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Pencils of cubic curves and rational elliptic surfaces

By

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Pencils of cubic curves and rational elliptic surfaces[*]

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In this paper, we study rational elliptic surfaces $f : S \to \mathbb{P}^1$ over a field k, which is algebraically closed and of characteristic 0. We always assume that f is relatively minimal and has a section. We denote by T the direct sum of root lattices of type A, D, E corresponding to all reducible fibers of f. T is called the *trivial lattice* of f and has a natural embedding into E_8. The rational elliptic surfaces are classified in terms of the lattice embedding $T \hookrightarrow E_8$ by Oguiso-Shioda[5].

It is well known that almost all pencils of cubic curves give rational elliptic surfaces by blowing up the projective plane $\mathbb{P}^2$ at nine base points. Conversely any rational elliptic surface arises in this way ([4, Section 2]). Naruki[4] gave explicit equations over $\mathbb{Q}$ of such pencils for each rational elliptic surface of Mordell-Weil rank $r = 0$. Our first result is to generalize his result to the case of $r = 1$ (Theorem 2.1), which are classified into 19 types in [5]. Moreover, we give a 1-parameter family over $\mathbb{Q}$ of pencils of cubic curves (Corollary 2.2). In order to find the equations of pencils of cubic curves, we completely determine the configurations of rational curves of negative self-intersection number on such rational elliptic surfaces (Theorem 1.1). In principle, we can find all pencils of cubic curves by this method.

In the final section, we show Theorem 3.1 which generalizes both Naruki's result and our Theorem 2.1. Namely for every primitive embedding $T \subseteq T'$ between trivial lattices of rank 7, 8, we give explicit specializations/generalizations between corresponding pencils of cubic curves of Mordell-Weil rank $r =$ 1, 0.

1 Configurations of rational curves on rational elliptic surfaces

The main result in this section is as follows.

Theorem 1.1. When $r = 0, 1$, configurations of singular fibers and sections on rational elliptic surfaces are as below (see Table 1, 2). In particular, their dual graphs are *unique*.

Note that we define the label of irreducible components in Kodaira fibers as follows :

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[*] This paper is based on my master thesis[2] except for Theorem 3.1.

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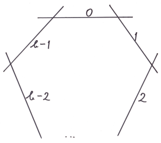

0
1
2
$\ell-1$
$\ell-2$
…

Figure1 I_b

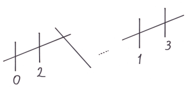


Figure2 I_b^*

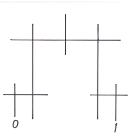


Figure3 III^*

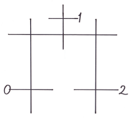
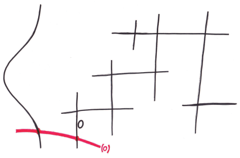
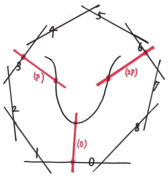
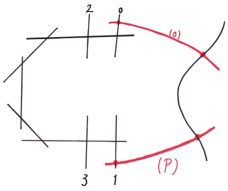
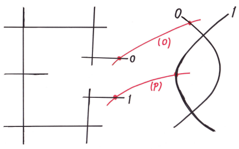
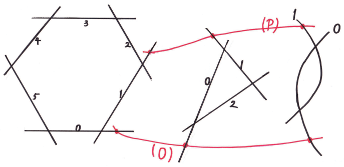
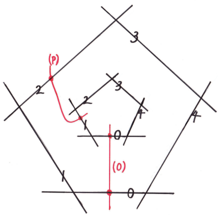
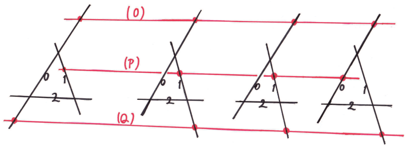
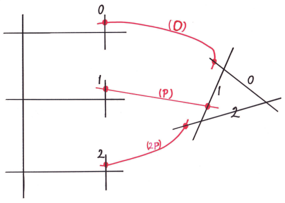
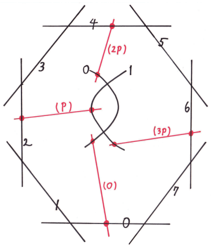


Figure4 IV^*

In the following table, $P = [i_1, i_2, \ldots]$ means that the section (P) passes the irreducible component with label i_k in the k-th reducible fiber.

Table1 case of $r = 0$

No. T $E(K)$	maximal disjoint configurations of (-1)-curves	sections disjoint from (O) [ ] of generators of $E(K)$
62 E_8 $\{0\}$		$\emptyset$
63 A_8 $\mathbb{Z}/3\mathbb{Z}$	 0 1 2 3 4 5 6 7 8 (P) (2P) (O)	$\{P \quad 2P\}$ $P = [3]$; order 3
64 D_8 $\mathbb{Z}/2\mathbb{Z}$		$\{P\}$ $P = [1]$; order 2
65 $E_7 \oplus A_1$ $\mathbb{Z}/2\mathbb{Z}$		$\{P\}$ $P = [1, 1]$; order 2

66 $A_5 \oplus A_2 \oplus A_1$ $\mathbb{Z}/6\mathbb{Z}$	 4 3 2 5 1 0 (P) 1 0 0 1 2 (0)	$\{P \quad 2P \quad 3P \quad 4P \quad 5P \quad 6P\}$ $P = [1,1,1]$; order 6
67 ${A_4}^{\oplus 2}$ $\mathbb{Z}/5\mathbb{Z}$	 (P) 3 2 2 3 1 4 0 4 1 (0) 0	$\{P \quad 2P \quad 3P \quad 4P \quad 5P\}$ $P = [1,2]$; order 5
68 ${A_2}^{\oplus 4}$ $(\mathbb{Z}/3\mathbb{Z})^2$	 (0) (P) 0 1 2 (Q) 0 1 2 0 1 2 0 1 2	$\left\{\begin{array}{ccc} & P & 2P \\ Q & P+Q & 2P+Q \\ 2Q & P+2Q & 2P+2Q \end{array}\right\}$ $P = [1,1,1,0]$; order 3 $Q = [0,1,1,1]$; order 3
69 $E_6 \oplus A_2$ $\mathbb{Z}/3\mathbb{Z}$	 0 (0) 1 (P) 1 0 2 2 (2P)	$\{P \quad 2P\}$ $P = [1,1]$; order 3
70 $A_7 \oplus A_1$ $\mathbb{Z}/4\mathbb{Z}$	 4 (2P) 5 3 0 1 6 (P) (3P) 2 (0) 7 1 0	$\{P \quad 2P \quad 3P\}$ $P = [2,1]$; order 4

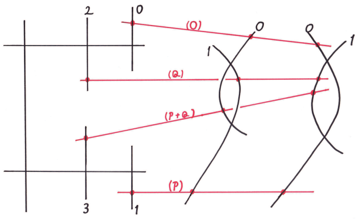
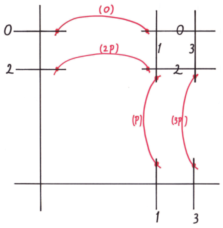
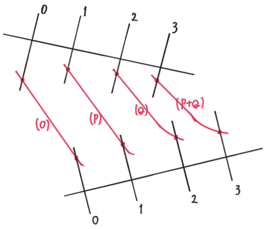
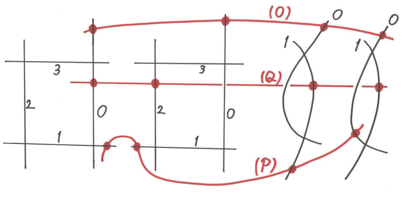
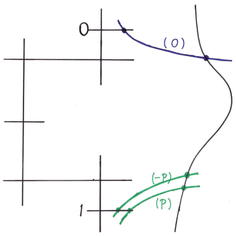
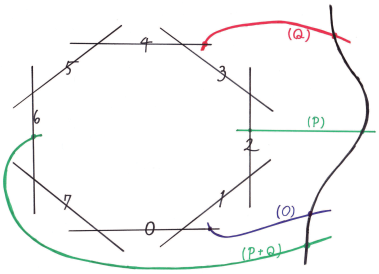
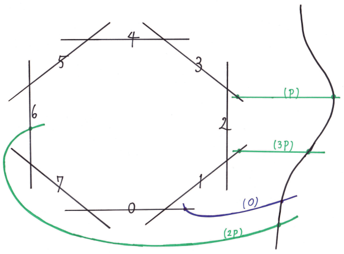
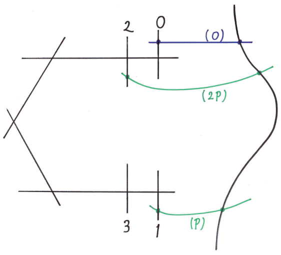
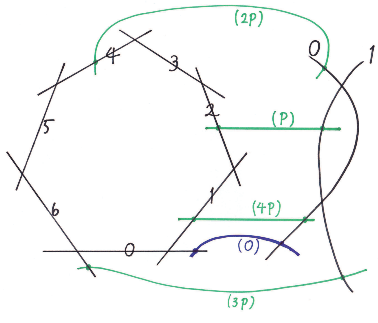
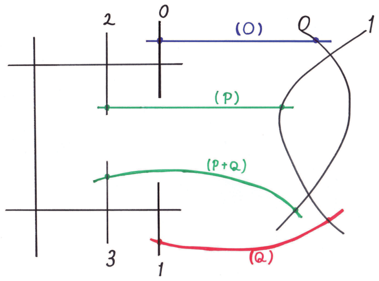
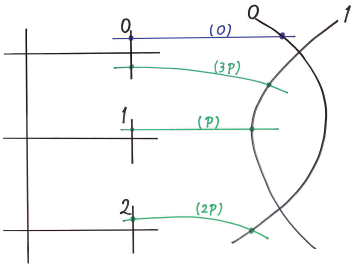
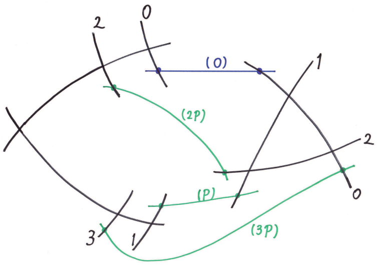
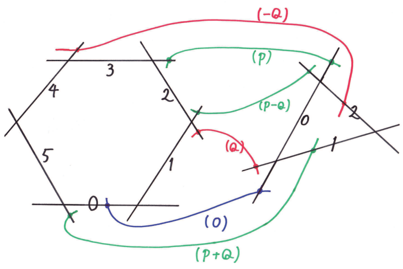
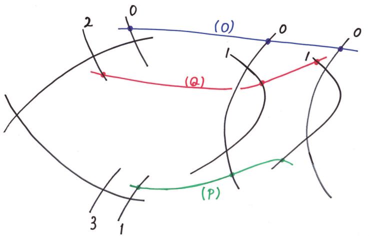
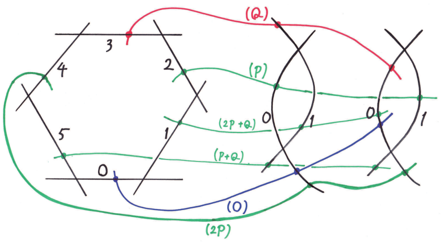
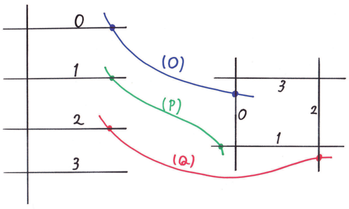
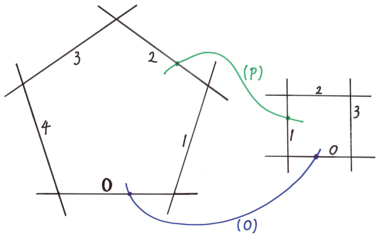
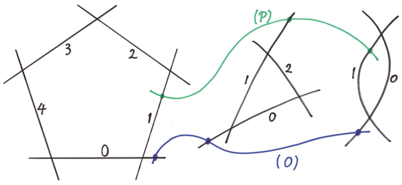
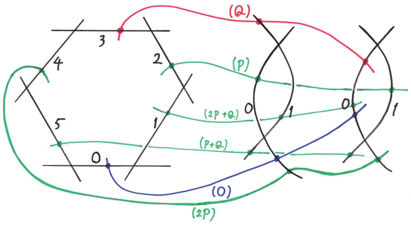
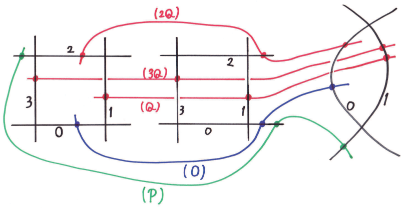
71 $D_6 \oplus {A_1}^{\oplus 2}$ $(\mathbb{Z}/2\mathbb{Z})^2$	 2 0 (O) 0 0 1 1 (Q) (P+Q) (P) 3 1	$\left\{\begin{matrix} & P \\ Q & P+Q \end{matrix}\right\}$ $P = [1,1,0]$; order 2 $Q = [2,1,1]$; order 2
72 $D_5 \oplus A_3$ $\mathbb{Z}/4\mathbb{Z}$	 0 (O) 0 1 3 2 (2P) 2 (P) (3P) 1 3	$\{P \quad 2P \quad 3P\}$ $P = [1,1]$; order 4
73 ${D_4}^{\oplus 2}$ $(\mathbb{Z}/2\mathbb{Z})^2$	 0 1 2 3 (O) (P) (Q) (P+Q) 0 1 2 3	$\left\{\begin{matrix} & P \\ Q & P+Q \end{matrix}\right\}$ $P = [1,1]$; order 2 $Q = [2,2]$; order 2
74 $(A_3 \oplus A_1)^{\oplus 2}$ $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	 (O) 0 0 1 1 3 3 (Q) 2 O 2 O 1 1 (P)	$\left\{\begin{matrix} & P & 2P & 3P \\ Q & P+Q & 2P+Q & 3P+Q \end{matrix}\right\}$ $P = [1,1,1,0]$; order 4 $Q = [0,2,1,1]$; order 2

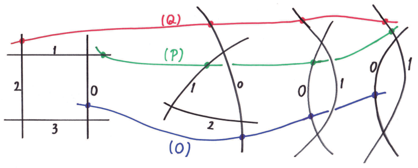
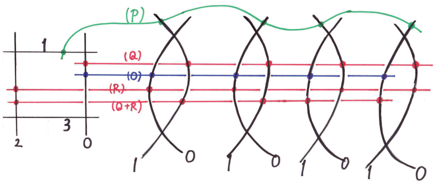
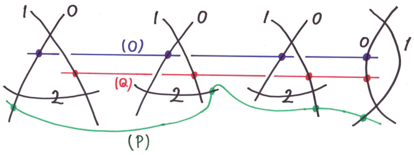
Table2 case of $r = 1$

No. T $E(K)$	maximal disjoint configurations of (-1)-curves	sections disjoint from (O) [ ] of generators of $E(K)$
43 E_7 $\left\langle \frac{1}{2} \right\rangle$		$\{-2P \quad -P \quad P \quad 2P\}$ $P = [1]$; $\langle P,\ P\rangle = \dfrac{1}{2}$
44 A_7 $\left\langle \frac{1}{2} \right\rangle$	 4 5 3 (Q) 6 (P) 2 7 1 0 (O) (P+Q)	$\left\{ \begin{array}{ccccc} & -P & & P & \\ -2P+Q & -P+Q & Q & P+Q & 2P+Q \end{array} \right\}$ $P = [2]$; $\langle P,\ P\rangle = \dfrac{1}{2}$ $Q = [4]$; order 2
45 A_7 $\left\langle \frac{1}{8} \right\rangle$	 4 5 3 6 (P) 2 (3P) 7 1 0 (O) (2P)	$\{-3P \quad -2P \quad -P \quad P \quad 2P \quad 3P\}$ $P = [3]$; $\langle P,\ P\rangle = \dfrac{1}{8}$
46 D_7 $\left\langle \frac{1}{4} \right\rangle$		$\{-2P \quad -P \quad P \quad 2P\}$ $P = [1]$; $\langle P,\ P\rangle = \dfrac{1}{4}$

47 $A_6 \oplus A_1$ $\left\langle \frac{1}{14} \right\rangle$	 (2P) 4 3 0 1 5 2 (P) 1 6 (4P) 0 (O) (3P)	$\left\{ \begin{matrix} -P & -2P & -3P & -4P \\ P & 2P & 3P & 4P \end{matrix} \right\}$ $P = [2,1]$; $\langle P,\ P\rangle = \frac{1}{14}$
48 $D_6 \oplus A_1$ $\left\langle \frac{1}{2} \right\rangle \oplus \mathbb{Z}/2\mathbb{Z}$	 2 0 (O) Q 1 (P) (P+Q) 3 1 (Q)	$\left\{ \begin{matrix} -2P & -P & & P & 2P \\ & -P+Q & Q & P+Q & \end{matrix} \right\}$ $P = [2,1]$; $\langle P,\ P\rangle = \frac{1}{2}$ $Q = [1,1]$; order 2
49 $E_6 \oplus A_1$ $\left\langle \frac{1}{6} \right\rangle$	 0 (O) Q 1 (3P) 1 (P) 2 (2P)	$\left\{ \begin{matrix} -P & -2P & -3P \\ P & 2P & 3P \end{matrix} \right\}$ $P = [1,1]$; $\langle P,\ P\rangle = \frac{1}{6}$
50 $D_5 \oplus A_2$ $\left\langle \frac{1}{12} \right\rangle$	 2 0 (O) 1 (2P) 2 (P) 0 3 1 (3P)	$\left\{ \begin{matrix} -P & -2P & -3P \\ P & 2P & 3P \end{matrix} \right\}$ $P = [1,1]$; $\langle P,\ P\rangle = \frac{1}{12}$

51 $A_5 \oplus A_2$ $\left\langle \frac{1}{2} \right\rangle \oplus \mathbb{Z}/3\mathbb{Z}$	 (−Q) 3 4 2 (P) (P−Q) 0 2 5 (Q) 1 1 0 (0) (P+Q)	$\left\{ \begin{array}{ccccc} & -P-Q & -Q & P-Q & \\ -2P & -P & & P & 2P \\ & -P+Q & Q & P+Q & \end{array} \right\}$ $P=[3,0]$; $\langle P,\ P\rangle = \frac{1}{2}$ $Q=[2,1]$; order 3
52 $D_5 \oplus {A_1}^{\oplus 2}$ $\left\langle \frac{1}{4} \right\rangle \oplus \mathbb{Z}/2\mathbb{Z}$		$\left\{ \begin{array}{ccccc} -2P & -P & & P & 2P \\ -2P+Q & -P+Q & Q & P+Q & 2P+Q \end{array} \right\}$ $P=[1,0,1]$; $\langle P,\ P\rangle = \frac{1}{4}$ $Q=[2,1,1]$; order 2
53 $A_5 \oplus {A_1}^{\oplus 2}$ $\left\langle \frac{1}{6} \right\rangle \oplus \mathbb{Z}/2\mathbb{Z}$	 (Q) 3 4 2 (P) 5 1 (2P+Q) 0 0 1 (P+Q) 0 (0) (2P)	$\left\{ \begin{array}{ccccccc} -3P & -2P & -P & & P & 2P & 3P \\ & -2P+Q & -P+Q & Q & P+Q & 2P+Q & \end{array} \right\}$ $P=[2,0,1]$; $\langle P,\ P\rangle = \frac{1}{6}$ $Q=[3,1,0]$; order 2
54 $D_4 \oplus A_3$ $\left\langle \frac{1}{4} \right\rangle \oplus \mathbb{Z}/2\mathbb{Z}$		$\left\{ \begin{array}{ccccc} -2P & -P & & P & 2P \\ 2P+Q & -P+Q & Q & P+Q & 2P+Q \end{array} \right\}$ $P=[1,1]$; $\langle P,\ P\rangle = \frac{1}{4}$ $Q=[2,2]$; order 2

55 $A_4 \oplus A_3$ $\left\langle \frac{1}{20} \right\rangle$		$\left\{ \begin{array}{ccccc} P & 2P & 3P & 4P & 5P \\ -P & -2P & -3P & -4P & -5P \end{array} \right\}$ $P = [2,1] \quad ; \quad \langle P,\ P \rangle = \frac{1}{20}$
56 $A_4 \oplus A_2 \oplus A_1$ $\left\langle \frac{1}{30} \right\rangle$	 (P) 3 2 4 1 2 1 0 1 0 0 (0)	$\left\{ \begin{array}{cccccc} P & 2P & 3P & 4P & 5P & 6P \\ -P & -2P & -3P & -4P & -5P & -6P \end{array} \right\}$ $P = [1,1,1] \quad ; \quad \langle P,\ P \rangle = \frac{1}{30}$
57 $D_4 \oplus {A_1}^{\oplus 3}$ $\left\langle \frac{1}{2} \right\rangle \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	 (Q) 3 2 4 (P) 5 1 (2P+Q) 0 0 1 1 (P+Q) 0 (0) (2P)	$\left\{ \begin{array}{ccccc} -2P & -P & & P & 2P \\ & -P+Q & Q & P+Q & \\ & -P+R & R & P+R & \\ & -P+Q+R & Q+R & P+Q+R & \end{array} \right\}$ $P = [0,1,1,1] \quad ; \quad \langle P,\ P \rangle = \frac{1}{2}$ $Q = [1,1,1,0] \quad ; \quad$ order 2 $R = [2,1,0,1] \quad ; \quad$ order 2
58 ${A_3}^{\oplus 2} \oplus A_1$ $\left\langle \frac{1}{2} \right\rangle \oplus \mathbb{Z}/4\mathbb{Z}$	 (2Q) 2 2 (3Q) 3 1 (Q) 3 1 0 0 0 1 (0) (P)	$\left\{ \begin{array}{ccccc} -2P & -P & & P & 2P \\ & -P+Q & Q & P+Q & \\ & -P+2Q & 2Q & P+2Q & \\ & -P+3Q & 3Q & P+3Q & \end{array} \right\}$ $P = [2,0,1] \quad ; \quad \langle P,\ P \rangle = \frac{1}{2}$ $Q = [1,1,1] \quad ; \quad$ order 4

59 $A_3 \oplus A_2 \oplus {A_1}^{\oplus 2}$ $\left\langle \dfrac{1}{12} \right\rangle \oplus \mathbb{Z}/2\mathbb{Z}$	 (Q) (P) (O) 1 2 3 0 1 0 2 0 1 0 1	$\left\{ \begin{array}{ccccccccc} -4P & -3P & -2P & -P & & P & 2P & 3P & 4P \\ & -3P+Q & -2P+Q & -P+Q & Q & P+Q & 2P+Q & 3P+Q & \end{array} \right\}$ $P = [1,1,0,1]$; $\langle P,\ P \rangle = \dfrac{1}{12}$ $Q = [2,0,1,1]$; order 2
60 $A_3 \oplus {A_1}^{\oplus 4}$ $\left\langle \dfrac{1}{4} \right\rangle \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	 (P) (Q) (O) (R) (Q+R) 1 2 3 0 1 0 1 0 1 0 1 0	$\left\{ \begin{array}{ccccc} -2P & -P & & P & 2P \\ & -P+Q & Q & P+Q & \\ -2P+R & -P+R & R & P+R & 2P+R \\ -2P+Q+R & -P+Q+R & Q+R & P+Q+R & 2P+Q+R \end{array} \right\}$ $P = [1,1,0,1,0]$; $\langle P,\ P \rangle = \dfrac{1}{4}$ $Q = [0,1,1,1,1]$; order 2 $R = [2,0,0,1,1]$; order 2
61 ${A_2}^{\oplus 3} \oplus A_1$ $\left\langle \dfrac{1}{6} \right\rangle \oplus \mathbb{Z}/3\mathbb{Z}$	 1 0 1 0 1 0 0 1 (O) (Q) 2 2 2 (P)	$\left\{ \begin{array}{ccccccc} -3P & -2P & -P & & P & 2P & 3P \\ & -2P+Q & -P+Q & Q & P+Q & 2P+Q & \\ & -2P+2Q & -P+2Q & 2Q & P+2Q & 2P+2Q & \end{array} \right\}$ $P = [0,2,1,1]$; $\langle P,\ P \rangle = \dfrac{1}{6}$ $Q = [1,1,1,0]$; order 3

1.1 Specialization map of Mordell-Weil group

A rational elliptic surface $f : S \to \mathbb{P}^1$ is identified with its generic fiber E which is an elliptic curve over the function field $K := k(\mathbb{P}^1)$, and also the Mordell-Weil group $MW(f)$ is identified with the group $E(K)$ of K-rational points of the generic fiber E. We denote by (P) the section corresponding to a rational point $P \in E(K)$.

$S^\sharp := \{x \in S \mid f : \text{smooth at } x\}$ and $\left(f^{-1}(v)\right)^\sharp := f^{-1}(v) \cap S^\sharp$. It is well known that $f^\sharp : S^\sharp \to C$ is a group scheme.

$$\begin{tikzcd}
E(K) \arrow[r, "sp_v"] & \left(f^{-1}(v)\right)^\sharp \\
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\
P \arrow[r, mapsto] & (P) \cap f^{-1}(v)
\end{tikzcd}$$

This homomorphism satisfies the important property as follows.

Proposition 1.2. The restriction of sp_v to $E(K)_{tor}$ is injective.

Proof. See [3, Proposition 8]. □

Let $\left(f^{-1}(v)\right)^\sharp_0$ be the neutral component, and $G_v := \left(f^{-1}(v)\right)^\sharp / \left(f^{-1}(v)\right)^\sharp_0$ be the group of connected components of $\left(f^{-1}(v)\right)^\sharp$. We denote the composite

$$\begin{tikzcd}
E(K) \arrow[r, "sp"] & \prod_{v \in \mathbb{P}^1} \left(f^{-1}(v)\right)^\sharp \arrow[r, two heads] & \prod_{v \in \mathbb{P}^1} G_v \\
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \\
P \arrow[r, mapsto] & \left((P) \cap f^{-1}(v)\right)_v &
\end{tikzcd}$$

by $\overline{sp}$. This homomorphism also satisfies the important property as follows.

Proposition 1.3. The restriction of $\overline{sp}$ to $E(K)_{tor}$ is injective.

To prove this, we need the *height formula*. In general, the Mordell-Weil group $E(K)$ is equipped with a natural $\mathbb{Z}$-valued bilinear pairing $\langle\ ,\rangle$ called the *height pairing*. The height pairing can be expressed more explicitly as follows.

Proposition 1.4 (Height Formula)**.** (1) $\langle P,\ P\rangle = 2 + 2(P).\,(O) - \sum_{v \in R} contr_v(P)$ where R is the set of $v \in \mathbb{P}^1$ whose fiber $f^{-1}(v)$ is reducible singular fibers, and $contr_v$ is the local contribution of P at v (see AppendixA).

(2) Any two torsion sections of f do not intersect each other. In particular we have

$$\forall P \in E(K)_{tor} - \{O\}, \quad \sum_{v \in R} contr_v(P) = 2.$$

Proof. (1) See [6, Theorem8.6].

(2) It follows from Proposition 1.2.

□

Proof of Proposition 1.3. Assume that $P \in E(K)_{tor}$ and $\overline{sp}(P) = 0$, then we have $\sum_{v \in R} contr_v(P) = 0$, which proves that $P = O$ by Proposition 1.4(2). □

1.2 Label

We define the label of irreducible components in Kodaira fibers (see Figure1-Figure4 below Theorem 1.1) so that the group structure of G_v is compatible with that of $E(K)$ when cyclic.

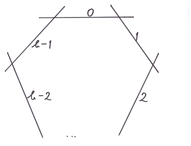


Figure5 I_b

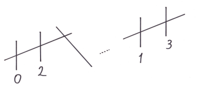


Figure6 I_b^*

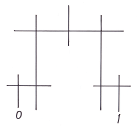


Figure7 III^*

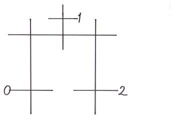


Figure8 IV^*

For type I_n^* (n : even), G_v is $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$, but we give the same labeling as type I_n^* (n : odd) for which G_v is $\mathbb{Z}/4\mathbb{Z}$. When n is even, the group structure of $G_v = \{0, 1, 2, 3\}$ is given as follows :

$1 + 1 = 0 \quad 2 + 2 = 0 \quad 3 + 3 = 0$

$1 + 2 = 3 \quad 2 + 3 = 1 \quad 3 + 1 = 2$

We now make the following definition. For every rational point $P \in E(K)$, let $\overline{sp}(P) = [i_1, i_2, \ldots]$ be the label of irreducible component through which the section (P) passes. If $\overline{sp}(P) = [i_1, i_2, \ldots]$, then we call i_k *the* i_k*-th component of* P, $[i_1, i_2, \ldots]$ *the coordinate of* P. For convenience, we denote it by $P = [i_1, i_2, \ldots]$. By definition, we have $P = [0, 0, \ldots, 0] \iff P \in E(K)_0$.

$$\begin{tikzcd}
E(K) \arrow[r, "sp"] & \prod_v \left(f^{-1}(v)\right)^\sharp \arrow[r, two heads] & \prod_v G_v \\
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\
P \arrow[r, mapsto] & \left((P) \cap f^{-1}(v)\right)_v \arrow[r, mapsto] & \overline{sp}(P) = [i_1, i_2, \ldots]
\end{tikzcd}$$

1.3 Proof of Theorem1.1

What we have to do is as follows:

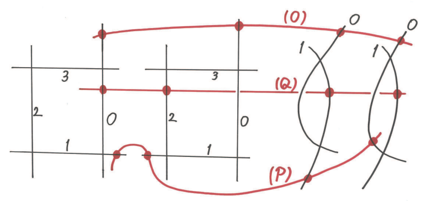
① to compute the coordinate $[i_1, i_2, \ldots,]$ of section P (especially coordinates of generators),
② to find all sections which are disjoint from (O),
③ to find maximal disjoint configurations $\mathfrak{S}'$ of (-1)-curves.

For the case of $r = 0$,

- ② is equivalent to find all non-zero sections.
- ③ $\mathfrak{S}'$ coincides with $E(K)$

by Proposition 1.4(2). As for ①, we prove it in the case No.74. Other cases are similar.

Example 1.5 (No.74 $(A_3 \oplus A_1)^{\oplus 2}$ $E(K) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$)**.**



The Mordell-Weil group $E(K)$ is generated by a section P of order 4 and Q of order 2. For a suitable choice of labeling, the coordinates of P and Q are $[1, 1, 1, 0]$ and $[0, 2, 1, 1]$, respectively.

i_1	0	1	2	3	i_2	0	1	2	3	i_3	0	1	i_4	0	1
contr	0	3/4	1	3/4	*contr*	0	3/4	1	3/4	*contr*	0	1/2	*contr*	0	1/2

Proof. By symmetry, we can assume that the coordinate $[i_1, i_2, i_3, i_4]$ of P satisfies $0 \leq i_1 \leq i_2 \leq 2$ and $i_3 \geq i_4$. Moreover, since P is of order 4, we have $(i_1, i_2) = (0, 1), (1, 1), (1, 2)$. However by Proposition 1.4(2), we only have $P = [1, 1, 1, 0]$.

By symmetry again, we can assume that the coordinate $[i_1, i_2, i_3, i_4]$ of Q satisfies $0 \leq i_1 \leq i_2 \leq 3$. Moreover, since Q is of order 2, we have $(i_1, i_2) = (0, 0), (0, 2), (2, 2)$. However by Proposition 1.4(2), we have $Q = [0, 2, 1, 1]$ or $[2, 2, 0, 0] (= 2P)$. However $Q = 2P$ would contradict how we choose P and Q, hence we have $Q = [0, 2, 1, 1]$. □

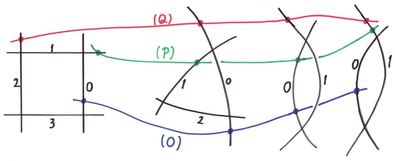
For the case of $r = 1$, we need the following lemma.

Lemma 1.6. A section (P) is disjoint from (O) if P attains the minimal height among all non-torsion sections.

Proof. This is implicitly proved in [5, Section 5]. □

As a sample, we prove the case No.59. Other cases are similar.

Example 1.7 (No.59 $A_3 \oplus A_2 \oplus {A_1}^{\oplus 2}$).



The Mordell-Weil group $E(K)$ is generated by a section P of height $1/12$ and Q of order 2. Note that section (P) does not intersect the zero-section (O) by Lemma 1.6. For a suitable choice of labeling, we have

$$① \begin{cases} E(K)_{tor} = \Big\{[0,0,0,0],\ [2,0,1,1]\Big\}, \\ P + E(K)_{tor} = \Big\{[1,1,0,1], [3,1,1,0]\Big\}. \end{cases}$$

② The sections which are disjoint from (O) are as follows.

$$\mathfrak{S} = \left\{ \begin{array}{ccccccccc} -4P = [0,2,0,0] & -3P = [1,0,0,1] & -2P = [2,1,0,0] & -P = [3,2,0,1] & & P = [1,1,0,1] & 2P = [2,2,0,0] & 3P = [3,0,0,1] & 4P = [0,1,0,0] \\ & -3P+Q = [3,0,1,0] & -2P+Q = [0,1,1,1] & -P+Q = [1,2,1,0] & Q = [2,0,1,1] & P+Q = [3,1,1,0] & 2P+Q = [0,2,1,1] & 3P+Q = [1,0,1,0] & \end{array} \right\}.$$

$$③\ \mathfrak{S}' = \left\{ \begin{array}{l} \left\{ \begin{array}{cccc} O = [0,0,0,0] & P = [1,1,0,1] & 2P = [2,2,0,0] & 3P = [3,0,0,1] \\ Q = [2,0,1,1] & P+Q = [3,1,1,0] & 2P+Q = [0,2,1,1] & 3P+Q = [1,0,1,0] \end{array} \right\} \\ \text{or} \\ \left\{ \begin{array}{ccccc} O = [0,0,0,0] & P = [1,1,0,1] & 2P = [2,2,0,0] & 3P = [3,0,0,1] & 4P = [0,1,0,0] \\ & P+Q = [3,1,1,0] & 2P+Q = [0,2,1,1] & 3P+Q = [1,0,1,0] & \end{array} \right\} \end{array} \right..$$

i_1	0	1, 3	2	i_2	0	1, 2	i_3	0	1	i_4	0	1
$contr_{v_1}$	0	3/4	1	$contr_{v_2}$	0	2/3	$contr_{v_3}$	0	1/2	$contr_{v_4}$	0	1/2

Proof. ① $\underline{\text{coordinate of } E(K)_{tor}}$: By Proposition 1.4(2), we have

$$0 = 2 - \sum_{v \in R} contr_v(Q).$$

Therefore,

$$\big(contr_{v_1}(Q), contr_{v_2}(Q), contr_{v_3}(Q), contr_{v_4}(Q)\big) = \left(1,\ 0,\ \frac{1}{2},\ \frac{1}{2}\right)$$

It follows that $E(K)_{tor} = \{[0,0,0,0],\ [2,0,1,1]\}\ \big(Q = [2,0,1,1]\big)$.

$\underline{\text{coordinate of } P + E(K)_{tor}}$: Since $\langle P,\ P\rangle = 1/12$ and $(P).(O) = 0$ by Lemma 1.6, we have

$$\frac{1}{12} = 2 - \sum_{v \in R} contr_v(P)$$

by Proposition 1.4(1). Hence we have

$$\big(contr_{v_1}(P), contr_{v_2}(P), contr_{v_3}(P), contr_{v_4}(P)\big) = \left(\frac{3}{4},\ \frac{2}{3},\ \frac{1}{2},\ 0\right), \left(\frac{3}{4},\ \frac{2}{3},\ 0,\ \frac{1}{2}\right).$$

Therefore $P+E(K)_{tor} \subseteq \Big\{[1 \text{ or } 3, 1 \text{ or } 2, 1, 0],\ [1 \text{ or } 3, 1 \text{ or } 2, 0, 1]\Big\}$. Since the order of $P+E(K)_{tor}$ is 2, we have

$$\begin{aligned} P + E(K)_{tor} = &\Big\{[1,1,1,0],\ [3,1,0,1]\Big\},\ \Big\{[1,1,0,1],\ [3,1,1,0]\Big\}, \\ &\Big\{[1,2,0,1],\ [3,2,1,0]\Big\},\ \Big\{[1,2,1,0],\ [3,2,0,1]\Big\} \end{aligned}$$

By symmetry, without loss of generality, we may assume that the right-hand contains $[1,1,0,1]$. It follows that $P + E(K)_{tor} = \Big\{[1,1,0,1],[3,1,1,0]\Big\}$ $\big(P = [1,1,0,1]\big)$.

② For all $m \in \mathbb{Z}$, if there exists an element in $mP + E(K)_{tor}$ which does not intersect (O), then

$$\frac{m^2}{12} = 2 - \sum_{v \in R} contr_v(mP) \leq 2$$

by Proposition 1.4(1). Therefore $m = 0, \pm 1, \pm 2, \pm 3, \pm 4$, and we have

$$\mathfrak{S} \subseteq \left\{ \begin{array}{ccccccccc} -4P=[0,2,0,0] & -3P=[1,0,0,1] & -2P=[2,1,0,0] & -P=[3,2,0,1] & & P=[1,1,0,1] & 2P=[2,2,0,0] & 3P=[3,0,0,1] & 4P=[0,1,0,0] \\ -4P+Q=[2,2,1,1] & -3P+Q=[3,0,1,0] & -2P+Q=[0,1,1,1] & -P+Q=[1,2,1,0] & Q=[2,0,1,1] & P+Q=[3,1,1,0] & 2P+Q=[0,2,1,1] & 3P+Q=[1,0,1,0] & 4P+Q=[2,1,1,1] \end{array} \right\}.$$

More precisely, $\pm 4P + Q$ are not contained in $\mathfrak{S}$. In fact, as for $-4P + Q = [2,2,1,1]$,

$$\frac{(-4)^2}{12} = 2 + 2(-4P+Q).\ (O) - 1 - \frac{2}{3} - \frac{1}{2} - \frac{1}{2}$$

by Proposition 1.4(1). Therefore we have $(-4P+Q).\ (O) = 1$. Other sections in the right-hand are contained in $\mathfrak{S}$ by similar argument. Hence, we have

$$\mathfrak{S} = \left\{ \begin{array}{ccccccccc} -4P=[0,2,0,0] & -3P=[1,0,0,1] & -2P=[2,1,0,0] & -P=[3,2,0,1] & & P=[1,1,0,1] & 2P=[2,2,0,0] & 3P=[3,0,0,1] & 4P=[0,1,0,0] \\ & -3P+Q=[3,0,1,0] & -2P+Q=[0,1,1,1] & -P+Q=[1,2,1,0] & Q=[2,0,1,1] & P+Q=[3,1,1,0] & 2P+Q=[0,2,1,1] & 3P+Q=[1,0,1,0] & \end{array} \right\}.$$

③ By using ②, we see that (Q) intersect $(4P)$. In fact, considering translation by $-Q$, what we have to show is equivalent to the fact that (O) intersects $(4P - Q) = (4P + Q)$, which we already have proved it in ②. By this argument, we have that $\mathfrak{S}'$ is one of the following modulo translation.

$$\mathfrak{S}' = \left\{ \begin{array}{l} \left\{ \begin{array}{cccc} O=[0,0,0,0] & P=[1,1,0,1] & 2P=[2,2,0,0] & 3P=[3,0,0,1] \\ Q=[2,0,1,1] & P+Q=[3,1,1,0] & 2P+Q=[0,2,1,1] & 3P+Q=[1,0,1,0] \end{array} \right\} \\ \text{or} \\ \left\{ \begin{array}{ccccc} O=[0,0,0,0] & P=[1,1,0,1] & 2P=[2,2,0,0] & 3P=[3,0,0,1] & 4P=[0,1,0,0] \\ & P+Q=[3,1,1,0] & 2P+Q=[0,2,1,1] & 3P+Q=[1,0,1,0] & \end{array} \right\} \end{array} \right. .$$

□

2 Determinations of pencils of cubic curves

It is well known that almost all pencils of cubic curves[*1] Λ induce the following diagram :

$$\begin{tikzcd} S \arrow[d, "\epsilon"] \arrow[dr, "f"] & \\ \mathbb{P}^2 \arrow[r, dotted, "\Lambda"] & \mathbb{P}^1 \end{tikzcd}$$

f : rational elliptic surface,

ϵ : blowing-up nine times.

More precisely, if we blow up the ambient plane $\mathbb{P}^2$ at the nine points of intersection of any two distinct members of the pencil, we obtain a rational elliptic surface. In other words, there exists a correspondence Φ between pencils of cubic curves and rational elliptic surfaces.

$$\begin{array}{cccc} \Phi : & \big\{\text{almost every cubic pencil on } \mathbb{P}^2\big\} & \longrightarrow & \big\{\text{rational elliptic surfaces}\big\} \\ & \Cup & & \Cup \\ & \Lambda & \longmapsto & f : S \to \mathbb{P}^1 \end{array}$$

Conversely, any rational elliptic surface with a section arises in this way, that is, Φ is surjective.

The structures of dual graphs in Theorem 1.1 tell us Φ^{-1} explicitly, that is, for a given rational elliptic surface S, we obtain pencils of cubic curves by blowing S down nine times. To sum up, we obtain the following :

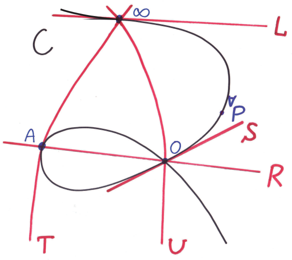
Theorem 2.1. When $r = 0$, 1, some examples of pencils of cubic curves corresponding to each rational elliptic surface are as follows (see Table 3, 4).

[*1] More precisely, generators C, C' of Λ satisfy the condition :

$$\forall p \in C \cap C', \min\big\{\mathrm{mult}_p C, \mathrm{mult}_p C')\big\} = 1.$$

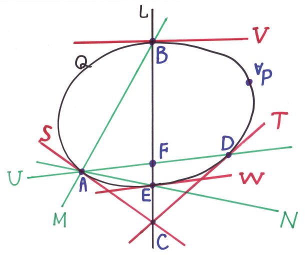
Each figure in Table 3, 4 is drawn based on one of the four patterns (i)-(iv).

(i)



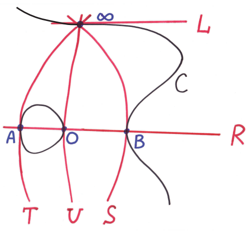
$C : y^2z - x^3 - x^2z = 0$ $\quad O = (0:0:1)$: node
$A = (-1:0:1)$ $\quad \infty = (0:1:0)$: inflexion
$P = (a^2-1 : a(a^2-1) : 1)$ $\quad \infty' \neq \infty$: inflexion
$L : z = 0$ $\quad R : y = 0$
$S : x - y = 0$ $\quad T : x + z = 0$
$U : x = 0$ $\quad \overline{PP} : (3a^2-1)x - 2ay - (a^2-1)^2z = 0$

(ii)



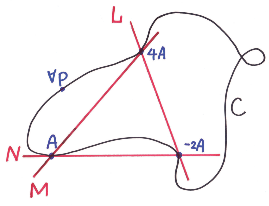
$Q : z^2 + xy = 0$ $\quad L : x + y = 0$ $\quad A = (0:1:0)$
$B = (1:-1:1)$ $\quad C = (0:0:1)$ $\quad D = (1:0:0)$
$E = (1:-1:-1)$ $\quad F = (1:-1:0)$ $\quad P = (-a^2:1:a)$
$M : x - z = 0$ $\quad N : x + z = 0$ $\quad S : x = 0$
$T : y = 0$ $\quad U : z = 0$ $\quad V : x - y - 2z = 0$
$W : x - y + 2z = 0$ $\quad \overline{PP} : x - a^2y + 2az = 0$

(iii)



$C : y^2z - x^3 - xz^2 = 0$ $\quad O = (0:0:1)$
$A = (-1:0:1)$ $\quad B = (1:0:1)$ $\quad \infty = (0:1:0)$: inflexion
$L : z = 0$ $\quad R : y = 0$ $\quad S : x - z = 0$
$T : x + z = 0$ $\quad U : x = 0$

(iv)

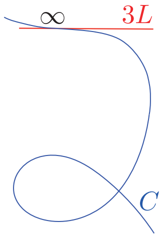


$C : x^2y + y^2z + z^2x - 3xyz = 0$ $\quad A = (1:0:0)$: order 9
$-2A = (0:1:0)$ $\quad 4A = (0:0:1)$
$L : x = 0$ $\quad M : y = 0$ $\quad N : z = 0$

Table3 case of $r = 0$

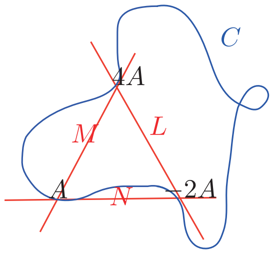
No.62 E_8

(i)



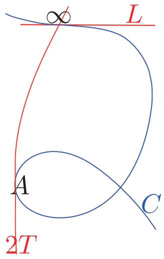
No.63 A_8

(iv)



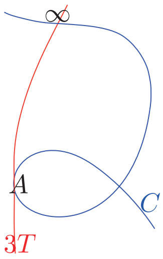
No.64 D_8

(i)

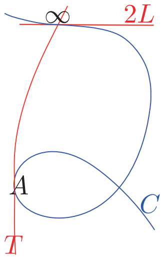


No.65 $E_7 \oplus A_1$

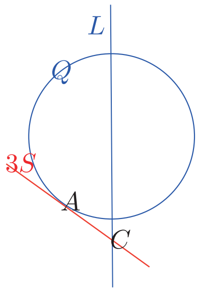
(i)



(i)

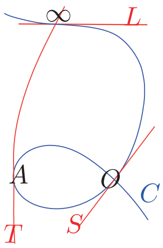


(ii)

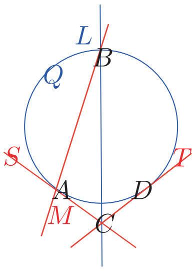


No.66 $A_5 \oplus A_2 \oplus A_1$

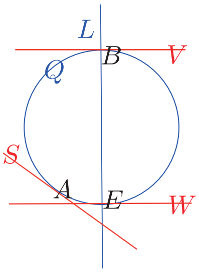
(i)



(ii)

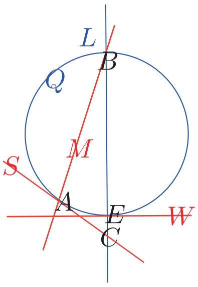


(ii)



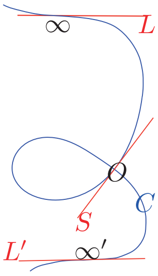
No.67 $A_4^{\oplus 2}$

(ii)



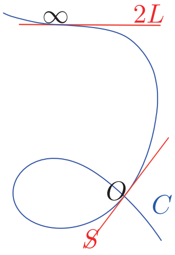
No.68 $A_2^{\oplus 4}$

(i)

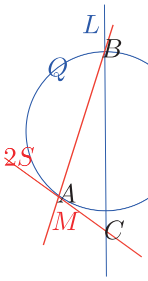


No.69 $E_6 \oplus A_2$

(i)

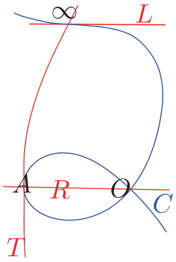


(ii)

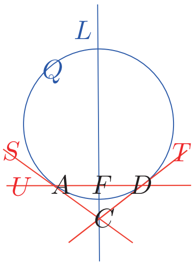


No.70 $A_7 \oplus A_1$

(i)

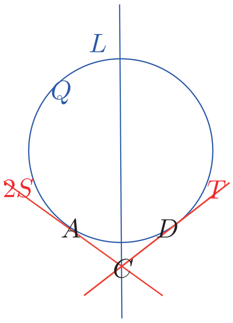


(ii)

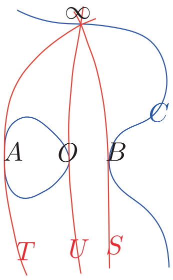


No.71 $D_6 \oplus {A_1}^{\oplus 2}$

(ii)

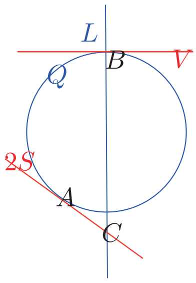


(iii)

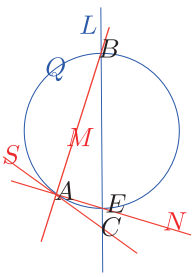


No.72 $D_5 \oplus A_3$

(ii)

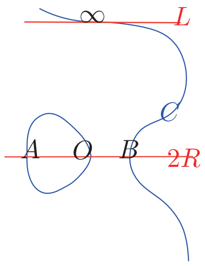


(ii)



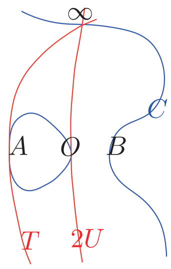
No.73 ${D_4}^{\oplus 2}$

(iii)



∞
L
C
A
O
B
$2R$

(iii)



∞
C
A
O
B
T
$2U$

No.74 $(A_3 \oplus A_1)^{\oplus 2}$

(ii)

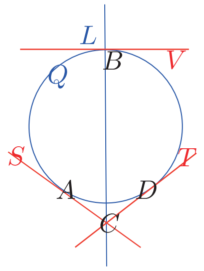
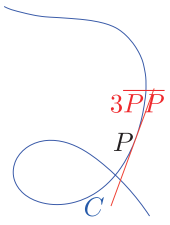


Table4 case of $r = 1$

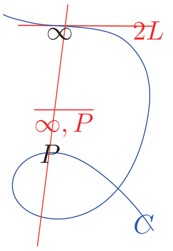
No.43 E_7

(i)



$3\overline{PP}$
P
C

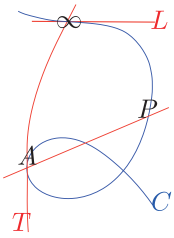
(i)



∞
$2L$
$\overline{\infty, P}$
P
C

No.44 A_7

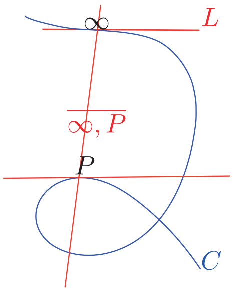
(i)



∞
L
P
A
T
C

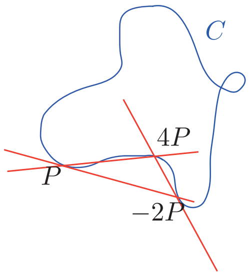
No.45 A_7

(i)



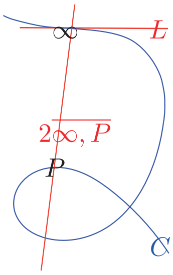
∞
L
$\overline{\infty, P}$
P
C

(iv)



No.46 D_7

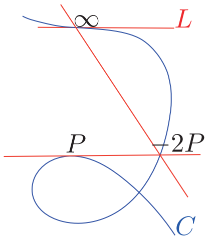
(i)



∞
L
$2\overline{\infty, P}$
P
C

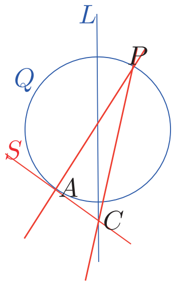
No.47 $A_6 \oplus A_1$

(i)

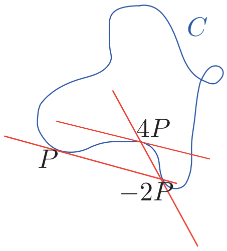


∞
L
P
$-2P$
C

(ii)

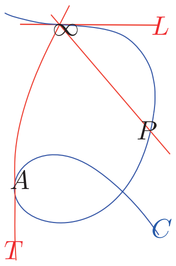


(iv)



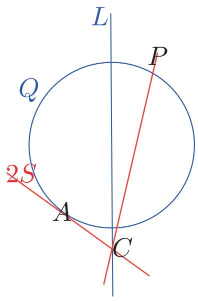
No.48 $D_6 \oplus A_1$

(i)

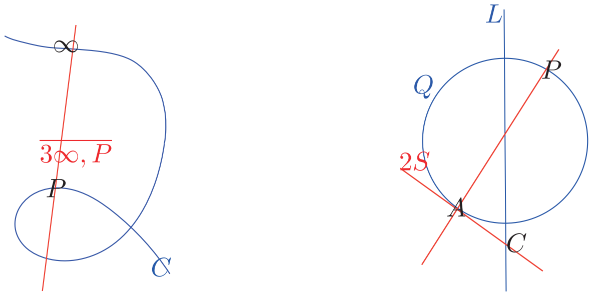


∞
L
P
A
T
C

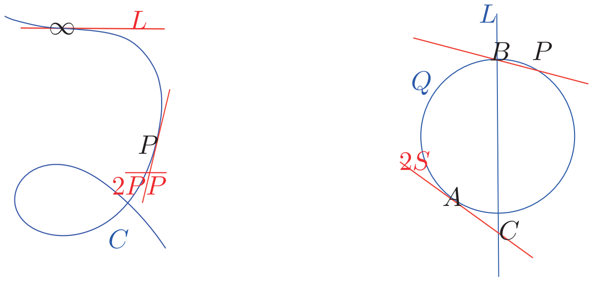
(ii)



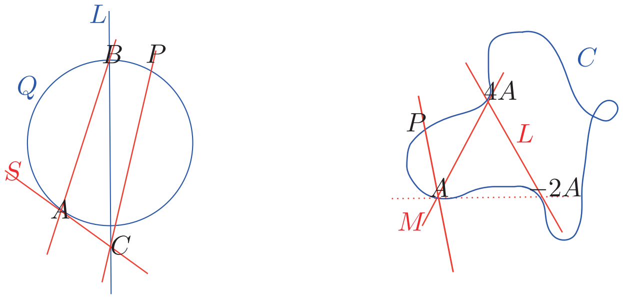
No.49 $E_6 \oplus A_1$


(i) (ii)
∞
$\overline{3\infty, P}$
P
C
L
Q
P
2S
A
C

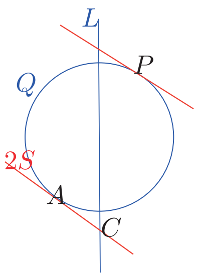
No.50 $D_5 \oplus A_2$


(i) (ii)
∞
L
P
$\overline{2P, P}$
C
L
B
P
Q
2S
A
C

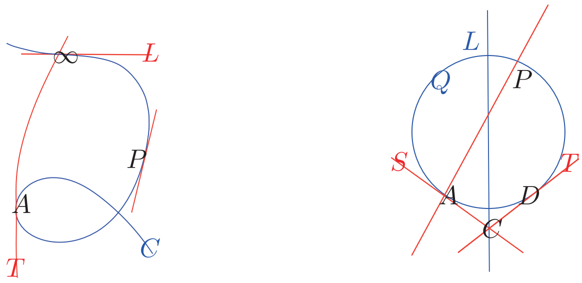
No.51 $A_5 \oplus A_2$


(ii) (iv)
L
B
P
Q
S
A
C
C
4A
P
L
A
2A
M

No.52 $D_5 \oplus {A_1}^{\oplus 2}$

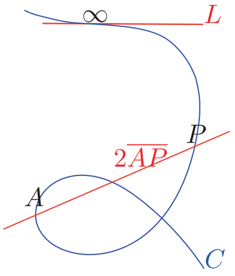

(ii)

No.53 $A_5 \oplus {A_1}^{\oplus 2}$

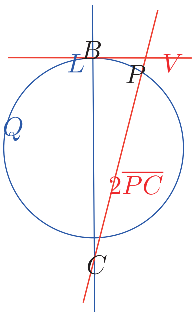

(i) (ii)
∞
L
P
A
C
T
L
Q
P
S
T
A
D
C

No.54 $D_4 \oplus A_3$

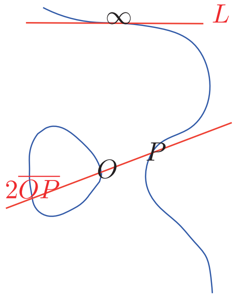
(i)


∞
L
P
$2\overline{AP}$
A
C

(ii)

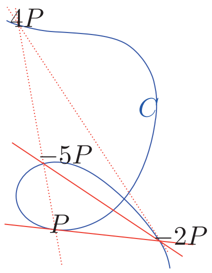

B
L
P
V
Q
$2\overline{PC}$
C

(iii)

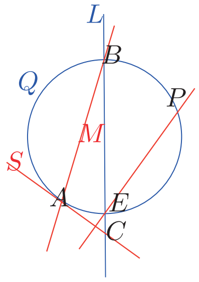

∞
L
$2\overline{OP}$
O
P

No.55 $A_4 \oplus A_3$

(i)

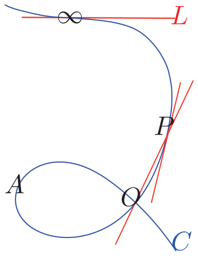


(ii)

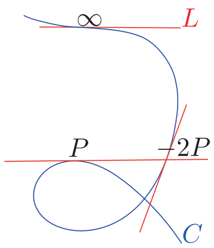


No.56 $A_4 \oplus A_2 \oplus A_1$

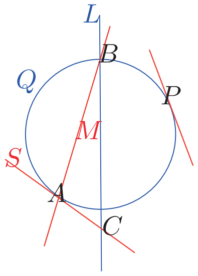
(i)


∞
L
P
A
O
C

(i)

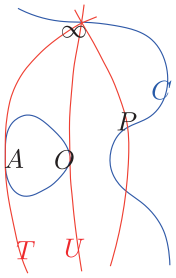

∞
L
P
$-2P$
C

(ii)



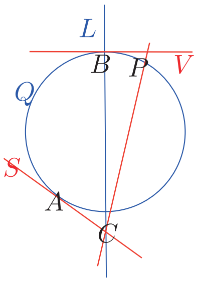
No.57 $D_4 \oplus {A_1}^{\oplus 3}$

(iii)

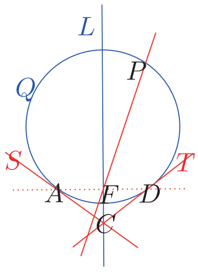

∞
C
P
A
O
T
U

No.58 ${A_3}^{\oplus 2} \oplus A_1$

(ii)

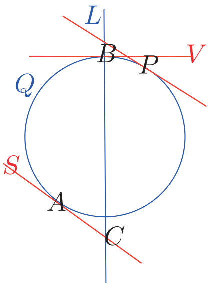


(ii)

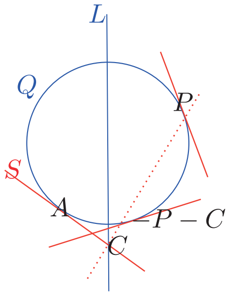


No.59 $A_3 \oplus A_2 \oplus {A_1}^{\oplus 2}$

(ii)

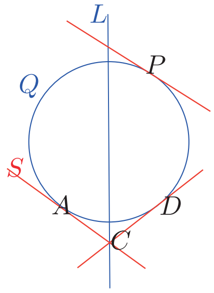


(ii)

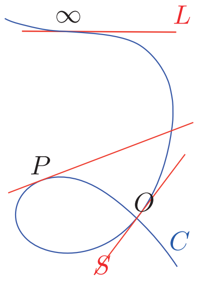


No.60 $A_3 \oplus {A_1}^{\oplus 4}$

(ii)



No.61 ${A_2}^{\oplus 3} \oplus A_1$
(i)



∞
L
P
O
C
S

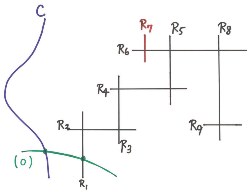
Corollary 2.2. For each class of rational elliptic surface of Mordell-Weil rank 1, we obtain a 1-parameter family defined over $\mathbb{Q}$.

The existence is already proved in Shioda [7], but explicit equations are not given there.

2.1 Proof of Theorem 2.1 : the construction of pencils of cubic curves

In this section, we explain the way of constructing pencils of cubic curves from the dual graphs which are determined in Theorem 1.1.

Example 2.3. (1) $\Phi^{-1}(E_8) = \{\langle C,\ 3L\rangle\}$



$\xrightarrow{\text{9 blow-downs}}$

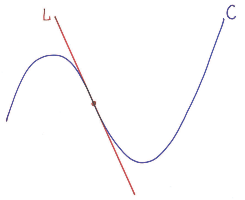
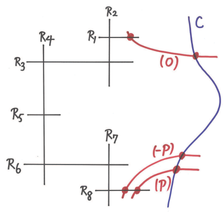


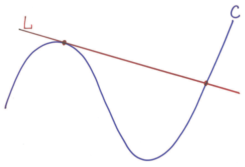
Figure9 graph of E_8

Figure10 pencil of cubic curves of E_8

(2) $\Phi^{-1}(E_7) = \Big\{\langle C,\ 3L\rangle\,,\ \langle C,\ 2L+T\rangle\Big\}$



$\xrightarrow{\text{9 blow-downs}}$



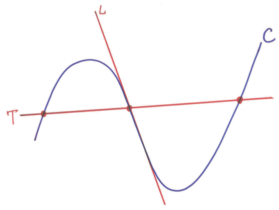


Figure11 dual graph of E_7

Figure12 $\langle C,\ 3L\rangle$

Figure13 $\langle C,\ 2L+T\rangle$

Proof. (1) Considering the self-intersection number of irreducible components, we can blow down in the order

$$(O) \to R_1 \to R_2 \to R_3 \to R_4 \to R_5 \to R_6 \to R_8 \to R_9$$

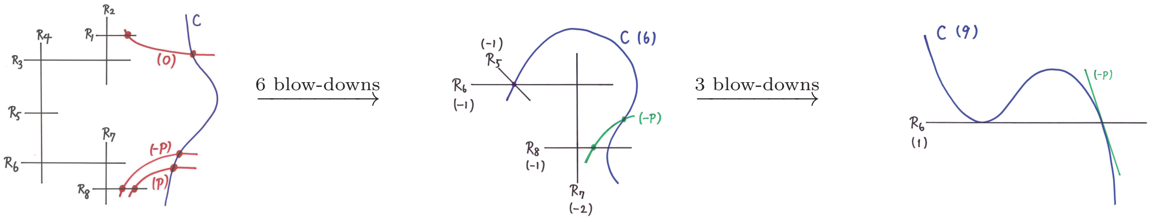
nine times and there is no other way (L corresponds to R_7 (multiplicity 3)).

(2) (i) When we contract only one section,

it is easy to see that we cannot blow S down nine times.

(ii) When we contract two sections,

In the following figures, ( ) denotes the self-intersection number of corresponding curve.


R2
R4
R1
R3
C
(O)
R5
R7
R6
(-P)
R8
(P)
6 blow-downs
(-1)
R5
R6
(-1)
C (6)
(-P)
R8
(-1)
R7
(-2)
3 blow-downs
C (9)
(-P)
R6
(1)

It is easy to see that if both (P) and $(-P)$ are contracted, we cannot blow S down nine times, so we can assume that both (O) and (P) are contracted. Moreover by symmetry, we can assume that we contract five sections (O), R_1,..., R_4, and the section (P).

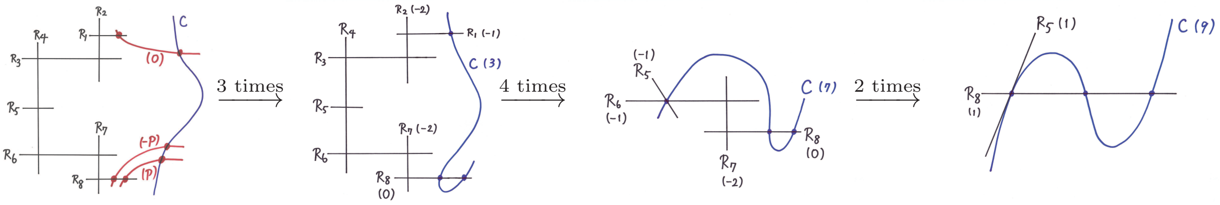
$$(O) \longrightarrow R_1 \longrightarrow R_2 \longrightarrow R_3 \longrightarrow R_4 \longrightarrow (P).$$

We have to blow S down three more times, but the way to blow down is only as follows :

$$R_5 \longrightarrow R_8 \longrightarrow R_7$$

(L corresponds to R_6).

(iii) When we contract three sections,


R2
R4
R1
R3
C
(O)
R5
R7
R6
(-P)
R8
(P)
3 times
R2 (-2)
R4
R1 (-1)
R3
C (3)
R5
R7 (-2)
R6
R8
(0)
4 times
(-1)
R5
R6
(-1)
C (7)
R8
(0)
R7
(-2)
2 times
R5 (1)
C (9)
R8
(1)

First, we contract three sections $(O), (P), (-P)$:

$$(O) \longrightarrow (P) \longrightarrow (-P).$$

Secondly, four sections $R_1, \ldots, R_4$:

$$R_1 \longrightarrow R_2 \longrightarrow R_3 \longrightarrow R_4.$$

Finally, two sections $R_6, \ldots, R_7$:

$$R_6 \longrightarrow R_7$$

($L := R_5,\ T := R_8,\ R_5$ is the tanengt line at the inflexion point of C).

□

2.2 Proof of Theorem 2.1 : verification with Magma

In the previous subsection 2.1, we gave how to obtain pencils of cubic curves, but they give little information on other singular fibers. Now we prove Theorem 2.1 with the computational algebra system *Magma*[*2]. For every pencil of cubic curves $\langle C, C' \rangle$, Magma tells us the Weierstrass equation, Kodaira symbols and Bad places of the rational elliptic surface corresponding to the pencil of cubic curves :

① Give a equation of $C,\ C'$ which are generators of the pencil of cubic curves.

$$\begin{cases} C \ : \ g(x,y,z) = 0 \\ C' \ : \ h(x,y,z) = 0 \end{cases}$$

② Find a $k(t)$-rational point on the elliptic curve $g + th = 0$. (t is a parameter in $\mathbb{P}^1$)

③ Input in Magma[*3] as follows.

```
> Q1 := RationalField();
> Q2<a> :=FunctionField(Q1,1);
> R<t> := FunctionField(Q2);
> P<x,y,z> := ProjectiveSpace(R,2);
> C := Curve(P, g + th = 0 );              %input g, h
> pt := C![ , , ];              %input a k(t)-rational point
> E := EllipticCurve(C, pt);
> E;                             %output Weierstrass equation of the elliptic curve C
> KodairaSymbols(E);             %output Kodaira Symbols of the elliptic curve C
> BadPlaces(E);                  %output the point t in P^1 whose fiber is singular
```

We prove Theorem 2.1 in the case No.62, 43 as a sample. Other cases are similar.

Example 2.4. We verify the pencils of cubic curves obtained in Example 2.3

(1) The pencil of cubic curves (i) $\langle C,\ 3L \rangle$ corresponds to E_8(No.62).

(2) The pencil of cubic curves (i) $\langle C,\ 3\overline{PP} \rangle$ corresponds to E_7(No.43).

(3) The pencil of cubic curves (i) $\langle C,\ 2L + \overline{\infty, P} \rangle$ also corresponds to E_7(No.43).

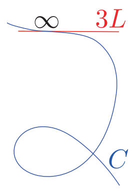


Figure14 (i) $\langle C,\ 3L \rangle$

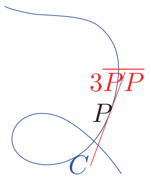


Figure15 (i) $\langle C,\ 3\overline{PP} \rangle$

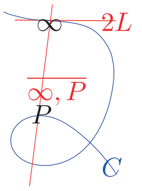


Figure16 (i) $\langle C,\ 2L + \overline{\infty, P} \rangle$

[*2] [1].

[*3] `http://magma.maths.usyd.edu.au/calc/`

Proof. (1) We give equations of this pencil of cubic curves as follows.

$$\begin{cases} C : y^2z - x^3 - x^2z = 0 \\ \infty := (0 : 1 : 0) \\ L : z = 0 \end{cases}$$

```
> Q1 := RationalField();
> Q2<a> :=FunctionField(Q1,1);
> R<t> := FunctionField(Q2);
> P<x,y,z> := ProjectiveSpace(R,2);
> C := Curve(P, y^2*z-x^3-x^2*z+t*z^3);
> pt := C![0, 1, 0];
> E := EllipticCurve(C, pt);
> E;
> KodairaSymbols(E);
> BadPlaces(E);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Elliptic Curve defined by y^2 = x^3 + x^2 - t over Univariate rational function
field over Multivariate rational function field of rank 1 over Rational Field
[ <II*, 1>, <I1, 1>, <I1, 1> ]
[
    1/t,
    t - 4/27,
    t
]
```

(2) We give equations of this pencil of cubic curves as follows (a is a parameter).

$$\begin{cases} C : y^2z - x^3 - x^2z = 0 \\ P := (a^2 - 1 : a(a^2 - 1) : 1) \\ \overline{PP} : (3a^2 - 1)x - 2ay - (a^2 - 1)^2z = 0 \end{cases}$$

```
> Q1 := RationalField();
> Q2<a> :=FunctionField(Q1,1);
> R<t> := FunctionField(Q2);
> P<x,y,z> := ProjectiveSpace(R,2);
> C := Curve(P, y^2*z-x^3-x^2*z+t*((3*a^2-1)*x-2*a*y-(a^2-1)^2*z)^3);
> pt := C![a^2-1, a*(a^2-1), 1];
> E := EllipticCurve(C, pt);
> E;
> KodairaSymbols(E);
> BadPlaces(E);
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Elliptic Curve defined by y^2 = x^3 + (9/16*a^4 + 3/8*a^2 + 1/16)/(a^18 - 6*a^16
    + 15*a^14 - 20*a^12 + 15*a^10 - 6*a^8 + a^6)/t^2*x^2 + (-729/256*a^12 -
    729/128*a^10 - 1215/256*a^8 - 135/64*a^6 - 135/256*a^4 - 9/128*a^2 -
    1/256)/(a^30 - 9*a^28 + 36*a^26 - 84*a^24 + 126*a^22 - 126*a^20 + 84*a^18 -
    36*a^16 + 9*a^14 - a^12)/t^3*x + (729/4096*a^12 + 729/2048*a^10 +
    1215/4096*a^8 + 135/1024*a^6 + 135/4096*a^4 + 9/2048*a^2 + 1/4096)/(a^42 -
    12*a^40 + 66*a^38 - 220*a^36 + 495*a^34 - 792*a^32 + 924*a^30 - 792*a^28 +
    495*a^26 - 220*a^24 + 66*a^22 - 12*a^20 + a^18)/t^5 over Univariate rational
function field over Multivariate rational function field of rank 1 over Rational
Field
[ <I1, 2>, <III*, 1>, <I1, 1> ]
[
    t^2 + (-1/108*a^12 + 1/54*a^8 + 13/81*a^6 - 17/108*a^4 + 19/243*a^2 -
        2/729)/(a^18 - a^16 - 4/3*a^14 + 20/27*a^12 + 26/27*a^10 + 2/81*a^8 -
        188/729*a^6 - 28/243*a^4 - 5/243*a^2 - 1/729)*t - 1/729/(a^18 - a^16 -
        4/3*a^14 + 20/27*a^12 + 26/27*a^10 + 2/81*a^8 - 188/729*a^6 - 28/243*a^4
        - 5/243*a^2 - 1/729),
    1/t,
    t
]
```

(3) We give equations of this pencil of cubic curves as follows (a is a parameter).

$$\begin{cases} C : y^2z - x^3 - x^2z = 0 \\ P := (a^2-1 : a(a^2-1) : 1) \\ \infty := (0:1:0) \\ L : z = 0 \\ \overline{\infty, P} : x - (a^2-1)z = 0 \end{cases}$$

```
> Q1 := RationalField();
> Q2<a> :=FunctionField(Q1,1);
> R<t> := FunctionField(Q2);
> P<x,y,z> := ProjectiveSpace(R,2);
> C := Curve(P, y^2*z-x^3-x^2*z+t*(z^2*(x-(a^2-1)*z)));
> pt := C![0, 1, 0];
> E := EllipticCurve(C, pt);
> E;
> KodairaSymbols(E);
> BadPlaces(E);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Elliptic Curve defined by y^2 = x^3 + x^2 - t*x + (a^2 - 1)*t over Univariate
```

```
rational function field over Multivariate rational function field of rank 1 over
Rational Field
[ <III*, 1>, <I1, 1>, <I1, 2> ]
[
    1/t,
    t,
    t^2 + (-27/4*a^4 + 9*a^2 - 2)*t - a^2 + 1
]
```

□

3 Specialization and generalization

3.1 The main result

By Example 2.3, we see that pencils of cubic curves of E_7 are generalized to that of E_8. This suggests us that for trivial lattices T, T' of rank 7, 8 respectively, if the diagram of trivial lattice T' is contained in that of T, there would exist specializations/generalizations between corresponding pencils of cubic curves, which is true as follows :

Theorem 3.1. Let T, T' be root lattices of rank 7, 8 respectively. If the diagram of T' is contained in that of $T \subset E_8$, then there exist pencils of cubic curves Λ, Λ' corresponding to $T \subset E_8$, $T' \subset E_8$ such that Λ specializes to Λ'. Note that the diagram inclusion induces a primitive embedding $T \hookrightarrow T'$ of root lattices.

Proof. See Table 5. □

How to read Table 5

- As for $C, L, Q, S, A, B, \ldots$, see the figures (i)-(iv) below Theorem 2.1.
- We give some examples.

  (1) $\langle C,\ 3\overline{PP}_\infty \rangle$ (see No.62 and No.43 in Table 5.)

  This means that the pencil of cubic curves $\langle C,\ 3\overline{PP} \rangle$ corresponds to the rational elliptic surface whose trivial lattice T is E_7(No.43), and as $P \to \infty$ along C, its limit $\langle C,\ 3L \rangle$ corresponds to the rational elliptic surface whose T is E_8(No.62), where $\overline{PP}$ is the tangent line of C at P.

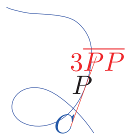


Figure17 (i)$\langle C,\ 3\overline{PP} \rangle$

$\xrightarrow{P \to \infty}$

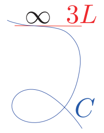


Figure18 (i)$\langle C,\ 3L \rangle$

  (2) $\langle Q + L,\ S + \left(\overline{A, P} + \overline{C, P}\right)_A \rangle$ (see No.65 and No.47 in Table 5.)

  This means that the pencil of cubic curves $\langle Q + L,\ S + \overline{A, P} + \overline{C, P} \rangle$ corresponds to the rational elliptic surface whose trivial lattice T is $A_6 \oplus A_1$(No.47), and as $P \to A$ along the conic Q, its limit $\langle Q + L,\ 3S \rangle$ corresponds to the rational elliptic surface whose T is $E_7 \oplus A_1$(No.65).

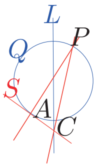


Figure19 (ii)$\langle Q + L,\ S + \overline{A, P} + \overline{C, P} \rangle$

$\xrightarrow{P \to A}$

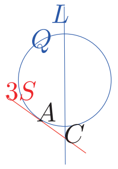


Figure20 (ii)$\langle Q + L,\ 3S \rangle$

Table5 specialization and generalization

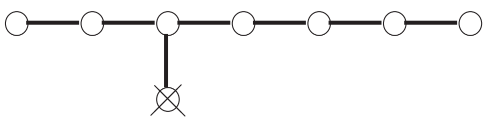
No.	$r = 0$	No.	$r = 1$	pencil of cubic curves (see the figures (i)-(iv) below Theorem2.1)
62	E_8	43	E_7	(i) $\langle C,\ 3\overline{P, P}_{\infty} \rangle$
		45	A_7	(i) $\langle C,\ L + (\overline{\infty, P} + \overline{P, P})_{\infty} \rangle$
		46	D_7	(i) $\langle C,\ L + 2\overline{\infty, P}_{\infty} \rangle$
		47	$A_6 \oplus A_1$	(i) $\langle C,\ L + (\overline{\infty, -2P} + \overline{P, P})_{\infty} \rangle$
		49	$E_6 \oplus A_1$	(i) $\langle C,\ 3\overline{\infty, P}_{\infty} \rangle$
		50	$D_5 \oplus A_2$	(i) $\langle C,\ L + 2\overline{P, P}_{\infty} \rangle$
		55	$A_4 \oplus A_3$	(i) $\langle C,\ (\overline{P, P} + \overline{4P, 4P} + \overline{-5P, -2P})_{\infty} \rangle$
		56	$A_4 \oplus A_2 \oplus A_1$	(i) $\langle C,\ L + (\overline{P, P} + \overline{-2P, -2P})_{\infty} \rangle$
63	A_8	45	A_7	(iv) $\langle C,\ (\overline{P, P} + \overline{-2P, -2P} + \overline{P, 4P})_{A} \rangle$
		47	$A_6 \oplus A_1$	(iv) $\langle C,\ (\overline{P, P} + \overline{-2P, -2P} + \overline{4P, 4P})_{A} \rangle$
		51	$A_5 \oplus A_2$	(iv) $\langle C,\ L + M + \overline{A, P}_{A} \rangle$
		55	$A_4 \oplus A_3$	(iv) $\langle C,\ (\overline{P, P} + \overline{4P, 4P} + \overline{-5P, -2P})_{A} \rangle$
64	D_8	44	A_7	(i) $\langle C,\ L + T + \overline{A, P}_{A} \rangle$
		45	A_7	(i) $\langle C,\ L + (\overline{\infty, P} + \overline{P, P})_{A} \rangle$
		46	D_7	(i) $\langle C,\ L + 2\overline{\infty, P}_{A} \rangle$
		48	$D_6 \oplus A_1$	(i) $\langle C,\ L + T + \overline{\infty, P}_{A} \rangle$
		50	$D_5 \oplus A_2$	(i) $\langle C,\ L + 2\overline{P, P}_{A} \rangle$
		53	$A_5 \oplus {A_1}^{\oplus 2}$	(i) $\langle C,\ L + T + \overline{P, P}_{A} \rangle$
		54	$D_4 \oplus A_3$	(i) $\langle C,\ L + 2\overline{A, P}_{A} \rangle$
		55	$A_4 \oplus A_3$	(i) $\langle C,\ (\overline{P, P} + \overline{4P, 4P} + \overline{-5P, -2P})_{A} \rangle$
65	$E_7 \oplus A_1$	43	E_7	(i) $\langle C,\ 3\overline{P, P}_{A} \rangle$
		47	$A_6 \oplus A_1$	(ii) $\langle Q + L,\ S + (\overline{A, P} + \overline{C, P})_{A} \rangle$
		48	$D_6 \oplus A_1$	(ii) $\langle Q + L,\ 2S + \overline{C, P}_{A} \rangle$
		49	$E_6 \oplus A_1$	(ii) $\langle Q + L,\ 2S + \overline{A, P}_{A} \rangle$
		52	$D_5 \oplus {A_1}^{\oplus 2}$	(ii) $\langle Q + L,\ 2S + \overline{P, P}_{A} \rangle$
		53	$A_5 \oplus {A_1}^{\oplus 2}$	(i) $\langle C,\ L + T + \overline{P, P}_{\infty} \rangle$
		56	$A_4 \oplus A_2 \oplus A_1$	(i) $\langle C,\ L + (\overline{P, P} + \overline{-2P, -2P})_{A} \rangle$
		59	$A_3 \oplus A_2 \oplus {A_1}^{\oplus 2}$	(ii) $\left\langle Q + L,\ S + \left(\overline{P, P} + \overline{(-P - C), (-P - C)}\right)_{A} \right\rangle$
66	$A_5 \oplus A_2 \oplus A_1$	51	$A_5 \oplus A_2$	(ii) $\langle Q + L,\ M + S + \overline{C, P}_{D} \rangle$
		53	$A_5 \oplus {A_1}^{\oplus 2}$	(ii) $\langle Q + L,\ S + T + \overline{A, P}_{B} \rangle$
		56	$A_4 \oplus A_2 \oplus A_1$	(ii) $\langle Q + L,\ M + S + \overline{PP}_{D} \rangle$
		59	$A_3 \oplus A_2 \oplus {A_1}^{\oplus 2}$	(ii) $\langle Q + L,\ V + S + \overline{P, P}_{E} \rangle$
		61	${A_2}^{\oplus 3} \oplus A_1$	(i) $\langle C,\ L + S + \overline{P, P}_{A} \rangle$

No.	$r=0$	No.	$r=1$	pencil of cubic curves (see the figures (i)-(iv) below Theorem2.1)
67	${A_4}^{\oplus 2}$	55	$A_4 \oplus A_3$	(ii) $\langle Q+L,\ M+S+\overline{E,P}_E \rangle$
		56	$A_4 \oplus A_2 \oplus A_1$	(ii) $\langle Q+L,\ M+S+\overline{P,P}_E \rangle$
68	${A_2}^{\oplus 4}$	61	${A_2}^{\oplus 3} \oplus A_1$	(i) $\langle C,\ L+S+\overline{P,P}_{\infty'} \rangle$
69	$E_6 \oplus A_2$	49	$E_6 \oplus A_1$	(ii) $\langle Q+L,\ 2S+\overline{A,P}_B \rangle$
		50	$D_5 \oplus A_2$	(ii) $\langle Q+L,\ 2S+\overline{B,P}_A \rangle$
		51	$A_5 \oplus A_2$	(ii) $\langle Q+L,\ M+S+\overline{C,P}_A \rangle$
		56	$A_4 \oplus A_2 \oplus A_1$	(ii) $\langle Q+L,\ M+S+\overline{P,P}_A \rangle$
		61	${A_2}^{\oplus 3} \oplus A_1$	(i) $\langle C,\ L+S+\overline{P,P}_{\infty} \rangle$
70	$A_7 \oplus A_1$	44	A_7	(i) $\langle C,\ L+T+\overline{A,P}_O \rangle$
		47	$A_6 \oplus A_1$	(ii) $\langle Q+L,\ S+\big(\overline{A,P}+\overline{C,P}\big)_D \rangle$
		53	$A_5 \oplus {A_1}^{\oplus 2}$	(ii) $\langle Q+L,\ S+T+\overline{A,P}_D \rangle$
		56	$A_4 \oplus A_2 \oplus A_1$	(i) $\langle C,\ L+\big(\overline{P,P}+\overline{O,P}\big)_A \rangle$
		58	${A_3}^{\oplus 2} \oplus A_1$	(ii) $\langle Q+L,\ S+T+\overline{F,P}_D \rangle$
71	$D_6 \oplus {A_1}^{\oplus 2}$	48	$D_6 \oplus A_1$	(ii) $\langle Q+L,\ 2S+\overline{C,P}_D \rangle$
		52	$D_5 \oplus {A_1}^{\oplus 2}$	(ii) $\langle Q+L,\ 2S+\overline{P,P}_D \rangle$
		53	$A_5 \oplus {A_1}^{\oplus 2}$	(ii) $\langle Q+L,\ S+T+\overline{A,P}_A \rangle$
		57	$D_4 \oplus {A_1}^{\oplus 3}$	(iii) $\langle C,\ T+U+\overline{B,P}_{\infty} \rangle$
		59	$A_3 \oplus A_2 \oplus {A_1}^{\oplus 2}$	(ii) $\Big\langle Q+L,\ S+\Big(\overline{P,P}+\overline{(-P-C),(-P-C)}\Big)_D \Big\rangle$
		60	$A_3 \oplus {A_1}^{\oplus 4}$	(ii) $\langle Q+L,\ S+T+\overline{P,P}_A \rangle$
72	$D_5 \oplus A_3$	50	$D_5 \oplus A_2$	(ii) $\langle Q+L,\ 2S+\overline{B,P}_B \rangle$
		52	$D_5 \oplus {A_1}^{\oplus 2}$	(ii) $\langle Q+L,\ 2S+\overline{P,P}_B \rangle$
		54	$D_4 \oplus A_3$	(ii) $\langle Q+L,\ V+2\overline{C,P}_A \rangle$
		55	$A_4 \oplus A_3$	(ii) $\langle Q+L,\ M+S+\overline{A,P}_E \rangle$
		58	${A_3}^{\oplus 2} \oplus A_1$	(ii) $\langle Q+L,\ V+S+\overline{C,P}_A \rangle$
		59	$A_3 \oplus A_2 \oplus {A_1}^{\oplus 2}$	(ii) $\langle Q+L,\ V+S+\overline{P,P}_A \rangle$
73	${D_4}^{\oplus 2}$	54	$D_4 \oplus A_3$	(iii) $\langle C,\ L+2\overline{O,P}_B \rangle$
		57	$D_4 \oplus {A_1}^{\oplus 3}$	(iii) $\langle C,\ T+U+\overline{O,P}_{\infty} \rangle$
74	$(A_3 \oplus A_1)^{\oplus 2}$	58	${A_3}^{\oplus 2} \oplus A_1$	(ii) $\langle Q+L,\ V+S+\overline{C,P}_D \rangle$
		59	$A_3 \oplus A_2 \oplus {A_1}^{\oplus 2}$	(ii) $\langle Q+L,\ V+S+\overline{P,P}_D \rangle$
		60	$A_3 \oplus {A_1}^{\oplus 4}$	(ii) $\langle Q+L,\ S+T+\overline{P,P}_B \rangle$

Remark 3.2. In the case of A_7 (No.44, 45), there exist exactly two inequivalent embeddings into E_8 modulo the action of the Weyl group $W(E_8)$. One is primitive and the other is imprimitive. The former is No.45 and the latter is No.44. The diagram of A_7 is contained in that of $E_8, A_8, D_8, A_7 \oplus A_1$.

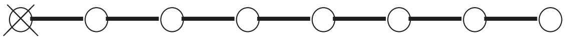
$$E_8 \supset A_7(\text{No.45}), \quad A_8 \supset A_7(\text{No.45}), \quad D_8 \supset A_7(\text{Nos.44, 45}), \quad A_7 \oplus A_1 \supset A_7(\text{No.44}).$$

(1) E_8 ($E(K) = \{O\}$)



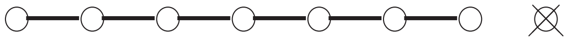
$A_7 \hookrightarrow E_8$: primitive (No.45).

(2) A_8 ($E(K) = \mathbb{Z}/3\mathbb{Z}$)



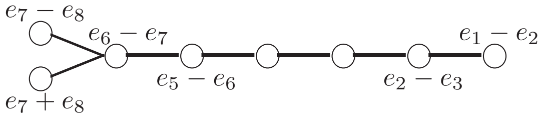
$A_7 \hookrightarrow A_8 \xrightarrow{\text{index 3}} E_8$: primitive (No.45).

(3) $A_7 \oplus A_1$ ($E(K) = \mathbb{Z}/4\mathbb{Z}$)



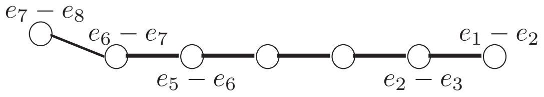
$A_7 \hookrightarrow A_7 \oplus A_1 \xrightarrow{\text{index 4}} E_8$: imprimitive (No.44).

(4) D_8 ($E(K) = \mathbb{Z}/2\mathbb{Z}$)



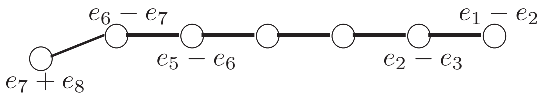
Lattice embeddings $D_8 \hookrightarrow E_8$ is obtained by ading $(\pm e_1 \pm e_2 \cdots \pm e_8)/2$ to the D_8 lattice. There are two such embeddings depending on the parity of numbers of “$-$″ (negative sign). We fix the embedding by adding $(e_1 - e_2 + e_3 - e_4 + e_5 - e_6 + e_7 - e_8)/2$. D_8 contains two subdiagrams of type A_7.

(i)



$A_7 \hookrightarrow D_8 \to E_8$: imprimitive (No.44). In fact, $(e_1 - e_2 + e_3 - e_4 + e_5 - e_6 + e_7 - e_8)/2 \in E_8$ does not belong to A_7, but its twice does.

(ii)



$A_7 \hookrightarrow D_8 \to E_8$: primitive (No.45).

3.2 Semi-Hesse pencil (another way to specialize ${A_2}^{\oplus 3} \oplus A_1$(No.61) to ${A_2}^{\oplus 4}$ (No.68))

According to the table in Theorem 1.1, the following nine sections on the rational elliptic surface type of No.61 (${A_2}^{\oplus 3} \oplus A_1$) are disjoint to each other :

$$\begin{array}{rrr} O = [0,0,0,0] & P = [0,2,1,1] & 2P = [0,1,2,0] \\ Q = [1,1,1,0] & P+Q = [1,0,2,1] & 2P+Q = [1,2,0,0] \\ 2Q = [2,2,2,0] & P+2Q = [2,1,0,1] & 2P+2Q = [2,0,1,0], \end{array}$$

where P and Q generate $E(K)$ and Q is a torsion of order 3. We obtain a canonical pencil of cubic curves by contracting these nine sections. In this section, we find its equation, and show that it specializes to *Hesse pencil*.

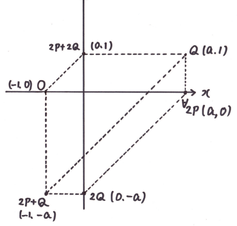
The pencil has four reducible members. Each of three members are composed of three lines (corresponding to ${A_2}^{\oplus 3}$), and the rest is composed of a conic and a line (corresponding to A_1). We call such pencils of cubic curves *semi-Hesse pencils*.

Proposition 3.3. For a suitable choice of coordinate on $\mathbb{P}^2$, the equations of a semi-Hesse pencil is

$$xy\left\{x - y - (a-1)\right\} + t(x+1)(y-1)(x-y-a) = 0.$$

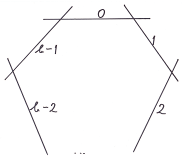
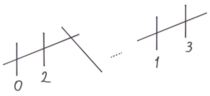
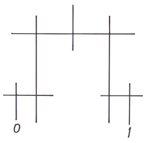
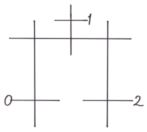
In fact, when $t = 0,\ -a/(a+1),\ \infty$, it is composed of three lines, and when $t = -1$, it is composed of a conic and the line at infinity.

Proof. Let $mP + nQ \in \mathbb{P}^2$ be the base point in $\mathbb{P}^2$ corresponding to the section $(mP + mQ)$. Three points $P, P+Q, P+2Q$ lie on the same line. In fact the fourth coordinate of these three points are all 1, but that of the other six points are all 0. Then we introduce coordinate on $\mathbb{P}^2$ so that this line is $z = 0$, $P = (1:0:0)$, $P+Q = (1:1:0)$ and $P+2Q = (0:1:0)$. For a suitable choice of affine coordinate, the remaining six points are as follows. □



When a is an imaginary cube root ω of unity , the ellipse splits into two lines. Therefore the limit of the semi-Hesse pencil is as $a \to \omega$ Hesse pencil.

AppendixA Table

Kodaira type	label	$contr_v(P)$	group structure of $\left(f^{-1}(v)\right)^\sharp$
I_b		$\dfrac{i(b-i)}{b}$	$\mathbb{G}_m \times \mathbb{Z}/b\mathbb{Z}$
I_b^*		$1 \ (i=2)$ $\frac{b+4}{4} \ (i=1,3)$	$\mathbb{G}_a \times \mathbb{Z}/4\mathbb{Z} \ \ (b: \text{odd})$ $\mathbb{G}_a \times (\mathbb{Z}/2\mathbb{Z})^2 \ (b: \text{even})$
III^*		$\dfrac{3}{2}$	$\mathbb{G}_a \times \mathbb{Z}/2\mathbb{Z}$
IV^*		$\dfrac{4}{3}$	$\mathbb{G}_a \times \mathbb{Z}/3\mathbb{Z}$

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References

[1] Bosma, W., Cannon, J., and Playoust, C.; The Magma algebra system. I. The user language, J. Symbolic Comput., **24** (1997), 235—265.

[2] Kurumadani, Y.; Configurations of rational curves on rational elliptic surfaces (in Japanese), Master thesis, Kyoto Univ., March 2014.

[3] Lang, S. and Tate, J.; Principal homogeneous spaces over abelian varieties, Amer. J. Math., **80** (1958), 659-684.

[4] Naruki, I.; Configurations related to maximal rational elliptic surfaces, Adv. Studies in Pure Math., **8** (1986), 314-340.

[5] Oguiso, K. and Shioda, T.; The Mordell-Weil lattice of a rational elliptic surface, Comment. Math. Univ. St. Pauli., **40** (1991), 83-99.

[6] Shioda, T.; On the Mordell-Weil lattices, Comment. Math. Univ. St. Pauli., **39** (1990), 211-240.

[7] Shioda, T.; Existence of rational elliptic surface with a given Mordell-Weil lattice, Proc. Japan Acad., **68** (1992), 251-255.