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THE WAVE EQUATIONS AS THE MODEL OF THE SCHRÖDINGER EQUATIONS

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Abstract. We discuss the analogies from classical mechanics in the modeling of the Schrödinger equations or the quantum equations.

The classical mechanics of model which they put in it are : the atomic motion of analogy from the Kepler motion, the eigenvalue problems and the perturbation problems on Kepler motion, the Huygens' principle of water wave, the Newtonian mechanics, Fresnel's corrective concept and theory of Huygens and so forth. After Lagrange defines the Kepler problems, Laplace, Poisson, Gauss, Bessel, et al. theorizes the various sort of classical principles and problems.

The gas theorists like Maxwell and Boltzmann, etc. introduce the transport and collision of gas particles and entropy, which are the first steps of quantum mechanics.

We like to document these sort of various materials of the modeling of the Schrödinger equations, above all, the essence of the Schrödinger equations as the wave equations, and as the Strum-Liouville type eigenvalue equations, which Schrödinger emphasizes.

1. Introduction

1.1. Outline from Kepler motion to the quantum mechanics.

Kepler (1571-1630) [17] proposes laws on the motions of planets in reserving many analytical open problems. Huygens (1625-95) proposes and Fresnel (1788-1827) corrects the wave principles. Newton (1643-1727) 1687 [47] also shows the wave principle. Euler(1707-1783) 1748 proposes the wave motion of string. Navier (1785-1836) and Poisson (1781-1840) propose wave equations in elasticity respectively.

Fourier (1768-1830) 1820 combines his communication theory with the Euler equation 1755 and puts the heat equation of motion in fluid, in which he expresses the molecular motion with communication and transportation of molecules before Boltzmann's modeling with collision and transportation.

Navier, Poisson, Cauchy, Stokes, et al. struggle to configurate the microscopically-descriptive fluid equations with mathematical and practical adaptation, to which Plandtl 1934 uses the nomenclature as the Navier-Stokes equations. Sturm (1803-55) and Liouville (1809-82) propose the differential equation of Strum-Liouville 1836-7 [28, 66], solving boundary value problem.

Boltzmann (1844-1904) 1895 proposes the gas theory, ending the microscopically descriptive equations such as the original Navier-Stokes equations. However, Boltzmann's motion theories aren't satisfied with the law of Newton (1643-1727) and are 'thrown into oblibion.'

> In my opinion it would be a great tragedy for science if the theory of gases were temporarily thrown into oblibion because of a momentary hostile attitude toward it, as was for example the wave theory because of Newtonian authority. Forward to Part II. [4, p.215]

1.2. Kepler motion, water wave of Huygens and Newtonian mechanics.

1.2.1. *Kepler motion.*

As we know, the Keplar's laws are as the three items :

1. the orbit of every planet is an ellipse with the sun at one of the two foci.
2. loi des aires (law of area) : constancy of areal velocity.
3. $n^2a^3 = \mu$, where, n : mean motion ($n = \frac{2\pi}{T}$), T : orbital period around the sun, a : long radius, μ : a constant.

The Kepler's equation : $l = u - e\sin u$, where, e : eccentric ratio, l : mean anomaly, [1] u : eccentric anomaly [2]. We can observe only the mean anomaly, then from this, we must deduce

[1] 平均近点角。

[2] 離心近点角。

the eccentric anomaly and true anomaly [3]. These calculations are tried by Lagrange at the first time. We introduce these mathematical history in our paper § 2.1. (cf. [42, pp.1022-4] Item number 322.「天体力学」.)

1.2.2. *Newtonian mechanics.*
Newton shows his principle on the wave motion in the water pressure [47].

> Pressio non propagatur per Fluidum secundum lineas rectas, nisi ubi particulæ Fluidi in directum jacent.
>
> - Si jaceant particulæ $a,\ b,\ c,\ d,\ e$ in linea recta, potest quidem pressio directe progari ab a ad e ;
> - at particula e urgebit particulas oblique positas f & g oblique, & particulæ illæ f & g non sustinebunt pressionem illatam, nisi fulciantur a particulis ulterioribus b & k ;
> - quantanus autem fulciuntur, premunt particulas fulcientes ;
> - & hæ non sustinebunt pressionem nisi fulciantur ab ulterioribus l & m easque premant, & sic deinceps in infinitum.
> - Pressio igitur, quam primum propagatur ad particulas quæ non in directum jacent, divaricare incipiet & oblique propagabitur in infinitum ;
> - & postquam incipit oblique propagari, si inciderit in particulas ulteriores, quæ non in directum jacent, iterum divaricabit ;
> - idque toties, quoties [4] in particulas non accurate in directum jacentes inciderit. Q.E.D. [47, pp.354-5]. (Itemization and indentation mine.)
>
> The pressure doesn't propagate by the fluid of the secondary linear strait, except for the particle of adjacent fluid.
>
> - If the adjacent particles a, b, c, d, e propagate in the straight line, press from a to e ;
> - the particle e progresses separately into the oblique points f and g, and without sustained pressure, and moreover, to the particles h and k ;
> - as it is fixed in another direction, it presses for the particle into propping up ;
> - the unsustained pressure goes separately into the particles l and m, and as this way, it follows successively and limitlessly.
> - Therefore, the pressure is propagated at first, to the particle in indirect adjacency, permanently ;
> - after it propagates obliquely, if it is far particle, then it will spread out iteratively into the next one ;
> - thus it will occur so many time, inaccurately, to the particle in the indirect adjacency. Q.E.D.
>
> (Translation mine.)

After Newton, we cite the Newtonian mechanics. For example, Laplace cites the expressin (4) :

$$\frac{3dt^2 \int \delta dR}{r^2 dr}$$

Gauss cites the expression (18) :

$$\frac{(1 - e\cos E)dE}{2\pi\rho^2}$$

[3]真近点角。

[4](⇓) toties, quoties = totiens, quotiens. so ofen, so many times, as often. [25, p.864, cf. totiens(toties)]

where, Laplace's r and Gauss' ρ are the distance between the planets.[5]

1.3. **The arguments between Euler and d'Alembert on the wave equations of cord.**

As Riemann[57, p.5] says, d'Alembert is the progenitor of the problem of the vibrating cord. For gaining a general solution of the differential equation, he proposes the series by trigonometric fonction, d'Alembert, Euler, D. Bernoulli and Lagrange extend each solution of the same problem. d'Alembert concludes after his observations in *Recherches sur les vibrations des Cordes sonores* [9, pp.1-64,65-73], priding the superiority to both Euler and Bernoulli, as follows :

> De toutes ces réfléxions je crois être en droit de conclure ;
> (1) que la solution que j'ai donnée la premier du Probleme des cordes vibrantes, n'est nullement renfermée dans la fonction de M. Taylor, s'étend beaucoup plus loin, & est aussi générale que la nature de la question le permet ;
> (2) que l'extension que M. Euler y a voulu donner, est capable d'conduire *en error* ;
> (3) que sa construction est contraire à ce qui avance lui-même en termes formels sur l'*identité* & l'*imparité* des fonctions $\varphi(x+t)$ & $\psi(x-t)$;
> (4) que cette construction ne satisfait point à l'équation $\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}$;
> (5) que dans l'équation $y = \varphi(x+t) + \psi(x-t)$, les fonctions doivent demontrer toujours de la même forme, comme M. Euler l'a tacitement supposé lui-même ;
> (6) que si on se permettoit[6] de faire varier la forme de ces fonctions, il faudroit renverser la principe de toutes les constructions & fonctions analytiques, & des démonstrations les plus généralement avouées ;
> (7) qu'en faisant varier cette forme, le Problême n'auroit plus de solution possible, & resteroit indéterminé ;
> (8) que cette solution ne represente pas mieux que la mienne , la vrai mouvement de la corde ;
> (9) enfin que la solution de M. Daniel Bernoulli, quelqu'ingénieuse qu'elle puisse être, est trop limitée, & n'ajoute absolument à la mienne aucune simplification ; qu'en un mot, le calcul analytique de Problême, & l'exemen synthétique que M. Bernoulli *m'accuse* à *tort de n'avoir point fait*, sont l'un & l'autre, ce me semble, à l'avantage de ma méthode.
>
> [9, pp.63-64]

1.4. **Sturm-Liouville type eigenvalue problem.**

At first, we introduce the Sturm-Liouville type eigenvalue problem as one of the essential Schrödinger equations. We cite Liouville [26], in which he introduces the initial value problem, which Schrödinger call it the Sturm-Liouville type eigenvalue problem, not citing [26] directly but via Courant-Hilbert [6]. The followings are one of the papers on the heat-wave initial-value equation of Liouville [26] 1836 :

> On connait la méthode dont les géomètres ont surtout fait usage dans ces dernières années, pour intégrer les équations aux différences partielles auxquelles on ramène la solution de la plupart des problèmes physico-mathematiques. Cette méthode consiste, comme on sait, à représenter l'intégrale complète de l'équation aux différences partielles par la somme d'un nombre infini d'intégrales particulières contenant chacune une ou plusieurs constantes arbitraires, et à disposer ensuite de ces constances de manière à satisfaire aux conditions définies propres à chaque cas.

[5]($\Downarrow$) This is due to the Newtonian mechanics.

[6]($\Downarrow$) sic. As today's usage, 'permettrions'. In bellow, as the same, faudroit($\Rightarrow$ fallût), auroit($\Rightarrow$ aurions), etc.

Table 1. Papers of Kepler problems

no	name/papers	lifetime	Thr first rule	The second rule	The third rule
1	Kepler [17]:1634	1571-1630	*	*	*
2	Lagrange [19]:1770	1736-1813	*		
3	Laplace [23]:1878	1736-1813	*	*	*
4	Poisson [49]:1809	1781-1840	*	*	*
5	Legendre [24]:1811	1752-1833	*		
6	Gauss [16]:1818	1777-1855		*	
7	Bessel [3]:1820	1784-1846			

Ainsi, dans le *Théorie de la chaleur* [7]que nous prendrons pour exemple, si on vent déterminer, en fonction du temps t et de l'abscisse x, la température u qui a lieu en chaque point d'une barre métallique homogène (recouvert d'une substance non conductrice) dont l'état initial est connu et dont les deux bouts sont entretenue à la température fixe 0°, on aura d'abord l'equation aux différences partielles $\frac{du}{dt} = a^2 \frac{d^2u}{dx^2}$, a^2 étant le rapport de la conductibilité intérieure de cette barre à sa chaleur spécifique.

En plaçant l'origine des coordinnées à l'une des extrémités des barre, et nommant l l'abscisse de l'autre extrémité, il faut ensuite satisfaire aux deux conditions défines

- $u = 0$ pour $x = 0$,
- $u = 0$ pour $x = l$,

et à la condition de l'état initial que l'on peut écrire ainsi $u = f(x)$ pour $t = 0$. [26, p.14]

In 1924, Courant-Hilbert [6, p.238] have read :

Fur die Function $y = f(x)$ ergibt sich dann die Differentialgleichungen

$$(py')' + \lambda gy = 0 \tag{18}$$

während g der Differentialgleichungen $\ddot{g} + \lambda g = 0$ genügen muß. Setzen wir $\lambda = \nu^2$ - daß negative Werte von λ nicht in Betracht kommen, wird sich bald von selbest ergeben -, so wird also u die Form haben

$$u = f(x)(a \cos \nu t + \sin \nu t),$$

während die Function $f(x)$ aus der Differentialgleichungen (18) gemaß den Randbedingungen zu bestimmen ist.

$\cdots\cdots$ (omitted) $\cdots\cdots$

Das so formulierte Problem heißt nach seinen ersten und erfolgreichsten Bearbeitern *das Sturm-Liouvillesche Eigenvertproblem.* Es laßt sich formal noch etwas verallgemeinern, wenn man statt (18) die Differentialgleichung

$$py'' + p'y' - qy + \lambda gy = 0 \tag{19}$$

betrachtet, wobei q eine gegebene stetige Function ist. [p.238][6](Italic mine.)

2. Definition and pioneering studies of the Kepler problems.

About the describability of the trigonometric series of an arbitrary function, Lagrange calculates the Kepler problem in 1770, and Gauss applies it in a case study of a planet in 1818. Gauss and Bessel devote in the planetary movement as well as Lagrange and Legendre.

[7]($\Downarrow$) This book seems to be one by Fourier 1822 [14] : *Théorie analytique de la chaleur*, not Poisson 1835 [53] : *Théorie mathématique de la chaleur*, although Liouvill cites Poisson's works frequently.

2.1. **Lagrange's pioneering solutions of Kepler problems.**
Lagrange introduces three types of studies of the Kepler problems.

> Ce Problème consiste, comme on sait, à couper l'aire elliptique en raison donnée, et sert principlement à déterminer l'anomalie vraie des planètes par leur anomalie moyenne. Depuis Képler, qui a le premier essayé de le résoudre, plusiers savants Gèomètres s'y sont appliqués et en ont donné différents solutions qu'on peu ranger dans trois classes.
>
> - Les unes sont simplement arithmétiques et sont fondées sur la règle de fausse possition : ce sont celles dont les Astronomes se servent ordinairement dans le calcul des éléments des planètes ;
> - les autres sont géométriques ou mécaniques, et dépendent de l'intersection des courbres : celles-ci son plutôt de simple curiosité que d'usage dans l'Astoronomie ;
> - le troisième classe enfin comprend les solutions algébriques, qui donnent l'expression analytique de l'anomalie vraire par l'anomalie moyanne, aussi bien que celle du rayon vecteur de l'orbite, expressions qui sont d'usage continuel et indispensabledans la théorie des perturbations des corps célestes. [19, p.113]

Lagrange's motivation on this paper is as follows :

> J'ai donné, dans un Mémoire imprimé dans le volume de l'année 1768 [8], une méthode particulière pour résoudre, par le moyen des series, toutes les équations, soit algébriques ou transcendentes ; comme cette méthode joint à l'avantage de la facilité et de la simplicité du calcul celui de donner toujours des séries régulières et dont le terme général soit connu, j'ai cru qu'il ne serait pas innutile d'en faire l'application au fameux Problème de Képler, et de fournir par là aux Astronomes des formules plus générales que celles qu'ils ont eues jusqu'à présent pour la solution de ce Problème : c'est là l'objet du présent Mémoire. [19, p.113]

We show the notation of the Kepler's orbital elements by Lagrange corresponding to the notation by MSJ [42, pp.1022-4]. [9]

- n : eccentricity ; (e) of MSJ.
- x : eccentric anomaly ; (u)
- t : mean anomaly ; (l)
- u : true anomaly ; (f)
- r : radius vector ;
- a : radius of circle

¶ 1.
$AD = 2a, \quad CB = ma, \quad CF = na = a\sqrt{1-m^2} \quad \Rightarrow \quad n = \sqrt{1-m^2}, \quad FL = ar$. The angle of *true anomaly* $\angle DFL = u$.
The demi-circle : AED, and with L, the strait line NLM is perpendiculer on AD, and cut at N by the strait lines NF and NC. We consider the ratio of the area of DFL to DFN equals to the ratio of the area of the whole of elliptic to the area of the whole of circle, which is $\frac{a^2\varpi}{2}$, namely

$$\varpi : t = \frac{a^2\varpi}{2} : DFN, \quad \Rightarrow \quad t = \frac{2DFN}{a^2} \tag{1}$$

We denote $\angle DCN \equiv x$: *the eccentric anomaly* due to Kepler, then

$$CM = a\cos x, \quad MN = a\sin x, \quad ML = ma\sin x$$

[8]sic. Œ*vres de Lagrange,* t.III,p.5.
[9]Item number 322.「天体力学」.

$$DFN = DCN + FCN = DCN + \frac{FC \cdot MN}{2} = \frac{a^2 x}{2} + \frac{na^2 \sin x}{2}$$

Hence, from (1),

$$t = \frac{2DFN}{a^2} = x + n \sin x \tag{2}$$

$$FL = ar = \sqrt{\overline{FM}^2 + \overline{ML}^2} = a\sqrt{(n + \cos x)^2 + m^2 \sin^2 x}$$

by $m^2 = 1 - n^2$, $\quad ar = a\sqrt{1 + 2n \cos x + n^2 \cos^2 x} = a(1 + n \cos x) \quad \Rightarrow \quad r = 1 + n \cos x$

$$\sin u = \frac{ML}{LF} = \frac{m \sin x}{1 + n \cos x}, \quad \cos u = \frac{FM}{LF} = \frac{n + \cos x}{1 + n \cos x}$$

$$\frac{\sin u}{1 + \cos u} = \frac{m}{1 + n} \cdot \frac{\sin x}{1 + \cos x}, \quad \tan \frac{u}{2} = \frac{m}{1 + n} \cdot \tan \frac{x}{2}$$

$$\frac{du}{\cos^2 \frac{u}{2}} = \frac{m}{1 + n} \cdot \frac{dx}{\cos^2 \frac{x}{2}}, \quad \Rightarrow \quad \frac{du}{1 + \cos u} = \frac{m}{1 + n} \cdot \frac{dx}{1 + \cos x}$$

Substituting to $1 + \cos u$ the value $(1 + n)\frac{1+\cos x}{1+n \cos x}$, then

$$du = \frac{mdx}{1 + n \cos x}$$

¶ 2.

Lagrange's aim of this paper is as follow :

> It must begin with to solve x of the equation (2), which is not able only by the approximation. Or, all the approximate methods known, I think, the most simplest and most general is one I have proposed in my *Mémoire* on the solution of this equation. I have proved in this *Mémoire* [10] that if we have a equation such that
>
> $$\alpha - x + \varphi(x) = 0$$

The serie :

$$\psi(x) + \varphi(x)\psi'(x) + \frac{1}{2} \cdot \frac{d[\varphi(x)]^2 \psi'(x)}{dx} - \frac{1}{2 \cdot 3} \cdot \frac{d^2[\varphi(x)]^3 \psi'(x)}{dx^2} + \cdots$$

explains the function to seek.

If $t = x + \varphi(x)$, then

$$\psi(x) = \psi(t) - \varphi(t)\psi'(t) + \frac{1}{2} \cdot \frac{d[\varphi(t)]^2 \psi'(t)}{dt} - \frac{1}{2 \cdot 3} \cdot \frac{d^2[\varphi(t)]^3 \psi'(t)}{dt^2} + \cdots$$

Hence, our equation (2) turns into

$$\psi(x) = \psi(t) - n \sin t \ \psi'(t) + \frac{n^2}{2} \cdot \frac{d \sin^2 t \ \psi'(t)}{dt} - \frac{n^3}{2 \cdot 3} \cdot \frac{d^2 \sin^3 t \ \psi'(t)}{dt^2} + \cdots \tag{3}$$

[10] (⇓) Lagrange doesn't identify this paper, however, it may be one of his several papers on these thema. In this expression, α isn't used in the followings.

¶ 3.
We suppose $\psi(x)=x$ to get the value of x by t, then we get $\psi(t)=t,\ \psi'(t)=1$, and then

$$x=t-n\sin t+\frac{n^2}{2}\cdot\frac{d\sin^2 t}{dt}-\frac{n^3}{2\cdot 3}\cdot\frac{d^2\sin^3 t}{dt^2}+\cdots$$

where,

$$\begin{aligned}
2\sin^2 t &= \frac{2}{2}-\cos 2t,\\
4\sin^3 t &= 3\sin t-\sin 3t,\\
8\sin^4 t &= \frac{4\cdot 3}{2\cdot 2}-4\cos 2t+\cos 4t,\\
16\sin^5 t &= \frac{5\cdot 4}{2}\sin t-5\sin 3t+\sin 5t,\\
32\sin^6 t &= \frac{6\cdot 5\cdot 4}{2\cdot 2\cdot 3}-\frac{6\cdot 5}{2}\cos 2t+6\cos 4t-\cos 6t,\\
\cdots
\end{aligned}$$

$$\begin{aligned}
x = &\ t-n\sin t\\
+ &\ \frac{n^2}{2\cdot 2}2\sin 2t\\
+ &\ \frac{n^3}{4\cdot 3!}\Big(3\sin t-3^2\sin 3t\Big)\\
- &\ \frac{n^4}{8\cdot 4!}\Big(4\cdot 2^3\sin 2t-4^3\sin 4t\Big)\\
- &\ \frac{n^5}{16\cdot 5!}\Big(\frac{5\cdot 4}{2}\sin t-5\cdot 3^4\cdot\sin 3t+5^4\sin 5t\Big)\\
+ &\ \frac{n^6}{32\cdot 6!}\Big(\frac{6\cdot 5}{2}2^5\sin 2t-6\cdot 4^5\sin 4t+6^5\sin 6t\Big)\\
+ &\ \frac{n^7}{64\cdot 7!}\Big(\frac{7\cdot 6\cdot 5}{2\cdot 3}\sin t-\frac{7\cdot 6}{2}3^6\sin 3t+7\cdot 5^6\sin 5t-7^6\sin 7t\Big)\\
\cdots
\end{aligned}$$

In this way, we get *the eccentric anomaly* x by *the mean anomaly* t. Next, we will get the radius vector r and *the true anomaly* u.

$$r=1+n\cos x,\quad \tan\frac{u}{2}=\frac{m}{1+n}\tan\frac{x}{2}$$

¶ 4. (The radius vector r.)
It is clear that to get the value $r+n\cos x$, we have to make the general formula in the art.2 $\psi(x)=n\cos x$, which donate $\psi(t)=n\cos t,\ \psi'(t)=-n\sin t$, so that, by (3), we have the following :

$$r=1+n\cos t+n^2\sin^2 t-\frac{n^3}{2}\cdot\frac{d\sin^3 t}{dt}+\frac{n^4}{2\cdot 3}\cdot\frac{d^2\sin^4 t}{dt^2}-\cdots$$

Replacing $\sin^2,\ \sin^3,\ \cdots$, with cos in respect to t, and definitely differenciating, then we get :

$$\begin{aligned} r \;=\; & 1 + n\cos t \\ - \; & \frac{n^2}{2}(-1+\cos 2t) \\ - \; & \frac{n^3}{4\cdot 2}\Big(3\cos t - 3\cos 3t\Big) \\ + \; & \frac{n^4}{8\cdot 3!}\Big(4\cdot 2^2\cos 2t - 4^2\cos 4t\Big) \\ + \; & \frac{n^5}{16\cdot 4!}\Big(\frac{5\cdot 4}{2}\cos t - 5\cdot 3^3\cdot\cos 3t + 5^3\cos 5t\Big) \\ - \; & \frac{n^6}{32\cdot 5!}\Big(\frac{6\cdot 5}{2}2^4\cos 2t - 6\cdot 4^4\cos 4t + 6^4\cos 6t\Big) \\ - \; & \frac{n^7}{64\cdot 6!}\Big(\frac{7\cdot 6\cdot 5}{2\cdot 3}\cos t - \frac{7\cdot 6}{2}3^5\cos 3t + 7\cdot 5^5\cos 5t - 7^5\cos 7t\Big) \\ \cdots \end{aligned}$$

[11] ¶ 5. (The value of $\tan\frac{u}{2}$.)

$$\psi(x) = \frac{m}{1+n}\tan\frac{x}{2} = \tan\frac{u}{2}, \quad \Rightarrow \quad \psi(t) = \frac{m}{1+n}\tan\frac{t}{2}$$

$$\psi'(t) = \frac{m}{1+n}\Big(\frac{1}{2\cos^2\frac{t}{2}}\Big) = \frac{m}{1+n}\cdot\frac{1}{\cos t} = \frac{m}{1+n}\cdot\frac{1-\cos t}{\sin^2}$$

$$\begin{aligned} \tan\frac{u}{2} \;=\; & \frac{m}{1+n}\Big[\tan\frac{t}{2} - n\frac{1-\cos t}{\sin t} + \frac{n^2}{2}\cdot\frac{d(1-\cos t)}{dt} - \frac{n^3}{2\cdot 3}\cdot\frac{d^2(1-\cos t)\sin t}{dt^2} \\ + \; & \frac{n^4}{2\cdot 3\cdot 4}\cdot\frac{d^3(1-\cos t)\sin^2 t}{dt^3} - \frac{n^5}{2\cdot 3\cdot 4\cdot 5}\cdot\frac{d^4(1-\cos t)\sin^3 t}{dt^4} + \cdots\Big] \end{aligned}$$

where, using the following :

$$\frac{1-\cos t}{\sin t} = \tan\frac{t}{2}, \quad \sin t\cos t = \frac{1}{2}\cdot\frac{d\sin^2 t}{dt}, \quad \sin^2 t\cos t = \frac{1}{3}\cdot\frac{d\sin^3 t}{dt}$$

then,

$$\begin{aligned} \tan\frac{u}{2} \;=\; & \frac{m}{1+n}\Big[(1-n)\tan\frac{t}{2} - \Big(\frac{n^2}{2} + \frac{n^3}{2\cdot 3}\Big)\cdot\frac{d^2\sin t}{dt^2} \\ + \; & \Big(\frac{n^3}{2\cdot 2\cdot 3} + \frac{n^4}{2\cdot 3\cdot 4}\Big)\cdot\frac{d^3\sin^2 t}{dt^3} - \Big(\frac{n^4}{3\cdot 2\cdot 3\cdot 4} + \frac{n^5}{2\cdot 3\cdot 4\cdot 5}\Big)\cdot\frac{d^4\sin^3 t}{dt^4} + \cdots\Big] \end{aligned}$$

Finally,

$$\begin{aligned} \tan\frac{u}{2} \;=\; & \frac{m}{1+n}\Big[(1-n)\tan\frac{t}{2} + \frac{n^2}{2}\Big(1+\frac{n}{3}\Big)\sin t - \frac{n^2}{2\cdot 2\cdot 3}\Big(\frac{1}{2}+\frac{n}{4}\Big)2^3\sin 2t \\ - \; & \frac{n^4}{4\cdot 4!}\Big(\frac{1}{3}+\frac{n}{5}\Big)\Big(3\sin t - 3^4\sin 3t\Big) + \frac{n^5}{8\cdot 5!}\Big(\frac{1}{4}+\frac{n}{6}\Big)\Big(4\cdot 2^5\sin 2t - 4^5\sin 4t\Big) \\ + \; & \frac{n^6}{16\cdot 6!}\Big(\frac{1}{5}+\frac{n}{7}\Big)\Big(\frac{5\cdot 4}{2}\sin t - 5\cdot 3^6\sin 3t - 5^6\sin 5t\Big)\cdots\Big] \end{aligned}$$

¶ 6. (*True anomaly u.*)

If we get the value of the same angle u, we must base on the equation :

$$\psi(x) = m\int\frac{dx}{1+n\cos x} = u$$

[11]($\Downarrow$) We correct Lagrange's expression of $-\frac{n^7}{64\cdot 7!}$ to $-\frac{n^7}{64\cdot 6!}$. [19, p.119]

where,

$$\psi(t) = m\int \frac{dt}{1+n\cos t}, \quad \psi'(t) = \frac{m}{1+n\cos t}$$

or,

$$\frac{1}{1+n\cos t} = 1 - n\cos t + n^2\cos^2 t - n^3\cos^3 t + n^4\cos^4 t - \cdots$$

Substituting this value in the formula in the art 2, and ordering the terms with respect to n, then

$$\begin{aligned} u = & \ m\Big[t - n\Big(\int \cos t dt + \sin t\Big) \\ + & \ n^2\Big(\int \cos^2 t dt + \cos t \sin t + \frac{1}{2}\cdot\frac{d\sin^2 t}{dt}\Big) \\ - & \ n^3\Big\{\int \cos^3 t dt + \cos^2 t\sin t + \frac{1}{2}\cdot\frac{d(\cos t\sin^2 t)}{dt} + \frac{1}{2\cdot 3}\cdot\frac{d^2\sin^3 t}{dt^2}\Big\} \\ + & \ n^4\Big\{\int \cos^4 t dt + \cos^3 t\sin t + \frac{1}{2}\cdot\frac{d(\cos^2 t\sin^2 t)}{dt} + \frac{1}{2\cdot 3}\cdot\frac{d^2\cos t\sin^3 t}{dt^2} + \frac{1}{2\cdot 3\cdot 4}\cdot\frac{d^3\sin^4 t}{dt^3}\Big\} + \cdots\Big] \end{aligned}$$

2.2. **Laplace's perturbative calculus on the third Kepler's law.**

Laplace produces the formulae of inegality of the planetary depending on the square and greater power of the eccetricities and inclination of the orbit.

§1. *Formules des inégalités planétaires dépendantes des carrés et des puissances supérieures des excentricités et des inclinations des orbites.*

We show the notation of the Kepler's orbital elements by Laplace corresponding to the notation by MSJ [42, pp.1022-4].

- e : eccentricity ; (e)
- n : mean motion ; ($n = 2\pi/T$)
- a : long radius ; (a)

Consider next the term

$$\frac{3dt^2\int \delta dR}{r^2 dr} \tag{4}$$

of the same formula (T). [12] If we regard it only with respect to the secular quantities, the squares and products of the eccentricities and the inclinations of orbits, depend, it will turn by the following numerical analysis,
Laplace calculates this numerical analysis. and as the result, he concludes as follows.

Recalling the preceding expression of δR, we will observe that the function

$$\frac{m'}{8}aa'B^{(1)}[(p-p')^2 + (q-q')^2] + \cdots$$

equals to a constant independent of time t, because its differential is null by the equation C of no.50 in vol. II ; If we don ' t consider only two planets : m and m', like we will make it in that comes next, $(p-p')^2 + (q-q')^2$ is with the same equation, a quantity independent of time ; the preceding function can not produce, in

$$\frac{2ndt\ a^2\cdot\frac{\partial\delta R}{\partial a}}{\sqrt{1-e^2}}$$

[12](⇓) This is due to the Newtonian mechanics. Gauss also calculates the perturbation of the planets with the Newtonian mechanics. cf. (18)

, only a quantity, in same way, independent of time, and can neglect hence, because it is able to be supposed probably to confound with the value of ndt. [23, p.15]

Poisson [49] discusses this part as defect.

2.3. **Poisson's perturbative calculus on the third Kepler's law.**

We show the notation of the Kepler's orbital elements by Poisson corresponding to the notation by MSJ [42, pp.1022-4]. Else notations are coincident with Laplace for sake of discussion.

- e : eccentricity ; (e)
- n : mean motion ; ($n = 2\pi/T$)
- ρ : (integrated) mean motion ;
- a : long radius ; (a)

Poisson [49] summarizes the Kepler's third low as follows :

> L'action réciproque des planètes produit, dans leurs mouvemens, des inégalitésque l'on distingue en deux espéce :
>
> - les unes sont périodiques et leurs périodes dépendent de la configuration des planètes entre elles ; de sorte qu'elles reprennent les mêmes valeurs toutes les fois que les planètes reviennent à la même position :
> - les autres sont encore périodiques ; mais leurs périodes sont incomparablement plus longues que celles des premières, et elles sont indépendentes de la position relative des planètes.
>
> On nomme ces inégalités à longues périodes, *inégalités séculaires* ; et, vu la lenteur avec laquelle elles croissent, on peut les considérer pendent plusieur siècles, comme propotionnelles au temps. Elles sont à-la-fois les plus difficiles et les plus importantes à déterminer. Ce sont elles qui font varier de siècle en siècle et par l'espace. On sait en effet qu'elles affectent les eccentricités, les inclinations, les longitudes des nœuds et des périhélies de ces orbites ; mais tandis que ces élémens varient, les grands axes restent constans, ainsi que les moyens mouvemens qui s'en déduisent par la *troisième loi de Képler*. [49, p.2]

According to the definitions of today's astoromy, the mean motion, which means average angle velocity, $n = \frac{2\pi}{T}$, where T : the revolution or revolutionary period of a planet. The Kepler's third low says : $n^2a^3 = \mu$, where a is the long radius of axis of the planet, and μ is a constant. He introduces the stability of planetary system :

> La stabilité du système planétaire tient à deux cause :
>
> - à l'invariabilité des grands axes
> - et à ce que les inégalités des eccentricites et des inclinations des orbites sont toujours renfernées dans des limites fort étroites ;
>
> de manière que ces orbites resteront dans tous les temps à-peu-près circulaires et peu inclinées les unes aux autres, comme elles le sont maintenant. Cette belle proposition a lieu quel que soit le nombre de planètes que l'on considère, pourve qu'elles tournent toutes dans le même sens autour du Soleil. [49, p.5]

M. Laplace est parvenu à la démontre (*Mémoires de Paris*, année 1784) en faisant usage du principe de la conservation des aires, et en supposant l'invariabilité des grands axes, qui n'était prouvée jusqu'ici que relativement aux premières puissances des masses. Nous avons repris cette démonstration à la fin de notre Mémoire, et nous avons fait voir que la stabilité du système planétaire n'est pas altérée, lorsqu'on a égard aux carrés des masses et à toutes les puissances des *eccentricités* et des *inclinaisons*. [49, p.5]

Mr. Laplace have reached to show in the *Mémoires de Paris* in 1784, by using the principle of the conservation of area and supposing *the invariability of long axes of planets*, which hasn't been proved until now *with respect to the first power of masses.* We have retried this proof in the last of our paper and show that the stability of planetary system is not changed, when we have regard for the square of masses, and for the total power of excetricities and inclinations. (Italic mine.)

Poisson examines the variations of mean movement of a planet :

> Avant de nous occuper de l'objet principal de ce Mémoire dans lequel nous proposons d'examiner les variations du moyen mouvment, il eat necessaire de rappeler les formules d'où dépendent les variations de tous les élémens elliptiques. [49, p.6]

$$(m)_P \quad \begin{cases} \dfrac{d^2x}{dt^2} + \dfrac{\mu x}{r^3} + \dfrac{dR}{dx} = 0, \\ \dfrac{d^2y}{dt^2} + \dfrac{\mu y}{r^3} + \dfrac{dR}{dy} = 0, \\ \dfrac{d^2z}{dt^2} + \dfrac{\mu z}{r^3} + \dfrac{dR}{dz} = 0, \end{cases}$$

where,

$$\mu = M + m, \quad r = \sqrt{x^2 + y^2 + z^2},$$

and

- M : the mass of the sun,
- $m,\ m',\ m'',\cdots$: masses of planets.
- $x,\ y,\ z,\ x',\ y',\ z',\ \cdots$: the coordinates of $m,\ m',\ m'',\cdots$ at the origin of the sun.

$$(C)_P \quad \begin{cases} Mm\dfrac{xdy - ydx}{dt} + Mm'\dfrac{x'dy' - y'dx'}{dt} + mm'\Big(\dfrac{(x-x')(dy-dy') - (y-y')(dx-dx')}{dt}\Big) = C, \\ Mm\dfrac{xdz - zdx}{dt} + Mm'\dfrac{x'dz' - z'dx'}{dt} + mm'\Big(\dfrac{(x-x')(dz-dz') - (z-z')(dx-dx')}{dt}\Big) = C', \\ Mm\dfrac{ydz - zdy}{dt} + Mm'\dfrac{y'dz' - z'dy'}{dt} + mm'\Big(\dfrac{(y-y')(dz-dz') - (z-z')(dy-dy')}{dt}\Big) = C'', \\ Mm\dfrac{dx^2 + dy^2 + dz^2}{dt^2} + Mm'\dfrac{d(x')^2 + d(y')^2 + d(z')^2}{dt^2} \\ +mm'\Big(\frac{(dx-dx')^2 + (dy-dy')^2 + (dz-dz')^2}{dt^2}\Big) \\ -2(M+m+m')\Big(\frac{Mm}{\sqrt{x^2+y^2+z^2}} + \frac{Mm'}{\sqrt{(x')^2+(y')^2+(z')^2}} + \frac{mm'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\Big) = A'' \end{cases}$$

where, $C,\ C',\ C'',\ A$: arbitrary constants. [13]

§1. *Variations des Élémens elliptiques.*

¶1. We have three resemble equations for arbitrary planets $m,\ m',\ m''$; This is the form of a system of the same differential equations of the *second order* which we have the coordinates $x,\ y,\ z$; $x',\ y',\ z'$; etc., to determine as the functions of time.

$$\begin{aligned} R = &\ m'\Big(\frac{xx'+yy'+zz'}{\sqrt{x'^2+y'^2+z'^2}}\Big) - \frac{m'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \\ + &\ m''\Big(\frac{xx''+yy''+zz''}{\sqrt{x''^2+y''^2+z''^2}}\Big) - \frac{m''}{\sqrt{(x-x'')^2+(y-y'')^2+(z-z'')^2}} \\ + &\ \cdots \end{aligned}$$

¶2. By multiplying the equations (m) the first with $2dx$, the second with rdy, the third with 2dz, adding them next, and integrating finally, it turns into

$$(1)_P \qquad \frac{dx^2+dy^2+dz^2}{dt^2} - \frac{2\mu}{r} + 2\int d'R$$

where, the symbol d' denotes a relative differential on the only coordinates $x,\ y,\ z$ of the planet m ; therefore, we get the followings

$$d'R = \frac{dR}{dx}dx + \frac{dR}{dy}dy + \frac{dR}{dz}dz.$$

¶4. (pp.13-14.)

We denote below $\int ndt$ with ρ, and we call this integral the mean motion of the perturbed planet, because it corresponds to the formula of elliptic motion of the perturbed motion of mean motion nt.

$$\int ndt = \rho, \quad n = \frac{\sqrt{\mu}}{a\sqrt{a}}, \quad \frac{\mu}{a} = 2\int d'R \quad \Rightarrow \quad dn = \frac{3an}{2}\cdot d\cdot\frac{1}{a} = \frac{3an}{\mu}\cdot d'R$$

where

$$dn = \mu^{\frac{1}{2}}\Big[\Big(\frac{1}{a}\Big)^{\frac{3}{2}}\Big]'\cdot d\Big(\frac{1}{a}\Big) = \Big(n^2a^3\Big)^{\frac{1}{2}}\cdot\frac{3}{2}\Big(\frac{1}{a}\Big)^{\frac{1}{2}}\cdot d\Big(\frac{1}{a}\Big), \quad d\Big(\frac{1}{a}\Big) = \frac{2d'R}{\mu}$$

Poisson proposes a formula of the mean integrated motion ρ, combining the mean motion n of the Kepler's third law : $n^2a^3 = \mu$, where μ is a constant, as follows :

$$\rho = \frac{3}{\mu}\iint an\ d'R\ dt$$

This is the formula which we will hold principally through the following of this Mémoire. [49, p.14.]

§2. *Variayions du Moyan mouvement. Première approximation.*

¶8. (p.23.)

Assuming thus, to take the value the differential of R with respect to the coordinates of m, without varying one of perturbed planets, it is necessary to differentiate with respect to the mean motion nt, and to treat the other mean motions as constants. This differentiation will disappear the constant term R, which responds to $i'=0,\ i=0$; and $d'R$ will be compose of only the periodic terms in the form :

$$d'R = m'Aindt\ \sin(i'n't - int + k)$$

[13](⇓) Poisson cites these expression $(C)_p$ in the Laplace's *Mécanique Céleste*, Libre II, chap.2., sic. These expression $C,\ C',\ C''$ and A correspond to (4), (5), (6) and (7), respectively in the Œ*uvres complète de Laplace*, Tome 1, pp.146-7.

as a result, it turns :

$$d^2\rho = \frac{3m'Ain^2adt^2}{\mu}\ \sin(i'n't - int + k) \quad \Rightarrow \quad \frac{d^2\rho}{dt^2} = \frac{3m'Ain^2a}{\mu}\ \sin(i'n't - int + k)$$

and by integrating with respect to t, we get the mean motion :

$$\rho = \frac{3m'Ain^2a}{\mu(i'n' - in)^2}\ \sin(i'n't - int + k)$$

¶10. (pp.26-27.)
However, Mr. Laplace, has regarded (MC, t. 3, p.14) that this part of R is constant with respect to the elements of the planet m ; from here, it follows that the differential d' of this part reduce into 0, at least when we neglect the quantities of forth order with respect to the *eccentricities* and to *inclinations*, as Mr. Laplace has supposed so. We will demonstrate later, (no.17), that this differential is 0, although we have regarded with all the powers of the eccentricities and inclinations ; the secular inequalities of elements of m, don't give any term non periodic in the value of $d'R$; it will be, as a result, no use to regard it ;
§3. *Variations Séculaires du Moyan mouvement. Seconde Approximation.* (pp.39-52)
¶18. (pp.46-47)
Due to the theorem of no.14, this mean motion will not contain any secular inequality, in spite of the number of perturbed planets ; and it is evident that it will be the same of the long axis, which is determined by the equation :

$$\frac{1}{a} = \frac{2}{\mu} \cdot \int d'R$$

We have deduced, by the preceding analysis, into this important theorem and which has been the principle object of our research : the mean motion and the long axis of the planets are invariables,

- while we ignore the periodic inequality, and
- while we neglect the quantities of the third order with respect to perturbed forces.

Finally, Poisson concludes his object of this paper with the following paragraph :

> Nous sommes donc conduits, par l'analyse précédente, à ce théorème important et qui était le but principal de nos recherches : *Les moyens mouvemens et les grands axes des planètes sont invariables, lorsqu'on fait abstraction des inégalités périodiques, et que l'on néglige les quantités du troisième order par rapport aux forces perturbatrices.* (itallic sic.) [49, p.47]
>
> The mean movements and long axes of planets are invariable, when we ignore the periodic inequality, and when we neglect the quantities of the *third order* with respect to the *perturbational forces.* (trans. mine.)

This result is the today's conventional sense in the astronomy.

2.4. **The perturbative calculus by Gauss and Bessel.**

Bessel, in [3] introduces Gauss' paper [16] in Latin entitled with lengthy title : *Determination of attraction in the point on an arbitrarily given position of the moving planet, of which their masses are described in the total orbit to be uniformly distributed by the time rule for each part,*[14] and hopes to his study in 1814, which is called by the announcement of the paper, is earlier than Lagrange and Legendre. It tells a extension of a scope of study in this series. Gauss' application is to calculate the pertubation of a planet in accordance with the Kepler second law. Bessel's paper is *Ueber die Entwicklung der Functionen zwier Winkel u und u′ in Reihen, welche nach*

[14](⇓) Translation mine.

den Cosinussen und Sinussen der Vierfachen von u und u′ fortgehen, 1820. Gauss' introduction says as follows :
¶1.

> The elements in the planetary orbit receive the strict variation, by the perturbation of the another planet, especially, we suppose that
> - after this position in the orbit is independent,
> - or the perturbing planet obeys the Kepler's second law in elliptic orbit,
> - or its mass per orbit considered in this range is equally distributed, and in the partial orbit, in other words, if,
> - the uniform interval of time are assumed, and
> - the same masses above mentioned are divided into parts, then, the time of the revolving planet perturbing and perturbed is not commensurable.
>
> For this elegant theorem, we don't give any sort of propositions, without proving almost perfectly from the astronomical physical principle. The problem has been offered by (Kepler) himself, or by the other, however, this solution needs to be the attraction : the orbital attraction of planets is, moreover, preferably, elliptic annulus, of which the thickness is infinitesimal, in the arbitrary point of given position exactly determined. [16, p.332]

¶2.
Gauss also struggles for perturbation with the given *eccentric anomaly*. We denote

- the *eccentric* ratio : e,
- the *eccentric anomaly* : E,
- the element of *eccentric anomaly* : dE,

then

- the mean element of *eccentric anomaly* : $(1 - e\cos E)dE$,
- the distance of the attraction point from a point : ρ,
- the product of orbit is

$$\frac{(1 - e\cos E)dE}{2\pi\rho^2} \tag{5}$$

[15]

We denote

- the long radius : a,
- the short radius : b,

then $a^2 - b^2 = a^2e^2$. And x value of the orbit : $a\cos E$, and y value of the orbit : $b\cos E$. the values on coordinates on x, y and z are A, B and C. From here, we can decompose into three elements

$$\begin{aligned} \frac{(A - a\cos E)(1 - e\cos E)dE}{2\pi\rho^2} &\equiv d\xi, \\ \frac{(B - b\sin E)(1 - e\cos E)dE}{2\pi\rho^2} &\equiv d\eta, \\ \frac{C(1 - e\cos E)dE}{2\pi\rho^2} &\equiv d\zeta \end{aligned}$$

where, $\rho = \sqrt{(A - a\cos E)^2 + (B - b\sin E)^2 + C^2}$.
We can get ξ, η and ζ by integrating these three differential.

[15](⇓) This is due to the Newtonian mechanics. Laplace also considers the perturbation of the planets owing to the Newtonian mechanics. cf. (4)

For lack of space, we omit the followings.
¶3.

$$\cos E = \frac{\alpha+\alpha'\cos T+\alpha''\cos T}{\gamma+\gamma'\cos T+\gamma''\cos T}, \quad \sin E = \frac{\beta+\beta'\cos T+\beta''\cos T}{\gamma+\gamma'\cos T+\gamma''\cos T}$$

where the coefficients $\alpha,\ \alpha',\ \alpha'',\ \ldots$ are not arbitary and must satisfy the conditions.$\alpha,\ \alpha',\ \alpha''$ or $\beta,\ \beta',\ \beta''$ or $\gamma,\ \gamma',\ \gamma''$ are respectively propotional : for it is avoidable for E not to be indeterminated. Clearly, the coefficients $\alpha,\ \alpha',\ \alpha'',\ \ldots$ are compararable,

$$(\alpha+\alpha'\cos T+\alpha''\cos T)^2+(\beta+\beta'\cos T+\beta''\cos T)^2-(\gamma+\gamma'\cos T+\gamma''\cos T)^2=0$$

and it is necessary for this function to have the form : $k(\cos^2 T+\sin^2 T-1)$. Hence, we get the following six conditional expressions $(I)_G$:

$$\begin{aligned}
-\alpha^2-\beta^2+\gamma^2 &= k, \\
-(\alpha')^2-(\beta')^2+(\gamma')^2 &= -k, \quad (6)\\
-(\alpha'')^2-(\beta'')^2+(\gamma'')^2 &= -k, \quad (7)\\
-\alpha'\alpha''-\beta'\beta''+\gamma'\gamma'' &= 0, \quad (8)\\
-\alpha''\alpha-\beta''\beta+\gamma''\gamma &= 0, \\
-\alpha\alpha'-\beta\beta'+\gamma\gamma' &= 0
\end{aligned}$$

$$\begin{aligned}
(II)_G \quad & \alpha\beta'\gamma''+\alpha'\beta''\gamma+\alpha''\beta\gamma'-\alpha\beta''\gamma'-\alpha'\beta\gamma''-\alpha''\beta'\gamma \\
&= \alpha(\beta'\gamma''-\beta''\gamma')+\alpha'(\beta''\gamma-\beta\gamma'')+\alpha''(\beta\gamma'-\beta'\gamma) \\
&\equiv \varepsilon
\end{aligned}$$

Combining $(I)_G$ and $(II)_G$, we make $(III)_G$:

$$\begin{aligned}
\varepsilon\alpha &= -k(\beta'\gamma''-\gamma'\beta''), \\
\varepsilon\beta &= -k(\gamma'\alpha''-\alpha'\gamma''), \quad (9)\\
\varepsilon\gamma &= k(\alpha'\beta''-\beta'\alpha''), \\
\varepsilon\alpha' &= k(\beta''\gamma-\gamma''\beta), \\
\varepsilon\beta' &= k(\gamma''\alpha-\alpha''\gamma), \quad (10)\\
\varepsilon\gamma' &= -k(\alpha''\beta-\beta''\alpha), \\
\varepsilon\alpha'' &= k(\beta\gamma'-\gamma\beta'), \\
\varepsilon\beta'' &= k(\gamma\alpha'-\alpha\gamma'), \quad (11)\\
\varepsilon\gamma'' &= -k(\alpha\beta'-\beta\alpha')
\end{aligned}$$

$$\begin{aligned}
& \varepsilon\alpha(\beta'\gamma''-\gamma'\beta'')+\varepsilon\beta(\gamma'\alpha''-\alpha'\gamma'')+\varepsilon\gamma(\alpha'\beta''-\beta'\alpha'') \\
&= -k(\beta'\gamma''-\gamma'\beta'')^2-k(\gamma'\alpha''-\alpha'\gamma'')^2+k(\alpha'\beta''-\beta'\alpha'')^2, \\
& \varepsilon\alpha'(\beta''\gamma-\gamma''\beta)+\varepsilon\beta'(\gamma\alpha''-\alpha''\gamma)-\varepsilon\gamma'(\alpha''\beta-\beta''\alpha) \\
&= k(\beta''\gamma-\gamma''\beta)^2+k(\gamma''\alpha-\alpha''\gamma)^2-k(\alpha''\beta-\beta''\alpha)^2, \\
& \varepsilon\alpha''(\beta\gamma'-\gamma\beta')+\varepsilon\beta''(\gamma\alpha'-\alpha\gamma')-\varepsilon\gamma''(\alpha\beta'-\beta\alpha') \\
&= k(\beta\gamma'-\gamma\beta')^2+k(\gamma\alpha'-\alpha\gamma')^2-k(\alpha\beta'-\beta\alpha')^2
\end{aligned}$$

$$\varepsilon^2 = k\big\{-(\alpha')^2-(\beta')^2+(\gamma')^2\big\}\big\{-(\alpha'')^2-(\beta'')^2+(\gamma'')^2\big\}-k\big\{-\alpha'\alpha''-\beta'\beta''+\gamma'\gamma''\big\}^2$$

$$(IV)_G \quad \varepsilon^2 = k^2$$

By adding (6), (7) and (8) in $(I)_G$, we get the following $(V)_G$:

$$\begin{aligned}
(\beta'\gamma''-\gamma'\beta'')^2 &= -k\{k-(\alpha')^2-(\alpha'')^2\},\\
(\gamma'\alpha''-\alpha'\gamma'')^2 &= -k\{k-(\beta')^2-(\beta'')^2\},\\
(\alpha'\beta''-\beta'\alpha'')^2 &= -k\{k+(\gamma')^2+(\gamma'')^2\},\\
(\beta''\gamma-\gamma''\beta)^2 &= k\{k+(\alpha)^2-(\alpha'')^2\},\\
(\gamma''\alpha'-\alpha''\gamma)^2 &= k\{k+(\beta)^2-(\beta'')^2\},\\
(\alpha''\beta-\beta''\alpha)^2 &= -k\{k-(\gamma)^2+(\gamma'')^2\},\\
(\beta\gamma'-\gamma\beta')^2 &= k\{k+(\alpha)^2-(\alpha')^2\},\\
(\gamma\alpha'-\alpha\gamma')^2 &= k\{k+(\beta)^2-(\beta')^2\},\\
(\alpha\beta'-\beta\alpha')^2 &= -k\{k-(\gamma)^2-(\gamma')^2\}
\end{aligned}$$

From (6), (7) and (8) in $(I)_G$, the following is deduced clearly

$$(\gamma'\gamma''-\beta'\beta'')^2-\{(\gamma')^2-(\beta')^2\}\{(\gamma'')^2-(\beta'')^2\}=(\alpha'\alpha'')^2-\{(\alpha')^2-k\}\{(\alpha'')^2-k\}$$

We get the following $(VI)_G$:

$$\begin{aligned}
\alpha^2-(\alpha')^2-(\alpha'')^2 &= -k,\\
\beta^2-(\beta')^2-(\beta'')^2 &= -k,\\
\gamma^2-(\gamma')^2-(\gamma'')^2 &= k
\end{aligned}$$

Combining rest expressions, we get $(VII)_G$:

$$\begin{aligned}
\beta\gamma-(\beta'\gamma')^2-(\beta''\gamma'')^2 &= 0,\\
\gamma\alpha-(\gamma'\alpha')^2-(\gamma''\alpha'')^2 &= 0,\\
\alpha\beta-(\alpha'\beta')^2-(\alpha''\beta'')^2 &= 0
\end{aligned}$$

From (9), (10) and (11) in $(III)_G$, we get the following :

$$\varepsilon\{\beta\gamma-\beta'\gamma'-\beta''\gamma''\}=-k\gamma(\gamma'\alpha''-\alpha'\gamma'')-k\gamma'(\gamma''\alpha-\alpha''\gamma)-k\gamma''(\gamma\alpha'-\alpha\gamma')=0,$$

where, we suppose $k\neq 0$. We suppose always $k=1$ and $\varepsilon=\pm 1$ in below.

$$\begin{aligned}
\alpha &= \cos L\tan N,\\
\beta &= \sin L\tan N,\\
\gamma &= \sec N,\\
\alpha' &= \cos L\cos M\sec N\pm\sin L\sin M,\\
\beta' &= \sin L\cos M\sec N\mp\cos L\sin M,\\
\gamma' &= \cos M\tan N,\\
\alpha'' &= \cos L\sin M\sec N\mp\sin L\cos M,\\
\beta'' &= \sin L\sin M\sec N\pm\cos L\cos M,\\
\gamma'' &= \sin M\tan N
\end{aligned}$$

¶4.

If the expression of distance ρ is substituted for the values $\cos E$ and $\sin E$, then

$$\rho=\frac{\Big(G+G'\cos^2 T+G''\sin^2 T+2H\cos T\sin T+2H'\sin T+2H''\cos T\Big)^{\frac{1}{2}}}{\gamma+\gamma'\cos T+\gamma''\sin T}$$

where, the coefficients $\alpha,\ \alpha',\ \alpha'',\cdots$ are determinated nicely by the following six conditional equations :

$$(1)_G \quad \begin{cases} -\alpha^2-\beta^2+\gamma^2=1, \\ -(\alpha')^2-(\beta')^2+(\gamma')^2=-1, \\ -(\alpha'')^2-(\beta'')^2+(\gamma'')^2=-1, \\ -\alpha'\alpha''-\beta'\beta''+\gamma'\gamma''=0, \\ -\alpha''\alpha-\beta''\beta+\gamma''\gamma=0, \\ -\alpha\alpha'-\beta\beta'+\gamma\gamma'=0 \end{cases}$$

We suppose $H=0,\ H'=0,\ H''=0$, then

$$(A^2+B^2+C^2)t^2+a^2t\cos^2 E+b^2t\sin^2 E-2aAt\cdot t\cos E-2bBt\cdot t\sin E$$

The coefficient matrix for transfer from T to E is expressed as follows :

$$\begin{cases} t\cos E=\alpha+\alpha'\cos T+\alpha''\sin T, \\ t\sin E=\beta+\beta'\cos T+\beta''\sin T, \\ t=\gamma+\gamma'\cos T+\gamma''\sin T \end{cases} \Rightarrow \begin{bmatrix} t\cos E \\ t\sin E \\ t \end{bmatrix} = \begin{bmatrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \end{bmatrix} \begin{bmatrix} 1 \\ \cos T \\ \sin T \end{bmatrix}$$

We substitute in the following expression :

$$G+G'\cos^2 T+G''\sin^2 T,$$

then

$$(W)_G \quad a^2x^2+b^2y^2+(A^2+B^2+C^2)z^2-2aAxz-2bByz$$

The coefficient matrix for transfer from $u,\ u',\ u''$ to $x,\ y,\ z$ is expressed as follows :

$$\begin{cases} x=\alpha u+\alpha'u'+\alpha''u'', \\ y=\beta u+\beta'u'+\beta''u'', \\ z=\gamma u+\gamma'u'+\gamma''u'' \end{cases} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \end{bmatrix} \begin{bmatrix} u \\ u' \\ u'' \end{bmatrix}$$

The expressions with undeterminated functions $u,\ u'$ and u'' are :

$$Gu^2+G'(u')^2+G''(u'')^2 \tag{12}$$

The (inverse) coefficient matrix for transfer from $x,\ y,\ z$ to $u,\ u',\ u''$ is expressed as follows :

$$\begin{cases} u=-\alpha x-\beta y+\gamma z, \\ u'=\alpha'x+\beta'y-\gamma'z, \\ u''=\alpha''x+\beta''y-\gamma''z \end{cases} \Rightarrow \begin{bmatrix} u \\ u' \\ u'' \end{bmatrix} = \begin{bmatrix} -\alpha & -\beta & \gamma \\ \alpha' & \beta' & -\gamma' \\ \alpha'' & \beta'' & -\gamma'' \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

By substituting $u,\ u',\ u''$ in (12), $(W)_G$ turns into :

$$(W')_G \quad G(-\alpha x-\beta y+\gamma z)^2+G'(\alpha'x+\beta'y-\gamma'z)^2+G''(\alpha''x+\beta''y-\gamma''z)^2,$$

and we get the following :

$$(2)_G \begin{cases} a^2=G\alpha^2+G'(\alpha')^2+G''(\alpha'')^2, \\ b^2=G\beta^2+G'(\beta')^2+G''(\beta'')^2, \\ A^2+B^2+C^2=G\gamma^2+G'(\gamma')^2+G''(\gamma'')^2, \\ bB=G\beta\gamma+G'\beta'\gamma'+G''\beta''\gamma'', \\ aA=G\gamma\alpha+G'\gamma'\alpha'+G''\gamma''\alpha'', \\ 0=G\alpha\beta+G'\alpha'\beta'+G''\alpha''\beta'' \end{cases} \Rightarrow \begin{bmatrix} a^2 \\ b^2 \\ A^2+B^2+C^2 \\ bB \\ aA \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha^2 & (\alpha')^2 & (\alpha'')^2 \\ \beta^2 & (\beta')^2 & (\beta'')^2 \\ \gamma^2 & (\gamma')^2 & (\gamma'')^2 \\ \beta\alpha & \beta'\alpha' & \beta''\alpha'' \\ \gamma\alpha & \gamma'\alpha' & \gamma''\alpha'' \\ \alpha\beta & \alpha'\beta' & \alpha''\beta'' \end{bmatrix} \begin{bmatrix} G \\ G' \\ G'' \end{bmatrix}$$

¶5. (The proof of satisfication for roots $G,\ G',\ G''$.)
By combinating $(1)_G$ and $(2)_G$,

$$-\alpha a^2+\gamma aA=\alpha G, \quad -\beta b^2+\gamma bB=\beta G, \quad \gamma(A^2+B^2+C^2)-\alpha aA-\beta bB=\gamma G$$

From these, we get :

$$(3)_G,\quad \alpha=\frac{\gamma aA}{a^2+G},\quad (4)_G\quad \beta=\frac{\gamma bB}{b^2+G},\quad A^2+B^2+C^2-\frac{a^2A^2}{a^2+G}-\frac{b^2B^2}{b^2+G}=G,\quad \Rightarrow$$

$$(5)_G\quad \frac{A^2}{a^2+G}+\frac{B^2}{b^2+G}+\frac{C^2}{G}=1$$

By combinating $(1)_G$ and $(2)_G$,

$$\alpha'a^2-\gamma'aA=\alpha'G',\quad \beta'b^2-\gamma'bB=\beta'G',\quad -\gamma'(A^2+B^2+C^2)+\alpha'aA+\beta'bB=\gamma'G'$$

From these, we get :

$$(6)_G\quad \alpha'=\frac{\gamma'aA}{a^2-G'},\quad (7)_G\quad \beta'=\frac{\gamma'bB}{b^2-G'},\quad (8)_G\quad \frac{A^2}{a^2-G'}+\frac{B^2}{b^2-G'}-\frac{C^2}{G'}=1$$

$$(9)_G\quad \alpha''=\frac{\gamma''aA}{a^2-G''},\quad (10)_G\quad \beta''=\frac{\gamma''bB}{b^2-G''},\quad (11)_G\quad \frac{A^2}{a^2-G''}+\frac{B^2}{b^2-G''}-\frac{C^2}{G''}=1$$

Hence, it is clear that $G,\ -G',\ -G''$ are roots of the equation :

$$(12)_G\quad \frac{A^2}{a^2+x}+\frac{B^2}{b^2+x}+\frac{C^2}{x}=1,$$

which is extended as the following cubic equation :

$$(13)_G\quad x^3-(A^2+B^2+C^2-a^2-b^2)x^2+(a^2b^2-a^2B^2-a^2C^2-b^2A^2-b^2C^2)x-a^2b^2C^2=0$$

¶6. (The relation between roots and the coefficients of $(13)_G$.)

I. From the last term $a^2b^2C^2$, we can conclude the roots are real, and positive, if $C=0$. We denote non-negative, real root by g.

II.

$$x=\frac{A^2x}{a^2+x}+\frac{B^2x}{b^2+x}+C^2,\quad g=\frac{A^2g}{a^2+g}+\frac{B^2g}{b^2+g}+C^2$$

Subtracting $(x-g)$ and deviding with $(x-g)$, then

$$1=\frac{a^2A^2}{(a^2+x)(b^2+g)}+\frac{b^2B^2}{(b^2+x)(b^2+g)}$$

$$(14)_G\quad 2x=\frac{a^2A^2}{(a^2+g)}+\frac{b^2B^2}{(b^2+g)}-a^2-b^2\pm\Big\{\big(a^2-b^2-\frac{a^2A^2}{(a^2+g)}+\frac{b^2B^2}{(b^2+g)}\big)^2+\frac{4a^2b^2A^2B^2}{(a^2+g)(b^2+g)}\Big\}^{\frac{1}{2}}$$

III.

$$gx=\frac{A^2gx}{a^2+x}+\frac{B^2gx}{b^2+x}+gC^2,\quad gx=\frac{A^2gx}{a^2+g}+\frac{B^2gx}{b^2+g}+xC^2$$

Subtracting the both hands of side, then

$$0=\frac{A^2gx}{(a^2+g)(a^2+x)}+\frac{B^2gx}{(b^2+g)(b^2+x)}+C^2$$

IV.

$$x^2-\underbrace{(A^2+B^2-a^2-b^2)}_{b}x+\underbrace{a^2b^2-a^2B^2-b^2A^2}_{c}=0$$

$$\frac{1}{2}(A^2+B^2-a^2-b^2)\pm\frac{1}{2}\Big\{\big(A^2+B^2-a^2-b^2\big)^2+4A^2B^2\Big\}^{\frac{1}{2}}=0$$

We observe the following three cases.

- The first case : $a^2b^2 - a^2B^2 - b^2A^2 > 0$, namely, the attractive perturbating in the ellipse is pulled inward curving.

$$(a^2b^2 - a^2B^2 - b^2A^2)\Big(\frac{1}{a^2} + \frac{1}{b^2}\Big) + \frac{b^2A^2}{a^2} + \frac{a^2B^2}{b^2} > 0$$

- The second case : $a^2b^2 - a^2B^2 - b^2A^2 < 0$, namely, the attractive perturbating in the ellipse is pulled outward curving.
- The third case : $a^2b^2 - a^2B^2 - b^2A^2 = 0$, the second root becomes 0, then the third root is

$$= -\frac{b^2A^2}{a^2} - \frac{a^2B^2}{b^2} < 0$$

This case is to be excluded in our problem, because this physical case doesn't occur.

¶7.

For determinating the coefficients $\gamma,\ \gamma',\ \gamma''$, from $(1)_G,\ (3)_G,\ (4)_G,\ (6)_G,\ (7)_G,\ (9)_G,\ (10)_G$ we can get

$$(15)_G \quad \begin{cases} \gamma = \Big\{1 - \big(\frac{aA}{a^2+G}\big)^2 - \big(\frac{bB}{b^2+G}\big)^2\Big\}^{-\frac{1}{2}}, \\ \gamma' = \Big\{\big(\frac{aA}{a^2-G'}\big)^2 + \big(\frac{bB}{b^2-G'}\big)^2 - 1\Big\}^{-\frac{1}{2}}, \\ \gamma'' = \Big\{\big(\frac{aA}{a^2-G''}\big)^2 + \big(\frac{bB}{b^2-G''}\big)^2 - 1\Big\}^{-\frac{1}{2}} \end{cases}$$

Combining $(5)_G,\ (8)_G,\ (11)_G$

$$(16)_G \quad \begin{cases} \gamma = \Big\{\frac{G}{\big(\frac{AG}{a^2+G}\big)^2 + \big(\frac{BG}{b^2+G}\big)^2 + C^2}\Big\}^{\frac{1}{2}}, \\ \gamma' = \Big\{\frac{G'}{\big(\frac{AG'}{a^2-G'}\big)^2 + \big(\frac{BG'}{b^2-G'}\big)^2 + C^2}\Big\}^{\frac{1}{2}}, \\ \gamma'' = \Big\{\frac{G''}{\big(\frac{AG''}{a^2-G''}\big)^2 + \big(\frac{BG''}{b^2-G''}\big)^2 + C^2}\Big\}^{\frac{1}{2}} \end{cases}$$

When we suppose $C = 0$ in $(13)_G$, the two roots are $-G'$ and $-G''$, for $G = 0$, γ in $(15)_G$ turns into :

$$\gamma = \Big\{1 - \frac{A^2}{a^2} - \frac{B^2}{b^2}\Big\}^{-\frac{1}{2}},$$

¶8.

Multiplying $(5)_G$ with $a^2b^2 - G^2$, we get the following expression :

$$\Big\{\frac{a^2A^2(b^2+G)}{a^2+G} - A^2G\Big\} + \Big\{\frac{b^2B^2(a^2+G)}{b^2+G} - B^2G\Big\} + \Big\{\frac{a^2b^2C^2}{G} - C^2G\Big\} = a^2b^2 - G^2 \qquad (13)$$

For we can consider (13) as a cubic equation,

$$\begin{cases} \text{the sum of roots:} \quad G - G' - G'' = A^2 + B^2 + C^2 - a^2 - b^2, \\ \text{production of roots :} \quad GG'G'' = a^2b^2C^2 \end{cases}$$

$$\frac{a^2A^2(b^2+G)}{a^2+G} + \frac{b^2B^2(a^2+G)}{b^2+G} + G'G'' - G(G - G' - G'' + a^2 + b^2) = a^2b^2 - G^2 \qquad (14)$$

The expression (14) turns into :

$$\frac{a^2A^2(b^2+G)}{a^2+G} + \frac{b^2B^2(a^2+G)}{b^2+G} - (a^2+G)(b^2+G) + (G+G')(G+G'') = 0$$

This coefficients $\gamma, \gamma', \ \gamma''$ of $(15)_G$ are transformed as follows :

$$\begin{cases} (17)_G \quad \gamma = \left\{ \frac{(a^2+G)(b^2+G)}{(G+G')(G+G'')} \right\}^{\frac{1}{2}}, \\ (18)_G \quad \gamma' = \left\{ \frac{(a^2-G')(b^2-G')}{(G+G')(G''-G')} \right\}^{\frac{1}{2}}, \\ (19)_G \quad \gamma'' = \left\{ \frac{(a^2-G'')(b^2-G'')}{(G+G'')(G'-G'')} \right\}^{\frac{1}{2}} \end{cases}$$

¶9.
We can accept easily that when we pay attention that $\gamma, \ \gamma', \ \gamma'''$ determine the solution. From III in art. 3,

$$\varepsilon\gamma = \alpha'\beta'' - \beta'\alpha'' \tag{15}$$

By $(6)_G, \ (7)_G, \ (9)_G, \ (19)_G$, transformed into as follow :

$$\begin{aligned} \varepsilon\gamma &= \frac{abAB\gamma'\gamma''}{(a^2-G')(b^2-G'')} - \frac{abAB\gamma'\gamma''}{(a^2-G'')(b^2-G')} \\ &= \frac{ab(a^2-b^2)AB(\gamma''-G'')\gamma'\gamma''}{(a^2-G')(a^2-G'')(b^2-G')(b^2-G'')} \end{aligned}$$

Here, we consider the expression $(13)_G$, we can deduce easily as follow,

$$\begin{aligned} &(a^2+G)(a^2-G')(a^2-G'') = a^2(a^2-b^2)A^2, \\ &(b^2+G)(b^2-G')(b^2-G'') = -b^2(a^2-b^2)B^2 \end{aligned}$$

Then we deduce (16),

$$\varepsilon\gamma = \frac{(a^2+G)(b^2+G)(G'-G'')\gamma'\gamma''}{ab(a^2-b^2)AB}$$

Combinning with $(17)_G$,

$$\gamma\gamma'\gamma'' = \frac{\varepsilon ab(a^2-b^2)AB}{(G+G')(G+G'')(G'-G'')}$$

Similarly,

$$\alpha\alpha'\alpha'' = \frac{\varepsilon a^2bA^2B}{(G+G')(G+G'')(G'-G'')}, \quad \beta\beta'\beta'' = -\frac{\varepsilon ab^2AB^2}{(G+G')(G+G'')(G'-G'')},$$
$$\alpha\beta = \frac{abAB}{(G+G')(G+G'')}, \quad \alpha'\beta' = -\frac{abAB}{(G+G')(G'-G'')}, \quad \alpha''\beta'' = \frac{abAB}{(G+G'')(G'-G'')}$$

¶10.
Our formulae are to be determine, we consider such occasions. At first, we discuss where the negative roots of the cubic equation $-G', \ -G''$ equally,and by the formulae $(18)_G, \ (19)_G$, the values γ' and γ'' become infinitesimal, other become indeterminating. Supposing $g = G$ in the formula $(14)_G$, then two values of x are clear, i.e. G' and $-G''$ are equal, it is necessary to be due to the following :

$$AB = 0, \quad a^2 - b^2 - \frac{a^2A^2}{a^2+G} + \frac{b^2B^2}{b^2+G} = 0$$

$a^2 - b^2 \geq 0$ is due to the following

$$B = 0, \quad a^2 - b^2 = \frac{a^2A^2}{a^2+G}, \quad \text{or} \quad a^2 + G = \frac{a^2A}{a^2-b^2} \tag{16}$$

Substituing these values to the equation $(14)_G$, then it turns into

$$G' = G'' = b^2$$

Substituing next $x = -b^2$ in the cubic equation $(13)_G$, it turns into

$$(a^2 - b^2)(C^2 + b^2) = b^2 A^2$$

From (16), $B = 0$

$$G = \frac{a^2 A^2}{a^2 - b^2} - a^2 = \frac{a^2 C^2}{b^2}$$

From $(17)_G$,

$$\gamma = \left\{ \frac{a^2 b^2 A^2}{(a^2 - b^2)(a^2 C^2 + b^2)} \right\}^{\frac{1}{2}} = \left\{ \frac{a^2 C^2 + a^2 b^2}{a^2 C^2 + b^2} \right\}^{\frac{1}{2}}$$

From the formulae$(3)_G$ and $(4)_G$,

$$\alpha = \frac{\gamma(a^2 - b^2)}{aA} = \frac{\gamma b^2 A}{a(C^2 + b^2)} = \left\{ \frac{b^2(a^2 - b^2)}{a^2 C^2 + b^2} \right\}^{\frac{1}{2}} = \left\{ \frac{b^2 A^2}{(C^2 + b^2)(a^2 C^2 + b^2)} \right\}^{\frac{1}{2}}, \quad \beta = 0$$

From $(18)_G$, $(19)_G$, γ' and γ'' in this case are indetermine, and the rest values of coefficients α', β', α'', β'' are the same. From $(6)_G$,

$$\alpha' = \frac{\gamma' a A}{a^2 - b^2}$$

and

$$\beta' = \sqrt{(1 - (\alpha')^2 + (\gamma')^2}, \quad \gamma'' = \sqrt{(\gamma^2 - 1 - (\gamma')^2}, \quad \alpha'' = \frac{\gamma'' a A}{a^2 - b^2}, \quad \beta'' = \sqrt{(1 - (\alpha'')^2 + (\gamma'')^2}$$

From

$$\gamma^2 = 1 + \alpha^2, \quad (\alpha')^2 = \gamma\gamma', \quad 1 = (\alpha')^2 + (\beta')^2 - (\gamma')^2$$

$$(\beta')^2 + \frac{(\gamma')^2}{\alpha^2} = 1 - (\alpha')^2 + \frac{(\gamma\gamma')^2}{\alpha^2} = 1$$

$$\beta' = \cos f, \quad \gamma' = \alpha \sin f, \quad \alpha' = \gamma \sin f$$

$$\varepsilon\alpha'' = \beta\gamma' - \gamma\beta', \quad \varepsilon\beta'' = \gamma\alpha' - \alpha\gamma', \quad \varepsilon\gamma'' = \beta\alpha' - \alpha\beta', \quad \varepsilon^2 = 1$$

$$\alpha'' = -\varepsilon\gamma \cos f, \quad \beta'' = \varepsilon \sin f, \quad \gamma'' = -\varepsilon\alpha \cos f$$

The value of angle f is arbitrary, and moreover, $\varepsilon = \pm 1$.
¶11.

- If $G' \neq G''$, the values γ, γ', γ'' are able to be determined by the formula : $(17)_G$, $(18)_G$, $(19)_G$, however, are evaporated by $a^2 - G'$, $b^2 - G'$, $a^2 - G''$, $b^2 - G''$.
- If α', β', α'', γ'' are indetermined respectively by the formula : $(6)_G$, $(7)_G$, $(9)_G$, $(10)_G$, for example, by supposing that $a^2 - G' = 0$, by $(18)_G$, $\gamma' = 0$, and moreover, by $(7)_G$, $\beta' = 0$ (as a result, $a^2 = b^2$), and is necessary that $\alpha' = \pm 1$.
- If $a^2 = b^2$, the formula, which have proceded by 6, in art 5, supplies $\alpha' A + \beta' B = 0$, which expression connecting with $(\alpha')^2 + (\beta')^2 = 1$, turns into

$$\alpha' = \frac{B}{\sqrt{A^2 + B^2}}, \quad \beta' = \frac{-A}{\sqrt{A^2 + B^2}}$$

These expressions are indeterminated when $A = 0$, $B = 0$ simultaneously.
¶12.
We would determine completely to evolve the differential dE, the following 12 quantities : G, G', G'', α, α', α'', β, β', β'', γ, γ', γ''.

$$(20)_G \quad t = \gamma + \gamma' \cos T + \gamma'' \sin T,$$

then we get

$$(21)_G \quad t\cos E = \alpha + \alpha' \cos T + \alpha'' \sin T, \quad (22)_G \quad t\sin E = \beta + \beta' \cos T + \beta'' \sin T$$

$$\begin{aligned} tdE &= \cos E d.T \sin E - \sin E d.T \cos E \\ &= \cos E(\beta'' \cos T - \beta' \sin T)dT - \sin E(\alpha'' \cos T - \alpha' \sin T)dT \end{aligned}$$

$$\begin{aligned} ttdE &= (\alpha\beta'' - \alpha''\beta)\cos T dT + (\alpha'\beta - \beta'\alpha)\sin T dT + (\alpha'\beta'' - \beta'\alpha'')dT \\ &= \varepsilon\gamma' \cos T dT + \varepsilon\gamma'' \sin T dT + \varepsilon\gamma dT = \varepsilon t dT \end{aligned}$$

namely,

$$(23)_G \quad tdE = \varepsilon dT$$

¶13.
We denote $\sqrt{\gamma^2 - 1} \equiv \delta$, namely $\delta^2 = \alpha^2 + \beta^2 = (\gamma')^2 + (\gamma'')^2$, from $(20)_G$, $(21)_G$ and $(22)_G$, the following is deduced

$$\begin{aligned} & t(\delta + \alpha \cos E + \beta \sin E) \\ &= \gamma\delta + \alpha^2 + \beta^2 + (\gamma\delta + \alpha\alpha' + \beta\beta')\cos T + (\gamma''\delta + \alpha\alpha'' + \beta\beta'')\sin T \\ &= (\gamma + \delta)(\delta + \gamma' \cos T + \gamma'' \sin T) \end{aligned}$$

From $(21)_G$ and $(22)_G$,

$$t(\alpha \sin E - \beta \cos E) = \varepsilon(\gamma' \sin T - \gamma'' \cos T)$$

By substituting these, these equations turn into :

$$\frac{\alpha}{\delta} = \cos L, \ \frac{\beta}{\delta} = \sin L, \ \frac{\gamma'}{\delta} = \cos M, \ \frac{\gamma''}{\delta} = \sin M$$

$$t(1 + \cos(E - L)) = (\gamma + \delta)\big(1 + \cos(T - M)\big), \quad t\sin(E - L) = \varepsilon \sin(T - M)$$

By deviding moreover, where $(\gamma + \delta)(\gamma - \delta) = 1$,

$$\tan\frac{1}{2}(E - L) = \varepsilon(\gamma - \delta)\frac{1}{2}\tan(T - M), \quad \tan\frac{1}{2}(T - M) = \varepsilon(\gamma + \delta)\tan\frac{1}{2}(E - L)$$

$$\gamma - \delta = \tan(\frac{\pi}{4} - \frac{N}{2}), \quad \gamma + \delta = \tan(\frac{\pi}{4} + \frac{N}{2})$$

¶14.
Combining $(20)_G$, $(21)_G$ and $(22)_G$

$$\begin{cases} at(A - a\cos E) = \alpha G - \alpha' G' \cos T - \alpha'' G'' \sin T, \\ (W)_G \quad bt(B - b\sin E) = \beta G - \beta' G' \cos T - \beta'' G'' \sin T \end{cases}$$

We assume that

$$\begin{cases} (\alpha G - \alpha' G' \cos T - \alpha'' G'' \sin T)\Big(\gamma - e\alpha + (\gamma' - e\alpha')\cos T + (\gamma'' - e\alpha'')\sin T\Big) \equiv aX, \\ (\beta G - \beta' G' \cos T - \beta'' G'' \sin T)\Big(\gamma - e\alpha + (\gamma' - e\alpha')\cos T + (\gamma'' - e\alpha'')\sin T\Big) \equiv bY, \\ C(\gamma + \gamma' \cos T + \gamma'' \sin T)\Big(\gamma - e\alpha + (\gamma' - e\alpha')\cos T + (\gamma'' - e\alpha'')\sin T\Big) \equiv Z \end{cases}$$

It turns :

$$d\xi = \frac{\varepsilon X dT}{2\pi t^2 \rho^2}, \quad d\eta = \frac{\varepsilon Y dT}{2\pi t^2 \rho^2}, \quad d\zeta = \frac{\varepsilon Z dT}{2\pi t^2 \rho^2}$$

Here,

$$t\rho = \pm\sqrt{G + G' \cos^2 T + G'' \sin^2 T}$$

$$\frac{\varepsilon dT}{2\pi t^2\rho^2} = \pm\frac{dT}{\big(G + G'\cos^2 T + G''\sin^2 T\big)^{\frac{3}{2}}}$$

$$\begin{cases} \xi = \oint \frac{\varepsilon X dT}{2\pi t^2\rho^2} = \oint \frac{X\ dT}{2\pi(G + G'\cos^2 T + G''\sin^2 T)^{\frac{3}{2}}}, \\ \eta = \oint \frac{\varepsilon Y dT}{2\pi t^2\rho^2} = \oint \frac{Y\ dT}{2\pi(G + G'\cos^2 T + G''\sin^2 T)^{\frac{3}{2}}}, \\ \zeta = \oint \frac{\varepsilon Z dT}{2\pi t^2\rho^2} = \oint \frac{Z\ dT}{2\pi(G + G'\cos^2 T + G''\sin^2 T)^{\frac{3}{2}}} \end{cases}$$

¶15.

$$\begin{cases} \oint \frac{\cos T\ dT}{(G + G'\cos^2 T + G''\sin^2 T)^{\frac{3}{2}}} = 0, \\ \oint \frac{\sin T\ dT}{(G + G'\cos^2 T + G''\sin^2 T)^{\frac{3}{2}}} = 0, \\ \oint \frac{\cos T\ \sin\ T\ dT}{(G + G'\cos^2 T + G''\sin^2 T)^{\frac{3}{2}}} = 0 \end{cases}$$

$$\begin{cases} \xi = \oint \frac{\Big((\gamma - e\alpha)\alpha G - (\gamma' - e\alpha')\alpha' G'\cos^2 T - (\gamma'' - e\alpha'')\alpha'' G''\sin^2 T\Big)\ dT}{2\pi(G + G'\cos^2 T + G''\sin^2 T)^{\frac{3}{2}}}, \\ \eta = \oint \frac{\Big((\gamma - e\alpha)\beta G - (\gamma' - e\alpha')\beta' G'\cos^2 T - (\gamma'' - e\alpha'')\beta'' G''\sin^2 T\Big)\ dT}{2\pi(G + G'\cos^2 T + G''\sin^2 T)^{\frac{3}{2}}}, \\ \zeta = \oint \frac{\Big((\gamma - e\alpha)\gamma + (\gamma' - e\alpha')\gamma'\cos^2 T + (\gamma'' - e\alpha'')\gamma''\sin^2 T\Big)C\ dT}{2\pi(G + G'\cos^2 T + G''\sin^2 T)^{\frac{3}{2}}} \end{cases}$$

By denoting $P,\ Q$ as follows :

$$\oint \frac{\cos^2 T\ dT}{2\pi\big((G+G')\cos^2 T + (G+G'')\sin^2 T\big)^{\frac{3}{2}}} \equiv P, \quad \oint \frac{\sin^2 T\ dT'}{2\pi\big((G+G')\cos^2 T' + (G+G'')\sin^2 T'\big)^{\frac{3}{2}}} \equiv Q,$$

then

$$\begin{cases} a\xi = \Big((\gamma - e\alpha)\alpha G - (\gamma' - e\alpha')\alpha' G'\Big)P + \Big((\gamma - e\alpha)\alpha G - (\gamma'' - e\alpha'')\alpha'' G''\Big)Q, \\ b\eta = \Big((\gamma - e\alpha)\beta G - (\gamma' - e\alpha')\beta' G'\Big)P + \Big((\gamma - e\alpha)\beta G - (\gamma'' - e\alpha'')\beta'' G''\Big)Q, \\ \zeta = \Big((\gamma - e\alpha)\gamma + (\gamma' - e\alpha')\gamma'\Big)CP + \Big((\gamma - e\alpha)\gamma + (\gamma'' - e\alpha'')\gamma''\Big)CQ \end{cases}$$

The expression with the matrix is :

$$\begin{bmatrix} a\xi \\ b\eta \\ \zeta \end{bmatrix} = \begin{bmatrix} (\gamma - e\alpha)\alpha G - (\gamma' - e\alpha')\alpha' G' & (\gamma - e\alpha)\alpha G - (\gamma'' - e\alpha'')\alpha'' G'' \\ (\gamma - e\alpha)\beta G - (\gamma' - e\alpha')\beta' G' & (\gamma - e\alpha)\beta G - (\gamma'' - e\alpha'')\beta'' G'' \\ \big((\gamma - e\alpha)\gamma + (\gamma' - e\alpha')\gamma'\big)C & \big((\gamma - e\alpha)\gamma + (\gamma'' - e\alpha'')\gamma''\big)C \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix}$$

¶16.

$P = Q = \frac{1}{2(G+G')^{\frac{3}{2}}}$, whenever $G' = G''$. These are deduced into a transcendental and are able to be expressed by the series. The clever readers will expect that if, in this occasion, we can determine these transcendental by the special algorithm without any exception, we get which we use frequently, and which is widely proposed. We set m' and n' by $m,\ n > 0$ as follows :

$$m' = \frac{1}{2}(m+n),\ n' = \sqrt{mn}, \quad m'' = \frac{1}{2}(m'+n'),\ n'' = \sqrt{m'n'}, \quad m''' = \frac{1}{2}(m''+n''),\ n''' = \sqrt{m''n''}$$

The common limit of $m,\ n$ in the following integral converges rapidly,

$$\oint \frac{dT}{2\pi\sqrt{m^2\cos^2 T + n^2\sin^2 T}} \equiv \frac{1}{\mu} \tag{17}$$

Demonstr.

$$\sin T = \frac{2m\sin T'}{(m+n)\cos^2 T' + 2m\sin^2 T'}$$

$$\frac{dT}{\sqrt{m^2\cos^2 T + n^2\sin^2 T}} = \frac{dT'}{\sqrt{(m')^2\cos^2 T' + (n')^2\sin^2 T'}}$$

$$\oint \frac{dT}{2\pi\sqrt{m^2\cos^2 T + n^2\sin^2 T}}, \quad \oint \frac{dT'}{2\pi\sqrt{(m')^2\cos^2 T' + (n')^2\sin^2 T'}}$$

$$\oint \frac{d\theta}{2\pi\sqrt{\mu^2\cos^2\theta + \mu^2\sin^2\theta}} = \frac{1}{\mu}. \qquad \square$$

¶17.

$$(m-n)\sin T\sin^2 T' = 2m\sin T' - (m+n)\sin T$$

$$\sqrt{m^2\cos^2 T + n^2\sin^2 T} = m - (m-n)\sin T\sin T'$$

$$\sqrt{(m')^2\cos^2 T' + (n')^2\sin^2 T'} = m\cot T\tan T'$$

$$\begin{aligned}&\sin T\sin T'\sqrt{m^2\cos^2 T + n^2\sin^2 T} + m'(\cos^2 T - \sin^2 T)\\ =\ \ &\cos T\cos T'\sqrt{(m')^2\cos^2 T' + (n')^2\sin^2 T'} - \frac{1}{2}(m-n)\sin^2 T'\end{aligned} \tag{18}$$

Multiplying (18) by

$$\frac{dT}{\sqrt{m^2\cos^2 T + n^2\sin^2 T}} = \frac{dT'}{\sqrt{(m')^2\cos^2 T' + (n')^2\sin^2 T'}}$$

then we get

$$\frac{m'(\cos^2 T - \sin^2 T)\ dT}{\sqrt{m^2\cos^2 T + n^2\sin T}} = -\frac{\frac{1}{2}(m-n)\sin^2 T'\ dT'}{\sqrt{(m')^2\cos^2 T + (n')^2\sin^2 T}} + d.\sin\ T'\cos\ T \tag{19}$$

Multiplying the both hand-sides of (19) by $\frac{m-n}{\pi}$, and substituting the following terms respectively

- $m'(m-n) = \frac{1}{2}(m^2 - n^2)$,
- $(m-n)^2 = 4\{(m')^2 - (n')^2\}$ and
- $\sin^2\ T' = \frac{1}{2} - \frac{1}{2}(\cos^2\ T' - \sin^2\ T')$

and integrating the both hand-side with T and T' respectively from 0 to 2π, then

$$\begin{aligned}
&(m^2-n^2)\oint \frac{(\cos^2 T-\sin^2 T)\ dT}{2\pi\sqrt{m^2\cos^2 T+n^2\sin^2 T}} \\
&= \frac{m-n}{\pi}\Big[-\oint \frac{\frac{1}{2}(m-n)(\frac{1}{2}-\frac{1}{2}(\cos^2\ T'-\sin^2\ T')\ dT'}{\sqrt{(m')^2\cos^2 T'+(n')^2\sin^2 T'}}\Big] \\
&= \frac{(m-n)^2}{2}\Big[-\oint \frac{\big\{1-(\cos^2\ T'-\sin^2\ T')\big\}\ dT'}{2\pi\sqrt{(m')^2\cos^2 T'+(n')^2\sin^2 T'}}\Big] \\
&= 2\{(m')^2-(n')^2\}\Big[-\oint \frac{dT'}{2\pi\sqrt{(m')^2\cos^2 T'+(n')^2\sin^2 T'}}+\oint \frac{(\cos^2\ T'-\sin^2\ T')\ dT'}{2\pi\sqrt{(m')^2\cos^2 T'+(n')^2\sin^2 T'}}\Big] \\
&= 2\{(m')^2-(n')^2\}\Big[-\frac{1}{\mu}+\oint \frac{(\cos^2\ T'-\sin^2\ T')\ dT'}{2\pi\sqrt{(m')^2\cos^2 T'+(n')^2\sin^2 T'}}\Big]
\end{aligned}$$

where, we used (18). The integral :

$$\int \frac{(\cos^2 T-\sin^2 T)\ dT}{2\pi\sqrt{m^2\cos^2 T+n^2\sin^2 T}}$$

is expressed by

$$-\frac{2\big\{(m')^2-(n')^2\big\}+4\big\{(m'')^2-(n'')^2\big\}+8\big\{(m''')^2-(n''')^2\big\}+\cdots}{(m^2-n^2)\mu}=-\frac{\nu}{\mu}$$

We suppose that

$$\sqrt{(m^2-n^2)}\equiv\lambda,\quad \sqrt{(m')^2-(n')^2}\equiv\lambda',\quad \sqrt{(m'')^2-(n'')^2}\equiv\lambda'',\quad \sqrt{(m''')^2-(n''')^2}\equiv\lambda''',\cdots,\tag{20}$$

then [16]

$$\lambda'=\frac{\lambda^2}{m'},\quad \lambda''=\frac{(\lambda')^2}{m''},\quad \lambda'''=\frac{(\lambda'')^2}{m'''},\quad \cdots$$

moreover,

$$\nu=\frac{2(\lambda')^2+4(\lambda'')^2+8(\lambda''')^2\cdots}{\lambda^2}\tag{21}$$

¶18.

By the method explained here, in addition, infinite integral (start from the variable value = 0) allows to assign the maxima. Namely, if T'' is supposed to be determine in the same way by m', n', T', T' is also by m, n, T in the same way, and T''' by m'', n'', T'', and so on, $\cdots$, namely, for any value is determined by T itself, the limit value of series T, T', T'', $T'''\cdots$, rapidly converge to the limit θ, it will become as follows :

$$\int_0^\infty \frac{dT}{\sqrt{m^2\cos^2\ T+n^2\sin^2\ T}}=\frac{\theta}{\mu}$$

$$\begin{aligned}
&\int_0^\infty \frac{(\cos^2\ T-\sin^2\ T)dT}{\sqrt{m^2\cos^2\ T+n^2\sin^2\ T}} \\
&= -\frac{\nu\theta}{\mu}+\frac{1}{\lambda^2}\Big(\lambda'\cos\ T\ \sin\ T'+2\lambda''\cos\ T'\ \sin\ T''+4\lambda'''\cos\ T''\ \sin\ T'''+\cdots\Big)
\end{aligned}$$

[16] We correct (20), for the relation with (21). Gauss puts as follows :

$$\frac{1}{4}\sqrt{(m^2-n^2)}\equiv\lambda,\quad \frac{1}{2}\sqrt{(m')^2-(n')^2}\equiv\lambda',\quad \frac{1}{4}\sqrt{(m'')^2-(n'')^2}\equiv\lambda'',\quad \frac{1}{8}\sqrt{(m''')^2-(n''')^2}\equiv\lambda''',\cdots,$$

[16, p.354]

¶19.
If we suppose $m = \sqrt{(G+G')}$, $n = \sqrt{G+G''}$, then the value quantity P and Q are reduced to μ and ν.

$$P = \oint \frac{\cos^2 T dT}{2\pi(m^2\cos^2 T + n^2 \sin^2 T)^{\frac{3}{2}}}, \quad Q = \oint \frac{\sin^2 T dT}{2\pi(m^2\cos^2 T + n^2 \sin^2 T)^{\frac{3}{2}}}$$

$$(24)_G \quad m^2P + n^2Q = \frac{1}{\mu}$$

$$\frac{(\cos^2 T - \sin^2 T)dT}{2\pi(m^2\cos^2 T + n^2\sin^2 T)} + \frac{(m^2\cos^2 T - n^2\sin^2 T)dT}{2\pi(m^2\cos^2 T + n^2\sin^2 T)^{\frac{3}{2}}}$$
$$= \quad (\frac{m^2\cos^2 T - n^2\sin^2 T)dT}{\pi(m^2\cos^2 T + n^2\sin^2 T)^{\frac{3}{2}}} = d.\frac{\cos T\sin T}{\pi\sqrt{(m^2\cos^2 T + n^2\sin^2 T)}}$$

$$(25)_G \quad \oint \frac{(\cos^2 T - \sin^2 T)dT}{2\pi(m^2\cos^2 T + n^2\sin^2 T)} + \oint \frac{(m^2\cos^2 T - n^2\sin^2 T)dT}{2\pi(m^2\cos^2 T + n^2\sin^2 T)^{\frac{3}{2}}} = -\frac{\nu}{\mu} + m^2P - n^2Q = 0$$

From $(24)_G$ and $(25)_G$, we get

$$P = \frac{1+\nu}{2m^2\mu}, \quad Q = \frac{1-\nu}{2n^2\mu}$$

3. The equations on elastic-solid wave and the equations of fluid dynamics

We show the the kinetic equations of the elastic-solid and hydrodynamics until the fixity of "Navier-Stokes equations" by Prandtl 1934 [55], in the Table 2, in which they discusses at first, the elastic-solid wave equations, for the physico-mathematicians need to consider the elastic-solid wave equations as the essential problems on our nature, and the elastic fluid as the special case, before discussing the equations of fluid dynamics, which is more difficult to understand and experiment. In fact, Poisson aimes to discuss these problems as the general equations in the same paper. In Euler's case also, he discusses with d'Alembert about the equations of vibration of cord in 1748 [12] after d'Alembert's proposal, [17] and the Euler equations of fluid in 1752-55 [13]. cf. §1.3.

We discussed the theory of mathematical physics in classical fluid dynamics in [39], so we omit the details here.

4. Partial differential equations on heat wave by Sturm and Liouville

4.1. **Liouville** [26]**, 1836.**

Liouville (1809-82) 1836 [26] introduces Poisson's works of proving the trigonometric series for an arbitrary function as follows :

> In regard to the equality of the form, $f(x) = \sum A_i \ \sin\frac{i\pi x}{l}$, however, serving as the result of the partial differential equation to solve a physico-mathematics, we have proposed to consider it by itself, abstraction made with the particular question where it presents ; And this idea have brought up the excellent theory of periodic series which Mr. Poisson have exposed at first in the 19th cahier of JEP, and after it, recently, in his works on the heat theory. [26, p.16] (trans. mine.)

[17] d'Alembert publishes his book 1761 [9], in which he didn't approve Euler's assertions.

Table 2. The kinetic HD equations until the fixed NS equations. (Rem. HD : hydro-dynamics, N : non-linear, gr.dv : grad.div, E : $\frac{\Delta}{gr.dv}$ of elastic, F : in fluid)

no	name/prob	the kinetic equations	Δ	gr.dv	E	F
1 N	Euler (1752-55) [13, p.127] fluid(E226)	$\begin{cases} X - \frac{1}{h}\frac{dp}{dx} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}, \\ Y - \frac{1}{h}\frac{dp}{dy} = \frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}, \\ Z - \frac{1}{h}\frac{dp}{dz} = \frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}, \end{cases} \quad \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$				
2	Navier (1827)[43] elastic solid	(6-1)$_{N^e}$ $\begin{cases} \frac{\Pi}{g}\frac{d^2x}{dt^2} = \varepsilon\left(3\frac{d^2x}{da^2} + \frac{d^2x}{db^2} + \frac{d^2x}{dc^2} + 2\frac{d^2y}{dbda} + 2\frac{d^2z}{dcda}\right), \\ \frac{\Pi}{g}\frac{d^2y}{dt^2} = \varepsilon\left(\frac{d^2y}{da^2} + 3\frac{d^2y}{db^2} + \frac{d^2y}{dc^2} + 2\frac{d^2x}{dadb} + 2\frac{d^2z}{dcdb}\right), \\ \frac{\Pi}{g}\frac{d^2z}{dt^2} = \varepsilon\left(\frac{d^2z}{da^2} + \frac{d^2z}{db^2} + 3\frac{d^2z}{dc^2} + 2\frac{d^2x}{dadc} + 2\frac{d^2y}{dbdc}\right) \end{cases}$ $\begin{cases} where, \\ \Pi : \text{density of the solid}, \\ g : \text{acceleration of gravity}. \end{cases}$	ε	2ε	$\frac{1}{2}$	
3 N	Navier (1827)[44] fluid	$\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = X + \varepsilon\left(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dxdz}\right) - \frac{du}{dt} - \frac{du}{dx}\cdot u - \frac{du}{dy}\cdot v - \frac{du}{dz}\cdot w\ ; \\ \frac{1}{\rho}\frac{dp}{dy} = Y + \varepsilon\left(\frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dydz}\right) - \frac{dv}{dt} - \frac{dv}{dx}\cdot u - \frac{dv}{dy}\cdot v - \frac{dv}{dz}\cdot w\ ; \\ \frac{1}{\rho}\frac{dp}{dz} = Z + \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2} + 2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz}\right) - \frac{dw}{dt} - \frac{dw}{dx}\cdot u - \frac{dw}{dy}\cdot v - \frac{dw}{dz}\cdot w\ ; \end{cases}$	ε	2ε		$\frac{1}{2}$
4	Cauchy (1828)[5] system of particles in elastic and fluid	$\begin{cases} (L+G)\frac{\partial^2\xi}{\partial x^2} + (R+H)\frac{\partial^2\xi}{\partial y^2} + (Q+I)\frac{\partial^2\xi}{\partial z^2} + 2R\frac{\partial^2\eta}{\partial x\partial y} + 2Q\frac{\partial^2\zeta}{\partial z\partial x} + X = \frac{\partial^2\xi}{\partial t^2}, \\ (R+G)\frac{\partial^2\eta}{\partial x^2} + (M+H)\frac{\partial^2\eta}{\partial y^2} + (P+I)\frac{\partial^2\eta}{\partial z^2} + 2P\frac{\partial^2\zeta}{\partial y\partial z} + 2R\frac{\partial^2\xi}{\partial x\partial y} + Y = \frac{\partial^2\eta}{\partial t^2}, \\ (Q+G)\frac{\partial^2\zeta}{\partial x^2} + (P+H)\frac{\partial^2\zeta}{\partial y^2} + (N+I)\frac{\partial^2\zeta}{\partial z^2} + 2Q\frac{\partial^2\xi}{\partial z\partial x} + 2P\frac{\partial^2\eta}{\partial y\partial z} + Z = \frac{\partial^2\zeta}{\partial t^2}, \end{cases}$ $G = H = I, \quad L = M = N, \quad P = Q = R, \quad L = 3R$	$R+$ G	$2R$	if G $=0$ $\frac{1}{2}$	
5	Poisson (1831)[52] elastic solid in general equations	$\begin{cases} X - \frac{d^2u}{dt^2} + a^2\left(\frac{d^2u}{dx^2} + \frac{2}{3}\frac{d^2v}{dydx} + \frac{2}{3}\frac{d^2w}{dzdx} + \frac{1}{3}\frac{d^2u}{dy^2} + \frac{1}{3}\frac{d^2u}{dz^2}\right) = \frac{\Pi}{\rho}\frac{d^2u}{dx^2}, \\ Y - \frac{d^2v}{dt^2} + a^2\left(\frac{d^2v}{dy^2} + \frac{2}{3}\frac{d^2u}{dxdy} + \frac{2}{3}\frac{d^2w}{dzdy} + \frac{1}{3}\frac{d^2v}{dx^2} + \frac{1}{3}\frac{d^2v}{dz^2}\right) = \frac{\Pi}{\rho}\frac{d^2v}{dy^2}, \\ Z - \frac{d^2w}{dt^2} + a^2\left(\frac{d^2w}{dz^2} + \frac{2}{3}\frac{d^2u}{dxdz} + \frac{2}{3}\frac{d^2v}{dydz} + \frac{1}{3}\frac{d^2w}{dx^2} + \frac{1}{3}\frac{d^2w}{dy^2}\right) = \frac{\Pi}{\rho}\frac{d^2w}{dz^2}, \end{cases}$	$\frac{a^2}{3}$	$\frac{2a^2}{3}$	$\frac{1}{2}$	
6	Poisson (1831)[52] fluid in general equations	$\begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} + \alpha(K+k)\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) + \frac{\alpha}{3}(K+k)\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} + \alpha(K+k)\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) + \frac{\alpha}{3}(K+k)\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} + \alpha(K+k)\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) + \frac{\alpha}{3}(K+k)\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \end{cases}$ $\begin{cases} \rho(X - \frac{d^2x}{dt^2}) = \frac{d\varpi}{dx} + \beta(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}), \\ \rho(Y - \frac{d^2y}{dt^2}) = \frac{d\varpi}{dy} + \beta(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}), \\ \rho(Z - \frac{d^2z}{dt^2}) = \frac{d\varpi}{dz} + \beta(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}) \end{cases}$ $\begin{cases} \text{where}, \\ \varpi \equiv p - \alpha\frac{d\psi t}{dt} - \frac{\beta+\beta'}{\chi t}\frac{d\chi t}{dt}, \\ \beta \equiv -\alpha(K+k) \end{cases}$	β	$\frac{\beta}{3}$		3
7	Stokes (1849)[61] fluid	(12)$_S$ $\begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \mu\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \mu\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0. \end{cases}$	μ	$\frac{\mu}{3}$		3
8	Maxwell (1865-66) [40] HD	$\begin{cases} \rho\frac{\partial u}{\partial t} + \frac{dp}{dx} - C_M\left[\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + \frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right] = \rho X, \\ \rho\frac{\partial v}{\partial t} + \frac{dp}{dy} - C_M\left[\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + \frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right] = \rho Y, \quad , C_M \equiv \frac{pM}{6k\rho\Theta_2} \\ \rho\frac{\partial w}{\partial t} + \frac{dp}{dz} - C_M\left[\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2} + \frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right] = \rho Z \end{cases}$	C_M	$\frac{C}{3}$		3
9	Kirchhoff (1876)[18] HD	$\begin{cases} \mu\frac{du}{dt} + \frac{\partial}{\partial x} - C_K\left[\Delta u + \frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \mu X, \\ \mu\frac{dv}{dt} + \frac{\partial p}{\partial y} - C_K\left[\Delta v + \frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \mu Y, \\ \mu\frac{dw}{dt} + \frac{\partial p}{\partial z} - C_K\left[\Delta z + \frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \mu Z, \end{cases}$ $\begin{cases} \frac{1}{\mu}\frac{d\mu}{dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \\ \text{where, } C_K \equiv \frac{1}{3\kappa}\frac{p}{\mu} \end{cases}$	C_K	$\frac{\Delta}{3}$		3
10 N	Rayleigh (1883)[56] HD	$\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = -\frac{du}{dt} + \nu\nabla^2 u - u\frac{du}{dx} - v\frac{du}{dy}, \\ \frac{1}{\rho}\frac{dp}{dy} = -\frac{dv}{dt} + \nu\nabla^2 v - u\frac{dv}{dx} - v\frac{dv}{dy} \end{cases} \quad , \quad \frac{du}{dx} + \frac{dv}{dy} = 0$	ν			
11	Boltzmann (1895)[4] HD	(221)$_B$ $\begin{cases} \rho\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} - \mathcal{R}\left[\Delta u + \frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \rho X, \\ \rho\frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} - \mathcal{R}\left[\Delta v + \frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \rho Y, \\ \rho\frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} - \mathcal{R}\left[\Delta w + \frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \rho Z \end{cases}$	$\mathcal{R}$	$\frac{\mathcal{R}}{3}$		3
12 N	Prandtl (1934)[55] HD	$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),$ for incompressible, it is simplified div $\mathbf{w} = 0, \quad \frac{D\mathbf{w}}{dt} = g - \frac{1}{\rho}\text{grad } p + \nu\Delta\mathbf{w}$	ν	$\frac{\nu}{3}$		3

§1.

$$f(x) = A_1 \sin \frac{\pi x}{l} + A_2 \sin \frac{2\pi x}{l} + \cdots + A_i \sin \frac{i\pi x}{l}$$

§2.

Mais au lieu de regarder les égalités de la forme

$$f(x) = \sum A_i \sin \frac{i\pi x}{l}$$

comme le résultat de la l'intégration d'une équation aux différences partielles servant à résoudre un probléme physico-mathématique, on s'est aussi proposé de la considérer en elle-mêmes, abstraction faite des questions particulières où elles se présentent et cette idée a donné naissance à la belle théorie des séries périodiques que M.Poisson a exposée d'abord dans le 19^e cahier du *Journal de l'Ecole polytechnique*,[18] et qu'il a reproduite récemment dans son ouvrage sur la chareur.

Cette théorie des séries périodiques, ainsi traitée comme un point d'analyse pure, en devient à la fois plus élégante et plus rigoureuse ; mais telle que M.Poisson l'ai donnée dans les mémoires cités, elle se borne aux développemens des fonctions ou tarties de fonctions d'une variable x en séries de sinus et cosinus des multiples entiers d'un arc proportionnel à x, et elle ne s'étend en aucune manière aux autres séries de sinus et cosinus que l'on rencontre aussi dans certains successifs s'obtiennent en multipliant la variable x par les diverses racines d'une équation transcendente.

Je me propose ici de faire connaitre une méthode au moyan de laquelle on effectuera d'une manière directe les développemens des fonctions ou parties de fonctions en séries de sinus et cosinus. Pour trouver cette méthode, il m'a suffi de modifier légèrement un procédé fort ingénieux dont M.Poisson a fait usage dans ses deux premiers Mémoires sur la *Théorie de la chaleur.* [19] La modification dont je parle consiste surtout en ce que j'ai pris pour point de départ formule

$$f(x) = \frac{1}{\pi} \int_0^\infty dx \int_{-\infty}^\infty \cos z(y-x) \cdot f(y) dy, \tag{22}$$

$$f(x) = \frac{1}{\pi} \int_0^\infty \cos\ zx\ dz \int_{-\infty}^\infty \cos zy \cdot f(y) dy + \frac{1}{\pi} \int_0^\infty \sin\ zx\ dz \int_{-\infty}^\infty \sin zy \cdot f(y) dy \tag{23}$$

§3.

$$(A)_L \quad f(x) = \sum (A \cos\ px + B \sin\ px)$$

$$f(x) = \int_0^\infty \cos\ zx\ U dz + \int_0^\infty \sin\ zx\ V dz$$

$$U = \frac{1}{\pi} \int_{-\infty}^{+\infty} \cos\ zy\ f(y)\ dy, \quad V = \frac{1}{\pi} \int_{-\infty}^{+\infty} \sin\ zy\ f(y)\ dy$$

§4.

$$(1)_L \quad f(x) = \frac{1}{\pi} \int_0^\infty dz \int_{-\infty}^{+\infty} \cos\ z(y-x)\ f(y)\ dy$$

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[18](⇓) Poisson [50].

[19](⇓) Poisson [50] and Poisson [51]. Poisson's another book on this theme : [53].

$$u = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin\ (y-x)}{y-x} f(y)\ dy$$

By $y = x + \frac{\theta}{z}$,

$$u = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin\ \theta}{\theta} f(x + \frac{\theta}{z})\ d\theta$$

If $z = \infty$, then $f(x + \frac{\theta}{z}) = f(x)$, moreover,

$$u = \frac{f(x)}{\pi} \int_{-\infty}^{+\infty} \frac{\sin\ \theta}{\theta}\ d\theta = f(x)$$

§5. [21]

$$\int_0^{\infty} e^{-hy}\ f(y)\ dy = p, \quad \int_0^{-\infty} e^{hy}\ f(y)\ dy = q,$$

$$p = \frac{\psi(h)}{\varphi(h)}, \quad q = \frac{\psi(-h)}{\varphi(-h)} \tag{24}$$

§6. We substitute $h = g + z\sqrt{-1}$ in p and $h = g - z\sqrt{-1}$ in q, then

$$p = \frac{\psi(z\sqrt{-1}+g)}{\varphi(z\sqrt{-1}+g)}, \quad q = \frac{\psi(z\sqrt{-1}-1)}{\varphi(z\sqrt{-1}-1)}$$

$$(2)_L \quad \int_{-\infty}^{+\infty} e^{-zy}\ f(y)\ dy = p - q = \frac{\psi(z\sqrt{-1}+g)}{\varphi(z\sqrt{-1}+g)} - \frac{\psi(z\sqrt{-1}-g)}{\varphi(z\sqrt{-1}-g)}$$

This intefral mean at first (22), next, (23) with

$$\cos zy \cdot f(y)dy, \quad \sin zy \cdot f(y)dy, \quad \cos z(y-x) \cdot f(y)dy,$$

§7. [22] We assume $z = \rho + z'$.

$$\int_{-\infty}^{+\infty} e^{-xy\sqrt{-1}}\ f(y)\ dy \equiv Z, \quad \int_{-\infty}^{+\infty} e^{xy\sqrt{-1}}\ f(y)\ dy \equiv Z'$$

$$\int_{-\infty}^{\infty} \cos z(y-x) \cdot f(y)dy = \frac{1}{2} Z e^{x(\rho+z')\sqrt{-1}} + \frac{1}{2} Z' e^{-x(\rho+z')\sqrt{-1}},$$

Substitute these value in $(1)_L$ and neglect z' except for Z and Z', then we get :

$$f(x) = \frac{1}{2\pi} \sum \Big(e^{\rho x\sqrt{-1}} \int Z dz' + e^{-\rho x\sqrt{-1}} \int Z' dz' \Big),$$

We define $\frac{d\varphi(h)}{dh} = \varphi'(h),\ z = \rho + z'$ and assume z' and g are infinitesimaly small.

$$\begin{cases} \varphi(z\sqrt{-1} \pm g) = (z'\sqrt{-1} \pm g)\ \varphi'(\rho\sqrt{-1}), \\ \psi(z\sqrt{-1} \pm g) = \psi(\rho\sqrt{-1}), \end{cases}$$

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[20](⇓) The expression of Poisson [50, p.29] correspoinding to $(1)_L$ is as follows :

$$(5)_P \quad u = \frac{e^{-bt}}{\pi} \int_0^{\infty} dz \Big(\int_{-\infty}^{+\infty} \cos\ (y-x)z\ f(y)\ dy \Big)\ e^{-a^2\ t\ z^2}$$

[21](⇓) Poisson [50, ¶17.]

[22](⇓) Poisson [50, ¶18.]

$$\begin{aligned}
Z = & \frac{\psi(z\sqrt{-1}+g)}{\varphi(z\sqrt{-1}+g)} - \frac{\psi(z\sqrt{-1}-1)}{\varphi(z\sqrt{-1}-1)} \\
= & \frac{\psi(\rho\sqrt{-1})}{(z'\sqrt{-1}+g)\ \varphi'(\rho\sqrt{-1})} - \frac{\psi(\rho\sqrt{-1})}{(z'\sqrt{-1}-g)\ \varphi'(\rho\sqrt{-1})} \\
= & \frac{\psi(\rho\sqrt{-1})}{\varphi'(\rho\sqrt{-1})}\ \Big(\frac{1}{(z'\sqrt{-1}+g)} - \frac{1}{(z'\sqrt{-1}-g)}\Big) \\
= & \frac{2g}{(g^2+z'^2)}\ \frac{\psi(\rho\sqrt{-1})}{\varphi'(\rho\sqrt{-1})}
\end{aligned}$$

$$\int_{-\delta}^{+\delta} Z dz' = 4\frac{\psi(\rho\sqrt{-1})}{\varphi'(\rho\sqrt{-1})} \arctan\frac{\delta}{g},$$

[23] If $\rho = 0$, we can integrate only the interval of $\int_0^\delta$, then the value reduces into incomplete. If $g = 0$ then we get :

$$\int_0^\delta Z dz' = 2\pi\ \frac{\psi(\rho\sqrt{-1})}{\varphi'(\rho\sqrt{-1})},$$

$$\int Z' dz' = 2\pi\ \frac{\psi(-\rho\sqrt{-1})}{\varphi'(-\rho\sqrt{-1})}$$

$$f(x) = \frac{1}{2\pi}\sum\Big[e^{\rho x\sqrt{-1}}\ \frac{\psi(\rho\sqrt{-1})}{\varphi'(\rho\sqrt{(-1)})} + e^{-\rho x\sqrt{-1}}\ \frac{\psi(-\rho\sqrt{-1})}{\varphi'(-\rho\sqrt{-1})}\Big],$$

$$(\alpha)_L \quad f(x) = \sum(A\cos\ \rho x + B\sin\ \rho x)$$

$$(\beta)_L \quad \begin{cases} A = \frac{\psi(\rho\sqrt{-1})}{\varphi'(\rho\sqrt{-1})} + \frac{\psi(-\rho\sqrt{-1})}{\varphi'(-\rho\sqrt{-1})}, \\ B = \sqrt{-1}\Big[\frac{\psi(\rho\sqrt{-1})}{\varphi'(\rho\sqrt{-1})} - \frac{\psi(-\rho\sqrt{-1})}{\varphi'(-\rho\sqrt{-1})}\Big] \end{cases}$$

§8. [24]

1. Multiplying the both hand-sides of an even function $f(y) = f(-y)$ with $e^{-hy}dy$ and integrate from $y = 0$ to $y = \infty$:

$$\int_0^\infty e^{-hy}\ f(y)\ dy = \int_0^\infty e^{-hy}\ f(-y)\ dy$$

2. Multiplying the both hand-sides of an even function :

$$f(l+y) = -f(l-y) \tag{25}$$

with $e^{-hy}dy$ and integrate from $y = 0$ to $y = \infty$:

$$\int_0^\infty e^{-hy}\ f(l+y)\ dy = -\int_0^\infty e^{-hy}\ f(l-y)\ dy$$

$$\int_0^\infty e^{-hy}\ f(l+y)\ dy = e^{hl}\Big[p - \int_0^l e^{-hy}\ f(y)\ dy\Big] \tag{26}$$

[23](⇓)

$$\int\frac{dx}{c^2+x^2} = \frac{1}{c}tan^{-1}\frac{x}{c}.$$

[24](⇓) Poisson [50, ¶17.]

$$\int_0^\infty e^{-hy}\ f(l-y)\ dy = -e^{hl}\Big[q - \int_0^l e^{hy}\ f(y)\ dy\Big] \tag{27}$$

From the equality in even function $f(y) = f(-y)$, between the right hand-side of (26) and the one of (27) are :

$$e^{hl}\Big[p - \int_0^l e^{-hy}\ f(y)\ dy\Big] = e^{hl}\Big[q - \int_0^l e^{hy}\ f(y)\ dy\Big]$$

By $p = -q$,

$$\int_0^l \Big(e^{h(l-y)}) - (e^{h(y-l)}\Big)\ f(y)\ dy = \psi(h), \quad e^{hl} + e^{-hl} = \varphi(h)$$

By ψ and φ', $\cos\ \rho l = 0$,

$$\begin{cases}\psi(\rho\sqrt{-1}) = 2\sqrt{-1}\int_0^l \sin\ \rho(l-y)\ f(y)\ dy, \\ \varphi'(\rho\sqrt{-1}) = 2l\sqrt{-1}\sin\rho l\end{cases}$$

By $\sin\ \rho(l-y) = \sin\rho l\ \ \cos\rho\ y$

$$\psi(\rho\sqrt{-1}) = 2\sqrt{-1}\sin\rho\ l\int_0^l \cos\ \ \rho y\ f(y)\ dy$$

From here, we get (α) :

$$f(x) = \frac{2}{l}\sum\cos\rho x\int_0^l \cos\ \ \rho y\ f(y)\ dy$$

§9. We consider the functionx in the outside of interval $[l', l],\ l' = -l$. We assume the next two conditions :

$$\begin{cases}f(l+y) + f(l-y) = 0, \\ f(-l+y) + f(-l-y) = 0,\end{cases}$$

where we contain implicitly the two special conditions : $f(l) = 0,\ f(-l) = 0$. Similarly, from (26) and (27) are :

$$\int_0^\infty e^{-hy}\ f(l+y)\ dy = e^{hl}\Big[p - \int_0^l e^{-hy}\ f(y)\ dy\Big] \tag{28}$$

$$\int_0^\infty e^{-hy}\ f(l-y)\ dy = -e^{hl}\Big[q - \int_0^l e^{hy}\ f(y)\ dy\Big] \tag{29}$$

Similarly, as (25)

$$f(l+y) = -f(l-y) \tag{30}$$

1. Multiplying the both hand-sides of a function (30) with $e^{-hy}dy$ and integrate from $y = 0$ to $y = \infty$:

$$\int_0^\infty e^{-hy}\ f(l+y)\ dy = -\int_0^\infty e^{-hy}\ f(l-y)\ dy$$

$$e^{hl}p - e^{-hl}q = e^{hl}\int_0^l e^{-hy}\ f(y)\ dy - e^{-hl}\int_0^l e^{hy} f(y)\ dy$$

$$f(-l+y) = -f(-l-y) \tag{31}$$

2. Multiplying the both hand-sides of a function (31) with $e^{-hy}dy$ and integrate from $y=0$ to $y=\infty$:

$$\int_0^\infty e^{-hy}\ f(l+y)\ dy = -\int_0^\infty e^{-hy}\ f(l-y)\ dy$$

$$e^{-hl}p - e^{hl}q = e^{-hl}\int_0^{-l} e^{-hy}\ f(y)\ dy - e^{hl}\int_0^{-l} e^{hy} f(y)\ dy$$

§10. [25]

$$(a)_L \quad \begin{cases} \beta f(l+y) + \frac{df(l+y)}{dy} + \beta f(l-y) + \frac{df(l-y)}{dy} = 0, \\ \beta' f(-l+y) + \frac{df(-l+y)}{dy} + \beta' f(-l-y) + \frac{df(-l-y)}{dy} = 0 \end{cases}$$

here, we consider similarly about $f(x)$

$$\begin{cases} \frac{df(x)}{dx} + \beta f(x) = 0, \quad \text{for} \quad x = l, \\ \frac{df(x)}{dx} + \beta' f(x) = 0, \quad \text{for} \quad x = -l \end{cases} \tag{32}$$

[26] Remenbering (28) and (29), shown by Poisson, put the equation $(\alpha)_L$. [27]

$$e^{-\beta y} d[e^{\beta y} f(l+y)] = e^{\beta y} d[e^{-\beta y} f(l-y)]$$

$$e^{-yh} f(l+y) + (h+\beta)\int e^{-yh} f(l+y)\ dy = C + e^{-yh} f(l-y) + (h-\beta)\int e^{-yh} f(l-y) dy$$

$$(h+\beta)\int e^{-yh} f(l+y)\ dy = (h-\beta)\int e^{yh} f(l-y) dy$$

$$(h+\beta)e^{-hl}p + (h-\beta)e^{-hl}q = (h+\beta)e^{hl}\int_0^l e^{-hy} f(y) dy + (h-\beta)e^{-hl}\int_0^l e^{hy} f(y) dy$$

Changing β with $-\beta'$ and l with $-l$, then

$$(h-\beta')e^{-hl}p + (h+\beta')e^{-hl}q = (h-\beta')e^{-hl}\int_0^{-l} e^{-hy} f(y) dy + (h+\beta')e^{hl}\int_0^{-l} e^{hy} f(y) dy$$

Hence, for abriviation :

$$\begin{cases} (h+\beta)(h+\beta')e^{-2hl}\int_0^{-l}\ e^{-hy} f(y)\ dy - (h-\beta)(h-\beta')e^{-2hl}\int_0^l\ e^{-hy} f(y)\ dy \\ +(h-\beta)(h+\beta')\Big[\int_0^l\ e^{hy} f(y)\ dy - \int_0^{-l}\ e^{hy} f(y)\ dy\Big] = \psi(h), \\ \\ (h+\beta)(h+\beta')e^{2hl} - (h-\beta)(h-\beta')e^{-2hl} = \varphi(h) \end{cases}$$

then we get (24). From this, the equation depending on the value of ρ is :

$$\frac{\varphi(\rho\sqrt{-1})}{2\sqrt{-1}} = (\beta\beta' - \rho^2)\sin\ 2pl + (\beta+\beta')\rho\cos\ 2pl = 0$$

§11.

$$(\beta\beta' - \rho^2)\sin\ 2pl + (\beta+\beta')\rho\cos\ 2pl = 0$$

$$[\beta+\beta' + 2l(\beta\beta' - \rho^2)]\cos\ 2pl - [2 + 2l(\beta+\beta')]\rho\sin\ 2pl = 0$$

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[25](⇓) Poisson [50, ¶17.]

[26](⇓) This is a boundary value problem of Sturm-Liouville type.

[27]sic. *Journal de l'Ecole Polytechnique*, 19^e cahier, page 30. ((⇓) cf. Poisson [50, p.30], §2, *Distribution dela Chaleur dans une Barre prismatique, d'une petite épaisseur*, ¶15.)

Above two expression consist of $\tan 2pl$ then

$$(\beta\beta' - \rho^2)[\beta + \beta' + 2l(\beta\beta' - \rho^2)] + (\beta + \beta')[2 + 2l(\beta + \beta')]\rho^2 = 0$$

or,

$$2l(\beta\beta' - \rho^2) + 2l(\beta + \beta')^2\rho^2 + (\beta + \beta')(\beta\beta' + \rho^2) = 0$$

§12.
We see $f(x)$ is to be developed in the series : $\sum(A\cos\rho x + B\sin\rho x)$ as restricted under the condition (32). We assume $v = A\cos\rho x + B\sin\rho x$, then

$$\frac{dv}{dx} + \beta v = 0, \quad \text{for} \quad x = l, \quad \& \quad \frac{dv}{dx} + \beta' v = 0, \quad \text{for} \quad x = -l$$

On the terms of the series : $A\cos\rho x + B\sin\rho x$, there are many other remarkable properties, which have been known since long before. v and v' are two terms of the serie corresponding to two roots : $\rho \neq \rho'$; $\int_{-l}^{+l} vv'dx = 0$, In fact,

$$\frac{d^2v}{dx^2} = -\rho^2 v, \quad \frac{d^2v'}{dx^2} = -(\rho')^2 v' \quad \Rightarrow \quad (\rho^2 - \rho'^2)vv' = v\frac{d^2v'}{dx^2} - v'\frac{d^2v}{dx^2}$$

$$(\rho^2 - \rho'^2)\int_{-l}^{+l} vv'dx = v\frac{dv'}{dx} - v'\frac{dv}{dx} \quad \Rightarrow \quad v\frac{dv'}{dx} - v'\frac{dv}{dx} = 0, \ \rho \neq \rho' \quad \Rightarrow \quad \int_{-l}^{+l} vv'dx = 0$$

Thus, as Poisson shows, we get this equality : $\int_{-l}^{+l} vv'dx = 0$.

4.2. **Liouvill** [28]**, 1836.**
§1.

$$(1)_{L_c} \quad g\frac{du}{dt} = \frac{d\left(k\frac{du}{dx}\right)}{dx} - lu$$

where, $g, \ k, \ l$: specific heat, interior conductivity and emmisive power, respectively.

$$(2)_{L_c} \quad \begin{cases} \frac{du}{dx} - hu = 0 & \text{for} \quad x = \text{x}, \\ \frac{du}{dx} + Hu = 0 & \text{for} \quad x = \text{X} \end{cases}$$

where, $h, \ H$: constants $0 \leq h, \ H \leq \infty$.

$$(3)_{L_c} \quad u = f(x) \qquad \text{for} \quad t = 0$$

$$\begin{cases} \frac{df(x)}{dx} - hf(x) = 0 & \text{for} \quad x = \text{x}, \\ \frac{df(x)}{dx} + Hf(x) = 0 & \text{for} \quad x = \text{X} \end{cases}$$

To form the value of u which satisfies with the equation $(1)_{L_c}$ and with the condition $(2)_{L_c}$ and $(3)_{L_c}$, we are conducted to develop the function $f(x)$, $\forall x \in$ [x, X], by the series which the successive terms are different each other by a parameter r, and has at the same time the property satisfied with the general differential equation :

$$-rg\frac{du}{dt} = \frac{d\left(k\frac{dV}{dx}\right)}{dx} - lV$$

$$\begin{cases} \frac{dV}{dx} - hV = 0 & \text{for} \quad x = \text{x}, \\ \frac{dV}{dx} + HV = 0 & \text{for} \quad x = \text{X} \end{cases}$$

On peut voir, dans l'ouvrage de M.Poisson sur la chaleur, comment on est porté, par la marche même du calcul, à admettre la possibilité de ce dévelopment pour une fonction quelque $f(x)$; mais jusqu'à ce jouril a paru difficile d'établir cette possibilité directement et d'une manière rigoureuse. Je me propose de donner ici une méthode très simple pour y parvenir. Je considère en elle-même la série par laquelle les géomètres ont représenté le développement de $f(x)$ dont il est question : sans rien supposer à *priori* sur l'origine de cette série ni sur sa nature, j'en cherche la valeur, et je trouve que cette valeur est précisément $f(x)$, du moins lorsque la variable x est compris entre les limites x at X. [28, pp.254-255.]

$$(A)_{L_c} \quad \frac{d\left(k\frac{dV}{dx}\right)}{dx} - (gr - l)V = 0$$

$$(B) \quad \begin{cases} \frac{dV}{dx} - hV = 0 & \text{for} \quad x = \mathrm{x}, \\ \frac{dV}{dx} + HV = 0 & \text{for} \quad x = \mathrm{X} \end{cases}$$

$$(C)_{L_c} \quad \varpi(r) = 0$$

Cela posé, notre but dans ce mémoire, est de trouver directement et par un procédérigoureux la valeur de la série

$$\sum \frac{V \int_{\mathrm{x}}^{\mathrm{X}} gV \ f(x) \ dx}{\int_{\mathrm{x}}^{\mathrm{X}} gV^2 \ dx},$$

where, the sign $\sum$ extend to all the value of r which satisfied with $(C)_{L_c}$. cf [66]. §5.

Problem : Find the value of series :

$$\sum \frac{V \int_{\mathrm{x}}^{\mathrm{X}} gV \ f(x) \ dx}{\int_{\mathrm{x}}^{\mathrm{X}} gV^2 \ dx}, \tag{33}$$

where, in this expression, the sign $\sum$ takes all the value of r which are roots of $(C)_{L_c}$. The valiable x is between x and X, and $f(x)$ is given arbitraly in this interval.

We extend the symbol : $\sum$ of (33) using the limitless series and assume the right hand-side of F.

$$F(x) = \frac{V_1 \int_{\mathrm{x}}^{\mathrm{X}} gV_1 \ f(x) \ dx}{\int_{\mathrm{x}}^{\mathrm{X}} gV_1^2 \ dx} + \frac{V_2 \int_{\mathrm{x}}^{\mathrm{X}} gV_2 \ f(x) \ dx}{\int_{\mathrm{x}}^{\mathrm{X}} gV_2^2 \ dx} + \cdots + \frac{V_m \int_{\mathrm{x}}^{\mathrm{X}} gV_m \ f(x) \ dx}{\int_{\mathrm{x}}^{\mathrm{X}} gV_m^2 \ dx} + \cdots \tag{34}$$

We multiply two members of (34) with $g \ V_m(x) \ dx$, and assume its indices $m \neq n, \ m < n$, then

$$\int_{\mathrm{x}}^{\mathrm{X}} g \ V_m(x) \ F(x) \ dx = \left(\frac{\int_{\mathrm{x}}^{\mathrm{X}} gV_1^2 \ dx}{\int_{\mathrm{x}}^{\mathrm{X}} gV_1^2 \ dx}\right) \int_{\mathrm{x}}^{\mathrm{X}} gV_1 \ f(x) \ dx + \left(\frac{\int_{\mathrm{x}}^{\mathrm{X}} gV_2^2 \ dx}{\int_{\mathrm{x}}^{\mathrm{X}} gV_2^2 \ dx}\right) \int_{\mathrm{x}}^{\mathrm{X}} gV_2 \ f(x) \ dx + \cdots$$
$$+ \left(\frac{\int_{\mathrm{x}}^{\mathrm{X}} gV_m^2 \ dx}{\int_{\mathrm{x}}^{\mathrm{X}} gV_m^2 \ dx}\right) \int_{\mathrm{x}}^{\mathrm{X}} gV_m \ f(x) \ dx + \cdots + \left(\frac{\int_{\mathrm{x}}^{\mathrm{X}} gV_m V_n \ dx}{\int_{\mathrm{x}}^{\mathrm{X}} gV_m V_n \ dx}\right) \int_{\mathrm{x}}^{\mathrm{X}} gV_m \ f(x) \ dx$$

$$\int_{\mathrm{x}}^{\mathrm{X}} g \ V_m(x) \ V_n(x) \ dx = 0$$

From here, we get only integral term of m :

$$\int_{\mathrm{x}}^{\mathrm{X}} g\ V_m(x)\ F(x)\ dx = \int_{\mathrm{x}}^{\mathrm{X}} g\ V_m(x)\ f(x)\ d$$

$$\int_{\mathrm{x}}^{\mathrm{X}} g\ [F(x) - f(x)]V_m(x)\ dx = 0$$

$g > 0,\ V_m(x) > 0$ then $F(x) = f(x)$.

> La valeur cherchée de la série est donc $f(x)$, entre ces limites de la variable, ce qui s'accorde avec le résultat que les géomètres ont obtenu *par d'autres méthodes moins directes et moins rigoureuses que la nôtre.* [28, p.263] (Italic mine.)

[28] §6.

$$\sigma_n \equiv \sum \frac{V \int_{\mathrm{x}}^{\mathrm{X}} gV\ f(x)\ dx}{\int_{\mathrm{x}}^{\mathrm{X}} gV^2\ dx},$$

$\rho_n \equiv f(x) - \sigma_n$. $Q \equiv A_1V_1(x) + A_2V_2(x) + \cdots + A_nV_n(x)$.

$$\int_{\mathrm{x}}^{\mathrm{X}} g\ \rho_n\ Q\ dx = 0 \tag{35}$$

Replecing $Q = \sigma_n$, then

$$\int_{\mathrm{x}}^{\mathrm{X}} g\ \rho_n\ \sigma_n\ dx = 0 \tag{36}$$

$$f(x) = \sigma_n + \rho_n$$

$$\int_{\mathrm{x}}^{\mathrm{X}} g\ \sigma_n\ f(x)dx = \int_{\mathrm{x}}^{\mathrm{X}} g\ \sigma_n\ (\sigma_n + \rho_n)\ dx = \int_{\mathrm{x}}^{\mathrm{X}} g\ \rho_n\sigma_n^2\ dx + \underbrace{\int_{\mathrm{x}}^{\mathrm{X}} g\ \rho_n\ \sigma_n\ dx}_{0 \text{ from } (35)} = \int_{\mathrm{x}}^{\mathrm{X}} g\ \rho_n\ \sigma_n^2\ dx$$

$$\int_{\mathrm{x}}^{\mathrm{X}} g\ f(x)^2\ dx = \int_{\mathrm{x}}^{\mathrm{X}} g\ (\sigma_n + \rho_n)^2\ dx = \int_{\mathrm{x}}^{\mathrm{X}} g(\sigma_n^2 + \rho_n^2)\ dx + 2\underbrace{\int_{\mathrm{x}}^{\mathrm{X}} g\ \sigma_n\ \rho_n\ dx}_{0 \text{ from } (36)} = \int_{\mathrm{x}}^{\mathrm{X}} g(\sigma_n^2 + \rho_n^2)dx$$

> Cette dernière formule nous prove que l'intégrale $\int_{\mathrm{x}}^{\mathrm{X}} g\ \sigma_n^2 dx$, quelque grand qu'on prenne l'indice n, ne peut jamais avoir une valeur numérique supérieure à la limite $\int_{\mathrm{x}}^{\mathrm{X}} g\ f(x)^2\ dx$ avec laquelle elle coïncide lorsque $n = \infty$. [28, p.265]

We see that $\lim_{n\to\infty} \int_{\mathrm{x}}^{\mathrm{X}} g\ \sigma_n^2 dx \leq \int_{\mathrm{x}}^{\mathrm{X}} g\ f(x)^2\ dx$, namely it means as follows :

$$\int_{\mathrm{x}}^{\mathrm{X}} g\ f(x)^2\ dx = \lim_{n\to\infty} \int_{\mathrm{x}}^{\mathrm{X}} g\ \sigma_n^2\ dx \quad \Rightarrow \quad \lim_{n\to\infty} \int_{\mathrm{x}}^{\mathrm{X}} g\ \rho_n^2\ dx = 0 \quad \Rightarrow \quad f(x) - \sigma_n = 0$$

[28]This is when he was 29 years old youth.

4.3. **Sturm-Liouvill** [66]**, 1837.**

$$(1)_{SL} \quad \frac{d(k\frac{dV}{dx})}{dx} + (gr - l)V = 0$$

and the define condition :

$$(2)_{SL} \quad \frac{dV}{dx} - hV = 0, \quad \text{for} \quad x = \mathrm{x}$$

$$(3)_{SL} \quad \frac{d(k\frac{dV_n}{dx})}{dx} + (gr_n - l)V_n = 0$$

$$\begin{cases} (4)_{SL} \quad \frac{dV_n}{dx} - hV_n = 0, \quad \text{for} \quad x = \mathrm{x}, \\ (5)_{SL} \quad \frac{dV_n}{dx} - HV_n = 0, \quad \text{for} \quad x = \mathrm{X} \end{cases}$$

$$(6)_{SL} \quad F(x) = \sum \Big\{ \frac{V_n \int_x^X g\ V_n\ f(x)\ dx}{\int_x^X g\ V_n^2\ dx} \Big\} \tag{37}$$

$$\int_x^X g\ V\ V_n\ dx = \frac{k}{r - r_n}\Big(V\frac{dV_n}{dx} - V_n\frac{dV}{dx}\Big)$$

$$(7)_{SL} \quad \int_x^X g\ V\ V_n\ dx = -KV_n(X)\frac{\varpi(r)}{r - r_n}$$

$$(8)_{SL} \quad \int_x^X g\ V^2\ dx = -KV_n(X)\varpi'(r_n)$$

$$\frac{V}{\varpi(r)} = \sum \Big\{ \frac{V_n}{(r - r_n)\varpi'(r_n)} \Big\}$$

$$(9)_{SL} \quad V = \sum \Big\{ \frac{\varpi(r)V_n}{(r - r_n)\varpi'(r_n)} \Big\}$$

$$\int_x^X g\ V\ f(x)\ dx = \sum \Big\{ \frac{\int_x^X g\ V\ V_n\ dx \cdot \int_x^X g\ V_n\ f(x)\ dx}{\int_x^X g\ V^2\ dx} \Big\} \tag{38}$$

From (37),

$$\int_x^X g\ V\ F(x)\ dx = \sum \Big\{ \frac{\int_x^X g\ V\ V_n\ dx \cdot \int_x^X g\ V_n\ f(x)\ dx}{\int_x^X g\ V^2\ dx} \Big\} \tag{39}$$

The right hand-side of (38) and of (39) are equal respectively, then

$$\int_x^X g\ V\ f(x)\ dx = \int_x^X g\ V\ F(x)\ dx \quad \Rightarrow \quad \int_x^X g\ V\ [F(x) - f(x)]\ dx = 0, \quad \Rightarrow \quad F(x) = f(x)$$

5. The modeling of Schrödinger equation in Schrödinger [58] and [59]

Schrödinger (1887-1961) [58, 59] bases his original quantum theory on the classic mechanics of Kepler motion, showing some examples to apply the eigenvalue problem by Sturm-Liouvill.[29]

$$(1) \quad L[y] = py'' + p'y' - qy, \qquad (2) \quad L[y] + E\rho y = 0 \tag{40}$$

where, E is an eigenvalue of constant to find, $y = y(x)$. p, p', q are unrelated functions with the variable x. $\rho = \rho(x)$ is a wide-ranging-continuous function. The solutions $y(x)$ relate to the equation (2), namely, the eigen function. Here, all the eigenvalues are real and positive. For example, the decay problem in Planck oscillator, supposing the potential energy $V(q) = 2\pi^2\nu_0^2 q^2$:

$$\frac{d^2\psi}{dq^2} + \frac{8\pi^2}{h^2}(E - 2\pi^2\nu_0^2 q^2)\psi = 0, \tag{41}$$

where, ν_0 : the eigenfrequency in the meaning of 'Mechanic', and the eigenvalues E_n and its eigenfunctions : Hermitian orthogonal function, made by Hermitian polynomial H_n are as follow :

$$e^{-\frac{x^2}{2}} H_n(x), \quad H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n},$$

then the eigenvalue problem (41) is deduced as follows

$$a \equiv \frac{8\pi^2 E}{h^2}, \quad b \equiv \frac{16\pi^4\nu_0^2}{h^2} \quad \Rightarrow \quad \frac{d^2\psi}{dq^2} + (a - bq^2)\psi = 0, \text{ if we set } x \equiv qb^{\frac{1}{4}} \ \Rightarrow$$

$$\frac{d^2\psi}{dx^2} + (\frac{a}{\sqrt{b}} - x^2)\psi = 0, \quad \frac{a}{\sqrt{b}} = 1, 3, 5, \cdots, (2n+1), \cdots, \ E_n = \frac{2n+1}{2} h\nu_0, \ n = 0, 1, 2, 3, \cdots$$

[58, (2), pp.514-5].
Schrödinger is necessary the new quantum mechanics based on the analogical ground from classical mechanics or the mathematics such as :

- the motion theory of planets by Kepler in classic principle for modeling the modern theory of atomic structure
- collision of electron with nucleus like Fourier's or gas-theorists' molecular collision
- entropy concept like energy conversion in gas theory unsatisfied with Newtonian mechanics since Clausius 1865
- light wave theory unsatisfied with Newtonian mechanics since Huygens' wave principle
- application of the Sturm-Liouville theory and its differential equation to the boundary value problem in atomic mechanics, etc.

We like to document especially the mathematical topics on the Sturm-Liouville type problem.

5.1. The first part of [58].

We show the *Hamiltonian* partial differential equation :

$$(1)_{S_1} \quad H(q, \frac{\partial\psi}{\partial q}) = E$$

$$(2)_{S_1} \quad S = K \log \psi$$

$$(1')_{S_1} \quad H(q, \frac{K}{\psi}\frac{\partial\psi}{\partial q}) = E$$

[29] Schrödinger gets this problem from Courant-Hilbert [6], Kap. V, §5, 1, p.238 f. [58, (3), p.440.].

where, H : Hamiltonian function of *Kepler motion.* e : charge of electron and m : mass of electron.

$$(1'')_{S_1} \quad \Big(\frac{\partial\psi}{\partial x}\Big)^2 + \Big(\frac{\partial\psi}{\partial y}\Big)^2 + \Big(\frac{\partial\psi}{\partial z}\Big)^2 - \frac{2m}{K^2}\Big(E + \frac{e^2}{r}\Big)\psi^2 = 0$$

Our variational problem :

$$(3)_{S_1} \quad \delta J = \delta \iiint dxdydz\Big[\Big(\frac{\partial\psi}{\partial x}\Big)^2 + \Big(\frac{\partial\psi}{\partial y}\Big)^2 + \Big(\frac{\partial\psi}{\partial z}\Big)^2 - \frac{2m}{K^2}\Big(E + \frac{e^2}{r}\Big)\psi^2\Big] = 0$$

Integrating under all spaces, we get :

$$(4)_{S_1} \quad \frac{1}{2}\delta J = \int df\delta\psi\frac{\partial\psi}{\partial n} - \iiint dxdydz\delta\psi\Big[\Delta\psi + \frac{2m}{K^2}\Big(E + \frac{e^2}{r}\Big)\psi\Big] = 0$$

$$(5)_{S_1} \quad \Delta\psi + \frac{2m}{K^2}\Big(E + \frac{e^2}{r}\Big)\psi = 0 \tag{42}$$

$$(6)_{S_1} \quad \int df\delta\psi\frac{\partial\psi}{\partial n} = 0$$

$$(7)_{S_1} \quad \frac{d^2\chi}{dr^2} + \frac{2}{r}\frac{d\chi}{dr} + \Big(\frac{2mE}{K^2} + \frac{2me^2}{K^2 r} - \frac{n(n+1)}{r^2}\Big)\chi = 0, \quad n = 0, 1, 2, 3, \cdots$$

$$(8)_{S_1} \quad \rho(\rho - 1) + 2\rho - n(n+1) = 0$$

The solutions :

$$(8')_{S_1} \quad \rho_1 = n, \quad \rho_2 = -n(n+1)$$

$$(9)_{S_1} \quad \chi = r^\alpha U$$

$$(7')_{S_1} \quad \frac{d^2U}{dr^2} + \frac{2(\alpha+1)}{r}\frac{dU}{dr} + \frac{2m}{K^2}\Big(E + \frac{e^2}{r}\Big)U = 0$$

$$(10)_{S_1} \quad \alpha = n$$

The general type of Laplacian equation $(7')_{S_1}$ is :

$$(7'')_{S_1} \quad U'' + \Big(\delta_0 + \frac{\delta_1}{r}\Big)U' + \Big(\varepsilon_0 + \frac{\varepsilon_1}{r}\Big)U = 0$$

where

$$(11)_{S_1} \quad \delta_0 = 0, \quad \delta_1 = 2(\alpha+1), \quad \varepsilon_0 = \frac{2mE}{K^2}, \quad \varepsilon_1 = \frac{2me^2}{K^2}$$

$$(12)_{S_1} \quad U = \int_L e^{zr}\ (z - c_1)^{\alpha_1 - 1}\ (z - c_2)^{\alpha_2 - 1} dz$$

is the solution of $(7'')_{S_1}$ on the integral way L, for

$$(13)_{S_1} \quad \int_L \frac{d}{dx}[e^{zr}\ (z - c_1)^{\alpha_1}\ (z - c_2)^{\alpha_2}]dz = 0$$

where, $c_1,\ c_2,\ \alpha_1,\ \alpha_2$ are as follows : $c_1,\ c_2$ are the two solutions of the next quadratic equation :

$$(14)_{S_1} \quad z^2 + \delta_0 z + \varepsilon_0 = 0$$

and

$$(14')_{S_1} \quad \alpha_1 = \frac{\varepsilon_1 + \delta_1 c_1}{c_1 - c_2}, \quad \alpha_2 = \frac{\varepsilon_1 + \delta_1 c_2}{c_2 - c_1}$$

By $(11)_{S_1}$ and $(10)_{S_1}$, in the case of equation $(7')_{S_1}$, $c_1,\ c_2,\ \alpha_1,\ \alpha_2$ turn into :

$$(14'')_{S_1} \quad \begin{cases} c_1 = +\sqrt{\frac{-2mE}{K^2}}, \quad c_2 = -\sqrt{\frac{-2mE}{K^2}}, \\ \alpha_1 = \frac{me^2}{K_+\sqrt{-2mE}} + n + 1, \quad \alpha_2 = -\frac{me^2}{K_+\sqrt{-2mE}} + n + 1, \end{cases}$$

$$(15)_{S_1} \quad \frac{me^2}{K_+\sqrt{-2mE}} = \text{real} \in \mathbb{Z}$$

$$(16)_{S_1} \quad \lim_{z \Rightarrow \infty} e^{zr} = 0$$

where, the real part of zr must be negative unlimit value.

$$(17)_{S_1} \quad \begin{cases} U_1 \sim e^{c_1 r} r^{-\alpha_1} (-1)^{\alpha_1} (e^{2\pi\alpha_1} - 1)\Gamma(\alpha_1)(c_1 - c_2)^{\alpha_2 - 1}, \\ U_2 \sim e^{c_2 r} r^{-\alpha_2} (-1)^{\alpha_2} (e^{2\pi\alpha_2} - 1)\Gamma(\alpha_2)(c_2 - c_1)^{\alpha_1 - 1}, \end{cases}$$

We classify the value $E > 0,\ E < 0$.

- $E > 0$.
- $E < 0$.

For the negative of E, it is not suitable to the condition $(15)_{S_1}$, as our solution of variational problem.

$$(15')_{S_1} \quad \frac{me^2}{K_+\sqrt{-2mE}} = l, \quad l = 1,\ 2,\ 3,\ \cdots$$

Hence, from $(14')_{S_1}$

$$(14'')_{S_1} \quad \alpha_1 - 1 = l + n, \quad \alpha_2 - 1 = -l + n$$

where, we classify the two cases : $l \leq n$ and $l > n$.

- $l \leq n$.
- $l > n$.

$$(18)_{S_1} \quad \chi = f\Big(r\frac{\sqrt{-2mE}}{K}\Big) \ ; \quad f(x) = x^n e^{-x} \sum_{k=0}^{l-n-1} \frac{(-2x)^k}{k!} \begin{pmatrix} l+n \\ l-n-1-k \end{pmatrix}$$

§2.
The condition $(15)_{S_1}$ show as follows :

$$(19)_{S_1} \quad -E_l = \frac{me^4}{2K^2 l^2}$$

$$(20)_{S_1} \quad K = \frac{h}{2\pi}$$

$$(19')_{S_1} \quad -E_l = \frac{2\pi^2 me^4}{h^2 l^2}$$

$$(21)_{S_1} \quad \frac{K}{\sqrt{-2mE}} = \frac{K^2 l}{me^2} = \frac{h^2 l}{4\pi^2 me^2} = \frac{a_l}{l}$$

$$(22)_{S_1} \quad \nu = C'\sqrt{C + E} = C'\sqrt{C} + \frac{C'}{2\sqrt{C}}E + \cdots$$

§3. Corrections.

$$(23)_{S_1} \quad \int d\tau \Big\{ K^2 T\Big(q,\ \frac{\partial\psi}{\partial q}\Big) + \psi^2 V \Big\},$$

makes stationary under the normalized subcondition :

$$(24)_{S_1} \quad \int \psi^2 d\tau = 1$$

The eigenvalue of variational problem is the stational value of $(23)_{S_1}$ and extend to our theme of 'Quanten niveau' of energy.

5.2. **The second part of** [58]**.**

§1 *The Hamiltonian analogy between mechanics and optic.*

$$(1)_{S_2} \quad \frac{\partial W}{\partial t} + T\Big(q_k,\ \frac{\partial W}{\partial q_k}\Big) + V(q_k) = 0$$

$$(2)_{S_2} \quad W = -Et + S(q_k)$$

$$(1')_{S_2} \quad 2T\Big(q_k,\ \frac{\partial W}{\partial q_k}\Big) = 2(E - V)$$

$$(3)_{S_2} \quad ds^2 = 2\bar{T}(q_k,\ \dot{q_k})dt^2$$

$$(1'')_{S_2} \quad (\text{grad } W)^2 = 2(E - V), \qquad (1''') \quad |\text{grad } W| = \sqrt{2(E - V)}$$

$$(4)_{S_2} \quad ds = \frac{dW_0}{\sqrt{2(E - V)}}$$

§2. *'Geometric' and 'undulatorische' (undulatory, wave) mechanics.*

$$(10)_{S_2} \quad \sin\Big(\frac{2\pi W}{h} + \text{const.}\Big) = \sin\Big(-\frac{2\pi Et}{h} + \frac{2\pi S(q_k)}{h} + \text{const.}\Big)$$

$$(11)_{S_2} \quad \nu = \frac{E}{h}$$

$$(12)_{S_2} \quad \lambda = \frac{u}{\nu} = \frac{h}{\sqrt{2(E - V)}}$$

$$\frac{h}{m\ v\ a}$$

This comparison of dimension between the atom and *Kepler orbit* is at least 10^{-27}.

$$(6')_{S_2} \quad u = \frac{h\nu}{\sqrt{2(h\nu - V)}}$$

$$(13)_{S_2} \quad v = \frac{d\nu}{d\Big(\frac{\nu}{u}\Big)}$$

$$(14)_{S_2} \quad W = W_0$$

$$(15)_{S_2} \quad W + \frac{\partial W}{\partial \alpha_1}d\alpha_1 + \frac{\partial W}{\partial \alpha_2}d\alpha_2 + \cdots + \frac{\partial W}{\partial \alpha_n}d\alpha_n = \text{Const.}$$

$$(15')_{S_2} \quad W + \frac{\partial W}{\partial \alpha_1}d\alpha_1 + \frac{\partial W}{\partial \alpha_2}d\alpha_2 + \cdots + \frac{\partial W}{\partial \alpha_n}d\alpha_n = W_0 + \Big(\frac{\partial W}{\partial \alpha_1}\Big)_0 d\alpha_1 + \Big(\frac{\partial W}{\partial \alpha_2}\Big)_0 d\alpha_2 + \cdots + \Big(\frac{\partial W}{\partial \alpha_n}\Big)_0 d\alpha_n$$

$$(16)_{S_2} \quad W = W_0, \quad \frac{\partial W}{\partial \alpha_1} = \Big(\frac{\partial W}{\partial \alpha_1}\Big)_0, \quad \frac{\partial W}{\partial \alpha_2} = \Big(\frac{\partial W}{\partial \alpha_2}\Big)_0, \quad \cdots, \quad \frac{\partial W}{\partial \alpha_n} = \Big(\frac{\partial W}{\partial \alpha_n}\Big)_0$$

$$(17)_{S_2} \quad \frac{\partial W}{\partial \alpha_1} = \Big(\frac{\partial W}{\partial \alpha_1}\Big)_0, \quad \frac{\partial W}{\partial \alpha_2} = \Big(\frac{\partial W}{\partial \alpha_2}\Big)_0, \quad \cdots, \quad \frac{\partial W}{\partial \alpha_n} = \Big(\frac{\partial W}{\partial \alpha_n}\Big)_0$$

$$(18)_{S_2} \quad \text{div grad } \psi - \frac{1}{u^2}\ddot{\psi} = 0 \tag{43}$$

$$(18')_{S_2} \quad \text{div grad } \psi + \frac{8\pi^2}{h^2}(h\nu - V)\psi = 0 \quad \Rightarrow \quad (18'')_{S_2} \quad \text{div grad } \psi + \frac{8\pi^2}{h^2}(E - V)\psi = 0 \tag{44}$$

$$(19)_{S_2} \quad f(q_k)\text{div}\Big(\frac{1}{f(q_k)} \text{ grad } \psi\Big)$$

§3 *Exsamples of application.*

¶1 *Planck Oscillator. The 'Entartung' (Contraction) problem.* We show the two forms of kinetic energy :

$$(20)_{S_2} \quad \bar{T} = \frac{1}{2}\dot{q}^2, \qquad T = \frac{1}{2}p^2$$

$$(21)_{S_2} \quad V(q) = 2\pi^2\nu_0^2 q^2$$

$$(22)_{S_2} \quad \frac{d^2\psi}{dq^2} + \frac{8\pi^2}{h^2}(E - 2\pi^2\nu_0^2 q^2)\psi = 0$$

$$(22')_{S_2} \quad \frac{d^2\psi}{dq^2} + (a - bq^2)\psi = 0, \qquad a = \frac{8\pi^2 E}{h^2}, \qquad b = \frac{16\pi^4\nu_0^2}{h^2}$$

$$(22'')_{S_2} \quad \frac{d^2\psi}{dq^2} + (\frac{a}{\sqrt{b}} - x^2)\psi = 0, \qquad x = qb^{\frac{1}{4}}$$

Eigenvalues and its eigenfunctions are :

$$(25)_{S_2} \quad \frac{a}{\sqrt{b}} = 1,\ 3,\ 5,\ \cdots, (2n+1),\ \cdots, \quad (26)_{S_2} \quad e^{-\frac{x^2}{2}}H_n(x),$$

where, H_n : *Hermite polynomial* :

$$(27)_{S_2} \quad H_n(x) = (-1)^n e^{x^2}\frac{d^n e^{-x^2}}{dx^n}$$

$$(27')_{S_2} \quad H_n(x) = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} - + \cdots$$

$$(27'')_{S_2} \quad \begin{cases} H_0(x) = 1, \\ H_1(x) = 2x, \\ H_2(x) = 4x^2 - 2, \\ H_3(x) = 8x^3 - 12x, \\ H_4(x) = 16x^4 - 48x^2 + 12, \\ \cdots \end{cases}$$

$$(25')_{S_2} \quad E_n = \frac{2n+1}{2}h\nu_0; \quad n = 1,\ 2,\ 3,\ \cdots$$

$$(26')_{S_2} \quad \psi_n = e^{-\frac{2\pi^2\nu_0^2 q^2}{h}}H_n\Big(2\pi q\sqrt{\frac{\nu_0}{h}}\Big),$$

$$(27)_{S_2} \quad q_n = \frac{\sqrt{E_n}}{2\pi\nu_0} = \frac{1}{2\pi}\sqrt{\frac{h}{\nu_0}}\sqrt{\frac{2n+1}{2}}$$

5.3. **The third part of** [58]**.**
I. *Perturbation theory.*
§1 *A unique independent variable.*

$$(1)_{S_3} \quad L[y] = py'' + p'y' - qy$$

We observe *Sturm-Liouville* type eigenvalue problem :

$$(2)_{S_3} \quad L[y] + E\rho y = 0 \tag{45}$$

We assume *Sturm-Liouville* type eigenfunctions: $y = u_i(x),\ i = 1, 2, 3, \cdots$, and to the eigenvalues, E_i or E_k, so we get :

$$(3)_{S_3} \quad \int \rho(x)u_i(x)u_k(x)dx = 0, \quad \forall i \neq k$$

We assign $(3)_{S_3}$ a perturbational term : $(4)_{S_3} \quad -\lambda r(x)y$

$$(2')_{S_3} \quad L[y] - \lambda ry + E\rho y = 0$$

$$(5)_{S_3} \quad E_k^* = E_k + \lambda\varepsilon_k; \quad u_k^* = u_k(x) + \lambda v_k(x)$$

$$(6)_{S_3} \quad L[v_k] + E\rho v_k = (r - \varepsilon_k\rho)u_k$$

$$(7)_{S_3} \quad \int (r - \varepsilon_k\rho)u_k^2 dx = 0, \quad (7')_{S_3} \quad \varepsilon_k = \frac{\int ru_k^2\ dx}{\int \rho u_k^2\ dx}$$

We normlize $(7')_{S_3}$, namely $\int \rho u_k^2\ dx \equiv 1$, then

$$(7'')_{S_3} \quad \varepsilon_k = \int ru_k^2\ dx$$

$$(8)_{S_3} \quad v_k(x) = \sum_{i=1}^{\infty} \gamma_{ki}\ u_i(x)$$

$$(9)_{S_3} \quad \Big(\frac{r(x)}{\rho(x)} - \varepsilon_k\Big)u_k(x) = \sum_{i=1}^{\infty} c_{ki}\ u_i(x),$$

where,

$$(10)_{S_3} \quad c_{ki} = \int (r - \varepsilon_k\rho)u_k u_i\ dx = \begin{cases} \int ru_k u_i\ dx & i \neq k, \\ 0 & i = k, \end{cases}$$

$$(11)_{S_3} \quad \sum_{i=1}^{\infty} \gamma_{ki}(L(u_i) + E_k\rho u_i) = \sum_{i=1}^{\infty} c_{ki}\ u_i(x),$$

Here, when $(2)_{S_3}$ satisfies with $E = E_i$, then

$$(12)_{S_3} \quad \sum_{i=1}^{\infty} \gamma_{ki}\rho(E_k - E_i)u_i = \sum_{i=1}^{\infty} c_{ki}\ u_i(x),$$

From $(10)_{S_3}$ and $(12)_{S_3}$, we get :

$$(13)_{S_3} \quad \gamma_{ki} = \frac{c_{ki}}{E_k - E_i} = \frac{\int ru_k u_i\ dx}{E_k - E_i}, \quad \forall i \neq k$$

$$(14)_{S_3} \quad u_k^*(x) = u_k(x) + \lambda{\sum_{i=1}^{\infty}}' \frac{u_i(x)\int ru_k u_i\ dx}{E_k - E_i}$$

$$(15)_{S_3} \quad E_k^* = E_k + \lambda\int ru_k^2\ dx$$

§2 *many dependent variables (partial differential equation).*
We observe *Sturm-Liouville* type eigenvalue problem (47). We consider α linear independant solutions, to which we put the boundary condition, and as α eigenfunctions as follows :

$$(16)_{S_3} \quad u_{k1},\ u_{k2},\ \cdots, u_{k\alpha}$$

$$(17)_{S_3} \quad \int \rho(x) u_{ki}(x) u_{k'i'}(x) dx \begin{cases} = 0, & (k,\ i) \neq (k',\ i') \\ = 1, & k = k' \ \&\ i = i' \end{cases}$$

$$(18)_{S_3} \quad E^*_{kl} = E_k + \lambda \varepsilon_l; \quad u^*_{kl} = \sum_{i=1}^{\alpha} \chi_{li} u_{ki}(x) + \lambda v_l(x), \quad (l = 1,\ 2,\ 3,\ \cdots,\ \alpha)$$

$$(19)_{S_3} \quad L[v_l] + E_k\ \rho\ v_l = \sum_{i=1}^{\alpha} \chi_{li}(r - \varepsilon_l \rho) u_{kl}$$

$$(20)_{S_3} \quad \sum_{i=1}^{\alpha} \chi_{li} \int (r - \varepsilon_l \rho) u_{kl}\ u_{km}, \quad (m = 1,\ 2,\ 3,\ \cdots,\ \alpha)$$

$$(21)_{S_3} \quad \chi_{lm} \varepsilon_l = \sum_{i=1}^{\alpha} \chi_{li} \int r\ u_{kl}\ u_{km}\ dx, \quad (m = 1,\ 2,\ 3,\ \cdots,\ \alpha)$$

$$(22)_{S_3} \quad \int r\ u_{kl}\ u_{km}\ dx = \varepsilon_{im}, \quad (i,\ m = 1,\ 2,\ 3,\ \cdots,\ \alpha)$$

$$(21')_{S_3} \quad \chi_{lm} \varepsilon_l = \sum_{i=1}^{\alpha} \chi_{li}\ \varepsilon_{im}, \quad (m = 1,\ 2,\ 3,\ \cdots,\ \alpha)$$

We get the secular equation :

$$(23)_{S_3} \quad \begin{bmatrix} \varepsilon_{11} - \varepsilon_l & \varepsilon_{12} & \varepsilon_{13} & \cdots & \varepsilon_{1\alpha} \\ \varepsilon_{21} & \varepsilon_{22} - \varepsilon_l & \varepsilon_{23} & \cdots & \varepsilon_{2\alpha} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} - \varepsilon_l & \cdots & \varepsilon_{3\alpha} \\ \cdots & & & & \\ \varepsilon_{\alpha 1} & \varepsilon_{\alpha 2} & \varepsilon_{\alpha 3} & \cdots & \varepsilon_{\alpha\alpha} - \varepsilon_l \end{bmatrix} = 0$$

Silmilarly with $(8)_{S_3}$,

$$(24)_{S_3} \quad v_l(x) = \sum_{(k',\ l')} \gamma_{l,\ k',\ i'}\ u_{k',\ l'}(x)$$

Silmilarly with $(9)_{S_3}$,

$$(25)_{S_3} \quad \sum_{i=1}^{\alpha} \chi_{li} \Big(\frac{r}{\rho} - \varepsilon_l \Big) u_{ki} = \sum_{(k',\ l')} c_{l,\ k'\ i'}\ u_{k'\ l'}$$

where, silmilarly with $(10)_{S_3}$,

$$(26)_{S_3} \quad c_{l,\ k'\ i'} = \sum_{i=1}^{\alpha} \chi_{li} \int (r - \varepsilon_l \rho) u_{ki}\ u_{k'i'}\ dx = \begin{cases} \sum_{i=1}^{\alpha} \chi_{li} \int r\ u_{ki}\ u_{k'i'}\ dx & k' \neq k, \\ 0 & k' = k, \end{cases}$$

Silmilarly with $(11)_{S_3}$,

$$(27)_{S_3} \quad \sum_{(k',\ l')} \gamma_{l,k'i'} (L(u_{k'i'}) + E_k \rho u_{k'i'}) = \sum_{(k',\ l')} c_{l,k'i'}\ u_{k'i'}(x),$$

Here, when $(2)_{S_3}$ satisfies with $E = E_{k'}$, then silmilarly with $(12)_{S_3}$

$$(28)_{S_3} \quad \sum_{(k',\ l')} \gamma_{l,k'i'}\rho(E_k - E_{k'})u_{k'i'} = \sum_{(k',\ l')} c_{l,k'i'}\ u_{k'i'}(x),$$

Silmilarly with $(13)_{S_3}$, we get from $(26)_{S_3}$ and $(28)_{S_3}$,

$$(29)_{S_3} \quad \gamma_{l,k'i'} = \frac{c_{l,k'i'}}{E_k - E_{k'}} = \frac{1}{E_k - E_{k'}} \sum_{i=1}^{\alpha} \chi_{li} \int r\ u_{ki}\ u_{k'i'}\ dx, \quad k' \neq k$$

Silmilarly with $(14)_{S_3}$,

$$(30)_{S_3} \quad u^*_{kl}(x) = \sum_{i=1}^{\alpha} \chi_{li}\Big(u_{ki}(x) + \lambda \sum_{(k',\ i')}{}^{\prime} \frac{u_{k'i'}(x)}{E_k - E'_k} \int r\ u_{ki}\ u_{k'i'}\ dx\Big), \quad (l = 1,\ 2,\ 3,\ \cdots,\ \alpha)$$

[30] From $(18)_{S_3}$,

$$(31)_{S_3} \quad E^*_{kl} = E_k + \lambda\varepsilon_l$$

II. *Application on the Starkeffect.*

§3 *Frequency condition on the method of Epstein interpritation.*

We begin with the snalysis of the wave equation of *Kepler problem* (42), here is the wave equation for the Starkeffect of hydrogens :

$$(32)_{S_3} \quad \Delta\psi + \frac{8\pi^2 m}{h^2}\Big(E + \frac{e^2}{r} - eFz\Big)\psi = 0$$

$$(33)_{S_3} \quad \begin{cases} x = (\sqrt{\lambda_1\lambda_2})^+\ \cos\varphi, \\ y = (\sqrt{\lambda_1\lambda_2})^+\ \sin\varphi, \\ z = \frac{1}{2}(\lambda_1 - \lambda_2) \end{cases}$$

We get the functional determinant :

$$(34)_{S_3} \quad \frac{\partial(x,\ y,\ z)}{\partial(\lambda_1,\ \lambda_2,\ \varphi)} = \frac{1}{4}(\lambda_1 + \lambda_2)$$

$$(35)_{S_3} \quad dx\ dy\ dz = \frac{1}{4}(\lambda_1 + \lambda_2)d\lambda_1\ d\lambda_2\ d\varphi$$

$$(36)_{S_3} \quad x^2 + y^2 = \lambda_1\lambda_2; \quad r^2 = x^2 + y^2 + z^2 = \Big[\frac{1}{2}(\lambda_1 + \lambda_2)\Big]^2$$

[31]

$$(32')_{S_3} \quad \frac{\partial}{\partial\lambda_1}\Big(\lambda_1\frac{\partial\psi}{\partial\lambda_1}\Big) + \frac{\partial}{\partial\lambda_2}\Big(\lambda_2\frac{\partial\psi}{\partial\lambda_2}\Big) + \frac{1}{4}\Big(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\Big)\frac{\partial^2\psi}{\partial\varphi^2} + \frac{2\pi^2 m}{h^2}\Big(E(\lambda_1 + \lambda_2) + 2e^2 - \frac{1}{2}eF(\lambda_1^2 - \lambda_2^2)\Big)\psi = 0$$

$$(37)_{S_3} \quad \psi = \Lambda_1\Lambda_2\Phi$$

$$(38)_{S_3} \quad \begin{cases} \frac{\partial^2\Phi}{\partial\varphi^2} = -n^2\Phi, \\ \frac{\partial}{\partial\lambda_1}\Big(\lambda_1\frac{\partial\Lambda_1}{\partial\lambda_1}\Big) + \frac{2\pi^2 m}{h^2}\Big(-\frac{1}{2}eF\lambda_1^2 + E\lambda_1 + e^2 - \beta - \frac{n^2h^2}{8\pi^2 m}\frac{1}{\lambda_1}\Big)\Lambda_1 = 0, \\ \frac{\partial}{\partial\lambda_2}\Big(\lambda_2\frac{\partial\Lambda_2}{\partial\lambda_2}\Big) + \frac{2\pi^2 m}{h^2}\Big(\ \frac{1}{2}eF\lambda_2^2 + E\lambda_2 + e^2 + \beta - \frac{n^2h^2}{8\pi^2 m}\frac{1}{\lambda_2}\Big)\Lambda_2 = 0 \end{cases}$$

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[30] The second sum $\sum'_{(k',\ i')}$ means $\sum'$, silmilarly with $(14)_{S_3}$.

[31] We correct that Schlödinger puts $r^2 = \frac{1}{2}(\lambda_1 + \lambda_2)$.

where, n : determining 'eigenvalue-like' integration constant with E. β : the constant according to *Sommerfeld*.

$$(39)_{S_3} \quad \Phi = \begin{cases} \sin \\ \cos \end{cases} n\varphi, \quad n = 0,\ 1,\ 2,\ 3,\ \cdots$$

$$(41)_{S_3} \quad \frac{\partial}{\partial \xi}\Big(\xi \frac{\partial \Lambda}{\partial \xi}\Big) + \Big(D\xi^2 + A\xi + 2B + \frac{C}{\xi}\Big)\Lambda = 0$$

$$(42)_{S_3} \quad \begin{cases} D_1 \\ D_2 \end{cases} = \mp \frac{\pi^2 meF}{h^2}, \quad A = \frac{2\pi^2 meF}{h^2}, \quad \begin{cases} B_1 \\ B_2 \end{cases} = \mp \frac{\pi^2 m}{h^2}(e^2 \mp \beta), \quad C = -\frac{n^2}{4}$$

$$(43)_{S_3} \quad \frac{B}{{}_+\sqrt{-A}} -_+ \sqrt{-C} = k + \frac{1}{2}, \quad k = 0,\ 1,\ 2,\ \cdots$$

$$(44)_{S_3} \quad \begin{cases} {}_+\sqrt{-A}(k_1 + \frac{1}{2} +_+ \sqrt{-C}) = B_1, \\ {}_+\sqrt{-A}(k_2 + \frac{1}{2} +_+ \sqrt{-C}) = B_2 \end{cases}$$

$$(45)_{S_3} \quad A = -\frac{4\pi^4 m^2 e^4}{h^4 l^2} \quad \text{or} \quad E = -\frac{2\pi^2 m^2 e^4}{h^2 l^2}$$

This is *Balmer-Bohr eliptic niveau*, where, the main quantum number is as follows :

$$(46)_{S_3} \quad l = k_1 + k_2 + n + 1$$

$$(47)_{S_3} \quad \Lambda = \xi^{\frac{n}{2}} u$$

$$(48)_{S_3} \quad 2\xi\sqrt{-A} = \eta$$

$$(41')_{S_3} \quad \frac{d^2 u}{d\eta^2} + \frac{n+1}{\eta}\ \frac{du}{d\eta} + \Big(\frac{D}{({}_+2\sqrt{-A})^3}\eta - \frac{1}{4} + \frac{B}{\sqrt{-A}}\ \frac{1}{\eta}\Big)u = 0$$

$$(103)_{S_3} \quad y'' + \frac{n+1}{x}\ y' + \Big[-\frac{1}{4} + \Big(k + \frac{n+1}{2}\Big)\frac{1}{x}\Big]y = 0$$

L^n_{n+k} means the operator of n-th derivative of $(n+k)$-th *Laguerre Polymomial.* [32] We show the eigenfunctions multiplied with $e^{-\frac{x}{2}}$, for infinitesimal D :

$$(49)_{S_3} \quad u_k(\eta) = e^{-\frac{\eta}{2}} L^n_{n+k}(\eta)$$

To the eigenwalues :

$$(50)_{S_3} \quad \frac{B}{{}_+\sqrt{-A}} = \frac{n+1}{2} + k, \quad (k = 0,\ 1,\ 2,\ \cdots)$$

$$(51)_{S_3} \quad -\frac{D}{({}_+2\sqrt{-A})^3}\eta^{n+2}$$

the perturbation of k-th eigenvalue are :

$$(52)_{S_3} \quad \varepsilon_k = -\frac{D}{({}_+2\sqrt{-A})^3}\frac{\int_0^\infty \eta^{n+2}e^{-\eta}\Big[L^n_{n+k}(\eta)\Big]^2 d\eta}{\int_0^\infty \eta^n e^{-\eta}\Big[L^n_{n+k}(\eta)\Big]^2 d\eta}$$

$$(53)_{S_3} \quad \frac{[(n+k)!]^3}{k!}$$

[32]Assosiated Laguerre polynomials.

$$(54)_{S_3}\quad \frac{[(n+k)!]^3}{k!}(n^2+6nk+6k^2+6k+3n+2)$$

$$(55)_{S_3}\quad \varepsilon_k=-\frac{D}{({}_{+}2\sqrt{-A})^3}(n^2+6nk+6k^2+6k+3n+2)$$

$$(56)_{S_3}\quad \frac{B}{{}_{+}\sqrt{-A}}=\frac{n+1}{2}+k+\varepsilon_k,\quad (k=0,\ 1,\ 2,\ \cdots)$$

$$(57)_{S_3}\quad \begin{cases}\frac{B}{{}_{+}\sqrt{-A}}=\frac{n+1}{2}+k_1+\varepsilon_{k_1},\\ \frac{B}{{}_{+}\sqrt{-A}}=\frac{n+1}{2}+k_2+\varepsilon_{k_2},\end{cases}\qquad (k_1,\ k_2=0,\ 1,\ 2,\ \cdots)$$

$$(58)_{S_3}\quad A=-\frac{(B_1+B_2)^2}{(l+\varepsilon_{k_1}+\varepsilon_{k_2})}$$

$$(59)_{S_3}\quad A=-\frac{(B_1+B_2)^2}{l^2}\Big[1-\frac{2}{l}(\varepsilon_{k_1}+\varepsilon_{k_2})\Big]$$

$$(60)_{S_3}\quad \begin{cases}\varepsilon_{k_1}=+\frac{Fh^4l^3}{64\pi^4m^2e^5}(n^2+6nk_1+6{k_1}^2+6k_1+3n+2),\\ \varepsilon_{k_2}=-\frac{Fh^4l^3}{64\pi^4m^2e^5}(n^2+6nk_2+6{k_2}^2+6k_2+3n+2)\end{cases}$$

From $(46)_{S_3}$,

$$(61)_{S_3}\quad \varepsilon_{k_1}+\varepsilon_{k_2}=\frac{3Fh^4l^4(k_1-k_2)}{32\pi^4m^2e^5}$$

We assign a_0 the radius of first orbit of hydrogen :

$$(63)_{S_3}\quad \frac{1}{2l\sqrt{-A}}=\frac{h^2}{4\pi^2me^2}\equiv a_0$$

The eigenfunctions (not yet normalized) are :

$$(64)_{S_3}\quad \psi_{nk_1k_2}=\lambda_1^{\frac{n}{2}}\lambda_2^{\frac{n}{2}}e^{-\frac{\lambda_1+\lambda_2}{2la_0}}L^n_{n+k_1}\Big(\frac{\lambda_1}{la_0}\Big)L^n_{n+k_2}\Big(\frac{\lambda_2}{la_0}\Big)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}n\varphi$$

§4. *Study of a calculation of intensity and polarization of an 'Aufspaltungsbild' (disolving structure).*

$$(65)_{S_3}\quad q^{rr'}=\int q\ \rho(x)\ \psi_r(x)\ \psi_{r'}(x)\ dx\cdot\Big\{\int\rho(x)[\psi_r(x)]^2\ dx\cdot\int\rho(x)[\psi_{r'}(x)]^2\ dx\Big\}^{-\frac{1}{2}}$$

§5. *The handling of Starkeffect on the method corresponding to Bohr.*

$$(66)_{S_3}\quad 2r\sqrt{-\frac{8\pi^2mE}{h^2}}=\eta$$

For one of the non-perturbed eigenvalues of $(45)_{S_3}$

$$(66')_{S_3}\quad \eta=\frac{2r}{la_0}$$

where, a_0 comes from $(63)_{S_3}$.

$$(67)_{S_3}\quad \Delta'\psi+\Big(-\frac{1}{4}-g\eta\cos\theta+\frac{l}{\eta}\Big)\psi=0,$$

with the abbreviation :

$$(68)_{S_3}\quad g=\frac{a_0Fl^3}{4e},$$

where, Δ' means *Laplacian operator* only for radius vector under the symbol of η.

$$(68)_{S_3} \quad l = 1,\ 2,\ 3,\ 4,\ \cdots$$

$$(69)_{S_3} \quad l = 1,\ 2,\ 3,\ 4,\ \cdots$$

The eigenfunctions (not yet normalized) are :

$$(70)_{S_3} \quad \psi_{l\ n\ m} = P_n^m(\cos\theta) \begin{cases} \sin \\ \cos \end{cases} (m\varphi) \cdot \eta^n\ e^{-\frac{\eta}{2}}\ L_{n+l}^{2n+1}(\eta), \quad n < l$$

where $\psi_{l\ n\ m}$: m-th sphere function of n-order and L_{n+l}^{2n+1} : $(2n+1)$-th derivative of $(n+l)$-th *Laguerre Polynomial*. The normalizing factor is :

$$(71)_{S_3} \quad \frac{1}{\sqrt{\pi}}\sqrt{\frac{2n+1}{2}}\sqrt{\frac{(n-m)!}{(n+m)!}}\sqrt{\frac{(l-n-1)!}{[(n+l)!]^3}}, \quad m \neq 0$$

If $m = 0$, this becomes $\sqrt{2}$ times smaller value. Two eigenfunctions $\psi_{l\ n\ m}$ and $\psi_{l\ n'\ m'}$ must be restricted as follows :

- the above indices of the 'zugeordneten Kugelfunctionen' (possesive sphere functions) are $m = m'$.
- the two spher functions must be $|n - n'| = 1$
- To each indices $l\ n\ m$, when $m \neq 0$ from $(70)_{S_3}$, must not be different with each choices : either $\cos m\varphi$ or $\sin m\varphi$. For example, it is allowed the combination of cos with cos, or sin with sin.

$$(72)_{S_3} \quad \varepsilon_{nm} = -6lg\sqrt{\frac{(l^2-n^2)(n^2-m^2)}{4n^2-1}}$$

$$(73)_{S_3} \quad \begin{bmatrix} -\varepsilon & \varepsilon_{m+1,m} & 0 & 0 & \cdots & 0 \\ \varepsilon_{m+1,m} & -\varepsilon & \varepsilon_{m+2,m} & 0 & \cdots & 0 \\ 0 & \varepsilon_{m+2,m} & -\varepsilon & \varepsilon_{m+3,m} & \cdots & 0 \\ 0 & 0 & \varepsilon_{m+3,m} & -\varepsilon & \cdots & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots \varepsilon_{l-1,m} & -\varepsilon \end{bmatrix}$$

$$(74)_{S_3} \quad k^* = -\frac{\varepsilon}{6lg}$$

$$(74)_{S_3} \quad k^* = \pm(l-m-1), \quad \pm(l-m-3), \quad \pm(l-m-5), \cdots$$

$$(76)_{S_3} \quad k^* = -6lgk^*$$

$$(77)_{S_3} \quad E = -\frac{2\pi^2 me^4}{h^2(l+\varepsilon)^2}$$

$$(77)_{S_3} \quad E = -\frac{2\pi^2 me^4}{h^2\ l^2} - \frac{3}{8}\frac{h^2 Flk^*}{\pi^2 me}$$

III. *Mathematical relation.*

1. *Generalized Laguerre Polynomials and orthogonal functions.*

We apply the k-th Laguerre polynomial $L_k(x)$ to the differential equation :

$$(101)_{S_3} \quad xy'' + (1-x)y' + ky = 0$$

In the case of n-th derivative of $(n+k)$-th *Laguerre Polymomial* by $L_{k+1}^n(x)$:

$$(102)_{S_3} \quad xy'' + (n+1-x)y' + ky = 0$$

$$(102)_{S_3} \quad xy'' + (n+1-x)y' + ky = 0$$

By the transform for $(49)_{S_3}$, we get as follows :

$$(103)_{S_3} \quad y'' + \frac{n+1}{x}\, y' + \Big[-\frac{1}{4} + \Big(k + \frac{n+1}{2}\Big)\frac{1}{x}\Big] y = 0$$

The relating generalized *laguerre Orthonalfunctions* are :

$$(104)_{S_3} \quad x^{\frac{n}{2}}\ e^{-\frac{\eta}{2}}\ L^{n}_{n+k}(x)$$

$$(105)_{S_3} \quad y'' + \frac{1}{x}\, y' + \Big[-\frac{1}{4} + \Big(k + \frac{n+1}{2}\Big)\frac{1}{x} - \frac{n^2}{4x^2}\Big] y = 0$$

where $n \in \mathbb{Z}$ (real) and k : eigenvalue parameter, $x >> 0$, then inthis case, the eigenfunctions are :

$$(106)_{S_3} \quad e^{-\frac{\eta}{2}}\ L^{n}_{n+k}(x)$$

Its eigenvalues are

$$(107)_{S_3} \quad k = 0,\ 1,\ 2,\ 3,\ \cdots$$

$$(108)_{S_3} \quad \frac{d^2y}{d\xi^2} + \frac{n+1}{\xi}\ \frac{dy}{d\xi} + \Big(-\frac{1}{(2k+n+1)^2} + \frac{1}{\xi}\Big) y = 0$$

In $(103)_{S_3}$, by the substitution :

$$(109)_{S_3} \quad \xi = \Big(k + \frac{n+1}{2}\Big) x$$

Thus it transfers to 'Streckenspectrum' (line spectrum), when we consider the following :

$$(110)_{S_3} \quad E = -\frac{1}{(2k+n+1)^2}$$

as the eigenvalue parameter. (See. The analysis of $(7)_{S_1}$.)

2. *Definite integral on the product two laguerre orthogonalfunctions.*

$$(111)_{S_3} \quad \sum_{k=0}^{\infty} L_k(x)\ \frac{t^k}{k!} = \frac{e^{-\frac{xt}{1-t}}}{1-t}$$

$$(112)_{S_3} \quad \sum_{k=0}^{\infty} L^{n}_{n+k}(x)\ \frac{t^k}{n+k!} = (-1)^n\ \frac{e^{-\frac{xt}{1-t}}}{(1-t)^{n+1}}$$

$$(113)_{S_3} \quad x^p\ e^{-x}$$

where $p \in \mathbb{Z},\ p > 0$, to satisfy our purpose.

$$(114)_{S_3} \quad \sum_{k=0}^{\infty}\sum_{k=0}^{\infty} \frac{t^k\ s^{k'}}{(n+k)!\ (n'+k')!} \int_0^{\infty} x^p\ e^{-x}\ L^{n}_{n+k_1}\ L^{n'}_{n'+k'}\ dx = (-1)^{n+n'}\ p!\ \frac{(1-t)^{p-n}(1-s)^{p-n'}}{(1-ts)^{p+1}}$$

$$(115)_{S_3} \quad \int_0^{\infty} x^p\ e^{-x}\ L^{n}_{n+k_1}\ L^{n'}_{n'+k'}\ dx$$

$$= \quad p!\ (n+k)!\ (n'+k')! \cdot \sum_{\tau=0}^{<k,\ k'} (-1)^{n+n'+k+k'+\tau} \begin{pmatrix} p-n \\ k-\tau \end{pmatrix} \begin{pmatrix} p-n' \\ k'-\tau \end{pmatrix} \begin{pmatrix} -p-1 \\ \tau \end{pmatrix}$$

$$(116)_{S_3} \quad J = \int_0^{\infty} x^p\ e^{-\frac{\alpha+\beta}{2}x}\ L^{n}_{n+k_1}(\alpha x)\ L^{n'}_{n'+k'}(\beta x)\ dx$$

$$(117)_{S_3} \quad -\frac{\alpha+\beta}{2} x \equiv y$$

$$(118)_{S_3} \quad \begin{cases} \alpha x = \left(1+\frac{\alpha-\beta}{\alpha+\beta}\right) y \\ \beta x = \left(1-\frac{\alpha-\beta}{\alpha+\beta}\right) y \end{cases}$$

$$(119)_{S_3} \quad \sigma \equiv \frac{2}{\alpha+\beta}, \quad \gamma \equiv \frac{\alpha-\beta}{\alpha+\beta}$$

5.4. **The fourth part of** [58]**.**

§1. *Elimination of eigen parameter from vibration equation. The eigen wave equation. Non-conservative system.*

From (43) and (44),

$$(1)_{S_4} \quad \Delta\psi - \frac{2(E-V)}{E^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

$$(1')_{S_4} \quad \Delta\psi + \frac{8\pi^2}{h^2}(E-V)\psi = 0$$

$$(2)_{S_4} \quad \psi \sim P.R.\left(e^{\pm\frac{2\pi iEt}{h}}\right)$$

[33] $(2)_{S_4}$ is equal to $(3)_{S_4}$:

$$(3)_{S_4} \quad \frac{\partial^2\psi}{\partial t^2} = -\frac{4\pi^2E^2}{h^2}\psi$$

$$(4)_{S_4} \quad \left(\Delta - \frac{8\pi^2}{h^2}V\right)^2\psi + \frac{16\pi^2}{h^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

$$(4')_{S_4} \quad \left(\Delta - \frac{8\pi^2}{h^2}V + \frac{8\pi^2}{h^2}E\right)\left(\Delta - \frac{8\pi^2}{h^2}V - \frac{8\pi^2}{h^2}E\right)\psi = 0$$

$$(3)_{S_4} \quad \frac{\partial\psi}{\partial t} = \pm\frac{2\pi}{h}E\psi$$

$$(4'')_{S_4} \quad \Delta\psi - \frac{8\pi^2}{h^2}V\psi \mp \frac{4\pi i}{h}\frac{\partial\psi}{\partial t} = 0$$

§2. *Extension of perturbation theory, including the time explicitely. Dispersion theory.*

$$(5)_{S_4} \quad V = V_0(x) + r(x,t)$$

Now, using 'Quadrature' (Integration), we like to solve the perturbation problem.

$$(6)_{S_4} \quad r(x,t) = A(x)\cos 2\pi\nu t$$

$$(5')_{S_4} \quad V = V_0(x) + A(x)\cos 2\pi\nu t$$

By $(5')_{S_4}$, $(4'')_{S_4}$, *Kepler problem* turns into :

$$(7)_{S_4} \quad \Delta\psi - \frac{8\pi^2}{h^2}\big(V_0(x) + A(x)\cos 2\pi\nu t\big)\psi \mp \frac{4\pi i}{h}\frac{\partial\psi}{\partial t} = 0$$

$$(8)_{S_4} \quad \psi = u(x)\ e^{\pm\frac{2\pi iEt}{h}}$$

$$(9)_{S_4} \quad u_k(x)\ e^{\pm\frac{2\pi iE_kt}{h}}, \quad k = 1,\ 2,\ 3,\ \cdots$$

$$(10)_{S_4} \quad \psi = u_k(x)\ e^{\pm\frac{2\pi iE_kt}{h}} + w(x,\ t)$$

[33]This means plus and real number.

$$(11)_{S_4} \quad \Delta w - \frac{8\pi^2}{h^2}V_0 w - \frac{4\pi i}{h}\frac{\partial\psi}{\partial t} = \frac{8\pi^2}{h^2}A\cos 2\pi\nu t \cdot u_k e^{\pm\frac{2\pi i E_k t}{h}}$$
$$= \frac{4\pi^2 i}{h^2}Au_k\Big(e^{\frac{2\pi it}{h}(E_k+h\nu)} + e^{\frac{2\pi it}{h}(E_k-h\nu)}\Big)$$

$$(12)_{S_4} \quad w = w_+(x)e^{\frac{2\pi it}{h}(E_k+h\nu)} + w_-(x)e^{\frac{2\pi it}{h}(E_k-h\nu)}$$

$$(13)_{S_4} \quad \Delta w_\pm + \frac{8\pi^2}{h^2}(E_k \pm h\nu - V_0)w_\pm = \frac{4\pi^2 i}{h^2}Au_k$$

$$(14)_{S_4} \quad w_\pm = \frac{1}{2}\sum_{i=1}^{\infty}\frac{(a_{kn})'u_n(x)}{E_k - E_n \pm h\nu}$$

where,

$$(15)_{S_4} \quad (a_{kn})' = \int A(x)u_k(x)u_n(x)\rho(x)\ dx$$

where, $\rho(x)$: density function. $(14)_{S_4}$ is from $(12)_{S_4}$ and $(10)_{S_4}$:

$$(16)_{S_4} \quad \psi = u_k(x)e^{\frac{2\pi it}{h}(E_k+h\nu)} + \frac{1}{2}\sum_{i=1}^{\infty}(a_{kn})'u_n(x)\Big(\frac{e^{\frac{2\pi it}{h}(E_k+h\nu)}}{E_k - E_n + h\nu} + \frac{e^{\frac{2\pi it}{h}(E_k-h\nu)}}{E_k - E_n - h\nu}\Big)$$

$$(17)_{S_4} \quad \psi\bar{\psi} = u_k(x)^2 + 2\cos 2\pi\nu t\sum_{i=1}^{\infty}\frac{(E_k - E_n)(a_{kn})'u_k(x)u_n(x)}{(E_k - E_n)^2 - h\nu}$$

Here, which classical mechanic gives the current dipole moment as the function of configuration of point-system as follows :

$$(18)_{S_4} \quad M_y = \sum e_i y_i$$

$$(19)_{S_4} \quad b_{kn} = \int M_y(x)u_k(x)u_n(x)\rho(x)\ dx$$

We assume the light vector T_z :

$$(20)_{S_4} \quad T_z = F\cos 2\pi\nu t$$

From $(20)_{S_4}$, we mean $A(x)$:

$$(21)_{S_4} \quad A(x) = -F \cdot M_z(x), \quad M_z = \sum e_i z_i$$

From $(19)_{S_4}$ similarly,

$$(22)_{S_4} \quad a_{kn} = \int M_z(x)u_k(x)u_n(x)\rho(x)\ dx$$

$$(23)_{S_4} \quad \int M_y\psi\bar{\psi}\rho\ dx = a_{kk} + 2\cos 2\pi\nu t\sum_{i=1}^{\infty}\frac{(E_k - E_n)a_{kn}b_{kn}}{(E_k - E_n)^2 - h^2\nu^2}$$

§3. *Addition to §2 : 'Angeregte' (activated) atom, 'entartete' (contracted) system, 'Strecken-spectrum' (partitioned spectrum).*

$$(24)_{S_4} \quad u_k(x)u_l(x)e^{\frac{2\pi it}{h}(E_k-E_l)t}$$

We consider not ν but as following term :

$$(25)_{S_4} \quad |\nu \pm (E_k - E_l)/h|$$

$$(26)_{S_4} \quad f(x) = \sum_{i=1}^{\infty}\varphi(x)\cdot u_n(x), \quad \varphi(x) = \int f(x)u_n(x)\rho(x)\ dx$$

$$(27)_{S_4} \quad f(x) = \sum_{i=1}^{\infty} \varphi_n \cdot u_n(x) + \int_a^b u(x,E)\varphi(E)\ dE, \quad a \leq E \leq b$$

$$(28)_{S_4} \quad \int dx\ \rho(x) \int_{E'}^{E'+\Delta} u(x,E)u(x,E)\ dE' \begin{cases} = 1, & \text{normalized} \\ = 0, & \text{else} \end{cases}$$

$$(29)_{S_4} \quad \varphi(E) = \lim_{\Delta \to 0} \frac{1}{\Delta} \int \rho(\xi) f(\xi) \cdot \int_E^{E+\Delta} u(\xi, E')\ dE' \cdot d\xi$$

$$(30)_{S_4} \quad \frac{4\pi^2}{h^2} A(x) u_k(x) = \frac{4\pi^2}{h^2} (a_{kn})' u_k(x) u_n(x) + \frac{4\pi^2}{h^2} \int_a^b u(x,E)\ (a_k)'(E)\ dE$$

$$(15')_{S_4} \quad \alpha'_k(E) = \lim_{\Delta \to 0} \frac{1}{\Delta} \int \rho(\xi) A(\xi) u_k(\xi) \cdot \int_E^{E+\Delta} u(\xi,\ E')\ dE' \cdot d\xi$$

From $(13)_{S_4}$,

$$\frac{8\pi^2}{h^2}(E_k \pm h\nu - E_n) u_n(x), \qquad \text{or} \qquad \frac{8\pi^2}{h^2}(E_k \pm h\nu - E) u(x,\ E),$$

$$(14')_{S_4} \quad w_{\pm} = \frac{1}{2} \sum_{i=1}^{\infty} \frac{\alpha'_{kn} u_n(x)}{E_k - E_n \pm h\nu} + \int_a^b \frac{\alpha'_k u(x,\ E)}{E_k - E \pm h\nu}$$

$$(23')_{S_4} \quad + 2\cos 2\pi\nu t \int d\xi\ \rho(\xi)\ M_y(\xi)\ u_k(\xi) \int_a^b \frac{(E_k - E_n)\alpha'_k(E)\ u(x,\ E)}{(E_k - E_n)^2 - h^2\nu^2} dE$$

$$(23'')_{S_4} \quad 2F \cos 2\pi\nu t \int_a^b \frac{(E_k - E_n)\alpha_k(E)\ \beta_k(E)}{(E_k - E_n)^2 - h^2\nu^2} dE$$

with the followings : from $(22)_{S_4}$,

$$(22')_{S_4} \quad \alpha'_k(E) = \lim_{\Delta \to 0} \frac{1}{\Delta} \int \rho(\xi)\ M_x(\xi)\ u_k(\xi) \cdot \int_E^{E+\Delta} u(\xi,\ E')\ dE' \cdot d\xi$$

and from $(19)_{S_4}$,

$$(19')_{S_4} \quad \beta'_k(E) = \lim_{\Delta \to 0} \frac{1}{\Delta} \int \rho(\xi)\ M_y(\xi)\ u_k(\xi) \cdot \int_E^{E+\Delta} u(\xi,\ E')\ dE' \cdot d\xi$$

§4. *Study of resonance case.*

$$(31)_{S_4} \quad h\nu = E_n - E_k > 0$$

$$(32)_{S_4} \quad E_k \pm E_n \mp E_k = \begin{cases} E_n, \\ 2E_k - En \end{cases}$$

From $(15)_{S_4}$,

$$(33)_{S_4} \quad \int A(x) u_k(x) u_n(x) \rho(x)\ dx = a'_{kn} = 0$$

§5. *Generalization for an arbitrary perturbation.*
§6. *Relativistic-magnetic generalization of basic equation.*
In the followings, we treat the relativistic quantum mechanics. In the case of *Kepler problem*, or, *Sommerfeld*-like 'Feinstrukturformel', or, with the 'half-integer' axis and 'Radialquant', will

be correctly convinced. The hamiltonian partial differential equation for Lorentz electron is as follows :

$$(34)_{S_4} \quad \Big(\frac{1}{c}\ \frac{\partial W}{\partial t} + \frac{e}{c}\mathfrak{A}\Big)^2 - \Big(\frac{\partial W}{\partial x} - \frac{e}{c}\mathfrak{A}\Big)^2 - \Big(\frac{\partial W}{\partial y} - \frac{e}{c}\mathfrak{A}\Big)^2 - \Big(\frac{\partial W}{\partial z} - \frac{e}{c}\mathfrak{A}\Big)^2 - m^2c^2 = 0$$

$$(35)_{S_4} \quad \frac{\partial W}{\partial t},\quad \frac{\partial W}{\partial x},\quad \frac{\partial W}{\partial y},\quad \frac{\partial W}{\partial z},\quad \text{or by the operations : } \pm\frac{h}{2\pi i}\frac{\partial}{\partial t},\quad \pm\frac{h}{2\pi i}\frac{\partial}{\partial x},\quad \pm\frac{h}{2\pi i}\frac{\partial}{\partial y},\quad \pm\frac{h}{2\pi i}\frac{\partial}{\partial z}$$

$$(36)_{S_4} \quad \Big(\Delta - \frac{1}{c^2}\ \frac{\partial^2\psi}{\partial t^2} \mp \frac{4\pi e}{hc}\Big(\frac{V}{c}\frac{\partial}{\partial t} + \mathfrak{A}\ \mathrm{grad}\psi\Big) + \frac{4\pi^2e^2}{h^2c^2}\Big(V^2 - \mathfrak{A}^2 - \frac{m^2c^4}{e^2}\Big)\psi = 0$$

§7. *On thev physical condition of field scaler.*
(Omitted.)

5.5. **Schlödinger** [59]**.**
We show Schlödinger [59] by S_5. This doesn't consist of the series : $S_1,\ S_2,\ S_3,\ S_4$, however is relating one.
§2. *The ordering of a operator and a matrix to a well-ordered function symbol and proof of production rule.*

$$(1)_{S_5} \quad \frac{\partial}{\partial q_l}q_l - q_l\frac{\partial}{\partial q_l}$$

$$(2)_{S_5} \quad F(q_k,\ p_k)f = (q_1\cdots q_n)p_rp_sp_tg(q_1\cdots q_n)p_{r'}h(q_1\cdots q_n)p_{r''}p_{s''}\cdots$$

$$(3)_{S_5} \quad [F,\ \bullet] = f(q_1\cdots q_n)K^3\frac{\partial^3}{\partial p_r\partial p_s\partial p_t}g(q_1\cdots q_n)K\frac{\partial}{q_r'}h(q_1\cdots q_n)K^2\frac{\partial^2}{q_{r''}q_{s''}}\cdots$$

$$(4)_{S_5} \quad u_1(x)\sqrt{\rho(x)},\quad u_2(x)\sqrt{\rho(x)},\quad u_3(x)\sqrt{\rho(x)},\quad \cdots\quad u_\infty(x)\sqrt{\rho(x)}$$

The complete orthogonal system normalized by 1 :

$$(5)_{S_5} \quad \int \rho(x)u_i(x)u_k(x)\ dx = \begin{cases} 0, & i \neq k \\ 1, & i = k \end{cases}$$

$$(6)_{S_5} \quad F^{kl} = \int \rho(x)\ u_k(x)\ [F,\ u_l(x)]\ dx$$

$$(7)_{S_5} \quad G^{lm} = \int \rho(x)\ u_l(x)\ [G,\ u_m(x)]\ dx$$

We introduce 'gewälzte' (circulated) operator : $[\bar{F},\ \bullet]$:

$$(3')_{S_5} \quad [\bar{F},\ \bullet] = (-1)^\tau\cdots K^2\frac{\partial^2}{q_{s''}q_{r''}}h(q_1\cdots q_n)K\frac{\partial}{q_r'}g(q_1\cdots q_n)K^3\frac{\partial^3}{\partial p_r\partial p_s\partial p_t}f(q_1\cdots q_n)$$

where τ : the number of derivative.

$$(6')_{S_5} \quad F^{kl} = \int u_l(x)\ [\bar{F},\ \rho(x)\ u_k(x)]\ dx$$

$$\begin{aligned}(8)_{S_5} \quad F^{kl}G^{lm} &= \sum_l \int u_l(x)\ [\bar{F},\ \rho(x)\ u_k(x)]\ dx \cdot \int \rho(x)\ u_l(x)\ [G,\ u_m(x)]\ dx \\ &= \int [\bar{F},\ \rho(x)\ u_k(x)]\ [G,\ u_m(x)]\ dx\end{aligned}$$

$$(9)_{S_5} \quad (FG) = \sum_l F^{kl}G^{lm} = \int \rho(x)\ u_k(x)\ [FG,\ u_m(x)]\ dx$$

§3. *Heisenberg's Quqntum condition and rule for the partial differential.*

$$(10)_{S_5} \quad p_l\ q_l - q_l\ p_l$$

$$(11)_{S_5} \quad (p_l\ q_l - q_l\ p_l)^{ik} = K\int \rho(x)u_i(x)u_k(x)\ dx = \begin{cases} 0, & i \neq k \\ K, & i = k \end{cases}$$

$$(12)_{S_5} \quad K = \frac{h}{2\pi\sqrt{-1}}$$

$$(13)_{S_5} \quad \begin{cases} q_l^{ik} = \quad \int q_l\ \rho(x)\ u_i(x)\ u_k(x)\ dx \\ p_l^{ik} = K\int \rho(x)\ u_i(x)\ \frac{\partial u_k(x)}{\partial q_l} dx \end{cases}$$

$$(14)_{S_5} \quad \left[\frac{\partial F}{\partial q_l},\ \bullet\right] = \frac{1}{K}\ [p_l F - F p_l,\ \bullet]$$

$$(15)_{S_5} \quad \left[\frac{\partial F}{\partial p_l},\ \bullet\right] = \frac{1}{K}\ [F q_l - q_l F,\ \bullet]$$

With this, we must not exchange q_l, but include in this operator :

$$(16)_{S_5} \quad \frac{\partial}{\partial q_l} \quad \text{by} \quad 1 + q_l \frac{\partial}{\partial q_l}$$

with a defined Hamiltonian function :

$$(17)_{S_5} \quad H(q_k,\ p_k)$$

It is 'normalized' or 'symmetrized' the function and with defined manner for purposes of quantummechanic, namely, the classical mechanics function : $q_k p_k^2$ is exchanged with $\frac{1}{2}(p_k^2 q_k + q_k p_k^2)$, or the function : $p_k\ q_k\ p_k$ with $\frac{1}{3}(p_k^2 q_k + p_k\ q_k\ p_k + q_k p_k^2)$.

$$(18)_{S_5} \quad \begin{cases} \left(\frac{dq_l}{dt}\right)^{ik} = \left(\frac{\partial H}{\partial p_l}\right)^{ik}, \\ \left(\frac{dp_l}{dt}\right)^{ik} = \left(-\frac{\partial H}{\partial q_l}\right)^{ik} \end{cases}, \quad l = 1,\ 2,\ 3,\ \cdots,\ n, \quad i,k = 1,\ 2,\ 3,\ \cdots,\ \infty$$

$$(19)_{S_5} \quad \nu_1,\ \nu_2,\ \nu_3, \nu_4, \cdots, \infty$$

$$(20)_{S_5} \quad \begin{cases} \left(\frac{dq_l}{dt}\right)^{ik} = 2\pi\sqrt{-1}(\nu_i - \nu_k)\ q_l^{ik}, \\ \left(\frac{dp_l}{dt}\right)^{ik} = 2\pi\sqrt{-1}(\nu_i - \nu_k)\ p_l^{ik} \end{cases}$$

$$(18')_{S_5} \quad \begin{cases} (\nu_i - \nu_k)\ q_l^{ik} = \frac{1}{h}(Hq_l - q_l H), \\ (\nu_i - \nu_k)\ p_l^{ik} = \frac{1}{h}(Hp_l - p_l H) \end{cases}$$

Here, we asset that :

1 $(21)_{S_5} \quad -[H,\ \psi] + E\psi = 0$

2 $(21)_{S_5}$ is equal to the wave equation, based on our undulatory machanic.

$$(22)_{S_5} \quad [H,\ u_i] = E_i\ u_i$$

From $(6)_{S_5}$, we get :

$$(23)_{S_5} \quad H^{kl} = \int \rho(x)\ u_k(x)\ [H,\ u_l(x)]\ dx = E_l \int \rho(x)\ u_k(x)\ u_l(x) = \begin{cases} 0, & l \neq k \\ E_l, & l = k \end{cases}$$

$$(24)_{S_5}\quad \begin{cases}(H\ q_l)^{ik} = \sum_m H^{im} q_l^{mk} = E_i\ q_l^{ik} \\ (q_l\ H)^{ik} = \sum_m q_l^{im}\ H^{mk} = E_k\ q_l^{ik}\end{cases}$$

From $(24)_{S_5}$ and the first statement of $(18')_{S_5}$,

$$(25)_{S_5}\quad \frac{E_i - E_k}{h} q_l^{ik}$$

$$(26)_{S_5}\quad H = \frac{1}{2}(p^2 + q^2)$$

$$(27)_{S_5}\quad H = \frac{1}{2}\Big(\frac{1}{f(q)} p f(q) p + q^2\Big)$$

$$(28)_{S_5}\quad H = T(q_k,\ p_k) + V(q_k)$$

Variational problem :

$$(29)_{S_5}\quad \begin{cases}\frac{1}{2}\delta J_1 = \delta \int \Big\{\frac{h^2}{4\pi^2} T\Big(q_k,\ \frac{\partial\psi}{\partial q_k}\Big) + \psi^2 V(q_k)\Big\}\Delta_p^{-\frac{1}{2}} dx = 0, \\ \text{with subcondition : } J_2 = \int \psi^2 \Delta_p^{-\frac{1}{2}} dx = 1\end{cases}$$

where $\int dx$ meanns $\int \cdots \int dq_1 \cdots dq_n$; $\Delta_p^{-\frac{1}{2}}$ means the reciprocal solution of quadratic from the discriminant of quadratic form T.

$$(30)_{S_5}\quad 0 = \frac{1}{2}(\delta J_1 - E\delta J_2) = \int \Big\{ -\frac{h^2}{8\pi^2} \sum_k \frac{\partial}{\partial q_k}\Big(\Delta_p^{-\frac{1}{2}} T_{p_k}\big(q_k,\ \frac{\partial\psi}{\partial q_k}\big)\Big) + \big(V(q_k) - E\big)\Delta_p^{-\frac{1}{2}}\psi\Big\}\delta\psi dx$$

The Euler variational equation turns into :

$$(31)_{S_5}\quad \frac{h^2}{8\pi^2}\Delta_p^{-\frac{1}{2}} \sum_k \frac{\partial}{\partial q_k}\Big\{\Delta_p^{-\frac{1}{2}} T_{p_k}\Big(q_k,\ \frac{\partial\psi}{\partial q_k}\Big)\Big\} - V(q_k)\psi + E\psi = 0$$

We see simply that the style : $(21)_{S_5}$ has the equation : $(31)_{S_5}$, when we apply the Euler equation for homogeneous function on the quadratic form T and pay attention to :

$$(32)_{S_5}\quad T\Big(q_k,\ p_k\Big) = \frac{1}{2}\sum_k p_k T_{p_k}\Big(q_k,\ p_k\Big)$$

§5. *The comparison of two theories. View on a classical understanding of intensity and polarization of emitting radiation.*

$$(33)_{S_5}\quad q_l^{ik} = \int u_i(x)\ u_k(x)\ dx$$

$$(34)_{S_5}\quad \int P(x)\ u_i(x)\ u_k(x)\ dx$$

Which is suitable in both theories of two new quantum theory under the viewpoint of merit ? We can not dare to answer it when we speak straitforwardly. We observe the undulatory machanical structure of hydrogen atom in a situation, where the mechanics fields scaler : ψ by a series of discrete eigenfunctions are given :

$$(35)_{S_5}\quad \psi = \sum_k c_k\ u_k(x)\ e^{\frac{2\pi\sqrt{-1}E_k}{h}}$$

$$(36)_{S_5}\quad \psi\frac{\partial\bar{\psi}}{\partial t}$$

[34]Schrödinger puts $E_i \int \rho(x)\ u_k(x)\ u_l(x)$ in the second right hand-side of $(23)_{S_5}$, then we correct $E_i \Rightarrow E_l$.

$$(37)_{S_5} \quad \text{space density} = 2\pi \sum_{k,\ m} c_k\ u_k(x)\ \frac{E_k - E_m}{h}\ u_k(x)\ u_m(x)\ \sin\frac{2\pi t}{h}(E_m - E_k)$$

We see from $(13)_{S_5}$ for the komponent of dipole moment in the direction q_l :

$$(38)_{S_5} \quad M_{q_l} = 2\pi \sum_{k,\ m} c_k\ u_k(x)\ q_l^{km}\ \frac{E_k - E_m}{h}\ \sin\frac{2\pi t}{h}(E_m - E_k)$$

6. The reference of Schrödinger equations and keywords

For we understand the modeling of the Schrödinger equation, we cite the reference of the equations and the keywords in Schrödinger [58] and [59]. We can pick up the equations and keywords from Schrödinger [58] and [59], as follows :

- Kepler problems
- wave equations
- Sturm-Liouville type partial differential equation for the eigenvalue problem
- boundary value problems
- perturbation problems
- variational problem as the solving method of eigenvalue problem

These are the essence the Schrödinger equation consistes of or the Schrödinger's theory do so. We can observe his geist in the followings.

6.1. **The Kepler problems.**

¶1.

$$(1')_{S_1} \quad H(q, \frac{K}{\psi}\ \frac{\partial\psi}{\partial q}) = E$$

where, H : Hamiltonian function of *Kepler motion.* e : charge of electron and m : mass of electron.

¶2.

$$(5)_{S_1} \quad \Delta\psi + \frac{2m}{K^2}\Big(E + \frac{e^2}{r}\Big)\psi = 0$$

¶3. $(p.372)_{S_1}$ relativistic *Kepler* problem.

¶4. This comparison of dimension between the atom and *Kepler orbit* is at least 10^{-27}.

¶5.

$$(6')_{S_2} \quad u = \frac{h\nu}{\sqrt{2(h\nu - V)}}$$

¶6.

$$(13)_{S_2} \quad v = \frac{d\nu}{d\Big(\frac{\nu}{u}\Big)}$$

¶7. $(p.518)_{S_2}$ *Kepler* problem.

¶8. $(p.519)_{S_2}$ *Kepler* motion.

¶9. We begin with the analysis of the wave equation of *Kepler problem* (42), here is the wave equation for the Starkeffect of hydrogens :

$$(32)_{S_3} \quad \Delta\psi + \frac{8\pi^2 m}{h^2}\Big(E + \frac{e^2}{r} - eFz\Big)\psi = 0$$

¶10. We assume the variation : V.

$$(4'')_{S_4} \quad \Delta\psi - \frac{8\pi^2}{h^2}\, V\psi \mp \frac{4\pi i}{h}\,\frac{\partial\psi}{\partial t} = 0$$

$$(5')_{S_4} \quad V = V_0(x) + A(x)\cos 2\pi\nu t$$

By $(5')_{S_4}$, $(4'')_{S_4}$, *Kepler problem* turns into :

$$(7)_{S_4} \quad \Delta\psi - \frac{8\pi^2}{h^2}\big(V_0(x) + A(x)\cos 2\pi\nu t\big)\psi \mp \frac{4\pi i}{h}\,\frac{\partial\psi}{\partial t} = 0$$

¶11.

$$(17)_{S_4} \quad \psi\bar{\psi} = u_k(x)^2 + 2\cos 2\pi\nu t \sum_{i=1}^{\infty} \frac{(E_k - E_n)(a_{kn})' u_k(x) u_n(x)}{(E_k - E_n)^2 - h\nu}$$

¶12. In the followings, we treat the relativistic quantum mechanics. In the case of *Kepler problem*, or, *Sommerfeld*-like 'Feinstrukturformel', or, with the 'half-integer' axis and 'Radialquant', will be correctly convinced. The hamiltonian partial differential equation for Lorentz electron is as follows :

$$(34)_{S_4} \quad \Big(\frac{1}{c}\,\frac{\partial W}{\partial t} + \frac{e}{c}\mathfrak{A}\Big)^2 - \Big(\frac{\partial W}{\partial x} - \frac{e}{c}\mathfrak{A}\Big)^2 - \Big(\frac{\partial W}{\partial y} - \frac{e}{c}\mathfrak{A}\Big)^2 - \Big(\frac{\partial W}{\partial z} - \frac{e}{c}\mathfrak{A}\Big)^2 - m^2c^2 = 0$$

¶13 $(p.749)_{S_5}$ *Kepler* problem.

6.2. **The wave equations.**

$$(1'')_{S_1} \quad \Big(\frac{\partial\psi}{\partial x}\Big)^2 + \Big(\frac{\partial\psi}{\partial y}\Big)^2 + \Big(\frac{\partial\psi}{\partial z}\Big)^2 - \frac{2m}{K^2}\Big(E + \frac{e^2}{r}\Big)\psi^2 = 0$$

$$(5)_{S_1} \quad \Delta\psi + \frac{2m}{K^2}\Big(E + \frac{e^2}{r}\Big)\psi = 0$$

$$(18)_{S_2} \quad \text{div grad } \psi - \frac{1}{u^2}\ddot{\psi} = 0$$

$$(18')_{S_2} \quad \text{div grad } \psi + \frac{8\pi^2}{h^2}(h\nu - V)\psi = 0$$

$$(18'')_{S_2} \quad \text{div grad } \psi + \frac{8\pi^2}{h^2}(E - V)\psi = 0$$

$$(22)_{S_2} \quad \frac{d^2\psi}{dq^2} + \frac{8\pi^2}{h^2}(E - 2\pi^2\nu_0^2 q^2)\psi = 0$$

$$(32)_{S_3} \quad \Delta\psi + \frac{8\pi^2 m}{h^2}\Big(E + \frac{e^2}{r} - eFz\Big)\psi = 0$$

$$(32')_{S_3} \quad \frac{\partial}{\partial\lambda_1}\Big(\lambda_1\frac{\partial\psi}{\partial\lambda_1}\Big) + \frac{\partial}{\partial\lambda_2}\Big(\lambda_2\frac{\partial\psi}{\partial\lambda_2}\Big) + \frac{1}{4}\Big(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\Big)\frac{\partial^2\psi}{\partial\varphi^2} + \frac{2\pi^2 m}{h^2}\Big(E(\lambda_1 + \lambda_2) + 2e^2 - \frac{1}{2}eF(\lambda_1^2 - \lambda_2^2)\Big)\psi = 0$$

$$(67)_{S_3} \quad \Delta'\psi + \Big(-\frac{1}{4} - g\eta\cos\theta + \frac{l}{\eta}\Big)\psi = 0$$

$$(1)_{S_4} \quad \Delta\psi - \frac{2(E - V)}{E^2}\,\frac{\partial^2\psi}{\partial t^2} = 0$$

$$(1')_{S_4} \quad \Delta\psi + \frac{8\pi^2}{h^2}\,(E-V)\psi = 0$$

$$(4)_{S_4} \quad \Big(\Delta - \frac{8\pi^2}{h^2}V\Big)^2\psi + \frac{16\pi^2}{h^2}\,\frac{\partial^2\psi}{\partial t^2} = 0$$

$$(4'')_{S_4} \quad \Delta\psi - \frac{8\pi^2}{h^2}\,V\psi \mp \frac{4\pi i}{h}\,\frac{\partial\psi}{\partial t} = 0$$

$$(21)_{S_5} \quad -[H,\ \psi] + E\psi = 0$$

6.3. **The perturbation problems.**
§2. *Extension of perturbation theory, including the time explicitely. Dispersion theory.*

$$(5)_{S_4} \quad V = V_0(x) + r(x,t)$$

Now, using 'Quadrature' (Integration), we like to solve the perturbation problem.

$$(6)_{S_4} \quad r(x,t) = A(x)\cos 2\pi\nu t$$

the perturbation of k-th eigenvalue are :

$$(52)_{S_3} \quad \varepsilon_k = -\frac{D}{(+2\sqrt{-A})^3}\,\frac{\int_0^\infty \eta^{n+2}e^{-\eta}\Big[L^n_{n+k}(\eta)\Big]^2 d\eta}{\int_0^\infty \eta^{n}e^{-\eta}\Big[L^n_{n+k}(\eta)\Big]^2 d\eta}$$

§5. $_{S_4}$ *Generalization for an arbitrary perturbation.*
We assign $(3)_{S_3}$ a perturbational term : $(4)_{S_3} \quad -\lambda r(x)y$

$$(2')_{S_3} \quad L[y] - \lambda r y + E\rho y = 0$$

6.4. **The Sturm-Liouville type partial differential equation for the eigenvalue equation.**
We observe *Sturm-Liouville* type eigenvalue problem :

$$(2)_{S_3} \quad L[y] + E\rho y = 0 \tag{46}$$

We assume *Sturm-Liouville* type eigenfunctions: $y = u_i(x),\ i = 1,2,3,\cdots$, and to the eigenvalues, E_i or E_k, so we get :

$$(3)_{S_3} \quad \int \rho(x)u_i(x)u_k(x)dx = 0, \quad \forall i \neq k$$

We observe *Sturm-Liouville* type eigenvalue problem $(2)_{S_3}$ [=(46)]. We consider α linear independant solutions, to which we put the boundary condition, and as α eigenfunctions as follows :

$$(16)_{S_3} \quad u_{k1},\ u_{k2},\ \cdots, u_{k\alpha}$$

6.5. **The boundary value problems.**
¶1 $(p.440)_{S_3}$ Boundary value problem.
We observe *Sturm-Liouville* type eigenvalue problem :

$$(2)_{S_3} \quad L[y] + E\rho y = 0$$

¶2 $(p.736)_{S_5}$ Boundary value problem.
¶3 $(p.745)_{S_5}$ Boundary value problem.

$$(21)_{S_5} \quad -[H,\ \psi] + E\psi = 0$$

¶4 $(p.746)_{S_5}$ Boundary value problem.

$$(22)_{S_5} \quad [H,\ u_i] = E_i\ u_i$$

From $(6)_{S_5}$, we get :

$$(23)_{S_5} \quad H^{kl} = \int \rho(x)\ u_k(x)\ [H,\ u_l(x)]\ dx = E_l \int \rho(x)\ u_k(x)\ u_l(x) = \begin{cases} 0, & l \neq k \\ E_l, & l = k \end{cases}$$

[35]

$$(24)_{S_5} \quad \begin{cases} (H\ q_l)^{ik} = \sum_m H^{im} q_l^{mk} = E_i\ q_l^{ik} \\ (q_l\ H)^{ik} = \sum_m q_l^{im}\ H^{mk} = E_k\ q_l^{ik} \end{cases}$$

From $(24)_{S_5}$ and the first statement of $(18')_{S_5}$,

$$(25)_{S_5} \quad \frac{E_i - E_k}{h} q_l^{ik}$$

¶5 $(p.749)_{S_5}$ Boundary value problem.

6.6. **The variational problem as the solving method of eigenvalue problem.**
Our variational problem :

$$(3)_{S_1} \quad \delta J = \delta \iiint dxdydz \Big[\Big(\frac{\partial \psi}{\partial x}\Big)^2 + \Big(\frac{\partial \psi}{\partial y}\Big)^2 + \Big(\frac{\partial \psi}{\partial z}\Big)^2 - \frac{2m}{K^2}\Big(E + \frac{e^2}{r}\Big)\psi^2 \Big] = 0$$

Integrating under all spaces, we get :

$$(4)_{S_1} \quad \frac{1}{2}\delta J = \int df \delta\psi \frac{\partial \psi}{\partial n} - \iiint dxdydz \delta\psi \Big[\Delta\psi + \frac{2m}{K^2}\Big(E + \frac{e^2}{r}\Big)\psi \Big] = 0$$

We assume the variation : V.

$$(4'')_{S_4} \quad \Delta\psi - \frac{8\pi^2}{h^2}\ V\psi \mp \frac{4\pi i}{h}\ \frac{\partial \psi}{\partial t} = 0$$

$$(5')_{S_4} \quad V = V_0(x) + A(x)\cos 2\pi\nu t$$

By $(5')_{S_4}$, $(4'')_{S_4}$, *Kepler problem* turns into :

$$(7)_{S_4} \quad \Delta\psi - \frac{8\pi^2}{h^2}\big(V_0(x) + A(x)\cos 2\pi\nu t\big)\psi \mp \frac{4\pi i}{h}\ \frac{\partial \psi}{\partial t} = 0$$

$$(29)_{S_5} \quad \begin{cases} \frac{1}{2}\delta J_1 = \delta \int \Big\{ \frac{h^2}{4\pi^2} T\Big(q_k,\ \frac{\partial \psi}{\partial q_k}\Big) + \psi^2 V(q_k) \Big\} \Delta_p^{-\frac{1}{2}} dx = 0, \\ \text{with subcondition : } J_2 = \int \psi^2 \Delta_p^{-\frac{1}{2}} dx = 1 \end{cases}$$

$$(30)_{S_5} \quad 0 = \frac{1}{2}(\delta J_1 - E\delta J_2) = \int \Big\{ -\frac{h^2}{8\pi^2} \sum_k \frac{\partial}{\partial q_k}\Big(\Delta_p^{-\frac{1}{2}} T_{p_k}\big(q_k,\ \frac{\partial \psi}{\partial q_k}\big)\Big) + \big(V(q_k) - E\big)\Delta_p^{-\frac{1}{2}}\psi \Big\} \delta\psi dx$$

7. Summary and conclusion

We show in our paper the mathematical history from Kepler to Schrödinger as follows :

1. We show that Kepler and Huygens propose planetary motion and wave phenomina, and Newton describes it by the expression, which become the model of future main principles of quantum theories.
2. Lagrange defined Kepler problem at the first time, of the orthodoxy calculus of true anomaly from mean anomaly with the equation by trigonometric series.
3. Laplace surveyed many problems. One of them is the perturbation problem of the planet by the first order.

[35]Schrödinger puts $E_i \int \rho(x)\ u_k(x)\ u_l(x)$ in the second right hand-side of $(23)_{S_5}$, then we correct $E_i \Rightarrow E_l$.

4. Poisson resurveyed after Lagrange and Laplace, the perturbation problem of mean motion and long axis of the third law on planet by the second order : today's criterion owes to him.
5. Gauss surveyed the perturbation of the planet on the second law.
6. Schrödinger proposed his equations from the analogy of Kepler motion with atomic motion, and his modeling are made on the very various arenas and scopes of problems.

We would like to conclude our paper including some speculations from our short studies as follows :

1. Schrödinger seeks the essential nature of various sort of wave phenomena, and the classical theory of Kepler motion and Newtonian mechanics to make quantum mechanics.
2. For the initial value problem of wave equation on the eigenvalue problem, Schrödinger must use the Sturm-Liouville type eigenvalue equation as the most suitable one, and his mathematical knowledge on the partial differential equations may come from via the mathematical theory by Courant and Hilbelt.
3. Schrödinger doesn't cite the original papers of Sturm-Liouville, and didn't investigate deeply Sturm and Liouville's papers, but through the books by Courant and Hilbelt, on whom Schrdinger depends mathematically in the theoretical modeling of the quantum equation.
4. Sturm and Liouville's interests are the actual and practical problem of heat diffusion equations (or heat wave equations) by Fourier and Poisson, and they are necessary the theoretical basement of the method of demonstration of convergence on trigonometric series.
5. To make the equation of wave mechanics including the electric wave and the electromagnetic wave, Schrödinger approaches only from the Sturm and Liouville's the actual and practical problem of heat diffusion equations (or heat wave equations) by Fourier and Poisson. On the other wave equation, such as water wave, the theoretical equations are unknown for him, except for the fluid dynamic equation such as the Euler equations (1755), the Navier-Stokes equations (1823-28, proclaimed by Prandtl [55] in early of 20C.), the Stokes equations (1845-9). Schrödinger may not refere the heat equation in fluid by Fourier 1820 [15] (published in 1835).
6. Schrödinger cites, of course, the book Boltzmann 1895 [4] of gas transport equations, which is one of the earlier quantum dynamics.
7. We owe the physical history of this problem to Darrigol [11], we think, who is the unique mathematical-physical historian in this arena, however he doesn't emphasize the mathematical aspect, so we could contribute to it from this viewpoint.

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