$ of $\mathit{\Gamma}^{\text{rel-}l}_{1,1} \to SL_2(\mathbb{F}_l)$, it follows immediately that the homomorphism $\varphi^l_{SL} \colon SL_2(\mathbb{F}_l) \to \mathrm{Out}(\mathit{\Delta}^l_{(Y(l))_k})$ factors through $PSL_2(\mathbb{F}_l)$. Hence, it follows from the *simplicity* of the group $PSL_2(\mathbb{F}_l)$ (cf., e.g., [5, Chapter II, §10, Exercise 14]) that the image of φ^l_{SL} is isomorphic to either $\{1\}$ or $PSL_2(\mathbb{F}_l)$, which thus implies that $\mathrm{im}(\varphi^l_{SL})$ is *center-free*. In particular, by Lemma 4.1, together with the fact that $\rho^l_{(Y(l))_k}(\sigma) \in \mathrm{im}(\varphi^l_{SL})$ (already verified above), we conclude that σ is contained in $\ker(\rho^l_{(Y(l))_k})$. This completes the proof of the case where $l > 3$.

Next, suppose that $l = 3$. Let us first recall that $(Y(3))_k$ is a curve of type $(0, 4)$ over k, and every cusp of $(Y(3))_k$ is *k-rational* (cf. [17, (A1.5.1)], [17, §1.4]). Note that $\varphi^3_{SL} \colon SL_2(\mathbb{F}_3) \to \mathrm{Out}(\mathit{\Delta}^3_{(Y(3))_k})$ factors through the natural homomorphism $\mathrm{Aut}_k((Y(3))_k) \to \mathrm{Out}(\mathit{\Delta}^3_{(Y(3)_k)})$ by the various definitions involved. Thus, by comparing the natural actions of $\mathrm{Aut}_k((Y(3))_k) \subseteq \mathrm{Aut}_k((X(3))_k) \simeq \mathrm{Aut}_k(\mathbb{P}^1_k)$ and $\mathrm{im}(\rho^3_{(Y(3))_k})$ on the set of conjugacy classes of cuspidal inertia subgroups of $\Delta^3_{(Y(3))_k}$ (cf. the fact that every cusp of $(Y(3))_k$ is defined over k), we conclude that σ is contained in $\ker(\rho^3_{(Y(3))_k})$. This completes the proof of the case where $l = 3$, hence also of Proposition 4.2. □

Theorem 4.3 (cf. [15, Corollary 3.8])**.** *Let* (g, r) *be a pair of nonnegative integers such that* $3g - 3 + r > 0$*. Then the kernel of the homomorphism* $(\rho^{\text{rel-}l}_{g,r})_k$ *is contained in the kernel of the homomorphism*

$$\rho^l_{\mathbb{P}^1_k \setminus \{0,1,\infty\}} : G_k \longrightarrow \mathrm{Out}(\Delta^l_{\mathbb{P}^1_k \setminus \{0,1,\infty\}}).$$

Proof. If either $(g, r) \neq (1, 1)$ or $l = 2$, then Theorem 4.3 follows from [15, Corollary 3.8]. Thus, to verify Theorem 4.3, we may assume that $(g, r) = (1, 1)$ and $l > 2$. Next, let us observe that one verifies immediately that, to complete the verification of Theorem 4.3, it suffices to verify Theorem 4.3 in the case where we take "k" to be $\mathbb{Q}$. Moreover, we claim that

> Theorem 4.3 in the case where we take "k" to be $\mathbb{Q}(\zeta_l)$ implies Theorem 4.3 (i.e., Theorem 4.3 in the case where we take "k" to be $\mathbb{Q}$).

Indeed, suppose that $\ker((\rho^{\text{rel-}l}_{1,1})_{\mathbb{Q}(\zeta_l)})$ is contained in $\ker(\rho^l_{\mathbb{P}^1_{\mathbb{Q}(\zeta_l)} \setminus \{0,1,\infty\}})$. Then we have that

$$\ker((\rho^{\text{rel-}l}_{1,1})_{\mathbb{Q}(\zeta_l)}) = \ker((\rho^{\text{rel-}l}_{1,1})_{\mathbb{Q}}) \cap G_{\mathbb{Q}(\zeta_l)} \subseteq \ker(\rho^l_{\mathbb{P}^1_{\mathbb{Q}(\zeta_l)} \setminus \{0,1,\infty\}}) \subseteq \ker(\rho^l_{\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}}).$$

In particular, the image of $\ker((\rho^{\text{rel-}l}_{1,1})_{\mathbb{Q}})$ by the natural homomorphism $G_{\mathbb{Q}} \twoheadrightarrow G_{\mathbb{Q}} / \ker(\rho^l_{\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}})$ is a finite normal subgroup of $G_{\mathbb{Q}} / \ker(\rho^l_{\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}})$. On the other hand, it follows from [10, Lemma 4.3, (ii)] that $G_{\mathbb{Q}} / \ker(\rho^l_{\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}})$ is *slim*. Thus, the above claim follows from the well-known fact that any finite normal closed subgroup of a slim profinite group is *trivial* (cf., e.g., [22, §0]). This completes the proof of the claim. It follows from the above claim that, to complete the verification of Theorem 4.3, we may assume without loss of generality that $k = \mathbb{Q}(\zeta_l)$.

Since $(Y(l))_k$ is a hyperbolic curve over k, it follows from [12, Theorem C] that

$$\ker(\rho^l_{(Y(l))_k}) \subseteq \ker(\rho^l_{\mathbb{P}^1_k \setminus \{0,1,\infty\}}).$$

Thus, since (we have assumed that) $l > 2$, Theorem 4.3 follows immediately from Proposition 4.2. This completes the proof of Theorem 4.3. □

Remark 4.4. Let (g, r) be a pair of nonnegative integers such that $3g-3+r > 0$. In the summer of 2011, Makoto Matsumoto gave the second author the following problem:

> $(M_{g,r,l})$: Does the kernel of the homomorphism $(\rho^{\text{rel-}l}_{g,r})_k$ *coincide* with the kernel of the homomorphism
>
> $$\rho^l_{\mathbb{P}^1_k \setminus \{0,1,\infty\}} : G_k \longrightarrow \mathrm{Out}(\Delta^l_{\mathbb{P}^1_k \setminus \{0,1,\infty\}})?$$

The second author proved, in response to this problem, the following Theorem (cf. [15, Theorem 3.4]):

Suppose that either $(g, r) \neq (1, 1)$ or $l = 2$. Then the kernel of the homomorphism $(\rho^{\text{geo-}l}_{g,r})_k$ coincides with the kernel of the homomorphism

$$\rho^{l}_{\mathbb{P}^1_k \setminus \{0,1,\infty\}} \colon G_k \longrightarrow \mathrm{Out}(\Delta^{l}_{\mathbb{P}^1_k \setminus \{0,1,\infty\}}).$$

However, $(M_{g,r,l})$ is answered in the *negative* if $g = 1$ and $l \neq 2, 3, 5, 7$ (cf. Theorem 4.5, Remark 4.6, below).

Theorem 4.5. *The equality*

$$\ker((\rho^{\text{rel-}l}_{1,1})_{\mathbb{Q}}) = \ker(\rho^{l}_{\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}})$$

holds if and only if

$$l \in \{2, 3, 5, 7\}.$$

In particular, if r is a positive integer, and

$$l \notin \{2, 3, 5, 7\},$$

then

$$\ker((\rho^{\text{rel-}l}_{1,r})_{\mathbb{Q}}) \subsetneq \ker(\rho^{l}_{\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}}).$$

Proof. Let us first observe that, for any positive integer r, we have the following commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma^{\text{rel-}l}_{1,r+1} & \longrightarrow & \Pi^{\text{rel-}l}_{(\mathcal{M}_{1,r+1})_{\mathbb{Q}}} & \longrightarrow & G_{\mathbb{Q}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Gamma^{\text{rel-}l}_{1,r} & \longrightarrow & \Pi^{\text{rel-}l}_{(\mathcal{M}_{1,r})_{\mathbb{Q}}} & \longrightarrow & G_{\mathbb{Q}} & \longrightarrow & 1 \end{array}$$

where the horizontal sequences are exact, the vertical arrows are surjective, and the right-hand vertical arrow is the identity morphism of $G_{\mathbb{Q}}$. Thus, it follows immediately from Theorem 4.3, together with Lemma 3.3, (i) (in the case where condition (a) is satisfied), that, to verify Theorem 4.5, it suffices to verify the first equivalence of the statement of Theorem 4.5.

First, suppose that $l = 2$. Then it follows from Corollary 2.3 and [15, Theorem 3.4] that the equalities

$$\ker((\rho^{\text{rel-}2}_{1,1})_{\mathbb{Q}}) = \ker((\rho^{\text{geo-}2}_{1,1})_{\mathbb{Q}}) = \ker(\rho^{2}_{\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}})$$

hold. This completes the proof of the case where $l = 2$.

Next, suppose that $l \in \{3, 5, 7\}$. Then let us recall that since l is *regular* and *odd*, Ihara's problem concerning the pro-l outer Galois action associated to a tripod (cf., e.g., [14, Lecture I, §2], [25, Introduction]) is answered in the *affirmative* (cf. the main result of [6], together with [26, Theorem 1.1]). Thus, since $(\rho^{\text{rel-}l}_{1,1})_{\mathbb{Q}}(G_{\mathbb{Q}(\zeta_l)})$ is *pro-l* (cf. the equivalence (P) $\Leftrightarrow$ (Y) of Theorem 3.13, Proposition 4.2), it follows immediately from [7, Theorem 3] that the equality

$$\ker((\rho^{\text{rel-}l}_{1,1})_{\mathbb{Q}}) = \ker(\rho^{l}_{\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}})$$

holds. This completes the proof of the case where $l \in \{3, 5, 7\}$.

Finally, suppose that $l \notin \{2, 3, 5, 7\}$. Then it follows from Proposition 4.2 that, to complete the verification of the case where $l \notin \{2, 3, 5, 7\}$, it suffices to prove that

$$\ker(\rho^{l}_{Y(l)}) \neq \ker(\rho^{l}_{\mathbb{P}^1_{\mathbb{Q}(\zeta_l)} \setminus \{0,1,\infty\}}).$$

On the other hand, if

$$\ker(\rho^{l}_{Y(l)}) = \ker(\rho^{l}_{\mathbb{P}^1_{\mathbb{Q}(\zeta_l)} \setminus \{0,1,\infty\}}),$$

then it follows from [10, Lemma 4.3, (ii)] that the image of $\rho^{l}_{Y(l)}$ is *pro-l*, which contradicts the equivalence (P) $\Leftrightarrow$ (Y) of Theorem 3.13. This completes the proof of the case where $l \notin \{2, 3, 5, 7\}$, hence also of Theorem 4.5. $\square$

Remark 4.6. Let (g, r) be a pair of nonnegative integers such that $3g - 3 + r > 0$.

(i) Let us recall that, as is well-known, there exists an isomorphism of $(\mathcal{M}_{0,4})_k$ with $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$ over k. Thus, it follows immediately from Definition 1.1 that we have an equality

$$\ker(\rho^{l}_{\mathbb{P}^1_k \setminus \{0,1,\infty\}}) \ = \ \ker((\rho^{\text{rel-}l}_{0,4})_k).$$

In particular, the problem $(M_{g,r,l})$ of Remark 4.4 is equivalent to the following problem:

Does the equality $\ker((\rho^{\text{rel-}l}_{g,r})_k) = \ker((\rho^{\text{rel-}l}_{0,4})_k)$ hold?

That is to say, roughly speaking, the problem $(M_{g,r,l})$ of Remark 4.4 concerns the issue of whether or not the kernel $\ker((\rho^{\text{rel-}l}_{g,r})_k)$ is *independent of the pair* (g, r).

(ii) We prove, in Theorem 4.5, that the problem $(M_{g,r,l})$ of Remark 4.4 has a *negative* answer for some triple (g, r, l).

(iii) From the point of view of the discussion of (i), one can pose the following problem, which may be regarded as a *weaker version* of the problem $(M_{g,r,l})$ of Remark 4.4:

$(M^{w}_{g,r,l})$: Does the kernel of the homomorphism $(\rho^{\text{rel-}l}_{g,r})_k$ *coincide* with the kernel of the homomorphism $(\rho^{\text{rel-}l}_{g,r+1})_k$?

That is to say, roughly speaking, this problem $(M^{w}_{g,r,l})$ concerns the issue of whether or not the kernel $\ker((\rho^{\text{rel-}l}_{g,r})_k)$ is *independent of* r.

Now let us observe that we have the following commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathit{\Gamma}^{\text{rel-}l}_{g,r+1} & \longrightarrow & \mathit{\Pi}^{\text{rel-}l}_{(\mathcal{M}_{g,r+1})_k} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathit{\Gamma}^{\text{rel-}l}_{g,r} & \longrightarrow & \mathit{\Pi}^{\text{rel-}l}_{(\mathcal{M}_{g,r})_k} & \longrightarrow & G_k & \longrightarrow & 1
\end{array}$$

where the horizontal sequences are exact, the vertical arrows are surjective, and the right-hand vertical arrow is the identity morphism of G_k. In particular, it follows from Lemma 3.3, (i) (in the case where condition (a) is satisfied), that

$$\ker((\rho^{\text{rel-}l}_{g,r+1})_k) \subseteq \ker((\rho^{\text{rel-}l}_{g,r})_k).$$

Thus, we conclude that

the problem $(M^w_{g,r,l})$ has an *affirmative* answer if and only if the inclusion

$$\ker((\rho^{\text{rel-}l}_{g,r})_k) \subseteq \ker((\rho^{\text{rel-}l}_{g,r+1})_k).$$

holds.

(iv) By [13, the Galois Kernel Theorem] or [21, Corollary 4.2, (ii)], $(M_{0,r,l})$ is answered in the *affirmative* for any integer $r > 3$, which thus implies that $(M^w_{0,r,l})$ is answered in the *affirmative* for any integer $r > 3$. Also, by Corollary 2.3 and [15, Theorem 3.4], $(M_{1,r,2})$ is answered in the *affirmative* for any positive integer r, which thus implies that $(M^w_{1,r,2})$ is answered in the *affirmative* for any positive integer r.

(v) Moreover, we can prove the following assertion:

Suppose that

$$r > \begin{cases} 3 & \textit{if } g = 0, \\ 1 & \textit{if } g = 1, \\ 0 & \textit{if } g > 1. \end{cases}$$

Then $(M^w_{g,r,l})$ is answered in the affirmative.

Indeed, let γ be an element of $\ker((\rho^{\text{rel-}l}_{g,r})_k)$, X a curve of type $(g, r-1)$ over $\overline{k}$, X_2 the 2-nd configuration space of X, and Y the curve of type (g, r) over $\overline{k}$ obtained by removing a $\overline{k}$-rational point from X. Let us first observe that it follows from Proposition 1.2 that we have the following commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Delta^l_Y & = & \Delta^l_Y & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Gamma^{\text{rel-}l}_{g,r+1} & \longrightarrow & \Pi^{\text{rel-}l}_{(\mathcal{M}_{g,r+1})_k} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle p'_{g,r+1}} & & \| & & \\
1 & \longrightarrow & \Gamma^{\text{rel-}l}_{g,r} & \longrightarrow & \Pi^{\text{rel-}l}_{(\mathcal{M}_{g,r})_k} & \xrightarrow{p_{g,r}} & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

where the vertical and horizontal sequences are exact, the top horizontal arrow is the identity morphism of Δ^l_Y, and the right-hand vertical arrow is the identity morphism of G_k. Since γ is an element of $\ker((\rho^{\text{rel-}l}_{g,r})_k)$, there exists an element γ' of $p^{-1}_{g,r}(\{\gamma\}) \cap Z_{\Pi^{\text{rel-}l}_{(\mathcal{M}_{g,r})_k}}(\Gamma^{\text{rel-}l}_{g,r})$. Let $\tilde{\gamma}$ be an element of $(p'_{g,r+1})^{-1}(\{\gamma'\})$. Then since Δ^l_Y is center-free and topologically finitely generated (cf., e.g., [22, Remark 1.2.2], [22, Proposition 1.4]), it follows from [9, Lemma

4.10] that, to verify the inclusion $\ker((\rho^{\text{rel-}l}_{g,r})_k) \subseteq \ker((\rho^{\text{rel-}l}_{g,r+1})_k)$, it suffices to prove that, after possibly multiplying $\tilde{\gamma}$ by a suitable element of Δ^l_Y, $\tilde{\gamma}$ is contained in $Z_{\Pi^{\text{rel-}l}_{(\mathcal{M}_{g,r+1})_k}}(\Delta^l_Y)$.

Now, by Proposition 1.2 and [22, Proposition 2.2, (i)], we have the following commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta^l_Y & \longrightarrow & \Delta^l_{X_2} & \longrightarrow & \Delta^l_X & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta^l_Y & \longrightarrow & \Gamma^{\text{rel-}l}_{g,r+1} & \longrightarrow & \Gamma^{\text{rel-}l}_{g,r} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta^l_Y & \longrightarrow & \Pi^{\text{rel-}l}_{(\mathcal{M}_{g,r+1})_k} & \xrightarrow{p'_{g,r+1}} & \Pi^{\text{rel-}l}_{(\mathcal{M}_{g,r})_k} & \longrightarrow & 1
\end{array}$$

where the horizontal sequences are exact, the vertical arrows are injective, and the left-hand vertical arrows are the identity morphisms of Δ^l_Y. Thus, since the image of $\Delta^l_{X_2}$ in $\Pi^{\text{rel-}l}_{(\mathcal{M}_{g,r+1})_k}$ is normal in $\Pi^{\text{rel-}l}_{(\mathcal{M}_{g,r+1})_k}$, $\gamma' \in Z_{\Pi^{\text{rel-}l}_{(\mathcal{M}_{g,r})_k}}(\Gamma^{\text{rel-}l}_{g,r})$, and Δ^l_X is center-free, it follows from [28, Theorem 0.1] or [12, Theorem B] that, after possibly multiplying $\tilde{\gamma}$ by a suitable element of Δ^l_Y, $\tilde{\gamma}$ is contained in $Z_{\Pi^{\text{rel-}l}_{(\mathcal{M}_{g,r+1})_k}}(\Delta^l_Y)$. This completes the proof of the inclusion $\ker((\rho^{\text{rel-}l}_{g,r})_k) \subseteq \ker((\rho^{\text{rel-}l}_{g,r+1})_k)$, hence also (cf. the final portion of the discussion of (iii)) of the above assertion.

Corollary 4.7. *Let r be a positive integer. Suppose that*

$$l \notin \{2, 3, 5, 7\}.$$

Then the homomorphism

$$\rho^{\text{univ-}l}_{1,r} \colon \Gamma^{\text{rel-}l}_{1,r} \longrightarrow \mathrm{Out}(\Delta^l_{1,r})$$

is not injective.

In particular, the problem $(\mathrm{CSP})^{\text{pro-}l}_{1,r}$ in the Introduction has a negative answer.

Proof. Let us first observe that it follows from Theorem 4.5 that

$$\ker((\rho^{\text{rel-}l}_{1,r})_{\mathbb{Q}}) \subsetneq \ker(\rho^l_{\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}}).$$

On the other hand, since Oda's problem is answered in the *affirmative* (cf. [28, Theorem 0.5, (2)]), we have that

$$\ker(\rho^l_{\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}}) \subseteq \ker((\rho^{\text{geo-}l}_{1,r})_{\mathbb{Q}}).$$

Thus, since the injectivity of $\rho^{\text{univ-}l}_{1,r}$ implies, by definition, the equality

$$\ker(\rho^{\text{rel-}l}_{1,r})_{\mathbb{Q}} = \ker(\rho^{\text{geo-}l}_{1,r})_{\mathbb{Q}},$$

we conclude that $\rho_{1,r}^{\text{univ-}l}$ is not injective. This completes the proof of the first portion of Corollary 4.7. Thus, the final portion of Corollary 4.7 follows immediately from the discussion following the statement of the problem $(\mathrm{CSP})_{g,r}^{\text{pro-}l}$ in the Introduction. This completes the proof of Corollary 4.7. $\square$

Lemma 4.8. *Let $f\colon G \to H$ be a homomorphism of profinite groups. For $i = 1, 2$, let $N_i \subseteq G$ be a normal open subgroup of G, N_i^l the maximal pro-l quotient of N_i, and $G_{N_i}^l$ the maximal almost pro-l quotient of G with respect to N_i (cf. "Profinite groups" in "Notations and Conventions"). Suppose that N_1^l is torsion-free, and that $f\colon G \to H$ factors through $G_{N_1}^l$ and $G_{N_2}^l$. Write $f_{N_i}\colon G_{N_i}^l \to H$ for the resulting homomorphism for $i = 1, 2$. Suppose, moreover, that the kernel of $f\colon G \to H$ is contained in N_1, and that $f_{N_2}\colon G_{N_2}^l \to H$ is injective. Then $f_{N_1}\colon G_{N_1}^l \to H$ is injective.*

Proof. Write $N_3 := N_1 \cap N_2$ and $G_{N_3}^l$ for the maximal almost pro-l quotient of G with respect to N_3. Let us observe that since $f_{N_2}\colon G_{N_2}^l \to H$ is injective, one verifies easily that $\ker(f) = \ker(N_2 \twoheadrightarrow N_2^l)$, which thus implies that $N_2/\ker(f)$ is *pro-l*. Thus, it follows that N_3 is a normal open subgroup of G which contains the kernel of $f\colon G \to H$. In particular, the quotient N_2/N_3, hence also $N_2/\ker(N_3 \twoheadrightarrow N_3^l)$, is *pro-$l$*. Therefore, by considering the natural exact sequence of profinite groups

$$1 \longrightarrow \ker(N_3 \twoheadrightarrow N_3^l) \longrightarrow \ker(N_2 \twoheadrightarrow N_2^l) \longrightarrow N_2/\ker(N_3 \twoheadrightarrow N_3^l)\ ,$$

we conclude that $\ker(N_3 \to N_3^l) = \ker(N_2 \to N_2^l)$, i.e., $f\colon G \to H$ determines an *injection* $G_{N_3}^l \to H$. In particular, by replacing N_2 by N_3, we may assume that $N_2 \subseteq N_1$. Then since $f_{N_2}\colon G_{N_2}^l \to H$ factors through $G_{N_1}^l$, and $f_{N_2}\colon G_{N_2}^l \to H$ is injective, we have the following commutative diagram of profinite groups

$$\begin{array}{ccc} N_2^l & \longrightarrow & N_1^l \\ & \searrow^{f_{N_2}|_{N_2^l}} & \downarrow^{f_{N_1}|_{N_1^l}} \\ & & H. \end{array}$$

Thus, since the top arrow $N_2^l \to N_1^l$ is an *open* injection, and N_1^l is *torsion-free*, it holds that $f_{N_1}|_{N_1^l}$, hence also f_{N_1}, is injective. This completes the proof of Lemma 4.8. $\square$

Lemma 4.9. *Let r be a positive integer. Suppose that $l > 2$. Then $(\mathit{\Gamma}_{1,r}[l])^l$ (cf. Definition 1.1, (ii)) is slim and torsion-free.*

Proof. Let us first observe that it follows from Proposition 1.2 and the definition of $\mathit{\Gamma}_{1,r}^{\text{rel-}l}$ that we have the following exact sequence

$$1 \longrightarrow \mathit{\Delta}_{1,r}^l \longrightarrow (\mathit{\Gamma}_{1,r+1}[l])^l \longrightarrow (\mathit{\Gamma}_{1,r}[l])^l \longrightarrow 1\ .$$

Thus, since $\mathit{\Delta}_{1,r}^l$ is *slim* and *torsion-free* (cf., e.g., [22, Proposition 1.4], [22, Remark 1.2.2]), it follows from *induction on* r that, to verify Lemma 4.9, we may assume without loss of generality that $r = 1$. Then it follows from the various definitions involved that $(\mathit{\Gamma}_{1,1}[l])^l$ is isomorphic to $\mathit{\Delta}_{Y(l)}^l$.

In particular, by [17, §1.4], [22, Remark 1.2.2], and [22, Proposition 1.4], $(\Gamma_{1,1}[l])^l$ is *slim* and *torsion-free*. This completes the proof of Lemma 4.9. □

Corollary 4.10. *Let r be a positive integer, $N \subseteq \Gamma_{1,r}$ a normal open subgroup of $\Gamma_{1,r}$, and $(\Gamma_{1,r})^l_N$ the maximal almost pro-l quotient of $\Gamma_{1,r}$ with respect to N (cf. "Profinite groups" in "Notations and Conventions"). Suppose that*

$$l \notin \{2, 3, 5, 7\},$$

and that the homomorphism $(\rho^{\text{puni-}l}_{1,r})_k|_{\Gamma_{1,r}}$ factors through $(\Gamma_{1,r})^l_N$. Then the resulting homomorphism

$$(\rho^{\text{univ-}l}_{1,r})_N \colon (\Gamma_{1,r})^l_N \longrightarrow \mathrm{Out}(\Delta^l_{1,r})$$

is not injective.

Proof. Corollary 4.10 follows immediately (in light of Lemma 4.9) from Corollary 4.7, together with Lemma 4.8 in the case where we take "(G, H, N_1, N_2)" in the statement of Lemma 4.8 to ($\Gamma_{1,r}$, $\mathrm{Out}(\Delta^l_{1,r})$, $\Gamma_{1,r}[l]$, N). □

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