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On Higher Fitting Ideals of Certain Iwasawa Modules Associated with Galois Representations and Euler Systems

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ON HIGHER FITTING IDEALS OF CERTAIN IWASAWA MODULES ASSOCIATED WITH GALOIS REPRESENTATIONS AND EULER SYSTEMS

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Abstract. By using "Gauss sum type" Kolyvagin systems, Kurihara studied the higher Fitting ideals of Iwasawa modules, and he obtained a refinement of the minus part of the Iwasawa main conjecture over totally real fields ([Ku]). In this paper, we study the higher Fitting ideals of Iwasawa modules arising from the dual fine Selmer groups of general Galois representations which have Euler systems of "Rubintype", like circular units or Beilinson–Kato elements. By using Kolyvagin derivatives, we construct an ascending filtration $\{\mathfrak{C}_i(\mathbf{c})\}_{i\geq 0}$ of the Iwasawa algebra, and show that the filtration $\{\mathfrak{C}_i(\mathbf{c})\}_{i\geq 0}$ gives good approximation of the higher Fitting ideals of the Iwasawa module under the assumption of "Iwasawa main conjecture". Our results can be regarded as analogues of Kurihara's results, and a refinement of "Iwasawa main conjecture" and Mazur–Rubin theory in certain cases.

Contents

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1. Introduction

By the theory of Euler systems, a norm compatible system of Galois cohomology classes called Euler system give a lower bound of the characteristic ideal of a certain Iwasawa module. (For instance, see Theorem 2.3.3 in [Ru2].) The characteristic ideals are an important invariants of finitely generated torsion Iwasawa modules, but in general, we cannot determine the pseudo-isomorphism classes of Iwasawa modules completely by the characteristic ideals. The higher Fitting ideals have more refined information on the structure of Iwasawa modules. For example, we can determine the pseudo-isomorphism class and the cardinality of the minimal system of generators of an Iwasawa module by the higher Fitting ideals. (For the definition and some basic properties of the higher Fitting ideals, see, for incetance, [Oh2] §2.)

In [MR], Mazur and Rubin established the theory of Kolyvagin systems, and obtained a refinement of "Iwasawa main conjecture" in certain situations. They does not write explicitly, but we can deduce, via their arguments in [MR] §5.3, that Λ-primitive

Kolyvagin systems determine the pseudo-isomorphism class of Iwasawa modules arising from dual fine Selmer groups of p-adic representations of the absolute Galois group of $\mathbb{Q}$ satisfying certain conditions. However in [MR], they do not obtain any explicit bound of the higher Fitting ideals of Iwasawa modules.

In [Ku], Kurihara studied the higher Fitting ideals of the minus-part of the Iwasawa module defined by the inverse limit of the p-Sylow subgroups of the ideal class groups along the cyclotomic $\mathbb{Z}_p$-extension of a CM-field K satisfying certain conditions. By using Kolyvagin systems of "Gauss sum type", he constructed an ascending filtration $\{\Theta_i\}_{i\in\mathbb{Z}_{\geq 0}}$ of Iwasawa algebra called *the higher Stickelberger ideals*, which are defined by analytic objects arising from p-adic L-functions, and he proved that higher Fitting ideals coincide with the higher Stickelberger ideals. (For details, see [Ku] Theorem 1.1.) His results give a refinement of the minus-part of the Iwasawa main conjecture for totally real number fields. In the proof of results, he developed new Euler system arguments, which can deal with more refined informations on the structure of Iwasawa modules than usual arguments.

In the paper [Oh2], the higher Fitting ideals of the plus-part of the Iwasawa modules of ideal class groups (over abelian fields) are studied. By using circular units, we constructed the ideals $\mathfrak{C}_i$ of the Iwasawa algebra, which are analogues of Kurihara's higher Stickelberger ideals, and proved that $\mathfrak{C}_i$ give "upper bounds" and "lower bounds" of the higher Fitting ideals in certain senses. (For details, see [Oh2] Theorem 1.1 and §10.1 in this paper.) The results in [Oh2] can be regarded as analogues of Kurihara's results and a refinement of the plus-part of the Iwasawa main conjecture. Note that in [Oh1], we also obtained similar results for the Iwasawa modules of ideal class groups over abelian extension fields of imaginary quadratic fields by using elliptic units. (See [Oh1] and Remark 10.1 in this paper.)

In this article, we study higher Fitting ideals of an Iwasawa module $X = X(T)$ arising from "dual fine Selmer groups" of a lattice T of a general p-adic Galois representation along the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Here, let us state our main theorem roughly. Under the assumption of the existence of a "non-vanishing" Euler system $\mathbf{c}$ of Rubin type (see the condition (NV) in §2), by using Kolyvagin derivatives of the Euler system, we shall construct ideals $\mathfrak{C}_i(\mathbf{c})$ of Iwasawa algebra Λ, which can be regarded as generalizations of ideals $\mathfrak{C}_i$ in [Oh2] and analogues of Kurihara's higher Stickelberger ideals. Under certain assumptions, we shall prove the following assertions, which are the main results in this article.

- In §2, we shall "explicitly" construct an ideal $I(\mathbf{c})$ of Λ, which satisfies the following properties.
  - If the Euler system $\mathbf{c}$ satisfies "Iwasawa main conjecture" (see the condition (MC) in §2), then the heiget of $I(\mathbf{c})$ is at least two.
  - Moreover, under the assumption of the Iwasawa main conjecture, we have $I(\mathbf{c}) = \Lambda$ in certain practical situations. For details, see Remark 2.3.

  Let X_{fin} be the maximal pseudo-null Λ-submodule of X, and put $X' := X/X_{\mathrm{fin}}$. Then, for any $i \in \mathbb{Z}_{\geq 0}$, we have

$$\mathrm{ann}_\Lambda(X_{\mathrm{fin}}) I(\mathbf{c}) \cdot \mathrm{Fitt}_{\Lambda,i}(X') \subseteq \mathfrak{C}_i(\mathbf{c}).$$

- For any $i \in \mathbb{Z}_{\geq 0}$, there exists a height-two ideal I_i of Λ satisfying
$$I_i \mathfrak{C}_i(\mathbf{c}) \subseteq \mathrm{Fitt}_{\Lambda,i}(X).$$
(However, in present, we do not have any explicit description of the "error factors" I_i.)

(For the precise statement of our main results, see Theorem 2.4.) In particular, under the assumption of the Iwasawa main conjecture, our main results implies that the filtration $\{\mathfrak{C}_i(\mathbf{c})\}_{i\geq 0}$ of Λ determines the perudo-isomorphism classe of X (see Corollary 2.7). Our results can be regarded as a generalization of the results in [Oh2] for general Galois representations and analogues of Kurihara's results. Moreover, our results can also be regarded as a refinement of the "Iwasawa main conjecture" and the results by Mazur and Rubin in [MR] §5.3.

In §2, we state our main results. (See Theorem 2.4 and its corollaries.) In §3, we set the local conditions on Galois cohomology groups, and give another description of the Iwasawa module X in terms of "Selmer groups" by using the Global duality theorem of Galois cohomology. In this section, we also recall some Iwasawa theoretical results which control the behaviour of Selmer groups along the $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty/\mathbb{Q}$. In §4, we recall the definition and some properties of Euler systems of Rubin type. In §5, we define the ideal $\mathfrak{C}_i(\mathbf{c})$, and prove Theorem 2.4 (i). In §6, we recall some results on Kolyvagin systems established by Mazur and Rubin. In this section, we prove the assertion (iii) of Theorem 2.4 by using Mazur–Rubin's arguments. In §7, by using Chebotarev density theorem we show a preliminary results which is used in Euler system arguments in the next section 8. Then, we complete the proof of Theorem 2.4 by using Kurihara's Euler system arguments in §8. In §9, we give some remarks on the structure of dual fine Selmer groups over the ground level $\mathbb{Q}_0 = \mathbb{Q}$. In the last section (§10), we apply our results to particular Euler systems: circular units and Kato's Euler systems.

Notation. Let K be a field, and fix a separable closure $\overline{K}$ of K. Then, we put $G_K := \mathrm{Gal}(\overline{K}/K)$. For a topological abelian group M with a continuous G_K-action, let $H^*(K, M) = H^*(G_K, M)$ be the continuous Galois cohomology group.

In this paper, an algebraic number field K is a finite extension of $\mathbb{Q}$ in this fixed algebraic closure $\overline{\mathbb{Q}}$. Let L/K be a finite extension of number fields. For a finite set Σ of places of K, we denote by L_Σ/L the maximal extension unramified outside Σ, and put $G_{L,\Sigma} := \mathrm{Gal}(L_\Sigma/L)$. We denote the ring of integers of a number field K by $\mathcal{O}_K$.

Let ℓ be a prime number, and L a finite extension field of $\mathbb{Q}_\ell$. We denote the Weil group of L by W_L, and the inertia subgroup of W_L by I_L.

Let L/K be a finite Galois extension of algebraic number fields. Let λ be a prime ideal of K, and λ' a prime ideal of L above λ. We denote the completion of K at λ by K_λ. If λ is unramified in L/K, the arithmetic Frobenius at λ' is denoted by $(\lambda', L/K) \in \mathrm{Gal}(L/K)$. We fix a family of embeddings $\{\ell_{\overline{\mathbb{Q}}} \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell\}_{\ell:\mathrm{prime}}$ satisfying the condition (Chb) as follows:

(Chb) *For any subfield $F \subset \overline{\mathbb{Q}}$ which is a finite Galois extension of $\mathbb{Q}$ and any element $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$, there exist infinitely many prime numbers ℓ such that ℓ is unramified in $F/\mathbb{Q}$ and $(\ell_F, F/\mathbb{Q}) = \sigma$, where ℓ_F is the prime ideal of F corresponding to the embedding $\ell_{\overline{\mathbb{Q}}}|_F$.*

The existence of a family satisfying the condition (Chb) follows easily from the Chebotarev density theorem.

For any prime number ℓ, we regard $W_{\mathbb{Q}_\ell} \subseteq G_{\mathbb{Q}_\ell}$ as a subgroup of $G_{\mathbb{Q}}$ via the embedding $\ell_{\overline{\mathbb{Q}}} \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$.

We also fix an embedding $\infty_{\overline{\mathbb{Q}}} \colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, and let $c \in G_{\mathbb{Q}}$ be the complex conjugation corresponding to this embedding. For any abelian group M with action of $G_{\mathbb{Q}}$, we denote by M^- the subgroup of M consisting of all elements on which c acts via -1. For any positive integer n, let $\mu_n := \boldsymbol{\mu}_n(\overline{\mathbb{Q}})$ be the group of n-th roots of unity in $\overline{\mathbb{Q}}$, and define an element $\zeta_n \in \mu_n$ by $\infty_{\overline{\mathbb{Q}}}(\zeta_n) = e^{2\pi i/n}$.

Let K be a finite extension field of $\mathbb{Q}_p$, and $\mathcal{O}$ the ring of integer of K. We fix a uniformizer $\pi \in \mathcal{O}$. For any $\mathcal{O}$-module M, we define the dual $\mathcal{O}$-module $M^\vee$ by $M^\vee := \mathrm{Hom}_{\mathcal{O}}(M, K/\mathcal{O})$. In this paper, we identify the $\mathcal{O}$-module $M^\vee$ with $\mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ by the isomomorphism

$$M^\vee := \mathrm{Hom}_{\mathcal{O}}(M, K/\mathcal{O}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p),$$

induced by

$$K \longrightarrow \mathbb{Q}_p;\ a \longmapsto \mathrm{Tr}_{K/\mathbb{Q}_p}(\pi^{-d_{K/\mathbb{Q}_p}} \cdot a),$$

where we denote the different of $K/\mathbb{Q}_p$ by $\mathfrak{d}_{K/\mathbb{Q}_p} = \pi^{d_{K/\mathbb{Q}_p}}\mathcal{O}$. If M has an $\mathcal{O}$-linear action of a group G, we define the action

$$G \times M^\vee \longrightarrow M^\vee;\ (g, f) \longmapsto gf$$

of G on $M^\vee$ by $(gf)(m) = f(g^{-1}m)$ for any $m \in M$.

Let R be a commutative ring, and M an R-module. For any $a \in R$, let $M[a]$ be the R-submodule of M consisting of all a-torsion elements. We denote the ideal of R consisting of all annihilators of M by $\mathrm{ann}_R(M)$. For any sheaf $\mathcal{F}$ of abelian groups on $(\mathrm{Spec}\, R)_{\text{ét}}$, and $i \in \mathbb{Z}_{\geq 0}$, we put

$$H^i_{\text{ét}}(R, \mathcal{F}) := H^i_{\text{ét}}(\mathrm{Spec}\, R, \mathcal{F}).$$

Let G be a group, and M an abelian group with an action of G. Then, we denote by M^G the maximal subgroup of M fixed by the action of G.

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2. Main results

In this section, we state the precise statement of our main results.

First, we set some terminologies. Let p be an odd prime number, and $\mathbb{Q}_\infty/\mathbb{Q}$ the cyclotomic $\mathbb{Z}_p$-extension. For any $m \in \mathbb{Z}_{\geq 0}$, we denote by $\mathbb{Q}_m$ the unique intermediate field of $\mathbb{Q}_\infty/\mathbb{Q}$ satisfying $[\mathbb{Q}_m : \mathbb{Q}] = p^m$. We put $\Gamma := \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$, and define the Iwasawa algebra Λ by

$$\Lambda := \mathbb{Z}_p[[\Gamma]] = \varprojlim \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})].$$

Let $K/\mathbb{Q}_p$ be a finite extension, and $\mathcal{O}$ the ring of integers of K. Fix a uniformizer $\pi \in \mathcal{O}$, and put $k := \mathcal{O}/\pi\mathcal{O}$. Let us consider a free $\mathcal{O}$-module T of finite rank d with a continuous $\mathcal{O}$-linear action of $G_\mathbb{Q}$ unramified outside a finite set Σ of places of $\mathbb{Q}$ containing $\{p, \infty\}$. We regard T as an étale $\mathcal{O}$-sheaf on $\mathrm{Spec}\, \mathcal{O}_{\mathbb{Q}_m,\Sigma}$, where $\mathcal{O}_{\mathbb{Q}_m,\Sigma}$ is the ring of Σ-integers of $\mathbb{Q}_m$. We denote the action of $G_\mathbb{Q}$ on T by

$$\rho_T \colon G_\mathbb{Q} \longrightarrow \mathrm{Aut}_\mathcal{O}(T) \simeq \mathrm{GL}_d(\mathcal{O}).$$

We put $V := T \otimes_\mathcal{O} K$, $A := T \otimes_\mathcal{O} K/\mathcal{O}$, and $A^* := \mathrm{Hom}_\mathcal{O}(T, K/\mathcal{O}(1))$. Here, we let $K/\mathcal{O}(1)$ be the Tate twist of the trivial $G_\mathbb{Q}$-module $K/\mathcal{O}$. In this article, we always assume the following conditions.

- (C1) The $G_{\mathbb{Q}_\infty}$-representation $A[\pi]$ over k is absolutely irreducible.
- (C2) There exists an element $\tau \in G_{\mathbb{Q}(\mu_{p^\infty})}$ which make $T/(\tau-1)T$ a free $\mathcal{O}$-module of rank one.
- (C3) The $\mathbb{F}_p[G_{\mathbb{Q}_\infty}]$-module $A[\pi]$ is *not* isomorphic to $A^*[\pi]$.
- (C4) If the rank of T is one, then $G_{\mathbb{Q}_\infty}$ does not act on $A[\pi]$ via the trivial character $\mathbf{1}$ or the Teichmüller character ω.
- (C5) Let $\Omega = \mathbb{Q}(\mu_p^\infty, A)$ be the maximal subfield of $\overline{\mathbb{Q}}$ fixed by the subgroup
$$\ker\left(G_{\mathbb{Q}(\mu_{p^\infty})} \longrightarrow \mathrm{Aut}(A)\right)$$
of $G_\mathbb{Q}$. Then, we have
$$H^1(\Omega/\mathbb{Q}_\infty, A) = H^1(\Omega/\mathbb{Q}_\infty, A^*) = 0.$$
- (C6) The torsion $\mathbb{Z}_p$-module $H^0_{\text{ét}}(\mathbb{Q}_\infty \otimes_\mathbb{Q} \mathbb{Q}_p, A^*)$ is divisible.
- (C7) Let $\ell \in \Sigma \setminus \{p, \infty\}$ be any element. We denote by
$$(r_\ell \colon W_{\mathbb{Q}_\ell} \longrightarrow \mathrm{GL}_d(K), N_\ell)$$
the Weil-Deligne representation corresponding to $(V, \rho_T|_{W_{\mathbb{Q}_\ell}})$, and let L_ℓ be the intermediate field of $\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell^{\mathrm{ur}}$ fixed by $\mathrm{Ker}(r_\ell|_{I_{\mathbb{Q}_\ell}})$. Then, the following holds.
  - (i) We have $p \nmid \# r_\ell(I_{\mathbb{Q}_\ell}) = [L_\ell : \mathbb{Q}_\ell^{\mathrm{ur}}]$.
  - (ii) The $\mathcal{O}$-module $H^1_{\mathrm{cont}}(G_{L_\ell}, T)$ is torsion-free.

In particular, the assumption (C7) implies that for any $\ell \in \Sigma \setminus \{p, \infty\}$, the $\mathcal{O}$-module $H^1_{\mathrm{cont}}(I_{\mathbb{Q}_\ell}, T)$ is torsion-free. The following lemma gives a sufficient condition for the condition (ii) of (C7).

Lemma 2.1. *Let $\ell \in \Sigma \setminus \{p, \infty\}$ be any element, and (r_ℓ, N_ℓ) and L_ℓ as in (C7). Fix a topological generator g_ℓ of the tame inertia group $I^t_{L_\ell}$ of L_ℓ. Suppose that the $\mathcal{O}$-module the $\mathcal{O}$-module $T/(g_\ell - 1)T$ is torsion-free. Then, the $\mathcal{O}$-module $H^1_{\mathrm{cont}}(I_{L_\ell}, T)$ is torsion-free.*

Proof. The proof of Lemma 2.1 is a routaine, and not difficult. So here, we only give a sketch of it. Assume that the $\mathcal{O}$-module the $\mathcal{O}$-module $T/(g_\ell - 1)T$ is torsion-free. Then, it is sufficient to show that the condition (ii) of (C7) holds. By our assumption, the ascending filtration

$$\{T_i^{(\ell)} := \mathrm{Ker}\left((g_\ell - 1)^i \colon T \longrightarrow T\right)\}_{i \in \mathbb{Z}_{\geq 0}}$$

of T satisfies that for any $i \in \mathbb{Z}_{\geq 0}$,

- G_{L_ℓ} acts trivially on $T_i^{(\ell)}/T_{i-1}^{(\ell)}$, and
- the $\mathcal{O}$-module $T_{i-1}^{(\ell)}/(g_\ell - 1)T_i^{(\ell)}$ is torsion-free.

By induction on i, we can deduce that the the $\mathcal{O}$-module $H^1_{\mathrm{cont}}(G_{L_\ell}, T_i^{(\ell)})$ is torsion-free for any $i \in \mathbb{Z}_{\geq 0}$, so in particular, $H^1_{\mathrm{cont}}(G_{L_\ell}, T)$ is a torsion-free $\mathcal{O}$-module. □

Now, we introduce an Iwasawa module $X = X(T)$, which we study in this article.

Definition 2.2. We define

$$\mathbb{H}^i_\Sigma(T) := \varprojlim H^i_{\text{ét}}(\mathcal{O}_{\mathbb{Q}_m,\Sigma}, T).$$

for any prime number ℓ, we put

$$\mathbb{H}^i_{\mathrm{loc},\ell}(T) := \varprojlim H^i_{\text{ét}}(\mathbb{Q}_m \otimes_{\mathbb{Q}} \mathbb{Q}_p, T)$$

Then, we define

$$X(T) := \ker\left(\mathbb{H}^2_\Sigma(T) \longrightarrow \bigoplus_{\ell \in \Sigma} \mathbb{H}^2_{\mathrm{loc},\ell}(T)\right).$$

It is well-known that $\mathbb{H}^i_\Sigma(T) = 0$ for any $i \geq 3$, and the Λ-module $\mathbb{H}^i_\Sigma(T)$ is finitely generated for any $i \in \mathbb{Z}_{\geq 0}$. (Recall that here, we assume p is odd, so the p-cohomological dimension of $G_{\mathbb{Q}_m,\Sigma}$ is two.) We denote the maximal pseudo-null Λ-submodule of X by $X_{\mathrm{fin}}(T)$.

For simplicity, we write $X := X(T)$ and $X_{\mathrm{fin}} := X_{\mathrm{fin}}(T)$. In fact, the Λ-module X is independent of the choice of Σ, and it is isomorphic to the Pontrjagin dual of the "dual fine Selmer group" $\mathcal{S}_{\Sigma_p}(\mathbb{Q}_\infty, A^*)$ in the sense of [Ru2] Definition 2.3.1. (See Proposition 3.7.) In this article, we study the higher Fitting ideals of the Λ-module

$$X' = X'(T) := X(T)/X_{\mathrm{fin}}(T)$$

under the assumption of the existence of a "non-vanishing" Euler system for T.

In order to mention Euler systems, we need to introduce some abelian extension fields of $\mathbb{Q}$. For each prime number ℓ not contained in Σ, we denote by $\mathbb{Q}(\ell)$ the maximal subfield of $\mathbb{Q}(\mu_\ell)$ whose extension degree over $\mathbb{Q}$ is a p-power. Let $\mathcal{N}(\Sigma)$ be the set of all positive integers decomposed into square-free products of prime numbers not contained in Σ. Here, we promise $1 \in \mathcal{N}(\Sigma)$. Let $n \in \mathcal{N}(\Sigma)$ be any element, and assume that n has a prime factorization $n = \prod_{i=1}^r \ell_i$. Then, we define the composite field

$$\mathbb{Q}_m(n) := \mathbb{Q}_m\mathbb{Q}(\ell_1)\cdots\mathbb{Q}(\ell_r)$$

for any $m \geq 0$.

In this paper, we assume that there exists an Euler system

$$\mathbf{c} := \left\{c_m(n) \in H^1(\mathbb{Q}_m(n), T)\right\}_{m\geq 0, n\in\mathcal{N}(\Sigma)}$$

in the sense of [Ru2] Remark 2.1.4 satisfying the following "non-vanishing" conditions.

(NV) The element $\mathbf{c}(1) := (c_m(1))_{m\geq 0} \in \mathbb{H}^1_\Sigma(T)$ is not Λ-torsion.

(For details of the definition of Euler systems in our terminology, see Definition 4.2.)

We define the ideal $\mathrm{Ind}(\mathbf{c})$ of Λ by

$$\mathrm{Ind}(\mathbf{c}) := \left\{\varphi\left(\mathbf{c}(1)\right) \mid \varphi \in \mathrm{Hom}_\Lambda\left(\mathbb{H}^1_\Sigma(T), \Lambda\right)\right\},$$

and denote by $\mathrm{Ind}_0(\mathbf{c})$ the minimal principal ideal of Λ containing $\mathrm{Ind}(\mathbf{c})$. By usual Euler system arguments, the assumption (NV) implies that X is a torsion Λ-module, and we have

$$\mathrm{char}_\Lambda(X) \supseteq \mathrm{Ind}_0(\mathbf{c}). \tag{1}$$

(See Theorem 2.3.2 and Theorem 2.3.3 in [Ru2].) We define the ideal $I_\varphi(\mathbf{c})$ of Λ by

$$I_\varphi(\mathbf{c}) := \left\{a \in \Lambda \mid a \cdot \mathrm{char}_\Lambda(X) \subseteq \varphi(\mathbf{c}(1)) \cdot \Lambda\right\}.$$

for any Λ-linear homomorphism $\varphi \in \mathrm{Hom}_\Lambda\left(\mathbb{H}^1_\Sigma(T), \Lambda\right)$, and put

$$I(\mathbf{c}) := \bigcup_{\varphi\in\mathrm{Hom}_\Lambda\left(\mathbb{H}^1_\Sigma(T),\Lambda\right)} I_\varphi(\mathbf{c}).$$

By the definition of $I(\mathbf{c})$ and (1), we have

$$\mathrm{Ind}(\mathbf{c}) = I(\mathbf{c}) \cdot \mathrm{char}_\Lambda(X).$$

Under the assumption (NV), we sometimes consider the following condition (MC), which is "Iwasawa main conjecture" for $(T, \mathbf{c})$.

(MC) The characteristic ideal of the Λ-module X coincides with $\mathrm{Ind}(\mathbf{c})$, that is, we have

$$\mathrm{char}_\Lambda(X) = \mathrm{Ind}_0(\mathbf{c}).$$

Remark 2.3. Assume that the pair $(T, \mathbf{c})$ satisfies the conditions (C1), (C4) and (NV), and that T^- is a free $\mathcal{O}$-module of rank one. Then, [Ru2] Theorem 2.3.2 and the formula on the global Euler–Poincaré characterisitic (for instance, see [Ta1] Theorem 2.2) imply that the Λ-module $\mathbb{H}^1_\Sigma(T)$ is generically of rank one, namely we have

$$\dim_{\mathrm{Frac}(\Lambda)} \mathbb{H}^1_\Sigma(T) \otimes_\Lambda \mathrm{Frac}(\Lambda) = 1.$$

Hence in this situation, we have

$$\mathrm{Ind}(\mathbf{c}) = \mathrm{Ind}_0(\mathbf{c}) = \mathrm{char}_\Lambda\left(\frac{\mathbb{H}^1_\Sigma(T)}{\mathbb{H}^1_\Sigma(T)_{\mathrm{tors}} + \Lambda\mathbf{c}(1)}\right),$$

where $\mathbb{H}^1_\Sigma(T)_{\mathrm{tors}}$ denotes the maximal torsion Λ-submodule of $\mathbb{H}^1_\Sigma(T)$. In particular, if we also assume that the pair $(T, \mathbf{c})$ satisfies the Iwasawa main conditions (MC), then we have $I(\mathbf{c}) = \Lambda$.

In order to state our main theorem, it is convenient to introduce the following notation. Let I and J be ideals of Λ. We write $I \prec J$ if there exists a height two ideal $\mathcal{A}$ of Λ (called an "error factor") satisfying $\mathcal{A}I \subseteq J$. Note that for two ideals I and J of Λ, we have $I \prec J$ if and only if $I\Lambda_\mathfrak{p} \subseteq J\Lambda_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of height one, where we denote the localization of Λ at $\mathfrak{p}$ by $\Lambda_\mathfrak{p}$. We write $I \sim J$ if $I \prec J$ and $J \prec I$. The relation $\sim$ is an equivalence relation on ideals of Λ.

We shall define ideals $\mathfrak{C}_i(\mathbf{c})$ of Λ, which are analogues of Kurihara's higher Stickelberger ideals. In §5 we define them by using Kolyvagin derivatives of the Euler system $\mathbf{c}$ (as in [Oh1] and [Oh2]). For details, see Definition 5.1 and Definition 5.4. Note that the definition of the ideals $\mathfrak{C}_i(\mathbf{c})$ is one of the key of our results. The following theorem is our main results.

Theorem 2.4. *Assume that T and $\mathbf{c}$ satisfy the conditions* (C1)–(C7) *and* (NV). *Then, we have the following.*

(i) *We have*
$$\mathrm{ann}_\Lambda(X_{\mathrm{fin}})I(\mathbf{c}) \cdot \mathrm{Fitt}_{\Lambda,0}(X') \subseteq \mathfrak{C}_0(\mathbf{c}).$$
(ii) *Assume $\mathcal{O} = \mathbb{Z}_p$. Then, we have*
$$\mathrm{ann}_\Lambda(X_{\mathrm{fin}})I(\mathbf{c}) \cdot \mathrm{Fitt}_{\Lambda,i}(X') \subseteq \mathfrak{C}_i(\mathbf{c})$$
for any $i \in \mathbb{Z}_{\geq 0}$.

(iii) *Assume that T^- is a free $\mathcal{O}$-module of rank one. Then for any $i \in \mathbb{Z}_{\geq 0}$, we have*
$$\mathfrak{C}_i(\mathbf{c}) \prec \mathrm{Fitt}_{\Lambda,i}(X)$$

Remark 2.5. We assume that T and $\mathbf{c}$ satisfy the conditions (C1)–(C7), (NV) and (MC). (Then, we have $I(\mathbf{c}) = \Lambda$.) Note that we have
$$\mathrm{Fitt}_{\Lambda,0}(X) \subseteq \mathrm{ann}_\Lambda(X_{\mathrm{fin}}) \cdot \mathrm{Fitt}_{\Lambda,0}(X').$$
So, in this case, Theorem 2.4 (i) implies
$$\mathrm{Fitt}_{\Lambda,0}(X) \subseteq \mathfrak{C}_0(\mathbf{c}).$$

Note that the higher Fitting ideals determine the pseudo-isomorphism class of a finitely generated torsion Λ-module. More precisely, we have the following lemma.

Lemma 2.6. *Let M be a finitely generated torsion Λ-module. Assume that M is pseudo-isomorphic to an elementary Λ-module $\bigoplus_{i=1}^n \Lambda/f_i\Lambda$, where $\{f_i\}_{i=1}^n$ is a sequence of non-zero elements of Λ satisfying $f_i \mid f_{i+1}$, then we have*
$$\mathrm{Fitt}_{\Lambda,i}(M) \sim \begin{cases} \left(\prod_{k=1}^{n-i} f_k\right)\Lambda & (\text{if } i < n) \\ \Lambda & (\text{if } i \geq n) \end{cases}$$
for any non-negative integer i (cf. [Ku] *Lemma 8.2). This implies that the pseudo-isomorphism class of M is determined by the higher Fitting ideals $\{\mathrm{Fitt}_{\Lambda,i}(M)\}_{i\geq 0}$.*

By Theorem 2.4, we immediately obtain the following corollaries.

Corollary 2.7. *Assume that T and $\mathbf{c}$ satisfy the conditions* (C1)–(C7), (NV) *and* (MC). *We also assume $\mathcal{O} = \mathbb{Z}_p$ and $\mathrm{rank}_{\mathbb{Z}_p} T^- = 1$. Then, we have*

$$\mathrm{Fitt}_{\Lambda,i}(X) \sim \mathfrak{C}_i(\mathbf{c})$$

for any $i \in \mathbb{Z}_0$. In other words, the ascending filtration $\{\mathfrak{C}_i(\mathbf{c})\}_{i\in\mathbb{Z}_{\geq 0}}$ of Λ determines the pseudo-isomorphism class of X. Moreover, we have

$$\mathrm{ann}_\Lambda(X_{\mathrm{fin}}) I(\mathbf{c}) \cdot \mathrm{Fitt}_{\Lambda,i}(X') \subseteq \mathfrak{C}_i(\mathbf{c})$$

for any $i \in \mathbb{Z}_{\geq 0}$.

Corollary 2.8. *Suppose $\mathcal{O} = \mathbb{Z}_p$. We assume that T and $\mathbf{c}$ satisfy the conditions* (C1)–(C7), (NV) *and* (MC). *We also assume that $X(T)$ has no non-trivial pseudo-null Λ-submodules (namely, $X_{\mathrm{fin}} = 0$), and $\mathrm{rank}_{\mathbb{Z}_p} T^- = 1$. Then, we have*

$$\mathrm{Fitt}_{\Lambda,i}(X') \subseteq \mathfrak{C}_i(\mathbf{c})$$

for any $i \in \mathbb{Z}_{\geq 0}$.

Remark 2.9. In certain nice cases, we can show that the ideals $\{\mathfrak{C}_{i,0,N}(\mathbf{c})\}_{i\geq 0}$ of $\mathcal{O}/\pi^N\mathcal{O}$ (for sufficiently large N) determine the isomorphism classes of dual fine Selmer groups over the ground level $\mathbb{Q}_0 = \mathbb{Q}$. For details, see Theorem 9.1. Note that this result itself is not so new because it is only a translation of Mazur–Rubin's results in [MR] §5.2 into the context of higher Fitting ideals and our ideals $\mathfrak{C}_i(\mathbf{c})$. As a corollary of this result, we shall see that in certain situations, the ideals $\{\mathfrak{C}_{i,0,N}(\mathbf{c})\}_{i\geq 0}$ determines the cardinality of the minimal system of generators of the Λ-module X. (For details, see Corollary 9.7.)

3. Fine Selmer groups and Iwasawa theory

Here, we use the similar notation to that in §2. Let Σ be a finite set of containing $\{p, \infty\}$, and T a free $\mathcal{O}$-module of finite rank d with a continuous $\mathcal{O}$-linear $G_{\mathbb{Q},\Sigma}$-action satisfying the conditions (C1)–(C7). We define A, T^* and A^* by similar manner to that in §2. In this section, we introduce the "fine" Selmer group $H^1_{\mathcal{F}_{\mathrm{can}}}(F, A^*)$, which is our main interest. Here, we also review some Iwasawa theoretical results.

3.1. Local conditions and Selmer groups.

In the first subsection, we introduce the "fine" Selmer group and some related Selmer groups.

First, we define Selmer groups for "general" local conditions. Let F be a number field, and Σ_F be a set of all places of F above Σ. Consider a topological $\mathbb{Z}_p$-module with an G_{F,Σ_F}-action. We assume that M is a discrete group (resp. a pro-p-group or a finite dimensional $\mathbb{Q}_p$-vector space), and we regard M as an étale sheaf (resp. étale pro-p-sheaf or étale $\mathbb{Q}_p$-sheaf) on $\mathrm{Spec}\, F$. A local condition $\mathcal{F}$ on M is a collection

$$\left\{ H^1_{\mathcal{F}}(F \otimes \mathbb{Q}_v, M) \subseteq H^1_{\text{ét}}(F \otimes \mathbb{Q}_v, M) \right\},$$

where v runs through all places of $\mathbb{Q}$. Note that we assume $p \neq 2$ in this paper, so we have automatically

$$H^1_{\mathcal{F}}(F \otimes \mathbb{R}, M) = H^1_{\text{ét}}(F \otimes \mathbb{R}, M) = 0$$

For such pair $(M, \mathcal{F})$, we define the Selmer group $H^1_{\mathcal{F}}(F, M)$ by

$$H^1_{\mathcal{F}}(F, M) := \ker\left(H^1(F, M) \longrightarrow \prod_{v \in P_{\mathbb{Q}}} \frac{H^1_{\text{ét}}(F \otimes \mathbb{Q}_v, M)}{H^1_{\mathcal{F}}(F \otimes \mathbb{Q}_v, M)}\right).$$

For a finite set Σ' of places of $\mathbb{Q}$, we define

$$H^1_{\mathcal{F}^{\Sigma'}}(F, M) : = \ker\left(H^1(F, M) \longrightarrow \prod_{v \in P_{\mathbb{Q}} \setminus \Sigma'} \frac{H^1_{\text{ét}}(F \otimes \mathbb{Q}_v, M)}{H^1_{\mathcal{F}}(F \otimes \mathbb{Q}_v, M)}\right),$$

$$H^1_{\mathcal{F}_{\Sigma'}}(F, M) : = \ker\left(H^1_{\mathcal{F}}(F, M) \longrightarrow \prod_{v \in \Sigma'} H^1_{\text{ét}}(F \otimes \mathbb{Q}_v, M)\right).$$

For any $n \in \mathbb{Z}_{\geq 0}$, we denote by $\mathcal{F}^n$ (resp. $\mathcal{F}_n$) the local condition $\mathcal{F}^{\text{prime}(n)}$ (resp. $\mathcal{F}_{\text{prime}(n)}$), where prime$(n)$ is the set of prime divisors of n.

From now on, let us consider local conditions and Selmer groups on the free $\mathcal{O}$-module T. Here, we assume $F = \mathbb{Q}_m$ is a subfield of $\mathbb{Q}_\infty$. Let $\mathcal{F}$ be a local condition on T, and N a positive integer. For any prime number ℓ, by local duality Theorem, we have a diagram

$$\begin{tikzcd}
H^i_{\text{ét}}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T) \arrow[d, "\pi_N"] & \times & H^{2-i}_{\text{ét}}(\mathbb{Q}_m \otimes \mathbb{Q}_v, A^*) \arrow[r, "(\cdot,\cdot)_\ell"] & K/\mathcal{O} \\
H^i_{\text{ét}}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/\pi^N T) & \times & H^{2-i}_{\text{ét}}(\mathbb{Q}_m \otimes \mathbb{Q}_v, A^*[\pi^N]) \arrow[u, "i_N"] \arrow[r, "(\cdot,\cdot)_\ell"] & \frac{1}{\pi^N}\mathcal{O}/\mathcal{O} \arrow[u, hook, "\subseteq"]
\end{tikzcd}$$

whose horizontal arrows are perfect pairings, and satisfy

$$(\pi_N(a), b)_\ell = (a, i_N(b))_\ell \in K/\mathcal{O}$$

for any $a \in H^i_{\text{ét}}(F \otimes \mathbb{Q}_\ell, T)$ and $b \in H^{2-i}_{\text{ét}}(F \otimes \mathbb{Q}_v, A^*[\pi^N])$. We denote the orthogonal component of $H^1_{\mathcal{F}}(F \otimes \mathbb{Q}_\ell, T)$ (resp. $H^1_{\mathcal{F}}(F \otimes \mathbb{Q}_\ell, T/\pi^N T)$) with respect to the above pairing $(\cdot, \cdot)_\ell$ by $H^1_{\mathcal{F}^*}(F \otimes \mathbb{Q}_\ell, A^*)$ (resp. $H^1_{\mathcal{F}^*}(F \otimes \mathbb{Q}_\ell, A^*)$). Then, we obtain the *dual* local condition $\mathcal{F}^*$ of A^* and $A^*[\pi^N]$.

Definition 3.1. Let ℓ be a prime number distinct from p, and $\mathbb{Q}_\ell^{\text{ur}}$ the maximal unramified extension of $\mathbb{Q}_\ell$.

- We define
$$\begin{aligned} H^1_f(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T \otimes K) &= H^1_{ur}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T \otimes K) \\ &:= \ker\left(H^1_{\text{ét}}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T \otimes K) \longrightarrow H^1_{\text{ét}}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell^{\text{ur}}, T \otimes K)\right). \end{aligned}$$
- We denote by $H^1_f(F \otimes \mathbb{Q}_\ell, T)$ the inverse image of $H^1_f(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T \otimes K)$ with respect to the natural map
$$H^1_{\text{ét}}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T) \longrightarrow H^1_{\text{ét}}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T \otimes K).$$
- We denote by $H^1_f(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T \otimes K/\mathcal{O})$ the image of $H^1_f(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T \otimes K)$ with respect to the natural map
$$H^1_{\text{ét}}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T \otimes K) \longrightarrow H^1_{\text{ét}}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T \otimes K/\mathcal{O}).$$

- We denote by $H^1_f(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/\pi^N T)$ the image of $H^1_f(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T)$ with respect to the natural map
$$H^1_{\text{ét}}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T) \longrightarrow H^1_{\text{ét}}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/\pi^N T).$$
Note that $H^1_f(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/\pi^N T)$ coincides with the inverse image of $H^1_f(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T \otimes K/\mathcal{O})$ with respect to the map
$$H^1_{\text{ét}}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/\pi^N T) \longrightarrow H^1_{\text{ét}}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T \otimes K/\mathcal{O})$$
induced by
$$T/\pi^N T \xrightarrow[\simeq]{\times(1/\pi^N)} \frac{1}{\pi^N} T/T \subseteq T \otimes K/\mathcal{O}.$$
(See [Ru2] Lemma 1.3.8.)

Then, we define the local condition $\mathcal{F}_{\text{can}}$ on T by
$$H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_v, T) := \begin{cases} H^1_f(\mathbb{Q}_m \otimes \mathbb{Q}_v, T) & \text{if } v \text{ is a finite place distinct from } p; \\ H^1_{\text{ét}}(\mathbb{Q}_m \otimes \mathbb{Q}_p, T) & \text{if } v = p; \\ 0 & \text{if } v = \infty. \end{cases}$$
For any $N > 0$, we define the *induced* local condition $\{H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_m \otimes \mathbb{Q}_v, T/\pi^N T)\}_v$ on $T/\pi^N T$ by the image of $H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_m \otimes \mathbb{Q}_v, T)$ for any place v of $\mathbb{Q}$. In this paper, we call the group $H^1_{\text{can}^*}(F, A^*)$ *the dual fine Selmer group* of A^*.

Remark 3.2. By the local duality, for any prime number ℓ distinct from p, $H^1_f(F \otimes \mathbb{Q}_\ell, T)$ and $H^1_f(F \otimes \mathbb{Q}_\ell, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ are orthogonal component each other with respect to the pairing $(\cdot, \cdot)_\ell$. In other words, the pairing induces the natural isomorphism
$$\begin{aligned} H^1_s(F \otimes \mathbb{Q}_\ell, T) &:= H^1_{\text{ét}}(F \otimes \mathbb{Q}_\ell, T)/H^1_f(F \otimes \mathbb{Q}_\ell, T) \\ &\simeq H^1_f(F \otimes \mathbb{Q}_\ell, A^*)^\vee, \\ H^1_s(F \otimes \mathbb{Q}_\ell, A^*) &:= H^1_{\text{ét}}(F \otimes \mathbb{Q}_\ell, A^*)/H^1_f(F \otimes \mathbb{Q}_\ell, A^*) \\ &\simeq H^1_f(F \otimes \mathbb{Q}_\ell, T)^\vee. \end{aligned}$$
The similar orthogonality holds for $H^1_f(F \otimes \mathbb{Q}_\ell, T/p^N T)$ and $H^1_f(F \otimes \mathbb{Q}_\ell, A^*[p^N]) = H^1_f(F \otimes \mathbb{Q}_\ell, T^*/p^N T^*)$.

By the orthogonality of the local condition f, the dual local condition $\mathcal{F}^*_{\text{can}}$ on A^* is as follows:
$$H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}_v, A^*) = \begin{cases} H^1_f(F \otimes \mathbb{Q}_v, A^*) & \text{if } v \text{ is a finite place distinct from } p; \\ 0 & \text{if } v = p, \infty. \end{cases}$$

In this paper, we often use the following elementary fact which immediately follows from the assumption (C1) and (C4).

Lemma 3.3. *For any integers $m \in \mathbb{Z}_{\geq 0}$ and $N \in \mathbb{Z}_{\geq 0}$, the natural homomorphism*
$$H^1(\mathbb{Q}_m, A[\pi^N]) \longrightarrow H^1(\mathbb{Q}_m, A)[\pi^N],$$
$$H^1(\mathbb{Q}_m, A^*[\pi^N]) \longrightarrow H^1(\mathbb{Q}_m, A^*)[\pi^N]$$
are isomorphisms.

We also note that the hypothesis (C6) implies $H^1_{\mathcal{F}^*_{\mathrm{can}}}(F \otimes \mathbb{Q}_p, A^*) = 0$. Then, by Lemma 3.3 and [Ru2] Proposition 7.4.4, we obtain the following proposition.

Proposition 3.4. *Let m be a non-negative integer, and N a positive integer. Then, we have a natural isomorphism*

$$H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}_m, A^*[\pi^N]) \simeq H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}_m, A^*)[\pi^N].$$

3.2. **Preliminaries on Iwasawa theoretical results.** Here, we recall some Iwasawa theoretical results which control Iwasawa modules arising from Galois cohomology groups and certain Selmer groups. Recall that in §2, we put $\Gamma := \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \simeq \mathbb{Z}_p$ and $\Lambda := \mathbb{Z}_p[[\Gamma]]$. For any non-negative m, we define

$$R_m := \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})] \simeq \Lambda/(\gamma^{p^m} - 1).$$

Recall that we have defined the Λ-module X by

$$X = X(T) := \ker\left(\mathbb{H}^2_\Sigma(T) \longrightarrow \bigoplus_{\ell \in \Sigma} \mathbb{H}^2_{\mathrm{loc},\ell}(T)\right).$$

Here, we assume that X is a torsion Λ-module. Let ℓ be any prime number contained in Σ. Since ℓ does not split completely in $\mathbb{Q}_\infty/\mathbb{Q}$, the $\mathbb{Z}_p$-module

$$\mathbb{H}^2_{\mathrm{loc},\ell}(T) \simeq \left(\varinjlim_m H^0_{\mathrm{\acute{e}t}}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, A^*)\right)^\vee$$

is finitely generated. This implies that $\mathbb{H}^2_\Sigma(T)$ is also a torsion Λ-module. We need the following lemma which follows from the assumptions (C6) and (C7).

Lemma 3.5. *Let ℓ be a prime number contained in Σ. Then, $\mathbb{H}^2_{\mathrm{loc},\ell}(T)$ is a torsion-free $\mathcal{O}$-module.*

Proof. If $\ell = p$, then it immediately follows from (C6) and the local duality theorem that $\mathbb{H}^2_{\mathrm{loc},p}(T)$ is a torsion-free $\mathbb{Z}_p$-module. So, we suppose $\ell \neq p$. Let $\mathbb{Q}_{\ell,\infty}$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}_\ell$. Here, we regard $W_{\mathbb{Q}_\ell} \subseteq G_{\mathbb{Q}_\ell}$ as a subgroup of $G_\mathbb{Q}$ via the embedding $\ell_{\overline{\mathbb{Q}}} \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$ fixed in §1. Note that as an $\mathcal{O}$-module, $H^0_{\mathrm{\acute{e}t}}(\mathbb{Q}_\infty \otimes \mathbb{Q}_\ell, A^*)$ is isomorphic to the direct product of finitely many copies of $H^0(\mathbb{Q}_{\ell,\infty}, A^*)$. So by the local duality theorem, in order to show Lemma 3.5, it is sufficient to show that the $\mathcal{O}$-module $H^0(\mathbb{Q}_{\ell,\infty}, A^*)$ is divisible. Let (r_ℓ, N_ℓ), L_ℓ and g_ℓ be as in (C7) and Lemma 2.1. Note that $T^*/(g_\ell - 1)T^* \simeq (T[g_\ell - 1])^*$ is a torsion-free $\mathcal{O}$-module. So we apply the snake lemma to the diagram

$$\begin{tikzcd}
0 \arrow[r] & T^* \arrow[r] \arrow[d, "g_\ell - 1"] & T^* \otimes_\mathcal{O} K \arrow[r] \arrow[d, "g_\ell - 1"] & A^* \arrow[r] \arrow[d, "g_\ell - 1"] & 0 \\
0 \arrow[r] & T^* \arrow[r] & T^* \otimes_\mathcal{O} K \arrow[r] & A^* \arrow[r] & 0
\end{tikzcd}$$

with exact rows, and we deduce that the $\mathcal{O}$-module

$$H^0(L_\ell, A^*) \simeq A^*[g_\ell - 1]$$

is divisible.

Let $N \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, and define the subgroup $\mathfrak{H}_{\ell,N}$ of $\mathfrak{G}_\ell := \mathrm{Gal}(L_\ell/\mathbb{Q}_{\ell,\infty})$ of by

$$\mathfrak{H}_{\ell,N} := \mathrm{Ker}\left(\mathfrak{G}_\ell \longrightarrow \mathrm{Aut}\left(H^0\left(L_\ell, A^*[\pi^N]\right)\right)\right).$$

The hypothesis (C7) implies that $\mathfrak{G}_\ell$ does not have pro-ℓ quotient. So for any finite N, the order of $\mathfrak{G}_{\ell,N} := \mathfrak{G}_{\ell,\infty}/\mathfrak{H}_{\ell,N}$ is (finite and) prime to p. Then, we can define an element

$$\mathfrak{e}_N := \frac{1}{\#\mathfrak{G}_\ell} \sum_{\sigma \in \mathfrak{G}_{\ell,N}} \sigma \in \mathcal{O}[\mathfrak{G}_{\ell,N}]$$

for any finite N, and obtain an idempotent element

$$\mathfrak{e} := (\mathfrak{e}_N)_{N\geq 0} \in \mathcal{O}[[\mathfrak{G}_{\ell,\infty}]] := \varprojlim_N \mathcal{O}[\mathfrak{G}_{\ell,N}].$$

We have $H^0(\mathbb{Q}_{\ell,\infty}, A^*) = \mathfrak{e}H^0(L_\ell, A^*)$, so the divisibility of $H^0(L_\ell, A^*)$ implies that the $\mathcal{O}$-module $H^0(\mathbb{Q}_{\ell,\infty}, A^*)$ is divisible. □

By Lemma 3.5, we immediately obtain the following corollary.

Corollary 3.6. *Let $\tilde{X}_{\mathrm{fin}}$ be the maximal pseudo-null Λ-submodule of $\mathbb{H}^2_\Sigma(T)$. Then, we have $\tilde{X}_{\mathrm{fin}} = X_{\mathrm{fin}}$.*

We define a Λ-module $H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}_\infty, A^*)$ by

$$H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}_\infty, A^*) := \varinjlim_{m\geq 0} H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}_m, A^*).$$

Note that $H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}_\infty, A^*)$ is a cofinitely generated Λ-module. The following proposition gives another description of the Λ-module X.

Proposition 3.7. *There exists a natural isomorphism*

$$X(T) \simeq H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}_\infty, A^*)^\vee$$

of Λ-module.

Proof. It follows from Proposition B.3.4 in [Ru2] that we have

$$\varprojlim_m H^1(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T) \simeq \varprojlim_m H^1_{ur}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T) = \varprojlim_m H^1(\mathcal{O}_{\mathbb{Q}_m} \otimes \mathbb{Z}_\ell, T^{I_\ell}) \tag{2}$$

for any prime number ℓ distinct from p, where $I_\ell := G_{\mathbb{Q}_\ell^{\mathrm{ur}}}$ is the inertia subgroup of $G_{\mathbb{Q}_\ell}$. Then, the isomorphism in Proposition 3.7 immediately follows from the limit of the Poitou–Tate exact sequence, the orthogonality of the local conditions and the equality (2). □

By our assumption (C6), we have the following proposition.

Proposition 3.8 ([Ru2] Proposition 7.4.4)**.** *Let m be a non-negative integer. Then, we have a natural isomorphism*

$$X(T) \otimes_\Lambda R_m \simeq H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}_m, A^*)^\vee.$$

In our paper, the following proposition plays important roles.

Proposition 3.9 ([Ne] Proposition 8.4.8.1)**.** *We have a spectral sequence*

$$(E_\Sigma)_2^{p,q} = \mathrm{Tor}_{-p}^{\Lambda}\left(R_m, \mathbb{H}^q_\Sigma(T)\right) \Longrightarrow H^{q-p}_{\acute{e}t}(\mathcal{O}_{\mathbb{Q}_m,\Sigma}, T).$$

Especially, by proposition 3.9, we have a short exact sequence

$$0 \longrightarrow \mathbb{H}^1_\Sigma(T) \otimes_\Lambda R_m \longrightarrow H^1_{\text{ét}}(\mathcal{O}_{\mathbb{Q}_m,\Sigma}, T) \longrightarrow \mathbb{H}^2_\Sigma(T)[\gamma^{p^m}-1] \longrightarrow 0$$

for any $m \in \mathbb{Z}_{\geq 0}$ be any element. By this fact and Corollary 3.6, we obtain the following corollary.

Corollary 3.10. *If $X = X(T)$ is a torsion Λ-module, and if*

$$\mathrm{char}_\Lambda(\mathbb{H}^2_\Sigma(T)) \not\subseteq (\gamma^{p^m}-1)\Lambda,$$

then the cokernel of the natural homomorphism

$$\mathbb{H}^1_\Sigma(T) \otimes_\Lambda R_m \longrightarrow H^1_{\acute{e}t}(\mathcal{O}_{\mathbb{Q}_m,\Sigma}, T)$$

is annihilated by $\mathrm{ann}_\Lambda(X_{\mathrm{fin}})$.

4. Euler systems of Rubin type

The axiomatic framework of Euler systems for general p-adic representations of $G_\mathbb{Q}$ are established in [P-R], [Ka1] and [Ru2]. Here, we recall the notion of Euler systems and some of their basic properties introduced in [Ru2].

4.1. **Euler systems.**

Throughout this section, we use the same notations as the previous section. In particular, we assume that T is a free $\mathcal{O}$-module of finite rank d with a continuous $\mathcal{O}$-linear $G_{\mathbb{Q},\Sigma}$-action, and satisfies the conditions (C1)–(C7) in §2.

Definition 4.1. Let M a free $\mathcal{O}$-module of finite rank with a $\mathcal{O}$-linear action of $G_\mathbb{Q}$. Then, for each element $\sigma \in G_\mathbb{Q}$, we define a polynomial $P(\sigma|M;x)$ by

$$P(\sigma|M;x) := \det_\mathcal{O}(1-\sigma x \mid M) \in \mathcal{O}[x].$$

Definition 4.2. Recall that we denote by $\mathcal{N}(\Sigma)$ the set of all positive integers decomposed into square-free products of prime numbers not contained in Σ. If no confusion arises, we write $\mathcal{N} := \mathcal{N}(\Sigma)$ for simplicity. For any $n \in \mathcal{N}$ and any non-negative integer m, we defined a field $\mathbb{Q}_m(n)$ in §2. In this paper, we call a family

$$\mathbf{c} := \left\{c_m(n) \in H^1(\mathbb{Q}_m(n), T)\right\}_{m\geq 0, n \in \mathcal{N}(\Sigma)}$$

of cohomology classes *an Euler system for* (T, Σ) if $\mathbf{c}$ satisfies the following conditions:

(ES1) For any $n \in \mathcal{N}$ and any non-negative integer m, we have

$$\mathrm{Cor}_{\mathbb{Q}_{m+1}(n)/\mathbb{Q}_m(n)}\left(c_{m+1}(n)\right) = c_m(n).$$

(ES2) Let $n \in \mathcal{N}$ and m a non-negative integer. Then, for any prime divisor ℓ of n, we have

$$\mathrm{Cor}_{\mathbb{Q}_m(n)/\mathbb{Q}_m(n/\ell)}\left(c_m(n)\right) = P(\mathrm{Fr}_\ell^{-1}|T^*;\mathrm{Fr}_\ell^{-1}) \cdot c_m(n/\ell),$$

where $\mathrm{Fr}_\ell \in \mathrm{Gal}\left(\mathbb{Q}_m(n/\ell)/\mathbb{Q}\right)$ is the arithmetic Frobenius element at ℓ.

We denote the set of all Euler systems for (T,Σ) by $\mathrm{ES}_\mathcal{O}(T,\Sigma)$.

In order to refer results in [MR], it is convenient to introduce the notion of "modified" Euler systems in the following sense.

Definition 4.3. Let $\mathrm{ES}'_{\mathcal{O}}(T,\Sigma)$ be the set of all families

$$\mathbf{c}' := \left\{ c'_m(n) \in H^1(\mathbb{Q}_m(n),T) \right\}_{m\geq 0, n\in\mathcal{N}(\Sigma)}$$

of cohomology classes satisfying (ES1) and the following condition (ES2)$'$:

(ES2)$'$ For any non-negative integer m, any $n \in \mathcal{N}$ and any prime divisor ℓ of n, we have

$$\mathrm{Cor}_{\mathbb{Q}_m(n)/\mathbb{Q}_m(n/\ell)}\left(c'_m(n)\right) = P(\mathrm{Fr}_\ell|T;\mathrm{Fr}_\ell^{-1})\cdot c'_m(n/\ell).$$

We call an element of $\mathrm{ES}'_{\mathcal{O}}(T,\Sigma)$ *a modified Euler system.*

Remark 4.4. Let $\mathbf{c} = \{c_m(n)\}_{m,n} \in \mathrm{ES}_{\mathcal{O}}(T,\Sigma)$ and $\mathbf{c}' = \{c'_m(n)\}_{m,n} \in \mathrm{ES}'_{\mathcal{O}}(T,\Sigma)$ be arbitrary elements. Regard T as a pro-p sheaf on $(\mathrm{Spec}\ \mathcal{O}_{\mathbb{Q}_m(n),\Sigma})_{\text{ét}}$. Then, t is known that $c_m(n)$ and $c'_m(n)$ both belong to $H^1_{\text{ét}}(\mathcal{O}_{\mathbb{Q}_m(n)}[1/p], j_*T)$, where

$$j\colon \mathrm{Spec}\ \mathcal{O}_{\mathbb{Q}_m(n),\Sigma} \longrightarrow \mathrm{Spec}\ \mathcal{O}_{\mathbb{Q}_m(n)}[1/p]$$

is the natural open immersion. For details, see [Ru2] Corollary B.3.5.

Note that in [MR], "an Euler system" does not mean an element of $\mathrm{ES}_{\mathcal{O}}(T,\Sigma)$ but $\mathrm{ES}'_{\mathcal{O}}(T,\Sigma)$. The following proposition relates $\mathrm{ES}_{\mathcal{O}}(T,\Sigma)$ to $\mathrm{ES}'_{\mathcal{O}}(T,\Sigma)$.

Proposition 4.5 ([Ru2] Lemma 9.6.1)**.** *There exists an isomorphism*

$$i_{\mathrm{ES}}\colon \mathrm{ES}_{\mathcal{O}}(T,\Sigma) \longrightarrow \mathrm{ES}'_{\mathcal{O}}(T,\Sigma); \quad \mathbf{c} \longmapsto i_{\mathrm{ES}}(\mathbf{c}) = \{c'_m(n)\}$$

of $\mathcal{O}[G_{\mathbb{Q}}]$-modules satisfying the following property:

> *"For any $\mathbf{c} := \{c_m(n)\} \in \mathrm{ES}_{\mathcal{O}}(T,\Sigma)$, any non-negative integer m and any $n \in \mathcal{N}$, there exists a unit $u_{T,n}$ is a unit of $\mathcal{O}[\mathrm{Gal}(\mathbb{Q}_m(n)/\mathbb{Q})]$ such that*
>
> $$i_{\mathrm{ES}}(c)_m(n) \equiv u_{T,n}\cdot c_m(n) \mod M_0(T;\mathbf{c},n).$$
>
> *Here, $M_m(T;\mathbf{c},n)$ is a $\mathcal{O}[\mathrm{Gal}(\mathbb{Q}_m(n)/\mathbb{Q})]$-submodule of $H^1(\mathbb{Q}_m(n),T)$ generated by*
>
> $$\left\{ \text{the image of } c_m(a) \;\middle|\; 0 < a \mid n \right\}$$
>
> *where ℓ runs through all prime divisors of n."*

Note that we can construct the map i_{ES} in Proposition 4.5 explicitly. For details of its construction, see the proof of Lemma 9.6.1.

4.2. **Localization maps and finite-singular comparison maps.** Here, we introduce two types of homomorphisms, namely *localization maps* and *finite-singular comparison maps*, which play key roles in Euler system arguments. Let $e := e_{K/\mathbb{Q}_p}$ be the absolute ramification index of K. In this and the next subsections, we fix integers m and N satisfying $N > em \geq 0$. We define

$$R_{m,N} := R_m/\pi^N R_m = \mathcal{O}/\pi^N\mathcal{O}[\mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})].$$

Definition 4.6. We fix an element $\tau \in G_{\mathbb{Q}(\mu_{p^\infty})}$ in the condition (C2) and an isomorphism

$$\Phi^*\colon T^*/(\tau-1)T^* \xrightarrow{\simeq} \mathcal{O}.$$

By taking $\mathrm{Hom}_{\mathcal{O}}(-, K/\mathcal{O}(1))$, we obtain the isomorphism

$$\Phi\colon K/\mathcal{O}(1) \longrightarrow (A)^{\tau=1}.$$

Then, we define an isomorphism

$$\Psi_{m,N}\colon R_{m,N} \otimes_{\mathbb{Z}/\pi^N\mathbb{Z}} (T/\pi^N T)^{\tau=1} = R_{m,N} \otimes_{\mathcal{O}/\pi^N\mathcal{O}} \left(A[\pi^N]\right)^{\mathrm{Fr}=1} \longrightarrow R_{m,N}$$

of $R_{m,N}$-modules by

$$\Psi_{m,N}(x \otimes a) \otimes (1/\pi^N) \otimes \zeta_{p^{[N/e]}} := x \cdot (\Phi)^{-1}(a) \in R_{m,N} \otimes_{\mathcal{O}/\pi^N\mathcal{O}} K/\mathcal{O}[\pi^N](1).$$

(Here, we declare that $\mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$ acts trivially on $T/\pi^N T$.)

For any prime number $\ell \notin \Sigma$, we denote by I_ℓ the ideal of $\mathcal{O}$ generated by $\ell-1$ and $P(\mathrm{Fr}_\ell^{-1}|T^*;1)$. Let n be a square-free product $n := \ell_1 \times \cdots \times \ell_r$, where ℓ_i is a prime number not contained in Σ for $i = 1, ..., r$. Then, we define an ideal

$$I_n := \sum_{i=1}^r I_{\ell_i} \subseteq \mathcal{O}.$$

Definition 4.7. We define a set $\mathcal{P}_N(\Sigma;T)_{\mathcal{O}}$ of prime numbers by

$$\mathcal{P}_N(\Sigma;T)_{\mathcal{O}} := \left\{ \ell \;\middle|\; \begin{array}{l} \ell \notin \Sigma,\ I_\ell \subseteq \pi^N\mathbb{Z}_p, \text{ and } T/(\pi^N T + (\mathrm{Fr}_\ell - 1)T) \text{ is} \\ \text{a free } \mathcal{O}/\pi^N\mathcal{O}\text{-module of rank one.} \end{array} \right\},$$

where $\mathrm{Fr}_\ell \in G_{\mathbb{Q}}$ is an arithmetic Frobenius element at ℓ. Then, we put

$$\mathcal{N}_N(\Sigma;T)_{\mathcal{O}} := \left\{ \prod_{i=1}^r \ell_i \;\middle|\; \begin{array}{l} r \in \mathbb{Z}_{>0},\ \ell_i \in \mathcal{P}_N(\Sigma;T)_{\mathcal{O}}\ (i=1,\dots,r) \\ \text{and } \ell_i \neq \ell_j \text{ if } i \neq j \end{array} \right\} \cup \{1\}.$$

We define subsets $\mathcal{P}_N^\tau(\Sigma;T)_{\mathcal{O}} \subseteq \mathcal{P}_N(\Sigma;T)_{\mathcal{O}}$ and $\mathcal{N}_N^\tau(\Sigma;T)_{\mathcal{O}} \subseteq \mathcal{N}_N^\tau(\Sigma;T)_{\mathcal{O}}$ by

$$\mathcal{P}_N^\tau(\Sigma;T)_{\mathcal{O}} : = \left\{\ell \in \mathcal{P}_N(\Sigma;T)_{\mathcal{O}} \mid \mathrm{Fr}_\ell \text{ coincides with } \tau \text{ on } \mathbb{Q}(\mu_{p^N}, A[\pi^N])\right\},$$

$$\mathcal{N}_N^\tau(\Sigma;T)_{\mathcal{O}} : = \left\{ \prod_{i=1}^r \ell_i \in \mathcal{N}_N(\Sigma;T)_{\mathcal{O}} \;\middle|\; \ell_i \in \mathcal{P}_N^\tau(\Sigma;T)_{\mathcal{O}}\ (i=1,\dots,r) \right\} \cup \{1\}$$

respectively. For simplicity, we write $\mathcal{P}_N := \mathcal{P}_N(\Sigma;T)_{\mathcal{O}}$, $\mathcal{P}_N^\tau := \mathcal{P}_N^\tau(\Sigma;T)_{\mathcal{O}}$, $\mathcal{N}_N := \mathcal{N}_N(\Sigma;T)_{\mathcal{O}}$ and $\mathcal{N}_N^\tau := \mathcal{N}_N^\tau(\Sigma;T)_{\mathcal{O}}$.

We define $H_n := \mathrm{Gal}\left(\mathbb{Q}(n)/\mathbb{Q}\right)$ for any $n \in \mathcal{N}_N$. If n is decomposed as $n = \prod_{i=1}^r \ell_i$, where $\ell_1, \dots, \ell_r$ are distinct prime numbers, then we have natural isomorphisms

$$\mathrm{Gal}(\mathbb{Q}_m(n)/\mathbb{Q}_m) \simeq H_n \simeq H_{\ell_1} \times \cdots \times H_{\ell_r}$$

for any integer $m \geq 0$. We identify these groups by the above natural isomorphisms.

Let ℓ be a prime number contained in $\mathcal{P}_N$. We shall take a generator σ_ℓ of the cyclic group H_ℓ as follows. We define a positive integer $N_{\{\ell\}}$ by

$$I_\ell = \pi^{N_{\{\ell\}}}\mathcal{O}.$$

Then, by definition, we have $N_{\{\ell\}} \geq N$. By the fixed embedding $\ell_{\overline{\mathbb{Q}}} \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, we regard $\mu_{p^{N\{\ell\}}}$ as a subset of $\mathbb{Q}_\ell$. Let $\lambda := \ell_{\mathbb{Q}(\ell)}$ be the place of $\mathbb{Q}(\ell)$ below $\ell_{\overline{\mathbb{Q}}}$, and identify $\mathrm{Gal}\left(\mathbb{Q}(\mu_\ell)_\lambda/\mathbb{Q}_\ell\right)$ with H_ℓ by the isomorphism induced from the natural embedding. Let π be a uniformizer of $\mathbb{Q}(\ell)_\lambda$. We fix a generator σ_ℓ of H_ℓ such that

$$\pi^{\sigma_\ell - 1} \equiv \zeta_{p^{\mathrm{ord}_p(\ell-1)}} \pmod{\mathfrak{m}_\lambda},$$

where $\mathfrak{m}_\lambda$ is the maximal ideal of $\mathbb{Q}(\ell)_\lambda$, and

$$\mathrm{ord}_p \colon \mathbb{Q}_p^\times \twoheadrightarrow \mathbb{Z}$$

is the normalized valuation at p. Note that the definition of σ_ℓ does not depend on the choice of π. We have the following Lemma.

Lemma 4.8 ([Ru2] Lemma 1.4.7)**.** *Let $\ell \in \mathcal{P}_N$. The following hold.*

(i) *The $\mathcal{O}/\pi^N\mathcal{O}$-modules $T/\left(\pi^N T + (\mathrm{Fr}_\ell - 1)T\right)$ and $(T/\pi^N T)^{\mathrm{Fr}_\ell = 1}$ are free of rank one.*

(ii) *Evaluating cocycles on Fr_ℓ and σ_ℓ induces isomorphisms*

$$H^1_f(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/\pi^N T) \xrightarrow{\simeq} R_{m,N} \otimes_{\mathcal{O}/\pi^N\mathcal{O}} T/\left(\pi^N T + (\mathrm{Fr}_\ell - 1)T\right)$$
$$H^1_s(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/\pi^N T) \xrightarrow{\simeq} R_{m,N} \otimes_{\mathcal{O}/\pi^N\mathcal{O}} (T/\pi^N T)^{\mathrm{Fr}_\ell = 1}$$

of R_m-modules respectively.

If $\ell \in \mathcal{P}_N^\tau$, then the isomorphism $\Psi_{m,N}$ and Lemma 4.8 induce an isomorphism

$$\Psi^\ell_{m,N} \colon H^1_s(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/\pi^N T) \simeq R_{m,N} \otimes_{\mathcal{O}/\pi^N\mathcal{O}} (T/\pi^N T)^{\mathrm{Fr}_\ell = 1} \longrightarrow R_{m,N}$$

of $R_{m,N}$-modules. Here, we define the "localization" map.

Definition 4.9 (Localization map)**.** For any $\ell \in \mathcal{N}_N$, we call the composite of natural maps

$$(\cdot)^{\ell,s}_{m,N} \colon H^1(\mathbb{Q}_m, T/\pi^N T) \longrightarrow H^1_{\text{ét}}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/\pi^N T) \longrightarrow H^1_s(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/\pi^N T)$$

the *localization* map. If $\ell \in \mathcal{P}_N^\tau$, we define the composite map

$$(\cdot)^{\ell,s}_{m,N,\Phi^*} := \Psi^\ell_{m,N} \circ (\cdot)^\ell_{m,N} \colon H^1(\mathbb{Q}_m, T/\pi^N T) \longrightarrow R_{m,N}.$$

In order to define the "finite-singular" map, we need to introduce a new local condition on $T/\pi^N T$.

Definition 4.10 ($\mathbb{Q}_m(\ell) \otimes \mathbb{Q}_\ell$-transverse condition)**.** Let $\ell \in \mathcal{P}_N$. Then, we define

$$H^1_{\mathrm{tr}}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/\pi^N T) := \mathrm{Ker}\begin{pmatrix} H^1_{\text{ét}}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/\pi^N T) \\ \longrightarrow H^1_{\text{ét}}(\mathbb{Q}_m(\ell) \otimes \mathbb{Q}_\ell, T/\pi^N T) \end{pmatrix}.$$

This local condition is called *the $\mathbb{Q}_m(\ell) \otimes \mathbb{Q}_\ell$-transverse condition* (cf. [MR] Definition 1.1.6).

Lemma 4.11 ([MR] Lemma 1.2.4). *Let $\ell \in \mathcal{P}_N$. Then, we have a direct sum decomposition*

$$H^1_{\acute{e}t}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/\pi^N T) = H^1_f(\mathbb{Q}_\ell, T/\pi^N T) \oplus H^1_{\mathrm{tr}}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/\pi^N T).$$

So, the natural projection

$$H^1_{\mathrm{tr}}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/\pi^N T) \longrightarrow H^1_s(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/\pi^N T)$$

is an isomorphism.

Let $\ell \in \mathcal{P}_N$. Since $\ell \equiv 1 \mod p^N$, we have

$$P(\mathrm{Fr}_\ell | T; x) \equiv P(\mathrm{Fr}_\ell^{-1} | T^*; x) \equiv 0 \mod \pi^N.$$

Then, by definition of $\mathcal{P}_N$, we have

$$P(\mathrm{Fr}_\ell^{-1} | T^*; 1) \equiv 0 \mod \pi^N,$$

so there exists a unique polynomial $Q(x) \in \mathcal{O}/\pi^N\mathcal{O}$ satisfying

$$(x-1)Q(x) \equiv P(\mathrm{Fr}_\ell | T; x) := \det(1 - x\mathrm{Fr}_\ell | T) \mod \pi^N.$$

By the Cayley–Hamilton Theorem, we have a group homomorphism

$$T/\left(\pi^N T + (\mathrm{Fr}_\ell - 1)T\right) \xrightarrow{\times Q(\mathrm{Fr}_\ell^{-1})} (T/\pi^N T)^{\mathrm{Fr}_\ell - 1}. \tag{3}$$

Thus we obtain the "finite-singular comparison" homomorphism as follows.

Definition 4.12 (Finite-singular comparison map). Let $\ell \in \mathcal{N}_N$. The homomorphism (3) and Lemma 4.8 induce the homomorphism

$$\phi^\ell_{\mathrm{fs}} \colon H^1_f(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/\pi^N T) \longrightarrow H^1_s(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/\pi^N T)$$

of $R_{m,N}$-modules called the *finite-singular comparison map.* We define the composite map

$$\begin{aligned} \phi^\ell_{m,N} \colon H^1(\mathbb{Q}_m, T/\pi^N T) &\longrightarrow \frac{H^1_{\acute{e}t}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/\pi^N T)}{H^1_{tr}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/\pi^N T)} \\ &\xrightarrow{\simeq} H^1_f(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/\pi^N T) \\ &\xrightarrow{\phi^\ell_{\mathrm{fs}}} H^1_s(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/\pi^N T) \end{aligned}$$

If $\ell \in \mathcal{P}^\tau_N$, we define the composite map

$$\phi^\ell_{m,N,\Phi^*} := \Psi^\ell_{m,N} \circ \phi^\ell_{m,N} \colon H^1(\mathbb{Q}_m, T/\pi^N T) \longrightarrow R_{m,N}.$$

4.3. **Kolyvagin derivatives.** In this subsection, we recall the notion of Kolyvagin derivatives briefly. As in the previous subsection, we fix integers $N > m \geq 0$. Let $\mathbf{c} = \{c_m(n)\}_{m,n} \in \mathrm{ES}_{\mathcal{O}}(T, \Sigma)$ be an Euler system, and $\mathbf{c}' = \{c'_m(n)\}_{m,n} \in \mathrm{ES}'_{\mathcal{O}}(T, \Sigma)$ a modified Euler system corresponding to $\mathbf{c}$.

Definition 4.13. For $\ell \in \mathcal{P}_N$, we define

$$D_\ell := \sum_{k=1}^{\ell-2} k\sigma_\ell^k \in \mathbb{Z}[H_\ell].$$

Let $n = \prod_{i=1}^{r} \ell_i \in \mathcal{N}_N$, where $\ell_i \in \mathcal{P}_N$ for each i. Then, we define

$$D_n := \prod_{i=1}^{r} D_{\ell_i} \in \mathbb{Z}[H_n].$$

By the formal arguments using "the universal Euler system", we have the following lemma. (For details, see [Ru2] §4.)

Lemma 4.14 ([Ru2] Lemma 4.4.2)**.** *For any $n \in \mathcal{N}_N$, the images of $D_n c_m(n)$ and $D_n c'_m(n)$ in $H^1(\mathcal{O}_{\mathbb{Q}_m(n)}[1/p], j_*T) \otimes_{\mathcal{O}} \mathcal{O}/\pi^N\mathcal{O}$ are fixed by the action of H_n, where*

$$j\colon \operatorname{Spec} \mathcal{O}_{\mathbb{Q}_m(n),\Sigma} \longrightarrow \operatorname{Spec} \mathcal{O}_{\mathbb{Q}_m(n)}[1/p]$$

is the natural inclusion.

Let $n \in \mathcal{N}_N$, and put $\Sigma_n := \Sigma \cup \mathrm{prime}(n)$. Let

$$j_n\colon \mathcal{O}_{\mathbb{Q}_m,\Sigma_n} \longrightarrow \operatorname{Spec} \mathcal{O}_{\mathbb{Q}_m}[1/pn]$$

be the natural open immersion. Later (especially in §8), we need the following lemma.

Lemma 4.15. *Under the assumption* (C7)*, the natural homomorphism*

$$j_{n*}T/\pi^N j_{n*}T \longrightarrow j_{n*}(T/\pi^N T)$$

of $\mathcal{O}$-sheaves on $(\operatorname{Spec} \mathcal{O}_{\mathbb{Q}_m}[1/pn])_{ét}$ is an isomorphism.

Proof. By the short exact sequence

$$0 \longrightarrow T \xrightarrow{\pi^N \times} T \longrightarrow T/\pi^N T \longrightarrow 0,$$

we obtain the exact sequence

$$0 \longrightarrow j_{n*}T/\pi^N j_{n*}T \longrightarrow j_{n*}(T/\pi^N T) \longrightarrow (R^1 j_{n*}T)[\pi^N] \longrightarrow 0$$

of $\mathcal{O}$-sheaves on $(\operatorname{Spec} \mathcal{O}_{\mathbb{Q}_m}[1/pn])_{\text{ét}}$. So, in order to show the lemma. it is sufficient to show $(R^1 j_{n*}T)[\pi^N] = 0$. For any place λ of $\mathbb{Q}_m$, we denote by $k_m(\lambda)$ the residue field of $\mathcal{O}_{\mathbb{Q}_m}$ at λ, and let

$$i_\lambda\colon \operatorname{Spec} k_m(\lambda) \longrightarrow \operatorname{Spec} \mathcal{O}_{\mathbb{Q}_m}$$

be the natural closed immersion. Then, we have the natural isomorphism

$$R^1 j_{n*}T \simeq \bigoplus_{\lambda} (i_\lambda)_* H^1_{\mathrm{cont}}(I_{\mathbb{Q}_{m,\lambda}}, T),$$

where λ runs through all places of $\mathbb{Q}_m$ above elements of $\Sigma \setminus \{p, \infty\}$, and we regard $H^1_{\mathrm{cont}}(I_{\mathbb{Q}_{m,\lambda}}, T)$ as a sheaf on $(\operatorname{Spec} k_m(\lambda))_{\text{ét}}$ via its $G_{k_m(\lambda)}$-module structure. Then, by the assumption (C7), it follows that the $\mathcal{O}$-module $H^1_{\mathrm{cont}}(I_{\mathbb{Q}_{m,\lambda}}, T)$ is torsion-free. Hence the $\mathcal{O}$-sheaf $R^1 j_{n*}T$ is torsion-free. □

Note that for any prime divisor ℓ of n, the action of $G_{\mathbb{Q}_\ell}$ on T is unramified. So for any $n \in \mathcal{N}_N$, the assumptions (C1) and (C4) imply that

$$H^0(\mathbb{Q}_m(n), T/\pi^N T) = H^0(\mathbb{Q}_m, T/\pi^N T) = 0.$$

In particular, we have

$$H^0_{\text{ét}}(\mathcal{O}_{\mathbb{Q}_m(n)}[1/pn], j_{n*}T/\pi^N j_{n*}T) = H^0_{\text{ét}}(\mathcal{O}_{\mathbb{Q}_m(n)}[1/pn], j_{n*}(T/\pi^N T)) = 0.$$

Thus by Hochschild–Serre spectral sequence, the restriction map

$$R^{(n)}_{m,N,T} \colon H^1_{\text{ét}}(\mathcal{O}_{\mathbb{Q}_m}[1/pn], j_{n*}T/\pi^N j_{n*}T) \longrightarrow H^1_{\text{ét}}(\mathcal{O}_{\mathbb{Q}_m(n)}, j_{n*}T/\pi^N j_{n*}T)^{H_n}$$

is an isomorphism.

Definition 4.16 (Kolyvagin derivative)**.** Let $\mathbf{z} = \{z_m(n)\}$ be $\mathbf{c}$ or $\mathbf{c}'$. Take any element of $n \in \mathcal{N}_N$. Then, we put

$$\kappa_{m,N}(n; \mathbf{z}) := (R^{(n)}_{m,N,T})^{-1}\,(D_n z_m(n)) \in H^1_{\text{ét}}(\mathcal{O}_{\mathbb{Q}_m}[1/pn], j_{n*}T/\pi^N j_{n*}T)$$

The cohomology classes $\kappa_{m,N}(n; \mathbf{z})$ is called *Kolyvagin derivatives.* Note that the Kolyvagin derivative $\kappa_{m,N}(n; \mathbf{z})$ can be regarded as an element of $H^1_{\mathcal{F}^n_{\text{can}}}(\mathbb{Q}_m, T/\pi^N T)$ since the natural map

$$H^1_{\text{ét}}(\mathcal{O}_{\mathbb{Q}_m}[1/pn], j_{n*}(T/\pi^N T)) \longrightarrow H^1_{\mathcal{F}^n_{\text{can}}}(\mathbb{Q}_m, T/\pi^N T).$$

is injective by Lemma 4.15.

For any $n \in \mathcal{N}$, we define the local condition $\mathcal{F}_{\text{can}}(n)$ on T by

$$H^1_{\mathcal{F}_{\text{can}}(n)}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T) := \begin{cases} H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_\ell, T) & \text{if } \ell \nmid n; \\ H^1_{\text{tr}}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T) & \text{if } \ell \mid n. \end{cases}$$

The following proposition is one of the essence in induction steps of the Euler system arguments.

Proposition 4.17 ([Ru2] Theorem 3.5.1 and [Ru2] Theorem 3.5.4)**.** *Let* $\mathbf{z} = \{z_m(n)\}$ *be* $\mathbf{c}$ *or* $\mathbf{c}'$*, and* n *any element of* $\mathcal{N}_N$*. Then, for any prime divisor* ℓ *of* n*, we have*

$$(\kappa_{m,N}(n; \mathbf{z}))^{\ell,s}_{m,N} = \phi^\ell_{m,N}\left(\kappa_{m,N}(n/\ell; \mathbf{z})\right).$$

In order to discuss the Kurihara's Euler system arguments, we need the notion of *well-ordered* integers, which is introduced in [Ku].

Definition 4.18. Let $n \in \mathcal{N}_N$. We call n *well-ordered* if n has a factorization $n = \prod_{i=1}^r \ell_i$ with $\ell_i \in \mathcal{P}_N$ such that ℓ_{i+1} splits in $\mathbb{Q}_m\big(\mu_{\prod_{j=1}^i \ell_j}\big)/\mathbb{Q}$ for any i satisfying $1 \leq i \leq r-1$. In other words, n is well-ordered if and only if n has a factorization $n = \prod_{i=1}^r \ell_i$ such that

$$\ell_{i+1} \equiv 1 \pmod{\prod_{j=1}^i \ell_j}$$

for $i = 1, \ldots, r-1$. We denote by $\mathcal{N}^{\text{w.o.}}_N$ the set of all elements in $\mathcal{N}_N$ which are well-ordered.

In Kurihara's Euler system arguments, the following Proposition is another essence.

Proposition 4.19 ([MR] Theorem A.4)**.** *Let* $n \in \mathcal{N}^{\text{w.o.}}_N$ *be any element. Then, the cohomology class* $\kappa_{m,N}(n; \mathbf{c}')$ *belongs to* $H^1_{\mathcal{F}_{\text{can}}(n)}(\mathbb{Q}_m, T/\pi^N T)$*. In particular, we have*

$$\phi^\ell_{m,N}\left(\kappa_{m,N}(n; \mathbf{c}')\right) = 0$$

for any prime divisor ℓ *of* n*.*

5. Construction of the ideal $\mathfrak{C}_i(\mathbf{c})$

Assume that T satisfies (C1)–(C7). Here, fix an Euler system $\mathbf{c} = \{c_m(n)\}_{m,n} \in \mathrm{ES}_{\mathcal{O}}(T, \Sigma)$ satisfying (NV), and let $\mathbf{c}' = \{c'_m(n)\}_{m,n} \in \mathrm{ES}'_{\mathcal{O}}(T, \Sigma)$ be the modified Euler system corresponding to $\mathbf{c}$. We denote $\mathbf{c}$ or $\mathbf{c}'$ by $\mathbf{z} = \{z_m(n)\}$.

In this section, we construct the ideals $\mathfrak{C}_i(\mathbf{c})$ of Λ, and prove Theorem 2.4 (i).

5.1. **Construction of $\mathfrak{C}_i(\mathbf{c})$.** First, we fix integers m, N satisfying $N > em \geq 0$, and construct an ideal $\mathfrak{C}_{i,m,N}(\mathbf{c})$ of $R_{m,N}$ for any $i \in \mathbb{Z}_{\geq 0}$.

Let $n \in \mathcal{N}_N^{\mathrm{w.o.}}$ with the prime decomposition $n = \prod_{j=1}^r \ell_j$, where $\ell_i \in \mathcal{P}_N$ for each j. We denote the number of prime divisors of n by $\epsilon(n)$, that is, $\epsilon(n) := r$. We define an ideal $\mathfrak{C}_{m,N}(n; \mathbf{z})$ of $R_{m,N}$ by

$$\mathfrak{C}_{m,N}(n; \mathbf{z}) := \left\{ f(\kappa_{m,N}(n; \mathbf{z})) \mid f \in \mathrm{Hom}_{R_{m,N}}(H^1(\mathbb{Q}_m, T/\pi^N T), R_{m,N}) \right\}.$$

Definition 5.1. Let $i \in \mathbb{Z}_{\geq 0}$. We denote by $\mathfrak{C}_{i,m,N}(\mathbf{z})$ the ideal of $R_{m,N}$ generated by $\bigcup_n \mathfrak{C}_{m,N}(n; \mathbf{z})$, where n runs through all elements of $\mathcal{N}_N^{\mathrm{w.o.}}$ satisfying $\epsilon(n) \leq i$.

Remark 5.2. By Proposition 4.5, we have

$$\mathfrak{C}_{i,m,N}(\mathbf{c}) = \mathfrak{C}_{i,m,N}(\mathbf{c}').$$

Now vary m and N, and let us construct the ideal $\mathfrak{C}_i(\mathbf{c})$ of Λ. As [Oh1] Claim 4.4, the following lemma holds.

Lemma 5.3. *Let m_1, m_2, N_1 and N_2 be positive integers satisfying $m_2 \geq m_1$ and $N_2 \geq N_1$. Take any element $n \in \mathcal{N}_{N_2}$. Then, the following hold.*

(i) *For any R_{m_2,N_2}-homomorphism*

$$f_2 \colon H^1(\mathbb{Q}_{m_2}, T/\pi^{N_2} T) \longrightarrow R_{m_2,N_2},$$

there exists an R_{m_1,N_1}-homomorphism

$$f_1 \colon H^1(\mathbb{Q}_{m_1}, T/\pi^{N_1} T) \longrightarrow R_{m_1,N_1},$$

which makes the diagram

$$\begin{tikzcd} H^1(\mathbb{Q}_{m_2}, T/\pi^{N_2}T) \arrow[r, "f_2"] \arrow[d, "\mathrm{Cor}_{\mathbb{Q}_{m_2}/\mathbb{Q}_{m_1}}"'] & R_{m_2,N_2} \arrow[d] \\ H^1(\mathbb{Q}_{m_1}, T/\pi^{N_1}T) \arrow[r, dashed, "f_1"] & R_{m_1,N_1} \end{tikzcd}$$

commute, where the left vertical arrow $\mathrm{Cor}_{\mathbb{Q}_{m_2}/\mathbb{Q}_{m_1}}$ is the corestriction map, and the right one is the natural projection.

(ii) *Assume $N_1 = N_2 =: N$. Then, for any $R_{m_1,N}[H_n]$-homomorphism*

$$g_1 \colon H^1(\mathbb{Q}_{m_1}(n), T/\pi^N T) \longrightarrow R_{m_1,N}[H_n],$$

there exists an $R_{m_2,N}[H_n]$-homomorphism

$$g_2 \colon H^1(\mathbb{Q}_{m_2}(n), T/\pi^N T) \longrightarrow R_{m_2,N,\chi}[H_n]$$

which makes the diagram

$$\begin{tikzcd}
H^1(\mathbb{Q}_{m_2}(n), T/\pi^N T) \arrow[r, dashed, "g_2"] \arrow[d, "\mathrm{Cor}_{\mathbb{Q}_{m_2}/\mathbb{Q}_{m_1}}"'] & R_{m_2,N,\chi}[H_n] \arrow[d] \\
H^1(\mathbb{Q}_{m_1}(n), T/\pi^N T) \arrow[r, "g_1"] & R_{m_1,N,\chi}[H_n]
\end{tikzcd}$$

commute.

***Proof*.** Proof of this lemma is completely to that of [Oh2] Lemma 4.13. First, we shall give preliminary remarks. Let n be as in the assertion (ii). When we treat the situation in the assertion (i), we assume that $n = 1$. We put $R_1 = R_{m_1,N_1}[H_n]$, $R_2 = R_{m_2,N_2}[H_n]$, and the natural surjection $\mathrm{pr}\colon R_2 \longrightarrow R_1$. $\Gamma_{m_2,m_1} := \mathrm{Gal}(\mathbb{Q}_{m_2}/\mathbb{Q}_{m_1})$. We assume that the action of $G_{\mathbb{Q}}$ is unramified at any prime ℓ dividing n, so it follows from the hypotheses (C1) and (C4) that we have

$$H^0(\mathbb{Q}_{m_2}(n), T/\pi^{N_2}T) = 0.$$

So, the Hochschild–Serre spectral sequence implies that the restriction map

$$H^1(\mathbb{Q}_{m_1}(n), T/\pi^{N_2}T) \longrightarrow H^1(\mathbb{Q}_{m_2}(n), T/\pi^{N_2}T)^{\Gamma_{m_2,m_1}}$$

is an isomorphism, and we identify these two R_{m_2,N_2}-modules by this isomorphism. W also note that

$$R_1 \simeq \mathrm{Hom}_{\mathbb{Z}_p}(R_1, \mathbb{Q}_p/\mathbb{Z}_p)$$

is an injective R_1-module,

Let us prove the assertion (i). Note that we can easily reduce the proof of this claim to the following two cases:

(A) $(m_2, N_2) = (m_1, N_1 + 1)$;
(B) $(m_2, N_2) = (m_1 + 1, N_1)$.

First, we consider the case (A). Here, we put $m = m_1 = m_2$ and $N = N_1$. By Lemma 3.3, the map

$$T/\pi^N T \xrightarrow{\times\pi} T/\pi^{N+1}T$$

induces an isomorphism

$$H^1(\times\pi)\colon H^1(\mathbb{Q}_m, T/\pi^N T) \xrightarrow{\simeq} H^1(\mathbb{Q}_m, T/\pi^{N+1}T)[\pi^N].$$

Note that we have an isomorphism

$$\times\pi^{-1}\colon R_2[\pi^N] = \pi R_2 \xrightarrow{\simeq} R_1,$$

so we can define the R_1-linear homomorphism

$$f_1 := (\times\pi^{-1}) \circ f_2 \circ H^1(\times\pi)\colon H^1(\mathbb{Q}_m, T/\pi^N T) \longrightarrow R_1.$$

The map f_1 is what we desired.

Let us consider the case (B). Put

$$N_{m+1/m} := \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}_{m+1}/\mathbb{Q}_m)} \sigma \in R_2,$$

and consider the isomorphism

$$\nu_{m+1/m} \colon R_1 \xrightarrow{\simeq} N_{m+1,m}R_2 = (R_2)^{\Gamma_{m+1,m}}$$

of R_1-modules defined by $1 \mapsto N_{m+1/m}$. We define the composite map

$$\begin{aligned} f_1 \colon H^1(\mathbb{Q}_m, T/\pi^N T) &= H^1(\mathbb{Q}_{m+1}, T/\pi^N T)^{\Gamma_{m_2,m_1}} \\ &\xrightarrow{f_2} (R_2)^{\Gamma_{m+1/m}} \xrightarrow{\nu_{m+1/m}^{-1}} R_1. \end{aligned}$$

Let $\mathcal{NH}$ be the image of $H^1(\mathbb{Q}_{m+1}, T/\pi^N T)$ in

$$H^1(\mathbb{Q}_m, T/\pi^N T) = H^1(\mathbb{Q}_{m+1}, T/\pi^N T)^{\Gamma_{m+1,m}}$$

by the norm map. Then, the diagram

$$\begin{tikzcd}
 & H^1(\mathbb{Q}_{m_2}, T/\pi^N T) \arrow[r, "f_2"] \arrow[dl, "\mathrm{Cor}_{\mathbb{Q}_{m+1}/\mathbb{Q}_m}"'] \arrow[d, "N_{m+1,m}\times(\cdot)"] & R_2 \arrow[r, "\times N_{m/m+1}"] \arrow[d, "\mathrm{pr}"] & R_2 \\
\mathcal{NH} \arrow[r, equal] & N_{m+1/m} \cdot H^1(\mathbb{Q}_{m_2}, T/\pi^N T) \arrow[r, dashed, "f_1"] & R_1 \arrow[ur, hook, "\nu_{m+1/m}"'] &
\end{tikzcd}$$

commutes. This completes the proof of the assertion (i).

Now, let us show the assertion (ii). It is sufficient to show in the case of $(m_2, N) = (m_1 + 1, N)$. Suppose that an arbitrary R_1-linear homomorphism

$$g_1 \colon H^1(\mathbb{Q}_{m_1}(n), T/\pi^{N_2} T) \longrightarrow R_1$$

is given. Since R_2 is injective as an R_2-module, we can extend the homomorphism

$$\nu_{m+1/m} \circ g_1 \colon H^1(\mathbb{Q}_m(n), T/\pi^N T) = H^1(\mathbb{Q}_{m+1}(n), T/\pi^N T)^{\Gamma_{m+1/m}} \longrightarrow R_2$$

to an R_2-linear homomorphism $g_2 \colon H^1(\mathbb{Q}_{m+1}(n), T/\pi^N T) \longrightarrow R_2$. Then, we have the commutative diagram

$$\begin{tikzcd}
H^1(\mathbb{Q}_{m+1}(n), T/\pi^N T) \arrow[r, dashed, "g_2"] \arrow[d, "\mathrm{Cor}_{\mathbb{Q}_{m+1}/\mathbb{Q}_m} = N_{m+1/m}\times(\cdot)"'] & R_2 \arrow[r, "\times N_{m+1/m}"] \arrow[d, "\mathrm{pr}"'] & R_2 \\
H^1(\mathbb{Q}_m(n), T/\pi^N T) \arrow[r, "g_1"] & R_1 \arrow[ur, hook, "\nu_{m+1/m}"'] &
\end{tikzcd}$$

of R_2-modules, and the assertion (ii) of Lemma 5.3 follows. □

Let m_1, m_2, N_1, N_2 and n be as above, and assume $N_i > em_i$ for each $i = 1, 2$. Then, Lemma 5.3 and "norm compatibility" of the Euler system $\mathbf{c}$ imply that the image of $\mathfrak{C}_{m_2,N_2}(\mathbf{c})$ in R_{m_1,N_1} is contained in $\mathfrak{C}_{m_1,N_1}(\mathbf{c})$. We obtain the projective system of the natural homomorphisms

$$\left\{ \mathfrak{C}_{i,m_2,N_2}(\mathbf{c}) \longrightarrow \mathfrak{C}_{i,m_1,N_1}(\mathbf{c}) \;\middle|\; N_2 \geq N_1 > em_1 \text{ and } N_2 > em_2 \geq em_1. \right\}$$

Finally, we define $\mathfrak{C}_i(\mathbf{c})$ as follows.

Definition 5.4. For any $i \in \mathbb{Z}_{\geq 0}$, we define the ideal $\mathfrak{C}_i(\mathbf{c})$ of $\Lambda = \varprojlim R_{m,N}$ by the projective limit

$$\mathfrak{C}_i(\mathbf{c}) := \varprojlim \mathfrak{C}_{i,m,N}(\mathbf{c}).$$

5.2. **Results on principal Fitting ideals.** In this subsection, we shall prove the assertion (i) of Theorem 2.4. Recall that we denote the maximal pseudo-null Λ-submodule of $X := X(T)$ by X_{fin}, and put $X' = X'(T) := X/X_{\mathrm{fin}}$. The goal of this subsection is the following theorem.

Theorem 5.5. *Assume that the pair $(T, \mathbf{c})$ satisfies the conditions* (C1)–(C7) *and* (NV)*. Then, we have*

$$\mathrm{ann}_\Lambda(X_{\mathrm{fin}})I(\mathbf{c}) \cdot \mathrm{Fitt}_{\Lambda,0}(X') \subseteq \mathfrak{C}_0(\mathbf{c}).$$

Later (including the proof of Theorem 5.5), we often use the following facts on twists of T by characters of Γ.

Lemma 5.6 ([Ru2] Lemma 6.1.3)**.** *The following hold.*

(i) *Let $\gamma_1, \ldots, \gamma_r$ be elements of $G_{\mathbb{Q}}$ whose image in Γ are non-trivial. Then, the set*
$$\left\{ \rho \in \mathrm{Hom}_{cont}(\Gamma, \mathcal{O}^\times) \;\middle|\; \begin{array}{l} (T \otimes \rho)^{\gamma_i^{p^n}=1} = 0 \textit{ for any } i \in \mathbb{Z} \cap [1, k] \\ \textit{and any } n \in \mathbb{Z}_{\geq 0}. \end{array} \right\}$$
is open and dense in $\mathrm{Hom}_{cont}(\Gamma, \mathcal{O}^\times)$.

(ii) *For any $m \in \mathbb{Z}_{\geq}$, the set*
$$\left\{ \rho \in \mathrm{Hom}_{cont}(\Gamma, \mathcal{O}^\times) \;\middle|\; \textit{the order of } \mathbb{H}^2_\Sigma(T \otimes \rho) \otimes_\Lambda R_m \textit{ is finite} \right\}$$
is open and dense in $\mathrm{Hom}_{cont}(\Gamma, \mathcal{O}^\times)$.

Remark 5.7. Assume that $(T, \mathbf{c})$ satisfies the conditions (C1)–(C7), (NV) and (MC). Let $\rho \colon \Gamma \longrightarrow 1 + \pi\mathcal{O}$ be any continuous character, and $(\mathcal{O}, \rho)$ be the free $\mathcal{O}$-module of rank one on which Γ acts via ρ. We put

$$T \otimes \rho := T \otimes_{\mathcal{O}} (\mathcal{O}, \rho),$$

and let $\mathbf{c} \otimes \rho \in \mathrm{ES}_{\mathcal{O}}(T \otimes \rho; \Sigma)$ be the twist of the Euler system $\mathbf{c}$ by the character ρ in the sense of in the [Ru2] Chapter 6. Note that both $\rho|_{G_{\mathbb{Q}_\infty}} \in \mathrm{Hom}(G_{\mathbb{Q}_\infty}, \mathcal{O}^\times)$ and $(\rho \bmod \pi\mathcal{O}) \in \mathrm{Hom}(G_{\mathbb{Q}}, k^\times)$ are trivial characters, so the pair $(T \otimes \rho, \mathbf{c} \otimes \rho)$ also satisfies the conditions (C1)–(C7), (NV) and (MC).

Proof of Theorem 5.5. Let m and N be integers satisfying $N > em \geq 0$. Since Fitting ideals are compatible with respect to base change, it is sufficient to show that

$$\mathrm{ann}_\Lambda(X_{\mathrm{fin}})I(\mathbf{c}) \cdot \mathrm{Fitt}_{R_{m,N},0}(X' \otimes_\Lambda R_{m,N}) \subseteq \mathfrak{C}_{0,m,N}(\mathbf{c}).$$

By Lemma 5.6, there exists a character $\rho \in \mathrm{Hom}_{\mathrm{cont}}(\Gamma, 1 + p^N\mathbb{Z}_p)$ which makes the order of

$$\left(\mathbb{H}^2_\Sigma(T) \otimes \rho\right) \otimes_\Lambda R_m \simeq \mathbb{H}^2_\Sigma(T \otimes \rho) \otimes_\Lambda R_m$$

finite. Let $\rho \colon \Gamma \longrightarrow 1 + p^N\mathbb{Z}_p$ be such a character, and $\mathbf{c} \otimes \rho$ be the twist of $\mathbf{c}$ by ρ. Note that the image of $\mathrm{ann}_\Lambda(X(T \otimes \rho)_{\mathrm{fin}})$ (resp. $I(\mathbf{c} \otimes \rho)$ and $\mathrm{Fitt}_{\Lambda,0}(X'(T \otimes \rho))$)

in $R_{m,N}$ coincides with the image of $\mathrm{ann}_\Lambda(X_{\mathrm{fin}})$ (resp. $I(\mathbf{c})$ and $\mathrm{Fitt}_{\Lambda,0}(X'))$. By the construction of Kolyvagin derivatives and the ideal $\mathfrak{C}_{i,m,N}(\mathbf{c})$, we also have

$$\mathfrak{C}_{i,m,N}(\mathbf{c}\otimes\rho)=\mathfrak{C}_{i,m,N}(\mathbf{c}).$$

So, we may replace T with $T\otimes\rho$, and assume that

$$\mathrm{char}_\Lambda(\mathbb{H}^2_\Sigma(T))\not\subseteq(\gamma^{p^m}-1)\Lambda.$$

Fix a Λ-linear map $\varphi\colon\mathbb{H}^1_\Sigma(T)\longrightarrow\Lambda$. This map induces a homomorphism

$$\bar{\varphi}_{m,N}\colon\mathbb{H}^1_\Sigma(T)\otimes_\Lambda R_{m,N}\longrightarrow R_{m,N}.$$

We take an arbitrary elements $\delta\in\mathrm{ann}_\Lambda(X_{\mathrm{fin}})$. By Corollary 3.10, we obtain the following lemma.

Lemma 5.8. *Let $\mathcal{NH}_{m,N}$ be the image of the natural homomorphism*

$$\mathbb{H}^1_\Sigma(T)\otimes_\Lambda R_{m,N}\longrightarrow H^1_{\acute{e}t}(\mathcal{O}_{\mathbb{Q}_m,\Sigma},T/\pi^NT).$$

Then, the kernel of this homomorphism is annihilated by δ, and there exists a homomorphism $\psi\colon\mathcal{NH}_{m,N}\longrightarrow R_{m,N,\chi}$ which makes the diagram

$$\begin{tikzcd}
\mathbb{H}^1_\Sigma(T)\otimes_\Lambda R_{m,N} \arrow[r, "\delta\bar{\varphi}_{m,N}"] \arrow[d] & R_{m,N} \\
\mathcal{NH}_{m,N} \arrow[ur, dashed, "\psi"'] &
\end{tikzcd}$$

commute.

By Lemma 5.8, we obtain

$$\delta\bar{\varphi}_{m,N}(\text{the image of } c_m(1))=\psi(\kappa_{m,N}(1;\mathbf{c}))\in\mathfrak{C}_{0,m,N}(\mathbf{c}).$$

By the definition of the ideal $I_\varphi(\mathbf{c})$, we have

$$\delta I_\varphi(\mathbf{c})\mathrm{Fitt}_{\Lambda,0}(X')=\delta I_\varphi(\mathbf{c})\,\mathrm{char}_\Lambda(X)=\delta\varphi(c_m(1))\Lambda,$$

so we obtain

$$\delta I(\mathbf{c})\cdot\mathrm{Fitt}_{R_{m,N},0}(X'\otimes_\Lambda R_{m,N})=\delta\bar{\varphi}_{m,N}(\text{the image of } c_m(1))R_{m,N}.$$

This completes the proof. □

6. Kolyvagin systems and lower bounds of higher Fitting ideals

Let $(T,\mathbf{c},\mathbf{c}')$ be as in the previous section. Here, we briefly recall the definition and some known results of Kolyvagin systems established in [MR], and prove the inequality

$$\mathfrak{C}_i(\mathbf{c})\prec\mathrm{Fitt}_{\Lambda,i}(X). \tag{4}$$

for any non-negative integer i under the assumption $\mathrm{rank}_{\mathcal{O}}T^-=1$..

6.1. **Review of Kolyvagin systems.** Here, we recall the notion and some results of Kolyvagin systems for discrete valuation rings. For any $n \in \mathcal{N}_1$, we put $I_n := \pi^{N_{\{n\}}}\mathcal{O}$. First, we recall the definition of Kolyvagin systems.

Definition 6.1. Let R be $\mathcal{O}$ or $\mathcal{O}/\pi^N\mathcal{O}$ for some $N \in \mathbb{Z}_{>0}$, and put $M := T \otimes_{\mathcal{O}} R$. Consider a local condition $\mathcal{F}$ on M satisfying $H^1_{\mathcal{F}}(\mathbb{Q}_\ell, M) = H^1_f(\mathbb{Q}_\ell, M)$ for any prime number ℓ not contained in Σ.

- For any $n \in \mathcal{N}_1(\Sigma; T)_{\mathcal{O}}$. we define a new local condition $\mathcal{F}(n)$ by
$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell, M) = \begin{cases} H^1_{tr}(\mathbb{Q}_\ell, M) & (\text{if } \ell \mid n). \\ H^1_f(\mathbb{Q}_\ell, M) & (\text{if } \ell \nmid n). \end{cases}$$
(For the definition of $H^1_{tr}(\mathbb{Q}_\ell, M)$, see Definition 4.10.)
- A *Kolyvagin system* for the triple $(M, \mathcal{F}, \mathcal{P}_1(T; \Sigma))_{\mathcal{O}}$ is a family of cohomology classes
$$\kappa = \{\kappa_n \in H^1_{\mathcal{F}(n)}(\mathbb{Q}, M/I_nM)\}_{n \in \mathcal{N}_1(\Sigma;T)_{\mathcal{O}}}$$
satisfying
$$(\kappa_n)^{\ell,s}_{0,\bar{N}_{\{n\}}} = \phi^\ell_{0,\bar{N}_{\{n\}}}(\kappa_{n/\ell}) \quad \text{in} \quad H^1_s(\mathbb{Q}_\ell, M/I_nM)$$
for any $n \in \mathcal{N}_1$ and any prime divisor ℓ of n, where we put $\bar{N}_{\{n\}} := N_{\{n\}}$ (resp. $N' := \min\{N, N_{\{n\}}\}$) if $M = T$ (resp. $M = T/\pi^N T$). We denote the set of all Kolyvagin systems for $(M, \mathcal{F}, \mathcal{P}_1(T; \Sigma))_{\mathcal{O}}$ by $\mathrm{KS}_R(M; \mathcal{F}, \Sigma)$. In particular, if $\mathcal{F}$ is the canonical local condition $\mathcal{F}_{\mathrm{can}}$, we put $\mathrm{KS}_R(M; \Sigma) := \mathrm{KS}_R(M; \mathcal{F}_{\mathrm{can}}, \Sigma)$ for simplicity.

Here, we remark on some relations between Euler systems and Kolyvagin systems.

Proposition 6.2 ([MR] Theorem 3.2.4)**.** *Assume the following two conditions.*

(K1) *The action of* $\mathrm{Fr}_\ell^{p^a} - 1$ *on* T *is injective for any* $\ell \in \mathcal{P}_1(\Sigma; T)_{\mathcal{O}}$ *and any* $a \in \mathbb{Z}_{\geq 0}$.
(K2) *It holds that*
$$\mathrm{rank}_{\mathcal{O}} T^- + \mathrm{corank}_{\mathcal{O}} H^0(\mathbb{Q}_p, A^*) = 1.$$

Then, there exists an $\mathcal{O}$*-linear map*
$$\mathrm{ES}'_{\mathcal{O}}(T, \Sigma) \longrightarrow \mathrm{KS}_{\mathcal{O}}(T, \Sigma); \quad \mathbf{z}' = \{z'_m(n)\}_{m,n} \longmapsto \kappa(\mathbf{z}') := \{\kappa(\mathbf{z}')_n\}_n$$
satisfying the following property.

(EK) *Let* $n \in \mathcal{N}_1(\Sigma; T)_{\mathcal{O}}$ *be an arbitrary well-ordered element, and put* $I_n = \pi^{N_{\{n\}}}\mathcal{O}$. *Then, for any* $\mathbf{z}' \in \mathrm{ES}_{\mathcal{O}}(T, \Sigma)$*, we have*
$$\kappa(\mathbf{z}')_n = \kappa_{0,N_{\{n\}}}(n; \mathbf{z}'),$$
where $\kappa_{0,N_{\{n\}}}(n; \mathbf{z}') \in H^1(\mathbb{Q}, T/I_nT)$ *is the Kolyvagin derivative of* $\mathbf{z}'$ *at* n.

For details of the construction of the map in Proposition 6.2, see [MR] Appendix A, in particular pp. 80–81.

Corollary 6.3. *Assume that* $\mathrm{rank}_{\mathcal{O}} T^{-} = 1$*. Let* $n_0 \in \mathcal{N}_1(\Sigma; T)_{\mathcal{O}}$ *be an arbitrary well-ordered element. Then, for any* $\mathbf{z}' \in \mathrm{ES}_{\mathcal{O}}(T, \Sigma)$*, there exists a Kolyvagin system*

$$\kappa(\mathbf{z}') = \{\kappa(\mathbf{z}')_n\}_n \in \mathrm{KS}_{\mathcal{O}/\pi^{N_{\{n_0\}}}\mathcal{O}}(T/\pi^{N_{\{n_0\}}}T, \Sigma)$$

satisfying

$$\kappa(\mathbf{z}')_{n_0} = \kappa_{0,N_{\{n_0\}}}(n_0; \mathbf{z}').$$

***Proof*.** Fix an well-ordered element $n_0 \in \mathcal{N}_N(\Sigma, T)_{\mathcal{O}}$. By Lemma 5.6, there exists a character $\rho \in \mathrm{Hom}_{\mathrm{cont}}(\Gamma, 1 + \pi^{N_{\{n_0\}}}\mathcal{O})$ satisfying the following conditions.

- The action of $\mathrm{Fr}_\ell^{p^a} - 1$ on $T \otimes \rho$ is injective for any $\ell \in \mathcal{P}_1$ and any $a \in \mathbb{Z}_{\geq 0}$.
- The order of $H^0(\mathbb{Q}_p, A^*)$ is finite.

Let $\mathbf{z}' \otimes \rho \in \mathrm{ES}'_{\mathcal{O}}(T \otimes \rho, \Sigma)$ be the twist of the Euler system $\mathbf{z}'$ by ρ. Note that $T \otimes \rho$ satisfies the conditions (K1) and (K2) in Proposition 6.2, so we can apply Proposition 6.2 for $T \otimes \rho$. Let $\tilde{\kappa} = \{\tilde{\kappa}_n\}_n \in \mathrm{KS}_{\mathcal{O}}(T \otimes \rho, \Sigma)$ be the Kolyvagin system corresponding to $\mathbf{z}' \otimes \rho$, and denote the image of $\tilde{\kappa}$ in $\mathrm{KS}_{\mathcal{O}}\big((T \otimes \rho)/\pi^{N_{n_0}}(T \otimes \rho), \Sigma\big)$ by $\kappa = \{\kappa_n\}_n$. Note that by definition, we have

$$\kappa_{0,N_{\{n_0\}}}(n; \mathbf{z}' \otimes \rho) = \kappa_{0,N_{\{n_0\}}}(n; \mathbf{z}')$$

for any $n \in \mathcal{N}_1(\Sigma; T)_{\mathcal{O}}$. So, the condition (EK) in Proposition 6.2 implies

$$\kappa_n = \kappa_{0,\min\{N_{n_0}, N_{\{n\}}\}}(n; \mathbf{z}') \tag{5}$$

for any well-ordered $n \in \mathcal{N}_1(\Sigma; T)_{\mathcal{O}}$.

Here, we denote by $\mathcal{F}_{\mathrm{can},1}$ (resp. $\mathcal{F}_{\mathrm{can},\rho}$) the local condition on the $G_{\mathbb{Q}}$-module $M := T/\pi^{N_{\{n_0\}}}T$ arising from the canonical local condition on T (resp. $T \otimes \rho$). In order to complete the proof of Corollary 6.3, it suffices to show $\kappa \in \mathrm{KS}_{\mathcal{O}/\pi^{N_{\{n\}}}\mathcal{O}}(M; \mathcal{F}_{\mathrm{can},1}, \Sigma)$. Fix an integer $n \in \mathcal{N}_1(\Sigma; T)_{\mathcal{O}}$, and let us show

$$\kappa_n \in H^1_{\mathcal{F}_{\mathrm{can},1}(n)}(\mathbb{Q}, M/I_n M).$$

By the definition of the local condition *tr* and [Ru2] Lemma 1.3.8, we have

$$\kappa_n \in H^1_{\mathcal{F}^{\Sigma}_{\mathrm{can},\rho}(n)}(\mathbb{Q}, M/I_n M) = H^1_{\mathcal{F}^{\Sigma}_{\mathrm{can},1}(n)}(\mathbb{Q}, M/I_n M).$$

On the other hand, by the definition of Kolyvagin derivatives and the equality (5) imply

$$\kappa_n \in H^1_{\mathcal{F}^n_{\mathrm{can},1}}(\mathbb{Q}, M/I_n M).$$

Since any prime divisor of n is not contained in Σ, we obtain

$$\kappa_n \in H^1_{\mathcal{F}_{\mathrm{can},1}(n)}(\mathbb{Q}, M/I_n M).$$

This completes the proof. □

Note that by [MR] Theorem 5.2.12, in certain good situations, Kolyvagin systems determine the isomorphism class of the Selmer groups over complete discrete valuation rings with finite residue fields. Here, we briefly review this result. Recall that for any element $n \in \mathcal{N}_1(\Sigma; T)_{\mathcal{O}}$, we denote the number of prime divisors of n by $\epsilon(n)$. Namely, we put $\epsilon(n) := r$ if n is decomposed into the product of r prime numbers. For any non-zero element $\kappa = \{\kappa_n\} \in \mathrm{KS}_{\mathcal{O}}(T, \Sigma)$ and any non-negative integer i, we denote the maximum (accepting ∞) of the set

$$\left\{ j \in \mathbb{Z}_{\geq 0} \mid \kappa_n \in \pi^j H_{\mathcal{F}(n)}(\mathbb{Q}, T/I_n T) \text{ for all } n \in \mathcal{N}_1(\Sigma; T)_{\mathcal{O}} \text{ with } \epsilon(n) = i \right\}$$

by $\partial_i(\kappa; T)$. We also define

$$\partial_i(T) := \min\{\partial_i(\kappa; T) \mid \kappa = \{\kappa_n\} \in \mathrm{KS}_{\mathcal{O}}(T, \Sigma)\}.$$

Note that $\partial_i(T) = 0$ for sufficiently large i, and $\partial_j(T) \geq \partial_{j+1}(T)$ for any $j \in \mathbb{Z}_{\geq 0}$. (See [MR] Theorem 5.10 (ii) and Theorem 5.12).

Proposition 6.4 ([MR] Theorem 5.2.12)**.** *Assume that T satisfies the condition (K2) in Proposition 6.2. We put*

$$X_0 = X_0(T) := \mathrm{Hom}\left(H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, A^*), \mathbb{Q}_p/\mathbb{Z}_p\right).$$

Then, we have

$$\mathrm{Fitt}_{\mathcal{O},i}(X_0) = \pi^{\partial_i(T)}\mathcal{O}$$

for any $i \in \mathbb{Z}_{\geq 0}$. Here, we put $\pi^\infty \mathcal{O} := \{0\}$.

6.2. **Lower bounds of higher Fitting ideals.** Here, let us prove Theorem 2.4 (iii). In this subsection, we always assume that T^- is a free $\mathcal{O}$-module of rank one. We fix an integer $i \in \mathbb{Z}_{\geq 0}$ and a height one prime ideal $\mathfrak{P}$ of Λ containing $\mathrm{Fitt}_{\Lambda,i}(X)$. We define two integers $\alpha = \alpha_i(\mathfrak{P})$ and $\beta = \beta_i(\mathfrak{P})$ by

$$\mathrm{Fitt}_{\Lambda_{\mathfrak{P}},i}(X_{\mathfrak{P}}) = \mathfrak{P}^\alpha \Lambda_{\mathfrak{P}},$$
$$\mathfrak{C}_i \Lambda_{\mathfrak{P}} = \mathfrak{P}^\beta \Lambda_{\mathfrak{P}}.$$

In order to prove Theorem 2.4 (iii), it is sufficient to show the following theorem.

Theorem 6.5. *We have $\beta_i(\mathfrak{P}) \geq \alpha_i(\mathfrak{P})$.*

We shall prove Theorem 6.5 by the parallel arguments to that in [Oh2] §8.3, but here, we also treat the cases when the μ-invariant of the Iwasawa module $X(T)$ is not zero. We identify $\Lambda = \mathcal{O}[[\Gamma]]$ with the ring $\mathcal{O}[[T]]$ of formal power series by an isomorphism $\mathcal{O}[[\Gamma]] \simeq \mathcal{O}[[T]]$ defined by $\gamma \mapsto 1 + T$. We assume that $G_{\mathbb{Q}}$ acts on Λ by the tautological action, and put

$$\mathbf{T} := T \otimes_{\mathcal{O}} \Lambda.$$

By Shapiro's lemma and limit arguments (cf. [Ta2] Corollary 2.2), we have the natural isomorphism

$$H^i(G_{\mathbb{Q},\Sigma}, \mathbf{T}) \simeq \mathbb{H}^i_{\Sigma}(T),$$
$$H^i(G_{\mathbb{Q}_p}, \mathbf{T}) \simeq H^i_{\mathrm{loc}}(T).$$

As in [MR] §5.3, we define the exceptional set Σ_Λ of hight-one prime ideals of Λ by

$$\Sigma_\Lambda := \left\{\mathfrak{P} \mid \#\left(\mathbb{H}^2_{\Sigma}(T)[\mathfrak{P}]\right) < \infty\right\}$$
$$\cup \left\{\mathfrak{P} \mid \#\left(\mathbb{H}^2_{\mathrm{loc}}(T)[\mathfrak{P}]\right) < \infty\right\} \cup \{\pi\Lambda\}$$

Note that Σ_Λ is a finite set. (See [MR] Lemma 5.3.13.) For any positive integer j, we define an element $f_j(T) \in \mathcal{O}[T] \subset \Lambda$ as follows.

- Suppose $\mathfrak{P} \neq (\pi)$, and let $f(T) = f_\infty(T)$ be the Weierstrass polynomial which generates $\mathfrak{P}$. Then, for any $j \in \mathbb{Z}_{\geq 0}$, we put $f_j(T) = f(T) + p^j$.
- Suppose $\mathfrak{P} = (\pi)$. Then, we put $f(T) = f_\infty(T) := \pi$, and $f_j(T) = \pi + T^j$ for any $j \in \mathbb{Z}_{\geq 0}$.

For any $j \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, let $\mathfrak{P}_j$ be the principal ideal of $\Lambda = \mathcal{O}[[T]]$ generated by $f_j(T)$. (So, especially, we have $\mathfrak{P}_\infty = \mathfrak{P}$.) Then, there exists a positive integer $N(\mathfrak{P})$ satisfying the following properties.

(i) The ideal $\mathfrak{P}_j$ is a prime ideal for any $j \in \mathbb{Z}_{\geq N(\mathfrak{P})}$.
(ii) The ideal $\mathfrak{P}_j$ is not contained in Σ_Λ for any $j \in \mathbb{Z}_{\geq N(\mathfrak{P})}$.
(iii) If $\mathfrak{P} \neq (\pi)$, then the ring $\Lambda/\mathfrak{P}_j$ is (non-canonically) isomorphic to $\Lambda/\mathfrak{P}$ as an $\mathcal{O}$-algebra for any $j \in \mathbb{Z}_{\geq N(\mathfrak{P})}$.

(For detail, see [MR] p. 66.) As in [Oh2], it is convenient to introduce the following notation.

Definition 6.6. Let M be an integer, and $\{x_j\}_{j \geq M}$ and $\{y_j\}_{j\geq M}$ sequences of real numbers. We write $x_j \succ y_j$ if $\liminf_{j\to\infty}(x_j - y_j) \neq -\infty$. We write $x_N \sim y_N$ if $x_j \succ y_j$ and $y_j \succ x_j$.

For any $j \in \mathbb{Z}_{\geq 0}$, we denote the normalization of $\Lambda/\mathfrak{P}_j$ by $\mathcal{O}_j$. Note that if $j \geq N(\mathfrak{P})$, the ring $\mathcal{O}_j$ is a complete discrete valuation ring, and we fix a uniformizer π_j of $\mathcal{O}_j$. We put $\mathrm{Frac}(\mathcal{O}_j) := K_j$. We define a non-negative integer s by

$$p^s = (\mathcal{O}_\infty : \Lambda/\mathfrak{P}).$$

(For instance, if $\mathfrak{P} = (\pi)$, then we have $\Lambda/\mathfrak{P} \simeq k[[T]]$. So, in this case, we can take $\pi_\infty = T$, and $s = 0$.) We define an integer e_∞ as follows.

- If $\mathfrak{P} \neq (\pi)$, we denote the ramification index of $K_\infty/\mathbb{Q}_p$ by e_∞.
- If $\mathfrak{P} = (\pi)$, we put $e_\infty = 1$

As in [Oh2] Lemma 8.10 and [Oh2] Corollary 8.11, via the observations in [MR], we obtain the following lemma.

Lemma 6.7. *Let M be a finitely generated torsion Λ-module. We define a non-negative integer C by*

$$\mathrm{Fitt}_{\Lambda_{\mathfrak{P}},i}(M_{\mathfrak{P}}) = \mathfrak{P}^C \Lambda_{\mathfrak{P}}.$$

For any $j \in \mathbb{Z}_{\geq N(\mathfrak{P})}$, we define a non-negative integer c_j by

$$\mathrm{Fitt}_{\mathcal{O}_j,i}(M \otimes_\Lambda \mathcal{O}_j) = \pi_j^{c_j}\mathcal{O}_j.$$

Then, we have $c_j \sim Ce_\infty j$.

Definition 6.8. For any $j \in \mathbb{Z}_{\geq N(\mathfrak{P})}$, we define $a_j, b_j \in \mathbb{Z}_{\geq 0}$ by

$$\pi_j^{a_j}\mathcal{O}_j = \mathrm{Fitt}_{\mathcal{O}_j,i}(X \otimes_\Lambda \mathcal{O}_j),$$
$$b_j = \mathrm{length}_{\mathcal{O}_j}\big((\Lambda/\mathfrak{C}_i(\mathbf{c})) \otimes_\Lambda \mathcal{O}_j\big).$$

By Lemma 6.7, we have $a_j \sim \alpha e_\infty j$ and $b_j \sim \beta e_\infty j$.

Let j be an integer satisfying $j \geq N(\mathfrak{P})$. We define a continuous character

$$\rho_j \colon \Gamma \longrightarrow 1 + \pi_j \mathcal{O}_j$$

by the composite

$$\Gamma \xrightarrow{\subseteq} \Lambda \longrightarrow \Lambda/\mathfrak{P}_j \xrightarrow{\subseteq} \mathcal{O}_j.$$

We put $T_j := \mathcal{O}_j \otimes_{\mathcal{O}} T$, and define the action of $G_{\mathbb{Q}}$ on it by

$$G_{\mathbb{Q}} \times T_j \longrightarrow T_j;\ (g, a \otimes m) \longmapsto \rho_j(g) a \otimes gm.$$

for any $g \in G_{\mathbb{Q}}$, $a \in \mathcal{O}_j$ and $m \in T$. We put $A_j^* := \mathrm{Hom}_{\mathcal{O}_j}(T_j, K_j/\mathcal{O}_j(1))$, and

$$X_0(T_j) := \mathrm{Hom}_{\mathbb{Z}_p}\left(H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, A_j^*), \mathbb{Q}_p/\mathbb{Z}_p\right).$$

Proposition 6.9 ([MR] Proposition 5.3.14)**.** *Take any $j \in \mathbb{Z}_{\geq N(\mathfrak{P})}$, and let $\pi_j \colon X \otimes_\Lambda \mathcal{O}_j \longrightarrow X_0(T_j)$ be a natural homomorphism. Then, the kernel and cokernel of π_j are both finite. Moreover, the orders of kernel and cokernel of π_j are bounded by a constant independent of $j \in \mathbb{Z}_{\geq N(\mathfrak{P})}$.*

As in [Oh2] Corollary 8.14, we deduce the following corollary from Proposition 6.9 and Proposition 6.4 for the $\mathcal{O}_j$-module T_j.

Corollary 6.10. *We have $a_j \sim \partial_i(T_j)$.*

For any $j \in \mathbb{Z}_{\geq N(\mathfrak{P})}$, we take an integer N'_j satisfying

- $e_\infty N'_j \geq \partial_i(T_j) + 4se_\infty$, and
- $p^{N'_j} \in \mathfrak{C}_i(\mathbf{c}) + \mathfrak{P}_j$.

Note that there exist such an integer N'_j since the order of $X_0(T_j)$ is finite, and since the ideal $\mathfrak{C}_i(\mathbf{c}) + \mathfrak{P}_j$ has finite index in Λ. Then, we take an integer N''_j satisfying

- $eN'' \geq \max\{e, e_\infty\} N'$,
- $\gamma^{p^{N''_j - 1}} - 1 \in \mathfrak{P}_j + p^{N'_j}\Lambda$, and
- $\mathcal{P}_{eN''_j}(\Sigma; T)_{\mathcal{O}} \subseteq \mathcal{P}_{e_\infty N'_j}(\Sigma; T_j)_{\mathcal{O}_j} \cap \mathcal{P}_{e_\infty N'_j}(\Sigma; (\mathcal{O}_j, \rho_j))_{\mathcal{O}_j}$,

where $(\mathcal{O}_j, \rho_j)$ is a free $\mathcal{O}_j$-module of rank one on which $G_{\mathbb{Q}}$ acts via the character ρ_j. We put $m_j := N''_j - 1$.

Proof of Theorem 6.5. Now, let us prove the inequality $\beta \geq \alpha$. Note that it is sufficient to show $\beta e_\infty j \succ \alpha e_\infty j$. Let $j \in \mathbb{Z}_{\geq N(\mathfrak{P})}$. Then, we have

$$\begin{aligned}
\beta e_\infty j \sim b_j &= \mathrm{length}_{\mathcal{O}_j}\big((\Lambda/(\mathfrak{C}_i(\mathbf{c}) + \mathfrak{P}_j)) \otimes_\Lambda \mathcal{O}_j\big) \\
&= \mathrm{length}_{\mathcal{O}_j}\left(\left(\Lambda/(\mathfrak{C}_i(\mathbf{c}) + \mathfrak{P}_j + p^{N'_j}\Lambda)\right) \otimes_\Lambda \mathcal{O}_j\right) \\
&= \mathrm{length}_{\mathcal{O}_j}\left(\left(\Lambda/(\mathfrak{C}_i(\mathbf{c}) + \mathfrak{P}_j + (p^{N'_j}, \gamma^{p^{m_j}} - 1))\right) \otimes_\Lambda \mathcal{O}_j\right) \\
&= \mathrm{length}_{\mathcal{O}_j}\left(\left(R_{m_j, eN'_j}/(\text{the image of } \mathfrak{C}_j(\mathbf{c}))\right) \otimes_\Lambda \mathcal{O}_j\right) \\
&\geq \mathrm{length}_{\mathcal{O}_j}\left(\left(R_{m_j, eN'_j}/(\text{the image of } \mathfrak{C}_{i, m_j, eN''_j}(\mathbf{c}))\right) \otimes_\Lambda \mathcal{O}_j\right).
\end{aligned}$$

Note that by Remark 5.2, we have

$$\mathfrak{C}_{i,m_j,eN''_j}(\mathbf{c}) = \mathfrak{C}_{i,m_j,eN''_j}(\mathbf{c}').$$

Since the ring $R_{m_j,eN'_j} \otimes_\Lambda \mathcal{O}_j$ is a quotient of the discrete valuation ring $\mathcal{O}_j$, the image of $\mathfrak{C}_{i,m_j,eN''_j}(\mathbf{c}')$ in $R_{m_j,eN'_j} \otimes_\Lambda \mathcal{O}_j$ is a principal ideal. So, there exist a well-ordered integer $n_j \in \mathcal{N}_{eN''_j}(\Sigma;T)_{\mathcal{O}}$ and a homomorphism

$$h_j \colon H^1(\mathbb{Q}_{m_j}, T/p^{N''_j}T) \longrightarrow R_{m_j,eN''_j}$$

such that the image of $\mathfrak{C}_{i,m_j,eN''_j}(\mathbf{c}')$ in $R_{m_j,eN'_j} \otimes_\Lambda \mathcal{O}_j$ is generated by the image of $h_j(\kappa_{m_j,eN''_j}(n_j;\mathbf{c}'))$. Therefore, we obtain

$$\beta e_\infty j \succ \operatorname{length}_{\mathcal{O}_j}\left(\left(R_{m_j,eN'_j}/(\text{the image of } h_j(\kappa_{m_j,eN''_j}(n_j;\mathbf{c}')))\right)\otimes_\Lambda \mathcal{O}_j\right). \tag{6}$$

By Lemma 5.3, there exists an R_{m_j,eN'_j}-linear homomorphism

$$\bar{h}_j \colon H^1(\mathbb{Q}_{m_j}, T/p^{N'_j}T) \longrightarrow R_{m_j,eN'_j}$$

which makes the diagram

$$\begin{tikzcd} H^1(\mathbb{Q}_{m_j}, T/p^{N''_j}T) \arrow[r, "h_j"] \arrow[d] & R_{m_j,eN''_j} \arrow[d] \\ H^1(\mathbb{Q}_{m_j}, T/p^{N_j}T) \arrow[r, dashed, "\bar{h}_j"] & R_{m_j,eN'_j} \end{tikzcd}$$

commute.

For a moment, we fix an integer $j \geq N(\mathfrak{P})$, and put $N' := N_j$, $N'' := N''_j$, $m := m_j$, $n = n_j$ and $\bar{h}_j := \bar{h}$ for simplicity. We put

$$N_{H_n} := \sum_{\sigma \in H_n} \sigma \in \mathbb{Z}[H_n].$$

Let $\nu_{H_n} \colon R_{m,eN'} \longrightarrow R_{m,eN'}[H_n]^{H_n}$ be an isomorphism of $R_{m,eN'}[H_n]$-module defined by $1 \mapsto N_{H_n}$. Note that the natural map

$$H^1(\mathbb{Q}_m, T/p^{N'}T) \longrightarrow H^1(\mathbb{Q}_m(n), T/p^{N'}T)$$

is injective by the assumption (C1) and (C4), and $R_{m,eN'}[H_n]$ is an injective R_{m_k,eN'_k}-module, so there exist an R_{m_k,eN'_k}-linear map

$$\tilde{h} \colon H^1(\mathbb{Q}_m(n), T/p^{N'}T) \longrightarrow R_{m,eN'}[H_n]$$

which makes the diagram

$$\begin{tikzcd} H^1(\mathbb{Q}_m, T/p^{N'}T) \arrow[r, "\bar{h}"] \arrow[d, hook] & R_{m,eN'} \arrow[d, "\nu_{H_n}"] \\ H^1(\mathbb{Q}_m(n), T/p^{N'}T) \arrow[r, "\tilde{h}"] & R_{m,eN'}[H_n] \end{tikzcd}$$

commute.

Proposition 6.11. *The following hold.*

(i) *There exists a homomorphism*

$$\tilde{h}_\infty \colon H^1(\mathbb{Q}(n), \mathbf{T}/p^{N'}\mathbf{T}) \longrightarrow \Lambda[H_n]/(p^{N'}) = \varprojlim_{m'} R_{m',eN'}[H_n]$$

of $\Lambda[H_n]/(p^{N'})$*-modules which makes the diagram*

$$\begin{tikzcd} H^1(\mathbb{Q}(n), \mathbf{T}/p^{N'}\mathbf{T}) \arrow[r, dashed, "\tilde{h}_\infty"] \arrow[d] & \Lambda[H_n]/(p^{N'}) \arrow[d, "\mathrm{mod}\ (\gamma^{p^m}-1)"] \\ H^1(\mathbb{Q}_m(n), T/p^{N'}T) \arrow[r, "\tilde{h}"] & R_{m,eN'}[H_n] \end{tikzcd}$$

commute.

(ii) *There exists an* $\mathcal{O}_j[H_n]$*-linear map*

$$\tilde{h}_{\mathfrak{P}_j,N'} \colon H^1(\mathbb{Q}(n), T_j/p^{N'}T_j) \longrightarrow \mathcal{O}_j[H_n]/(p^{N'})$$

which makes the diagram

$$\begin{tikzcd} H^1(\mathbb{Q}(n), \mathbf{T}/p^{N'}T) \arrow[r, "p^{4s}\tilde{h}_\infty"] \arrow[d] & \Lambda[H_n]/(p^{N'}) \arrow[d] \\ H^1(\mathbb{Q}(n), T_j/p^{N'}T_j) \arrow[r, dashed, "\tilde{h}_{\mathfrak{P}_j}"] & \mathcal{O}_j[H_n]/(p^{N'}) \end{tikzcd}$$

commute. Here, the vertical maps in this diagram are the natural ones, and $\tilde{h}_\infty$ *denotes the map in the assertion* (i)*.*

Proof. By Shapiro's lemma and limit arguments (cf. [Ta2] Corollary 2.3), we have a natural isomorphism

$$H^1(G_{\mathbb{Q}(n)}, \mathbf{T}/p^{N'}\mathbf{T}) \xrightarrow{\simeq} \varprojlim_{m'} H^1(\mathbb{Q}_{m'}(n), T/p^{N'}T).$$

Then, the assetion (i) follows from Lemma 5.3 (ii). The assertion (ii) is proved by the similar arguments to that in the proof of [Oh2] Proposition 8.16. For details, see loc. cit.. □

The map $\tilde{h}_{\mathfrak{P}_j}$ introduced in Proposition 6.11 (ii) induces a homomorphism

$$h_{\mathfrak{P}_j} \colon H^1(\mathbb{Q}, T_j/p^{N'}T_j) \longrightarrow (\mathcal{O}_j[H_n]/(p^{N'}))^{H_n} \xleftarrow[\simeq]{\nu_{H_n} := N_{H_n}\times} \mathcal{O}_j/p^{N'}\mathcal{O}_j.$$

Recall that here, we assume the ideal $\mathfrak{P}_j + p^{N'}\Lambda$ contains $\gamma^{p^m} - 1$, so the natural homomorphism

$$H^1(\mathbb{Q}(n), \mathbf{T}/p^{N'}\mathbf{T}) \longrightarrow H^1(\mathbb{Q}(n), T_j/p^{N'}T_j)$$

factors through

$$H^1\big(\mathbb{Q}(n), \mathbf{T}/((\gamma^{p^m} - 1)\mathbf{T} + p^{N'}\mathbf{T})\big) \simeq H^1(\mathbb{Q}_m(n), T/p^{N'}T).$$

We denote by $\mathcal{NH}_{m,eN',(n)}$ the image of the natural map

$$H^1(\mathbb{Q}(n), \mathbf{T}/p^{N'}\mathbf{T}) \longrightarrow H^1(\mathbb{Q}_m(n), T/p^{N'}T).$$

Note that since $\mathbf{T}$ is a free Λ-module, and since $\gamma^{p^m}-1=(1+T)^{p^m}-1$ is a monic polynomial, we have an exact sequence

$$0 \longrightarrow \mathbf{T}/p^{N'}\mathbf{T} \xrightarrow{\times(\gamma^{p^m}-1)} \mathbf{T}/p^{N'}\mathbf{T} \longrightarrow \mathbf{T}\otimes_\Lambda R_{m,N'} \longrightarrow 0.$$

So, the natural map

$$H^1(\mathbb{Q}(n),\mathbf{T}/p^{N'}\mathbf{T})\otimes_\Lambda R_{m,eN'} \longrightarrow \mathcal{NH}_{m,eN',(n)}$$

is an isomorphism, and the map $\tilde{h}_\infty$ in Proposition 6.11 (i) induces a homomorphism

$$\tilde{h}=\tilde{h}_\infty\otimes R_{m,eN'}\colon \mathcal{NH}_{m,eN',(n)} \longrightarrow R_{m,eN'}[H_n].$$

By Proposition 6.11 (ii), we obtain the commutative diagram

$$\begin{tikzcd}
H^1(\mathbb{Q}(n),\mathbf{T}/p^{N'}\mathbf{T}) \arrow[r, "p^{4s}\tilde{h}_\infty"] \arrow[d] & \Lambda[H_n]/(p^{N'}) \arrow[d] \\
\mathcal{NH}_{m,eN',(n)} \arrow[r, "p^{4s}\tilde{h}"] \arrow[d] & R_{m,eN'}[H_n] \arrow[d] \\
H^1(\mathbb{Q}(n),T_j/p^{N'}T_j) \arrow[r, "\tilde{h}_{\mathfrak{P}_j}"] & \mathcal{O}_j[H_n]/(p^{N'}) \\
H^1(\mathbb{Q},T_j/p^{N'}T_j) \arrow[r, "h_{\mathfrak{P}_k}"] \arrow[u, hook] & \mathcal{O}_j/p^{N'}\mathcal{O}_j. \arrow[u, hook, "\nu_{H_n}^{-1}"']
\end{tikzcd}$$

We denote the image of $c'_m(n)$ in $H^1(\mathbb{Q}_m(n),T/p^{N'}T)$ by $\bar{c}'_m(n)$, and put

$$D_nc'(n):=(D_nc'_m(n))_m\in H^1(\mathbb{Q}(n),\mathbf{T}/p^{N'}\mathbf{T})=\varprojlim_m H^1(\mathbb{Q}_m(n),T/p^{N'}T).$$

Note that we have $D_n\bar{c}'_m(n)\in\mathcal{NH}_{m,eN',(n)}$. Let

$$\mathbf{c}'\otimes\rho_j:=\{(c'\otimes\rho_j)_{m'}(n')\}_{m',n'}\in \mathrm{ES}_{\mathcal{O}_j}(T_j,\Sigma)$$

be a modified Euler system for the $\mathcal{O}_j$-module T_j which is the twist of the modified Euler system $\mathbf{c}'$ by the character ρ_j. Since we assume that n is an well-ordered integer satisfying

$$n\in\mathcal{P}_{eN''_j}(\Sigma;T)_{\mathcal{O}}\subseteq\mathcal{P}_{e_\infty N'_j}(\Sigma;T_j)_{\mathcal{O}_j}\cap\mathcal{P}_{e_\infty N'_j}(\Sigma;(\mathcal{O}_j,\rho_j))_{\mathcal{O}_j}$$

we can define the Kolyvagin derivative

$$\kappa_{0,e_\infty N'}(n;\mathbf{c}'\otimes\rho_j)\in H^1(\mathbb{Q},T_j/p^{N'}T_j)=H^1(\mathbb{Q},(T/p^{N'}T)\otimes_{\mathcal{O}}(\mathcal{O}_j,1))$$

whose image in $H^1(\mathbb{Q}(n),(T/p^{N'}T)\otimes_{\mathcal{O}}(\mathcal{O}_j,1))$ coincides with the image of

$$D_n\bar{c}'_m(n)\in H^1(\mathbb{Q}_m(n),(T/p^{N'}T)\otimes_{\mathcal{O}}(\mathcal{O}_j,1)).$$

Here, $(\mathcal{O}_j,1)$ is a free $\mathcal{O}_j$-module of rank one on which $G_{\mathbb{Q}}$ acts via the trivial character. By Corollary 6.3, there exists a Kolyvagin system

$$\kappa(\mathbf{c}'\otimes\rho_j)=\{\kappa(\mathbf{c}'\otimes\rho_j)_{n'}\}_{n'}\in\mathrm{KS}_{\mathcal{O}_j}(T_j,\Sigma)$$

such that the image of $\kappa(\mathbf{c}'\otimes\rho_j)_n$ in $H^1(\mathbb{Q}, T_j/p^{N'}T_j)$ coincides with the Kolyvagin derivative $\kappa_{0,e_\infty N'}(n;\mathbf{c}'\otimes\rho)$. Therefore, we obtain

$$\begin{aligned}\beta e_\infty j &\succ \mathrm{length}_{\mathcal{O}_j}\left(\frac{\mathcal{O}_j/p^{N'_j}\mathcal{O}_j}{h_j(\kappa_{m_j,eN''_j}(n_j;\mathbf{c}'))\cdot(\mathcal{O}_j/p^{N'_j}\mathcal{O}_j)}\right)\\ &\sim \mathrm{length}_{\mathcal{O}_j}\left(\frac{\mathcal{O}_j/p^{N'_j}\mathcal{O}_j}{\nu_{H_n}^{-1}\left(p^{4s}\tilde{h}(D_{n_j}c'_{m_j}(n_j))\right)\cdot(\mathcal{O}_j/p^{N'_j}\mathcal{O}_j)}\right).\end{aligned}$$

So, we obtain the inequality

$$\beta e_\infty j \succ \mathrm{length}_{\mathcal{O}_j}\left(\frac{\mathcal{O}_j/p^{N'_j}\mathcal{O}_j}{h_{\mathfrak{P}_j}(p^{4s}\kappa(\mathbf{c}'\otimes\rho_j)_n)\cdot(\mathcal{O}_j/p^{N'_j}\mathcal{O}_j)}\right). \tag{7}$$

By the estimate (7) and Corollary 6.10, if $\mathfrak{P}\neq(\pi)$, then we have

$$\begin{aligned}\beta e_\infty j &\succ \min\{\partial_i(T_j)+4se_\infty, e_\infty N'_j\}\\ &= \partial_i(T_j)+4se_\infty \sim a_j \sim \alpha e_\infty j.\end{aligned}$$

Let us consider the case when $\mathfrak{P}=(\pi)$. Note that in this case, we have $s=0$. So, the inequality (7) and Corollary 6.10 imply that we have

$$\beta e_\infty j \succ \min\{\partial_i(T_j), N'j\} = \partial_i(T_j) \sim a_j \sim \alpha e_\infty j.$$

Thus, we obtain $\beta\geq\alpha$ in any case, and this completes the proof of Theorem 6.5. □

7. Evaluation maps and the Chebotarev density theorem

In this section, we briefly recall the definitions and some properties of "evaluation maps" induced in [Ru2] §7.2. Then, by using the Chebotarev density theorem, we shall prove a proposition which plays a key role in our Euler system arguments in §8. (See Proposition 7.6.) In this and the next sections, we assume $\mathcal{O}=\mathbb{Z}_p$.

Through out this section, we fix integers m and N satisfying $N>m\geq 0$. We assume that the $\mathbb{Z}_p[G_\mathbb{Q}]$-module T satisfies the conditions (C1)–(C7) $\tau\in$ in introduction. In particular, we fix an element $\tau\in G_{\mathbb{Q}(\mu_{p^\infty})}$ in the condition (C2).

7.1. **Evaluation maps.** Here, we introduce "evaluation maps" defined in [Ru2] §7.2. First, we recall the definition of them. By the assumption (C2), the divisible $\mathbb{Z}_p$-module $A/(\tau-1)A$ is cofree of rank one. Recall that we have fixed an $\mathbb{Z}_p$-linear isomorphism

$$\Phi^*\colon T/(\tau-1)T \xrightarrow{\simeq} \mathbb{Z}_p.$$

in Definition 4.6. By taking $(-)\otimes_{\mathbb{Z}_p}\mathbb{Q}/\mathbb{Z}$, this isomorphism induces an $\mathbb{Z}_p$-linear isomorphism

$$\theta^*\colon A/(\tau-1)A \xrightarrow{\simeq} \mathbb{Q}_p/\mathbb{Z}_p.$$

By taking $\mathrm{Hom}(-,\mu_{p^\infty})$, we obtain the $\mathbb{Z}_p$-linear isomorphism

$$\theta^{**}\colon \mathbb{Z}_p(-1) \xrightarrow{\simeq} T^{\tau=1}.$$

Recall that we have fixed a $\mathbb{Z}_p$-basis $(\zeta_{p^n})_n$ of $\mathbb{Z}_p(1)$. By this basis, we identify $\mathbb{Z}_p(1)$ with $\mathbb{Z}_p$ as an $\mathbb{Z}_p$-module. Then, we put

$$\theta := (\theta^{**})^{-1} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \colon A^{\tau=1} \xrightarrow{\simeq} \mathbb{Q}_p/\mathbb{Z}_p.$$

The assumption (C2) also implies that we have

$$q_\tau(x) := \frac{\det_{\mathbb{Z}_p}(1-\tau^{-1}x \mid T^*)}{x-1} = \frac{\det_{\mathbb{Z}_p}(1-\tau x \mid T)}{x-1} \in \mathbb{Z}_p[x].$$

We denote the composite map

$$A/(\tau-1)A \xrightarrow{q_\tau(\tau^{-1})} A^{\tau=1} \xrightarrow{\theta} \mathbb{Q}_p/\mathbb{Z}_p$$

by $\bar{\theta}$. Note that the $\mathbb{Z}_p$-linear map $\bar{\theta}$ is an isomorphism. (See [Ru2] Corollary A.2.7.) We define the evaluation maps $\mathrm{Ev}^*_{m,N}$ and $\mathrm{Ev}_{m,N}$ as follows.

Definition 7.1. Let Σ' be a finite set of prime numbers, and write

$$X^{\Sigma'}_{m,N} := \mathrm{Hom}_{\mathbb{Z}_p}\left(H^1_{(\mathcal{F}^*_{\mathrm{can}})^{\Sigma'}}(\mathbb{Q}_m, A^*[p^N]), \mathbb{Z}/p^N\mathbb{Z}\right).$$

Let θ^* and θ be as above. We put $\Omega_{m,N} := \mathbb{Q}_m(\mu_{p^N}, A[p^N])$, and let $\Omega^{\tau=1}_{m,N}$ be the maximal subfield of $\Omega_{m,N}$ fixed by τ. We define group homomorphisms

$$\mathrm{Ev}^*_{m,N,\Sigma'} \colon G_{\Omega^{\tau=1}_{m,N}} \longrightarrow X^{\Sigma'}_{m,N}$$
$$\mathrm{Ev}_{m,N} \colon G_{\Omega^{\tau=1}_{m,N}} \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}\left(H^1(\mathbb{Q}_m, A[p^N]), \mathbb{Z}/p^N\mathbb{Z}\right)$$

by $\left(\mathrm{Ev}^*_{m,N}(\sigma)\right)(c) := \theta^*(c(\sigma))$ and $\left(\mathrm{Ev}_{m,N}(\sigma)\right)(c) := \bar{\theta}(c(\sigma))$. Here, we identify $\mathbb{Z}/p^N\mathbb{Z}$ with $p^{-N}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$ by the isomorphism

$$p^{-N}\mathbb{Z}/\mathbb{Z} \xrightarrow[\simeq]{\times p^N} \mathbb{Z}/p^N\mathbb{Z}.$$

For simplicity, we put $\mathrm{Ev}^*_{m,N} := \mathrm{Ev}^*_{m,N,\emptyset}$.

In the next section, we need the surjectivity of evaluation maps in some sense. (See §§8.2–8.3.) The following proposition ensures it.

Proposition 7.2. *For any finite set Σ' of prime numbers and for any integer N_0 satisfying $N \geq N_0 > m$, we have*

$$\mathrm{Hom}_{\mathbb{Z}_p}\left(H^1(\mathbb{Q}_m, A[p^{N_0}]), \mathbb{Z}/p^{N_0}\mathbb{Z}\right) = \mathrm{Ev}_{m,N_0}(G_{\Omega_{m,N}})$$

and

$$X^{\Sigma'}_{m,N_0} = \mathrm{Ev}^*_{m,N_0}(G_{\Omega_{m,N}}).$$

Proof. By Lemma 3.3 and the assumptions (C1), (C2) and (C5), we can deduce the composite

$$H^1(\mathbb{Q}_m, A[p^N]) \xrightarrow{\mathrm{Res}} \mathrm{Hom}(G_{\Omega_{m,N}}, A[p^N])^{G_{\mathbb{Q}_m}}$$
$$\longrightarrow \mathrm{Hom}(G_{\Omega_{m,N}}, A[p^N]/(\tau-1)A[p^N]) \xrightarrow{\bar{\theta}} p^{-N\mathbb{Z}}/\mathbb{Z}.$$

is injective. (For details, see the arguments in the proof of [Ru2] Lemma7.2.4.) We apply $\mathrm{Hom}(-, \mathbb{Q}_p/\mathbb{Z}_p)$ to this injection, and obtain the surjection

$$G_\Omega \twoheadrightarrow \mathrm{Hom}(H^1(\mathbb{Q}_m, A[p^N]), \mathbb{Q}_p/\mathbb{Z}_p).$$

The first assertion immediately follows from this surjectivity. The second assertion follows from similar arguments for (A^*, τ, θ^*). □

Remark 7.3. In the arguments in the proof of Proposition 7.2 (the surjectivity of evaluation maps), we use the assumption $\mathcal{O} = \mathbb{Z}_p$. In our paper, this is the only part which requires $\mathcal{O} = \mathbb{Z}_p$.

Let ℓ be a prime number contained in $\mathcal{P}^{\tau}_N(\Sigma; T)_{\mathcal{O}}$. By definition, we have $\mathrm{Frob}_\ell \in G_{\Omega^{\tau=1}_N}$ and $I_\ell \subseteq G_{\Omega^{\tau=1}_N}$, where Frob_ℓ is a lift of the arithmetic Frobenius element at $\ell_{\overline{\mathbb{Q}}}/\ell$, and $I_\ell \subseteq G_{\mathbb{Q}}$ is the inertia subgroup at $\ell_{\overline{\mathbb{Q}}}/\ell$. Then, by the definition of maps $(-)^{\ell,s}_{m,N,\Phi^*}$ and ϕ^{ℓ}_{m,N,Φ^*} introduced in §4.2, we immediately obtain the following lemma.

Lemma 7.4. *Let $\ell \in \mathcal{P}^{\tau}_N(\Sigma; T)_{\mathcal{O}}$. Recall that we put $R_{m,N} = \mathbb{Z}/p^N\mathbb{Z}[\mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})]$.*

(i) *For any element $x \in H^1(\mathbb{Q}_m, A[p^N])$, we have*
$$(x)^{\ell,s}_{m,N,\Phi^*} = \sum_{g \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})} (\mathrm{Ev}_{m,N}(\tilde{\sigma}_\ell))\,(g^{-1}x) \cdot g \in R_{m,N},$$
where $\tilde{\sigma}_\ell \in I_\ell$ is a lift of a generator σ_ℓ of the cyclic group H_ℓ.

(ii) *For any element $x \in \mathrm{Ker}(-)^{\ell,s}_{m,N,\Phi^*}$, we have*
$$\phi^{\ell}_{m,N,\Phi^*}(x) = \sum_{g \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})} (\mathrm{Ev}_{m,N}(\mathrm{Frob}_\ell))\,(g^{-1}x) \cdot g \in R_{m,N}.$$

By Tate's local duality theorem, we have the following proposition.

Proposition 7.5. *Let $\ell \in \mathcal{P}^{\tau}_N(\Sigma; T)_{\mathcal{O}}$, and λ a place of $\mathbb{Q}_m$ above ℓ. Consider the local pairing*
$$(\cdot,\cdot)_\lambda \colon H^1_f(\mathbb{Q}_{m,\lambda}, A^*[p^N]) \times H^1_s(\mathbb{Q}_{m,\lambda}, A[p^N]) \longrightarrow \mathbb{Z}/p^N\mathbb{Z}.$$
Let $\bar{g} \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$ be an element satisfying $\lambda = \ell_{\mathbb{Q}_m} \circ \bar{g}^{-1}$. Then, for any $x \in H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}_m, A^*[p^N])$ and $y \in H^1(\mathbb{Q}_m, A[p^N])$, we have
$$(x, y)_\lambda = \mathrm{Ev}^*_{m,N}(\mathrm{Frob}_\ell)(g^{-1}x) \cdot \mathrm{Ev}_{m,N}(\tilde{\sigma}_\ell)(g^{-1}y) \in \mathbb{Z}/p^N\mathbb{Z}.$$

7.2. **Application of the Chebotarev density theorem.** Here, by using the Chebotarev density theorem, we shall show a key proposition for our Euler system arguments (See Proposition 7.6.) Recall that we have fixed a collection of embeddings $\{\ell_{\overline{\mathbb{Q}}} \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell\}$ in the introduction. Note that the existence of such a family follows from the Chebotarev density theorem. The goal of this subsection is the following proposition.

Proposition 7.6. *Let $q \in \mathcal{P}^{\tau}_N(\Sigma; T)_{\mathcal{O}}$ be any prime number, and $n \in \mathcal{N}^{\tau}_N(\Sigma; T)_{\mathcal{O}}$ by any integer prime to q. We assume that n has a decomposition $n = \prod_{i=1}^{r} \ell_i$, where $\ell_1, \ldots, \ell_r$ are prime numbers. Let N_0 be any integer satisfying $N \geq N_0 > m$. Suppose the following are given:*

- *an R_{m,N_0}-submodule W of $H^1(\mathbb{Q}_m, A[p^{N_0}])$ of finite order;*
- *an R_{m,N_0}-homomorphism $\psi \colon W \longrightarrow R_{m,N_0}$.*

Then, there exist infinitely many $q' \in \mathcal{P}_N^{\tau}(\Sigma; T)_{\mathcal{O}}$ which split completely in $\mathbb{Q}_m(\mu_{qn})/\mathbb{Q}$, and satisfy all of the following properties.

(i) *We have*
$$\mathrm{Ev}^*_{m,N}(\mathrm{Frob}_{q'}) = \mathrm{Ev}^*_{m,N}(\mathrm{Frob}_q),$$
where $\mathrm{Frob}_{q'} \in G_{\mathbb{Q}_m}$ (resp. $\mathrm{Frob}_q \in G_{\mathbb{Q}_m}$) is an arithmetic Frobenius element at $q'_{\overline{\mathbb{Q}}}/q'_{\mathbb{Q}_m}$ (resp. at $q_{\overline{\mathbb{Q}}}/q_{\mathbb{Q}_m}$).

(ii) *there exists an element $z \in H^1(\mathbb{Q}_m, A[p^N])$ satisfying the following conditions*
- *We have $(z)^{q',s}_{m,N,\Phi^*} = 1$ and $(z)^{q,s}_{m,N,\Phi^*} = -1$.*
- *For any prime number ℓ not contained in $\{q, q'\}$, the image of z in $H^1_s(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, A[p^N])$ is zero. Moreover, if $\ell \in \Sigma$, then the image of the image of z in $H^1(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, A[p^N])$ is zero.*
- *We have $\phi^{\ell_i}_{m,N,\Phi^*}(z) = 0$ for each $i = 1, \dots, r$.*

(iii) *the group W is contained in the kernel of $(-)^{q',s}_{m,N_0,\Phi^*}$, and*
$$\psi(x) = \phi^{q'}_{m,N_0,\Phi^*}(x)$$
for any $x \in W$.

Proof. First, let us define an element g_1 of the Galois group of a certain finite extension over $\Omega_{m,N}$ related to the conditions (i) and (ii). We put $\Sigma' := \{q, \ell_1, \dots, \ell_r\} \cup \Sigma$. Let L_0 be the maximal subfield of $\overline{\mathbb{Q}}$ fixed by the kernel of which is the evaluation map
$$e^* \colon G_{\Omega_{m,N}} \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}\left(\mathrm{Hom}(G_{\Omega_{m,N}}, A^*[p^N])^{\mathrm{Gal}(\Omega_{m,N}/\mathbb{Q}_m)}, A^*[p^N]\right),$$
and we put $L := L_0(\mu_{qn})$. By definition, the homomorphism $\mathrm{Ev}^*_{m,N,\Sigma'}|_{G_{\Omega_{m,N}}}$ factors through e^*. Note that L_0 and L are Galois over $\mathbb{Q}_m$. On the one hand, all Jordan–Hölder constituents of $\mathrm{Gal}(L_0/\Omega_{m,N})$ as a $\mathbb{Z}_p[G_{\mathbb{Q}_m}]$-module are subquotients of $A^*[p]$. On the other hand, the action of $G_{\mathbb{Q}_m}$ on $\mathrm{Gal}(\Omega_{m,N}(\mu_{qn})/\Omega_{m,N})$ is trivial. So by the assumption (C4), L_0 and $\Omega_{m,N}(\mu_{qn})$ are linearly disjoint over $\Omega_{m,N}$. Hence we can take an element $g_1 \in \mathrm{Gal}(L/\Omega_{m,N})$ which satisfies
$$g_1|_{L_0} = \tau^{-1}\mathrm{Frob}_q|_{L_0}.$$

Next, we shall take an element g_2 of the Galois group of a certain finite extension over $\Omega_{m,N}$ related to the conditions (i) and (ii).

We define a surjective homomorphism
$$P_{m,N_0} \colon R_{m,N_0} \longrightarrow \mathbb{Z}/p^{N_0}\mathbb{Z}$$
of abelian groups by $\sum_g a_g \cdot g \longmapsto a_1$, where $1 \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$ is the identity element. Note that the map
$$\mathrm{Hom}_{R_{m,N_0}}(W, R_{m,N_0}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/p^{N_0}\mathbb{Z});\ f \longmapsto P_{m,N_0} \circ f$$
is bijective. Indeed, its inverse is given by
$$h \longmapsto \left(x \longmapsto \sum_{g \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})} h(g^{-1}x)g\right) \in \mathrm{Hom}_{R_{m,N_0}}(W, R_{m,N_0}),$$

for any $h \in \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/p^{N_0}\mathbb{Z})$. We define an element $\bar{\psi}_0 \in \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/p^{N_0}\mathbb{Z})$ by

$$\bar{\psi}_0 := P_{m,N_0} \circ \psi - \mathrm{Ev}_{m,N_0}(\tau)|_W.$$

Then, by Proposition 7.2, we have $\bar{\psi}_0$ is the restriction of an element contained in $\mathrm{Ev}_{m,N_0}(G_{\Omega_{m,N}})$.

Recall that by Lemma 3.3 and the assumption (C5), the restriction map

$$H^1(\mathbb{Q}_m, A[p^{N_0}]) \longrightarrow \mathrm{Hom}(G_{\Omega_{m,N}}, A[p^{N_0}])^{\mathrm{Gal}(\Omega_{m,N}/\mathbb{Q}_m)}$$

is injective. By this injection, we regard W as an $R_m := \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})]$-submodule of $\mathrm{Hom}(G_{\Omega_{m,N}}, A[p^{N_0}])^{\mathrm{Gal}(\Omega_{m,N}/\mathbb{Q}_m)}$. We denote by M the maximal subfield of $\overline{\mathbb{Q}}$ fixed by the kernel of which is the evaluation map

$$e \colon G_{\Omega_{m,N}} \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}\left(W, A[p^{N_0}]\right),$$

Note that $M/\mathbb{Q}$ is a Galois extension, and all Jordan–Hölder constituents of $\mathrm{Gal}(M/\mathbb{Q})$ as a $\mathbb{Z}_p[G_{\mathbb{Q}}]$-module are subquotients of W. Let $\sigma \in G_{\Omega_{m,N}}$ be an element satisfying $\mathrm{Ev}_{m,N}(\sigma) = \bar{\psi}_0$. Then, we denote the image of σ in $\mathrm{Gal}(M/\Omega_{m,N})$ by g_2.

Here, we consider the composite field LM. Note that L and M are linearly disjoint over $\Omega_{m,N}$ since by the assumptions (C3) and (C4), the set of Jordan–Hölder constituents of $\mathrm{Gal}(L/M)$ as a $\mathbb{Z}_p[G_{\mathbb{Q}}]$-module is disjoint from that of $\mathrm{Gal}(M/\Omega_{m,N})$.

Let M' be the maximal subfield of $\overline{\mathbb{Q}}$ fixed by $\mathrm{Ker}\,\mathrm{Ev}_{m,N}|_{G_{\Omega_{m,N}}}$, and Σ_W the set of all prime numbers which ramifies in the extension $M'/\mathbb{Q}$. Let q' be a prime number not contained in $\Sigma \cup \Sigma' \cup \Sigma_W$ satisfying

$$\begin{cases} (q'_L, L/\mathbb{Q}) = \tau g_1; \\ (q'_M, M/\mathbb{Q}) = \tau g_2. \end{cases}$$

Note that our choice of the family of embeddings $\{\ell_{\overline{\mathbb{Q}}} \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell\}$ ensures that there exist infinitely may prime numbers q'. By the definition of q', we deduce that q' splits completely in $\mathbb{Q}_m(\mu_{qn})/\mathbb{Q}$, and q' belongs to $\mathcal{P}_N^\tau(\Sigma; T)_{\mathcal{O}}$. In order to prove Proposition 7.6, it is sufficient to show that all but finitely many such prime numbers q' satisfy all conditions (i)–(iii).

The condition (i) follows from the definition of q' since $\mathrm{Ev}^*_{m,N}|_{G_{\Omega_{m,N}}}$ factors through the map e^*. We shall consider the condition (ii). Let $\ell \in \mathcal{P}_N^\tau(\Sigma; T)_{\mathcal{O}}$ be any element. Recall that in §4.2, we have defined an isomorphism

$$\Psi^\ell_{m,N} \colon H^1_{\mathrm{tr}}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, A[p^N]) \simeq H^1_s(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, A[p^N]) \xrightarrow{\simeq} R_{m,N}.$$

We take an element $c_\ell \in H^1_{\mathrm{tr}}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, A[p^N])$ satisfying $\Psi^\ell_{m,N}(c_\ell) = 1$. Then, we define an element $x := (x_\ell)_{\ell \in \Sigma' \cup \{q'\}} \in \bigoplus_{\ell \in \Sigma'} H^1_{\text{ét}}(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, A[p^N])$ by

$$x_\ell := \begin{cases} c_{q'} & (\ell = q'); \\ -c_q & (\ell = q); \\ 0 & (\ell \neq q', q). \end{cases}$$

Then, Lemma 7.4, Proposition 7.5 and the Poitou–Tate exact sequence

$$H^1(\mathcal{O}_{\mathbb{Q}_m,\Sigma'\cup\{q'\}}, A[p^N]) \longrightarrow \bigoplus_{\ell\in\Sigma'\cup\{q'\}} H^1_{\text{ét}}(\mathbb{Q}_m\otimes\mathbb{Q}_\ell, A[p^N]) \longrightarrow X^{\Sigma'\cup\{q'\}}_{m,N}.$$

imply that there exists an element $z \in H^1(\mathcal{O}_{\mathbb{Q}_m,\Sigma'\cup\{q'\}}, A[p^N])$ whose image in

$$\bigoplus_{\ell\in\Sigma'\cup\{q'\}} H^1_{\text{ét}}(\mathbb{Q}_m\otimes\mathbb{Q}_\ell, A[p^N])$$

coincides with x. Such z clearly satisfies all the properties required in (ii).

Now let us consider the condition (iii). Since $q' \notin \Sigma_W$, Lemma 7.4 implies that the group W is contained in the kernel of $(-)^{q',s}_{m,N,\Phi^*}$. It is sufficient to show

$$P_{m,N}\circ\psi(x) = P_{m,N}\circ\phi^{q'}_{m,N,\Phi^*}(x)$$

for any $x \in W$. This equality follows from the choice of q' (in particular, see the definition of g_2) and Lemma 7.4. □

8. Upper bounds of higher Fitting ideals

In this section, assume that T satisfies (C1)–(C7), and fix an Euler system $\mathbf{c} = \{c_m(n)\}_{m,n} \in \mathrm{ES}_{\mathcal{O}}(T,\Sigma)$ satisfying (NV). Let $\mathbf{c}' = \{c'_m(n)\}_{m,n} \in \mathrm{ES}'_{\mathcal{O}}(T,\Sigma)$ be a modified Euler system corresponding to $\mathbf{c}$. Throughout this section, we assume $\mathcal{O} = \mathbb{Z}_p$. Here, by using Kurihara's Euler system arguments, we shall prove Theorem 2.4 (ii), which asserts that the ideals $\mathfrak{C}_i(\mathbf{c})$ give "upper bounds" of $\mathrm{Fitt}_{\Lambda,i}(X)$.

8.1. **Setting.** Recall that we denote the maximal pseudo-null Λ-submodule of $X := X(T)$ by X_{fin}, and put $X' = X'(T) := X/X_{\mathrm{fin}}$. If $X' = 0$, the assertion (ii) of Theorem 2.4 immediately follows from Theorem 5.5. So, we assume $X' \neq 0$ here. The projective dimension of the Λ-module X' is one, so there exists an exact sequence

$$0 \longrightarrow \Lambda^h \xrightarrow{f} \Lambda^h \xrightarrow{g} X' \longrightarrow 0, \tag{8}$$

where h is the minimal of the cardinalities of sets of generators of X'. We denote by M the matrix corresponding to f with respect to the standard basis $\mathbf{e} := (e_i)_{i=1}^r$ of the free Λ-module Λ^h.

Let $\{m_1,\dots,m_h\}$ and $\{n_1,\dots,n_h\}$ be permutations of $\{1,\dots,h\}$, and let i be an integer satisfying $1 \le i \le h-1$. Let us consider the matrix M_i which is obtained from M by eliminating the n_j-th rows ($j = 1,\dots,i$) and the m_k-th columns ($k = 1,\dots,i$). If $\det(M_i) = 0$, we clearly have $\det(M_i) \in \mathfrak{C}_i(\mathbf{c})$, so we assume that $\det(M_i) \neq 0$. If necessary, we permute $\{m_1,\dots,m_i\}$, and assume $\det(M_j) \neq 0$ for all integers j satisfying $0 \le j \le i$.

For a while, we fix integers m and N satisfying $N > m \ge 0$, and we put

$$X_{m,N} := \mathrm{Hom}_{\mathbb{Z}_p}\left(H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}_m, A^*[p^N]), \mathbb{Z}/p^N\mathbb{Z}\right).$$

(Note that our tentative goal is Proposition 8.6 in §8.3, which states that certain equalities in $R_{m,N}$ related to $\mathfrak{C}_{i,m,N}(\mathbf{c})$ and $X \otimes_\Lambda R_{m,N}$ hold.) By Proposition 3.8, we have the natural isomorphism

$$X \otimes_\Lambda R_{m,N} \simeq X_{m,N}.$$

Let $X_{m,N,\mathrm{fin}}$ be the image of X_{fin} in $X_{m,N}$, and write $X'_{m,N} := X_{m,N}/X_{m,N,\mathrm{fin}}$. We shall consider the ideal $\mathfrak{C}_{i,m,N}(\mathbf{c}) = \mathfrak{C}_{i,m,N}(\mathbf{c}')$ and the image of $\det(M_i)\,\mathrm{ann}_\Lambda(X_{\mathrm{fin}})I(\mathbf{c})$ in $R_{m,N}$.

By Lemma 5.6, there exists a character $\rho \in \mathrm{Hom}_{\mathrm{cont}}(\Gamma, 1 + p^N\mathbb{Z}_p)$ which satisfies

$$\mathrm{char}_\Lambda(\mathbb{H}^2_\Sigma(T \otimes \rho)) \not\subseteq (\gamma^{p^m} - 1)\Lambda.$$

Let $\rho\colon \Gamma \longrightarrow 1 + p^N\mathbb{Z}_p$ be such a character, and $\mathbf{c} \otimes \rho$ be an Euler system of $T \otimes \rho$ which is the twist of $\mathbf{c}$ by the character ρ. In particular, we have

$$\mathrm{char}_\Lambda(X(T \otimes \rho)) = \mathrm{char}_\Lambda(X(T) \otimes \rho) \not\subseteq (\gamma^{p^m} - 1)\Lambda.$$

Note that by the construction of the ideal $\mathfrak{C}_{i,m,N}(\mathbf{c})$ implies

$$\mathfrak{C}_{i,m,N}(\mathbf{c} \otimes \rho) = \mathfrak{C}_{i,m,N}(\mathbf{c}).$$

We define an endomorphism

$$\iota_\rho\colon \Lambda \longrightarrow \Lambda;\ \gamma \longmapsto \rho(\gamma)^{-1} \cdot \gamma$$

of a topological $\mathcal{O}$-algebra Λ. By the exact sequence (8), we have an exact sequence

$$0 \longrightarrow \Lambda^h \xrightarrow{\iota_\rho(M)} \Lambda^h \longrightarrow X'(T \otimes \rho) \longrightarrow 0.$$

In order to study $\mathfrak{C}_{i,m,N}(\mathbf{c})$ and $X \otimes_\Lambda R_{m,N}$, we may replace T (resp. M_i) with $T \otimes \rho$ (resp. $\iota_\rho(M_i)$). So, from now on, we assume that the order of $X \otimes_\Lambda R_m$ is finite.

We apply $(-) \otimes_\Lambda R_m$ to the short exact sequence (8), then we obtain an exact sequence

$$0 \longrightarrow R_m^h \xrightarrow{\bar{f}} R_m^h \xrightarrow{\bar{g}} X' \otimes_\Lambda R_m \longrightarrow 0. \tag{9}$$

Here, the injectivity of $\bar{f}$ follows from the assumption $\#(X' \otimes_\Lambda R_m) < \infty$. We define an integer N' by

$$p^{N'} = \max\left\{\#H^2_{\text{ét}}(\mathcal{O}_m[1/p], j_*T)_{\mathrm{tor}}, \#(X' \otimes_\Lambda R_m)\right\},$$

where $j\colon \mathrm{Spec}\,\mathcal{O}_{m,\Sigma} \longrightarrow \mathrm{Spec}\,\mathcal{O}_m[1/p]$ is the natural inclusion, and

$$H^2_{\text{ét}}(\mathcal{O}_m[1/p], j_*T)_{\mathrm{tor}} \subseteq H^2_{\text{ét}}(\mathcal{O}_m[1/p], j_*T)_{\mathrm{tor}}$$

denotes the $\mathcal{O}$-torsion part. We apply $(-) \otimes_{R_m} R_{m,N+N'}$ to the short exact sequence (9), and consider the exact sequence

$$R^h_{m,N+N'} \xrightarrow{\bar{f}_{N+N'}} R^h_{m,N+N'} \xrightarrow{\bar{g}_{N+N'}} X'_{m,N+N'} \longrightarrow 0. \tag{10}$$

We put $\mathfrak{K} := \mathrm{Ker}\, f_{N+N'}$. Note that $X' \otimes_\Lambda R_m \simeq X'_{m,N+N'}$ is annihilated by $p^{N'}$, so $\mathfrak{K}$ is annihilated by $p^{N'}$.

For each integer j with $1 \leq j \leq h$, we denote the image of e_j in $R^h_{m,N+N'}$ by $\bar{e}_j$, and fix a lift $x_j \in X_{m,N}$ of $\bar{g}_{N+N'}(e_j)$. By Proposition 7.2, we have

$$x_j - \mathrm{Ev}_{m,N+N'}(\tau) \in \mathrm{Ev}^*_{m,N+N'}(G_{\Omega_{m,N}}).$$

Note that sine we assume that h is minimal, Nakayama's lemma implies $x_j \neq x_{j'}$ if $j \neq j'$. For each integer j with $1 \leq j \leq h$, we fix a prime number $q_j \in \mathcal{P}^{\tau}_{N+2N'}(\Sigma;T)_{\mathbb{Z}_p}$ satisfying

$$\mathrm{Ev}^*_{m,N+N'}(\mathrm{Frob}_{q_{j,\overline{\mathbb{Q}}}}) = x_j,$$

and define a subset P_j of $\mathcal{P}^{\tau}_{N}(\Sigma;T)_{\mathbb{Z}_p}$ by

$$P_j := \left\{ \ell \in \mathcal{P}^{\tau}_{N+2N'}(\Sigma;T)_{\mathbb{Z}_p} \;\middle|\; \mathrm{Ev}^*_{m,N+2N'}(\mathrm{Frob}_{\ell_{\overline{\mathbb{Q}}}}) = \mathrm{Ev}^*_{m,N+2N'}(\mathrm{Frob}_{q_{j,\overline{\mathbb{Q}}}}) \right\}.$$

By assumption (Chb), there exists infinitely many elements in P_j.

Here, let us construct 3 maps α_0, α_1 and β_0, which play a key role in this section. For any integer j with $1 \leq j \leq h$ and any $t \in \{0, 1, 2\}$, we put

$$J_{j,t} := \bigoplus_{\ell \in P_j} H^1_s(\mathbb{Q}_m \otimes \mathbb{Q}_\ell, T/p^{N+tN'}T)$$

define an $R_{m,N+tN'}$-linear map $\alpha_{j,t} \colon J_{j,t} \longrightarrow R_{m,N+tN'}$ by

$$\alpha_{j,t} := \oplus(-)^{\ell,s}_{m,N+tN',\Phi^*}$$

We put $J_t := \bigoplus_{j=1}^h J_{j,t}$ and consider the map

$$\alpha_t := (\alpha_{j,t})_{j=1}^h \colon J_t \longrightarrow R^h_{m,N+tN'}.$$

The construction of the map β_0 is slightly complicated. By (the direct limit of) the Poitou–Tate exact sequence and the exact sequence (10), we deduce that there exists a homomorphism

$$\tilde{\beta}_1 \colon H^1_{\mathcal{F}^P_{\mathrm{can}}}(\mathbb{Q}_m, T/p^{N+N'}T) \longrightarrow R^h_{m,N+N'}/\mathfrak{K}$$

which makes the diagram

$$\begin{tikzcd}
 & H^1_{\mathcal{F}^P_{\mathrm{can}}}(\mathbb{Q}_m, T/p^{N+N'}T) \arrow[r] \arrow[d, dashed, "\tilde{\beta}_1"] & J_1 \arrow[r] \arrow[d, "\alpha_1"] & X'_{m,N+N'} \arrow[d, equal] \\
0 \arrow[r] & R^h_{m,N+N'}/\mathfrak{K} \arrow[r, "\bar{f}_{N+N'}"] & R^h_{m,N+N'} \arrow[r] & X'_{m,N+N'}
\end{tikzcd} \tag{11}$$

commutes. Note that the bottom low in the diagram (11) is exact, and the top low may not be exact but a complex. Recall that $\mathfrak{K}$ is annihilated by $p^{N'}$, so $\mathfrak{K}$ is contained in $p^N R^h_{N+N'}$. We define β_0 to be the composite map

$$\beta_1 \colon H^1_{\mathcal{F}^P_{\mathrm{can}}}(\mathbb{Q}_m, T/p^{N+N'}T) \xrightarrow{\tilde{\beta}_1} R^h_{m,N+N'}/\mathfrak{K} \xrightarrow{\bmod p^N} R^h_{m,N}.$$

Take an integer n which is a product of some prime numbers contained in $P := \coprod_{j=1}^h P_j$, and put $\Sigma_n := \Sigma \cup \mathrm{prime}(n)$. Let

$$j_n \colon \mathrm{Spec}\, \mathcal{O}_{\mathbb{Q}_m,\Sigma_n} \longrightarrow \mathrm{Spec}\, \mathcal{O}_{\mathbb{Q}_m}[1/pn]$$

be the natural open immersion.

Definition 8.1. Let $t \in \{0,1,2\}$. We denote the image of the natural homomorphism

$$H^1_{\text{ét}}(\mathcal{O}_{\mathbb{Q}_m}[1/pn], j_{n*}T/p^{N+2N'}j_{n*}T) \longrightarrow H^1_{\text{ét}}(\mathcal{O}_{\mathbb{Q}_m}[1/pn], j_{n*}T/p^{N+tN'}j_{n*}T)$$

by $\mathcal{H}_t(n)$. Note that by Lemma 4.15, we can naturally regard

$$\mathcal{H}_t(n) \subseteq H^1_{\mathcal{F}^n_{\text{can}}}(\mathbb{Q}_m, T/p^{N+tN'}T).$$

In order to construct the map $\beta_0 \colon \mathcal{H}_0(n) \longrightarrow R^h_N$, we need the following lemma.

Lemma 8.2. *The natural map*

$$\mathcal{H}_1(n) \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p/p^N\mathbb{Z}_p) \longrightarrow \mathcal{H}_0(n)$$

is an isomorphism.

Proof. For any $t \in \{0,1,2\}$ and $a \in \mathbb{Z}_{\geq 0}$, we put

$$\begin{aligned} \tilde{\mathcal{H}}_t(n) &:= H^1_{\text{ét}}(\mathcal{O}_{\mathbb{Q}_m}[1/pn], j_{n*}T/p^{N+tN'}j_{n*}T), \\ H^a(j_{n*}T) &:= H^a_{\text{ét}}(\mathcal{O}_{\mathbb{Q}_m}[1/pn], j_{n*}T). \end{aligned}$$

The exact sequence

$$0 \longrightarrow j_{n*}T \xrightarrow{\times p^{N+tN'}} j_{n*}T \longrightarrow j_{n*}T/p^{N+tN'}j_{n*}T \longrightarrow 0,$$

induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(j_{n*}T)/p^{N+2N'} \arrow[r] \arrow[d, "\text{mod } p^{N+N'}"'] & \tilde{\mathcal{H}}_2(n) \arrow[r] \arrow[d] & H^2(j_{n*}T)[p^{N+2N'}] \arrow[r] \arrow[d, "p^{N'}\times"] & 0 \\
0 \arrow[r] & H^1(j_{n*}T)/p^{N+N'} \arrow[r] \arrow[d, "\text{mod } p^{N}"'] & \tilde{\mathcal{H}}_1(n) \arrow[r] \arrow[d] & H^2(j_{n*}T)[p^{N+N'}] \arrow[r] \arrow[d, "p^{N'}\times"] & 0 \\
0 \arrow[r] & H^1(j_{n*}T)/p^{N} \arrow[r] & \tilde{\mathcal{H}}_0(n) \arrow[r] & H^2(j_{n*}T)[p^{N}] \arrow[r] & 0,
\end{tikzcd}$$

whose rows are exact. (Here, for simplicity, we put $M/a := M/aM$ for any $\mathbb{Z}_p$-module M and any element $a \in \mathbb{Z}_p$.) So, we obtain a commutative diagram

$$\begin{tikzcd}
 & H^1(j_{n*}T)/p^N \arrow[r] \arrow[d, equal] & \mathcal{H}_1(n)/p^N \arrow[r] \arrow[d] & \left(p^{N'}H^2(j_{n*}T)[p^{N+2N'}]\right)/p^N \arrow[r] \arrow[d, "\bar{P}:=p^{N'}\times"] & 0 \\
0 \arrow[r] & H^1(j_{n*}T)/p^N \arrow[r] & \tilde{\mathcal{H}}_0(n) \arrow[r] & H^2(j_{n*}T)[p^N] \arrow[r] & 0,
\end{tikzcd} \tag{12}$$

with exact rows. In order to prove Lemma 8.2, it is sufficient to show that the right vertical map $\bar{P}$ in the diagram (12) is injective.

Here, let us consider $H^2(j_{n*}T)$. Let

$$j = j_1 \colon \operatorname{Spec} \ \mathcal{O}_{\mathbb{Q}_m,\Sigma} \longrightarrow \operatorname{Spec} \mathcal{O}_{\mathbb{Q}_m}[1/p]$$

be the natural open immersion. By Leray spectral sequence, we obtain the exact sequence

$$H^2_{\text{ét}}(\mathcal{O}_{\mathbb{Q}_m}[1/p], j_*T) \longrightarrow H^2_{\text{ét}}(j_{n*}T) \longrightarrow H^1_{\text{ét}}(\mathcal{O}_{\mathbb{Q}_m}[1/pn], R^1j_{n*}T) \longrightarrow 0. \tag{13}$$

Let we have natural isomorphisms

$$H^1_{\text{ét}}(\mathcal{O}_{\mathbb{Q}_m}[1/pn], R^1 j_{n*}T) \simeq \bigoplus_{\lambda|n} H^1(k_m(\lambda), H^1(\mathbb{Q}^{\mathrm{ur}}_{m,\lambda}, T)) \simeq \bigoplus_{\lambda|n} H^2(\mathbb{Q}_{m,\lambda}, T)$$
$$\simeq \bigoplus_{\lambda|n} H^0(\mathbb{Q}_{m,\lambda}, A^*),$$

where λ runs through all places of $\mathbb{Q}_m$ above prime divisors of n. Since we assume $n \in \mathcal{N}_{N+2N'}(\Sigma; T)$, for any place λ of $\mathbb{Q}_m$ dividing n, there exists an integer $M_\lambda \geq N + 2N'$ such that

$$H^0(\mathbb{Q}_{m,\lambda}, A^*) \simeq \mathbb{Z}_p/p^{M_\ell}\mathbb{Z}_p.$$

So, the exact sequence (13) and the choice of N' imply that we have a decomposition

$$H^2(j_{n*}T)[p^{N+2N'}] \simeq L_0 \oplus L_1,$$

where L_0 is an abelian group annihilated by $p^{N'}$, and L_1 is a free $\mathbb{Z}/p^{N+2N'}\mathbb{Z}$-module. In particular,

$$p^{N'} H^2(j_{n*}T)[p^{N+2N'}] = p^{N'} L_1$$

is a free $\mathbb{Z}/p^{N+N'}\mathbb{Z}$-module. Hence the right vertical map $\bar{P}$ in the diagram (12) is injective, and this completes the proof. □

By Lemma 8.2, the map $\beta_1|_{\mathcal{H}_1(n)}$ factors through $\mathcal{H}_0(n)$. Namely, there exist a unique map $\beta_0 = (\beta_{j,0})_{j=1}^h \colon \mathcal{H}_0(n) \longrightarrow R_N^h$ which makes the diagram

$$\begin{tikzcd}
\mathcal{H}_1(n) \arrow[r, "\beta_1"] \arrow[d] & R_{N+N'} \arrow[d, "\bmod p^N"] \\
\mathcal{H}_0(n) \arrow[r, dashed, "\beta_0"'] & R_N
\end{tikzcd}$$

commute, where the left vertical arrow is the natural map. Summary, we obtain the following lemma.

Lemma 8.3. *Let n, α_0 and β_0 be as above. Then, the diagram*

$$\begin{tikzcd}
\mathcal{H}_0(n) \arrow[r] \arrow[d, "\beta_0"'] & J_0 \arrow[d, "\alpha_0"] \\
R^h_{m,N} \arrow[r, "\bar{f}_N"'] & R^h_{m,N}
\end{tikzcd}$$

commutes, where $\bar{f}_N$ is the map induced by f.

8.2. **Analogue of Kurihara's element.** The arguments in the rest of the proof of Theorem 2.4 are similar to those in [Oh2] §§7.2–7.4. In this subsection, as in [Oh2] §7.2, we shall introduce elements $x(\nu; q) \in \mathcal{H}_2(q\nu)$ which are analogues of Kurihara's elements, and which become a key of the proof of Theorem 2.4. In the present and the next subsection, for simplicity, we put

$$\begin{aligned}(-)_t^{s,\ell} &:= (-)_{m,N+tN'}^{s,\ell}\\ \phi_t^{\ell} &:= \phi_{m,N+tN',\Phi^*}^{\ell}\\ \kappa_t(n) &:= \kappa_{m,N+tN'}(n;\mathbf{c}')\end{aligned}$$

Let $q\nu \in \mathcal{N}_N^{\tau}(\Sigma;T)_{\mathbb{Z}_p}$ and assume that $q\nu$ is *well-ordered*. Suppose that for each prime number ℓ dividing ν, an element $w_\ell \in R_{m,N+2N'}$ is given. (Later, we shall choose $q\nu$ and $\{w_\ell\}_{\ell|\nu}$ explicitly, but we take arbitrary elements here.) For any $e \in \mathbb{Z}_{\geq 0}$ dividing ν, we define $w_e := \prod_{\ell|e} w_\ell$. Since we have $\kappa_2(q\nu/e) \in \mathcal{H}_2(q\nu)$ for any positive divisor e of ν, we can define the element $x(\nu;q)$ by

$$x(\nu;q) := \prod_{e|\nu} w_e \otimes \kappa_2(q\nu/e) \in \mathcal{H}_2(q\nu).$$

For any $t \in \{0,1,2\}$, we denote by $x_t(\nu;q)$ the image of $x(\nu;q)$ by the natural homomorphism

$$H^1(\mathbb{Q}_m, T/p^{N+2N'}T) \longrightarrow H^1(\mathbb{Q}_m, T/p^{N+tN'}T).$$

The following proposition immediately follow from Proposition 4.17 and Proposition 4.19.

Proposition 8.4 (cf. [Ku] Proposition 6.1)**.** *Let $q\nu$ be an integer which is decomposed into the square-free product of some prime numbers contained in P. We assume that $q\nu$ is well-ordered. Let $t \in \{0,1,2\}$.*

(i) *Let ℓ be a prime number dividing ν. Then, we have*

$$(x_t(\nu;q))_t^{\ell,s} = \phi_t^{\ell}\left(x_t(\nu/\ell;q)\right).$$

(ii) *Let ℓ be a prime number dividing ν. Then, we have*

$$\phi_t^{\ell}\left(x_t(\nu;q)\right) = w_\ell \cdot \phi_t^{\ell}\left(x_t(\nu/\ell;q)\right).$$

Here, let us take q, ν, and $\{w_\ell\}_{\ell|\nu}$. First, let us take a prime number q as follows. Fix a non-zero element $\delta_A \in \mathrm{ann}_\Lambda(X_{\mathrm{fin}})$. Recall that for each integer r with $1 \leq r \leq h$, we have fixed a prime number $q_r \in P_{n_r}$ in the previous subsection. We put

$$Q := \prod_{r=1}^{h} q_r \in \mathcal{N}_N.$$

We fix a homomorphism $\varphi\colon \mathbb{H}^1_\Sigma(T) \longrightarrow \Lambda$ of Λ-modules satisfying $\varphi(\mathbf{c}(1)) \neq 0$. Note that we have

$$\mathbf{c}(1) = \mathbf{c}'(1) := (c'_m(1))_{m\geq 0} \in \mathbb{H}^1_\Sigma(T).$$

By the definition of $I_\varphi(\mathbf{c})$, it holds that

$$\varphi(\mathbf{c}(1)) \cdot \Lambda = \det(M) \cdot I_\varphi(\mathbf{c}),$$

where $M \in M_h(\Lambda)$ is the matrix defined in the previous subsection. We take an arbitrary element $\delta_\varphi \in I_\varphi(\mathbf{c})$. If necessary, we replace φ with $a\varphi$ for some $a \in \Lambda$, and we may assume that we have $\varphi(\mathbf{c}(1)) = \delta_\varphi \det(M)$. Let

$$\bar{\varphi}\colon \mathbb{H}^1_\Sigma(T) \otimes_\Lambda R_{m,N} \longrightarrow R_{m,N}$$

be the homomorphism induced by φ. Recall that in the proof of Theorem 5.5, we denote by $\mathcal{NH}$ the image of the natural homomorphism

$$\mathbb{H}^1_\Sigma(T) \otimes_\Lambda R_{m,N} \longrightarrow H^1_{\text{ét}}(\mathcal{O}_{\mathbb{Q}_m,\Sigma}, T/p^N T).$$

By the similar argument to that in Lemma 5.8, there exists an $R_{m,N}$-linear map $\psi\colon \mathcal{NH} \longrightarrow R_{m,N}$ which makes the diagram

$$\begin{tikzcd} \mathbb{H}^1_\Sigma(T) \otimes_\Lambda R_{m,N} \arrow[r, "\delta_A\bar{\varphi}"] \arrow[d] & R_{m,N} \\ \mathcal{NH} \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

commute. By Proposition 7.6, we can take a prime number q satisfying the following two conditions:

(q1) $q \in P_{n_1} \setminus \{q_{n_1}\}$;
(q2) $\mathcal{NH}$ is contained in the kernel of $(-)_0^{q,s}$, and for all $x \in \mathcal{NH}$, we have

$$\phi_0^q(x) = \psi(x).$$

In particular, we have

$$\begin{aligned} \bar{\phi}^q(\text{the image of } \bar{\mathbf{c}}(1)) &= \psi(\text{the image of } c_m(1)) \\ &= \delta\bar{\varphi}(\text{the image of } c_m(1)) \\ &= \delta \det(\bar{M}), \end{aligned}$$

where $\bar{M} \in M_h(R_{m,N})$ is the image of M.

Next, let us take ν and $\{w_\ell\}_{\ell|\nu}$. First, we consider $\beta_{m_1,1}\colon \mathcal{H}_1(Qq) \longrightarrow R_N$. By Proposition 7.6, we can take a prime number ℓ_2 and an element $b_2 \in \mathcal{H}_0(q_{n_2}\ell_2)$ satisfying the following conditions.

- The prime number ℓ_2 splits completely in $\mathbb{Q}_m(\mu_q)/\mathbb{Q}$, and $\ell_2 \in P_{n_2} \setminus \{q_{n_2}\}$.
- For all $x \in \mathcal{H}_0(qQ)$, we have $\phi_0^{\ell_2}(x) = \beta_{m_1,0}(x)$.
- We have $(b_2)_2^{\ell_j,s} = 1$ and $(b_2)_2^{q_{n_j},s} = -1$.

Then, we put $\nu_1 := 1$, and $w_{\ell_2} := \phi_2^{\ell_j}(b_2) \in R_{m,N+2N'}$.

If $i = 1$, we put $\nu := \nu_1 = 1$, and $x(\nu; q) = x(1; q) = \kappa_2(q)$.

Suppose $i \geq 2$. In order to take ν and $\{w_\ell\}_{\ell|\nu}$, we choose prime numbers ℓ_j for any integer j with $2 \leq j \leq i+1$ by induction on j as follows. Let j be an integer satisfying $2 < j \leq i+1$, and suppose that we have chosen distinct prime numbers $\ell_1, \dots, \ell_{j-1}$ contained in P such that $\ell_{j'}$ splits completely in $\mathbb{Q}_m(\mu_{q\nu_{j'-1}})/\mathbb{Q}$ for any $2 \leq j' \leq j-1$, where we put $\nu_{j'-1} := \prod_{r=2}^{j'-1} \ell_r$. Let us consider the R_m-linear homomorphism

$$\beta_{m_{j-1},0}\colon \mathcal{H}_0(Qq\nu_{j-1}) \longrightarrow R_{m,N}.$$

Applying Proposition 7.6, we can take a prime number ℓ_j which splits completely in $\mathbb{Q}_m(\mu_{q\nu_{j-1}})/\mathbb{Q}$, and satisfies the following conditions:

(x1) $\ell_j \in P_{n_j} \setminus \{q_{n_j}\}$;

(x2) There exists an element $b_j \in \mathcal{H}_2(q_{n_j}\ell_j)$ satisfying the following conditions.

(x2.1) We have $(b_j)_2^{\ell_j,s} = 1$ and $(b_j)_2^{q_{n_j},s} = -1$.

(x2.2) We have $\phi_2^{\ell_{j'}}(b_j) = 0$ for each j' satisfying $2 \leq j' \leq j-1$.

(x3) $\phi_0^{\ell_j}(x) = \beta_{m_{j-1},0}(x)$ for any $x \in \mathcal{H}_0(Qq\nu_{j-1})$.

Thus, we have taken $\ell_2, \ldots, \ell_{i+1}$, and we put $\nu := \nu_i = \prod_{j=2}^{i} \ell_j \in \mathcal{N}_N^{\tau}(\Sigma; T)_{\mathcal{O}}$. For each j with $2 \leq j \leq i$, we put

$$w_{\ell_j} := \phi_2^{\ell_j}(b_j) \in R_{m,N+2N'},$$

and we obtain $x(\nu; q) \in \mathcal{H}_2(q\nu)$. Note that $q\nu$ is *well-ordered.*

8.3. **Computation of the minors.** In this subsection, we observe two homomorphism α_i and β_i by using $x_{\nu,q}$, and describe the image of $\det(M_i)$ in $R_{m,N}$. (The goal of this subsection is Proposition 8.6.) Recall that we fix non-zero elements $\delta_A \in \mathrm{ann}_\Lambda(X_{\mathrm{fin}})$ and $\delta_\varphi \in I_\varphi(\mathbf{c})$. In order to compute the image of $\det(M_i)$, we need the following lemma.

Lemma 8.5 (cf. [Ku] Lemma 10.2)**.** *Suppose $i \geq 2$. Then,*

(i) $\beta_{m_{j-1},0}(x_0(\nu; q)) = 0$ *for all j with $2 \leq j \leq i$.*

(ii) $\bar{\alpha}_{j,0}(x_0(\nu; q)) = 0$ *for any $j \neq n_1, \ldots, n_i$.*

Proof. Let us prove the first assertion. Here, for any $a \in \mathcal{H}_2(Qq\nu)$, we denote the image of a in $\mathcal{H}_0(Qq\nu)$ by $\bar{a}$. Let j be an integer satisfying $2 \leq j \leq i$, and define an element $y_j \in \mathcal{H}_2(Qq\nu)$ by

$$y_j := x(\nu; q) - \sum_{j'=j}^{i} \phi_2^{\ell_{j'}}(x(\nu/\ell_{j'}; q)) \cdot b_{j'}.$$

Note that by the diagram (11) and the condition (x2) for the elements b_j, we have

$$\beta_{m_{j-1},0}(x_0(\nu; q)) = \beta_{m_{j-1},0}(\bar{y}_j).$$

(Recall the construction of the map $\beta_0 = (\beta_{j,0})_{j=1}^{h}$.) Then, by the similar arguments to that in the proof of [Oh2] Lemma 7.6, we obtain

$$\beta_{m_{j-1},0}(\bar{y}_j) = \phi_0^{\ell_j}(\bar{y}_j) = 0,$$

and this completes the proof of the assertion (i). Since $x_0(\nu; q)$ belongs to $\mathcal{H}_0(q\nu)$, the assertion (ii) of this lemma is clear. $\square$

By the similar arguments to those in the proof of [Oh2] Proposition 7.7, we can deduce the following proposition from the above lemmas.

Proposition 8.6 (cf. [Ku] pp.763–764 and [Oh2] Proposition 7.7)**.** *The following equalities in $R_{m,N}$ hold.*

(i) *We have*

$$\det(M) \cdot \phi_0^{\ell_2}(x_0(1; q)) = \pm \delta_A \det(M_1) \cdot \bar{\varphi}(\bar{\mathbf{c}}(1)),$$

where $\bar{\varphi} \colon \mathbb{H}^1_\Sigma(T) \otimes_\Lambda R_{m,N} \longrightarrow R_{m,N}$ is the homomorphism induced by φ.

(ii) *Assume $i \geq 2$. Then, we have*

$$\det(M_{j-1}) \cdot \phi_0^{\ell_{j+1}}(x_0(\nu_j; q)) = \pm \det(M_j) \cdot \phi_0^{\ell_j}(x_0(\nu_{j-1}; q))$$

for any integer j with $2 \leq j \leq i$.

The signs $\pm$ in (i) *and* (ii) *do not depend on m.*

Proof. All steps in arguments of the proof of [Oh2] Proposition 7.7 works in our setting. Here, we only show the assertion (ii). (Note that the arguments in the proof of the assertion (ii) contains an essence how to use Kurihara's elements in the computation of the minors of the matrix M.)

We assume $i \geq 2$. We denote by $\mathbf{e}^{(m)} := (e_j^{(m)})_{j=1}^h$ be the standard $R_{m,N}$-basis of $R_{m,N}^h$. For each j satisfying $1 \leq j \leq i$ we put

$$\mathbf{x}^{(j)} := \beta_0(x_0(\nu_j; q)) \in R_{m,N}^h;$$
$$\mathbf{y}^{(j)} := \alpha_0(x_0(\nu_j; q)) \in R_{m,N}^h,$$

and regard them as column vectors. By the commutative diagram in Lemma 8.3, we have $\mathbf{y}^{(j)} = M\mathbf{x}^{(j)}$ in $R_{m,N}^h$.

It is sufficient to prove the assertion when $j = i$. We write $\mathbf{x} = \mathbf{x}^{(i)}$ and $\mathbf{y} = \mathbf{y}^{(i)}$. Let $\mathbf{x}' \in R_N^{h-i+1}$ be the vector obtained from $\mathbf{x}$ by eliminating the m_k-th rows for $k = 1, \ldots, i-1$, and $\mathbf{y}'$ the vector obtained from $\mathbf{y}$ by eliminating the $n_{k'}$-th rows for $k' = 1, \ldots, i-1$. By Lemma 8.5 (i), we have $\mathbf{y}' = M_{i-1}\mathbf{x}'$. We assume the m_i'-th component of $\mathbf{x}'$ corresponds to the m_i-th component of $\mathbf{x}$, and the n_i'-th component of $\mathbf{y}'$ corresponds to the n_i-th component of $\mathbf{y}$. By Lemma 8.5 (ii) and Proposition 8.4 (ii), we have

$$\mathbf{y}' = \phi_0^{\ell_i}(x_0(\nu_{i-1}; q)) {e'}_{n_i'}^{(m)},$$

where $({\mathbf{e}'}_i^{(m)})_{i=1}^{h-i+1}$ denotes the standard basis of $R_{m,N}^{h-i+1}$. Let $\widetilde{M}_{i-1}$ be the matrix of cofactors of M_{i-1}. Multiplying the both sides of

$$\mathbf{y}' = M_{i-1}\mathbf{x}'$$

by $\widetilde{M}_{i-1}$, and comparing the m_i'-th components, we obtain

$$(-1)^{n_i'+m_i'} \det(M_i) \cdot \phi_0^{\ell_i}(x_0(\nu_{i-1}; q)) = \det(M_{i-1}) \cdot \beta_{m_i,0}(x_0(\nu; q)).$$

By condition (x3) for ℓ_{i+1}, we have

$$\beta_{m_i,0}(x_0(\nu; q)) = \phi_0^{\ell_{i+1}}(x_0(\nu; q)).$$

This completes the proof. □

8.4. **Proof of the theorem.** Now let us complete the proof of Theorem 2.4 by the similar arguments to that in [Oh2] §7.4. Fix a strictly increasing sequence $\{N_m\}_{m \geq 0} \subseteq \mathbb{Z}$ satisfying $N_m > m$ for any $m \in \mathbb{Z}_{\geq 0}$. In this subsection, we vary m, and denote the element

$$\phi_0^{\ell_{j+1}}(x_0(\nu_j; q)) \in R_{m,N_m} = (\mathbb{Z}/p^{N_m}\mathbb{Z})[\mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})]$$

defined in §6.2 by $\phi_0^{\ell_{j+1}}(x_0(\nu_j; q))_m$.

Proof of Theorem 2.4 . As in [Oh2] §7.4, by induction on j, we shall prove that the sequence $(\phi_0^{\ell_{j+1}}(x_0(\nu_j;q))_m)_{m\geq 0}$ converges to

$$\pm\delta_A\delta_\varphi \det(M_r) \in \Lambda$$

in the sense of [Oh2] Definition 7.8, for any integer j satisfying $0 \leq j \leq i$. First, let us show it when $j = 0$. By proposition 8.6 (i), we have

$$\det(M) \cdot \phi_0^{\ell_2}(x_0(1;q))_m = \pm\delta_A \det(M_1)\bar{\varphi}(\bar{\mathbf{c}}(1)) \in R_{m,N_m}.$$

The right hand side of this equality converges to $\pm\delta_A\delta_\varphi \det(M_1)\det(M)$, and since X is a torsion Λ-module, we have $\det(M) \neq 0$. Hence the sequence $(\phi_0^{\ell_2}(x_0(1;q))_m)_{m\geq 0}$ converges to $\delta_A \det(M_1)$. (Recall that the sign $\pm$ does not depend on m, see Proposition 8.6).

Next, we assume that the sequence $(\phi_0^{\ell_j}(x_0(\nu_{j-1};q))_m)_{m\geq 0}$ converges to

$$\pm\delta_A\delta_\varphi \det(M_{j-1}) \in \Lambda.$$

Then, the right hand side of the equality

$$\det(M_{r-1}) \cdot \phi_0^{\ell_{j+1}}(x_0(\nu_j;q))_m = \pm \det(M_r) \cdot \phi_0^{\ell_j}(x_0(\nu_{j-1};q))_m \in R_{m,N_m}$$

converges to $\pm\delta_A\delta_\varphi \det(M_j)\det(M_{j-1})$. Since we take $\det(M_{j-1}) \neq 0$, the sequence $(\bar{\phi}_0^{\ell_{j+1}}(x_0(\nu_j;q))_m)_{m\geq 0}$ converges to $\pm\delta_A\delta_\varphi \det(M_r)$.

By induction on j, the above arguments imply that the sequence $(\phi_0^{\ell_{i+1}}(x_0(\nu;q))_m)$ converges to $\pm\delta_A\delta_\varphi \det(M_i)$. Since

$$\phi_0^{\ell_{i+1}}(x_0(\nu;q))_m \in \mathfrak{C}_{i,m,N_m}(\mathbf{c}') = \mathfrak{C}_{i,m,N_m}(\mathbf{c})$$

for any $m \in \mathbb{Z}_{\geq 0}$, we have

$$\pm\delta_A\delta_\varphi \det(M_i) \in \mathfrak{C}_i(\mathbf{c}).$$

This completes the proof of theorem. □

9. Remarks on the ground level

In this section, we assume that $\mathcal{O} = \mathbb{Z}_p$ and T satisfies (C1)–(C7). Further, we assume the hypothesis (K1) and (K2) in Proposition 6.2. Fix an Euler system $\mathbf{c} = \{c_m(n)\}_{m,n} \in \mathrm{ES}_{\mathbb{Z}_p}(T,\Sigma)$ satisfying (NV). Let $\mathbf{c}' \in \mathrm{ES}'_{\mathbb{Z}_p}(T,\Sigma)$ be the modified Euler system corresponding to $\mathbf{c}$. We also assume the following hypothesis.

- (G1) The ideal $\mathrm{pind}(\mathbf{c})$ does not contained in $(\gamma-1)\Lambda$.
- (G2) The Kolyvagin system $\kappa(\mathbf{c}') = \{\kappa(\mathbf{c}')_n\}_n \in \mathrm{KS}_{\mathbb{Z}_p}(T,\Sigma)$ corresponding to $\mathbf{c}'$ is primitive in the sense of [MR] Definition 4.5.5. In particular, for any $i \in \mathbb{Z}_{\geq 0}$, we have $\partial_i(\kappa(\mathbf{c}');T) = \partial_i(T)$.

Here, we give some remarks on the assumptions (G1)–(G3). The assumption (G1) implies that the order of $X_0 := H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q},A^*)^\vee$ is finite. By Proposition 6.4, we have

$$\mathrm{Fitt}_{\mathcal{O},i}(X_0) = p^{\partial_i(\kappa(\mathbf{c}');T)}\mathbb{Z}_p \tag{14}$$

for any $i \in \mathbb{Z}_{\geq 0}$. (Note that since X_0 has finite order, we have $\partial_i(\kappa(\mathbf{c}');T) < \infty$ for any $i \in \mathbb{Z}_{\geq 0}$.)

By combining the "standard" Euler system arguments like [Ru1] §4 (without Kurihara's elements) and the equality (14), we obtain the following theorem.

Theorem 9.1. *Assume that* $(T, \mathbf{c})$ *satisfies the conditions* (C1)–(C7)*,* (K1)–(K2)*,* (NV) *and* (G1)–(G2)*. Let N be an integer satisfying $p^N > \# X_0$. Then, we have*

$$\mathrm{Fitt}_{R_{0,N},i}(X_0) = \mathfrak{C}_{i,0,N}(\mathbf{c})$$

for any $i \in \mathbb{Z}_{\geq 0}$.

Proof. Let N and i be as in the assertion of Theorem 9.1. By definition of the ideal $\mathfrak{C}_{0,N}(n; \mathbf{c}')$, for any well-ordered $n \in \mathcal{N}_N(\Sigma; T)_{\mathbb{Z}_p}$, we have

$$\mathfrak{C}_{0,N}(n; \mathbf{c}') \subseteq p^{\partial(\kappa(\mathbf{c}')_n; T)} R_{0,N}$$

where we put $\partial(\kappa(\mathbf{c}')_n; T) := \min \left\{ j \in \mathbb{Z}_{\geq 0} \mid \kappa_n \in p^j H_{\mathcal{F}(n)}(\mathbb{Q}, T/I_nT) \right\}$. So, by the equality (14), we obtain

$$\mathfrak{C}_{i,0,N}(\mathbf{c}) \subseteq p^{\partial_i(\kappa(\mathbf{c}'); T)} R_{0,N} = \mathrm{Fitt}_{R_{0,N},i}(X_0). \tag{15}$$

Let us show $\mathfrak{C}_{i,0,N}(\mathbf{c}) \supseteq \mathrm{Fitt}_{R_{0,N},i}(X_0)$. Here, we use similar notation to that in the previous section. Recall that $R_{0,N} = \mathbb{Z}/p^N\mathbb{Z}$ is a quotient of the discrete valuation ring $\mathbb{Z}_p$. Since X_0 is a finitely generated torsion $\mathbb{Z}_p$-module, we have an exact sequence

$$0 \longrightarrow \mathbb{Z}_p^h \xrightarrow{f} \mathbb{Z}_p^h \xrightarrow{g} X_0 \longrightarrow 0$$

of $\mathbb{Z}_p$-modules, where the matrix M_f associated with f for the standard basis $\mathbf{e} := (e_j)_{j=1}^h$ of $\mathbb{Z}_p^r$ is a diagonal matrix

$$M_f := \begin{pmatrix} p^{d_1} & & & \\ & p^{d_2} & & \\ & & \ddots & \\ & & & p^{d_h} \end{pmatrix}$$

satisfying $d_1 \geq d_2 \geq \cdots \geq d_h$. Note that by the equality (14), we have $\partial_i(\kappa(\mathbf{c}'); T)\mathbb{Z}_p = \sum_{j=i+1}^h d_j$ for any $i \in \mathbb{Z}_{\geq 0}$. (If $i > h$, we put $\sum_{j=i+1}^h d_j := 0$.) We apply $(-) \otimes_{\mathbb{Z}_p} R_{0,2N}$ to the above short exact sequence, and obtain the exact sequence

$$R_{0,2N}^h \xrightarrow{\bar{f}_{2N}} R_{0,2N}^h \xrightarrow{\bar{g}_{2N}} X_0 \longrightarrow 0.$$

For any integer j with $1 \leq j \leq h$, we denote the image of e_j in $R_{0,2N}^h$ by $\bar{e}_j$. For each integer j with $1 \leq j \leq h$, we define a set P'_j of prime numbers by

$$P'_j := \left\{ \ell \in \mathcal{P}_{3N}^{\tau}(\Sigma; T)_{\mathbb{Z}_p} \mid \mathrm{Ev}_{m,2N}^*(\mathrm{Frob}_{\ell_{\overline{\mathbb{Q}}}}) = \bar{g}(\bar{e}_j) \right\},$$

and put $P' := \coprod_{j=1}^h P'_j$. For each integer j with $1 \leq j \leq h$, we put

$$J_j := \bigoplus_{\ell \in P_j} H_s^1(\mathbb{Q}_\ell, T/p^NT)$$

and put $J := \bigoplus_{j=1}^h J_j$.

Definition 9.2. Let n be an integer which is a product of some prime numbers contained in P', and put $\Sigma_n := \Sigma \cup \mathrm{prime}(n)$. Let

$$j_n \colon \operatorname{Spec} \mathbb{Z}[1/\Sigma_n] \longrightarrow \operatorname{Spec} \mathbb{Z}[1/pn]$$

be the natural open immersion. We denote the image of the natural homomorphism

$$H^1_{\text{ét}}(\mathbb{Z}[1/pn], j_{n*}T/p^{3N}j_{n*}T) \longrightarrow H^1_{\text{ét}}(\mathbb{Z}[1/pn], j_{n*}T/p^{N}j_{n*}T)$$

by $\mathcal{H}(n)$. Then, we have a commutative diagram

$$\begin{tikzcd} \mathcal{H}(n) \arrow[r, "(\cdot)^s_n"] \arrow[d, "\beta"'] & J \arrow[d, "\alpha"] \\ R^h_{m,N} \arrow[r, "\bar{f}"'] & R^h_{m,N} \end{tikzcd}$$

commutes, where the top horizontal arrow $(\cdot)^s_n$ is the localization map, the bottom horizontal arrow $\bar{f}$ is the map induced by f, and the vertical arrows α (resp. β) are homomorphisms defined by the same manner to α_1 (resp. β_1) in the previous section. For each integer j with $1 \le i \le h$, let

$$\mathrm{pr}_j \colon R^h_{m,N} \longrightarrow R_{m,N}$$

be the j-th projector, and put $\alpha_j := \mathrm{pr}_j \circ \alpha$ and $\beta_j := \mathrm{pr}_j \circ \beta$.

By the commutative diagram in Definition 9.2 and the matrix representation of f, we have

$$\beta_j \circ (\cdot)^s_n = p^{d_j}\alpha_j \tag{16}$$

for any integer j with $1 \le i \le h$. We need the following lemma.

Lemma 9.3. *There exists a homomorphism*

$$\bar{\psi} \colon H^1_{\acute{e}t}(\mathbb{Z}[1/\Sigma], T/p^N T) \longrightarrow R_{0,N}$$

which satisfies

$$\bar{\psi}(c'_0(1))R_{0,N} = p^{\partial_0(T)}R_{0,N}.$$

Proof of 9.3. We denote by $H^2_{\text{ét}}(\mathbb{Z}[1/\Sigma], T)_{\mathrm{tor}}$ the maximal torsion $\mathbb{Z}_p$-submodule of $H^2_{\text{ét}}(\mathbb{Z}[1/\Sigma], T)$, and let M be an integer satisfying $p^M \geq \# H^2_{\text{ét}}(\mathbb{Z}[1/\Sigma], T)_{\mathrm{tor}}$. Then, we have a commutative diagram

$$\begin{tikzcd} H^1_{\text{ét}}(\mathbb{Z}[1/\Sigma], T) \otimes_{\mathbb{Z}_p} R_{0,N+M} \arrow[r, hook] \arrow[d] & H^1_{\text{ét}}(\mathbb{Z}[1/\Sigma], T/p^{N+M}T) \arrow[r] \arrow[d, "\mathrm{pr}_{N,N+M}"] & H^2_{\text{ét}}(\mathbb{Z}[1/\Sigma], T)_{\mathrm{tor}} \arrow[d, "\times p^M = 0"] \\ H^1_{\text{ét}}(\mathbb{Z}[1/\Sigma], T) \otimes_{\mathbb{Z}_p} R_{0,N} \arrow[r, hook] & H^1_{\text{ét}}(\mathbb{Z}[1/\Sigma], T/p^{N}T) \arrow[r] & H^2_{\text{ét}}(\mathbb{Z}[1/\Sigma], T)_{\mathrm{tor}} \end{tikzcd} \tag{17}$$

whose rows are exact. We denote the image of $c_0(1)$ in $H^1_{\text{ét}}(\mathbb{Z}[1/\Sigma], T/p^{N+M}T)$ (resp. $H^1_{\text{ét}}(\mathbb{Z}[1/\Sigma], T/p^N T)$) by $\bar{c}_0(1)_{N+M}$ (resp. $\bar{c}_0(1)_N$). By the assumption (G2), there exists an element $\bar{y} \in H^1_{\text{ét}}(\mathbb{Z}[1/\Sigma], T/p^{N+M}T)$ satisfying

$$\bar{c}_0(1)_{N+M} = p^{\partial_0(T)}\bar{y}.$$

Note that by the commutative diagram (17), we have

$$\mathrm{pr}_{N,N+M}(\bar{y}) \in H^1_{\text{ét}}(\mathbb{Z}[1/\Sigma], T) \otimes_{\mathbb{Z}_p} R_{0,N},$$

so we obtain

$$\bar{c}_0(1)_N \in p^{\partial_0(T)} H^1_{\text{ét}}(\mathbb{Z}[1/\Sigma], T) \otimes_{\mathbb{Z}_p} R_{0,N}.$$

By assumption (C1) and (C4), we have $H^0(\mathbb{Q}, A) = 0$, so $H^1_{\text{ét}}(\mathbb{Z}[1/\Sigma], T)$ is a free $\mathbb{Z}_p$-module (of finite rank). Hence, $H^1_{\text{ét}}(\mathbb{Z}[1/\Sigma], T) \otimes_{\mathbb{Z}_p} R_{0,N}$ is a free $R_{0,N}$-module of finite rank. This implies that there exists a homomorphism

$$\bar{\psi}_0 \colon H^1_{\text{ét}}(\mathbb{Z}[1/\Sigma], T) \otimes_{\mathbb{Z}_p} R_{0,N} \longrightarrow R_{0,(h+1)N}$$

which satisfies

$$\bar{\psi}_0(\bar{c}_0(1)_N) R_{0,N} = p^{\partial_0(T)} R_{0,N}.$$

Let $\bar{\psi} \colon H^1_{\text{ét}}(\mathbb{Z}[1/\Sigma], T/p^N T) \longrightarrow R_{0,N}$ be an extension of $\bar{\psi}_0$, then $\bar{\psi}$ is the map as desired. $\square$

By the similar arguments to that in §8.2 using Proposition 7.6, we obtain the following lemma.

Lemma 9.4. *There exists a well-ordered integer $n \in \mathcal{N}^{\tau}_{3N}(\Sigma; T)_{\mathbb{Z}_p}$ with a prime decomposition $n = \ell_1 \times \cdots \times \ell_{i+1}$ satisfying the following conditions.*

- *For any $j \in \mathbb{Z}$ satisfying $1 \le j \le i$, we have $\ell_j \in \mathcal{N}^{\tau}_{3N}(\Sigma; T)_{bbZ_p}$, and*
$$\mathrm{Ev}^*_{m,2N}(\mathrm{Frob}_{\ell_j}) = \bar{g}(\bar{e}_j),$$
where $\mathrm{Frob}_{\ell_j} \in G_{\mathbb{Q}}$ is an arithmetic Frobenius element at $\ell_{j,\overline{\mathbb{Q}}}/\ell_j$.
- *We have*
$$\phi^{\ell_j}_{m,N,\Phi^*}|_{H^1_{\acute{e}t}(\mathbb{Z}[1/\Sigma], T/p^N T)} = \bar{\psi}|_{H^1_{\acute{e}t}(\mathbb{Z}[1/\Sigma], T/p^N T)},$$
where $\bar{\psi}$ is the homomorphism in Lemma 9.3.
- *Let j be an integer satisfying $2 \le j \le i+1$. We put $n_{j-1} := \prod_{s=1}^{j-1} \ell_s$. Then, we have*
$$\phi^{\ell_j}_{m,N,\Phi^*}|_{\mathcal{H}(n_{j-1})} = \beta_{j-1}|_{\mathcal{H}(n_{j-1})}.$$

By Lemma 9.4 and the equality (16), we obtain

$$\begin{aligned} p^{\sum_{j=1}^{i-1} d_j} \beta_i(\kappa_{0,N}(n_i; \mathbf{c}')) R_{0,N} &= p^{\sum_{j=1}^{i-2} d_j} \beta_{i-1}(\kappa_{0,N}(n_{i-1}, \mathbf{c}')) R_{0,N} \\ &= \cdots \\ &= \bar{\psi}(\bar{\mathbf{c}}_0(1)) R_{0,N} = p^{\partial_0(T)} R_{0,N} \\ &= p^{\sum_{j=1}^{h} d_j} R_{0,N}. \end{aligned}$$

Since we assume that

$$\sum_{j=1}^{i-1} d_j \le \partial_0(T) = \# X_0 < N,$$

we have

$$\mathrm{Fitt}_{R_{0,N},i}(X_0) = p^{\sum_{j=i+1}^{h} d_j} R_{0,N} = \beta_i(\kappa_{0,N}(n_i; \mathbf{c}')) R_{0,N}.$$

This completes the proof of Theorem 9.1. $\square$

Corollary 9.5. *Let $(T, \mathbf{c})$ be as in Theorem 9.1. Then, for any $i \in \mathbb{Z}_{\geq 0}$, the following holds.*

(i) *The image of $\mathfrak{C}_i(\mathbf{c})$ in $R_{0,N}$ coincides with the ideal $\mathfrak{C}_{i,0,N}(\mathbf{c})$ for any positive integer N.*
(ii) *The image of $\mathfrak{C}_i(\mathbf{c})$ in $\mathbb{Z}_p = \Lambda/(\gamma - 1)\Lambda$ coincides with the ideal $\mathfrak{C}_{i,0}(\mathbf{c}) := \varprojlim_N \mathfrak{C}_{i,0,N}(\mathbf{c})$.*

Proof. Fix $i \in \mathbb{Z}_{\geq 0}$. Let m, N and N' be integers satisfying $p^{N'} \geq p^N > \# X_0$ and $N > m \geq 0$. Then, Lemma 5.3 (ii) implies that the image of $\mathfrak{C}_{i,m,N'}(\mathbf{c})$ in $R_{0,N'}$ coincides with $\mathfrak{C}_{i,0,N'}(\mathbf{c})$. The image of $\mathfrak{C}_{i,0,N'}(\mathbf{c})$ in $R_{0,N}$ coincides with $\mathfrak{C}_{i,0,N}(\mathbf{c})$ since Theorem 9.1 implies they are both equal to $\mathrm{Fitt}_{R_{0,N},i}(X_0)$. Hence the image of $\mathfrak{C}_{i,m,N'}(\mathbf{c})$ in $R_{0,N}$ coincides with $\mathfrak{C}_{i,0,N}(\mathbf{c})$. By varying m and N', Corollary 9.5 follows. $\square$

Note that the higher Fitting ideals determines the the cardinality of the minimal system of generators of a finitely presented module over a local ring. Precisely speaking, we can easily show the following lemma.

Lemma 9.6. *Let R be a commutative local ring, and M a finitely presented R-module. Then, the cardinality of the minimal system of generators of M is $i+1$ if and only if $\mathrm{Fitt}_{R,i}(M) \neq R$ and $\mathrm{Fitt}_{R,i+1}(M) = R$. (Note that by Nakayama's lemma, the cardinarity is independent of the choice of the minimal system of generators of M.)*

By Lemma 9.6 and the results in this section, we deduce the following corollary.

Corollary 9.7. *Let $(T, \mathbf{c})$ be as in Theorem 9.1. Further, we assume $\mathrm{rank}_{\mathbb{Z}_p} T^- = 1$. Let r be a non-negative integer. Then, the following two conditions are equivalent.*

(i) *The cardinality of the minimal system of generators of the Λ-module X is r.*
(ii) *$\mathfrak{C}_{r-1}(\mathbf{c}) \neq \Lambda$ and $\mathfrak{C}_r(\mathbf{c}) = \Lambda$.*

Proof. Put $R_0 := \Lambda/(\gamma - 1)\Lambda \simeq \mathbb{Z}_p$. Then, by Proposition 3.8, we have a natural isomorphism $X \otimes_\Lambda R_0 \simeq X_0$. So, by Nakayama's lemma, the cardinality of the minimal system of generators of the Λ-module X coincides with the cardinality of the minimal system of generators of the R_0-module X_0. By Theorem 9.1 and Corollary 9.5, the image of $\mathfrak{C}_i(\mathbf{c})$ in R_0 coincides with $\mathrm{Fitt}_{R_0,i}(X_0)$ for any $i \in \mathbb{Z}_{\geq 0}$. Therefore, Corollary 9.7 follows from Lemma 9.6. $\square$

Corollary 9.8. *Let $(T, \mathbf{c})$ be as in Theorem 9.1. We also assume that $\mathrm{rank}_{\mathbb{Z}_p} T^- = 1$. Further, we assume the condition* (MC) *and that X is a pseudo-null Λ-module. Then, the image of $\mathrm{Fitt}_{\Lambda,0}(X)$ in $R_0 = \Lambda/(\gamma - 1)\Lambda$ coincides with the image of $\mathrm{ann}_\Lambda(X)$.*

Proof. By our assumption, we have $I(\mathbf{c}) = 0$. So, by Theorem 2.4 (i), we have

$$\mathrm{Fitt}_{\Lambda,0}(X) \subseteq \mathrm{ann}_\Lambda(X) \subseteq \mathfrak{C}_0(\mathbf{c}).$$

On the other hand, by Theorem 9.1, we have

$$\mathrm{Fitt}_{R_0,0}(X_0) = \mathfrak{C}_{0,0}(\mathbf{c}).$$

Note that the image of $\mathrm{Fitt}_{\Lambda,0}(X)$ in R_0 coincides with $\mathrm{Fitt}_{R_0,0}(X_0)$, and by Corollary 9.5, image of $\mathfrak{C}_0(\mathbf{c})$ in R_0 coincides with $\mathfrak{C}_{0,0}(\mathbf{c})$. Hence we obtain Corollary 9.8. □

10. Examples

In this section, we study application of our results for two well-known Euler systems: circular units (for one dimensional cases) and Beilinson–Kato elements (for one dimensional cases). Recall that we have fixed embeddings $p_{\overline{Q}}\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and $\infty_{\overline{Q}}\colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ in §1. We regard $\overline{\mathbb{Q}}$ as a subfield of $\overline{\mathbb{Q}}_p$ and $\mathbb{C}$ by these embeddings. We fix an isomorphism $\iota\colon \overline{\mathbb{Q}}_p \xrightarrow{\simeq} \mathbb{C}$ of fields over $\overline{\mathbb{Q}}$.

10.1. **Circular units.** Let $K/\mathbb{Q}$ be an abelian extension satisfying $p \nmid [K:\mathbb{Q}]$ and unramified at p. We put $\Delta := \mathrm{Gal}(K\mathbb{Q}(\mu_p)/\mathbb{Q})$, and fix an even character $\chi \in \mathrm{Hom}(\Delta, \mathbb{Z}_p^\times)$ satisfying $\chi|_{G_{\mathbb{Q}_p}} \neq 1$. We define a $\mathbb{Z}_p[G_\mathbb{Q}]$-module T_χ by

$$T_\chi := \mathbb{Z}_p(1) \otimes \chi^{-1}.$$

For any $m \in \mathbb{Z}_{\geq 0}$, let $A_{m,\chi}$ be the χ-part of the p-Sylow subgroup of the ideal class group of $K\mathbb{Q}(\mu_{p^{m+1}})$, and define a Λ-module X_χ by $X_\chi := \varprojlim A_{m,\chi}$. Then, we have a natural isomorphism $X_\chi \simeq X(T_\chi)$ of Λ-modules. Note that $\mathbb{Z}_p(1) \otimes \chi^{-1}$ satisfies (C1)–(C7), and we have an Euler system $\mathbf{c}_\chi^{\mathrm{cyc}}$ of "circular units" for $\mathbb{Z}_p(1) \otimes \chi^{-1}$ satisfying (NV) and (MC). For details on $\mathbf{c}_\chi^{\mathrm{cyc}}$, see [Oh2] Proposition 4.5 and Remark 4.6. (For the Iwasawa main conjecture (MC) for this case, see [MW], [Ru1] or [Gre].) So we can apply Theorem 2.4 to the pair $(T_\chi, \mathbf{c}_\chi^{\mathrm{cyc}})$, and obtain $\mathrm{Fitt}_{\Lambda,i}(X_\chi) \sim \mathfrak{C}_i(\mathbf{c}_\chi^{\mathrm{cyc}})$ and

$$\mathrm{ann}_\Lambda(X_{\chi,\mathrm{fin}}) \cdot \mathrm{Fitt}_{\Lambda,i}(X'_\chi) \subseteq \mathfrak{C}_i(\mathbf{c}_\chi^{\mathrm{cyc}})$$

for any $i \geq 0$ ([Oh2] Theorem 1.1). Moreover, in this case, we have the following results:

- Note that the pair $(T_\chi, \mathbf{c}_\chi^{\mathrm{cyc}})$ also satisfies the conditions (K1)–(K2) and (G1)–(G2), so we can apply the results in the previous section. (Note that in this situation, the condition (G1) follows from the Leopoldt's conjecture for abelian fields, and (G2) follows from [MW] Theorem 1.10.1.) In particular, the cardinality of the minimal system of generators of the Λ-module X_χ is r if and only if we have $\mathfrak{C}_{r-1}(\mathbf{c}_\chi^{\mathrm{cyc}}) \neq \Lambda$ and $\mathfrak{C}_r(\mathbf{c}_\chi^{\mathrm{cyc}}) = \Lambda$.
- If the Λ-module X is pseudo-null, then we have

$$\mathrm{Fitt}_{\Lambda,0}(X_\chi) = \mathrm{ann}_\Lambda(X_\chi) = \mathfrak{C}_0(\mathbf{c}_\chi^{\mathrm{cyc}}).$$

For details of such results on circular units, see [Oh2]. Note that we also treat an arbitrary non-trivial character $\chi \in \mathrm{Hom}(\Delta, \overline{\mathbb{Q}}_p^\times)$ in [Oh2].

Remark 10.1 (Elliptic units)**.** For the classical Iwasawa module of ideal class groups associated with (not necessary cyclotomic) $\mathbb{Z}_p$-extension of certain abelian extension field of imaginary quadratic fields and Euler systems of elliptic units, we have similar results to the first and second assertions of Theorem 2.4. For details, see [Oh1] Theorem 1.1.

10.2. **Beilinson-Kato elements.** In this subsection, we study the Iwasawa module arising from elliptic modular forms by using the Euler systems of Belinson–Kato elements introduced by Kato.

10.2.1. *Basic setting.* Fix integers $k \in \mathbb{Z}_{\geq 2}$, $N \in \mathbb{Z}_{\geq 1}$ and an even Dirichlet character $\varepsilon\colon (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \overline{\mathbb{Q}}^\times$. In this paper, via the isomorphism $\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) \simeq (\mathbb{Z}/N\mathbb{Z})^\times$ induced by the global reciprocity map of the class field theory, we often regard ε as a character $\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) \longrightarrow \overline{\mathbb{Q}}^\times$.

We denote by $S_k(N)$ the $\mathbb{C}$-vector space of all cuspforms of weight k and level $\Gamma_1(N)$, and by $S_k(N,\varepsilon)$ the $\mathbb{C}$-subspace of $S_k(N)$ consisting of all forms with nebentypus ε. For any subring $R \subseteq \mathbb{C}$, we define the Hecke algebra $\mathfrak{h}_k(N;R) \subseteq \mathrm{End}_{\mathbb{C}}(S_k(N;\mathbb{C}))$ by

$$\mathfrak{h}_k(N;R) := R\left[T(\ell), S(\ell') \;\middle|\; \begin{array}{l} \ell\text{: any prime number,} \\ \ell'\text{: prime number not dividing } N \end{array}\right],$$

where $T(\ell)$ and $S(\ell')$ are operators given by

$$T(\ell) := \Gamma_1(N)\begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix}\Gamma_1(N),$$
$$S(\ell') := \Gamma_1(N)\begin{pmatrix} \ell' & 0 \\ 0 & \ell' \end{pmatrix}\Gamma_1(N).$$

We fix a normalized eigen newform $f = \sum_{n=1}^\infty a_n(f)q^n \in S_k(N;\varepsilon)$, and assume that f does not have complex multiplication. We put $F := \mathbb{Q}(\{a_n\}_{n\geq 1}, \mathrm{Im}\,\varepsilon) \subseteq \overline{\mathbb{Q}}$, and $p_F := p_{\overline{\mathbb{Q}}}|_F$. Let

$$\lambda_f\colon \mathfrak{h}_k(N;\mathbb{Q}_p) := \mathfrak{h}_k(N;\iota(\mathbb{Q}_p)) \longrightarrow F_{p_F};\; \begin{cases} T(\ell) \longmapsto a_\ell(f) \\ S(\ell') \longmapsto \varepsilon(\ell')\ell'^{k-2} \end{cases}$$

be the ring homomorphism corresponding to f.

10.2.2. *Construction of a lattice.* For the normalized eigen newform f fixed above, Deligne have constructed a two dimensional representation $V(f)$ of $G_{\mathbb{Q}}$ over F_{p_F}. (See [De].) In this subsection §10.2, by using the Euler systems of Beilinson–Kato elements

$$\left\{ {}_{c,d}z_{p^m n}(f,1,r',\xi,S) \in H^1(\mathbb{Q}(\mu_{p^m n}), T(f)) \right\}_{m\geq 0}$$

intoroduced in [Ka2], we study the Λ-module $X := X(T(f))$ arising from a special lattice $T(f)$ of $V(f)$. Here, let us recall the construction of the spcial lattice $T(f)$ briefly. (For details, see [Ka2] §8.3. The special lattice $T(f)$ is denoted by $V_{\mathcal{O}_{p_F}}(f)$ in [Ka2].) Note that by Remark 10.7 (ii) below, the choice of the lattice is not essential for our main results for modular forms, namely Thoerem 10.14. However, when we state Thoerem 10.14 precisely, the choice of the lattice $T(f)$ makes it easy to list up a class of Euler systems which we need.

In order to construct the special lattice $T(f)$, we need to recall the construction of $V(f)$. First, we assume $N \geq 4$. Let $Y_1(N)$ be the (open) modular curve over $\mathbb{Q}$, and $\lambda^{\mathrm{univ}}\colon E^{\mathrm{univ}} \longrightarrow Y_1(N)$ the universal elliptic curve. (Note that $Y_1(N)$ is an algebraic stack in general, but it is a scheme if $N \geq 4$.) We define a pro-p-sheaf $\mathcal{H}_p^1$ on $Y_1(N)_{\text{ét}}$ by

$$\mathcal{H}_p^1 := R^1\lambda_*^{\mathrm{univ}}\mathbb{Z}_p.$$

Then, we define a free $\mathbb{Z}_p$-module $V_{k,\mathbb{Z}_p}(Y_1(N))$ of finite rank with a continuous $G_{\mathbb{Q}}$-action by

$$V_{k,\mathbb{Z}_p}(Y_1(N)) := H^1_{\text{ét}}\left(Y_1(N)_{\overline{\mathbb{Q}}}, \mathrm{Sym}^{k-2}\mathcal{H}_p^1\right)$$

and put $V_{k,\mathbb{Q}_p}(Y_1(N)) := V_{k,\mathbb{Z}_p}(Y_1(N)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Note that by geometrically interpretation of Hecke actions via Hecke correspondence of modular curves, $V_{k,\mathbb{Q}_p}(Y_1(N))$ admits a natural $\mathfrak{h}_k(N;\mathbb{Q}_p)$-action which commutes with the action of $G_{\mathbb{Q}}$. We define the F_{p_F}-vector space $V(f)$ by

$$V(f) := V_{k,\mathbb{Q}_p}(Y_1(N)) \otimes_{\mathfrak{h}_k(N;\mathbb{Q}_p)} (F_{p_F}, \lambda_f),$$

and denote the image of $V_{k,\mathbb{Z}_p}(Y_1(N))$ in $V(f)$ by $T(f)$. Note that $V(f)$ is a two dimensional F_{p_F}-vector space with a continuous $G_{\mathbb{Q}}$-action ρ_f unramified outside pN, and satisfying

$$P(\mathrm{Fr}_\ell^{-1}|V(f);x) := \det{}_{F_{p_F}}(1 - \mathrm{Fr}_\ell^{-1}\cdot x|V(f)) = 1 - a_\ell x + \varepsilon(\ell)\ell^{k-1}x^2$$

for any prime number ℓ not dividing pN, where $\mathrm{Fr}_\ell^{-1} \in G_{\mathbb{Q}}$ is a geometric Frobenius element at ℓ.

Now let $N \leq 3$. In this case, we define $V(f)$ and $T(f)$ as follows. Let $L \in N\mathbb{Z}_{\geq 0}$ be an element satisfying $L \geq 4$, and regard $Y_1(N)$ as a quotient stack $G\backslash Y_1(L)$ of $Y_1(N)$ by a subgroup G of $\mathrm{GL}_2(\mathbb{Z}/L\mathbb{Z})$. Then, we define $V_{k,\mathbb{Q}_p}(Y_1(N)) := V_{k,\mathbb{Q}_p}(Y_1(L))^G$, and denote by $V_{k,\mathbb{Z}_p}(Y_1(N))$ the image of the trace map

$$N_G\colon V_{k,\mathbb{Q}_p}(Y_1(L)) \longrightarrow V_{k,\mathbb{Q}_p}(Y_1(N));\ v \longmapsto \sum_{g\in G} gv.$$

Note that $V_{k,\mathbb{Q}_p}(Y_1(N))$ and $V_{k,\mathbb{Q}_p}(Y_1(N))$ are independent of the choice of L. Then, we define the F_{p_F}-vector space $V(f)$ by

$$V(f) := V_{k,\mathbb{Q}_p}(Y_1(N)) \otimes_{\mathfrak{h}_k(N;\mathbb{Q}_p)} (F_{p_F}, \lambda_f),$$

and denote the image of $V_{k,\mathbb{Z}_p}(Y_1(N))$ in $V(f)$ by $T(f)$. Thus we obtain the two dimensional $G_{\mathbb{Q}}$-representation $(V(f), \rho_f)$ over F_{p_F} and its $G_{\mathbb{Q}}$-stable lattice $T(f)$ for any $N \in \mathbb{Z}_{\geq 0}$.

Definition 10.2. In this subsection, we study the $\mathcal{O}_{F_{p_F}}[G_{\mathbb{Q}}]$-module

$$(T, \rho_T) := \left(T(f)(k-1), \rho_f \otimes \chi_{\mathrm{cyc}}^{k-1}\right),$$

where χ_{cyc} denotes the cyclotomic character at p. In this subsection we denote by Σ a finite set of places of $\mathbb{Q}$ consisting of ∞, p and all finite places v dividing N.

Note that $\mathrm{rank}_{\mathcal{O}_{F_{p_F}}} T = 2$, and

$$\det \rho_T = \varepsilon^{-1}\chi_{\mathrm{cyc}}^{k-1}$$

is an odd character on $G_{\mathbb{Q}}$, so we have $\mathrm{rank}_{\mathcal{O}_{F_{p_F}}} T^- = 1$.

10.2.3. *Assumptions on T.* In this paper, we assume the followin condition on the pair (f, p):

(MF1) We have $F_{p_F} = \mathbb{Q}_p$.
(MF2) Under the assumption (MF1), the homomorphism
$$\rho_f \colon G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{\mathbb{Z}_p}(T(f)) \simeq \mathrm{GL}_2(\mathbb{Z}_p)$$
is surjective.
(MF3) We have
$$(\varepsilon^{-1} \cdot \chi_{\mathrm{cyc}}^{k-1})|_{G_{\mathbb{Q}_\infty}} \not\equiv (\varepsilon \cdot \chi_{\mathrm{cyc}}^{3-k})|_{G_{\mathbb{Q}_\infty}} \mod \mathfrak{m}_{F_{p_F}},$$
where $\mathfrak{m}_{F_{p_F}}$ is the maximal ideal of $\mathcal{O}_{F_{p_F}}$. (This condition implies that $T(f)$ is not "mod p selfdual".)
(MF4) The $\mathcal{O}_{F_{p_F}}[G_{\mathbb{Q}}]$-module $T := T(f)(k-1)$ satisfies the condition (C6).
(MF5) The $\mathcal{O}_{F_{p_F}}[G_{\mathbb{Q}}]$-module $T := T(f)(k-1)$ satisfies the condition (C7).

Here, let us check that $T := T(f)(k-1)$ satisfies the conditions (C1)–(C7) in §2 under the assumption (MF1)–(MF5). First, the condition (C4) for $T(f)$ clearly holds. The conditions (MF4) and (MF5) tautologically imply (C6) and (C7). The condition (MF3) implies
$$(\det \rho_T)|_{G_{\mathbb{Q}_\infty}} \not\equiv (\det \rho_T)|_{G_{\mathbb{Q}_\infty}} \mod \mathfrak{m}_{F_p},$$
so (C3) holds. Let us consider the assumption (MF2) and the conditions (C1), (C2) and (C5). We need the following lemma.

Lemma 10.3. *Under the assumption* (MF2)*, the subgroup $\rho_f(G_{\mathbb{Q}(\mu_{p^\infty})})$ of* $\mathrm{GL}_2(\mathbb{Z}_p)$ *contains* $\mathrm{SL}_2(\mathbb{Z}_p)$*. Note that the assertion of this lemma is independent of the choice of the basis $V(f)$ since* $\mathrm{SL}_2(\mathbb{Z}_p)$ *is a normal subgroup of* $\mathrm{GL}_2(\mathbb{Z}_p)$*.*

Proof. Since $G_{\mathbb{Q}(\mu_{p^\infty})}$ contains the commutator subgroup of $G_{\mathbb{Q}}$, the surjectivity of ρ_f implies that the image of $G_{\mathbb{Q}(\mu_{p^\infty})}$ by ρ_f contains the commutator subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$. It is known that the commutator subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ coincides with $\mathrm{SL}_2(\mathbb{Z}_p)$. (For the proof of this fact, instance, see [Ro] Proposition 2.1.4 and the proof of [Ro] Proposition 2.2.2. Note that the statement of [Ro] Proposition 2.2.2 treats only fields but similar proof works for the local rings.) □

By Lemma 10.3, the conditions (C1) and (C2) follow from the assumption (MF2). For the condition (C5), we need the following lemma.

Lemma 10.4. *The following hold.*

(i) *We have $H^1_{\mathrm{cont}}(\mathrm{GL}_2(\mathbb{Z}_p), (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus 2}) = 0$, where we regard $(\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus 2}$ as a discrete $\mathbb{Z}_p[\mathrm{GL}_2(\mathbb{Z}_p)]$-module by the standard matrix action of $\mathrm{GL}_2(\mathbb{Z}_p)$.*
(ii) *Let G be a closed subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ which acts on $\mathbb{Q}^{\oplus 2}$ irreducibly. Then the order of $H^1_{\mathrm{cont}}(G, (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus 2})$ is finite.*

Proof. Let us show the first assertion. Since $H^1_{\mathrm{cont}}(\mathrm{GL}_2(\mathbb{Z}_p), \mathbb{F}_p^2)$ is isomorphic to
$$\mathrm{Ker}\left(H^1_{\mathrm{cont}}(\mathrm{GL}_2(\mathbb{Z}_p), (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus 2}) \xrightarrow{\times p} H^1_{\mathrm{cont}}(\mathrm{GL}_2(\mathbb{Z}_p), (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus 2})\right),$$

it is sufficient to show $H^1_{\mathrm{cont}}(\mathrm{GL}_2(\mathbb{Z}_p), \mathbb{F}_p^2) = 0$. We denote the kernel of the natural projection $\mathrm{GL}_2(\mathbb{Z}_p) \longrightarrow \mathrm{GL}_2(\mathbb{F}_p)$ by G_0. Then, we have the inflation-restriction exact sequence

$$H^1(\mathrm{GL}_2(\mathbb{F}_p), \mathbb{F}_p^2) \longrightarrow H^1_{\mathrm{cont}}(\mathrm{GL}_2(\mathbb{Z}_p), \mathbb{F}_p^2) \longrightarrow \mathrm{Hom}_{\mathrm{cont}}(G_0, \mathbb{F}_p^2)^{\mathrm{GL}_2(\mathbb{F}_p)}. \tag{18}$$

First, we shall show $H^1(\mathrm{GL}_2(\mathbb{F}_p), \mathbb{F}_p^2) = 0$. Let us consider the Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(\mathrm{PGL}_2(\mathbb{F}_p), H^q(\mathbb{F}^\times, \mathbb{F}_p^2)) \Longrightarrow H^1(\mathrm{GL}_2(\mathbb{F}_p), \mathbb{F}_p^2). \tag{19}$$

Since the order of $\mathbb{F}^\times$ is prime to p, we have $H^q(\mathbb{F}^\times, \mathbb{F}_p^2) = 0$ for any $q \in \mathbb{Z}_{\geq 0}$. Clearly, we also have $H^0(\mathbb{F}^\times, \mathbb{F}_p^2) = 0$. (Recall that we always assume $p \neq 2$ in this paper.) Hence by the spectral sequence (19), we obtain

$$H^1(\mathrm{GL}_2(\mathbb{F}_p), \mathbb{F}_p^2) = 0.$$

Next, let us show $\mathrm{Hom}_{\mathrm{cont}}(G_0, \mathbb{F}_p^2)^{\mathrm{GL}_2(\mathbb{F}_p)} = 0$. Note that the homomorphism

$$G_0 \longrightarrow M_2(\mathbb{F}_p);\ \begin{pmatrix} 1+px & py \\ pz & 1+pw \end{pmatrix} \longmapsto \begin{pmatrix} x & y \\ z & w \end{pmatrix} \bmod p$$

induces an isomorphism $(G_0/[G_0, G_0]) \otimes_{\mathbb{Z}} \mathbb{F}_p \simeq M_2(\mathbb{F}_p)$. So, we obtain

$$\mathrm{Hom}_{\mathrm{cont}}(G_0, \mathbb{F}_p^2)^{\mathrm{GL}_2(\mathbb{F}_p)} \simeq \mathrm{Hom}_{\mathbb{F}_p[\mathrm{GL}_2(\mathbb{F}_p)]}(\mathrm{ad}_{\mathrm{GL}_2(\mathbb{F}_p)}, \mathbb{F}_p^2),$$

where $\mathrm{ad}_{\mathrm{GL}_2(\mathbb{F}_p)} := (M_2(\mathbb{F}_p), \mathrm{ad}_{\mathrm{GL}_2(\mathbb{F}_p)})$ is the adjoint representation, which is a representation defined by the conjugation of matrices. The set of the Jordan–Hölder constituents of the $\mathbb{F}_p[\mathrm{GL}_2(\mathbb{F}_p)]$-module $\mathrm{ad}_{\mathrm{GL}_2(\mathbb{F}_p)}$ consists of two elements: one element is the one dimensional (trivial) representation, and the other is $\mathfrak{sl}_2(\mathbb{F}_p)$, which is three dimensional. In particular, the two dimensional irreducible representation $\mathbb{F}_p^2$ (with the standard $\mathrm{GL}_2(\mathbb{F}_p)$-action) does not appear as a quotient of $\mathrm{ad}_{\mathrm{GL}_2(\mathbb{F}_p)}$. This implies

$$\mathrm{Hom}_{\mathbb{F}_p[\mathrm{GL}_2(\mathbb{F}_p)]}(\mathrm{ad}_{\mathrm{GL}_2(\mathbb{F}_p)}, \mathbb{F}_p^2) = 0.$$

By the exact sequence (18), this completes the proof Lemma 10.4 (i).

The second assertion follows from Theorem C.1.1 and the arguments in the proof of Corollary C.2.2 in [Ru2] Appendix C. For details, see loc. cit.. □

Corollary 10.5. *Let G be a subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ containing $\mathrm{SL}_2(\mathbb{Z}_p)$. We also assume that $Q := \mathrm{GL}_2(\mathbb{Z}_p)/G \simeq \mathbb{Z}_p$. Then, we have $H^1_{\mathrm{cont}}(G, (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus 2}) = 0$*

Proof. By Lemma 10.4 (i) and the Hochschild–Serre spectral sequence, we have an injection

$$H^1_{\mathrm{cont}}\left(G, (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus 2}\right)^Q \hookrightarrow H^2_{\mathrm{cont}}\left(Q, H^0\left(G, (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus 2}\right)\right).$$

Since G contains $\mathrm{SL}_2(\mathbb{Z}_p)$, we have $H^0_{\mathrm{cont}}\left(G, (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus 2}\right) = 0$. So, we obtain

$$H^1_{\mathrm{cont}}\left(G, (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus 2}\right)^Q = 0. \tag{20}$$

Fix a topological generator $q \in Q$, and consider the complete group ring

$$\Lambda_Q := \mathbb{Z}_p[[Q]] \xrightarrow{\simeq} \mathbb{Z}_p[[t]];\ q \longmapsto 1+t.$$

By Lemma 10.4 (ii), the length of the Λ_Q-module $H^1_{\mathrm{cont}}(G, (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus 2})$ is finite. On the other hand, by (20), the map

$$(q-1)\times\colon H^1_{\mathrm{cont}}\left(G, (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus 2}\right) \longrightarrow H^1_{\mathrm{cont}}\left(G, (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus 2}\right)$$

is injective (and also surjective). Hence by Nakayama's lemma, we conclude

$$H^1_{\mathrm{cont}}\left(G, (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus 2}\right) = 0.$$

This completes the proof. □

Corollary 10.6. *The hypothesis* (MF2) *implies* (C5).

Proof. Recall that we put $A := T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ Note that clearly we have

$$\mathrm{Ker}(\rho_T|_{G_{\mathbb{Q}_\infty}}) = \mathrm{Ker}(G_{\mathbb{Q}_\infty} \longrightarrow \mathrm{Aut}(A)).$$

Let Ω' be the subfield of Ω fixed by $\mathrm{Ker}(\rho_{T^*}|_{\mathbb{Q}_\infty})$. Then, $[\Omega' : \Omega]$ is prime to p, so we have $H^1(\Omega/\Omega', A) = 0$. By the inflation-restriction exact sequence, we obtain the isomorphism

$$H^1_{\mathrm{cont}}(\rho_T(G_{\mathbb{Q}_\infty}), A) \xrightarrow{\simeq} H^1(\Omega/\mathbb{Q}_\infty, A).$$

By the assumption (MF2), the images of $G_{\mathbb{Q}_\infty}$ by ρ_T contains $\mathrm{SL}_2(\mathbb{Z}_p)$. So we can apply Corollary 10.5 for $\rho_T(G_{\mathbb{Q}_\infty})$, and obtain

$$H^1(\Omega/\mathbb{Q}_\infty, A) = 0$$

The hypothesis (MF2) implies that ρ_{T^*} is also surjective. So by similar arguments, we obtain

$$H^1(\Omega/\mathbb{Q}_\infty, A) = 0.$$

Thus, we completes the corollary 10.6. □

Remark 10.7. Here, we give some remarks on the assumptions (MF1)–(MF5).

(i) The choice of the family $\{\ell_{\overline{\mathbb{Q}}}\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell\}_\ell$ of embeddings satisfying the assumption (Chb) ensures that for a given modular form f, there exists infinitely many prime number p satisfying the condition (MF1).

(ii) By the work of Ribet (cf. [Ri1] and [Ri2], generalization of the results by Serre [Se1] and Swinnerton-Dyer [S-D]), if the given eigen cuspform f does not have complex multiplication, the condition (MF2) holds for all but finitely many prime number p. Under the assumption (MF2), $T(f)$ is essentially the only one $G_{\mathbb{Q}}$-stable lattice of $V(f)$ in the following sense.

(*) *If T' is a $G_{\mathbb{Q}}$-stable lattice of $V(f)$, then there exists an element $a \in F_{p_F}^\times$ such that $T' = a \cdot T(f)$.*

For the proof of the fact (*), see the proof of Lemma 14.7 in [Ka2].

(iii) Skinner and Urban have completed the proof of Iwasawa main conjectures of elliptic modular forms under certain conditions which require $\varepsilon = 1$ and

$$\chi_{\mathrm{cyc}}^{k-2}|_{G_{\mathbb{Q}_\infty}} \equiv 1 \mod \mathfrak{m}_{F_{p_F}}$$

in our notation. (See [SU] Theorem 3.6.4.) Because of the assumption (MF3), we cannot treat the case they studied. In particular, we have to exclude the case when $T(f)$ is a Tate module of an elliptic curve defined over $\mathbb{Q}$.

(iv) Let us consider the condition (MF4). Here, we assume $p \geq 5$, and let I_p be the inertia subgroup of $G_{\mathbb{Q}_p}$. For any (f,p), the representation $(T^* \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p, \bar{\rho}_{T^*}|_{I_p})$ of I_p over $\overline{\mathbb{F}}_p$ is given either of the following two forms:

- If the representation $(T^* \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p, \bar{\rho}_{T^*}|_{I_p})$ is irreducible, then
$$\bar{\rho}_{T^*}|_{I_p} \simeq \begin{pmatrix} \phi & 0 \\ 0 & \phi' \end{pmatrix},$$
where ϕ and ϕ' are characters $I_p \longrightarrow \mathbb{F}_{p^2}^\times$ of level two in the sense of [Ed] §2.4.
- If the representation $(T^* \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p, \bar{\rho}_{T^*}|_{I_p})$ is reducible, then
$$\bar{\rho}_{T^*}|_{I_p} \simeq \begin{pmatrix} \bar{\chi}_{\mathrm{cyc}}^a & * \\ 0 & \bar{\chi}_{\mathrm{cyc}}^b \end{pmatrix},$$
where $\bar{\chi}_{\mathrm{cyc}}$ denotes the mod p cyclotomic character, and a and b are some integers.

(For details of this fact, see [Se2].) Hence there exists an integer i satisfying
$$H^0(\mathbb{Q}_\infty, A^* \otimes \bar{\chi}_{\mathrm{cyc}}^{-i}) = 0.$$
In particular, for such i, the pair $(f \otimes \omega^i, p)$ satisfies the condition (MF4), where
$$\omega\colon (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mathbb{Z}_p^\times \overset{\iota}{\hookrightarrow} \mathbb{C}^\times$$
is the Teichmüller character. In Example 10.8, we shall study more details on the condition (MF4) in certain special situations.

(v) Let N be the level of a fixed newform f, and p a prime number not dividing N. We denote by $\pi_f = \bigotimes_v' \pi_{f,v}$ the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ corresponding to f. For any prime number ℓ, we denote by $(r_{f,\ell}, N_{f,\ell})$ the Weil-Deligne representation of $W_{\mathbb{Q}_\ell}$ corresponding to $V(f)$. Suppose that for each prime divisor ℓ of N, one of the following conditions holds.

- We have $N_{f,\ell} = 0$, namely $\pi_{f,\ell}$ is not special, and the order of $r_{f,\ell}(I_{\mathbb{Q}_\ell})$ is prime to p.
- We have $\pi_{f,\ell} = \mathrm{St} \otimes (\varepsilon' \circ \det)$, where St is the Steinberg representation and $\varepsilon'\colon \mathbb{Q}_\ell^\times \longrightarrow \mathbb{C}^\times$ is a continuous character such that the order of $\varepsilon'|_{\mathbb{Z}_\ell^\times}$ is prime to p. Further, the action of $I_{\mathbb{Q}_\ell}$ on the mod $\mathfrak{m}_{F_{p_F}}$ reduction of $T(f)$ is non-trivial.

Then, Lemma 2.1 implies that the condition (MF5) is satisfied. Note that if $\pi_{f,\ell}$ is not special for any prime number ℓ dividing N, then the pair (f, p') satisfies the condition (MF5) for all but finitely many prime number p'.

Example 10.8. Here, we consider sufficient conditions for (MF4) in the following special situations (i) and (ii).

(i) Suppose that $p \nmid N$ and $p_F \nmid a_p(f)$, namely the p-adic representation $(V(f), \rho_f)$ of $G_\mathbb{Q}$ is crystalline and ordinary at p. Then, by Deligne's unpublished work, the mod p representation $(T \otimes_{\mathbb{Z}_p} \mathbb{F}_p, \bar{\rho}_{T^*}|_{G_{\mathbb{Q}_p}})$ is given by
$$\bar{\rho}_{T^*}|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \bar{\chi}_{\mathrm{cyc}} \cdot \lambda\left(\bar{\varepsilon}(p) \cdot \bar{a}_p(f)\right) & * \\ 0 & \bar{\chi}_{\mathrm{cyc}}^k \cdot \lambda\left(1/\bar{a}_p(f)\right) \end{pmatrix}$$

where we denote the image of an element $x \in \mathbb{Z}_p$ in $\mathbb{F}_p$ by $\bar{x}$, and for any $\bar{a} \in \mathbb{F}_p^\times$, we define an unramified character $\lambda(\bar{a})\colon G_{\mathbb{Q}_p} \longrightarrow \mathbb{F}_p^\times$ by

$$\lambda(\bar{a})(\mathrm{Fr}_p^{-1}) := \bar{a}.$$

(For the proof of this fact in under assumption $k \leq p+1$, see [Gro] Proposition 12.1.) So in particular, if we have $a_p(f) \not\equiv \varepsilon^{-1}(p) \mod \mathfrak{m}_{p_F}$, then the condition (MF4) holds.

(ii) Let us consider the non-ordinary cases. Assume that $p \nmid N$ and $p_F \mid a_p(f)$. We also assume that $k \leq p+1$. Then, by unpublished Fontaine's work, it holds that the representation $(T^* \otimes_{\mathbb{Z}_p} \mathbb{F}_{p^2}, \bar{\rho}_{T^*}|_{I_p})$ of I_p over $\mathbb{F}_{p^2}$ is given by

$$\bar{\rho}_{T^*}|_{I_p} \simeq \bar{\chi}_{\mathrm{cyc}} \otimes \begin{pmatrix} \psi^{1-k} & 0 \\ 0 & \psi'^{1-k} \end{pmatrix},$$

where ψ and $\psi' := \psi^p$ are the fundamental characters $I_p \longrightarrow \mathbb{F}_{p^2}^\times$ of level two. (For details on this fact, see [Ed] Theorem 2.6.) In particular, we have

$$H^0(\mathbb{Q}_\infty, A^*) \subseteq H^0(\mathbb{Q}_\infty \otimes \mathbb{Q}_p, A^*) = 0.$$

So, in this case, the condition (MF4) always holds.

Example 10.9. Let us consider the Ramanujan's delta

$$\Delta := q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n \in S_{12}(1,1).$$

Here, let $p = 13$. The condition (MF1), (MF2) and (MF5) for $(\Delta, 13)$ hold clearly. By the Swinnerton-Dyer's work, it is known that $(\Delta, 13)$ satisfies the condition (MF2). (See [S-D] Corollary of Theorem 4.) We have

$$\tau(13) = -577738 \equiv 8 \mod 13,$$

so by Remark 10.7 (iv), the pair $(\Delta, 13)$ satisfies the condition (MF4). Therefore, the pair $(\Delta, 13)$ satisfies the conditions (MF1)–(MF5).

10.2.4. *Kato's Euler systems.* In order to introduce Euler systems of Beilinson–Kato elements for $T := T(f)(k-1)$, we need to define an "index" set $\mathrm{Index}(f)$. We denote by $\mathrm{Index}(f)$ the set of 5-ples (r', ξ, S, c, d) consisting of the following data:

- r' is an integer satisfying $1 \leq r' \leq k-1$.
- (ξ, S) is either of the following two:
  - "ξ is a symbol $a(A)$, where $(a, A) \in \mathbb{Z} \times \mathbb{Z}_{\geq 1}$, and S is a non-empty finite set of prime numbers containing $\mathrm{prime}(pA)$, where $\mathrm{prime}(pA)$ is the set of prime divisors of N" or
  - "ξ is a matrix $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, and S is a non-empty finite set of prime numbers containing $\mathrm{prime}(pN)$".
- c and d are integers satisfying $\mathrm{prime}(cd) \cap S = \emptyset$, $(c, 6) = 1$ and $(d, N) = 1$. If $\xi \in \mathrm{SL}_2(\mathbb{Z})$, we also assume $c \equiv d \equiv 1 \mod N$.

Here, let us recall Kato's Euler systems for T. In [Ka2] Chapter I, by using Siegel units ${}_c g_{\alpha,\beta} \in \mathcal{O}(Y(N)_{\mathbb{Q}})$, Kato introduced elements

$$ {}_{c,d}z_{M,N} ;= \{ {}_c g_{1/M,0}, {}_d g_{0,1/N} \} \in K_2(Y(Mp^m, Np^m)) $$

called "zeta elements", which satisfy certain good norm relations with respect to the pushforward maps

$$ K_2(Y(M', N')_{\mathbb{Q}}) \longrightarrow K_2(Y(M, N)_{\mathbb{Q}}) $$

of K_2-groups induced by the natural projections for $M \mid M'$ and $N \mid N'$. Then, in [Ka2] Chapter II, by using the image of zeta elements by (the limit of) the Chern character map

$$ \varprojlim K_2(Y(Mp^m, Np^m)) \longrightarrow \varprojlim H^2_{\text{ét}}(Y(Mp^m, Np^m), \boldsymbol{\mu}_{p^m}^{\otimes 2}), $$

Kato constructed an Euler system

$$ z_J := \{ z_{J,m}(n) \in H(\mathbb{Q}_m(n), T) \} \in \mathrm{ES}_{\mathbb{Z}_p}(T, S) $$

for any $J = (r', \xi, S, c, d) \in \mathrm{Index}(f)$. For details of the construction of z_I, see [Ka2]. Note that our $z_{J,m}(n)$ is the image of the ω^0-component of

$$ {}_{c,d}z_{p^m n}(f, 1, r', \xi, S) \in H^1(\mathbb{Q}(\mu_{p^m n}), T) = \bigoplus_{i=0}^{p-2} H^1(\mathbb{Q}_m(\mu_n), T \otimes \omega^i) $$

(in Kato's notation) by the corestriction map $H^1(\mathbb{Q}_m(\mu_n), T) \longrightarrow H(\mathbb{Q}_m(n), T)$. We define a subset $\mathrm{Index}_+(f)$ of $\mathrm{Index}(f)$ by

$$ \left\{ J \in \mathrm{Index}(f) \;\middle|\; z_J \text{ satisfies the condition (NV) in §2} \right\}. $$

Then, by Kato, the set $\mathrm{Index}_+(f)$ is not empty. (See Theorem 12.5 and Theorem 12.6. in [Ka2].) Therefore, $X(T)$ is a *torsion* Λ-module.

Definition 10.10. We denote the ideal of Λ generated by

$$ \bigcup_{J \in \mathrm{Index}_+(f)} \mathrm{Ind}(z_J) \quad (\text{resp. } \bigcup_{J \in \mathrm{Index}_+(f)} I(z_J)) $$

by $\mathrm{Ind}(z; f)$ (resp. $I(z; f)$). Note that by definition, we have

$$ \mathrm{Ind}(z; f) = I(z; f) \cdot \mathrm{char}_\Lambda(X(T)). $$

We also denote the minimal principal ideal of Λ containing $\mathrm{Ind}(z; f)$ by $\mathrm{Ind}_0(z; f)$.

Now, we can state the Iwasawa main conjecture for the dual fine Selmer groups of modular forms.

Conjecture 10.11 (See Conjecture 12.10 in [Ka2])**.** *Let $T := T(f)(k-1)$. Here, we may not assume the hypothesis* (MF1)–(MF5)*. Then, it should be hold*

$$ \mathrm{char}_\Lambda(X(T)) = \mathrm{Ind}_0(z; f). $$

Remark 10.12. Note that "Conjecture 12.10 in [Ka2] for $T := T(f)(k-1)$" is equivalent to "Conjecture 10.11 for the modular forms $f \otimes \omega^i$ for all $i \in \mathbb{Z}$ with $0 \le i \le p-2$". As remarked in 10.7 (iii), Skinner and Urban proved Conjecture 10.11 under certain assumptions (cf. [SU] Theorem 3.6.4.), but we cannot treat their cases since our assumption (MF3) excludes the conditions on the weight of f required in [SU].

10.2.5. *Main results on modular forms.* In order to state our main results on modular forms, we need to introduce ideals $\mathfrak{C}_i(z; f)$, which arise from Kolyvagin derivatives of Kato's Euler systems.

Definition 10.13. For any $i \in \mathbb{Z}_{\geq 0}$, we denote by $\mathfrak{C}_i(z; f)$ the ideal of Λ generated by $\bigcup_{\mathrm{Index}_+(f)} \mathfrak{C}_i(z_J)$.

Now, we can state our main results on modular forms. Under the assumption (MF1)–(MF5), we can apply Theorem 2.4 for $T := T(f)(k-1)$.

Theorem 10.14. *Fix integers $k \in \mathbb{Z}_{\geq 2}$, $N \in \mathbb{Z}_{\geq 1}$ and an even Dirichlet character $\varepsilon\colon (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \overline{\mathbb{Q}}^\times$. Let p be an odd prime, and $f = \sum_{n=1}^\infty a_n(f) q^n \in S_k(N; \varepsilon)$ a normalized eigen newform. Assume that the pair (f, p) satisfies* (MF1)–(MF5). *We put $T := T(f)(k-1)$. Then, we have the following.*

(i) *For any $i \in \mathbb{Z}_{\geq 0}$, we have*
$$\mathrm{ann}_\Lambda(X_{\mathrm{fin}}) I(z; f) \cdot \mathrm{Fitt}_{\Lambda,i}(X'(T)) \subseteq \mathfrak{C}_i(z; f).$$
Recall that X_{fin} is the maximal pseudo-null Λ-submodule of $\mathbb{H}^2_\Sigma(T)$.

(ii) *For any $i \in \mathbb{Z}_{\geq 0}$, we have*
$$\mathfrak{C}_i(z; f) \prec \mathrm{Fitt}_{\Lambda,i}(X(T)).$$

In particular, if Conjecture 10.11 holds, then we have
$$\mathrm{Fitt}_{\Lambda,i}(X(T)) \sim \mathfrak{C}_i(z; f),$$
and the pseudo-isomorphism class of $X(T)$ is determined by $\mathfrak{C}_i(z; f)$.

References

[De] Deligne, P., *Formes modulaires et représentation ℓ-adiques*, LNM **179**, Springer (1971), 139–172.

[Ed] Edixhoven, B., *The weight in Serre's conjectures on modular forms*, Invent. Math. **109**, (1992), 563–594.

[Gre] Greither, C., *Class groups of abelian fields and the main conjecture*, Ann. Inst. Fourier (Grenoble) **42** (1992), 449–499.

[Gro] Gross, B. H., *A tameness criterion for Galois representations associated to modular forms (mod p)*, Duke Math. J. **61**, No. 2 (1990), 445–517.

[Ka1] Kato, K., *Euler systems, Iwasawa theory, and Selmer groups*, Kodai math. J. Volume **22**, Number **3** (1999), 313–372.

[Ka2] Kato, K., *p-adic Hodge theory and values of zeta functions of modular forms*, Astérisque **295** (2004), 117–290.

[Ku] Kurihara, M., *Refined Iwasawa theory and Kolyvagin systems of Gauss sum type*, Proceedings of the London Mathematical Society (3) **104** (2012), 728–769.

[MR] Mazur, B., and Rubin, K., *Kolyvagin systems*, Memoirs of the AMS Vol **168**, Number **799** (2004).

[MW] Mazur, B., and Wiles, A., *Class fields of abelian extension of* **Q**, Invent. math. **76** (1984), 179–330.

[Ne] J. Nekovár *Selmer complexes*, Astérisque **310** (2006).

[Oh1] Ohshita, T., *On higher Fitting ideals of Iwasawa modules of ideal class groups over imaginary quadratic fields and Euler systems of elliptic units*, to appear in Kyoto Journal of Mathematics.

[Oh2] Ohshita, T., *On the higher Fitting ideals of Iwasawa modules of ideal class groups over real abelian fields*, to appear in Journal of Number Theory **135** (2014), 67-138, http://dx.doi.org/10.1016/j.jnt.2013.08.002.

[P-R] Perrin-Riou, B., *Systèmes d'Euler p-adiques et théorie d'Iwasawa*, Ann. Inst. Fourier (Grenoble) **48** (1998), 1231–1307.

[Ri1] Ribrt, K., *On ℓ-adic representations attached to modular forms*, Invent. Math. **28** (1975), 245–275.

[Ri2] Ribrt, K., *On ℓ-adic representations attached to modular forms II*, Glasgow Math. J. **27** (1985), 185–194.

[Ru1] Rubin, K., The main conjecture, Appendix to *Cyclotomic fields I and II* by S. Lang, Graduate Texts in mathematics **121**, Springer-Verlag (1990), 397–419.

[Ru2] Rubin, K., *Euler systems*, Hermann Weyl lectures, Ann. of Math. Studies, vol. **147**, Princeton Univ. Press (2000).

[S-D] Swinnerton-Dyer, H. P. V., *On ℓ-adic representations and congruences for coefficients of modular forms*, International Summer School on Modular Functions; Antwerp, 1972, Lecture Notes in Mathematics **350** (1973), 1–55.

[Se1] Serre, J.-P., *Propriétés galoisiennes des points d'ordre fini des courbes elliptiques*, Invent. Math. **15** (1972), 259–331.

[Se2] Serre, J.-P., *Sur les represéntations modulaires de degré 2 de* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, Duke Math. J. **54**, No. 1 (1987), 179–230.

[SU] Skinner, C., Urban, E., *The Iwasawa main conjectures for* GL_2, to appear in Invent. Math..

[Ro] Rosenberg, J., *Algebraic K-Theory and Its Applications*, Graduate Texts in mathematics **147**, Springer-Verlag (1994).

[Ta1] TaTe, J., *On the conjectures of Birch and Swinnerton-Dyer and a geometric analog*, Sem. Bourbaki, 1965–66, n^{o} 306, Bemjamin, 1966.

[Ta2] Tate, J., *Relation between K_2 and Galois cohomology* Invent. Math. **36** (1976), 257–274.

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