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THE EARLIER TOIL AND MOIL IN PROVING ON THE DESCRIBABILITY OF TRIGONOMETRIC SERIES

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Contents

Abstract. After Lagrange expressed the theory of propagation of sound 1759-61 by the trigonometric series, Fourier 1822 proposes the analytical theory of heat, including the trigonometric series without proving the convergence. Since then, many mathematicians, like Poisson 1823, Cauchy 1823, et al. try the proof problem on the describability of trigonometric series until the success by Carlson 1966 of L^2 and by Hunt 1968 of L^p. At first, Dirichlet 1837 introduces especially Cauchy 1823 as the only challenging one, however, falls himself into a circular argument. Liouville 1836 introduces Poisson 1823 as the first study of this sort. Kummer 1860, in the mourning paper of Dirichlet, evaluates Dirichlet's work 1837 on this problem. We focus on the earlier triers, such as Lagrange, Fourier, Poisson, Cauchy, Dirichlet, Liouville, et al., of proving trials on the describability of trigonometric series.

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1. Introduction

[1,2,3,4] In the early days before and after Fourier, many matimaticians begin to try the proof on the describability of trigonometric series until the success by Carlson 1966 of L^2 and by Hunt 1968 of L^p, for example, Cauchy (1789-1857) 1823 [1], Poisson (1781-1840) 1823 [35], 1835 [36], Dirichlet (1805-59) 1837 [4], Liouville (1809-82) 1836 [14]. They discuss this problem as the 'mathematical theory of heat' like Fourier, to solve the heat diffusion problems, not as the pure mathematical theory directly.

2. Lagrange [8] : 1759, [12] : 1762

2.1. ***Recherches sur la Nature et la Propagation du Son*** by Lagrange [8], 1759.

Lagrange explains the motion of sound diffusing along with time t by the trigonometric series of the original sample which the after ages, such as Fourier, Poisson, Dirichlet, et al. refer to it. Here, $\varpi = \pi$.

¶ 23. (pp.79-81).

$$P_\nu \equiv Y_1 \sin\frac{\varpi}{2m} + Y_2 \sin\frac{2\varpi}{2m} + Y_3 \sin\frac{3\varpi}{2m} + \cdots + Y_{m-1} \sin\frac{(m-1)\varpi}{2m}$$

$$Q_\nu \equiv V_1 \sin\frac{\varpi}{2m} + V_2 \sin\frac{2\varpi}{2m} + V_3 \sin\frac{3\varpi}{2m} + \cdots + V_{m-1} \sin\frac{(m-1)\varpi}{2m}$$

$$\begin{aligned} & y_1 \sin\frac{\varpi}{2m} + y_2 \sin\frac{2\varpi}{2m} + y_3 \sin\frac{3\varpi}{2m} + \cdots + y_n \sin\frac{(m-1)\varpi}{2m} \\ = \; & P_\nu \cos\left(2t\sqrt{e}\sin\frac{\nu\varpi}{4m}\right) + \frac{Q_\nu \sin\left(2t\sqrt{e}\sin\frac{\nu\varpi}{4m}\right)}{2\sqrt{e}\frac{\nu\varpi}{4m}} \equiv S_\nu \end{aligned}$$

2.1. Transfer array by Lagrange.

$$y_1 \sin\frac{\varpi}{2m} + y_2 \sin\frac{2\varpi}{2m} + y_3 \sin\frac{3\varpi}{2m} + \cdots + y_{m-1} \sin\frac{(m-1)\varpi}{2m} = S_1$$

$$y_1 \sin\frac{2\varpi}{2m} + y_2 \sin\frac{4\varpi}{2m} + y_3 \sin\frac{6\varpi}{2m} + \cdots + y_{m-1} \sin\frac{2(m-1)\varpi}{2m} = S_2$$

$$y_1 \sin\frac{3\varpi}{2m} + y_2 \sin\frac{6\varpi}{2m} + y_3 \sin\frac{9\varpi}{2m} + \cdots + y_{m-1} \sin\frac{3(m-1)\varpi}{2m} = S_3$$

$$\cdots\cdots\cdots\cdots$$

$$y_1 \sin\frac{(m-1)\varpi}{2m} + y_2 \sin\frac{2(m-1)\varpi}{2m} + y_3 \sin\frac{3(m-1)\varpi}{2m} + \cdots + y_{m-1} \sin\frac{(m-1)^2\varpi}{2m} = S_{m-1}$$

[1] Basically, we treat the exponential / trigonometric / logarithmic / π / et al. / functions as the transcendental functions.

[2] Translation from Latin/French/German into English mine, except for Boltzmann.

[3] To establish a time line of these contributor, we list for easy reference the year of their birth and death: Euler(1707-83), d'Alembert(1717-83), Lagrange(1736-1813), Laplace(1749-1827), Fourier(1768-1830), Poisson(1781-1840), Cauchy(1789-1857), Dirichlet(1805-59), Riemann(1826-66), Boltzmann(1844-1906), Schrödinger (1887-1961).

[4] The symbol (⇓) means our remark not original, when we want to avoid the confusions between our opinion and sic.

Here, we can show with a today's style of $(m-1)\times(m-1)$ transform matrix :[5]

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_{m-1} \end{bmatrix} = \begin{bmatrix} \sin\frac{\varpi}{2m} & \sin\frac{2\varpi}{2m} & \sin\frac{3\varpi}{2m} & \cdots & \sin\frac{(m-1)\varpi}{2m} \\ \sin\frac{2\varpi}{2m} & \sin\frac{4\varpi}{2m} & \sin\frac{6\varpi}{2m} & \cdots & \sin\frac{2(m-1)\varpi}{2m} \\ \sin\frac{3\varpi}{2m} & \sin\frac{6\varpi}{2m} & \sin\frac{9\varpi}{2m} & \cdots & \sin\frac{3(m-1)\varpi}{2m} \\ \cdots \\ \sin\frac{(m-1)\varpi}{2m} & \sin\frac{2(m-1)\varpi}{2m} & \sin\frac{3(m-1)\varpi}{2m} & \cdots & \sin\frac{(m-1)^2\varpi}{2m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{m-1} \end{bmatrix} \quad (1)$$

Lagrange continues as follows : It must now, by the ordinary rules, substitute the values of unknown with an equation in the other successively, to reach to one which contains no more than only one of these variables ; however, it is clear to see if we take this manner, we will fail in the unpractical calculus by reason of undetermined number of equation and unknowns ; it is necessary, therefore, to take another route : this is one which seems to us to be the best. We show the Lagrange's bibliography [8] adding our comments to understand easily as possible, with ¶ : the article number and pages of it, in the following :

¶ 24. (pp.81-82).
We assume $D_1 = 1$.

$$\begin{aligned} & y_1\Big[D_1\sin\frac{\varpi}{2m} + D_2\sin\frac{2\varpi}{2m} + D_3\sin\frac{3\varpi}{2m} + \cdots + D_{m-1}\sin\frac{(m-1)\varpi}{2m}\Big] \\ + \ & y_2\Big[D_1\sin\frac{2\varpi}{2m} + D_2\sin\frac{4\varpi}{2m} + D_3\sin\frac{6\varpi}{2m} + \cdots + D_{m-1}\sin\frac{2(m-1)\varpi}{2m}\Big] \\ + \ & y_3\Big[D_1\sin\frac{3\varpi}{2m} + D_2\sin\frac{6\varpi}{2m} + D_3\sin\frac{9\varpi}{2m} + \cdots + D_{m-1}\sin\frac{3(m-1)\varpi}{2m}\Big] \\ + \ & \cdots\cdots\cdots\cdots \\ + \ & y_{m-1}\Big[D_1\sin\frac{(m-1)\varpi}{2m} + D_2\sin\frac{2(m-1)\varpi}{2m} + D_3\sin\frac{3(m-1)\varpi}{2m} + \cdots + D_{m-1}\sin\frac{(m-1)^2\varpi}{2m}\Big] \\ = \ & D_1S_1 + D_2S_2 + D_3S_3 + \cdots + D_{m-1}S_{m-1} \end{aligned}$$

That is,

$$\begin{aligned} & \begin{bmatrix} D_1 & D_2 & D_3 \cdots & D_{m-1} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_{m-1} \end{bmatrix} \\ = & \begin{bmatrix} D_1\sin\frac{\varpi}{2m} & D_2\sin\frac{2\varpi}{2m} & D_3\sin\frac{3\varpi}{2m} & \cdots & D_{m-1}\sin\frac{(m-1)\varpi}{2m} \\ D_1\sin\frac{2\varpi}{2m} & D_2\sin\frac{4\varpi}{2m} & D_3\sin\frac{6\varpi}{2m} & \cdots & D_{m-1}\sin\frac{2(m-1)\varpi}{2m} \\ D_1\sin\frac{3\varpi}{2m} & D_2\sin\frac{6\varpi}{2m} & D_3\sin\frac{9\varpi}{2m} & \cdots & D_{m-1}\sin\frac{3(m-1)\varpi}{2m} \\ \cdots \\ D_1\sin\frac{(m-1)\varpi}{2m} & D_2\sin\frac{2(m-1)\varpi}{2m} & D_3\sin\frac{3(m-1)\varpi}{2m} & \cdots & D_{m-1}\sin\frac{(m-1)^2\varpi}{2m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{m-1} \end{bmatrix} \quad (2) \end{aligned}$$

In general, we may state as follows : [6]

$$\begin{aligned} & y_\mu\Big[D_1\sin\frac{\mu\varpi}{2m} + D_2\sin\frac{2\mu\varpi}{2m} + D_3\sin\frac{3\mu\varpi}{2m} + \cdots + D_{m-1}\sin\frac{(m-1)\mu\varpi}{2m}\Big] \\ = \ & D_1S_1 + D_2S_2 + D_3S_3 + \cdots + D_{m-1}S_{m-1}, \quad (3) \end{aligned}$$

Generally speaking,

$$D_1\sin\frac{\lambda\varpi}{2m} + D_2\sin\frac{2\lambda\varpi}{2m} + D_3\sin\frac{3\lambda\varpi}{2m} + \cdots + D_{m-1}\sin\frac{(m-1)\lambda\varpi}{2m} = 0,$$

[5]Lagrange didn't use the transform-matrix symbol, but mine. cf. Poisson's expression (32).
[6]Dirichlet's (74) also uses the same style of expression with (3).

where, for $0 \leq \lambda \leq m-1,\ \lambda \in \mathbb{Z}$.

¶ 25. (p.82).
To deduce the values of the quantities D from this equation, I remark at first all sins of a multiple angle reduces to a series of integer power and positives of cosine of simple angle, which the largest exponential is equal to the number which I denote the multiple decreasing by 1, all the series being still multiplied by the sin of the simple angles.
Therefore,

- if we develop from this manner, all the sins of multiple angle of $\frac{\lambda\varpi}{2m}$ and
- which we divide the equation with $\sin\frac{\lambda\varpi}{2m}$,

then we reach another equation, which will contain only the power of $\cos\frac{\lambda\varpi}{2m}$, and which will be $m-2$ in degree ; from here, it follows that by regarding $\cos\frac{\lambda\varpi}{2m}$ as the unknown of this equation, these roots are

$$\cos\frac{\varpi}{2m},\quad \cos\frac{2\varpi}{2m},\quad \cos\frac{3\varpi}{2m},\quad \cdots,\quad \cos\frac{(m-1)\varpi}{2m}$$

except for $\cos\frac{\mu\varpi}{2m}$.
As the result, all the equations are only the continue products of factors :

$$A-\cos\frac{\varpi}{2m},\quad A-\cos\frac{2\varpi}{2m},\quad A-\cos\frac{3\varpi}{2m},\quad \cdots,\quad A-\cos\frac{(m-1)\varpi}{2m}$$

where, $A \equiv \cos\frac{\lambda\varpi}{2m}$, and omitting the middle term : $A-\cos\frac{\mu\varpi}{2m}$. Hence, if L is a constant, then

$$\begin{aligned} &\sin\frac{\lambda\varpi}{2m}\Big[D_1\sin\frac{\lambda\varpi}{2m}+D_2\sin\frac{2\lambda\varpi}{2m}+D_3\sin\frac{3\lambda\varpi}{2m}+\cdots+D_{m-1}\sin\frac{(m-1)\lambda\varpi}{2m}\Big] \\ = \;& L\Big(A-\cos\frac{\varpi}{2m}\Big)\Big(A-\cos\frac{2\varpi}{2m}\Big)\Big(A-\cos\frac{3\varpi}{2m}\Big)\cdots\Big(A-\cos\frac{(m-1)\varpi}{2m}\Big) \end{aligned} \tag{4}$$

¶ 25. (p.83.) (This step corresponds to S6 in the Table 1.)
According to the theorem, cited by R.Cotes, we consider the followings : [7]

$$\begin{aligned} p^{2m}-q^{2m} = \;& (p^2-q^2)\Big(p^2-2pq\cos\frac{\varpi}{2m}+q^2\Big)\Big(p^2-2pq\cos\frac{2\varpi}{2m}+q^2\Big)\Big(p^2-2pq\cos\frac{3\varpi}{2m}+q^2\Big)\cdots \\ & \Big(p^2-2pq\cos\frac{(m-1)\varpi}{2m}+q^2\Big) \end{aligned} \tag{5}$$

$$p^2+q^2 \equiv \cos\frac{\lambda\varpi}{2m},\quad 2pq\equiv 1, \tag{6}$$

$$p^2+2pq+q^2 = 1+\cos\frac{\lambda\varpi}{2m} = 2\cos^2\frac{\lambda\varpi}{4m},\quad p^2-2pq+q^2=\cos\frac{\lambda\varpi}{2m}-1=-2\sin^2\frac{\lambda\varpi}{4m}$$

$$p+q=\pm\sqrt{2}\Big(\cos\frac{\lambda\varpi}{2m}\Big),\quad p-q=\pm\sqrt{2}\Big(\sin\frac{\lambda\varpi}{2m}\Big)\sqrt{-1}$$

$$p=\pm\frac{1}{\sqrt{2}}\Big(\cos\frac{\lambda\varpi}{4m}+\sin\frac{\lambda\varpi}{4m}\sqrt{-1}\Big),\quad q=\pm\frac{1}{\sqrt{2}}\Big(\cos\frac{\lambda\varpi}{4m}-\sin\frac{\lambda\varpi}{4m}\sqrt{-1}\Big)$$

$$p^2=\frac{1}{2}\Big(\cos\frac{\lambda\varpi}{4m}+\sin\frac{\lambda\varpi}{4m}\sqrt{-1}\Big)^2=\frac{1}{2}\Big(\cos\frac{\lambda\varpi}{2m}+\sin\frac{\lambda\varpi}{2m}\sqrt{-1}\Big) \tag{7}$$

$$q^2=\frac{1}{2}\Big(\cos\frac{\lambda\varpi}{4m}-\sin\frac{\lambda\varpi}{4m}\sqrt{-1}\Big)^2=\frac{1}{2}\Big(\cos\frac{\lambda\varpi}{2m}-\sin\frac{\lambda\varpi}{2m}\sqrt{-1}\Big) \tag{8}$$

[7](⇓) (1682-1716).

(7)-(8) :

$$p^2 - q^2 = \sin\frac{\lambda\varpi}{2m}\sqrt{-1}$$

Similarly,

$$p^{2m} = 2^{-m}\Big(\cos\frac{\lambda\varpi}{4m} + \sin\frac{\lambda\varpi}{4m}\sqrt{-1}\Big)^{2m} = 2^{-m}\Big(\cos\frac{\lambda\varpi}{2} + \sin\frac{\lambda\varpi}{2}\sqrt{-1}\Big) \tag{9}$$

$$q^{2m} = 2^{-m}\Big(\cos\frac{\lambda\varpi}{4m} - \sin\frac{\lambda\varpi}{4m}\sqrt{-1}\Big)^{2m} = 2^{-m}\Big(\cos\frac{\lambda\varpi}{2} - \sin\frac{\lambda\varpi}{2}\sqrt{-1}\Big) \tag{10}$$

(9)-(10) :

$$p^{2m} - q^{2m} = 2^{1-m}\sin\frac{\lambda\varpi}{2}\sqrt{-1} \tag{11}$$

Using (6) and dividing (11) with $(p^2-q^2)\Big(p^2 - 2pq\cos\frac{\mu\varpi}{2m} + q^2\Big)$, then we get the right hand-side of (5) except for the first and middle $\mu-$th factor :

$$\frac{\sin\frac{\lambda\varpi}{2}}{2^{m-1}\sin\frac{\lambda\varpi}{2m}\Big(\cos\frac{\lambda\varpi}{2m} - \cos\frac{\mu\varpi}{2m}\Big)}$$

Namely,

$$D_1\sin\frac{\lambda\varpi}{2m} + D_2\sin\frac{2\lambda\varpi}{2m} + D_3\sin\frac{3\lambda\varpi}{2m} + \cdots + D_{m-1}\sin\frac{(m-1)\lambda\varpi}{2m} = \frac{L}{2^{m-1}}\frac{\sin\frac{\lambda\varpi}{2}}{\Big(\cos\frac{\lambda\varpi}{2m} - \cos\frac{\mu\varpi}{2m}\Big)}$$

¶ 25. (pp.84-85).

- If we multiply all the equation by $\cos\frac{\lambda\varpi}{2m} - \cos\frac{\mu\varpi}{2m}$, and
- if, after having reduced the products of sins by cosines with simple sins, we make the comparing with terms,

then we will get the values sought from undetermined quantities. To make this operation more easily, we should begin with to multiply the sequence that forms the left hand-side of the equation connected by $2\cos\frac{\lambda\varpi}{2m}$; by developing any particular product, and by ordering the terms, it turns into :

$$D_2\sin\frac{\lambda\varpi}{2m} + (D_3 - D_1)\sin\frac{2\lambda\varpi}{2m} + (D_4 - D_2)\sin\frac{3\lambda\varpi}{2m} + \cdots + D_{m-1}\sin\frac{(m-1)\lambda\varpi}{2m} + D_{m-1}\sin\frac{\lambda\varpi}{2}$$

Next,

- if we multiplying the same series with $2\cos\frac{\mu\varpi}{2m}$ and
- if we cut the last product of other

then we get :

$$\Big(D_2 - 2D_1\cos\frac{\mu\varpi}{2m}\Big)\sin\frac{\lambda\varpi}{2m} + \Big(D_3 - 2D_2\cos\frac{\mu\varpi}{2m} + D_1\Big)\sin\frac{2\lambda\varpi}{2m} + \Big(D_4 - 2D_3\cos\frac{\mu\varpi}{2m} + D_2\Big)\sin\frac{3\lambda\varpi}{2m}$$
$$+ \quad \cdots + \Big(-2D_{m-1}\cos\frac{\mu\varpi}{2m} + D_{m-2}\Big)\sin\frac{(m-1)\lambda\varpi}{2m} + D_{m-1}\sin\frac{\lambda\varpi}{2} = \frac{L}{2^{m-1}}\sin\frac{\lambda\varpi}{2}$$

$$D_2 - 2D_1\cos\frac{\mu\varpi}{2m} = 0, \quad D_3 - 2D_2\cos\frac{\mu\varpi}{2m} + D_1 = 0, \quad D_4 - 2D_3\cos\frac{\mu\varpi}{2m} + D_2 = 0,$$
$$\cdots, \ -2D_{m-1}\cos\frac{\mu\varpi}{2m} + D_{m-2} = 0, \quad D_{m-1} = \frac{L}{2^{m-1}}.$$

From here, we have to get the value of D.
¶ 25. (pp.85-86).

It is clear that the quantity D constitute a recursive progression, which begins with the bottom, it is as follows :

$$D_m = 0, \quad D_{m-1} = \frac{L}{2^{m-1}}, \quad D_{m-2} = 2D_{m-1}\cos\frac{\mu\varpi}{2m} - D_m, \quad D_{m-3} = 2D_{m-2}\cos\frac{\mu\varpi}{2m} - D_{m-1}, \quad \cdots$$

$$D_{m-n} = Aa^n + Bb^n$$

where, a and b are the roots of the quadratic :

$$z^2 - 2z\cos\frac{\mu\varpi}{2m} + 1 = 0$$

To solve the coefficients A and B, we assume $n = 0,\ m = 1$.

$$A + B = 0, \quad Aa + Bb = \frac{L}{2^{m-2}}$$

$$B = -A, \quad A = \frac{L}{2^{m-2}(a-b)}, \quad B = -\frac{L}{2^{m-2}(a-b)}$$

$$D_{m-n} = \frac{L}{2^{m-2}}\frac{a^n - b^n}{a-b}$$

$$\frac{a^n - b^n}{a-b} = \frac{2^{m-2}}{L}D_{m-n} = \frac{\sin\frac{n\mu\varpi}{2m}}{\sin\frac{\mu\varpi}{2m}}$$

¶ 25. (p.87).

From here,

$$D_{m-n} = L2^{2-m}\frac{\sin\frac{n\mu\varpi}{2m}}{\sin\frac{\mu\varpi}{2m}}$$

For convenience sake, we assume $m - n = s$, then

$$D_s = \frac{L}{2^{m-2}}\sin\frac{(m-s)\mu\varpi}{2m}\Big/\sin\frac{\mu\varpi}{2m}$$

However,

$$\sin(m-s)\frac{\mu\varpi}{2m} = \sin\left(\frac{\mu\varpi}{2} - \frac{s\mu\varpi}{2m}\right) = \pm\sin\frac{s\mu\varpi}{2m}, \quad m, s, \mu \in \mathbb{Z}$$

where,

$$\begin{cases} + & \bmod\ (\mu, 2) = 1, \\ - & \bmod\ (\mu, 2) = 0 \end{cases}$$

Assuming L : const, then

$$D_s = \pm\Big(\frac{L}{2^{m-2}}\Big)\sin\frac{s\mu\varpi}{\sin}\Big/\sin\frac{\mu\varpi}{2m}$$

¶ 26. (p.87).

$$\begin{aligned} & y_\mu\Big[D_1\sin\frac{\mu\varpi}{2m} + D_2\sin\frac{2\mu\varpi}{2m} + D_3\sin\frac{3\mu\varpi}{2m} + \cdots + D_{m-1}\sin\frac{(m-1)\mu\varpi}{2m}\Big] \\ = \ & \pm\frac{L}{2^{m-2}\sin\frac{\mu\varpi}{2m}}\Big[S_1\sin\frac{\mu\varpi}{2m} + S_2\sin\frac{2\mu\varpi}{2m} + S_3\sin\frac{3\mu\varpi}{2m} + \cdots + S_{m-1}\sin\frac{(m-1)\mu\varpi}{2m}\Big] \end{aligned} \quad (12)$$

We put the value of the bracket in the left hand-side of (12) by Y. From the observation in ¶ 25, Y turns into :

$$Y = \frac{L}{2^{m-1}}\ \frac{\sin\frac{\lambda\varpi}{2}}{\cos\frac{\lambda\varpi}{2m} - \cos\frac{\mu\varpi}{2m}}$$

¶ 26. (pp.88-89.)
Here, if we assume $\lambda = \mu$, then

$$Y = \frac{L}{2^{m-1}} \frac{\sin \frac{\mu\varpi}{2}}{\cos \frac{\mu\varpi}{2m} - \cos \frac{\mu\varpi}{2m}} \tag{13}$$

By $\sin \frac{\mu\varpi}{2} = 0$, Y turns into $\frac{L}{2^{m-1}} \frac{0}{0}$. To seek this exact value of the last factor, we differentiate the last factor of (13) :

$$\frac{\sin \frac{\lambda\varpi}{2}}{\cos \frac{\lambda\varpi}{2m} - \cos \frac{\mu\varpi}{2m}}$$

then

$$\frac{m \cos \frac{\lambda\varpi}{2}}{-\sin \frac{\lambda\varpi}{2m}}$$

Considering $\mu \in \mathbb{Z}$, $\cos \frac{\mu\varpi}{2} = \pm 1$, where,

$$\begin{cases} + & \mathrm{mod}\ (\mu, 2) = 0, \\ - & \mathrm{mod}\ (\mu, 2) = 1 \end{cases}$$

then

$$Y = \frac{L}{2^{m-1}} \frac{m}{\sin \frac{\mu\varpi}{2m}}$$

Hence, (5) turns into :

$$\pm y_\mu \frac{Lm}{2^{m-1}} = \pm \frac{L}{2^{m-2}} \Big[S_1 \sin \frac{\mu\varpi}{2m} + S_2 \sin \frac{2\mu\varpi}{2m} + S_3 \sin \frac{3\mu\varpi}{2m} + \cdots + S_{m-1} \sin \frac{(m-1)\mu\varpi}{2m} \Big]$$

Finally, Lagrange gets the coefficient y_μ :

$$y_\mu = \frac{2}{m} \Big[S_1 \sin \frac{\mu\varpi}{2m} + S_2 \sin \frac{2\mu\varpi}{2m} + S_3 \sin \frac{3\mu\varpi}{2m} + \cdots + S_{m-1} \sin \frac{(m-1)\mu\varpi}{2m} \Big] \tag{14}$$

[8, ¶23-26, pp.79-89]
The above mentioned Lagrange's long steps (¶25-26, pp.79-89) correspond to Poisson's only few steps : (33)-(34)-(35) or Dirichlet's one. [8]

Lagrange states the next steps of deduction of integral in the next section 2.2.

2.2. *Solution de différents problèmes de calcul intégral. Des vibrations d'une corde tendue et changée d'un nombre quelconque de poids* by Lagrange [12], 1762-65.

We can see *Miscellanea Taurinensia, III*, which Poisson [35] and Riemann [37] cite as the alledged 'original' trigonometric series is (17).
¶ 40. (The n-body model of the sonic cord.)

> Supposons présentement que le nombre n des corps soit très grand, et que, par conséquent, la distance a d'un corps à l'autre soit très-petit, la longeur de toute la corde étant égale à 1 ; il est clair que les différences $\Delta^2 Y$, $\Delta^4 Y$, $\cdots$ deviendront très-petite du second ordre, du quatrième, $\cdots$; donc, puisque $k = \sqrt{\frac{nc^2}{a}} = \frac{c}{a}$, à cause de $n = \frac{1}{a}$, les quantités $k\Delta^2 Y$, $k\Delta^4 Y$, $k^2\Delta^6 Y$, $\cdots$ seront très-petite du second ordre, du quatrième, $\cdots$; et par conséquent les quantités P et Q pourront être regardées et traitées comme nulles sans erreur sensible.
>
> Ainsi, dans cette hypothèse, on aura à très-peu près le mouvement de la corde, en faisant passer par les sommets des ordonnées très-proches Y', Y'', Y''', $\cdots$,

[8] cf. Dirichlet's (74) and (75), which seem Dirichlet obeys and cites Lagrange's and Poisson's mathematical sense. cf. § 10.

lesquelles représentent la figure initial du polygone vibrant, une courbe dont l'équation sont

$$y = \alpha \sin \pi x + \beta \sin 2\pi x + \gamma \sin 3\pi x + \cdots + \omega \sin n\pi x, \quad (15)$$

et que j'appellerai *génératrice*, et prenant ensuit pour l'ordonnée du polygone vibrant, qui répond à une abscisse quelconque $\frac{s}{n+1} = x$, la demi-somme de deux ordonnées de cette courbe, desquelle l'une réponde à l'abscisse $\frac{s+kt}{n+1} = x + ct$, et l'autrer éponde à l'abscisse $\frac{s-kt}{n+1} = x - ct$; et cette détermination sera toujours d'autant plus exacte que le nombre n sera plus grand. Or il est évident que plus le nombre des poids est grand, plus le polygone initial doit s'approcher de la courbe circonscrite ; d'où il s'ensuit qu'en supposant le nombre des poids infini, ce qui est le cas de la corde vibrante, on pourra regarder la figure initiale même de la corde comme une branche de la courbe génératrice, et qu'ainsi pour avoir cette courbe il n'y aura qu'à transporter la coubre initial alternativement au-desus et au-dessus de l'axe à l'infini (numéro précédent). [12, ¶ 40, p.551-2]

¶ 41. (Deduction of trigonometric series and its coefficients.)

Pour confirmer ce que je viens de dire, je vais faire voir comment on peut trouver une infinité de telles courbes, qui coincident avec une courbe donnée en un nombre quelconque de poids aussi près les uns des autres qu'on voudra. Pour cela je prends l'équation

$$y = \frac{2Y_1}{n+1} \sin x\pi + \frac{2Y_2}{n+1} \sin 2x\pi + \frac{2Y_3}{n+1} \sin 3x\pi + \cdots + \frac{2Y_n}{n+1} \sin nx\pi$$

et, par ce que j'ai démontré dans le n^o 39, j'aurai, lorsque $x = \frac{s}{n+1}$, $y = Y^{(')}$.

Soient maintenant $n + 1 = \frac{1}{dX}$, $\quad \frac{s}{n+1} = X$, on aura

$$y_m = \int Y \sin mX\pi = (n+1) \int Y \sin mX\pi dX, \quad (16)$$

cette intégral étant prise depuis $X = 0$ jusqu'à $X = 1$; par conséquent

$$\begin{aligned} y = \; & 2 \int Y \sin X\pi dX \;\; \sin x\pi + 2 \int Y \sin 2X\pi dX \;\; \sin 2x\pi + 2 \int Y \sin 3x\pi \sin 3x\pi + \cdots \\ + \; & 2 \int Y \sin nX\pi dX \;\; \sin nx\pi \quad (17) \end{aligned}$$

de sorte que, lorsque $x = X$, on aura $y = Y$, Y étant l'ordonné qui réspond à l'sbscisse X.

[12, ¶ 41, p.553]

Lagrange's (1), (14) and (17) corresponds with Poisson's (32), (36) and (37), and Dirichlet's (73), (76) and (81)-(82)-(83) respectively. We can observe each sequential steps to deduce the trigonometric series by the Table 1, which tells each meticulousness.

3. Fourier [5], 1822

Chapter 3. *Propagation de la chaleur dans un solide rectangulaire infini*, pp.141-238.

§6 *Dévelopment d'une function arbitraire en séries trigonométriques*

¶ 219. An arbitrary function can be developed under the following form :

$$a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + a_4 \sin 4x \cdots \quad (18)$$

Fourier states his kernel in ¶ 219 − 235. He redescribes these articles from the corresponding of his first version. He announces these correction in 'Discours Preliminaire', however, the proof

TABLE 1. The expressions of deductive steps into trigonometric series in our paper

no	steps	Lagrange	Fourier manuscript	Poisson extract	Fourier prize paper	Fourier 2nd edition	Poisson	Dirichlet	Riemann
1	bibliography year	[8]1759, [12]1762-65	[6]1807	[24]1808	[6]1811	[2]1822	[33]1823	[4]1837	[37]1867
2	arbitrary function by trigonometric series : $f(x) =$	(15)	(18)		(18)	(18)	(31)		(??)
3	transfer array	§ 2.1				§ 3	§ 4.1.1	§ 10	
4	transfer matrix(mine)	(1)				(27)	(32)	(73)	
5	multiply $2\sin *$ and sum						(33)	(74)	
6	difference of term by term						(34)-(35)	(75)	
7	general coefficient expression 1	(3)							
8	general coefficient expression 2	(14)				(28)	(36)	(76)	
9	coefficient a_n, b_n by integral	(16)				(20)		(77),(81)	
10	expression by integral	(17)				(19),(23),(24)			
11	expression by sum		(21)						
12	final expression with sin	(17)	(21)		(19)	(19)	(37)		
13	combination of sin and cos in interval $(-\pi, \pi)$					(25)		(80)	
14	final expression with both series of sin and cos					(26)		(81),(82), (83)	

is completely same with the expression of first version, except the different expression between (19) and (21).

$$(D)_F \quad \frac{\pi}{2}\varphi(x) = \sin x \int \sin x\varphi(x)dx + \sin 2x \int \sin 2x\varphi(x)dx + \ \cdots \ + \sin ix \int \sin ix\varphi(x)dx + \cdots ;(19)$$

¶ 221. Fourier states only from the proving of orthonormal relation, so Poisson is disapointed with the lack of vigorousness and exactitude of the very mathematical importance in the future.

> Lagrange, dans les anciens Mémoires de Turin, et M. Fourier, dans ses Recherches sur la théorie de la chaleur, avaient déjà fait usage de sembles expressions ; mais il m'a semblé qu'elles n'avaient point encore été démonstrées d'une manière précise et rigoureuse ; [33, ¶28, p.46]

The following are Fourier's description about the proof of trigonometric series.

> On peut aussi vérifier l'équation précédente $(D)_F$ (art. 219), en déterminant immédiatement les quantités $a_1,\ a_2,\ a_3,\ \cdots,\ a_j,\ \cdots$ dans l'équation
>
> $$\varphi(x) = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots + a_j \sin jx \cdots$$
>
> pour cela on multipliera chacun des membres de dernière équation par $\sin ixdx$, i étant un nombre entier, et on prendra l'intégrale depuis $x = 0$ jusqu'à $x = \pi$,

on aura

$$\int \varphi(x)\sin ixdx = a_1 \int \sin x \sin ixdx + a_2 \int \sin 2x \sin ixdx + \cdots a_j \int \sin jx \sin ixdx + \cdots$$

Or on peut facilement prover :

1. Que toutes les intégrales qui entrent dans le second membre ont une valeur nulle, excepté le seul terme $a_i \int \sin ix \sin ixdx$;
2. Que la valeur de $\int \sin ix \sin ixdx$ est $\frac{\pi}{2}$.

Tout se réduit á considérer la valeur des intégrales qui entrent dans la second membre, et à démonstrer les deux propositions précédentes. L'intégrale $2\int \sin jx \sin ixdx$ prise depuis $x = 0$ jusqu'à $x = \pi$, et dans laquelle i et j sont des nombres entiers, est

$$\frac{1}{i-j}\sin(i-j)x - \frac{1}{i+j}\sin(i+j)x + C$$

L'intégrale devant commencer lorsque $x = 0$, la constante C est null, et les nombres i et j étant entiers, la valeur de l'intégrale deviendra null lorsqu'on fera $x = \pi$; il s'ensuit que chacun des termes tels que

$$a_1 \int \sin x \sin ixdx, \quad a_2 \int \sin 2x \sin ixdx, \quad a_3 \int \sin 3x \sin ixdx, \quad \cdots$$

s'evanouit, et que cela aura lieu toutes les fois que les nombres i et j seron différents. Il n'en est pas de même lorsque les nombres i et j sont égaux ; car le terme $\frac{1}{i-j}\sin(i-j)x$ auquel se réduit l'intégrale devient $\frac{0}{0}$, et sa valeur est π. On a, par conséquent,

$$2\int \sin\ ix\ \sin\ ix\ dx = \pi;$$

on obtient ainsi, de la manière la plus briéve, les valeurs de $a_1,\ a_2,\ a_3,\ \cdots,\ a_j,\ \cdots$ qui sont

$$a_1 = \frac{2}{\pi}\int \varphi(x)\sin xdx, \quad a_2 = \frac{2}{\pi}\int \varphi(x)\sin 2xdx, \quad a_3 = \frac{2}{\pi}\int \varphi(x)\sin 3xdx, \cdots, a_i = \frac{2}{\pi}\int \varphi(x)\sin ixdx \quad (20)$$

En les substituant, on a $(D)_F$ (=(19)). [2, ¶220-221, pp.210-212]

[9]Here, in the Fourier's first version, or, the manuscript in 1807, $(D)_F$ (=(19)) corresponds with (21)

$$\frac{\pi}{2}\varphi x = \sin x\ S(\varphi x \sin .xdx) + \sin 2x\ S(\varphi x \sin .2xdx) + \cdots + \sin ix\ S(\varphi x \sin .ixdx) \cdots; \quad (21)$$

where, S means a summation symbol for the trigonometric series in Fourier's n-body-model analysis.[10] [6, p.217]

¶ 224−231. (Trigonometric series by cosine with multiple angles.)

$$(m)_F \quad \varphi(x) = a_0 \cos 0x + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots + a_i \cos ix + \cdots \quad (22)$$

$\Rightarrow$

$$\begin{aligned}(\nu)_F \quad \frac{\pi}{2}\varphi(x) &= \frac{1}{2}\int_0^\pi \varphi(x)dx + \cos x\int_0^\pi \varphi(x)\cos x\ dx + \cos 2x\int_0^\pi \varphi(x)\cos 2x\ dx \\ &+ \cos 3x\int_0^\pi \varphi(x)\cos 3x\ dx + \cdots \end{aligned} \quad (23)$$

¶ 232. (Trigonometric series by sine with multiple angles.)

$$(\mu)_F \quad \frac{\pi}{2}\varphi(x) = \sin x\int_0^\pi \varphi(x)\sin x\ dx + \sin 2x\int_0^\pi \varphi(x)\sin 2x\ dx + \sin 3x\int_0^\pi \varphi(x)\sin 3x\ dx + \cdots \quad (24)$$

[9]cf. Grattan-Guinness [6, ¶63, p.216-7].

[10]cf. § 2.2. Grattan-Guinness discusses the n-body-model analysis in [6, pp.241-9].

¶ 233. (Trigonometric series in the interval $(-\pi, \pi)$.)

$F(x) = \varphi(x) + \psi(x), \quad f(x) = \varphi(x) - \psi(x) = F(-x), \quad \varphi(x) = \varphi(-x), \quad \psi(x) = -\psi(-x)$

$$\varphi(x) = \frac{F(x) + F(-x)}{2}, \qquad \psi(x) = \frac{F(x) - F(-x)}{2} \tag{25}$$

Combining $(\nu)_F$ and $(\mu)_F$ i.e. (23) and (24), we get the following last expression :

$$\begin{aligned}(p) \quad \pi F(x) &= \frac{1}{2}\int_{-\pi}^{\pi} F(x)\ dx \\ &+ \cos x \int_{-\pi}^{\pi} F(x)\cos x\ dx + \cos 2x \int_{-\pi}^{\pi} F(x)\cos 2x\ dx + \cdots \\ &+ \sin x \int_{-\pi}^{\pi} F(x)\sin x\ dx + \sin 2x \int_{-\pi}^{\pi} F(x)\sin 2x\ dx + \cdots \end{aligned} \tag{26}$$

The arts ¶ $231 - 237$ are the plus parts of edition in 1822 version by Fourier, who improves here his theories against Lagrange and Poisson.

¶ 235. (The development of function in the trigonometric series.)

Nous aurions à ajouter plusieurs remarques concernant l'usage et les propriété des séries trigonométriques ; nous nous bornerons à énoncer brièvement celles qui ont un rapport plus direct avec la théorie dont nous nous occupons.

1. Les séries ordonnées selon les cosinus ou les sinus des arcs multiples sont toujous convergentes, c'est-à-dire qu'en donnant à la variable une valeur quelque non imaginaire, la somme des termes converge de plus en plus vers une seul limite fixe, qui est la valeur de la fonction développée ;
2. Si l'on a l'expression de la fonction $f(x)$ qui répond à une série donnée

$$a + b\cos x + c\cos 2x + d\cos 3x + e\cos 4x + \cdots$$

et celle d'une autre fonction $\varphi(x)$, dont le développement donné est

$$\alpha + \beta\cos x + \gamma\cos 2x + \delta\cos 3x + \varepsilon\cos 4x + \cdots$$

il est facile de trouver en termes réels la somme de la série composée

$$a\alpha + b\beta + c\gamma + d\delta + e\varepsilon + \cdots$$

et, plus généralement, celle de la série

$$a\alpha + b\beta\cos x + c\gamma\cos 2x + d\delta\cos 3x + e\varepsilon\cos 4x + \cdots$$

que l'on forme en comparant terme à terme les duex série données. Cette remarque s'applique à un nombre quelconque de séries.

3. La série (p)(art.233) (26)[11] qui donne le développment d'une fonction $F(x)$ en un situe de sinus et de cosinus d'arcs multiples peut être mise sous cette forme

$$\begin{aligned}\pi F(x) = \frac{1}{2}\int F(\alpha)\ d\alpha + &\cos x \int F(\alpha)\cos\alpha\ d\alpha + \cos 2x \int F(\alpha)\cos 2\alpha\ d\alpha + \cdots \\ + &\sin x \int F(\alpha)\sin\alpha\ d\alpha + \sin 2x \int F(\alpha)\sin 2\alpha\ d\alpha + \cdots\end{aligned}$$

[11]cf. The series are the equation replaced α in the right-hand side of (26) with x.

α étant une nouvelle variable qui disparait après les intégrations. On a donc

$$\pi F(x) = \int_{-\pi}^{+\pi} F(\alpha)\ d\alpha \Big(\frac{1}{2} + \cos x \cos \alpha + \cos 2x \cos 2\alpha + \cdots + \sin x \sin \alpha + \sin 2x \sin 2\alpha + \cdots \Big)$$

ou

$$F(x) = \frac{1}{\pi} \int_{-\pi}^{+\pi} F(\alpha)\ d\alpha \Big(\frac{1}{2} + \cos(x-\alpha) + \cos 2(x-\alpha) + \cos 3(x-\alpha) + \cdots \Big)$$

Donc, en désignant par

$$\sum \cos\ i(x-\alpha)$$

aura

$$F(x) = \frac{1}{\pi} \int F(\alpha)\ d\alpha \Big[\frac{1}{2} + \sum \cos\ i(x-\alpha) \Big]$$

4. (citation omitted by the auther of this paper.)
[2, ¶ 235, p.232-3]

¶ 267. (The n equations : (m) with transfer matrix of $(n \times 2n)$.)

3. Transfer array by Fourier.

$$\begin{aligned}
a_1 &= A_1 \sin 0.0 \frac{2\pi}{n} + A_2 \sin 0.1 \frac{2\pi}{n} + A_3 \sin 0.2 \frac{2\pi}{n} + \cdots + A_n \sin 0.n \frac{2\pi}{n} \\
&+ B_1 \cos 0.0 \frac{2\pi}{n} + B_2 \cos 0.1 \frac{2\pi}{n} + B_3 \cos 0.2 \frac{2\pi}{n} + \cdots + B_n \cos 0.n \frac{2\pi}{n} \\
a_2 &= A_1 \sin 1.0 \frac{2\pi}{n} + A_2 \sin 1.1 \frac{2\pi}{n} + A_3 \sin 1.2 \frac{2\pi}{n} + \cdots + A_n \sin 1.n \frac{2\pi}{n} \\
&+ B_1 \cos 1.0 \frac{2\pi}{n} + B_2 \cos 1.1 \frac{2\pi}{n} + B_3 \cos 1.2 \frac{2\pi}{n} + \cdots + B_n \cos 1.n \frac{2\pi}{n} \\
&\cdots \\
a_n &= A_1 \sin(n-1)0 \frac{2\pi}{n} + A_2 \sin(n-1)1 \frac{2\pi}{n} + A_3 \sin(n-1).2 \frac{2\pi}{n} + \cdots \\
&+ B_1 \cos(n-1)0 \frac{2\pi}{n} + B_2 \cos(n-1)1 \frac{2\pi}{n} + B_3 \cos(n-1)2 \frac{2\pi}{n}\ \cdots
\end{aligned}$$

$$\begin{aligned}
&\left[\ a_1\ a_2\ a_3\ \cdots\ a_n\ \right]^T \\
&= \begin{bmatrix}
\sin 0.0\frac{2\pi}{n} & \sin 0.1\frac{2\pi}{n} & \sin 0.2\frac{2\pi}{n} & \cdots & \sin 0.n\frac{2\pi}{n} & \cos 0.0\frac{2\pi}{n} & \cos 0.1\frac{2\pi}{n} & \cos 0.2\frac{2\pi}{n} & \cdots & \cos 0.n\frac{2\pi}{n} \\
\sin 1.0\frac{2\pi}{n} & \sin 1.1\frac{2\pi}{n} & \sin 1.2\frac{2\pi}{n} & \cdots & \sin 1.n\frac{2\pi}{n} & \cos 1.0\frac{2\pi}{n} & \cos 1.1\frac{2\pi}{n} & \cos 1.2\frac{2\pi}{n} & \cdots & \cos 1.n\frac{2\pi}{n} \\
\sin 2.0\frac{2\pi}{n} & \sin 2.1\frac{2\pi}{n} & \sin 2.2\frac{2\pi}{n} & \cdots & \sin 2.n\frac{2\pi}{n} & \cos 2.0\frac{2\pi}{n} & \cos 2.1\frac{2\pi}{n} & \cos 2.2\frac{2\pi}{n} & \cdots & \cos 2.n\frac{2\pi}{n} \\
\cdots \\
\sin(n-1)0\frac{2\pi}{n} & \sin(n-1)1\frac{2\pi}{n} & \sin(n-1)2\frac{2\pi}{n} & \cdots & \cos(n-1)0\frac{2\pi}{n} & \cos(n-1)1\frac{2\pi}{n} & \cos(n-1)2\frac{2\pi}{n} & \cdots
\end{bmatrix} \\
&\times \left[\ A_1\ A_2\ A_3\ \cdots\ A_n\ B_1\ B_2\ B_3\ \cdots\ B_n\ \right]^T
\end{aligned} \tag{27}$$

¶ 270. (Introduction of trigonometric series.)

En general, la somme de produits terms à term est égale à 0, ou $\frac{1}{2}n$, ou n ; au reste, les formules connues conduiraient directment aux mêmes résultats. On les presente ici comme des conséquences évidentes des theorèmes élémentaires de la Trigonométrie.

¶ 271. (The coefficients A_j and B_j of the equation m.)

$$\frac{1}{2}nA_j = \sum_{i=1}^{n} a_i\ \sin\ (i-1)(j-1)\frac{2\pi}{n}, \quad \frac{1}{2}nB_j = \sum_{i=1}^{n} a_i\ \cos\ (i-1)(j-1)\frac{2\pi}{n}, \quad j = 1, \cdots, n \tag{28}$$

¶ 277. (Changing of the communicational analysis from the disjoint masses to continuum.)

$$\varphi(x,t) = v = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx + \frac{1}{\pi}\Big[\sin x\int_0^{2\pi} f(x)\sin xdx + \cos x\int_0^{2\pi} f(x)\cos xdx\Big]e^{-g\pi t}$$
$$+ \frac{1}{\pi}\Big[\sin 2x\int_0^{2\pi} f(x)\sin 2xdx + \cos 2x\int_0^{2\pi} f(x)\cos 2xdx\Big]e^{-2^2 g\pi t}$$
$$+ \cdots,$$

This expression coresponds to the final formation of trigonometric series (26). Substituting $g\pi$ with k,

$$(E)_F \quad \pi v = \frac{1}{2}\int_0^{2\pi} f(x)dx + \Big[\sin x\int_0^{2\pi} f(x)\sin xdx + \cos x\int_0^{2\pi} f(x)\cos xdx\Big]e^{-kt}$$
$$+ \Big[\sin 2x\int_0^{2\pi} f(x)\sin 2xdx + \cos 2x\int_0^{2\pi} f(x)\cos 2xdx\Big]e^{-2^2 kt}$$
$$+ \cdots,$$

4. Poisson

4.1. ***Suite du Mémoire sur les intégrales définies et sur la sommation des séries*, by Poisson** [35]**, 1823.**

Poisson has observed the problems on the definite integral during 12 years of 1811-23 in the series : [25], [26], [28], [30], and finally, [35].

4.1.1. Expression des Fonctions par des Séries de Quantités périodiques.

¶58. (pp.435-8).

$$(b)_P \quad fx = \frac{1}{2l}\int_{-l}^{l} fx'\,dx' + \frac{1}{l}\int_{-l}^{l}\Big[\sum\cos\frac{n\pi(x-x')}{l}\Big]fx'\,dx'$$

$$(f)_P \quad fx = \frac{1}{l}\int_0^{l} fx'\,dx' + \frac{2}{l}\int_0^{l}\Big[\sum\cos\frac{n\pi x}{l}\cos\frac{n\pi x'}{l}\Big]fx'\,dx'$$

$$(g)_P \quad fx = \frac{2}{l}\int_0^{l}\Big[\sum\sin\frac{n\pi x}{l}\sin\frac{n\pi x'}{l}\Big]fx'\,dx'$$

$$fx = \frac{1}{4l}\int_{-l}^{l} fx'\,dx' + \frac{1}{2l}\int_{-l}^{l}\Big[\sum\cos\frac{n\pi(x-x')}{2l}\Big]fx'\,dx' \tag{29}$$

We divide the second term of the right-hand side of (29) into even and odd part, then

$$fx = \frac{1}{4l}\int_{-l}^{l} fx'\,dx' + \frac{1}{2l}\int_{-l}^{l}\Big[\sum\cos\frac{n\pi(x-x')}{l}\Big]fx'\,dx' + \frac{1}{2l}\int_{-l}^{l}\Big[\sum\cos\frac{(2n-1)\pi(x-x')}{2l}\Big]fx'\,dx' \tag{30}$$

Multiplying (30) with 2 and subtract with $(b)_P$, then

$$fx = \frac{1}{l}\int_{-l}^{l}\Big[\sum\cos\frac{(2n-1)\pi(x-x')}{2l}\Big]fx'\,dx'$$

¶62. (pp.444-9). The integral known facts reduced to Lagrange's.
We suppose $n > 0 \in \mathbb{Z}$, $f(\frac{m}{n+1}) = y_m$, $\quad m = 1,\ 2,\ 3,\ \cdots,\ n$. We state n equations :

$$(i)_P \quad y = Y_1\sin\pi x + Y_2\sin 2\pi x + Y_3\sin 3\pi x + \cdots + Y_n\sin n\pi x \tag{31}$$

4.1.1. Transfer array by Poisson.

$$
\begin{aligned}
y_1 &= Y_1 \sin\frac{\pi}{n+1} + Y_2 \sin\frac{2\pi}{n+1} + Y_3 \sin\frac{3\pi}{n+1} + \cdots + Y_n \sin\frac{\pi n}{n+1}\\
y_2 &= Y_1 \sin\frac{2\pi}{n+1} + Y_2 \sin\frac{4\pi}{n+1} + Y_3 \sin\frac{6\pi}{n+1} + \cdots + Y_n \sin\frac{2\pi n}{n+1}\\
y_3 &= Y_1 \sin\frac{3\pi}{n+1} + Y_2 \sin\frac{6\pi}{n+1} + Y_3 \sin\frac{9\pi}{n+1} + \cdots + Y_n \sin\frac{3\pi n}{n+1}\\
&\cdots\cdots\cdots\cdots\\
y_n &= Y_1 \sin\frac{\pi n}{n+1} + Y_2 \sin\frac{2\pi n}{n+1} + Y_3 \sin\frac{3\pi n}{n+1} + \cdots + Y_n \sin\frac{\pi n^2}{n+1}
\end{aligned}
$$

Now, we can show with a today's style of $(n \times n)$ transform matrix :[12]

$$
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \sin\frac{\pi}{n+1} & \sin\frac{2\pi}{n+1} & \sin\frac{3\pi}{n+1} & \cdots & \sin\frac{\pi n}{n+1} \\ \sin\frac{2\pi}{n+1} & \sin\frac{4\pi}{n+1} & \sin\frac{6\pi}{n+1} & \cdots & \sin\frac{2\pi n}{n+1} \\ \sin\frac{3\pi}{n+1} & \sin\frac{6\pi}{n+1} & \sin\frac{9\pi}{n+1} & \cdots & \sin\frac{3\pi n}{n+1} \\ \cdots & & & & \\ \sin\frac{\pi n}{n+1} & \sin\frac{2\pi n}{n+1} & \sin\frac{3\pi n}{n+1} & \cdots & \sin\frac{\pi n^2}{n+1} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_n \end{bmatrix} \tag{32}
$$

Multiplying with $2\sin\frac{\pi m}{n+1}, \quad 2\sin\frac{2\pi m}{n+1}, \quad 2\sin\frac{3\pi m}{n+1}, \cdots, 2\sin\frac{n\pi m}{n+1}$, then the coefficient $Y_{m'}$, where $m' \neq m$, is as follows :

$$
2\sin\frac{\pi m'}{n+1}\sin\frac{\pi m}{n+1} + 2\sin\frac{2\pi m'}{n+1}\sin\frac{2\pi m}{n+1} + 2\sin\frac{3\pi m'}{n+1}\sin\frac{3\pi m}{n+1} + \cdots + 2\sin\frac{n\pi m'}{n+1}\sin\frac{n\pi m}{n+1}, \tag{33}
$$

This is the difference in term by term of two sums (34) and (35):

$$
1 + \cos\frac{\pi(m'-m)}{n+1} + \cos\frac{2\pi(m'-m)}{n+1} + \cos\frac{3\pi(m'-m)}{n+1} \cdots + \cos\frac{n\pi(m'-m)}{n+1}, \tag{34}
$$

$$
1 + \cos\frac{\pi(m'+m)}{n+1} + \cos\frac{2\pi(m'+m)}{n+1} + \cos\frac{3\pi(m'+m)}{n+1} \cdots + \cos\frac{n\pi(m'+m)}{n+1}, \tag{35}
$$

$$
\begin{cases} (34) = \frac{1}{2}[1-\cos(m'-m)\pi] = 1, \quad (35) = \frac{1}{2}[1-\cos(m'+m)\pi] = 1, & m' \neq m, \quad m',\ m \in \mathbb{Z}, \\ (34) = n+1, \quad (35) = \frac{1}{2}[1-\cos\ 2m\pi] = 0, & m' = m, \quad m',\ m \in \mathbb{Z} \end{cases}
$$

If $m' \neq m$, the difference is zero, if $m' = m$, $(34) - (35) = n + 1$. [13] Then we must divide Y_m by $n+1$:

$$
Y_m = \frac{2}{n+1}\Big(y_1 \sin\frac{\pi m}{n+1} + y_2 \sin\frac{2\pi m}{n+1} + y_3 \sin\frac{3\pi m}{n+1} + \cdots + y_n \sin\frac{n\pi m}{n+1}\Big) \tag{36}
$$

Here, Poisson explains the exchange the sum of Y_m with the integral $\int_0^1$ by a special technique of interpolation.

> Les coefficiens $Y_1,\ Y_2,\ Y_3,\ \cdots,\ Y_n$, étant ainsi déterminés, la formule $(i)_P$ coïncidera avec la fonction fx,
> - pour toutes les valeurs de x contenues depuis $x = 0$ jusqu'à $x = 1$, et qui sont des multiples exacts de la fraction $\frac{1}{n+1}$;
> - et pour les autres valeurs de x comprises dans le même intervalle,

[12] Poisson doesn't use the transform-matrix symbol, but mine.

[13] $n+1$ comes from $1 + n \times 1$ of (34).

on devra la regarder comme une formule d'interpolation d'une espèce particulière, qui pourra servir à calculer les valeurs approchées de fx, quand la forme de cette fonction ne sera pas connus. Si l'on construit deux coubres qui aient x et y pour coordonées, dont

- l'une ait $y = fx$ pour équation,
- et l'autre l'équation $(i)_P$,

ces deux coubres couperont l'axe des abscisses x aux deux points correspondans à $x = 0$ et $x = 1$; et dans l'intervalle compris entre ces deux points, elles auront un nombre n de points communs, dont les projections sur l'axe des x seront équidistantes. Ce resultat subsistera, quelque grand qu'on suppose le nombres n ; à mesure que ce nombre augmentera, les points communs aux deux coubres se rapprocheront ; et à la limite $n = \infty$, ces deux coubres coïncideront parfaitement dans toute la portion comprise depuis $x = 0$ jusqu'à $x = 1$. Or, à cette limite, la somme qui exprime la valeur de Y_m se changera en une integrale définie ; [35, pp.446-7]

If we suppose $\frac{m'}{n+1} = x'$, $\frac{1}{n+1} = dx'$, and $y_{m'} = fx'$, then [14]

$$Y_m = 2\int_0^1 \sin m\pi x' \cdot \ fx'\ dx', \quad m > 0, \ \in \mathbb{Z}$$

We extend $(i)_P$ to the infinite and replace y with fx, then

$$fx = \sum_{m=1}^{\infty} Y_m \sin m\pi x = 2\sum_{m=1}^{\infty}\Big(\int_0^1 \sin m\pi x' \cdot \ fx'\ dx'\Big)\sin m\pi x, \quad m > 0, \ \in \mathbb{Z} \tag{37}$$

This statement corresponds with $(g)_P$, by assuming $l = 1$ and replacing the order of simbol of sum $\int$ and $\sum$. Therefore, this statement means the Lagrange's statement of trigonometric series, which we cite with the equation (17).

La même méthod pourrait servir à démontrer directment toutes les autres formules de la même espèce ; il y a donc deux moyans de parvenir à ces formules ; celui que j'ai employé, et qui consiste à regarder la série périodique que chacune de ces expressions renferme, comme la limite d'une série convergente dont on peut avoir la somme ; et celui qui je viens d'exposer, d'après *Lagrange*, et dans lequel on coinsidère chacune de ces expressions comme la limite d'une formule d'interpolation. Les recherches de M.*Fourier* sur la distribution de la chaleur dans les corps solides, et mon premier Mémoire sur le même sujet, contiennent différents formules de cette espèce. [35, pp.447-8]

¶64. (pp.452-4). We assume $\frac{n\pi}{l} = a$, $\frac{\pi}{l} = da$, we get

$$(k)_P \quad fx = \frac{1}{\pi}\iint \cos a(x - x')\ fx'\ da\ dx' \equiv P$$

$$\int_0^{\infty} \cos a(x - x')\ da = \int_0^{\infty} e^{-ka^2}\cos a(x - x')\ da = \frac{1}{2}\sqrt{\frac{\pi}{k}}e^{\frac{(x-x')^2}{4k}}$$

$$P = \frac{1}{2\sqrt{k\pi}}\int_{-\infty}^{\infty} e^{-ka^2}\ fx'\ dx'$$

We assume $x' = x + z$, then

$$P = \frac{fx}{2\sqrt{k\pi}}\int e^{-ka^2}\ dz \quad \Rightarrow \quad P = \frac{fx}{2\sqrt{k\pi}}\int_{-\infty}^{\infty} e^{-ka^2}\ dz = fz$$

[14] fx, fx', φx, ψx, etc. mean the then usage of $f(x)$, $f(x')$, $\varphi(x)$, $\psi(x)$, etc.

Another method :

$$\int \cos a(x-x')\ da = \frac{\sin a(x-x')}{x-x'} \quad \Rightarrow \quad P = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin a(x-x')}{x-x'}\ fx'\ dx'$$

We assume $x' = x + \frac{z}{a}$, then

$$P = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin z}{z}\ f(x+\frac{z}{a})\ dz \quad \Rightarrow_{a\to\infty} \quad P = \frac{1}{\pi}\ fx \int_{-\infty}^{\infty} \frac{\sin z}{z}\ dz = fx$$

¶65. (pp.454-6). We assume φx, ψx are two functions of x, such as $\varphi x = \varphi(-x)$, $\psi x = -\psi(-x)$, namely implicit and explicit functions.

$$fx = \frac{1}{\pi}\iint \cos ax \cos ax'\ fx'\ dx' + \iint \frac{1}{\pi} \sin ax \sin ax'\ fx'\ dx' \tag{38}$$

$$\varphi x = \frac{1}{\pi}\iint \cos ax\ \ \cos ax'\ \varphi x'\ dx', \quad \psi x = \frac{1}{\pi}\iint \sin ax\ \ \sin ax'\ \psi x'\ dx' \tag{39}$$

> On pourra, si l'on veut, n'étandre l'equation relatives à x', que depuis $x' = 0$ jusqu'à $x' = \infty$, et doubler le facteur $\frac{1}{\pi}$; ces formules coincideront alors avec celle que M.*Fourier* a données dans son premier Mémoire sur la chaleur. [15] [35, p.455]

The equation $(k)_P$ is reciprocally deduced from (39), by conserving $x' = \pm\infty$, then

$$0 = \frac{1}{\pi}\iint \cos ax\ \ \cos ax'\ \psi x'\ dx', \quad 0 = \frac{1}{\pi}\iint \sin ax\ \ \sin ax'\ \varphi x'\ dx' \tag{40}$$

Adding (39) and (40), we get $(k)_P$.

$$\begin{aligned}
\varphi x + \psi x &= \frac{1}{\pi}\iint \Big(\cos ax \cos ax'\ \varphi x'\ dx' + \sin ax \sin ax'\ \varphi x'\ dx'\Big) \\
&+ \Big(\sin ax \sin ax'\ \psi x'\ dx' + \cos ax \cos ax'\ \psi x'\ dx'\Big) \\
&= \frac{1}{\pi}\iint (\varphi x' + \psi x') \cos a(x-x')\ da\ dx' = fx
\end{aligned}$$

$$fx = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} \cos ax\ \ \cos ax'\ fx'\ dx', \quad fx = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} \sin ax\ \ \sin ax'\ fx'\ dx'$$

4.2. **Poisson** [33, 35]**, 1823-35.**

Poisson [35], [36, pp.183-232] discuss this problem, reffering to Lagrange in both papers spanned twelve years, as the 'mathematical theory' of heat entitling his paper like Fourier. [35, §62, pp.444-449], [36, §101, pp.200-204]. Cauchy [1] struggles to find it reffering step by step his results to Poisson's.

§1, *Équations differentielles du Mouvement de la Chaleur dans une Barre d'une petit épaisseur* ¶4.

$$(1)_P \quad \frac{du}{dt} = a^2\frac{d^2u}{dx^2} - bu$$

$$(2)_P \quad \begin{cases} \frac{du}{dx} + \beta u = 0, & \text{for} \quad x = l, \\ \frac{du}{dx} + \beta' u = 0, & \text{for} \quad x = -l' \end{cases}$$

[15] sic. *Annales de physique et de chimie*, tome III, p.361.

§2, *Distribution de la Chaleur dans une Barre prismatique, d'une petite épaisseur*

¶11. The secondary differential equation : $(1)_P$, which Laplace [13] proposes with the definite integral : [16]

$$(3)_P \quad u = \frac{e^{-bt}}{\sqrt{\pi}} \int e^{-\alpha^2}\ f(x + 2\ a\ \alpha\sqrt{t})\ d\alpha$$

¶12. If we put $x + 2a\alpha\sqrt{t} = x'$, then

$$\alpha = \frac{x' - x}{2a\sqrt{t}}, \quad \Rightarrow \quad d\alpha = \frac{dx'}{2a\sqrt{t}}, \quad \Rightarrow \quad u = \frac{e^{-bt}}{2a\sqrt{\pi t}} \int e^{\frac{-x-x'^2}{4a^2t}}\ f(x')\ dx'$$

$$u = \frac{e^{-bt}\ e^{\frac{-x-x'^2}{4a^2t}}}{2a\sqrt{\pi t}} \int\ f(x')\ dx'$$

¶16.

$$x + 2a\alpha\sqrt{t} = y, \quad \Rightarrow \quad d\alpha = \frac{dy}{2a\sqrt{t}}$$

$$u = \frac{e^{-bt}}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4a^2t}} f(y)dy, \quad \Rightarrow \quad -\frac{du}{dx} = \frac{e^{-bt}}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4a^2t}} df(y)$$

The first equation $(2)_P$, which satisfies with $x = l$, becomes

$$\int_{-\infty}^{\infty} e^{-\frac{(l-y)^2}{4a^2t}} \Big(\beta f(y) + \frac{df(y)}{dy}\Big) dy = 0$$

$$(4)_P \quad \beta f(l + y) + \frac{df(l + y)}{dy} + \beta f(l - y) - \frac{df(l - y)}{dy} = 0$$

$$(5)_P \quad u = e^{-bt} \int_0^{\infty} \Big(\int_{\infty}^{\infty} \cos(y - x)z\ f(y)\ dy \Big)\ e^{-a^2\ t\ \rho^2},$$

¶17. [17]

$$\int_0^{\infty} e^{-hy}\ f(y)\ dy = p, \quad \int_0^{-\infty} e^{hy}\ f(y)\ dy = q,$$

1. Multiplying the both hand-sides of an even function $f(y) = f(-y)$ with $e^{-hy}dy$ and integrate from $y = 0$ to $y = \infty$:

$$\int_0^{\infty} e^{-hy}\ f(y)\ dy = \int_0^{\infty} e^{-hy}\ f(-y)\ dy$$

2. Multiplying the both hand-sides of an even function :

$$f(l + y) = -f(l - y) \tag{41}$$

with $e^{-hy}dy$ and integrate from $y = 0$ to $y = \infty$:

$$\int_0^{\infty} e^{-hy}\ f(l + y)\ dy = -\int_0^{\infty} e^{-hy}\ f(l - y)\ dy$$

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[16] (⇓) Laplace proposes

$$y = \frac{1}{\sqrt{\pi}} \int\ dz\ e^{-z^2}\ \varphi(x + 2\ z\ \sqrt{x'})$$

[13, p.241].

[17] (⇓) Liouville [14, §5, §8, §10]

$$\int_0^\infty e^{-hy}\ f(l+y)\ dy = e^{hl}\Big[p - \int_0^l e^{-hy}\ f(y)\ dy\Big] \tag{42}$$

$$\int_0^\infty e^{-hy}\ f(l-y)\ dy = -e^{hl}\Big[q - \int_0^l e^{hy}\ f(y)\ dy\Big] \tag{43}$$

From the equality in even function $f(y) = f(-y)$, between the right hand-side of (42) and the one of (43) are :

$$e^{hl}\Big[p - \int_0^l e^{-hy}\ f(y)\ dy\Big] = e^{hl}\Big[q - \int_0^l e^{hy}\ f(y)\ dy\Big]$$

By $p = -q$,

$$\int_0^l \Big(e^{h(l-y)}) - (e^{h(y-l)}\Big)\ f(y)\ dy = \psi(h), \quad e^{hl} + e^{-hl} = \varphi(h)$$

By ψ and φ', cos $\rho l = 0$,

$$\begin{cases} \psi(\rho\sqrt{-1}) = 2\sqrt{-1}\int_0^l \sin\ \rho(l-y)\ f(y)\ dy, \\ \varphi'(\rho\sqrt{-1}) = 2l\sqrt{-1}\sin\rho l \end{cases}$$

By $\sin\ \rho(l-y) = \sin\rho l\ \ \cos\rho\ y$

$$\psi(\rho\sqrt{-1}) = 2\sqrt{-1}\sin\rho\ l\int_0^l \cos\ \rho y\ f(y)\ dy$$

From here, we get (α) :

$$f(x) = \frac{2}{l}\sum\cos\rho x\int_0^l \cos\ \rho y\ f(y)\ dy$$

§9. We consider the functionx in the outside of interval $[l', l],\ l' = -l$. We assume the next two conditions :

$$\begin{cases} f(l+y) + f(l-y) = 0, \\ f(-l+y) + f(-l-y) = 0, \end{cases}$$

where we contain implicitly the two special conditions : $f(l) = 0,\ f(-l) = 0$. Similarly, from (42) and (43) are :

$$\int_0^\infty e^{-hy}\ f(l+y)\ dy = e^{hl}\Big[p - \int_0^l e^{-hy}\ f(y)\ dy\Big] \tag{44}$$

$$\int_0^\infty e^{-hy}\ f(l-y)\ dy = -e^{hl}\Big[q - \int_0^l e^{hy}\ f(y)\ dy\Big] \tag{45}$$

Similarly, as (41)

$$f(l+y) = -f(l-y) \tag{46}$$

1. Multiplying the both hand-sides of a function (46) with $e^{-hy}dy$ and integrate from $y = 0$ to $y = \infty$:

$$\int_0^\infty e^{-hy}\ f(l+y)\ dy = -\int_0^\infty e^{-hy}\ f(l-y)\ dy$$

$$e^{hl}p - e^{-hl}q = e^{hl}\int_0^l e^{-hy}\ f(y)\ dy - e^{-hl}\int_0^l e^{hy}f(y)\ dy$$

$$f(-l+y) = -f(-l-y) \tag{47}$$

2. Multiplying the both hand-sides of a function (47) with $e^{-hy}dy$ and integrate from $y=0$ to $y=\infty$:

$$\int_0^\infty e^{-hy}\ f(l+y)\ dy = -\int_0^\infty e^{-hy}\ f(l-y)\ dy$$

$$e^{-hl}p - e^{hl}q = e^{-hl}\int_0^{-l} e^{-hy}\ f(y)\ dy - e^{hl}\int_0^{-l} e^{hy} f(y)\ dy$$

$$(a)_L \quad \begin{cases} \beta f(l+y) + \frac{df(l+y)}{dy} + \beta f(l-y) + \frac{df(l-y)}{dy} = 0, \\ \beta' f(-l+y) + \frac{df(-l+y)}{dy} + \beta' f(-l-y) + \frac{df(-l-y)}{dy} = 0 \end{cases}$$

here, we consider similarly about $f(x)$

$$\begin{cases} \frac{df(x)}{dx} + \beta f(x) = 0, \quad \text{for} \quad x = l, \\ \frac{df(x)}{dx} + \beta' f(x) = 0, \quad \text{for} \quad x = -l \end{cases} \tag{48}$$

[18] Remenbering (44) and (45), shown by Poisson, put the equation $(\alpha)_L$. [19]

$$e^{-\beta y} d[e^{\beta y} f(l+y)] = e^{\beta y} d[e^{-\beta y} f(l-y)]$$

$$e^{-yh} f(l+y) + (h+\beta)\int e^{-yh} f(l+y)\ dy = C + e^{-yh} f(l-y) + (h-\beta)\int e^{-yh} f(l-y) dy$$

$$(h+\beta)\int e^{-yh} f(l+y)\ dy = (h-\beta)\int e^{yh} f(l-y) dy$$

$$(h+\beta)e^{-hl}p + (h-\beta)e^{-hl}q = (h+\beta)e^{hl}\int_0^l e^{-hy} f(y)dy + (h-\beta)e^{-hl}\int_0^l e^{hy} f(y)dy$$

Changing β with $-\beta'$ and l with $-l$, then

$$(h-\beta')e^{-hl}p + (h+\beta')e^{-hl}q = (h-\beta')e^{-hl}\int_0^{-l} e^{-hy} f(y)dy + (h+\beta')e^{hl}\int_0^{-l} e^{hy} f(y)dy$$

$$p = \frac{\psi(h)}{\varphi(h)}, \quad q = \frac{\psi(-h)}{\varphi(-h)} \tag{49}$$

Hence, for abriviation :

$$\begin{cases} (h+\beta)(h+\beta')e^{-2hl}\int_0^{-l}\ e^{-hy} f(y)\ dy - (h-\beta)(h-\beta')e^{-2hl}\int_0^l\ e^{-hy} f(y)\ dy \\ +(h-\beta)(h+\beta')\Big[\int_0^l\ e^{hy} f(y)\ dy - \int_0^{-l}\ e^{hy} f(y)\ dy\Big] = \psi(h), \\ \\ (h+\beta)(h+\beta')e^{2hl} - (h-\beta)(h-\beta')e^{-2hl} = \varphi(h) \end{cases}$$

then we get (49). We replace in (49), $h = g + z\sqrt{-1}$ in the value of p, and $h = g - z\sqrt{-1}$ in the value of q, then

$$p = \frac{\psi(z\sqrt{-1}+g)}{\varphi(z\sqrt{-1}+g)}, \quad q = \frac{\psi(z\sqrt{-1}-g)}{\varphi(z\sqrt{-1}-g)}$$

$$(6)_P \quad \int_{-\infty}^\infty e^{-zy\sqrt{-1}}\ f(y)\ dy = p - q = \frac{\psi(z\sqrt{-1}+g)}{\varphi(z\sqrt{-1}+g)} - \frac{\psi(z\sqrt{-1}-g)}{\varphi(z\sqrt{-1}-g)}$$

---

[18](⇓) This is a boundary value problem of Sturm-Liouville type.

[19]sic. *Journal de l'Ecole Polytechnique*, 19^e cahier, page 30. ((⇓) cf. Poisson [33, p.30], §2, *Distribution de la Chaleur dans une Barre prismatique, d'une petite épaisseur*, ¶15.)

¶18. [20] We assume $z = \rho + z'$.

$$\int_{-\infty}^{+\infty} e^{-xy\sqrt{-1}}\ f(y)\ dy \equiv Z, \quad \int_{-\infty}^{+\infty} e^{xy\sqrt{-1}}\ f(y)\ dy \equiv Z'$$

$$\int_{-\infty}^{\infty} \cos z(y-x)\cdot f(y)dy = \frac{1}{2}Ze^{x(\rho+z')\sqrt{-1}} + \frac{1}{2}Z'e^{-x(\rho+z')\sqrt{-1}},$$

Substitute these value in $(1)_L$ and neglect z' except for Z and Z', then we get :

$$u = e^{-bt}\sum\Big(e^{\rho x\sqrt{-1}}\int Zdz' + e^{-\rho x\sqrt{-1}}\int Z'dz'\Big)\ e^{-a^2\ t\ \rho^2}, \tag{50}$$

We define $\frac{d\varphi(h)}{dh} = \varphi'(h),\ z = \rho + z'$ and assume z' and g are infinitesimaly small.

$$\begin{cases}\varphi(z\sqrt{-1}\pm g) = (z'\sqrt{-1}\pm g)\ \varphi'(\rho\sqrt{-1}),\\ \psi(z\sqrt{-1}\pm g) = \psi(\rho\sqrt{-1}),\end{cases}$$

$$\begin{aligned} Z = &\ \frac{\psi(z\sqrt{-1}+g)}{\varphi(z\sqrt{-1}+g)} - \frac{\psi(z\sqrt{-1}-1)}{\varphi(z\sqrt{-1}-1)} \\ = &\ \frac{\psi(\rho\sqrt{-1})}{(z'\sqrt{-1}+g)\ \varphi'(\rho\sqrt{-1})} - \frac{\psi(\rho\sqrt{-1})}{(z'\sqrt{-1}-g)\ \varphi'(\rho\sqrt{-1})} \\ = &\ \frac{\psi(\rho\sqrt{-1})}{\varphi'(\rho\sqrt{-1})}\ \Big(\frac{1}{(z'\sqrt{-1}+g)} - \frac{1}{(z'\sqrt{-1}-g)}\Big) \\ = &\ \frac{2g}{(g^2+z'^2)}\ \frac{\psi(\rho\sqrt{-1})}{\varphi'(\rho\sqrt{-1})} \end{aligned}$$

$$\int_{-\delta}^{+\delta} Zdz' = 4\frac{\psi(\rho\sqrt{-1})}{\varphi'(\rho\sqrt{-1})}\arctan\frac{\delta}{g},$$

[21] If $\rho = 0$, we can integrate only the interval of $\int_0^\delta$, then the value reduces into incomplete. If $g = 0$ then we get :

$$\int_0^\delta Zdz' = 2\pi\ \frac{\psi(\rho\sqrt{-1})}{\varphi'(\rho\sqrt{-1})},$$

$$\int Z'dz' = 2\pi\ \frac{\psi(-\rho\sqrt{-1})}{\varphi'(\rho\sqrt{-1})}$$

From (50), we get :

$$u = e^{-bt}\sum\Big(\frac{e^{\rho x\sqrt{-1}}\ \psi(\rho\sqrt{-1}) + e^{-\rho x\sqrt{-1}}\ \psi(-\rho\sqrt{-1})}{\varphi'(\rho\sqrt{-1})}\Big)\ e^{-a^2\ t\ \rho^2}$$

¶19.

$$(7)_P \quad (\beta\beta' - \rho^2)\sin\ 2l\rho + (\beta+\beta')\rho\cos\ 2l\rho = 0$$

---

[20](⇓) Liouville [14, §7.]

[21](⇓)

$$\int\frac{dx}{c^2+x^2} = \frac{1}{c}tan^{-1}\frac{x}{c}.$$

$$\Big[(\beta\beta'-\rho^2)\cos\ 2l\rho-(\beta+\beta')\rho\sin\ 2l\rho-\beta\beta'-\rho^2\Big]\int_{-l}^{l}\cos\ \rho y\ f(y)\ dy+(\beta-\beta')\rho\int_{-l}^{l}\sin\ \rho y\ f(y)\ dy=P$$

$$\Big[(\beta\beta'-\rho^2)\cos\ 2l\rho-(\beta+\beta')\rho\sin\ 2l\rho+\beta\beta'+\rho^2\Big]\int_{-l}^{l}\sin\ \rho y\ f(y)\ dy+(\beta-\beta')\rho\int_{-l}^{l}\cos\ \rho y\ f(y)\ dy=Q$$

$$\Big[\beta+\beta'+2l(\beta\beta'-\rho^2)\Big]\cos\ 2l\rho-[2+2l(\beta+\beta')]\rho\sin\ 2l\rho=R$$

$$(8)_P\quad u=e^{-bt}\sum\Big(\frac{P\cos\rho x+Q\sin\ \rho x}{R}\Big)e^{-a^2\ t\ \rho^2}$$

¶22.

$$(9)_P\quad f(x)=\sum\Big(\frac{P\cos\rho x+Q\sin\ \rho x}{R}\Big)$$

> Comme l'équation $(9)_P$ est une suite nécessaire de notre analyse, il ne peut rester aucun doute sur son exactitude ; mais il serait difficile de l'obtenir *à priori*, ou de la verifier dans toute sa généralité. [33, p.38]

We can see the slite difference of conclusions between [35, §62, p.449] and [36, §101, p.204], which had brought during 12 years of study after [35].

> In addition, independently with the formulae which we have talked up to now, and which include the series of ordered sequence of the sins or cosines of multiplied of variable, it is deduced frequently, in the problems of physic or mechanics, into other expression of the same nature, containing the series of sins or cosines, where, the variable angle, multiplied by the roots of one of the transcendental equations which the form doesn't depend on the every particular question. But, two Memoire on the heat include many these formulae, which are given as the results necessary for the rigorous solutions of various problems which I am occupied ; however, I don't know any method to perform directly these expressions, to which, the method of this no. (of article) and that of no, and that of no. 57 aren't applicable. [35, §62, p.449]

> In addition, the formulae preceding and all of we have got in this chapter, are included in the equation (22) of no. 86 ; however, this equation contains a great number of other formulae of the same nature, which we must admit as the certain result from the general solution of every problem, and it will desire that it deduces to prove from a more direct method. Unfortunately, the mode of proof by Lagrange and that of no. 93 seem not to be able to apply to that of other formulae, in which an arbitrary function is not explained by the series of sins or cosines of multiplied by 1, 2, 3, 4, $\cdots$, or, 1, 3, 5, 7, $\cdots$. , of the variable, as in all the preceding formulae. [36, §101, p.204]

In the former, Poisson avoids to name Lagrange's fault and cites implicitely it, the latter cites explicitely Lagrange's difficulty. In both paper or book, Poisson recognizes the defect to apply to other formulae of his own method.

5. Cauchy [1], 1823

Cauchy says the following object of this paper in the top page :
§1.

L'objet que je me propose dans ce Mémoire est de résoudre la question suivante :

Étant donnée entre la variable principale φ et les variables indépendentes $x,\ y,\ z,\ \cdots,\ t$ une équation linéaire aux différences partielles et à coefficiens constans avec un dernier terme fonction des variables indépendentes, intégrer cette équation de manière que les quantités

$$\varphi,\ \frac{d\varphi}{dt},\ \frac{d^2\varphi}{dt^2},\ \cdots$$

se réduisent à des fonctions connues de $x,\ y,\ z,\cdots$, pour $t=0$. [1, p.511]

私のこの論文の目的とするところは次の問題を解くことである：
主変数 φ と独立変数 $x,\ y,\ z,\ \cdots,\ t$ の間に与えられたある偏微分方程式で最後の項に定数を持ったものがあるとして、この方程式を次の値：$\varphi,\ \frac{d\varphi}{dt},\ \frac{d^2\varphi}{dt^2},\ \cdots$ が $t=0$ とする $x,\ y,\ z,\cdots$ から成る既知関数に帰着するように積分せよ。

This is a what-is-called initial value problem.

In the second chapter, in which Cauchy intends to solve the problem, he says my logic is unavoidable to fall into a 'circular argument' of $(73)_C \Rightarrow (74)_C \sim (77)_C \Rightarrow (73)_C$.

$$(17)_C \quad R=-\Big(\frac{e^{\theta_0 t}}{\theta_0^m F(\theta_0)}+\frac{e^{\theta_1 t}}{\theta_1^m F'(\theta_1)}+\cdots+\frac{e^{\theta_{m-1} t}}{\theta_{m-1}^m F'(\theta_{m-1})}\Big)$$

If we suppose as follows :

$$(19)_C \quad Q=\Big(\frac{1}{2\pi}\Big)^n\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots R\ e^{\alpha(x-\mu)\sqrt{-1}}e^{\beta(y-\nu)\sqrt{-1}}e^{\gamma(z-\varpi)\sqrt{-1}}\cdots\ d\alpha\ d\beta\ d\gamma\cdots$$

$$\begin{aligned}
(20)_C\quad & \varphi \\
= \quad & \nabla_0\frac{d^{m-1}Q}{dt^{m-1}}\iiint\cdots Q\ f_0(\mu,\ \nu,\ \varpi,\cdots)d\mu\ d\nu\ d\varpi\cdots \\
+ \quad & \Big(\nabla_0\frac{d^{m-2}Q}{dt^{m-2}}\iiint\cdots Q\ f_1(\mu,\ \nu,\ \varpi,\cdots)d\mu\ d\nu\ d\varpi\cdots+\nabla_1\frac{d^{m-1}Q}{dt^{m-1}}\iiint\cdots Q\ f_1(\mu,\ \nu,\ \varpi,\cdots)d\mu\ d\nu\ d\varpi\cdots\Big) \\
+ \quad & \cdots \\
+ \quad & \Big(\nabla_0\iiint\cdots Q\ f_{m-1}(\mu,\ \nu,\ \varpi,\cdots)d\mu\ d\nu\ d\varpi+\nabla_1\frac{d}{dt}\iiint\cdots Q\ f_{m-1}(\mu,\ \nu,\ \varpi,\cdots)d\mu\ d\nu\ d\varpi+\cdots \\
+ \quad & \nabla_{m-1}\frac{d^{m-1}}{dt^{m-1}}\iiint\cdots Q\ f_{m-1}(\mu,\ \nu,\ \varpi,\cdots)d\mu\ d\nu\ d\varpi\cdots\Big)
\end{aligned}$$

$$= \Big[\ \nabla_0\tfrac{d^{m-1}Q}{dt^{m-1}}+\nabla_0\tfrac{d^{m-2}Q}{dt^{m-2}}+\cdots+\nabla_0+\cdots\nabla_{m-1}\tfrac{d^{m-1}}{dt^{m-1}}\ \Big]\begin{bmatrix}\iiint\cdots Q\ f_0(\mu,\ \nu,\ \varpi,\cdots)d\mu\ d\nu\ d\varpi\cdots \\ \iiint\cdots Q\ f_1(\mu,\ \nu,\ \varpi,\cdots)d\mu\ d\nu\ d\varpi\cdots \\ \iiint\cdots Q\ f_2(\mu,\ \nu,\ \varpi,\cdots)d\mu\ d\nu\ d\varpi\cdots \\ \cdots \\ \iiint\cdots Q\ f_{m-1}(\mu,\ \nu,\ \varpi,\cdots)d\mu\ d\nu\ d\varpi\cdots\end{bmatrix}$$

$$(21)_C \quad \nabla Q=0$$

$$(1)_C \quad \nabla\varphi=0$$

$$(22)_C \quad \varphi=U+V$$

$$(23)_C \quad \nabla V=0$$

$$(24)_C \quad V=0$$

§2.
§3.

$$(29)_C \quad R = -\frac{1}{m}\Big(\frac{e^{\theta_0 t}}{\theta_0^{2m-1}} + \frac{e^{\theta_1 t}}{\theta_1^{2m-1}} + \cdots + \frac{e^{\theta_{m-1} t}}{\theta_{m-1}^{2m-1}}\Big)$$

where, $\theta_0,\ \theta_1,\ \cdots\ ,\theta_{m-1}$: m roots of the equation :

$$(30)_C \quad \theta^m = A_0$$

Replacing φ with 1, $\frac{d\varphi}{dx}$ with $\alpha\sqrt{-1}$, $\cdots$, and generally

$$\frac{d^{p+q+r+\cdots}\varphi}{dx^p\ dy^q\ dz^r\ \cdots}$$

by $(\alpha\sqrt{-1})^p\ (\beta\sqrt{-1})^q\ (\gamma\sqrt{-1})^r \cdots$, then we get :

$$(19)_C \quad Q = \Big(\frac{1}{2\pi}\Big)^n \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots R\ e^{\alpha(x-\mu)\sqrt{-1}} e^{\beta(y-\nu)\sqrt{-1}} e^{\gamma(z-\varpi)\sqrt{-1}} \cdots\ d\alpha\ d\beta\ d\gamma \cdots$$

$$\begin{aligned}(31)_C \quad \varphi \ &= \ \nabla_0 \frac{d^{m-1}Q}{dt^{m-1}} \iiint \cdots Q\ f_0(\mu,\ \nu,\ \varpi,\cdots) d\mu\ d\nu\ d\varpi \cdots \\ &+ \ \nabla_0 \frac{d^{m-2}Q}{dt^{m-2}} \iiint \cdots Q\ f_1(\mu,\ \nu,\ \varpi,\cdots) d\mu\ d\nu\ d\varpi \cdots \\ &+ \ \cdots \\ &+ \ \nabla_0 \iiint \cdots Q\ f_{m-1}(\mu,\ \nu,\ \varpi,\cdots) d\mu\ d\nu\ d\varpi \cdots\end{aligned}$$

$$= \Big[\ \nabla_0 \tfrac{d^{m-1}Q}{dt^{m-1}} + \nabla_0 \tfrac{d^{m-2}Q}{dt^{m-2}} + \cdots + \nabla_0\ \Big] \begin{bmatrix} \iiint \cdots Q\ f_0(\mu,\ \nu,\ \varpi,\cdots) d\mu\ d\nu\ d\varpi \cdots \\ \iiint \cdots Q\ f_1(\mu,\ \nu,\ \varpi,\cdots) d\mu\ d\nu\ d\varpi \cdots \\ \iiint \cdots Q\ f_2(\mu,\ \nu,\ \varpi,\cdots) d\mu\ d\nu\ d\varpi \cdots \\ \cdots \\ \iiint \cdots Q\ f_{m-1}(\mu,\ \nu,\ \varpi,\cdots) d\mu\ d\nu\ d\varpi \cdots \end{bmatrix}$$

$$(32)_C \quad \nabla_0 Q = \frac{d^m Q}{dt^m}, \quad \Rightarrow \quad \nabla_0\Big(\frac{d^m Q}{dt^m}\Big) = \frac{d^{m+m-1}Q}{dt^{m+m-1}} = \frac{d^{2m-1}Q}{dt^{2m-1}}$$

$$= \Big(\frac{1}{2\pi}\Big)^n \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \frac{d^{2m-1}R}{dt^{2m-1}}\ e^{\alpha(x-\mu)\sqrt{-1}} e^{\beta(y-\nu)\sqrt{-1}} e^{\gamma(z-\varpi)\sqrt{-1}} \cdots\ d\alpha\ d\beta\ d\gamma \cdots$$

$$= \Big(\frac{1}{2\pi}\Big)^n \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \frac{e^{\theta_0 t} + e^{\theta_1 t} + \cdots + e^{\theta_{m-1} t}}{m}\ e^{\alpha(x-\mu)\sqrt{-1}} e^{\beta(y-\nu)\sqrt{-1}} e^{\gamma(z-\varpi)\sqrt{-1}} \cdots\ d\alpha\ d\beta\ d\gamma \cdots$$

$$(33)_C \quad T = \frac{e^{\theta_0 t} + e^{\theta_1 t} + \cdots + e^{\theta_{m-1} t}}{m}$$

$$(34)_C \quad P = \Big(\frac{1}{2\pi}\Big)^n \iiint \cdots T\ e^{\alpha(x-\mu)\sqrt{-1}} e^{\beta(y-\nu)\sqrt{-1}} e^{\gamma(z-\varpi)\sqrt{-1}} \cdots\ d\alpha\ d\beta\ d\gamma \cdots$$

$$(35)_C \quad \nabla_0 \frac{d^{m-1}Q}{dt^{m-1}} = P$$

The general value of φ :

$$
\begin{aligned}
(36)_C \quad \varphi \; = \; & \iiint \cdots P\ f_0(\mu,\ \nu,\ \varpi,\cdots)d\mu\ d\nu\ d\varpi\cdots \\
+ \; & \int dt \iiint \cdots P\ f_1(\mu,\ \nu,\ \varpi,\cdots)d\mu\ d\nu\ d\varpi\cdots \\
+ \; & \int^2 dt^2 \iiint \cdots P\ f_2(\mu,\ \nu,\ \varpi,\cdots)d\mu\ d\nu\ d\varpi\cdots \\
+ \; & \cdots \\
+ \; & \int^{m-1} dt^{m-1} \iiint \cdots P\ f_{m-1}(\mu,\ \nu,\ \varpi,\cdots)d\mu\ d\nu\ d\varpi\cdots
\end{aligned}
$$

$$
= \left[\ 1+\int dt+\int^2 dt^2+\cdots+\int^{m-1} dt^{m-1}\ \right]
\begin{bmatrix}
\iiint\cdots P\ f_0(\mu,\ \nu,\ \varpi,\cdots)d\mu\ d\nu\ d\varpi\cdots \\
\iiint\cdots P\ f_1(\mu,\ \nu,\ \varpi,\cdots)d\mu\ d\nu\ d\varpi\cdots \\
\iiint\cdots P\ f_2(\mu,\ \nu,\ \varpi,\cdots)d\mu\ d\nu\ d\varpi\cdots \\
\cdots \\
\iiint\cdots P\ f_{m-1}(\mu,\ \nu,\ \varpi,\cdots)d\mu\ d\nu\ d\varpi\cdots
\end{bmatrix}
$$

From $(32)_C$

$$(37)_C \quad \nabla_0 P = \frac{d^m P}{dt^m}$$

§4.
§5.
§6.

$$(69)_C \quad \frac{d^m\varphi}{dt^m} = a\frac{d^l\varphi}{dx^l}$$

Regarding to $(33)_C$, $(34)_C$ and $(36)_C$, we get :

$$(70)_C \quad P = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{m}\Big(\exp\{\theta_0 t\}+\exp\{\theta_1 t\}+\cdots+\exp\{\theta_{m-1}t\}\Big)\exp\{\alpha(x-\mu)\sqrt{-1}\}\ d\alpha$$

$$(71)_C \quad \varphi = \int Pf_0(\mu)\ d\mu + \int dt\int Pf_1(\mu)\ d\mu + \ \cdots\ + \int^{m-1} dt^{m-1}\int Pf_{m-1}(\mu)\ d\mu,$$

where $\theta_0,\ \theta_1,\ \theta_2,\ \cdots\theta_{m-1}$ are the roots of the equation, and $\int^{m-1}\ dt^{m-1}$ means the $m-1$ times integrals with respect to t :

$$(72)_C \quad \theta^m = a\Big(\alpha\sqrt{-1}\Big)^l$$

In the $(69)_C$, we suppose $l=m=2,\ a=-1$ then $(69)_C$ turns into

$$\Leftrightarrow \quad (73)_C \quad \frac{d^2\varphi}{dt^2}+\frac{d^2\varphi}{dx^2}=0.$$

$$\Updownarrow$$

$$(74)_C \quad P = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{\alpha t}+e^{-\alpha t}}{2}e^{\alpha(x-\mu)\sqrt{-1}}\ d\alpha = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{\alpha t}+e^{-\alpha t}}{2}\cos\ \alpha(x-\mu)\ d\alpha$$

$$
\begin{aligned}
\Rightarrow \quad (75)_C \quad \varphi \; = \; & \frac{1}{2\pi}\iint\frac{e^{\alpha t}+e^{-\alpha t}}{2}\cos\ \alpha(x-\mu)\ f_0(\mu)\ d\alpha\ d\mu \\
+ \; & \frac{1}{2\pi}\int dt\iint\frac{e^{\alpha t}+e^{-\alpha t}}{2}\cos\ \alpha(x-\mu)\ f_1(\mu)\ d\alpha\ d\mu \\
\Rightarrow \quad (71)_C \quad \; = \; & \int P\ f_0(\mu)\ d\mu + \int dt\ \int P\ f_1(\mu)\ d\mu, \quad (m=2)
\end{aligned}
$$

This integral of φ is not determined, however we can integrate φ when we multiply $e^{-k\alpha^2}$ under the inner sign of integral and assume $\mu = x + 2k^{\frac{1}{2}}u$, where k is an arbitrary, infinitesimally small number. Cauchy explains the difficulties of this integral as follows :

> La value précédente de φ est indéterminée. Mais l'indétermination cessera pour l'ordinaire, si, dans chaque intégrale relative à la variable α, on multiplie la fonction sous le signe $\int$ par $e^{-k\alpha^2}$, k désignant un nombre infiniment petit. Alors, en effectuant les intégrations relatives à cette variable, et posant $\mu = x + 2k^{\frac{1}{2}}u$, on obtiendra la fomule

$$(76)_C \quad \varphi = \frac{1}{\sqrt{\pi}} e^{\frac{t^2}{4k}} \int_{-\infty}^{\infty} e^{-u^2} \cos\Big(\frac{ut}{\sqrt{k}}\Big) f_0(x + 2k^{\frac{1}{2}}u)\ du + \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{\frac{t^2}{4k}}\ dt \int_{-\infty}^{\infty} e^{-u^2} \cos\Big(\frac{ut}{\sqrt{k}}\Big) f_1(x + 2k^{\frac{1}{2}}u)\ du$$

where, k never become evapolate only due to the integral in respecting to u. We regard that $u = \frac{\mu - x}{2\sqrt{k}}$, $u^2 = \frac{(x-\mu)^2}{4k}$, $d\mu = 2\sqrt{k}\ du$. We use the formula :[22]

$$\int_0^{\infty} e^{-a^2x^2} \cos\ bx\ dx = \frac{\sqrt{\pi} \cdot e^{-\frac{b^2}{4a^2}}}{2a} \quad \Rightarrow \quad \int_{-\infty}^{\infty} e^{-a^2x^2} \cos\ bx\ dx = \frac{\sqrt{\pi} \cdot e^{-\frac{b^2}{4a^2}}}{a}$$

where $x = \alpha,\ b = x - \mu,\ x = b + \mu$. The first integral with respect to α, $a = k^{\frac{1}{2}},\ b = -2k^{\frac{1}{2}}u,\ x = \alpha$, and for the second with respect to t, $a = k^{\frac{1}{2}},\ b = \alpha,\ x = t$, namely :

$$\int_0^{\infty} e^{-k\alpha^2} \cos\ \alpha\big(-2k^{\frac{1}{2}}u\big)\ d\alpha = \frac{\sqrt{\pi} \cdot e^{\frac{-4ku^2}{4k}}}{2\alpha} = \frac{\sqrt{\pi} \cdot e^{-u^2}}{2\alpha}$$

$$\int_0^{\infty} e^{-\alpha\ t} \cos\ t\big(-2k^{\frac{1}{2}}u\big)\ dt = \frac{\sqrt{\pi} \cdot e^{\frac{t^2}{4k}}}{2\alpha}$$

Regarding $x - \mu = -2k^{\frac{1}{2}}u$, we may follow the integral of $(75)_C$ as follows :

$$\begin{aligned}
(75') \quad \varphi &= \frac{\sqrt{k}}{\pi} \int_{-\infty}^{+\infty} f_0(x + 2k^{\frac{1}{2}}u)\ du \int_{-\infty}^{+\infty} e^{-k\alpha^2}\Big(\frac{e^{\alpha t} + e^{-\alpha t}}{2}\Big) \cos\ \alpha(-2k^{\frac{1}{2}}u)\ d\alpha \\
&+ \frac{\sqrt{k}}{\pi} \int_0^{\infty} dt \int_{-\infty}^{+\infty} f_1(x + 2k^{\frac{1}{2}}u)\ du \int_{-\infty}^{+\infty} e^{-k\alpha^2}\Big(\frac{e^{\alpha t} + e^{-\alpha t}}{2}\Big) \cos\ \alpha(-2k^{\frac{1}{2}}u)\ d\alpha \\
&= \frac{1}{2}\frac{\sqrt{k}}{\pi} \int_{-\infty}^{+\infty} f_0(x + 2k^{\frac{1}{2}}u)\ du \int_{-\infty}^{+\infty} \Big(e^{-k\alpha^2 + \alpha t} + e^{-k\alpha^2 - \alpha t}\Big) \cos\ \alpha(-2k^{\frac{1}{2}}u)\ d\alpha \\
&+ \frac{1}{2}\frac{\sqrt{k}}{\pi} \int_0^{\infty} dt \int_{-\infty}^{+\infty} f_1(x + 2k^{\frac{1}{2}}u)\ du \int_{-\infty}^{+\infty} \Big(e^{-k\alpha^2 + \alpha t} + e^{-k\alpha^2 - \alpha t}\Big) \cos\ \alpha(-2k^{\frac{1}{2}}u)\ d\alpha \\
&= \frac{2}{2}\frac{\sqrt{k}}{\pi} e^{\frac{t^2}{4k}} \int_{-\infty}^{+\infty} f_0(x + 2k^{\frac{1}{2}}u)\ du\ 2\frac{\sqrt{\pi}\ e^{-\frac{4ku^2}{4k}}}{2\sqrt{k}} \cos\ \alpha(-2k^{\frac{1}{2}}u) \\
&+ \frac{2}{2}\frac{\sqrt{k}}{\pi} \int_0^{\infty} e^{\frac{t^2}{4k}}\ dt \int_{-\infty}^{+\infty} f_1(x + 2k^{\frac{1}{2}}u)\ du\ 2\frac{\sqrt{\pi}\ e^{-\frac{4ku^2}{4k}}}{2\sqrt{k}} \cos\ \alpha(-2k^{\frac{1}{2}}u)
\end{aligned}$$

[22] This integral is called Laplace integral, cf. *Iwanami Mathematical Formulae I*, [22, p.233]

where, top value of $\frac{2}{2}$ means the results of two integrals of $e^{-\alpha t}\ \cos\ \alpha(-2k^{\frac{1}{2}}u)t$ and $e^{\alpha t}\ \cos\ (-2k^{\frac{1}{2}}u)t$ with respect to t.

$$\Rightarrow \quad (76)_C \quad \varphi = \frac{1}{\sqrt{\pi}} e^{\frac{t^2}{4k}} \int_{-\infty}^{\infty} e^{-u^2} \cos\Big(\frac{ut}{\sqrt{k}}\Big) f_0(x + 2k^{\frac{1}{2}}u)\ du + \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{\frac{t^2}{4k}}\ dt \int_{-\infty}^{\infty} e^{-u^2} \cos\Big(\frac{ut}{\sqrt{k}}\Big) f_1(x + 2k^{\frac{1}{2}}u)\ du$$

$$(77)_C \quad \varphi = \frac{1}{2}\Big[f_0\big(x + t\sqrt{-1}\big) + f_0\big(x - t\sqrt{-1}\big)\Big] + \frac{1}{2}\int f_1\big(x + t\sqrt{-1}\big) + f_1\big(x - t\sqrt{-1}\big)\ dt$$

Mais, quoique cette derniér value de φ, substituée dans l'équation $(73)_C$, paraisse la verifier dans tous les cas, néanmoins on ne saurait la considére comme générale, tant que l'on n'aura pas donné de l'expression imaginaire $f(x+t\sqrt{-1})$ un definition indépendente de la forme de la fonction $f(x)$ supposée réelle. A la vérite, cette expression imaginaire se trouverait suffisamment définie, si l'on convenait de représenter par la notation $f(x+t\sqrt{-1})$ une fonction φ de x et de t, qui étant continue par rapport à ces deux variables, fût propre à remplir la double condition de se réduire à $f(x)$ pour $t = 0$, et de vérifier l'équation

$$(78)_C \quad \frac{d\varphi}{dt} + \frac{d\varphi}{dx}\sqrt{-1} = 0$$

Mais il est facil de voir que, dans ce cas, la fonction φ serait celle qui vérifie l'équation $(73)_C$ pour tous les valeurs possibles de t, et les équations de condition $\varphi = f(x)$, $\frac{d\varphi}{dt} = 0$, pour la valeur particulière $t = 0$.

Ainsi, la recherche de la fonction $f(x+t\sqrt{-1})$ se trouverait ramenée à l'intégration de la formule $(73)_C$, et l'on ne pourrait plus donner pour intégrale de cette formule l'équation $(77)_C$, sans tomber dans un cercle vicieux. [1, p.568]

こうして、関数 $f(x+t\sqrt{-1})$ を求めようとすれば (73) 式で積分することに帰着し、この式の積分のためには式 (77) を与えるしかないという循環論法に陥らざるを得ない。

We assume $m = 2,\ l = 1,\ a = b^2$, then

$$(69)_C \quad \Rightarrow \quad (79)_C \quad \frac{d^2\varphi}{dt^2} = b^2\frac{d\varphi}{dx}$$

$$(33)_C,\ (34)_C \quad \Rightarrow \quad (80)_C \quad P = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{(\alpha\sqrt{-1})^{\frac{1}{2}}bt} + e^{-(\alpha\sqrt{-1})^{\frac{1}{2}}bt}}{2}\ e^{\alpha(x-\mu)\sqrt{-1}}\ d\alpha$$

$$(36)_C \quad \Rightarrow \quad (81)_C \quad \varphi = \int P\ f_0(\mu)d\mu + \int dt \int P\ f_1(\mu)d\mu$$

Dirichlet doesn't miss Cauchy's description and that becoms Dirichlet's motivation for his following papers.

6. Dirichlet [3], 1829

Dirichlet's motivation in [3] to prove the unknown problem is due to Cauchy's confession about own defect of proving as follows :

Mais personne, que je sache, n'en a donné jusqu'à présent une démonstration générale. Je ne conais sur cet objet qu'un travail dù à M. Cauchy et qui fait partie des Mémoires de Académie des sciences de Paris pour l'année 1823. L'auteur de ce travail avoue lui-même que sa démonstration se trouve *en defaut* pour certaines fonctions pour lesquelles la convergence est poutant incontestable. [3, p.119]

しかし、私の知る限り今日まで、誰も一般的証明に成功していない。この目的では MAS に 1823 年に提出した Cauchy に拠る論文しか知らない。この論文の著者は自ら、自分の証明は、ある関数に対しては収束は議論の余地のないにも拘わらず破綻すると漏らしている。

$$f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_m \sin mx + \cdots, \text{ where, } a_m = \frac{2}{\pi}\int_0^\pi \sin\ mx\ f(x)dx \tag{51}$$

$$f(x) = \frac{1}{2}b_0 + b_1 \cos x + b_2 \cos 2x + \cdots + b_m \cos mx + \cdots, \text{ where, } b_m = \frac{2}{\pi}\int_0^\pi \cos\ mx\ f(x)dx \tag{52}$$

He binds (77) and (79), then gets the final series :

$$\begin{aligned}\varphi(x) = \frac{1}{2}b_0 \quad &+ \quad b_1 \cos x + b_2 \cos 2x + \cdots + b_m \cos mx + \cdots \\ &+ \quad a_1 \sin x + a_2 \sin 2x + \cdots + a_m \sin mx + \cdots \end{aligned} \tag{53}$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos mxg(x)dx, \qquad a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin mxg(x)dx$$

Dirichlet's target of proving is the convergence of following summation of the first $n+1$ terms of (81) :

$$\frac{1}{\pi}\int_{-\pi}^{\pi} d\alpha\ g(\alpha)\frac{\sin(2n+1)\frac{\alpha-x}{2}}{2\sin\frac{\alpha-x}{2}}$$

7. Liouville [14], 1836

Liouville (1809-82) 1836 [14] introduces Poisson's works of proving the trigonometric series for an arbitrary function as follows :

> In regard to the equality of the form, $f(x) = \sum A_i\ \sin\frac{i\pi x}{l}$, however, serving as the result of the partial differential equation to solve a physico-mathematics, we have proposed to consider it by itself, abstraction made with the particular question where it presents ; And this idea have brought up the excellent theory of periodic series which Mr. Poisson have exposed at first in the 19th cahier of JEP, and after it, recently, in his works on the heat theory. [14, p.16] (trans. mine.)

§1.

$$f(x) = A_1 \sin\frac{\pi x}{l} + A_2 \sin\frac{2\pi x}{l} + \cdots + A_i \sin\frac{i\pi x}{l}$$

§2.

> Mais au lieu de regarder les égalités de la forme
>
> $$f(x) = \sum A_i \sin\frac{i\pi x}{l}$$
>
> comme le résultat de la l'intégration d'une équation aux différences partielles servant à résoudre un probléme physico-mathématique, on s'est aussi proposé de la considérer en elle-mêmes, abstraction faite des questions particulières où elles se présentent et cette idée a donné naissance à la belle théorie des séries périodiques que M.Poisson a exposée d'abord dans le 19^e cahier du *Journal de*

l'Ecole polytechnique,[23] et qu'il a reproduite récemment dans son ouvrage sur la chareur.

Cette théorie des séries périodiques, ainsi traitée comme un point d'analyse pure, en devient à la fois plus élégante et plus rigoureuse ; mais telle que M.Poisson l'ai donnée dans les mémoires cités, elle se borne aux développemens des fonctions ou tarties de fonctions d'une variable x en séries de sinus et cosinus des multiples entiers d'un arc proportionnel à x, et elle ne s'étend en aucune manière aux autres séries de sinus et cosinus que l'on rencontre aussi dans certains successifs s'obtiennent en multipliant la variable x par les diverses racines d'une équation transcendente.

Je me propose ici de faire connaitre une méthode au moyan de laquelle on effectuera d'une manière directe les développemens des fonctions ou parties de fonctions en séries de sinus et cosinus. Pour trouver cette méthode, il m'a suffi de modifier légèrement un procédé fort ingénieux dont M.Poisson a fait usage dans ses deux premiers Mémoires sur la *Théorie de la chaleur.* [24] La modification dont je parle consiste surtout en ce que j'ai pris pour point de départ formule

$$f(x) = \frac{1}{\pi}\int_0^\infty dx \int_{-\infty}^\infty \cos z(y-x)\cdot f(y)dy, \tag{54}$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \cos\ zx\ dz \int_{-\infty}^\infty \cos zy\cdot f(y)dy + \frac{1}{\pi}\int_0^\infty \sin\ zx\ dz \int_{-\infty}^\infty \sin zy\cdot f(y)dy \tag{55}$$

§3.

$$(A)_L \quad f(x) = \sum (A\cos\ px + B\sin\ px)$$

$$f(x) = \int_0^\infty \cos\ zx\ U dz + \int_0^\infty \sin\ zx\ V dz$$

$$U = \frac{1}{\pi}\int_{-\infty}^{+\infty} \cos\ zy\ f(y)\ dy, \quad V = \frac{1}{\pi}\int_{-\infty}^{+\infty} \sin\ zy\ f(y)\ dy$$

§4.

$$(1)_L \quad f(x) = \frac{1}{\pi}\int_0^\infty dz \int_{-\infty}^{+\infty} \cos\ z(y-x)\ f(y)\ dy$$

25

$$u = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\sin\ (y-x)}{y-x} f(y)\ dy$$

By $y = x + \frac{\theta}{z}$,

$$u = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\sin\ \theta}{\theta} f(x+\frac{\theta}{z})\ d\theta$$

If $z=\infty$, then $f(x+\frac{\theta}{z}) = f(x)$, moreover,

$$u = \frac{f(x)}{\pi}\int_{-\infty}^{+\infty} \frac{\sin\ \theta}{\theta}\ d\theta = f(x)$$

---

[23](⇓) Poisson [33].

[24](⇓) Poisson [33] and Poisson [34]. Poisson's another book on this theme : [36].

[25](⇓) The expression of Poisson [33, p.29] correspoinding to $(1)_L$ is as follows :

$$(5)_P \quad u = \frac{e^{-bt}}{\pi}\int_0^\infty dz \Big(\int_{-\infty}^{+\infty} \cos\ (y-x)z\ f(y)\ dy\Big)\ e^{-a^2\ t\ z^2}$$

§5. [26]

$$\int_0^{\infty} e^{-hy}\ f(y)\ dy = p, \quad \int_0^{-\infty} e^{hy}\ f(y)\ dy = q,$$

$$p = \frac{\psi(h)}{\varphi(h)}, \quad q = \frac{\psi(-h)}{\varphi(-h)} \tag{56}$$

§6. We substitute $h = g + z\sqrt{-1}$ in p and $h = g - z\sqrt{-1}$ in q, then

$$p = \frac{\psi(z\sqrt{-1}+g)}{\varphi(z\sqrt{-1}+g)}, \quad q = \frac{\psi(z\sqrt{-1}-1)}{\varphi(z\sqrt{-1}-1)}$$

$$(2)_L \quad \int_{-\infty}^{+\infty} e^{-zy}\ f(y)\ dy = p - q = \frac{\psi(z\sqrt{-1}+g)}{\varphi(z\sqrt{-1}+g)} - \frac{\psi(z\sqrt{-1}-g)}{\varphi(z\sqrt{-1}-g)}$$

This intefral mean at first (54), next, (55) with

$$\cos zy \cdot f(y)dy, \quad \sin zy \cdot f(y)dy, \quad \cos z(y-x) \cdot f(y)dy,$$

§7. [27] We assume $z = \rho + z'$.

$$\int_{-\infty}^{+\infty} e^{-xy\sqrt{-1}}\ f(y)\ dy \equiv Z, \quad \int_{-\infty}^{+\infty} e^{xy\sqrt{-1}}\ f(y)\ dy \equiv Z'$$

$$\int_{-\infty}^{\infty} \cos z(y-x) \cdot f(y)dy = \frac{1}{2}Ze^{x(\rho+z')\sqrt{-1}} + \frac{1}{2}Z'e^{-x(\rho+z')\sqrt{-1}},$$

Substitute these value in $(1)_L$ and neglect z' except for Z and Z', then we get :

$$f(x) = \frac{1}{2\pi}\sum\Big(e^{\rho x\sqrt{-1}}\int Zdz' + e^{-\rho x\sqrt{-1}}\int Z'dz'\Big),$$

We define $\frac{d\varphi(h)}{dh} = \varphi'(h),\ z = \rho + z'$ and assume z' and g are infinitesimaly small.

$$\begin{cases}\varphi(z\sqrt{-1} \pm g) = (z'\sqrt{-1} \pm g)\ \varphi'(\rho\sqrt{-1}),\\ \psi(z\sqrt{-1} \pm g) = \psi(\rho\sqrt{-1}),\end{cases}$$

$$\begin{aligned} Z = &\ \frac{\psi(z\sqrt{-1}+g)}{\varphi(z\sqrt{-1}+g)} - \frac{\psi(z\sqrt{-1}-1)}{\varphi(z\sqrt{-1}-1)} \\ = &\ \frac{\psi(\rho\sqrt{-1})}{(z'\sqrt{-1}+g)\ \varphi'(\rho\sqrt{-1})} - \frac{\psi(\rho\sqrt{-1})}{(z'\sqrt{-1}-g)\ \varphi'(\rho\sqrt{-1})} \\ = &\ \frac{\psi(\rho\sqrt{-1})}{\varphi'(\rho\sqrt{-1})}\Big(\frac{1}{(z'\sqrt{-1}+g)} - \frac{1}{(z'\sqrt{-1}-g)}\Big) \\ = &\ \frac{2g}{(g^2+z'^2)}\ \frac{\psi(\rho\sqrt{-1})}{\varphi'(\rho\sqrt{-1})} \end{aligned}$$

$$\int_{-\delta}^{+\delta} Zdz' = 4\frac{\psi(\rho\sqrt{-1})}{\varphi'(\rho\sqrt{-1})}\arctan\frac{\delta}{g},$$

[26](⇓) Poisson [33, ¶17.]
[27](⇓) Poisson [33, ¶18.]

[28] If $\rho = 0$, we can integrate only the interval of $\int_0^\delta$, then the value reduces into incomplete. If $g = 0$ then we get :

$$\int_0^\delta Zdz' = 2\pi\ \frac{\psi(\rho\sqrt{-1})}{\varphi'(\rho\sqrt{-1})},$$

$$\int Z'dz' = 2\pi\ \frac{\psi(-\rho\sqrt{-1})}{\varphi'(-\rho\sqrt{-1})}$$

$$f(x) = \frac{1}{2\pi}\sum\left[e^{\rho x\sqrt{-1}}\ \frac{\psi(\rho\sqrt{-1})}{\varphi'(\rho\sqrt{(-1)})} + e^{-\rho x\sqrt{-1}}\ \frac{\psi(-\rho\sqrt{-1})}{\varphi'(-\rho\sqrt{-1})}\right],$$

$$(\alpha)_L \quad f(x) = \sum(A\cos\ \rho x + B\sin\ \rho x)$$

$$(\beta)_L \quad \begin{cases} A = \frac{\psi(\rho\sqrt{-1})}{\varphi'(\rho\sqrt{-1})} + \frac{\psi(-\rho\sqrt{-1})}{\varphi'(-\rho\sqrt{-1})}, \\ B = \sqrt{-1}\Big[\frac{\psi(\rho\sqrt{-1})}{\varphi'(\rho\sqrt{-1})} - \frac{\psi(-\rho\sqrt{-1})}{\varphi'(-\rho\sqrt{-1})}\Big] \end{cases}$$

§8. [29]

1. Multiplying the both hand-sides of an even function $f(y) = f(-y)$ with $e^{-hy}dy$ and integrate from $y = 0$ to $y = \infty$:

$$\int_0^\infty e^{-hy}\ f(y)\ dy = \int_0^\infty e^{-hy}\ f(-y)\ dy$$

2. Multiplying the both hand-sides of an even function :

$$f(l+y) = -f(l-y) \tag{57}$$

with $e^{-hy}dy$ and integrate from $y = 0$ to $y = \infty$:

$$\int_0^\infty e^{-hy}\ f(l+y)\ dy = -\int_0^\infty e^{-hy}\ f(l-y)\ dy$$

$$\int_0^\infty e^{-hy}\ f(l+y)\ dy = e^{hl}\Big[p - \int_0^l e^{-hy}\ f(y)\ dy\Big] \tag{58}$$

$$\int_0^\infty e^{-hy}\ f(l-y)\ dy = -e^{hl}\Big[q - \int_0^l e^{hy}\ f(y)\ dy\Big] \tag{59}$$

From the equality in even function $f(y) = f(-y)$, between the right hand-side of (58) and the one of (59) are :

$$e^{hl}\Big[p - \int_0^l e^{-hy}\ f(y)\ dy\Big] = e^{hl}\Big[q - \int_0^l e^{hy}\ f(y)\ dy\Big]$$

By $p = -q$,

$$\int_0^l \Big(e^{h(l-y)}) - (e^{h(y-l)}\Big)\ f(y)\ dy = \psi(h), \quad e^{hl} + e^{-hl} = \varphi(h)$$

[28](⇓)

$$\int \frac{dx}{c^2+x^2} = \frac{1}{c}tan^{-1}\frac{x}{c}.$$

[29](⇓) Poisson [33, ¶17.]

By ψ and φ', $\cos\ \rho l = 0$,

$$\begin{cases} \psi(\rho\sqrt{-1}) = 2\sqrt{-1}\int_0^l \sin\ \rho(l-y)\ f(y)\ dy, \\ \varphi'(\rho\sqrt{-1}) = 2l\sqrt{-1}\sin\rho l \end{cases}$$

By $\sin\ \rho(l-y) = \sin\rho l\ \ \cos\rho\ y$

$$\psi(\rho\sqrt{-1}) = 2\sqrt{-1}\sin\rho\ l\int_0^l \cos\ \ \rho y\ f(y)\ dy$$

From here, we get (α) :

$$f(x) = \frac{2}{l}\sum\cos\rho x\int_0^l \cos\ \ \rho y\ f(y)\ dy$$

§9. We consider the functionx in the outside of interval $[l', l]$, $l' = -l$. We assume the next two conditions :

$$\begin{cases} f(l+y) + f(l-y) = 0, \\ f(-l+y) + f(-l-y) = 0, \end{cases}$$

where we contain implicitly the two special conditions : $f(l) = 0$, $f(-l) = 0$. Similarly, from (58) and (59) are :

$$\int_0^\infty e^{-hy}\ f(l+y)\ dy = e^{hl}\Big[p - \int_0^l e^{-hy}\ f(y)\ dy\Big] \tag{60}$$

$$\int_0^\infty e^{-hy}\ f(l-y)\ dy = -e^{hl}\Big[q - \int_0^l e^{hy}\ f(y)\ dy\Big] \tag{61}$$

Similarly, as (57)

$$f(l+y) = -f(l-y) \tag{62}$$

1. Multiplying the both hand-sides of a function (62) with $e^{-hy}dy$ and integrate from $y = 0$ to $y = \infty$:

$$\int_0^\infty e^{-hy}\ f(l+y)\ dy = -\int_0^\infty e^{-hy}\ f(l-y)\ dy$$

$$e^{hl}p - e^{-hl}q = e^{hl}\int_0^l e^{-hy}\ f(y)\ dy - e^{-hl}\int_0^l e^{hy}f(y)\ dy$$

$$f(-l+y) = -f(-l-y) \tag{63}$$

2. Multiplying the both hand-sides of a function (63) with $e^{-hy}dy$ and integrate from $y = 0$ to $y = \infty$:

$$\int_0^\infty e^{-hy}\ f(l+y)\ dy = -\int_0^\infty e^{-hy}\ f(l-y)\ dy$$

$$e^{-hl}p - e^{hl}q = e^{-hl}\int_0^{-l} e^{-hy}\ f(y)\ dy - e^{hl}\int_0^{-l} e^{hy}f(y)\ dy$$

§10. [30]

$$(a)_L \quad \begin{cases} \beta f(l+y) + \frac{df(l+y)}{dy} + \beta f(l-y) + \frac{df(l-y)}{dy} = 0, \\ \beta' f(-l+y) + \frac{df(-l+y)}{dy} + \beta' f(-l-y) + \frac{df(-l-y)}{dy} = 0 \end{cases}$$

[30]($\Downarrow$) Poisson [33, ¶17.]

here, we consider similarly about $f(x)$

$$\begin{cases} \frac{df(x)}{dx} + \beta f(x) = 0, \quad \text{for} \quad x = l, \\ \frac{df(x)}{dx} + \beta' f(x) = 0, \quad \text{for} \quad x = -l \end{cases} \tag{64}$$

[31] Remenbering (60) and (61), shown by Poisson, put the equation $(\alpha)_L$. [32]

$$e^{-\beta y} d[e^{\beta y} f(l+y)] = e^{\beta y} d[e^{-\beta y} f(l-y)]$$

$$e^{-yh} f(l+y) + (h+\beta) \int e^{-yh} f(l+y)\ dy = C + e^{-yh} f(l-y) + (h-\beta) \int e^{-yh} f(l-y) dy$$

$$(h+\beta) \int e^{-yh} f(l+y)\ dy = (h-\beta) \int e^{yh} f(l-y) dy$$

$$(h+\beta) e^{-hl} p + (h-\beta) e^{-hl} q = (h+\beta) e^{hl} \int_0^l e^{-hy} f(y) dy + (h-\beta) e^{-hl} \int_0^l e^{hy} f(y) dy$$

Changing β with $-\beta'$ and l with $-l$, then

$$(h-\beta') e^{-hl} p + (h+\beta') e^{-hl} q = (h-\beta') e^{-hl} \int_0^{-l} e^{-hy} f(y) dy + (h+\beta') e^{hl} \int_0^{-l} e^{hy} f(y) dy$$

Hence, for abriviation :

$$\begin{cases} (h+\beta)(h+\beta') e^{-2hl} \int_0^{-l} e^{-hy} f(y)\ dy - (h-\beta)(h-\beta') e^{-2hl} \int_0^l e^{-hy} f(y)\ dy \\ +(h-\beta)(h+\beta') \Big[\int_0^l e^{hy} f(y)\ dy - \int_0^{-l} e^{hy} f(y)\ dy \Big] = \psi(h), \\ \\ (h+\beta)(h+\beta') e^{2hl} - (h-\beta)(h-\beta') e^{-2hl} = \varphi(h) \end{cases}$$

then we get (56). From this, the equation depending on the value of ρ is :

$$\frac{\varphi(\rho\sqrt{-1})}{2\sqrt{-1}} = (\beta\beta' - \rho^2) \sin\ 2pl + (\beta+\beta') \rho \cos\ 2pl = 0$$

§11.

$$(\beta\beta' - \rho^2) \sin\ 2pl + (\beta+\beta') \rho \cos\ 2pl = 0$$

$$[\beta + \beta' + 2l(\beta\beta' - \rho^2)] \cos\ 2pl - [2 + 2l(\beta+\beta')] \rho \sin\ 2pl = 0$$

Above two expression consist of $\tan 2pl$ then

$$(\beta\beta' - \rho^2)[\beta + \beta' + 2l(\beta\beta' - \rho^2)] + (\beta+\beta')[2 + 2l(\beta+\beta')]\rho^2 = 0$$

or,

$$2l(\beta\beta' - \rho^2) + 2l(\beta+\beta')^2 \rho^2 + (\beta+\beta')(\beta\beta' + \rho^2) = 0$$

§12.

We see $f(x)$ is to be developed in the series : $\sum(A\cos\rho x + B\sin\rho x)$ as restricted under the condition (64). We assume $v = A\cos\rho x + B\sin\rho x$, then

$$\frac{dv}{dx} + \beta v = 0, \quad \text{for} \quad x = l, \quad \& \quad \frac{dv}{dx} + \beta' v = 0, \quad \text{for} \quad x = -l$$

[31](⇓) This is a boundary value problem of Sturm-Liouville type.

[32]sic. *Journal de l'Ecole Polytechnique*, 19^e cahier, page 30. ((⇓) cf. Poisson [33, p.30], §2, *Distribution dela Chaleur dans une Barre prismatique, d'une petite épaisseur*, ¶15.)

On the terms of the series : $A\cos\rho x + B\sin\rho x$, there are many other remarkable properties, which have been known since long before. v and v' are two terms of the serie corresponding to two roots : $\rho \neq \rho'$; $\int_{-l}^{+l} vv'dx = 0$, In fact,

$$\frac{d^2v}{dx^2} = -\rho^2 v, \quad \frac{d^2v'}{dx^2} = -(\rho')^2 v' \quad \Rightarrow \quad (\rho^2 - \rho'^2)vv' = v\frac{d^2v'}{dx^2} - v'\frac{d^2v}{dx^2}$$

$$(\rho^2 - \rho'^2)\int_{-l}^{+l} vv'dx = v\frac{dv'}{dx} - v'\frac{dv}{dx} \quad \Rightarrow \quad v\frac{dv'}{dx} - v'\frac{dv}{dx} = 0,\ \rho \neq \rho' \quad \Rightarrow \quad \int_{-l}^{+l} vv'dx = 0$$

Thus, as Poisson shows, we get this equality : $\int_{-l}^{+l} vv'dx = 0$.

8. Liouvill [16], 1836

§1.

$$(1)_{L_c} \quad g\frac{du}{dt} = \frac{d\left(k\frac{du}{dx}\right)}{dx} - lu$$

where, $g,\ k,\ l$: specific heat, interior conductivity and emmisive power, respectively.

$$(2)_{L_c} \quad \begin{cases} \frac{du}{dx} - hu = 0 & \text{for} \quad x = \text{x}, \\ \frac{du}{dx} + Hu = 0 & \text{for} \quad x = \text{X} \end{cases}$$

where, $h,\ H$: constants $0 \leq h,\ H \leq \infty$.

$$(3)_{L_c} \quad u = f(x) \qquad \text{for} \quad t = 0$$

$$\begin{cases} \frac{df(x)}{dx} - hf(x) = 0 & \text{for} \quad x = \text{x}, \\ \frac{df(x)}{dx} + Hf(x) = 0 & \text{for} \quad x = \text{X} \end{cases}$$

To form the value of u which satisfies with the equation $(1)_{L_c}$ and with the condition $(2)_{L_c}$ and $(3)_{L_c}$, we are conducted to develop the function $f(x)$, $\forall x \in$ [x, X], by the series which the successive terms are different each other by a parameter r, and has at the same time the property satisfied with the general differential equation :

$$-rg\frac{du}{dt} = \frac{d\left(k\frac{dV}{dx}\right)}{dx} - lV$$

$$\begin{cases} \frac{dV}{dx} - hV = 0 & \text{for} \quad x = \text{x}, \\ \frac{dV}{dx} + HV = 0 & \text{for} \quad x = \text{X} \end{cases}$$

> On peut voir, dans l'ouvrage de M.Poisson sur la chaleur, comment on est porté, par la marche même du calcul, à admettre la possibilité de ce développment pour une fonction quelque $f(x)$; mais jusqu'à ce jouril a paru difficile d'établir cette possibilité directement et d'une manière rigoureuse. Je me propose de donner ici une méthode très simple pour y parvenir. Je considère en elle-même la série par laquelle les géomètres ont représenté le développement de $f(x)$ dont il est question : sans rien supposer à *priori* sur l'origine de cette série ni sur sa nature, j'en cherche la valeur, et je trouve que cette valeur est précisément $f(x)$, du moins lorsque la variable x est compris entre les limites x at X. [16, pp.254-255.]

$$(A)_{L_c} \quad \frac{d\left(k\frac{dV}{dx}\right)}{dx} - (gr - l)V = 0$$

$$(B) \quad \begin{cases} \frac{dV}{dx} - hV = 0 & \text{for} \quad x = \mathrm{x}, \\ \frac{dV}{dx} + HV = 0 & \text{for} \quad x = \mathrm{X} \end{cases}$$

$$(C)_{L_c} \quad \varpi(r) = 0$$

Cela posé, notre but dans ce mémoire, est de trouver directement et par un procédérigoureux la valeur de la série

$$\sum \frac{V \int_{\mathrm{x}}^{\mathrm{X}} gV \ f(x) \ dx}{\int_{\mathrm{x}}^{\mathrm{X}} gV^2 \ dx},$$

where, the sign $\sum$ extend to all the value of r which satisfied with $(C)_{L_c}$. cf [38].
§5.

Problem : Find the value of series :

$$\sum \frac{V \int_{\mathrm{x}}^{\mathrm{X}} gV \ f(x) \ dx}{\int_{\mathrm{x}}^{\mathrm{X}} gV^2 \ dx}, \tag{65}$$

where, in this expression, the sign $\sum$ takes all the value of r which are roots of $(C)_{L_c}$. The valiable x is between x and X, and $f(x)$ is given arbitraly in this interval.

We extend the symbol : $\sum$ of (65) using the limitless series and assume the right hand-side of F.

$$F(x) = \frac{V_1 \int_{\mathrm{x}}^{\mathrm{X}} gV_1 \ f(x) \ dx}{\int_{\mathrm{x}}^{\mathrm{X}} gV_1^2 \ dx} + \frac{V_2 \int_{\mathrm{x}}^{\mathrm{X}} gV_2 \ f(x) \ dx}{\int_{\mathrm{x}}^{\mathrm{X}} gV_2^2 \ dx} + \cdots + \frac{V_m \int_{\mathrm{x}}^{\mathrm{X}} gV_m \ f(x) \ dx}{\int_{\mathrm{x}}^{\mathrm{X}} gV_m^2 \ dx} + \cdots \tag{66}$$

We multiply two members of (66) with $g \ V_m(x) \ dx$, and assume its indices $m \neq n, \ m < n$, then

$$\int_{\mathrm{x}}^{\mathrm{X}} g \ V_m(x) \ F(x) \ dx = \Big(\frac{\int_{\mathrm{x}}^{\mathrm{X}} gV_1^2 \ dx}{\int_{\mathrm{x}}^{\mathrm{X}} gV_1^2 \ dx}\Big) \int_{\mathrm{x}}^{\mathrm{X}} gV_1 \ f(x) \ dx + \Big(\frac{\int_{\mathrm{x}}^{\mathrm{X}} gV_2^2 \ dx}{\int_{\mathrm{x}}^{\mathrm{X}} gV_2^2 \ dx}\Big) \int_{\mathrm{x}}^{\mathrm{X}} gV_2 \ f(x) \ dx + \cdots$$

$$+ \quad \Big(\frac{\int_{\mathrm{x}}^{\mathrm{X}} gV_m^2 \ dx}{\int_{\mathrm{x}}^{\mathrm{X}} gV_m^2 \ dx}\Big) \int_{\mathrm{x}}^{\mathrm{X}} gV_m \ f(x) \ dx + \cdots + \Big(\frac{\int_{\mathrm{x}}^{\mathrm{X}} gV_mV_n \ dx}{\int_{\mathrm{x}}^{\mathrm{X}} gV_mV_n \ dx}\Big) \int_{\mathrm{x}}^{\mathrm{X}} gV_m \ f(x) \ dx$$

$$\int_{\mathrm{x}}^{\mathrm{X}} g \ V_m(x) \ V_n(x) \ dx = 0$$

From here, we get only integral term of m :

$$\int_{\mathrm{x}}^{\mathrm{X}} g \ V_m(x) \ F(x) \ dx = \int_{\mathrm{x}}^{\mathrm{X}} g \ V_m(x) \ f(x) \ d$$

$$\int_{\mathrm{x}}^{\mathrm{X}} g \ [F(x) - f(x)] V_m(x) \ dx = 0$$

$g > 0, \ V_m(x) > 0$ then $F(x) = f(x)$.

La valeur cherchée de la série est donc $f(x)$, entre ces limites de la variable, ce qui s'accorde avec le résultat que les géomètres ont obtenu par d'autres méthodes moins directes et moins rigoureuses que la nôtre. [16, p.263]

§6.

$$\sigma_n \equiv \sum \frac{V \int_{\mathrm{x}}^{\mathrm{X}} gV\ f(x)\ dx}{\int_{\mathrm{x}}^{\mathrm{X}} gV^2\ dx},$$

$\rho_n \equiv f(x) - \sigma_n$. $Q \equiv A_1V_1(x) + A_2V_2(x) + \cdots + A_nV_n(x)$.

$$\int_{\mathrm{x}}^{\mathrm{X}} g\ \rho_n\ Q\ dx = 0 \tag{67}$$

Replecing $Q = \sigma_n$, then

$$\int_{\mathrm{x}}^{\mathrm{X}} g\ \rho_n\ \sigma_n\ dx = 0 \tag{68}$$

$$f(x) = \sigma_n + \rho_n$$

$$\int_{\mathrm{x}}^{\mathrm{X}} g\ \sigma_n\ f(x)dx = \int_{\mathrm{x}}^{\mathrm{X}} g\ \sigma_n\ (\sigma_n + \rho_n)\ dx = \int_{\mathrm{x}}^{\mathrm{X}} g\ \rho_n\sigma_n^2\ dx + \underbrace{\int_{\mathrm{x}}^{\mathrm{X}} g\ \rho_n\ \sigma_n\ dx}_{0 \text{ from } (67)} = \int_{\mathrm{x}}^{\mathrm{X}} g\ \rho_n\ \sigma_n^2\ dx$$

$$\int_{\mathrm{x}}^{\mathrm{X}} g\ f(x)^2\ dx = \int_{\mathrm{x}}^{\mathrm{X}} g\ (\sigma_n + \rho_n)^2\ dx = \int_{\mathrm{x}}^{\mathrm{X}} g(\sigma_n^2 + \rho_n^2)\ dx + 2\underbrace{\int_{\mathrm{x}}^{\mathrm{X}} g\ \sigma_n\ \rho_n\ dx}_{0 \text{ from } (68)} = \int_{\mathrm{x}}^{\mathrm{X}} g(\sigma_n^2 + \rho_n^2)dx$$

> Cette dernière formule nous prove que l'intégrale $\int_{\mathrm{x}}^{\mathrm{X}} g\ \sigma_n^2 dx$, quelque grand qu'on prenne l'indice n, ne peut jamais avoir une valeur numérique supérieure à la limite $\int_{\mathrm{x}}^{\mathrm{X}} g\ f(x)^2\ dx$ avec laquelle elle coïncide lorsque $n = \infty$. [16, p.265]

We see that $\lim_{n\to\infty} \int_{\mathrm{x}}^{\mathrm{X}} g\ \sigma_n^2 dx \leq \int_{\mathrm{x}}^{\mathrm{X}} g\ f(x)^2\ dx$, namely it means as follows :

$$\int_{\mathrm{x}}^{\mathrm{X}} g\ f(x)^2\ dx = \lim_{n\to\infty} \int_{\mathrm{x}}^{\mathrm{X}} g\ \sigma_n^2\ dx \quad \Rightarrow \quad \lim_{n\to\infty} \int_{\mathrm{x}}^{\mathrm{X}} g\ \rho_n^2\ dx = 0 \quad \Rightarrow \quad f(x) - \sigma_n = 0$$

9. Sturm-Liouvill [38], 1837

$$(1)_{SL} \quad \frac{d(k\frac{dV}{dx})}{dx} + (gr - l)V = 0$$

and the define condition :

$$(2)_{SL} \quad \frac{dV}{dx} - hV = 0, \quad \text{for} \quad x = \mathrm{x}$$

$$(3)_{SL} \quad \frac{d(k\frac{dV_n}{dx})}{dx} + (gr_n - l)V_n = 0$$

$$\begin{cases} (4)_{SL} \quad \frac{dV_n}{dx} - hV_n = 0, \quad \text{for} \quad x = \mathrm{x}, \\ (5)_{SL} \quad \frac{dV_n}{dx} - HV_n = 0, \quad \text{for} \quad x = \mathrm{X} \end{cases}$$

$$(6)_{SL} \quad F(x) = \sum \left\{ \frac{V_n \int_x^X g\ V_n\ f(x)\ dx}{\int_x^X g\ V_n^2\ dx} \right\} \tag{69}$$

$$\int_x^X g\ V\ V_n\ dx = \frac{k}{r - r_n}\Big(V\frac{dV_n}{dx} - V_n\frac{dV}{dx}\Big)$$

$$(7)_{SL} \quad \int_x^X g\ V\ V_n\ dx = -KV_n(X)\frac{\varpi(r)}{r-r_n}$$

$$(8)_{SL} \quad \int_x^X g\ V^2\ dx = -KV_n(X)\varpi'(r_n)$$

$$\frac{V}{\varpi(r)} = \sum\Big\{\frac{V_n}{(r-r_n)\varpi'(r_n)}\Big\}$$

$$(9)_{SL} \quad V = \sum\Big\{\frac{\varpi(r)V_n}{(r-r_n)\varpi'(r_n)}\Big\}$$

$$\int_x^X g\ V\ f(x)\ dx = \sum\Big\{\frac{\int_x^X g\ V\ V_n\ dx\cdot\int_x^X g\ V_n\ f(x)\ dx}{\int_x^X g\ V^2\ dx}\Big\} \tag{70}$$

From (69),

$$\int_x^X g\ V\ F(x)\ dx = \sum\Big\{\frac{\int_x^X g\ V\ V_n\ dx\cdot\int_x^X g\ V_n\ f(x)\ dx}{\int_x^X g\ V^2\ dx}\Big\} \tag{71}$$

The right hand-side of (70) and of (71) are equal respectively, then

$$\int_x^X g\ V\ f(x)\ dx = \int_x^X g\ V\ F(x)\ dx \quad\Rightarrow\quad \int_x^X g\ V\ [F(x)-f(x)]\ dx = 0, \quad\Rightarrow\quad F(x)=f(x)$$

10. Dirichlet [4], 1837

¶ 2.

$$(10)_{DU} \quad \int_0^{\frac{\pi}{2}} \cos\ 2mxdx = 0, \quad m\in\mathbb{Z}, m\neq 0.$$

Here, we remark $2m$ corresponds with $\frac{\pi}{2}$. We assume 'Fläschenraum' (area space) of $2m$ intervals.

$$\Big[0,\frac{\pi}{4m}\Big],\Big[\frac{\pi}{4m},\frac{2\pi}{4m}\Big],\Big[\frac{2\pi}{4m},\frac{3\pi}{4m}\Big],\cdots,\Big[\frac{(2m-1)\pi}{4m},\frac{2m\pi}{4m}\Big]$$

For example,

$$m=1\rightarrow\Big[0,\frac{\pi}{4}\Big],\Big[\frac{\pi}{4},\frac{\pi}{2}\Big]$$

$$m=2\rightarrow\Big[0,\frac{\pi}{8}\Big],\Big[\frac{\pi}{8},\frac{\pi}{4}\Big],\Big[\frac{\pi}{4},\frac{3\pi}{8}\Big],\Big[\frac{3\pi}{8},\frac{\pi}{2}\Big]$$

$$m=3\rightarrow\Big[0,\frac{\pi}{12}\Big],\Big[\frac{\pi}{12},\frac{\pi}{6}\Big],\Big[\frac{\pi}{6},\frac{\pi}{4}\Big],\Big[\frac{\pi}{4},\frac{\pi}{3}\Big],\Big[\frac{\pi}{3},\frac{5\pi}{12}\Big],\Big[\frac{5\pi}{12},\frac{\pi}{2}\Big]$$

$$m=4\rightarrow\Big[0,\frac{\pi}{16}\Big],\Big[\frac{\pi}{16},\frac{\pi}{8}\Big],\Big[\frac{\pi}{8},\frac{3\pi}{16}\Big],\Big[\frac{3\pi}{16},\frac{\pi}{4}\Big],\Big[\frac{\pi}{4},\frac{5\pi}{16}\Big],\Big[\frac{5\pi}{16},\frac{3\pi}{8}\Big],\Big[\frac{3\pi}{8},\frac{7\pi}{16}\Big],\Big[\frac{7\pi}{16},\frac{\pi}{2}\Big]$$

$$z = \cos\theta+\cos2\theta+\cdots+\cos\ n\theta$$

From the formula : $2\cos\beta\cos\gamma = \cos(\beta-\gamma)+\cos(\beta+\gamma)$,

$$\begin{aligned} 2z\cos\theta &= 1+\ \cos\theta+\ \cos2\theta+\cdots+\cos\ (n-1)\theta \\ &+\ \cos2\theta+\cos3\theta+\cos4\theta+\cdots+\cos\ (n+1)\theta \end{aligned}$$

$$z+1-\cos\ n\theta, \qquad z-\cos\theta+\cos(n+1)\theta$$

Adding both hand-sides, then we get :

$$2z\cos\theta = 2z + 1 - \cos\ \theta + \cos\theta + \cos(n+1)\theta - \cos\ n\theta$$

$$z = -\frac{1}{2} + \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}$$

$$(11)_{DU} \qquad \cos\theta + \cos 2\theta + \cdots + \cos\ n\theta = -\frac{1}{2} + \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta} \tag{72}$$

¶ 3. (Deduction of trigonometric series.)

10. Transfer array by Dirichlet.

$$f(\frac{\pi}{n}) = a_1\sin\frac{\pi}{n} + a_2\sin\frac{2\pi}{n} + a_3\sin\frac{3\pi}{n} + \cdots + a_n\sin\frac{\pi n}{n}$$

$$f(\frac{2\pi}{n}) = a_1\sin\frac{2\pi}{n} + a_2\sin\frac{4\pi}{n} + a_3\sin\frac{6\pi}{n} + \cdots + a_n\sin\frac{2\pi n}{n}$$

$$f(\frac{3\pi}{n}) = a_1\sin\frac{3\pi}{n} + a_2\sin\frac{6\pi}{n} + a_3\sin\frac{9\pi}{n} + \cdots + a_n\sin\frac{3\pi n}{n}$$

$$\cdots\cdots\cdots\cdots$$

$$f(\frac{(n-1)\pi}{n}) = a_1\sin(n-1)\frac{\pi}{n} + a_2\sin(n-1)\frac{2\pi}{n} + a_3\sin(n-1)\frac{\pi n}{n} + \cdots + a_n\sin(n-1)^2\frac{\pi}{n}$$

Here, these equations are shown with a today's style of $(n-1)\times(n-1)$ transform matrix :[33]

$$\begin{bmatrix} f(\frac{\pi}{n}) \\ f(\frac{2\pi}{n}) \\ f(\frac{2\pi}{n}) \\ \vdots \\ f(\frac{(n-1)\pi}{n}) \end{bmatrix} = \begin{bmatrix} \sin\frac{\pi}{n} & \sin\frac{2\pi}{n} & \sin\frac{3\pi}{n} & \cdots & \sin\ (n-1)\frac{\pi}{n} \\ \sin\frac{2\pi}{n} & \sin\frac{4\pi}{n} & \sin\frac{6\pi}{n} & \cdots & \sin\ 2(n-1)\frac{\pi}{n} \\ \sin\frac{3\pi}{n} & \sin\frac{6\pi}{n} & \sin\frac{9\pi}{n} & \cdots & \sin\ 3(n-1)\frac{\pi}{n} \\ \cdots & & & & \\ \sin(n-1)\frac{\pi}{n} & \sin(n-1)\frac{2\pi}{n} & \sin(n-1)\frac{3\pi}{n} & \cdots & \sin\ (n-1)^2\ \frac{\pi}{n} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix} \tag{73}$$

Multiplying with $2\sin\frac{\pi m}{n}$, $\quad 2\sin\frac{2\pi m}{n}$, $\quad 2\sin\frac{3\pi m}{n}, \cdots, 2\sin\ (n-1)\frac{\pi m}{n}$, where $m = 1,\ 2,\ 3,\ \cdots,\ n-1$, $h = 1,\ 2,\ 3,\ \cdots,\ n-1$, then the coefficient a_h is as follows :

$$a_h\Big[2\sin\frac{m\pi}{n}\sin\frac{h\pi}{n} + 2\sin\frac{2m\pi}{n}\sin\frac{2h\pi}{n} + 2\sin\frac{3m\pi}{n}\sin\frac{3h\pi}{n} + \cdots + 2\sin(n-1)\frac{m\pi}{n}\sin(n-1)\frac{h\pi}{n}\Big] \tag{74}$$

[34] If $m \neq h$, the value between the square brackets is null, we can express by the following difference replacing the products of sin with cos, then

$$(12)_{DU} \qquad \cos(m-h)\frac{\pi}{n} + \cos 2(m-h)\frac{\pi}{n} + \cdots + \cos(n-1)(m-h)\frac{\pi}{n}$$
$$- \quad \Big(\cos(m+h)\frac{\pi}{n} + \ \cos 2(m+h)\frac{\pi}{n} + \cdots + \cos(n-1)(m+h)\frac{\pi}{n}\Big) \tag{75}$$

From $(11)_{DU}$, assuming $\theta = (m-h)\frac{\pi}{n}$ and replacing n with $n-1$, and by $\sin(l\pi - \gamma) = \mp\sin\gamma, \quad l \in \mathbb{Z}$,

$$\sin(n-\frac{1}{2})(m-h)\frac{\pi}{n} = \sin\Big((m-h)\pi - (m-h)\frac{\pi}{2n}\Big) = \mp\sin(m-h)\frac{\pi}{2n}$$

[33]Dirichlet didn't use the transform-matrix symbol, but mine. cf. Lagrange's expression (1) and Poisson's expression (32)

[34]Here, Dirichlet's (74) comes from Lagrange's style : (3). cf. [8, p.81]

$$-\frac{1}{2}+\frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}=-\frac{1}{2}+\frac{\sin(n-\frac{1}{2})(m-h)\frac{\pi}{n}}{2\sin(m-h)\frac{\pi}{2n}}=-\frac{1}{2}\mp\frac{1}{2}$$

Each of $(12)_{DU}$ has

$$\begin{cases}-1 \quad \text{mod } (m-h,2)=0, \quad 0, \quad \text{mod } (m-h,2)=1\\ 1 \quad \text{mod } (m+h,2)=0, \quad 0, \quad \text{mod } (m+h,2)=1\end{cases}$$

Here, we have the sum $2m$.

$$na_m = 2\sin\frac{m\pi}{n}f\Big(\frac{\pi}{n}\Big)+2\sin\frac{2m\pi}{n}f\Big(\frac{2\pi}{n}\Big)+\cdots+2\sin\frac{(n-1)m\pi}{n}f\Big(\frac{(n-1)\pi}{n}\Big)$$

$$a_m = \frac{2}{n}\Big[\sin\frac{m\pi}{n}f\Big(\frac{\pi}{n}\Big)+\sin\frac{2m\pi}{n}f\Big(\frac{2\pi}{n}\Big)+\cdots+\sin\frac{(n-1)m\pi}{n}f\Big(\frac{(n-1)\pi}{n}\Big)\Big] \tag{76}$$

$$a_m=\frac{2}{\pi}\Big[\frac{\pi}{n}\sin\Big(\frac{0m\pi}{n}\Big)f\Big(\frac{0\pi}{n}\Big)+\frac{\pi}{n}\sin\Big(\frac{m\pi}{n}\Big)f\Big(\frac{\pi}{n}\Big)+\frac{\pi}{n}\sin\Big(\frac{2m\pi}{n}\Big)f\Big(\frac{2\pi}{n}\Big)+\cdots+\frac{\pi}{n}\sin\Big(\frac{(n-1)m\pi}{n}\Big)f\Big(\frac{(n-1)\pi}{n}\Big)\Big]$$

$$(13)_{DU}\ f(x)=a_1\sin x+a_2\sin 2x+\cdots+a_m\sin mx+\cdots,\ \text{where, } a_m=\frac{2}{\pi}\int_0^\pi \sin\ mx\ f(x)dx \tag{77}$$

[35] Replacing the left hand-side of $(13)_{DU}$ by $2\sin\ x\ f(x)$,

$$2\sin\ x\ f(x)=a_1\sin x+a_2\sin 2x+\cdots+a_m\sin mx+\cdots,\quad \text{where,}$$

$$a_m=\frac{2}{\pi}\int_0^\pi 2\sin\ mx\ \Big(\sin\ x\ f(x)\Big)\ dx=\frac{2}{\pi}\int_0^\pi \cos(m-1)\ x\ f(x)dx-\frac{2}{\pi}\int_0^\pi \cos(m+1)\ x\ f(x)dx$$

$$\frac{2}{\pi}\int_0^\pi \cos\ h\ x\ f(x)dx\equiv b_h,\quad \Rightarrow \quad a_m=b_{m-1}-b_{m+1},\quad m=1,2,3,\cdots,$$

$$\Rightarrow \quad 2\sin\ x\ f(x)=(b_0-b_1)\sin x+(b_1-b_2)\sin 2x+(b_2-b_3)\sin 3x+\cdots$$

$$\begin{aligned}2\sin\ x\ f(x) &= b_0\sin\ x+b_1\sin 2x+b_2(\sin\ 3x-\sin\ x)+b_3(\sin\ 4x-\sin\ 2x)+\cdots\\ &= b_0\sin\ x+b_1(2\sin\ x\cos\ x)+b_2(2\sin\ x\cos\ 2x)+b_3(2\sin\ x\cos\ 3x)+\cdots\end{aligned} \tag{78}$$

Dividing both hand-sides of (78) by $2\sin x$,

$$(14)_{DU}\quad f(x)=\frac{1}{2}b_0+b_1\cos x+b_2\cos 2x+b_3\cos 3x+\cdots,\ \text{where, } b_m=\frac{2}{\pi}\int_0^\pi \cos\ mx\ f(x)dx \tag{79}$$

Making the form :[36]

$$\varphi(x)=\frac{\varphi(x)+\varphi(-x)}{2}+\frac{\varphi(x)-\varphi(-x)}{2} \tag{80}$$

we bind $(13)_{DU}$ and $(14)_{DU}$,

$$\begin{aligned}(15)_{DU}\quad \varphi(x)=\frac{1}{2}b_0 &+ b_1\cos x+b_2\cos 2x+\cdots+b_m\cos mx+\cdots\\ &+ a_1\sin x+a_2\sin 2x+\cdots+a_m\sin mx+\cdots\end{aligned} \tag{81}$$

where,

$$b_m=\frac{1}{\pi}\int_0^\pi \cos\ mx\big[(\varphi(x))+(\varphi(-x))\big]\ dx,\quad a_m=\frac{1}{\pi}\int_0^\pi \sin\ mx\big[(\varphi(x))-(\varphi(-x))\big]\ dx$$

namely,

$$b_m=\frac{1}{\pi}\int_0^\pi \cos\ mx\varphi(x)+\frac{1}{\pi}\int_0^\pi \cos\ mx\varphi(-x)\ dx=\frac{1}{\pi}\int_{-\pi}^\pi \cos\ mx\ \varphi(x)\ dx \tag{82}$$

[35] cf. The eqxpression (77) corresponds with Poisson's expression (37).

[36] cf. Dirichlet cites Fourier's induction (25).

$$a_m = \frac{1}{\pi}\int_0^{\pi} \sin\ mx(\varphi(x)) - \frac{1}{\pi}\int_0^{\pi} \sin\ mx\big(\varphi(-x)\big)\ dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin\ mx\ \varphi(x)\ dx. \tag{83}$$

¶ 4. (The consideration of proof on convergence.)

> Die Betrachtung, die dem Verfahren, welches uns die Reihe $(13)_{DU}$ geliefert hat, die gehörige Strenge geben würden, sind so zusammengesetzter Art, daß wir lieber einen andern Weg der Bewiseführung einschlagen. Wir werden die Reihe $(15)_{DU}$, welch die beiden andern $(13)_{DU}$ und $(14)_{DU}$ als besondere Fälle in sieh begreift, an und für sich untersuchen und, ohne etwas von dem Früheren vorauszusetzen, direct nachweisen, daß diese Reihe :
>
> $$\begin{aligned}\frac{1}{2}b_0 + \quad & b_1\cos x + b_2\cos 2x + \cdots + b_m\cos mx + \cdots \\ + \quad & a_1\sin x + a_2\sin 2x + \cdots + a_m\sin mx + \cdots ,\end{aligned} \tag{84}$$
>
> wenn man ihre Coefficienten durch die Gleichungen :
>
> $$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos mxg(x)dx, \qquad a_m = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin mxg(x)dx$$
>
> bestimmt, immer convergirt und für alle zwischen $-\pi$ und π enthaltenen Werthe von x der Function $\varphi(x)$ gleich ist. [4, ¶ 4, p.146]
>
> この考察は、式 $(13)_{DU}$ の導出を我々に示したが、それなりの厳しさを与えたかも知れないし、一緒に併置した方法だったので、もっと別の方法で証明を与えるべきだろう。
>
> • 我々は級数 $(15)_{DU}$ をそれ自体が特別な場合として把握したが、それ自体を調べて見たいし、また、最初から何らの前提条件を与える事をせずに、二つの別々の式 $(13)_{DU}$ 及び $(14)_{DU}$ がこの級数 (84) で式 b_m 及び, a_m によって係数を決定する時、常に収束し、区間 $(-\pi, \pi)$ において全ての値で、x に関する関数 $\varphi(x)$ が等しいという事を直截に証明して見よう。 [4, ¶ 4, p.146]

Dirichlet's proving target is the following summation of the first $n+1$ terms of (84)

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha g(\alpha)\ + \quad & \frac{1}{\pi}\cos x\int_{-\pi}^{\pi} d\alpha\cos\alpha g(\alpha) + \cdots + \frac{1}{\pi}\cos nx\int_{-\pi}^{\pi} d\alpha\cos n\alpha g(\alpha) \\ + \quad & \frac{1}{\pi}\sin x\int_{-\pi}^{\pi} d\alpha\sin\alpha g(\alpha) + \cdots + \frac{1}{\pi}\sin nx\int_{-\pi}^{\pi} d\alpha\sin n\alpha g(\alpha)\end{aligned}$$

or,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} d\alpha g(\alpha)\Big[\frac{1}{2} + \cos(\alpha - x) + \cos 2(\alpha - x) + \cdots + \cos\ n(\alpha - x)\Big]$$

or,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} d\alpha g(\alpha)\frac{\sin(2n+1)\frac{\alpha - x}{2}}{2\sin\frac{\alpha - x}{2}}$$

¶ 5.

From $(11)_{DU}$(=(72)), we get :

$$\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)\beta}{\sin\beta}\ d\beta \tag{85}$$

,where, (72) is equivalent with

$$1 + 2\cos 2\beta + 2\cos 4\beta + \cdots + 2\cos\ 2n\theta$$

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$$\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)\beta}{\sin\beta}\ d\beta = \frac{\pi}{2}$$

[37] (85) is what is called the Dirichlet kernel.

Here, we assume $k \equiv 2n+1$.

$$\rho_r = \mp \int_{\frac{(\nu-1)\pi}{k}}^{\frac{\nu\pi}{k}} \frac{\sin\ k\beta}{\sin\beta} d\beta$$

$$\int_{\frac{(\nu-1)\pi}{k}}^{\frac{\nu\pi}{k}} \mp \sin\ k\beta d\beta = \frac{2}{k}$$

We assume $\Delta \equiv \int_0^\pi \sin\beta d\beta$

$$\frac{\Delta}{k}\frac{1}{\sin\frac{\nu\pi}{k}} < \rho_i < \frac{\Delta}{k}\frac{1}{\sin\frac{(\nu-1)\pi}{k}}$$

$$\rho_1 > \rho_2 > \rho_3 > \cdots > \rho_{2m+1}$$

$$\frac{\pi}{2} = \rho_1 - \rho_2 + \rho_3 - \cdots - \rho_{2m} \pm \rho_{2m+1}$$

$$(16)_{DU} \qquad \begin{cases} \frac{\pi}{2} > \rho_1 - \rho_2 + \rho_3 - \cdots - \rho_{2m}, \\ \frac{\pi}{2} < \rho_1 - \rho_2 + \rho_3 - \cdots - \rho_{2m} + \rho_{2m+1} \end{cases}$$

Aim : What is S ?

$$\lim_{n\to\infty} \int_0^h \frac{\sin\ k\beta}{\sin\beta} f(\beta) d\beta = S, \tag{86}$$

where $h < \frac{\pi}{2}$ is a constant, $k = 2n+1$ and $f(\beta)$ is a continuous function with respect to β.

$$R_r = \mp \int_{\frac{(\nu-1)}{k}}^{\frac{\nu\pi}{k}} \frac{\sin\ k\beta}{\sin\beta} f(\beta) d\beta,$$

$$R_r = R_1 - R_2 + R_3 - \cdots \pm R_{r+1}$$

$$R_r = \int_{\frac{(\nu-1)\pi}{k}}^{\frac{\nu\pi}{k}} \mp \frac{\sin\ k\beta}{\sin\beta} f(\beta) d\beta,$$

$$f\Big(\frac{r\pi}{k}\Big) \int_{\frac{(\nu-1)\pi}{k}}^{\frac{\nu\pi}{k}} \mp \frac{\sin\ k\beta}{\sin\beta} f(\beta) d\beta < R_i < f\Big(\frac{(r-1)\pi}{k}\Big) \int_{\frac{(\nu-1)\pi}{k}}^{\frac{\nu\pi}{k}} \mp \frac{\sin\ k\beta}{\sin\beta} f(\beta) d\beta,$$

$$\rho_r f\Big(\frac{r\pi}{k}\Big) < R_r < \rho_r f\Big(\frac{(r-1)\pi}{k}\Big),$$

$$\begin{cases} S > R_1 - R_2 + R_3 - \cdots - R_{2m}, \\ S < R_1 - R_2 + R_3 - \cdots - R_{2m} + R_{2m+1} \end{cases}$$

$$\begin{cases} S > (\rho_1 - \rho_2) f\Big(\frac{\pi}{k}\Big) + (\rho_3 - \rho_4) f\Big(\frac{3\pi}{k}\Big) \cdots (\rho_{2m-1} - \rho_{2m}) f\Big(\frac{(2m-1)\pi}{k}\Big), \\ S < \rho_1 f\Big(\frac{0\pi}{k}\Big) - (\rho_2 - \rho_3) f\Big(\frac{2\pi}{k}\Big) + (\rho_4 - \rho_5) f\Big(\frac{4\pi}{k}\Big) - \cdots - (\rho_{2m} - \rho_{2m-1}) f\Big(\frac{2m\pi}{k}\Big) \end{cases}$$

$$\begin{cases} S > (\rho_1 - \rho_2 + \rho_3 - \rho_4 \cdots + \rho_{2m-1} - \rho_{2m}) f\Big(\frac{(2m-1)\pi}{k}\Big), \\ S < \rho_1 f\Big(\frac{0\pi}{k}\Big) - (\rho_2 - \rho_3 + \rho_4 - \rho_5 - \cdots - \rho_{2m} - \rho_{2m-1}) f\Big(\frac{2m\pi}{k}\Big) \end{cases}$$

$$\begin{cases} \rho_2 + \rho_3 - \cdots - \rho_{2m} > \rho_1 - \frac{\pi}{2}, \\ \rho_1 - \rho_2 + \rho_3 - \cdots - \rho_{2m} > \frac{\pi}{2} - \rho_{2m+1} \end{cases}$$

$$\begin{cases} S > \frac{\pi}{2} f\Big(\frac{2m\pi}{k}\Big) - \rho_{2m+1} f\Big(\frac{2m\pi}{k}\Big), \\ S < \frac{\pi}{2} f\Big(\frac{2m\pi}{k}\Big) + \rho_{2m+1} f\Big(\frac{2m\pi}{k}\Big) + \rho_1 \Big| f\Big(0\Big) - f\Big(\frac{2m\pi}{k}\Big)\Big|, \end{cases}$$

$$0 < S < \rho_1 \Big| f\Big(0\Big) - f\Big(\frac{2m\pi}{k}\Big)\Big|,$$

Here, we can get easily our aim proving (86). We consider at first :

$$\frac{\pi}{2} f\Big(\frac{2m\pi}{k}\Big) - \rho_{2m+1} f\Big(\frac{2m\pi}{k}\Big)$$

ρ_{2m+1} locates as follows

$$\frac{\Delta}{k} \frac{1}{\sin \frac{2m\pi}{k}} < \rho_{2m+1} < \frac{\Delta}{k} \frac{1}{\sin \frac{(2m+1)\pi}{k}}$$

then we can state it as follow

$$\frac{\Delta}{2m\pi} \cdot \frac{2m\frac{\pi}{k}}{\sin\ 2m\frac{\pi}{k}} < \rho_{2m+1} < \frac{\Delta}{(2m+1)\pi} \cdot \frac{(2m+1)\frac{\pi}{k}}{\sin\ (2m+1)\frac{\pi}{k}} \tag{87}$$

$$\lim_{m\to\infty} \frac{\Delta}{2m\pi} = 0, \qquad \lim_{2m\frac{\pi}{k}\to\infty} \frac{2m\frac{\pi}{k}}{\sin\ 2m\frac{\pi}{k}} = 1$$

And two products of the inequality (87) become zero respectively.

Finally, in (86), S is $\frac{\pi}{2} f(0)$.

The term $f\Big(\frac{2m\pi}{k}\Big)$ in $\rho_1 \Big| f\Big(0\Big) - f\Big(\frac{2m\pi}{k}\Big)\Big|$ become null.

$$\rho_1 < \frac{\pi}{2} + \rho_2, \quad \rho_2 < \frac{\Delta}{k} \frac{1}{\sin \frac{\pi}{k}} \quad \text{then} \quad \rho_1 < \frac{\pi}{2} + \frac{\Delta}{k} \frac{1}{\sin \frac{\pi}{k}},$$

$$\frac{\Delta}{k} \frac{1}{\sin \frac{\pi}{k}} = \frac{\Delta}{\pi} \frac{\frac{\pi}{k}}{\sin \frac{\pi}{k}} \quad \text{and} \quad \lim_{k\to\infty} \frac{\Delta}{\pi} \frac{\frac{\pi}{k}}{\sin \frac{\pi}{k}} = \frac{\Delta}{\pi}$$

We get the following for $0 < h < \frac{\pi}{2},\ k = 2n+1,$

$$\lim_{n\to\infty} \int_0^h f([\beta) + c] \frac{\sin k\beta}{\sin \beta} d\beta = \frac{\pi}{2} \Big[f(0) + c \Big],$$

$$\lim_{n\to\infty} \int_0^h -f(\beta) \frac{\sin k\beta}{\sin \beta} d\beta = -\frac{\pi}{2} f(0),$$

namely

$$\lim_{n\to\infty} \int_0^h f(\beta) \frac{\sin k\beta}{\sin \beta} d\beta = \frac{\pi}{2} f(0)$$

Finally, Dirichlet deduces the result :

$$\lim_{n\to\infty}\int_0^h \frac{\sin(2n+1)\beta}{\sin\beta}f(\beta)d\beta = \frac{\pi}{2}f(0), \quad 0<h<\frac{\pi}{2},$$
$$(\Rightarrow \lim_{n\to\infty}\frac{2}{\pi}\int_0^h \frac{\sin(2n+1)\beta}{\sin\beta}f(\beta)d\beta = f(0), \quad 0<h<\frac{\pi}{2}),$$
$$\lim_{n\to\infty}\int_g^h \frac{\sin(2n+1)\beta}{\sin\beta}f(\beta)d\beta = 0, \quad 0<g<h<\frac{\pi}{2}$$

The original by Dirichlet is explained as the following two statements $(17)_{DU}$ and $(18)_{DU}$:

$(17)_{DU}$:
Ist $f(\beta)$ eine stetige Function von β, die, während β von 0 bis h wächst (wo die Constante $h>0$ und $<\frac{\pi}{2}$),[a] nie von Abnehmen ins Zunehmen oder umgekehrt übergeht, so wird das Integral :

$$\int_0^h \frac{\sin(2n+1)\beta}{\sin\beta}f(\beta)d\beta$$

wenn man darin der ganzen Zahl n immer größere positive Werthe heilegt, zu letzt immerfort weniger c als jede angebbare Größe von $\frac{\pi}{2}f(0)$ verschieden sein. [4, p.154]

[a]Riemenn[37, p.14] corrects this inequality as follows : $\frac{\pi}{2}\geq h>0$

$$\int_0^h \frac{\sin(2n+1)\beta}{\sin\beta}f(\beta)d\beta = \int_0^g \frac{\sin(2n+1)\beta}{\sin\beta}f(\beta)d\beta + \int_g^h \frac{\sin(2n+1)\beta}{\sin\beta}f(\beta)d\beta = \frac{\pi}{2}f(0),$$

and

$$\int_0^g \frac{\sin(2n+1)\beta}{\sin\beta}f(\beta)d\beta = \frac{\pi}{2}f(0),$$

then

$$\int_g^h \frac{\sin(2n+1)\beta}{\sin\beta}f(\beta)d\beta = 0$$

$(18)_{DU}$:
Sind g und h Constanten, welche den Bedingungen genügen $g>0$, $\frac{\pi}{2}>h>g$,[a] und geht die Function $f(\beta)$, wenn β von g bis h wächst, nie vom Abnehmen ins Zunehmen oder umgekehrt über, so wird das Integral :

$$\int_g^h \frac{\sin(2n+1)\beta}{\sin\beta}f(\beta)d\beta$$

für ein unendlich großes n der Null gleich. [4, p.155]

[a]Riemenn[37, p.14] corrects this inequality as follows : $\frac{\pi}{2}\geq h>g>0$

¶ 6.

$$\frac{1}{\pi}\int_{-\pi}^{\pi} d\beta\varphi(\beta)\frac{\sin(2n+1)\frac{\beta-x}{2}}{2\sin\frac{\beta-x}{2}}$$

$$\frac{1}{\pi}\int_{-\pi}^{x} d\beta\ \varphi(\beta)\frac{\sin(2n+1)\frac{\beta-x}{2}}{2\sin\frac{\beta-x}{2}}, \quad \frac{1}{\pi}\int_{x}^{\pi} d\beta\ \varphi(\beta)\frac{\sin(2n+1)\frac{\beta-x}{2}}{2\sin\frac{\beta-x}{2}}$$

$$\frac{1}{\pi}\int_{-(\pi+x)}^{0} d\beta\ \varphi(x+\beta)\frac{\sin(2n+1)\frac{\beta-x}{2}}{2\sin\frac{\beta-x}{2}}, \quad \frac{1}{\pi}\int_{0}^{(\pi-x)} d\beta\ \varphi(x+\beta)\frac{\sin(2n+1)\frac{\beta-x}{2}}{2\sin\frac{\beta-x}{2}}$$

$$(19)_{DU} \quad \frac{1}{\pi}\int_{0}^{\frac{\pi+x}{2}} d\beta\ \varphi(x-2\beta)\frac{\sin(2n+1)\beta}{\sin\ \beta}, \quad \frac{1}{\pi}\int_{0}^{\frac{\pi-x}{2}} d\beta\ \varphi(x+2\beta)\frac{\sin(2n+1)\beta}{\sin\ \beta}$$

$$\int_{\frac{\pi}{2}}^{\frac{\pi-x}{2}} d\beta\ \varphi(x+2\beta)\frac{\sin(2n+1)\beta}{\sin\ \beta}$$

$$-\int_{\frac{\pi}{2}}^{\frac{\pi+x}{2}} d\beta\ \varphi(x+2\pi-2\beta)\frac{\sin(2n+1)(\pi-\beta)}{\sin\ (\pi-\beta)}$$

$$\int_{\frac{\pi+x}{2}}^{\frac{\pi}{2}} d\beta\ \varphi(x+2\pi-2\beta)\frac{\sin(2n+1)\beta}{\sin\ \beta}$$

$$\int_{0}^{x} d\beta\ \varphi(x+2\pi-2\beta)\frac{\sin(2n+1)\beta}{\sin\ \beta} = \int_{0}^{x} d\beta\ \varphi(\pi-2\beta)\frac{\sin(2n+1)\beta}{\sin\ \beta}$$

$$\begin{aligned}(20)_{DU} \quad \frac{1}{2}b_0 + \quad & b_1\cos x + b_2\cos 2x + \cdots + b_m\cos mx + \cdots \\ + \quad & a_1\sin x + a_2\sin 2x + \cdots + a_m\sin mx + \cdots\end{aligned}$$

where, the coefficients are by the following expressions :

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos m\beta g(\beta)\ d\beta, \qquad a_m = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin m\beta g(\beta)\ d\beta$$

$$b_m = \frac{1}{\pi}\int_{\pi}^{0}\cos m\beta g(\beta)\ d\beta + \frac{1}{\pi}\int_{0}^{\pi}\cos m\beta g(\beta)\ d\beta$$

$$a_m = \frac{1}{\pi}\int_{\pi}^{0}\sin m\beta g(\beta)\ d\beta + \frac{1}{\pi}\int_{0}^{\pi}\sin m\beta g(\beta)\ d\beta$$

$$\varphi(-\beta) = \varphi(\beta), \quad \cos(-m\beta) = \cos(m\beta), \quad \sin(-m\beta) = -\sin(m\beta)$$

$$b_m = \frac{2}{\pi}\int_{0}^{\pi}\cos m\beta g(\beta)\ d\beta, \quad a_m = 0$$

$$\varphi(x) = \frac{1}{2}b_0 + b_1\cos x + b_2\cos 2x + \cdots + b_m\cos mx + \cdots$$

We introduce the corresponding paragraphs to this proving, from the mourning paper [7, p.325] to passing-away of Dirichlet by E.E.Kummer (1810-93) on Dirichlet's work [4].

> Die rein analytischen Arbeiten Dirichlet's über unendliche Reihen und bestimmte Integrale und über die in diesen Formen erscheinenden Functionen sind ursprünglich aus seinem Studium der mathematischen Physik, und namentlich der Fourier'schen Wärmetheorie hervor gegangen. Bei den Anwendungen dieser allgemeinen Formen auf die physikalischen Probleme konnte er sich aber nicht damit beruhigen, sie als fertige Hülfsmittel zu benutzen, und da eine nühere Pruffung ihm bald zeigte, dass sie selbst in den wichtigen Punkten noch der

strengen wissenschaftlichen Begründung ermangelten, so richtete er seine Arbeit zunächst auf die Sicherung dieser Fundamente. Die nach Sinus und Cosinus der Vierfachen eines Bogens fortschreitenden Reihen, welche von Fourier mit dem ausgezeichnetsten Erforge zur Darstellung der in der Wärmetheorie vorkommenden willkürlichen Functionen angewandt worden sind, hatten bisher in allen Fällen, wo die zu entwickelnde Function nicht unendlich wird, die ausgezeichnete Eigenschaft gezeigt, immer convergent zu sein, es war aber selbst Cauchy's Bemühungen nicht gelungen, [38] dieses allgemein und streng zu beweisen.

Der Weg, welchen dieser, nicht minder durch die Strenge, als durch die Originalität seiner Methoden berühmte Mathematiker hier eingeschlagen hatte, nur die Grössenverhältnisse der einzelnen Glieder dieser Reihen zu untersuchen und darauf seine Schlüsse zu gründen, führte aber nicht zur wahlen Erkentniss dieser verborgenen Eigenschaft, sondern nur ziemlich nahe bei derselben vorbei, weil die Convergenz dieser Reihen in gewissen Fällen auch von der besonderen Art und Weise abhängig ist, wie die positiven und negativen Glieder derselben sich gegenseitig aufheben. Aus diesem grunde untersuchte Dirichlet, auf den ursprünglichen Begriff der Convergenz der unendlichen Reihen zurückgehend, den Grenzwerth, welchen die Summe eine Anzahl Gliede erreicht, wenn diese Anzahl ins Unendliche wachsend angennommen wird, und diese Frage ergrändeter vollständig mittelst der genauen Bestimmung des Grenzwerthes eines einfachen bestimmten Integrales, welches, wegen der vierfachen Anwendungen, die es gestattet, seitdem zu den Grundlagen der Theorie der bestimmten Integrale gerechnet wird. [7, p.325]

We translate thess paragraphs as follows :

The pure analytical works by Dirichlet on the infinite series and defined integral and in this form of appearing function are properly originated from the study of mathematical physics, namely, the Fourier's heat theory. Because he (Dirichlet) can't handle the application of general form on the physical problem, he gets the useful help, and shows him soon a precise proof, in the most important points, for he lacks for even the exact scientific fundamentals, at first, he (Dirichlet) pays his work on the making sure this basement.

By the progressing series by sine and cosine with multivalent of an angle, which is due to Fourier, describing in the heat theory, using an arbitrary function, where a expanding function will be definite, however, it had failed even for Cauchy's painful work, to prove generally and exactly to be always convergent.

The method, which is by no means inferior to the exactitude, as by the originality of his method of this genius mathematician (Dirichlet) had taken here, to seek for the ratio of largeness of each term of this series, however, *he doesn't choose the understanding of unseen characteristics, but approaching it very closely, while the convergence of this series depends on even the special means and manners in the certain cases, for the positive and negative terms cancel out reciprocally.*

Dirichlet observes from this policy, the proper meaning of the convergence of infinite series, which he reaches to the sum of a term, when this value becomes infinitely extending, and he inquires extensively this question on the exact determination of the value of term of a simple definite integral, which for various applications, to which he dares to challenge, since the fundamental theory of the definite integral becomes clear. [7, p.325] (indention and underline is mine.)

[38]cf. We cite Cauchy's paper [1] in §5.

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