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Singular Solutions to the Bethe Ansatz Equations and Rigged Configurations

By

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Abstract

We provide a conjecture for the following two quantities related with the spin-$\frac{1}{2}$ isotropic Heisenberg model defined over rings of even lengths: (i) the number of the solutions to the Bethe ansatz equations which correspond to non-zero Bethe vectors; (ii) the number of physical singular solutions of the Bethe ansatz equations in the sense of Nepomechie–Wang. The conjecture is based on a natural relationship between the solutions to the Bethe ansatz equations and the rigged configurations.

1 Introduction

The problem of constructing "physical states" to the isotropic Heisenberg model on a ring (see Section 2 for the definition) by the so-called Bethe ansatz method has been extensively studied in both physical and mathematical literatures for more than 80 years after the seminal work [2] by Hans Bethe published in 1931. There exists enormous number of papers concerning this problem, but even for the simplest case of the $\mathfrak{sl}_2$ spin-$\frac{1}{2}$ isotropic Heisenberg model, it is widely regarded that there are still remaining unclear aspects about the problem. The goal of the present paper is to draw attention to a mysterious connection between "*physical solutions*" of the Bethe ansatz equations and combinatorial objects called the *rigged configurations*.

For our purpose, we find it is convenient to use the so-called algebraic Bethe ansatz method introduced by Faddeev's school ([7], see also [17]). Basic procedure is as follows (see Section 2 for details). We start from the state $|0\rangle$ in which all spins pointing up. The state $|0\rangle$ is the obvious eigenvector of the Hamiltonian. Then we construct a certain creation operator $B(\lambda)$ depending on a parameter $\lambda \in \mathbb{C}$ to construct the other states as $B(\lambda_1)\cdots B(\lambda_\ell)|0\rangle$ which we call the Bethe vectors. The main observation is that if the the parameters $\lambda, \ldots, \lambda_\ell$ satisfy the Bethe ansatz equation (see equation (11) below), then the corresponding Bethe vector (if non-zero) is an eigenvector of the Hamiltonian.

However it is well known that the Bethe ansatz equations admit too many solutions and many solutions correspond to the zero Bethe vector (see [8] and references therein). Therefore one can state the main problem as follows: Describe a set of solutions to the Bethe ansatz equations which provide non-zero Bethe vectors after certain regularization if necessary.

There is a common assumption such that it is enough to consider solutions with pairwise distinct components. Although there are no rigorous proofs of the assumption, in the physical literature usually it is motivated by the *Pauli exclusion principle*. However even if we impose this assumption, still there is a very subtle problem. Indeed, if we discard all the solutions corresponding to the zero Bethe vector, then some of the eigenvectors are missing from the Bethe vectors.

Let us call the solutions of the Bethe ansatz equations corresponding to the zero Bethe vector the singular solutions and those corresponding to the non-zero Bethe vectors the regular solutions. Then our task is to find a set of singular solutions which provide non-zero Bethe vectors after certain regularizations. There are a large number of works which try to solve this problem (see for example [6, 26, 1]). Among them we are interested in a method to introduce a higher order correction to the Bethe vectors. In particular, recently Nepomechie–Wang [23] proposed an explicit criterion under which one can pick all the missing physical states from the singular solutions. Following [23] we call such solution *physical singular solution*. Their conjecture is verified up to length 14 systems by an extensive numerical computation [8].

The main observation of the present paper is that a combination of both the regular solutions and the physical singular solutions has a mysterious but natural correspondence with the combinatorial objects called the rigged configurations. The rigged configurations have been introduced by A. N. Kirillov and N. Reshetikhin [11] (see also [12], [15], [16]) as an application of the so-called *string conjecture* [2, 28]. They have a canonical bijection with the tensor products of crystal bases and known to possess a deep structure related with subtle properties of finite dimensional representations of the quantum affine algebras. Moreover, they provide a complete set of the action-angle variables of the box-ball system which is a prototypical example of ultradiscrete (or tropical) soliton systems [18].

Motivated by the correspondence between the rigged configurations and the solutions to the Bethe ansatz equations, we propose a conjecture for the total number of the physical solutions when the system size N is even (see Conjecture 11). Since the total number of eigenvectors is well-known, the conjecture in turn provides a total number of the regular solutions. Our conjecture have a perfect agreement with the numerical data provided in [8].

We remark that another method to construct Bethe vectors corresponding to the so-called *admissible solutions* to the Bethe ansatz equations has been developed in [22].

The organization of the present paper is as follows. In Section 2, we provide a necessary background about the algebraic Bethe ansatz as well as description of the Nepomechie–Wang's prescription. In Section 3, we provide the definition of the rigged configurations and state our main results. In Section 4 we discuss a subtlety which appears when the system size is odd.

In the case when the number of sites N and the spin ℓ for the spin $\frac{1}{2}$ isotropic Heisenberg model are both even, we state conjectural formulas for the number of solutions with pairwise distinct roots, the number of singular solutions and the number of physical singular solutions, denoted respectively by $\mathcal{N}$, $\mathcal{N}_s$ and $\mathcal{N}_{sp}$ in [8].

2 Algebraic Bethe ansatz analysis

The space of states $\mathfrak{H}_N$ and the Hamiltonian $\mathcal{H}_N$ of the spin-$\frac{1}{2}$ isotropic Heisenberg model on a length N chain with periodic boundary condition are

$$\begin{aligned}\mathfrak{H}_N &= \bigotimes_{j=1}^{N} V_j, \quad V_j \simeq \mathbb{C}^2,\\ \mathcal{H}_N &= \frac{J}{4}\sum_{k=1}^{N}(\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + \sigma_k^z\sigma_{k+1}^z - \mathbb{I}_N), \qquad \sigma_{N+1}^a = \sigma_1^a. \end{aligned} \tag{1}$$

Here σ^a $(a = x, y, z)$ are the Pauli matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

and the operators σ_k^a $(a = x, y, z)$ act on $\mathfrak{H}_N$ as

$$\sigma_k^a = I \otimes \cdots \otimes \underbrace{\sigma^a}_{k} \otimes \cdots \otimes I, \tag{3}$$

that is, they act non trivially only on the space V_k. Here I is the 2×2 identity matrix and $\mathbb{I}_N$ is the identity operator on the space of states; $\mathbb{I}_N = I^{\otimes N}$.

Our task is to diagonalize the Hamiltonian exactly. Instead of following the original arguments by Bethe, we use the formalism called the algebraic Bethe ansatz [17]. Let us denote the canonical basis vectors of $\mathbb{C}^2$ as

$$v_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4}$$

Then the vector

$$|0\rangle_N = v_+ \otimes \cdots \otimes v_+ \in \mathfrak{H}_N \tag{5}$$

is an eigenvector of the Hamiltonian. The basic idea of the algebraic Bethe ansatz is to construct the remaining eigenvectors by using certain creation operators $B(\lambda)$ ($\lambda \in \mathbb{C}$) acting on the state $|0\rangle_N$;

$$\Psi_N(\lambda_1, \ldots, \lambda_\ell) := B_N(\lambda_1) \cdots B_N(\lambda_\ell)|0\rangle_N. \tag{6}$$

We call such vectors the Bethe vectors. The definition of the operators $B_N(\lambda)$ is as follows. We introduce (2×2)-size matrix operator $L_k(\lambda)$

$$L_k(\lambda) = \begin{pmatrix} \lambda\mathbb{I}_N + \frac{i}{2}\sigma_k^z & \frac{i}{2}\sigma_k^- \\ \frac{i}{2}\sigma_k^+ & \lambda\mathbb{I}_N - \frac{i}{2}\sigma_k^z \end{pmatrix}, \tag{7}$$

where $\sigma_k^\pm = \sigma_k^x \pm i\sigma_k^y$. The operator $L_k(\lambda)$ acts on $\mathbb{C}^2 \otimes \mathfrak{H}_N$ where $\mathbb{C}^2$ is an auxiliary space representing 2 by 2 matrix in (7) and operators like σ_k^z act on $\mathfrak{H}_N$. Then define the transfer matrix $T_N(\lambda)$ by

$$T_N(\lambda) = L_N(\lambda) L_{N-1}(\lambda) \cdots L_1(\lambda) \tag{8}$$

and define the operators $A_N(\lambda), B_N(\lambda), C_N(\lambda)$ and $D_N(\lambda)$ by

$$T_N(\lambda) = \begin{pmatrix} A_N(\lambda) & B_N(\lambda) \\ C_N(\lambda) & D_N(\lambda) \end{pmatrix}. \tag{9}$$

Since we can show that $[B_N(\lambda_1), B_N(\lambda_2)] = 0$, we only need the set $\{\lambda_1, \ldots, \lambda_\ell\}$ modulo permutations to specify the Bethe vector.

The fundamental observation is as follows. The non-zero Bethe vector

$$\Psi_N(\lambda_1, \ldots, \lambda_\ell) = B_N(\lambda_1) \cdots B_N(\lambda_\ell)|0\rangle_N \tag{10}$$

is an eigenvector of the Hamiltonian if and only if the numbers $\lambda_1, \ldots, \lambda_\ell$ satisfy the following system of algebraic equations

$$\left(\frac{\lambda_k + \frac{i}{2}}{\lambda_k - \frac{i}{2}}\right)^N = \prod_{\substack{j=1 \\ j\neq k}}^{\ell} \frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i}, \qquad (k = 1, \cdots, \ell). \tag{11}$$

These equations are the celebrated Bethe ansatz equations which we denote by BAE(N, ℓ). In the following, we only consider the solutions $\{\lambda_1, \ldots, \lambda_\ell\}$ to the Bethe ansatz equations which have pairwise distinct components; $\lambda_i \neq \lambda_j$ if $i \neq j$. If a solution $\{\lambda_1, \ldots, \lambda_\ell\}$ of the Bethe ansatz equations corresponds to a non-zero Bethe vector, we call such solution **regular**.

Recall that the space of the states $\mathfrak{H}_N$ have natural action of the Lie algebra $\mathfrak{sl}_2$ such that

$$S^a := \frac{1}{2} \sum_{k=1}^{N} \sigma_k^a, \qquad (a = x, y, z). \tag{12}$$

Then we can show that the model possesses the $\mathfrak{sl}_2$ symmetry $[\mathcal{H}_N, S^a] = 0$. Hence we can simultaneously diagonalize the Hamiltonian $\mathcal{H}_N$ and the operators S^z and $S^2 = (S^x)^2 + (S^y)^2 + (S^z)^2$. Moreover, we can show that the Bethe vectors are highest weight vectors

$$S^+ \Psi_N(\lambda_1, \ldots, \lambda_\ell) = 0, \qquad S^\pm = S^x \pm i S^y \tag{13}$$

if the parameters $\lambda_1, \ldots, \lambda_\ell$ satisfy the Bethe ansatz equations (11). In this case, the vector $\Psi_N(\lambda_1, \ldots, \lambda_\ell)$ is the eigenvector of S^z with the eigenvalue $\frac{N}{2} - \ell$. Therefore we can obtain the other vectors by acting the lowering operator S^- successively on the Bethe vectors. We denote by $\mathbf{m}$ the irreducible $\mathfrak{sl}_2$-module of dimension m.

Remark 1. Therefore it is enough to consider the case $\ell \leq \frac{N}{2}$. However it is interesting to note the following conjecture. If $\ell > \frac{N}{2}$, then for any solution $\lambda_1, \ldots, \lambda_\ell$ to the corresponding Bethe ansatz equations, we have $B_N(\lambda_1) \cdots B_N(\lambda_\ell)|0\rangle_N = 0$. If we use the precise form of the operators $B_N(\lambda)$ we see that the corresponding vector vanishes non-trivially. ■

However it is well known that there is a very subtle problem about the procedure. Indeed, as already Bethe himself realized, the number of pairwise distinct solutions to the Bethe ansatz equations is too large than the actual number of the eigenvectors, and also some eigenvectors are missing from the Bethe vectors.

Example 2. Consider the case $N = 4$. We use the lexicographic ordering for the 16 basis vectors of $\mathfrak{H}_4$; $|++++\rangle$, $|+++-\rangle$, $|++-+\rangle$, $|++--\rangle$, $\cdots$, where we have used an abbreviated notation like

$$|+++-\rangle = v_+ \otimes v_+ \otimes v_+ \otimes v_-.$$

If $\ell = 0$, the vector $|0\rangle_4$ provides the highest weight vector of the representation **5**. If $\ell = 1$, the Bethe ansatz equation

$$\left(\frac{\lambda_1 + \frac{i}{2}}{\lambda_1 - \frac{i}{2}} \right)^4 = 1,$$

has the solutions $\lambda_1 = 0, \pm\frac{1}{2}$. Then we have

$$\begin{aligned} 8B_4(0)|0\rangle_4 &= (0,-1,1,0,-1,0,0,0,1,0,0,0,0,0,0,0)^t, \\ \frac{4}{1-i}B_4\left(\frac{1}{2}\right)|0\rangle_4 &= (0,1,i,0,-1,0,0,0,-i,0,0,0,0,0,0,0)^t, \\ \frac{4}{1+i}B_4\left(-\frac{1}{2}\right)|0\rangle_4 &= (0,1,-i,0,-1,0,0,0,i,0,0,0,0,0,0,0)^t. \end{aligned}$$

By suitable combinations of these vectors, we have three highest weight vectors corresponding to $\mathbf{3}^{\oplus 3}$.

Let us consider the case $\ell = 2$. Then the Bethe ansatz equations for this case is

$$\left(\frac{\lambda_1 + \frac{i}{2}}{\lambda_1 - \frac{i}{2}}\right)^4 = \frac{\lambda_1 - \lambda_2 + i}{\lambda_1 - \lambda_2 - i}, \qquad \left(\frac{\lambda_2 + \frac{i}{2}}{\lambda_2 - \frac{i}{2}}\right)^4 = \frac{\lambda_2 - \lambda_1 + i}{\lambda_2 - \lambda_1 - i}.$$

If we assume that all λ_j are distinct, we have the following four solutions;

$$\{\lambda_1, \lambda_2\} = \{\frac{i}{2}, -\frac{i}{2}\},\ \{-\frac{i}{2}, \frac{i}{2}\},\ \{\frac{1}{\sqrt{12}}, -\frac{1}{\sqrt{12}}\},\ \{-\frac{1}{\sqrt{12}}, \frac{1}{\sqrt{12}}\}.$$

We can confirm that

$$B_4\left(\frac{i}{2}\right)B_4\left(-\frac{i}{2}\right) = B_4\left(-\frac{i}{2}\right)B_4\left(\frac{i}{2}\right) = 0.$$

On the other hand, we have

$$\frac{27}{2}B_4\left(\frac{1}{\sqrt{12}}\right)B_4\left(-\frac{1}{\sqrt{12}}\right)|0\rangle_4 = (0,0,0,1,0,-2,1,0,0,1,-2,0,1,0,0,0)^t.$$

Since $B_N(\lambda)$ are commutative, the remaining solution provides the same vector. This vector gives the representation $\mathbf{1}$. ∎

Recall that we have the irreducible decomposition $\mathfrak{H}_4 = \mathbf{5} \oplus \mathbf{3}^{\oplus 3} \oplus \mathbf{1}^{\oplus 2}$. In particular, the Hamiltonian $\mathcal{H}_4$ has one more eigenvector

$$(0,0,0,1,0,0,-1,0,0,-1,0,0,1,0,0,0)^t$$

or, equivalently,

$$|++--\rangle - |+--+\rangle - |-++-\rangle + |--++\rangle.$$

The missing eigenvector, which should correspond to the solutions $\{i/2, -i/2\}$ and $\{-i/2, i/2\}$, can be found by a "refinement" of the Bethe ansatz method. Below we follow Nepomechie–Wang's arguments [23][1].

[1] In the case of $\ell = 2$, their result coincides with the result of [6], equation (26).

We call the following type of solutions **singular**:

$$\left\{\frac{i}{2}, -\frac{i}{2}, \lambda_3, \dots, \lambda_\ell\right\}. \tag{14}$$

Recall that if $\{\lambda_1, \dots, \lambda_\ell\}$ is a solution to the Bethe ansatz equations, we have

$$\mathcal{H}_N \Psi_N(\lambda_1, \dots, \lambda_\ell) = \mathcal{E}_{\lambda_1,\dots,\lambda_\ell} \Psi_N(\lambda_1, \dots, \lambda_\ell), \qquad \mathcal{E}_{\lambda_1,\dots,\lambda_\ell} := -\frac{J}{2}\sum_{j=1}^{\ell} \frac{1}{\lambda_j^2 + \frac{1}{4}}. \tag{15}$$

Therefore singular solutions correspond to divergent energy eigenvalues. For singular solutions, let us consider the following regularization

$$\lambda_1 = \frac{i}{2} + \epsilon + c\,\epsilon^N, \qquad \lambda_2 = -\frac{i}{2} + \epsilon. \tag{16}$$

We remark that the same regularization method was also noted in [1], equation (3.4). In the case of $N = 4$, we obtain the following result

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon^4} B_4\left(\frac{i}{2} + \epsilon + c\,\epsilon^4\right) B_4\left(-\frac{i}{2} + \epsilon\right) |0\rangle_4 = (0,0,0,2,0,0,-2,0,0,ic,0,0,2,0,0,0)^t.$$

If $c = 2i$, we obtain the correct eigenvector.

In general, Nepomechie–Wang's prescription is as follows. We start from the general singular solution (14). If a singular solution satisfies the relation

$$\left(-\prod_{j=3}^{\ell} \frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}\right)^N = 1, \tag{17}$$

we call **physical singular solution**. We define the number c as follows

$$c = 2i^{N+1} \prod_{j=3}^{\ell} \frac{\lambda_j + \frac{3i}{2}}{\lambda_j - \frac{i}{2}}. \tag{18}$$

According to [23],

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon^N} B_N\left(\frac{i}{2} + \epsilon + c\,\epsilon^N\right) B_N\left(-\frac{i}{2} + \epsilon\right) B_N(\lambda_3)\cdots B_N(\lambda_\ell)|0\rangle_N \tag{19}$$

gives a non-zero eigenvector of the Hamiltonian.

Then the main conjecture of [23] is as follows.

Conjecture 3. *Regular solutions and physical singular solutions of the Bethe ansatz equations provide all the highest weight vectors of* $\mathfrak{H}_N$. ■

In [8], this conjecture is verified up to $N = 14$ by an extensive numerical computation.

3 Physical singular solutions to the Bethe ansatz equations and rigged configurations

3.1 Rigged Configurations—$\mathfrak{sl}_2$-case

Rigged configurations have been introduced by the first author and N. Reshetikhin in the beginning of 80's of the last century as a consequence of application of the *String Conjecture* to the problem of counting the number of "*physically interesting solutions*" to the Bethe ansatz equations, [10]. In this Section we remind to the reader a definition and some basic result in the case of $\mathfrak{sl}_2$ Heisenberg model.

In a few words, a rigged configuration (in the case of $\mathfrak{sl}_2$) is a pair (ν, J), where ν is a partition, and J is a weakly increasing sequence of non-negative integer numbers of the length equals to the number of parts of a partition ν. A pair (ν, J) has to satisfy certain conditions depending on a type of the Heisenberg model we are interested in. The starting data is a collection of positive integers (spins) $\mu = (\mu_1, \ldots, \mu_N)$ and a partition $\eta = (\eta_1, \eta_2)$ $(\eta_1, \eta_2 \geq 0)$ such that $\eta_1 + \eta_2 = \sum_{j=1}^{r} \mu_j$. In the language of the Heisenberg model, μ specifies the shape of the tensor product of the space of states and η specifies the type of the Bethe vectors. For example, if we consider a Bethe vector $B_N(\lambda_1) \cdots B_N(\lambda_\ell)|0\rangle_N$ of the tensor product of spin-$\frac{1}{2}$ representations $\mathfrak{H}_N = (\mathbb{C}^2)^{\otimes N}$, we have $\mu = (\overbrace{1, 1, \cdots, 1}^{N})$ and $\eta = (\eta_1, \eta_2) = (N - \ell, \ell)$.

Definition 4. Given μ and η as above, a configuration of type η is a partition $\nu = (\nu_1, \ldots, \nu_s)$ such that $\sum_{j=1}^{s} \nu_j = \eta_2$. ■

Definition 5. For a given configuration of type $\mu = (\mu_1, \ldots, \mu_N)$ and $\nu = (\nu_1, \ldots, \nu_s)$, define the so-called **vacancy numbers** $P_k(\nu)$ as follows

$$P_k(\nu) = \sum_{j=1}^{N} \min(k, \mu_j) - 2 \sum_{j=1}^{s} \min(k, \nu_j), \qquad k \in \mathbb{Z}_{\geq 0}. \tag{20}$$

A configuration ν of type (μ, η) is called **admissible**, if all vacancy numbers $\{P_k(\nu)\}$ are non-negative. ■

It is convenient to introduce numbers $m_k(\nu) = m_k = \{j \mid \nu_j = k\}$. In the following, we will freely identify the partitions and the Young diagrams. In the language of the Young diagrams, $m_k(\nu)$ is the number of length k rows of ν and $\sum_{j=1}^{s} \min(k, \nu_j)$ is the number of boxes within the first k columns of ν. In the case of the spin-$\frac{1}{2}$ model, the definition for the vacancy numbers (20) is simply

$$P_k(\nu) = N - 2 \sum_{j=1}^{s} \min(k, \nu_j), \qquad k \in \mathbb{Z}_{>0}. \tag{21}$$

In particular, it should be reminded that $P_k(\nu)$ depends on the data N.

Definition 6. A rigged configuration (ν, J) of type (μ, η) is an admissible configuration ν of type η together with a weakly increasing sequence of non-negative integers

$$0 \leq J_{k,1} \leq J_{k,2} \leq \cdots \leq J_{k,m_k} \leq P_k(\nu), \qquad k = 1, 2, \ldots. \tag{22}$$

We call the integers $J_{k,1}, \ldots, J_{k,m_k}$ the **riggings** associated to the length k rows of the partition ν. We denote by $\mathrm{RC}(\mu, \nu)$ the set of rigged configurations with specific μ and ν. ■

Definition 7. Define the flip map

$$\kappa : \mathrm{RC}(\mu, \nu) \longrightarrow \mathrm{RC}(\mu, \nu) \tag{23}$$

as follows: $\kappa(\nu) = \nu$, and

$$\kappa(J_{k,\alpha}) = P_k(\nu) - J_{k,m_k-\alpha+1} \tag{24}$$

for all $k \in \mathbb{Z}_{>0}$ and $\alpha = 1, \ldots, m_k$. ■

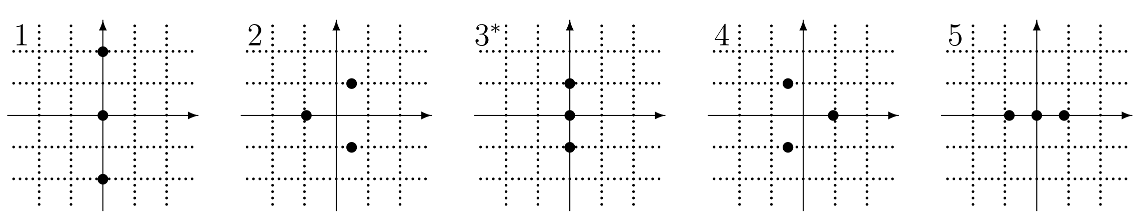
It is convenient to regard the rigged configuration (ν, J) as a collection of data $(k, J_{k,\alpha})$ $(k \in \mathbb{Z}_{>0},\ \alpha = 1, \ldots, m_k(\nu))$ which we call strings. Then the flip map is the map

$$(k, J_{k,\alpha}) \longmapsto (k, P_k(\nu) - J_{k,\alpha})$$

together with certain reordering of the strings to make the riggings satisfy the weakly increasing condition. However, we note that the order of the strings is not essential in the rigged configuration theory.

Our fundamental observation is that the rigged configurations provide a nice parameterization of the combination of both regular solutions and physical singular solutions to the Bethe ansatz equations. As we explain in the following examples, the basic idea is to identify the string $(k, J_{k,\alpha})$ of the rigged configuration with the collection of k solutions which have (almost) same real part specified by $J_{k,\alpha}$. We call such collection of roots as length k string of solutions. Let us tentatively suppose that the larger rigging corresponds to the rightwards string of solutions. However there is an ambiguity described in Conjecture 11 (**A**) below.

Example 8. Consider the spin-$\frac{1}{2}$ case of $N = 6$ and $\ell = 3$. Then we have the following five solutions to the Bethe ansatz equations (11) which are depicted on the complex plain.

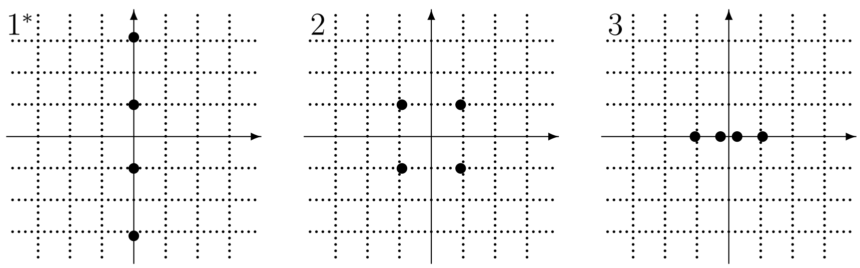


Here the spacing of the dotted lines is 0.5 and the label with asterisk (3^* in this case) means that the solution is singular and physical. These solutions correspond to the rigged configurations as in the following table.

label	values of the roots	rigged configurations
1	$0, \pm i$	0 □□□ 0
2	$-0.47, 0.24 \pm 0.5i$	0 □□ 0 2 □ 0
3*	$0, \pm 0.5i$	0 □□ 0 2 □ 1
4	$0.47, -0.24 \pm 0.5i$	0 □□ 0 2 □ 2
5	$0, \pm 0.43$	0 □ 0 0 □ 0 0 □ 0

Here we depict the configuration ν by the Young diagram. For the string $(k, J_{k,\alpha})$, we put $P_k(\nu)$ (resp. $J_{k,\alpha}$) on the left (resp. right) of the corresponding length k row of the diagram. ■

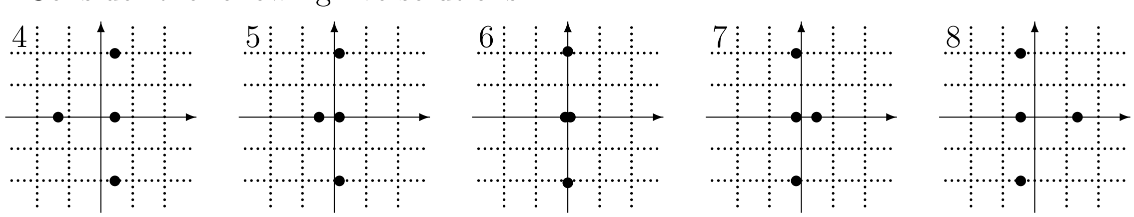
Example 9. Consider the spin-$\frac{1}{2}$ case of $N = 8$ and $\ell = 4$ [8]. We keep the notations of the previous example. The following solutions corresponds to rigged configurations with uniquely determined riggings.



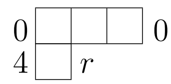
These solutions correspond to the following rigged configurations.

label	values of the roots	rigged configuration
1*	$\pm 0.5i, \pm 1.56i$	0 □□□□ 0
2	$\pm 0.46 \pm 0.5i$	0 □□ 0 0 □□ 0
3	$\pm 0.13, \pm 0.53$	0 □ 0 0 □ 0 0 □ 0 0 □ 0

Consider the following five solutions.

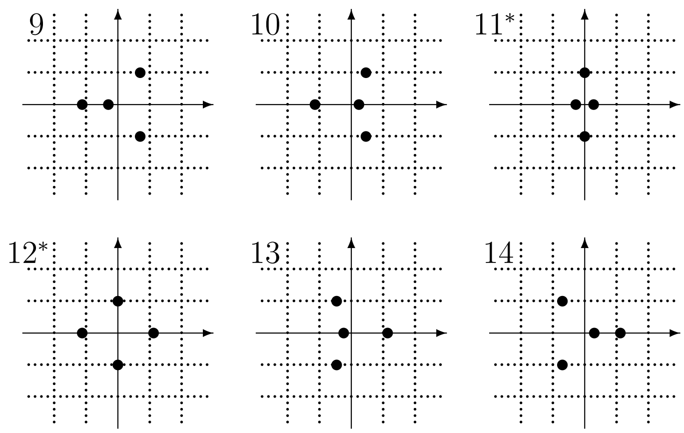


These solutions correspond to the following rigged configurations.

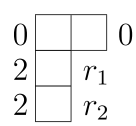


label	values of the roots	value of r
4	$-0.67, 0.22, 0.22 \pm i$	0
5	$-0.24, 0.08 \pm 1.01i, 0.08$	1
6	$\pm 0.04, \pm 1.03i$	2
7	$-0.08, -0.08 \pm 1.01i, 0.24$	3
8	$-0.22 \pm i, -0.22, 0.67$	4

Finally let us consider the following six solutions.



These solutions correspond to the following rigged configurations.



label	values of the roots	value of (r_1, r_2)
9	$-0.56, -0.14, 0.35 \pm 0.5i$	$(0,0)$
10	$-0.57, 0.12, 0.23 \pm 0.5i$	$(0,1)$
11*	$\pm 0.14, \pm 0.5i$	$(1,1)$
12*	$\pm 0.56, \pm 0.5i$	$(0,2)$
13	$-0.23 \pm 0.5i, -0.12, 0.57$	$(1,2)$
14	$-0.35 \pm 0.5i, 0.14, 0.56$	$(2,2)$

Again we have a perfect correspondence between the solutions to the Bethe ansatz equations and the rigged configurations. ■

3.2 Main Conjectures

Motivated by the examples in the previous subsection, we propose the following conjecture. Let $\mathrm{BA}(\ell)$ be the set of solutions $\{\lambda_1, \ldots, \lambda_\ell\}$ of the Bethe ansatz equations which are either regular or singular and physical.

Conjecture 10. *There exist a bijection between* $\mathrm{BA}(\ell)$ *and the set of the rigged configurations* $\mathrm{RC}(\mu, \nu)$ *where the total number of the boxes of* ν *is* ℓ. ■

Furthermore, we propose the following conjectures.

Conjecture 11.

(**A**) *The map* $\iota : \mathrm{BA}(\ell) \longrightarrow \mathrm{BA}(\ell)$ *given by*

$$(\lambda_1, \ldots, \lambda_\ell) \in \mathrm{BA}(\ell) \longmapsto (-\lambda_1, \ldots, -\lambda_\ell) \in \mathrm{BA}(\ell) \tag{25}$$

induces the flip map on the set of rigged configurations.

In the next two conjectures we assume that the generalized Heisenberg chain is defined on the length N tensor product of the spin s representation. In this case, we have $\mu = (\overbrace{2s, 2s, \cdots, 2s}^{N})$.

(**B**) *Assume that* N *is even.*

(**a**) *If* $2s$ *is* odd *and* ℓ *is even, then the set of physical singular solutions to* $\mathrm{BAE}(N, \ell)$ *is in one-to-one correspondence with the set of flip invariant rigged configurations* (ν, J) *such that partition* ν *contains odd number of* even *parts which are greater than or equal to* $2s + 1$.

(**b**) *If* $2s$ *is* even *and* ℓ *is odd, then the set of physical singular solutions to* $\mathrm{BAE}(N, \ell)$ *is in one-to-one correspondence with the set of flip invariant rigged configurations* (ν, J) *such that partition* ν *contains odd number of* odd *parts which are greater than or equal to* $2s + 1$.

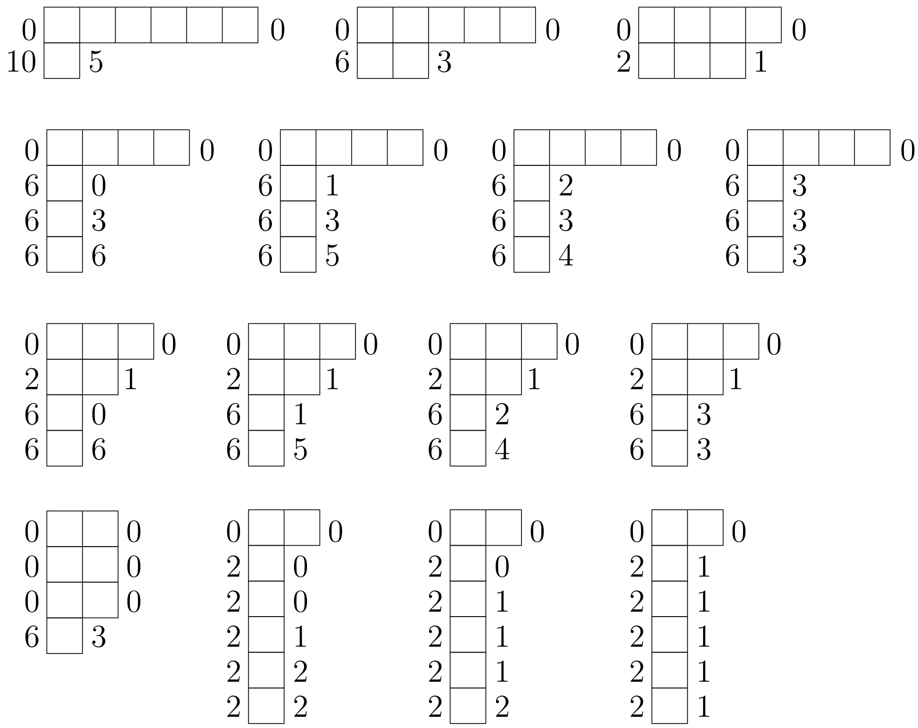
(**C**) *Assume that* $s = \frac{1}{2}$, N *is even and* ℓ *is* odd. *Then the set of physical singular solutions to* $\mathrm{BAE}(N, \ell)$ *is in one-to-one correspondence with the set of flip invariant rigged configurations* (ν, J) *such that partition* ν *contains odd number of even parts of the lengths longer than* $2s + 1$, and *if the number* $m_k(\nu) \geq 3$ *is odd and the corresponding vacancy number* $P_k(\nu) > 0$ *is divisible by 4, then the rigging*

$$J_{k,1} = J_{k,2} = \cdots = J_{k,m_k} = \frac{P_k(\nu)}{2} \tag{26}$$

is forbidden. ■

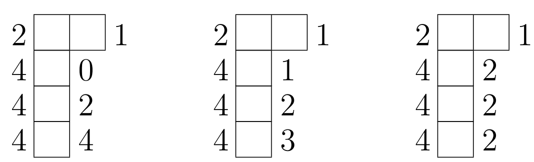
Example 12.

- Take $s = \frac{1}{2}$, $N = 14$ and $\ell = 7$. Then the following partitions satisfy the condition (**C**); $(6,1)$, $(5,2)$, $(4,3)$, $(4,1,1,1)$, $(3,2,1,1)$, $(2,2,2,1)$ and $(2,1^5)$. ¿From them we can construct the following 15 flip invariant rigged configurations.



Since there are no forbidden riggings, our result agrees with the result of [8].

- Take $s = \frac{1}{2}$, $N = 12$ and $\ell = 5$. Then the following partitions satisfy the condition (**C**); (5), $(3,2)$ and $(2,1,1,1)$. Corresponding to the partition $(2,1,1,1)$ we have the following three flip invariant rigged configurations.



Since $P_1(\nu) = 4$, the rigging $J_{1,1} = J_{1,2} = J_{1,3} = 2$ is forbidden. Therefore totally one has $1 + 1 + (3 - 1) = 4$ for the number of flip invariant rigged configurations satisfying the condition given in (**C**).

More generally, if $s = \frac{1}{2}$, N is even and $\ell = 5$ our conjecture predicts that the number of physical singular solutions should be

$$\begin{array}{ll} \frac{N-2}{2} & \text{if } N \equiv 2 \pmod 4, \\ \frac{N-4}{2} & \text{if } N \equiv 0 \pmod 4. \end{array}$$

- Take $s = 3/2$, even integer $N \geq 8$ and $\ell = 10$. Then the number of rigged configurations satisfying conditions of Conjecture 11 (**B-a**) is equal to

$$\frac{N-4}{N+2}\binom{\frac{N+6}{2}}{3} + 2.$$

  For $N = 8$ this number is equal to 16, cf [9], Table 1.

- Take $s = 3/2$, $N = 8$ and $\ell = 12$. Then the following partitions satisfy the condition (**B-a**); (12), $(10,2)$, $(10,1,1)$, $(8,3,1)$, $(8,2,2)$, $(8,2,1,1)$, $(7,4,1)$, $(6,5,1)$, $(6,3,3)$, $(6,3,2,1)$, $(6,2,2,2)$, $(5,4,3)$, $(5,4,2,1)$, $(4,4,4)$ and $(4,3,3,2)$. Then the number of flip invariant rigged configurations corresponding to these partitions is

$$(1+1+2+1+3+1+1+4+1+1+1+1+1+1+2) = 22.$$

  More generally, if $s = 3/2$, $N \geq 8$ is even and $\ell = 12$, the number of rigged configurations satisfying conditions of Conjecture 11, (**B-a**) is equal to

$$\frac{N-6}{N+2}\binom{\frac{N+8}{2}}{4} + 8.$$

- Take $s = 1$, even integer $N \geq 8$ and $\ell = 7$. Then the following partitions satisfy the condition (**B-b**); (7), $(5,2)$, $(5,1,1)$, $(4,3)$, and $(3,2,2)$. The number of flip invariant rigged configurations is

$$\frac{(N-2)(N+4)}{8}.$$

  For $N = 8$ this number is equal to 9, cf [9], Table 1.

- Take $s = 1$, even integer $N \geq 10$ and $\ell = 9$. Then the number of rigged configurations satisfying conditions of Conjecture 11 (**B-b**) is equal to

$$\frac{N-4}{N+2}\binom{\frac{N+6}{2}}{3} + 2 - \frac{N-2}{2}.$$

■

Corollary 13.

- *Suppose that N and ℓ are both even. The number of physical singular solutions to the Bethe ansatz equations* $\mathrm{BAE}(N,\ell)$ *for the homogeneous spin s Heisenberg chain is equal to*

$$\sum_{\nu}\prod_{k}\binom{[\frac{m_k}{2}] + \frac{P_k(\nu)}{2}}{[\frac{m_k}{2}]}, \tag{27}$$

  where the summation runs over either

– *the set of partitions ν which satisfies the condition* (**B-a**) *of Conjecture 11, if* $2s$ *is* **odd**, *or*

– *the set of partitions ν which satisfies the condition* (**B-b**) *of Conjecture 11, if* $2s$ *is* **even**;

and for any real number x the symbol $[x]$ means the **integer part** *of x, i.e. is a unique integer n such that $n \le x < n+1$;*

the symbol $\binom{n+m}{m} = \frac{(n+m)!}{n!\ m!}$ means the **binomial coefficient**.

- *If $s = \frac{1}{2}$, N is even, but ℓ is odd, the number of physical singular solutions to* $\mathrm{BAE}(N,\ell)$ *for the homogeneous spin-$\frac{1}{2}$ Heisenberg chain is equal to*

$$\sum_{\nu} \prod_{k} \left\{ \binom{[\frac{m_k}{2}] + \frac{P_k(\nu)}{2}}{[\frac{m_k}{2}]} - \chi_k(\nu) \right\}, \tag{28}$$

where the summation runs over the set of partitions ν which satisfies the condition (**C**) *of Conjecture 11;*

$\chi_k(\nu) = 1$ if $m_k(\nu) \ge 3$ is an odd integer and the vacancy number $P_k(\nu) > 0$ is divisible by 4, and $\chi_k(\nu) = 0$ otherwise.

The surprising thing is that the sum (27) can be computed.

Proposition 14.

- *If $s = \frac{1}{2}$, N and ℓ are both even, then the number of physical singular solutions predicted by Corollary 13, is equal to*

$$\binom{\frac{N-2}{2}}{\frac{\ell-2}{2}}. \tag{29}$$

- *If $s = \frac{1}{2}$, $N \equiv 2 \pmod 4$ and ℓ is an odd integer, then the number of physical singular solutions predicted by Corollary 13, is equal to*

$$\binom{\frac{N-2}{2}}{\frac{\ell-3}{2}}. \tag{30}$$

■

Note that for the spin-$\frac{1}{2}$ isotropic Heisenberg model, the total number of the highest weight states of $\mathfrak{H}_N$ is

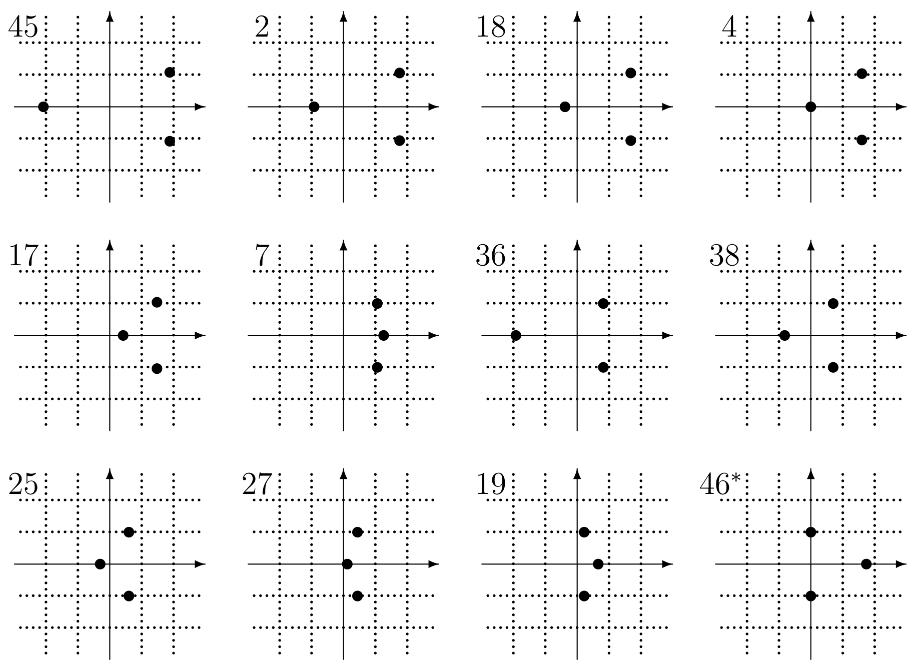
$$\binom{N}{\ell} - \binom{N}{\ell-1} \tag{31}$$

for the prescribed value of ℓ. Thus the above conjecture also provides a conjecture for the total number of regular solutions in this case.

4 Discussion

4.1 The case N and ℓ are both odd

So far we have concentrated on the case when the system length N is even. For the spin-$\frac{1}{2}$ Heisenberg model, the paper [8] discovered one exceptional case when both N and ℓ are odd. In such situation we do not have any flip invariant rigged configurations. Nevertheless there are two physical singular solutions when $N = 9$ and $\ell = 3$ which corresponds to the partition $\nu = (2, 1)$. Below we give a list of 12 regular and physical singular solutions in this case.



Here some remarks are in order;

- the above solutions are arranged according to the real parts of length 2 strings of solutions,
- the spacing of dotted lines is 0.5,
- the label of each solution corresponds to the label in the supplementary table of [8] (table $N = 9$, $M = 3$); label with asterisk (46^* in this case) means that the solution is singular and physical,
- the remaining 12 solutions are obtained by multiplying (-1) to each root in the above 12 solutions.

For each solution in the above table, it is natural to associate the following rigged configurations. The first six solutions correspond to

$$\begin{array}{r|c|c|l} \cline{2-3} 3 & & & 3 \\ \cline{2-3} 5 & & \multicolumn{2}{l}{r} \\ \cline{2-2} \end{array}$$

where $r = 0, \dots, 5$ according to the order of the above table. The next six solutions correspond to

$$\begin{array}{r|c|c|l} \cline{2-3} 3 & & & 2 \\ \cline{2-3} 5 & & \multicolumn{2}{l}{r} \\ \cline{2-2} \end{array}$$

where $r = 0, \dots, 5$ according to the order of the above table.

To summarize, the exceptional physical singular solutions in the case of $N = 9$ and $\ell = 3$ correspond to the following rigged configurations;

$$\begin{array}{r|c|c|l} \cline{2-3} 3 & & & 2 \\ \cline{2-3} 5 & & \multicolumn{2}{l}{5} \\ \cline{2-2} \end{array} \qquad \begin{array}{r|c|c|l} \cline{2-3} 3 & & & 1 \\ \cline{2-3} 5 & & \multicolumn{2}{l}{0} \\ \cline{2-2} \end{array}$$

It will be an interesting problem to find general rule to characterize the rigged configurations corresponding to physical singular solutions for the case when both N and ℓ are odd.

4.2 On the number of solutions to the Bethe ansatz equations

Follow [8], let us denote by $\mathcal{N}(N, \ell)$ the number of solutions with pairwise ℓ distinct roots, and by $\mathcal{N}_{sp}(N, \ell)$ the number of physical singular solutions to the Bethe ansatz equations for the spin-$\frac{1}{2}$ Heisenberg model of length N.

Conjecture 15.

- *Assume that N and $\ell \geq 3$ are both even, then*

$$\mathcal{N}(N, \ell) + \mathcal{N}_{sp}(N, \ell) = \binom{N-1}{\ell}, \qquad \textit{if} \quad 2\ell \leq N. \tag{32}$$

  In other words,

$$\mathcal{N}(N, \ell) = \binom{N-1}{\ell} - \binom{\frac{N-2}{2}}{\frac{\ell-2}{2}}. \tag{33}$$

- *Assume that $N \equiv 2 \pmod 4$, $N \geq 6$, but $\ell \geq 3$ is odd, then*

$$\mathcal{N}(N, \ell) + \mathcal{N}_{sp}(N, \ell) = \binom{N-1}{\ell}, \qquad \textit{if} \quad 2\ell \leq N. \tag{34}$$

  In other words,

$$\mathcal{N}(N, \ell) = \binom{N-1}{\ell} - \binom{\frac{N-2}{2}}{\frac{\ell-3}{2}}. \tag{35}$$

- *If $N \geq 3$ is odd, but $\ell \geq 4$ is even, then*

$$\mathcal{N}(N,\ell) + \mathcal{N}_{sp}(N-1,\ell-2) = \binom{N-1}{\ell}, \qquad \textit{if} \quad 2\ell \leq N. \tag{36}$$

In other words,

$$\mathcal{N}(N,\ell) = \binom{N-1}{\ell} - \binom{\frac{N-3}{2}}{\frac{\ell-4}{2}}, \quad \mathcal{N}_s(N,\ell) = \binom{N-1}{\ell-2} - \binom{\frac{N-3}{2}}{\frac{\ell-4}{2}}. \tag{37}$$

- *If $N \geq 3$ and $\ell \geq 3$ are both odd, then*

$$\mathcal{N}(N,\ell) + \mathcal{N}_{sp}(N,\ell) = \binom{N-1}{\ell}, \quad \mathcal{N}_s(N,\ell) = \binom{N-1}{\ell-2}. \tag{38}$$

■

Therefore, Conjecture 15 predicts that if $\ell \geq 2$ and $N \geq 2$ have the same parity, then the number of singular solutions $\mathcal{N}_s(N,\ell)$ to the Bethe equations in question, is equal to

$$\mathcal{N}_s(N,\ell) = \binom{N-1}{\ell-2}. \tag{39}$$

For example,

$$\mathcal{N}(14,6)+\mathcal{N}_{sp}(14,6) = 1716 = \binom{13}{6}, \ \mathcal{N}_{sp}(14,6) = 15 = \binom{6}{2}, \ \mathcal{N}_s(14,6) = 715 = \binom{13}{4},$$

$$\mathcal{N}(14,5)+\mathcal{N}_{sp}(14,5) = 1287 = \binom{13}{5}, \ \mathcal{N}_{sp}(14,5) = 6 = \binom{6}{1}, \ \mathcal{N}_s(14,5) = 286 = \binom{13}{3},$$

$$\mathcal{N}(13,6) + \mathcal{N}_{sp}(12,4) = 919 + 5 = \binom{12}{6}, \quad \mathcal{N}_s(13,6) = 490 = \binom{12}{4} - \binom{5}{1},$$

$$\mathcal{N}(9,3) + \mathcal{N}_{sp}(9,3) = 54 + 2 = \binom{8}{3}, \ \mathcal{N}_s(9,3) = 8 = \binom{8}{1}.$$

However,

$$\mathcal{N}(12,5) + \mathcal{N}_{sp}(12,5) = 456 + 4 = 460 < \binom{11}{5} = 462, \ \mathcal{N}_s(12,5) = 163 < \binom{11}{3} = 165.$$

4.3 On Conjecture 11 (C)

Probably, if $N \equiv 0 \pmod 2$ and ℓ is an odd number, then for k such that $m_k \geq 3$ and odd, and $P_k(\nu) \equiv 0 \pmod 4$ and $P_k(\nu) > 0$, it is more natural to allow only riggings with strict inequalities:

$$0 \leq J_{k,1} < J_{k,2} < \ldots < J_{k.m_k} \leq \frac{P_k(\nu)}{2}.$$

For example, if $\ell = 7$, the number of such rigged configurations is equal to

$$(N-2)(N-4)/8 - N + 9,$$

whereas the number of rigged configurations which satisfies conditions of Conjecture 11 (**C**) is equal to

$$(N-2)(N-4)/8 - 3.$$

4.4 Some related topics from mathematical physics

It should be worth while to mention that the theory of the rigged configurations is extensively studied from various points of view. Indeed, it is widely believed that the rigged configurations exist for finite dimensional representations of arbitrary quantum affine algebras. Especially, we would like to mention that there is a clear physical interpretation of the rigged configurations. See [25] for an introductory review related with spin-$\frac{1}{2}$ case of $\mathfrak{sl}_2$ which is the main case of the present paper.

The main point of the rigged configuration theory is the bijection between the rigged configurations and the tensor products of crystals. In the spin-$\frac{1}{2}$ case of $\mathfrak{sl}_2$, one can regard the latter objects as sequences of the letters 1 and 2 which we call crystal paths. On the crystal paths we can define a discrete soliton system called the box-ball system [29, 27]. Then the fundamental observation of [18] is that the rigged configurations provide a complete set of the action and angle variables of the box-ball systems. More precisely, each row of the configuration ν corresponds to a soliton whose position is specified by the corresponding riggings. This soliton picture is also confirmed from the point of view of the ordinary soliton theory (the KP equation) [20].

Finally, we would like to mention that in the spin-$\frac{1}{2}$ case of $\mathfrak{sl}_2$, there is a periodic version of the box-ball systems [30] which admit the rigged configuration approach [21]. By taking a suitable limit of initial value solutions of the linear box-ball systems [20, 24], we can solve the initial value problem for the periodic case in terms of the tropical Riemann theta functions [19]. This is a direct discrete analogue of the periodic solution for the KP equation obtained by B. A. Dubrovin, V. B. Matveev and S. P. Novikov [5].

4.5 Some related combinatorics

Finally, let us mention some related topics from the point of view of pure combinatorics. Here we consider only the case of $\mathfrak{gl}(N)$ case.

The starting data for definition of rigged configurations are:
a partition $\lambda = (\lambda_1, \ldots, \lambda_N \geq 0)$, and
a collection of rectangular shape partitions

$$R := \{R_a = (\underbrace{\mu_a, \ldots, \mu_a}_{\eta_a})\} \quad such \quad that \quad \sum_{j\geq 1} \lambda_j = \sum_a \mu_a \, \eta_a.$$

These data come from the analyses of the the Bethe ansatz equations corresponding to the $\mathfrak{gl}(N)$ XXX model of "spin" R, based on the use of the so-called *String Conjecture*, see e.g. [10].

The main results concerning the rigged configuration theory discovered in [11], [13], [15], [16] are

• The number of rigged configuration related with pair (λ, R) is equal to the tensor product multiplicity

$$\mathrm{Mult}[V_{\lambda}^{\mathfrak{gl}(N)} : \bigotimes_{a \geq 1} V_{R_a}^{\mathfrak{gl}(N)}], \tag{40}$$

where $V_{\mu}^{\mathfrak{gl}(N)}$ stands for the irreducible representation of the Lie algebra $\mathfrak{gl}(N)$ corresponding to partition λ.

• There exists a bijection, called *Rigged Configuration Bijection*, (*RC*-bijection for short), between the set of rigged configurations related with pair (λ, R), and the set of so-called *Littlewood–Richardson tableaux* which are some combinatorial objects describing the tensor product multiplicity introduced above.

The *RC*-bijection has a big parity of deep and sometimes unexpected properties related with Algebraic Combinatorics [13], Representation Theory [14], [18], Integrable Systems [19], [20], [21] and *etc*. In the present paper we state only one unexpected (at least for A.N.K.) result discovered by the first author, see [12], [16] for proofs, that the flip map on the set of rigged configurations related with pair (λ, R), corresponds to the so-called *Schützenberger involution* on the set of Littlewood–Richardson tableaux needed to describe the tensor product multiplicity (40).

In the case of $\mathfrak{sl}_2$ spin-$\frac{1}{2}$ Heisenberg model, the *RC*-bijection gives rise to a bijection between the set of rigged configurations $RC(\ell, N)$ and the set of standard Young tableaux of the shape $(N-\ell, \ell)$. It is well-known that if N is even, then the set of standard Young tableaux of shape $(N-\ell, \ell)$ which are invariant under the action of the Schützenberger involution, is in one-to-one correspondence with the set of standard *domino tableaux* of the same shape, see e.g. [4], [3]. Our Conjecture 11 holds that the set of singular physical solutions to the BAE is in a bijection with a set of special domino tableaux depending on a $\mathfrak{gl}(2) - XXX$ mode chosen. We expext a similar connection in general case.

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